QUASITRIANGULAR COIDEAL SUBALGEBRAS OF $U_q(\mathfrak{g})$ IN TERMS OF GENERALIZED SATAKE DIAGRAMS

VIDAS REGELSKIS AND BART VLAAR

Abstract. Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra and θ an involutive automorphism of $\mathfrak{g}$. It is well-known from works of Letzter, Kolb and Balagović that the fixed-point subalgebra $\mathfrak{k} = \mathfrak{g}^\theta$ has a quantum counterpart B, a coideal subalgebra of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ possessing a cylinder-twisted universal K-matrix $\mathcal{K}$. The objects θ, $\mathfrak{k}$, B and $\mathcal{K}$ can all be described in terms of a combinatorial datum, a Satake diagram. In the present work we extend this construction to generalized Satake diagrams, objects first considered by Heck. A generalized Satake diagram defines a semisimple automorphism of $\mathfrak{g}$ restricting to the standard Cartan subalgebra $\mathfrak{h}$ as an involution. We show that it naturally leads to a subalgebra $\mathfrak{k} \subset \mathfrak{g}$, not necessarily a fixed-point subalgebra, but still satisfying $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta$. Such a subalgebra $\mathfrak{k}$ can be quantized to a coideal subalgebra of $U_q(\mathfrak{g})$ endowed with a cylinder-twisted universal K-matrix. We conjecture that all such coideal subalgebras of $U_q(\mathfrak{g})$ arise from generalized Satake diagrams in this way.

Contents

1. Introduction

Given a finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$ and an involutive Lie algebra automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$, a *symmetric pair* is a pair $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{k} = \mathfrak{g}^\theta$ is the θ-fixed subalgebra of $\mathfrak{g}$, see [Ara62, Sat71]. *Quantum symmetric pairs* are their quantum analogons. That is to say, the enveloping algebra $U(\mathfrak{g})$ can be quantized to a quasitriangular Hopf algebra, the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ endowed with the universal R-matrix $\mathcal{R}$, see [Ji85, Dr87]. Similarly, the θ-fixed subalgebra $\mathfrak{k}$ can be quantized to a coideal subalgebra $B \subseteq U_q(\mathfrak{g})$ [Let99, Let02, Ko14] having a compatible quasitriangular structure, the cylinder-twisted universal K-matrix $\mathcal{K}$ [BK16, Ko17].

The involution θ, the corresponding fixed-point subalgebra $\mathfrak{k}$, the coideal subalgebra B and the universal object $\mathcal{K}$ are all defined in terms of a combinatorial data, the so-called Satake diagram (X, τ). Here X is a subdiagram of the Dynkin diagram of $\mathfrak{g}$ and τ is an involutive diagram automorphism stabilizing X and satisfying certain compatibility conditions, see [Let02, Ko14].

It is the aim of this paper to extend some of the above work to a more general setting than (quantizations of) fixed-point subalgebras. A direct motivation for this is the fact that the correct quantum group analogue of the fixed-point subalgebra in the Letzter-Kolb approach is not a fixed-point subalgebra itself, but merely tends to one as $q \to 1$, see [Ko14, Ch. 10]. This suggests that there may be a generalization of this approach that does not require a fixed-point subalgebra as input.

2010 *Mathematics Subject Classification.* Primary 81R10, 81R12; Secondary 16T05, 16T25.

A careful analysis of [Ko14, BK15, BK16] indeed indicates that the compatibility conditions for X and τ can be weakened, leading to the notion of a *generalized Satake diagram*, see Definition 2.2, and the whole theory survives in this setting with very minor adjustments. The resulting Lie subalgebra $\mathfrak{k} = \mathfrak{k}(X,\tau)$ is given in Definition 3.1 and the corresponding coideal subalgebra $B = B(X,\tau)$ in Definition 4.1. Indeed, in [BK15, Rmks. 2.6, 3.14] it is explicitly suggested that some key passages of the theory are amenable for generalizations.

Our proposed generalization of Satake diagrams can be traced back to the work of A. Heck [He84]. In this work Heck provides a classification of involutions of finite root systems such that the corresponding restricted Weyl group is the Weyl group of the restricted root system. We will review this point-of-view and make a connection with a theorem of Lusztig stating that the restricted Weyl group is in fact a Coxeter group.

The characterization in terms of the restricted Weyl group is relevant in the context of the universal R- and K-matrices for quantum symmetric pairs. The universal R-matrix $\mathcal{R}$ has a distinguished factor called quasi R-matrix playing an important role in the theory of canonical bases for $U_q(\mathfrak{g})$ developed by Kashiwara and Lusztig, see [Ka90] and [Lu94, Part IV]. This object possesses a remarkable factorization property expressed in terms of the braid group action on $U_q(\mathfrak{g})$ of the Weyl group associated to $\mathfrak{g}$, see e.g. [KR90, LS90]. Recently it has become clear that many of these properties extend to the cylinder-twisted universal K-matrix $\mathcal{K}$. It has a distinguished factor called quasi K-matrix introduced in [BW13] for certain coideal subalgebras of $U_q(\mathfrak{sl}_N)$ and in a more general setting in [BK15], and featuring prominently in the theory of canonical bases for quantum symmetric pairs [BW16]. In [DK18] a factorization property is established for the quasi K-matrix using a braid group action of the aforementioned restricted Weyl group. In the present work we argue that the factorization property extends to quasi K-matrices defined in terms of the generalized Satake diagrams.

A generalization of this approach to the Kac-Moody setting will be addressed in a future work. Another outstanding issue is a Lie-theoretic motivation of the subalgebra $\mathfrak{k}$, which we define in a rather *ad hoc* manner directly in terms of the combinatorial data (X,τ), see Definition 3.1.

Therefore let us end the introduction with an additional motivation for the study of the subalgebra $\mathfrak{k}$ and its quantization B by making some observations related to the representation theory of the pair $(U_q(\mathfrak{g}), B)$. Following [BK16, Ko17], there exists a suitable completion $\mathcal{U}$ of $U_q(\mathfrak{g})$ such that the objects $\mathcal{R} \in (\mathcal{U}\otimes\mathcal{U})^\times$ and $\mathcal{K} \in \mathcal{U}^\times$ have well-defined images under any finite-dimensional representation $\rho : U_q(\mathfrak{g}) \to \mathrm{End}(V)$. Furthermore, there exists an involutive Hopf algebra automorphism ϕ of $\mathcal{U}$ such that $(\phi\otimes\phi)(\mathcal{R}) = \mathcal{R}$ and the following quartic relation is satisfied, known as the *(universal) ϕ-twisted reflection equation* (see [Ko17, Eqs. (3.22-3.23)]):

$$\mathcal{R}_{21}\,\mathcal{K}_2\,(\phi\otimes\mathrm{id})(\mathcal{R})\,\mathcal{K}_1 = \mathcal{K}_1\,(\phi\otimes\mathrm{id})(\mathcal{R}_{21})\,\mathcal{K}_2\,\mathcal{R} \qquad \in \mathcal{U}\otimes\mathcal{U} \tag{1.1}$$

where $\mathcal{K}_1 = \mathcal{K}\otimes 1$, $\mathcal{K}_2 = 1\otimes\mathcal{K}$, $\mathcal{R}_{21} = \sigma(\mathcal{R})$ and $\sigma \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{U}\otimes\mathcal{U})$ is the flip map. Let $R \in \mathrm{GL}(V\otimes V)$ be proportional to $(\rho\otimes\rho)(\mathcal{R})$ and $K \in \mathrm{GL}(V)$ proportional to $\rho(\mathcal{K})$. In the case $\phi = \mathrm{id}$, applying $\rho\otimes\rho$ to (1.1) one obtains the matrix reflection equation

$$R_{21}\,K_2\,R\,K_1 = K_1\,R_{21}\,K_2\,R \qquad \in \mathrm{End}(V\otimes V) \tag{1.2}$$

where $K_1 = K\otimes\mathrm{Id}$, $K_2 = \mathrm{Id}\otimes K$ and $R_{21} = PRP$ with $P : V\otimes V \to V\otimes V$ the permutation operator. When $\phi \neq \mathrm{id}$ one naturally obtains the so-called twisted matrix reflection equation which we omit for simplicity, but this does not significantly affect any of the following remarks. In particular, starting with a Satake diagram, one will recover the solutions of (1.2) used in [NDS95, NS95] to define quantum symmetric pairs.

Treating the matrix R as given, one can of course solve (1.2) for $K \in \mathrm{GL}(V)$. For $U_q(\mathfrak{sl}_N)$ and $V = \mathbb{C}^N$ this was done by A. Mudrov [Mu02]. Based on this result and computations for $U_q(\mathfrak{g})$ with $\mathfrak{g}$ of types B_n, C_n, D_n ($n \le 4$) and G_2, and V the vector representation, we formulate the following conjecture.

Conjecture 1.1. *Let $\rho : U_q(\mathfrak{g}) \to \mathrm{End}(V)$ be the vector representation of $U_q(\mathfrak{g})$. If $K \in \mathrm{GL}(V)$ is a solution of (1.2) then there exists a generalized Satake diagram (X,τ) such that K is proportional to $\rho(\mathcal{K})$ where $\mathcal{K}$ is the universal K-matrix for the coideal subalgebra $B(X,\tau)$, i.e. the quantization of $U(\mathfrak{k}(X,\tau))$.*

Based on the available evidence in terms of solutions to (1.2) known to intertwine restrictions of ρ to coideal subalgebras, we also make the following claim.

Conjecture 1.2. *Let $\rho : U_q(\mathfrak{g}) \to \mathrm{End}(V)$ be the vector representation of $U_q(\mathfrak{g})$. Then ρ can be used to identify coideal subalgebras, i.e. if the distinct coideal subalgebras $B, B' \subseteq U_q(\mathfrak{g})$ possess the universal K-matrices $\mathcal{K}$ and $\mathcal{K}'$, respectively, then $\rho(\mathcal{K})$ and $\rho(\mathcal{K}')$ are not scalar multiples of each other.*

If these two conjectures are true, the only coideal subalgebras of $U_q(\mathfrak{g})$ which possess a universal K-matrix in the sense of [Ko17] are those which are quantizations of $U(\mathfrak{k}(X,\tau))$ with (X,τ) a generalized Satake diagram.

We should remark that coideal subalgebras B in the Letzter-Kolb approach carry additional parameters. The generators associated to the nodes $i \in I \backslash X$ depend on scalars $\gamma_i \neq 0$ and s_i, see Definition 4.1. We can thus sharpen Conjecture 1.1. Any invertible matrix solution K of (1.2) is proportional to $\rho(\mathcal{K})$ for some $B(X,\tau)$ with the additional parameters satisfying certain constraints. Most of these constraints were found in [Let03, Ko14] given in terms of the sets Γ_q and $\mathcal{S}_q$, see (4.3). Always, we must have $(\gamma_i)_{i \in I\backslash X} \in \Gamma_q$. For the conditions on s_i it is helpful to consider the set $I_{\mathrm{ns}} = \{i \in I \backslash X \,|\, i \text{ does not neighbour } X,\, \tau(i) = i\}$, see (3.16). The constraints on the s_i are as follows. If $i \notin I_{\mathrm{ns}}$ then $s_i = 0$. For all $(i,j) \in I_{\mathrm{ns}} \times I_{\mathrm{ns}}$ such that $i \neq j$ conjecturally one of three conditions must hold: the Cartan integer a_{ij} is even, $s_j = 0$, or s_i^2/γ_i lies in a particular finite subset of a quadratic completion of $\mathbb{C}(q)$. The defining condition of the set $\mathcal{S}_q$ does not cover the third possibility, which appeared in [BB10].

The paper is organized as follows. Section 2 contains the preliminaries and basic definitions. We define the necessary Lie-theoretic objects surrounding a finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$. We introduce the notion of a generalized Satake diagram as a decoration of the Dynkin diagram of $\mathfrak{g}$. We explain how the generalized Satake diagrams emerge in the work of A. Heck.

In Section 3 we define the main object of this paper, the subalgebra $\mathfrak{k} = \mathfrak{k}(X,\tau) \subseteq \mathfrak{g}$. Theorem 3.2 is the main result of this section. We show that $\mathfrak{k}$ satisfies the intersection condition $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta$ (which trivially holds when $\mathfrak{k} = \mathfrak{g}^\theta$ with $\theta^2 = \mathrm{id}_\mathfrak{g}$) precisely if (X,τ) is a generalized Satake diagram. We then study the derived subalgebra of $\mathfrak{k}$. When $\mathfrak{k}$ is not a reductive Lie algebra, Propositions 3.5 and 3.6 establish a semidirect product decomposition for $\mathfrak{k}$ in terms of a reductive subalgebra and a nilpotent ideal of class 2. We end this section with some results about the universal enveloping algebra $U(\mathfrak{k})$. (Appendix A contains three technical lemmas in aid of Section 3.)

In Section 4 we briefly review the quasitriangular structure behind the quantum symmetric pairs. We indicate the necessary modifications to the theory of Balagović-Kolb so that it would be applicable to the quantum pair algebras associated to the generalized Satake diagrams.

Acknowledgments. The authors are indebted to Stefan Kolb and Martina Balagović for helpful discussions. V.R. was supported in part by the European Social Fund, grant number 09.3.3-LMT-K-712-02-0017. B.V. was supported by the Engineering and Physical Sciences Research Council (EPSRC), grant numbers EP/N023919/1 and EP/R009465/1. The authors gratefully acknowledge the financial support.

2. Finite-dimensional semisimple Lie algebras and root system involutions

Let I be a finite set and $A = (a_{ij})_{i,j \in I}$ a Cartan matrix. In particular, there exist positive rationals d_i ($i \in I$) such that $d_i a_{ij} = d_j a_{ji}$. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding finite-dimensional semisimple Lie algebra over $\mathbb{C}$. More precisely, $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i\}_{i \in I}$ subject to

$$[h_i, h_j] = 0, \qquad [h_i, e_j] = a_{ij} e_j, \qquad [h_i, f_j] = -a_{ij} f_j, \qquad [e_i, f_j] = \delta_{ij} h_i \tag{2.1}$$

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \qquad \text{if } i \neq j, \tag{2.2}$$

for all $i, j \in I$. We denote the standard Cartan subalgebra by $\mathfrak{h} = \langle h_i \,|\, i \in I \rangle$ and also consider the corresponding nilpotent subalgebras $\mathfrak{n}^+ = \langle e_i \,|\, i \in I \rangle$, $\mathfrak{n}^- = \langle f_i \,|\, i \in I \rangle$.

The simple roots $\alpha_i \in \mathfrak{h}^*$ ($i \in I$) satisfy $\alpha_j(h_i) = a_{ij}$ for $i, j \in I$. Let $Q = \sum_{i \in I} \mathbb{Z}\alpha_i$ denote the root lattice. In terms of the root spaces $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \forall h \in \mathfrak{h}, [h,x] = \alpha(h)x\}$ ($\alpha \in Q$), $\mathfrak{g}$ is a Q-graded Lie algebra and we have the following identities for $\mathfrak{h}$-modules:

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \qquad \mathfrak{n}^\pm = \bigoplus_{\alpha \in Q^+} \mathfrak{g}_{\pm\alpha}, \qquad \mathfrak{h} = \mathfrak{g}_0. \tag{2.3}$$

Hence the root system $\Phi := \{\alpha \in Q \,|\, \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0\}$ satisfies $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^\pm = \pm(\Phi \cap Q^+)$ and $Q^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$.

The Weyl group W is a finite subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the simple reflections r_i ($i \in I$) acting via $r_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$ for all $i \in I$, $\alpha \in \mathfrak{h}^*$. More precisely, W is a normal subgroup of

$$\mathrm{Aut}(\Phi) := \{g \in \mathrm{GL}(\mathfrak{h}^*) \,|\, g(\Phi) = \Phi\}.$$

Since W induces a simple transitive action on the set of bases of Φ, one readily obtains that $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(A)$, where

$$\mathrm{Aut}(A) = \{\sigma : I \to I \text{ invertible} \,|\, a_{\sigma(i)\,\sigma(j)} = a_{ij} \text{ for all } i, j \in I\}$$

is the group of diagram automorphisms (acting by relabelling).

The following subgroup of $\mathrm{Aut}(\mathfrak{g})$ will be important in what follows:

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) = \{\sigma \in \mathrm{Aut}(\mathfrak{g}) \,|\, \sigma(\mathfrak{h}) = \mathfrak{h}\} < \mathrm{Aut}(\mathfrak{g}).$$

We briefly review some important subgroups of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$. A braid group action on $\mathfrak{g}$ which extends the dual action of W on $\mathfrak{h}$ is defined by $\mathrm{Ad}(r_i) = \exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-f_i))\exp(\mathrm{ad}(e_i)) \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ for $i \in I$, yielding $\mathrm{Ad}(W) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$. We also have $\mathrm{Aut}(A) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ (acting by relabelling). The Chevalley involution $\omega \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ is defined by swapping e_i and $-f_i$ for all $i \in I$; it commutes with $\mathrm{Ad}(W)$ and with $\mathrm{Aut}(A)$. Finally, the group $\widetilde{H} := \mathrm{Hom}(Q, \mathbb{C}^\times)$ naturally induces a subgroup $\mathrm{Ad}(\widetilde{H}) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ via $\mathrm{Ad}(\chi)|_{\mathfrak{g}_\alpha} = \chi(\alpha)\,\mathrm{id}_{\mathfrak{g}_\alpha}$ for all $\chi \in \widetilde{H}$, $\alpha \in Q$.

The elements of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ can be dualized to elements of $\mathrm{Aut}(\Phi)$. Conversely, given $g \in \mathrm{Aut}(\Phi)$ there are $\psi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ whose restriction to $\mathfrak{h}$ dualizes to g. Indeed, from $-\mathrm{id}_{\mathfrak{h}^*} \in \mathrm{Aut}(\Phi)$ and the direct product decomposition $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(A)$, there exist unique $(w, \tau) \in W \times \mathrm{Aut}(A)$ such that $g = -w\tau$. Then one easily checks that $\psi = \mathrm{Ad}(w)\omega\tau \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ satisfies $(\psi|_{\mathfrak{h}})^* = g$.

2.1. **Compatible decorations and involutions of Φ.** Given a subset $X \subseteq I$ denote the corresponding Cartan submatrix by $A_X = (a_{ij})_{i,j\in X}$ and consider the corresponding semisimple Lie algebra $\mathfrak{g}_X := \langle e_i, f_i, h_i \,|\, i \in X\rangle \subseteq \mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}_X = \mathfrak{h} \cap \mathfrak{g}_X$, dual Weyl vector $\rho^\vee_X \in \mathfrak{h}_X$ and Weyl group $W_X := \langle r_i \,|\, i \in X\rangle \leq W$. The unique longest element $w_X \in W_X$ is an involution and there exists $\tau_{0,X} \in \mathrm{Aut}(A_X)$ which satisfies

$$-\, w_X(\alpha_i) = \alpha_{\tau_{0,X}(i)} \qquad \text{for all } i \in X. \tag{2.4}$$

Note that $\mathrm{Ad}(w_X)|_{\mathfrak{g}_X} = \tau_{0,X}\,\omega|_{\mathfrak{g}_X}$ and $\mathrm{Ad}(w_X)^2|_{\mathfrak{g}_\alpha} = \zeta(\alpha)\,\mathrm{id}_{\mathfrak{g}_\alpha}$ for all $\alpha \in \Phi$, where $\zeta = \zeta(X) \in \widetilde{H}$ is defined by

$$\zeta(\alpha_i) := (-1)^{2\alpha_i(\rho^\vee_X)} \qquad \text{for } i \in I.$$

We will study

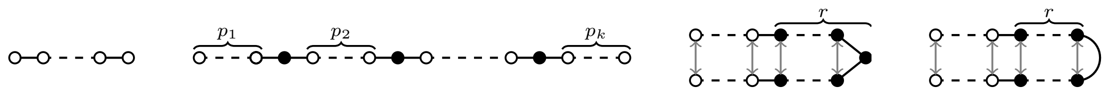
$$\mathrm{Aut}^{\mathrm{inv}}(\mathfrak{g}, \mathfrak{h}) := \{\psi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) \,|\, \psi^2|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\},$$
$$\mathrm{Aut}^{\mathrm{inv}}(\Phi) := \{g \in \mathrm{Aut}(\Phi) \,|\, g^2 = \mathrm{id}_{\mathfrak{h}^*}\}$$

by means of combinatorial data: we define

$$\mathrm{CDec}(A) = \{(X, \tau) \,|\, X \subseteq I, \tau \in \mathrm{Aut}(A), \tau^2 = \mathrm{id}_I, \tau(X) = X, \tau|_X = \tau_{0,X}\} \tag{2.5}$$

and call its elements *compatible decorations* (of A). In the Dynkin diagram associated to $\mathfrak{g}$ one marks this decoration by filling the nodes corresponding to X and drawing two-sided arrows for the nontrivial orbits of τ.

Example 2.1. Let A be of type A_n, $n \geq 2$. The compatible decorations $\mathrm{CDec}(A)$ are



where $p_1, p_k \in \mathbb{Z}_{\geq 0}$, $p_2, \ldots, p_{k-1} \in \mathbb{Z}_{\geq 1}$ for any $k \in \mathbb{Z}_{\geq 2}$ and $0 \leq r \leq \lceil n/2 \rceil$.

Given $(X,\tau) \in \mathrm{CDec}(A)$, we define

$$\theta = \theta(X,\tau) = -w_X \tau \in \mathrm{Aut}^{\mathrm{inv}}(\Phi). \tag{2.6}$$

As explained above, the map dual to θ can be extended to an element of $\mathrm{Aut}^{\mathrm{inv}}(\mathfrak{g},\mathfrak{h})$ which we shall also call θ. It is given by $\theta = \mathrm{Ad}(w_X)\tau\omega$ so that $\theta|_{\mathfrak{h}} = -w_X\tau$. Note that, as a consequence of properties of $\mathrm{Ad}(w_X)$ mentioned earlier, we have

$$\theta|_{\mathfrak{g}_X} = \mathrm{id}_{\mathfrak{g}_X}, \tag{2.7}$$

$$\theta^2|_{\mathfrak{g}_\alpha} = \zeta(\alpha)\,\mathrm{id}_{\mathfrak{g}_\alpha} \qquad \text{for all } \alpha \in \Phi. \tag{2.8}$$

2.2. **Generalized Satake diagrams and the restricted Weyl group.** We choose a subset $I^* \subseteq I\backslash X$ such that it contains precisely one element from each τ-orbit in $I\backslash X$. For $i \in I^*$ denote by $\check{X}(i) \subseteq X$ the union of connected components of X neighbouring $\{i,\tau(i)\}$ and $\check{X}[i] := \check{X}(i) \cup \{i,\tau(i)\}$. By a *minimal subdiagram* of $(X,\tau) \in \mathrm{CDec}(A)$ we mean any subdiagram of the form $\check{X}[i]$ for some $i \in I^*$. By definition $\check{X}[i]$ is a compatible decoration of $A_{\check{X}[i]}$; it is also known as a Satake diagram of (restricted) rank 1.

Definition 2.2. *Generalized Satake diagrams* are elements of the set

$$\mathrm{GSat}(A) := \{(X,\tau) \in \mathrm{CDec}(A) \,|\, (X,\tau) \text{ contains no minimal subdiagram of the form } \circ\!\!-\!\!\bullet\}.$$ ⌀

The compatible decorations in Example 2.1 are generalized Satake diagrams when $p_1 = p_k = 0$ and $p_2 = \ldots = p_{k-1} = 1$.

Remark 2.3. Generalized Satake diagrams were first considered by Heck in [He84] where they are shown to classify involutions of root systems such that the restricted Weyl group is the Weyl group of the restricted root system. Heck uses the symbol σ to denote the negative of our map θ. He also uses the term Satake diagram for any (X,τ) such that $X \subseteq I$, $\tau \in \mathrm{Aut}(A)$, $\tau^2 = \mathrm{id}_I$ and $\tau(X) = X$ (this properly contains the set $\mathrm{CDec}(A)$) and the elements of $\mathrm{GSat}(A)$ are called admissible Satake diagrams. However, the term Satake diagram has become reserved for those combinatorial data which classify involutions of $\mathfrak{g}$ up to conjugacy (and their fixed-point subalgebras), which is the reason for our nomenclature "compatible decoration" and "generalized Satake diagram". ⌀

Note that (X,τ) is a generalized Satake diagrams precisely if

$$\forall (i,j) \in I\backslash X \times X \,:\, \tau(i) = i,\, w_X(\alpha_i) = \alpha_i + \alpha_j \implies a_{ij} \neq -1, \tag{2.9}$$

which is precisely the condition needed in [Ko14, Proof of Lemma 5.11, Step 1] and [BK16, Proof of Lemma 6.4]. One can show that (2.9) is equivalent to either of the following more compact conditions:

$$\forall i,j \in I : \theta(\alpha_i) = -(\alpha_i + \alpha_j) \implies a_{ij} \neq -1,$$
$$\forall i \in I : (\theta(\alpha_i))(h_i) \neq -1.$$

Satake diagrams can be defined as the following subset of compatible decorations of A:

$$\mathrm{Sat}(A) = \{(X,\tau) \in \mathrm{CDec}(A) \,|\, \forall i \in I\backslash X : i = \tau(i) \implies \zeta(\alpha_i) = 1\}. \tag{2.10}$$

It is well-known that Satake diagrams classify involutive Lie algebra automorphisms up to conjugacy, see e.g. [Ara62]. More precisely, in the current setup, for $(X,\tau) \in \mathrm{Sat}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I^*}$ define $s_{\boldsymbol{\gamma}} \in \widetilde{H}$ by means of

$$s_{\boldsymbol{\gamma}}(\alpha_i) = \begin{cases} 1 & \text{if } i \in X, \\ \gamma_i & \text{if } i \in I^*, \\ \gamma_{\tau(i)}\zeta(\alpha_i) & \text{if } i \in (I\backslash X)\backslash I^*, \end{cases}$$

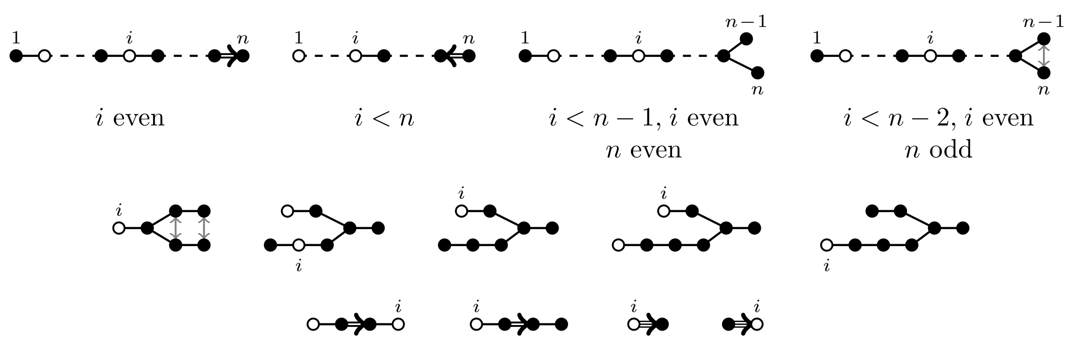
cf. [BK16, Eqs. (5.1-5.2)]. Then it follows from (2.8) that

$$\theta_{\boldsymbol{\gamma}} := \mathrm{Ad}(s_{\boldsymbol{\gamma}})\theta \tag{2.11}$$

satisfies $(\theta_{\boldsymbol{\gamma}})^2 = \mathrm{id}_{\mathfrak{g}}$.

If $(X,\tau) \in \mathrm{CDec}(A)\backslash\mathrm{GSat}(A)$ then there exists a pair $(i,j) \in I\backslash X \times X$ such that the union of connected components of X neighbouring i is simply $\{j\}$ and $a_{ji} = -1$. Hence $\rho_X^\vee = \frac{1}{2}h_j$ so that $\zeta(\alpha_i) = (-1)^{a_{ji}} = -1$ implying $(X,\tau) \notin \mathrm{Sat}(A)$. Consequently $\mathrm{Sat}(A) \subseteq \mathrm{GSat}(A)$. The complement $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ is empty if and only if A is of type A_n. We refer the reader to the classification in [He84, Table I], which does not explicitly distinguish between elements of $\mathrm{Sat}(A)$ and $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$. It is convenient for our purposes to list the elements of $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$, which we do in Table 1.

Table 1. All elements of $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ for indecomposable Cartan matrices A. By a case-by-case analysis there is a unique $i \in I$ such that $\zeta(\alpha_i) = -1$; we have indicated the corresponding node in the diagrams. The classical diagrams are labelled in the usual way. For types C_n and D_n upper bounds on i are imposed to avoid the cases when θ is an involution whose fixed-point subalgebra is isomorphic to $\mathfrak{gl}_n$.



Consider the real vector space $V = \mathbb{R}\Phi$. For a fixed $\theta \in \mathrm{Aut}^{\mathrm{inv}}(\Phi)$ we can decompose V into the positive and negative θ-eigenspaces, $V = V^\theta \oplus V^{-\theta}$. Denote by $\bar{\ } : V \to V$ the corresponding projection onto $V^{-\theta}$. The *restricted roots* are the elements of

$$\overline{\Phi} = \{\overline{\alpha} \,|\, \alpha \in \Phi\}\backslash\{0\}.$$

Given an arbitrary $\theta \in \mathrm{Aut}^{\mathrm{inv}}(\Phi)$, $\overline{\Phi}$ is not necessarily a root system in its own right. According to [He84, Thm. 6.1], $\overline{\Phi}$ is a (possibly non-reduced or empty) root system precisely if $\theta = \theta(X,\tau) = -w_X\tau$, where $(X,\tau) \in \mathrm{GSat}(A)$ or (X,τ) is the diagram ○—●.

Now consider the following groups:

$$W^\theta = \{w \in W \,|\, w = \theta w \theta\} = \{w \in W \,|\, w = w_X\tau(w)w_X\},$$
$$\overline{W} = \{w|_{V^{-\theta}} \,|\, w \in W,\, w(V^{-\theta}) \subseteq V^{-\theta}\}.$$

If $\theta = \theta(X,\tau)$ it follows straightforwardly that W_X is a subgroup of W^θ. Moreover, [He84, Prop. 3.1] implies that $\overline{W}$ is isomorphic to W^θ/W_X. For $i \in I^*$ we define $\widetilde{r}_i := w_X w_{X[i]} \in W$ where $X[i] = X \cup \{i,\tau(i)\}$ and set $s_i \in \mathrm{GL}(V^{-\theta})$ to be the unique element satisfying $s_i(\overline{\alpha}_i) = -\overline{\alpha}_i$ and $s_i(\beta) = \beta$ for all $\beta \in V^{-\theta}$ such that $\beta(h_i) = 0$. In [He84, Lemma 3.2, Thm. 3.3, Thm. 4.4] the following result is proved.

Theorem 2.4. *Let $(X,\tau) \in \mathrm{CDec}(A)$. The following conditions are equivalent:*

(i) $(X,\tau) \in \mathrm{GSat}(A)$.
(ii) For all $i \in I^*$, $s_i \in \overline{W}$.
(iii) For all $i \in I^*$, $\widetilde{r}_i$ lies in W^θ and satisfies $\widetilde{r}_i|_{V^{-\theta}} = s_i$.
(iv) For all $i \in I^*$, $\tau_{0,X[i]}$ preserves X.
(v) $\overline{W} = W(\overline{\Phi})$.

In [Lu76, 5.9 (i)] it is shown that $(\widetilde{W}, \{\widetilde{r}_i\}_{i\in I^*})$ with $\widetilde{W} = \langle \widetilde{r}_i \rangle_{i \in I^*}$ is a Coxeter system if condition (iv) in Theorem 2.4 holds (also see [Lu02, 25.1]). If condition (iv) fails then for some $i \in I^*$, $w_{X[i]}$ and w_X do not commute so that $\widetilde{r}_i^2 \neq \mathrm{id}_V$. Hence we obtain the following result.

Corollary 2.5. *Let $(X,\tau) \in \mathrm{CDec}(A)$. Then $(\widetilde{W}, \{\widetilde{r}_i\}_{i\in I^*})$ is a Coxeter system if and only if $(X,\tau) \in \mathrm{GSat}(A)$.*

3. The subalgebra $\mathfrak{k}$

For $(X,\tau) \in \mathrm{Sat}(A)$ and a suitable choice of $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I^*}$ the $\theta_{\boldsymbol{\gamma}}$-fixed subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ can be presented in terms of generators, see e.g. [Ko14, Lemma 2.8]. This motivates the following seemingly *ad hoc* definition, where we permit a more general $\boldsymbol{\gamma}$.

Definition 3.1. For $(X,\tau) \in \mathrm{CDec}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I\backslash X}$ define $\mathfrak{k}_{\boldsymbol{\gamma}} = \mathfrak{k}_{\boldsymbol{\gamma}}(X,\tau)$ to be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_X$, $\mathfrak{h}^\theta$ and

$$b_{i,\boldsymbol{\gamma}} = f_i + \gamma_i \theta(f_i) \qquad \text{for all } i \in I\backslash X. \tag{3.1}$$

$\varnothing$

It is convenient to suppress the dependence on $\boldsymbol{\gamma}$ and simply write b_i and $\mathfrak{k}$ if there is no cause for confusion. We denote $b_i = f_i$ if $i \in X$. Since $\mathfrak{h}_X \subseteq \mathfrak{h}^\theta$ it follows that $\mathfrak{k}$ is generated by $\mathfrak{n}_X^+ := \{e_i | i \in X\}$, $\mathfrak{h}^\theta$ and b_i for $i \in I$. Owing to (2.1-2.2), these satisfy

$$[e_i, b_j] = \delta_{ij} h_i \in \mathfrak{h}^\theta \qquad \text{for all } i \in X,\, j \in I, \tag{3.2}$$

$$[h, b_j] = -\alpha_j(h) b_j \qquad \text{for all } h \in \mathfrak{h}^\theta,\, j \in I, \tag{3.3}$$

$$[h, e_j] = \alpha_j(h) e_j \qquad \text{for all } h \in \mathfrak{h}^\theta,\, j \in X, \tag{3.4}$$

$$[h, h'] = 0 \qquad \text{for all } h, h' \in \mathfrak{h}^\theta, \tag{3.5}$$

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \qquad \text{for all } i,j \in X,\, i \neq j. \tag{3.6}$$

By setting $m = 1 - a_{ij}$ in Lemmas (A.1-A.3) one also obtains analogues of Serre relations among the generators b_i. Namely, for $i,j \in I$ such that $i \neq j$,

$$\mathrm{ad}(b_i)^{1-a_{ij}}(b_j) = \begin{cases} (1+\zeta(\alpha_i))\gamma_i [\theta(f_i),[f_i,f_j]] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i)+\alpha_i+\alpha_j \in \Phi^-,\, a_{ij}=-1, \\ -18\gamma_i^2 e_j & \text{if } \theta(\alpha_i)+\alpha_i+\alpha_j = 0,\, a_{ij}=-3, \\ -\gamma_i(2h_i+h_j) & \text{if } \theta(\alpha_i)+\alpha_i+\alpha_j = 0,\, a_{ij}=-1, \\ \big(\gamma_i + \zeta(\alpha_i)\gamma_j\big)[\theta(f_i),f_j] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i)+\alpha_j \in \Phi^-,\, a_{ij}=0, \\ \gamma_j h_i - \gamma_i h_j & \text{if } \theta(\alpha_i)+\alpha_j = 0,\, a_{ij}=0, \\ 2(\gamma_i+\gamma_j) b_i & \text{if } \theta(\alpha_i)+\alpha_j = 0,\, a_{ij}=-1, \\ -\gamma_i b_j & \text{if } \theta(\alpha_i)+\alpha_i = 0,\, j \in I\backslash X,\, a_{ij}=-1, \\ -3\gamma_i [b_i,b_j] & \text{if } \theta(\alpha_i)+\alpha_i = 0,\, j \in I\backslash X,\, a_{ij}=-2, \\ -6\gamma_i^2 b_j - 3\gamma_i [b_i,[b_i,b_j]] & \text{if } \theta(\alpha_i)+\alpha_i = 0,\, j \in I\backslash X,\, a_{ij}=-3, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

In order to state the main result of this section, we need some more notation. Consider the subsets

$$I_{\mathrm{diff}} = \{ i \in I^* \mid i \neq \tau(i) \text{ and } (\theta(\alpha_i))(h_i) \neq 0\} = \{ i \in I^* \mid i \neq \tau(i) \text{ and } \exists j \in X[i] \text{ s.t. } a_{ij} < 0\}$$

and

$$\Gamma = \Gamma(X,\tau) = \{\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I\setminus X} \,|\, \forall i \in I^* : \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\rm diff}\}.$$

For $\boldsymbol{i} \in I^\ell$ with $\ell \in \mathbb{Z}_{>0}$ we write $\alpha_{\boldsymbol{i}} = \sum_{r=1}^{\ell} \alpha_{i_r}$ and

$$b_{\boldsymbol{i}} = \mathrm{ad}(b_{i_1})\cdots \mathrm{ad}(b_{i_{\ell-1}})(b_{i_\ell}), \quad e_{\boldsymbol{i}} = \mathrm{ad}(e_{i_1})\cdots \mathrm{ad}(e_{i_{\ell-1}})(e_{i_\ell}), \quad f_{\boldsymbol{i}} = \mathrm{ad}(f_{i_1})\cdots \mathrm{ad}(f_{i_{\ell-1}})(f_{i_\ell}).$$

Observe that $\mathfrak{n}^- = \mathrm{Sp}\bigcup_{\ell>0}\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in I^\ell}$. Hence for all $\ell \in \mathbb{Z}_{>0}$ we can choose $\mathcal{J}_\ell \subseteq I^\ell$ such that $\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathcal{J}_\ell}$ is a basis for $\mathrm{Sp}\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in I^\ell}$. Then $\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathcal{J}}$ with $\mathcal{J} := \bigcup_{\ell\in\mathbb{Z}_{>0}}\mathcal{J}_\ell$ is a basis of $\mathfrak{n}^-$.

Theorem 3.2. *Let $(X,\tau) \in \mathrm{CDec}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I\setminus X}$. The following statements are equivalent:*

(i) $(X,\tau) \in \mathrm{GSat}(A)$ *and* $\boldsymbol{\gamma} \in \Gamma$.

(ii) For all $i,j \in I$ such that $i \neq j$ we have the following bounded Serre relations:

$$\mathrm{ad}(b_i)^{1-a_{ij}}(b_j) \in \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \oplus \bigoplus_{\substack{\boldsymbol{k}\in I^\ell \\ \alpha_{\boldsymbol{k}}<\lambda_{ij}}} \mathbb{C}b_{\boldsymbol{k}} \tag{3.8}$$

where $\lambda_{ij} := (1-a_{ij})\alpha_i + \alpha_j \in Q^+\setminus\Phi^+$.

(iii) We have the following identity for $\mathfrak{h}^\theta$-modules:

$$\mathfrak{k} = \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \oplus \bigoplus_{\boldsymbol{i}\in\mathcal{J}} \mathbb{C}b_{\boldsymbol{i}}. \tag{3.9}$$

(iv) We have

$$\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta. \tag{3.10}$$

Remark 3.3. In the fixed-point case $\mathfrak{k} = \mathfrak{g}^{\theta_{\boldsymbol{\gamma}}}$ (3.10) is trivially satisfied (note that $\mathfrak{h}^\theta = \mathfrak{h}^{\theta_{\boldsymbol{\gamma}}}$). ⌀

Proof of Theorem 3.2.

(i) $\iff$ (ii): This is a direct consequence of (3.7).

(ii) $\implies$ (iii): Owing to (3.3-3.5) it is sufficient to prove (3.9) as an identity for vector spaces. First we prove that $\mathfrak{k} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \mathrm{Sp}\{b_{\boldsymbol{i}} | \boldsymbol{i} \in \mathcal{J}\}$. From (3.2-3.3) it follows that, as vector spaces,

$$\mathfrak{k} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \langle b_j\rangle_{j\in I} = \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \sum_{\ell\in\mathbb{Z}_{>0}}\sum_{\boldsymbol{i}\in I^\ell} \mathbb{C}b_{\boldsymbol{i}}. \tag{3.11}$$

As a consequence of this, we see that it suffices to prove that for all $\boldsymbol{j} \in \cup_\ell I^\ell$ we have

$$b_{\boldsymbol{j}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \mathrm{Sp}\{b_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathcal{J}\}. \tag{3.12}$$

We will prove this by induction with respect to the height ℓ. Since for all $j \in I$ we have $\dim(\mathfrak{g}_{-\alpha_j}) = 1$ and hence $(j) \in \mathcal{J}$, the case $\ell = 1$ is trivial. Now fix $\ell \in \mathbb{Z}_{>1}$ and assume that (3.12) holds true for all smaller positive integers. Fix $\boldsymbol{j} \in I^\ell$ and repeatedly apply the Serre relations (2.2) to obtain that for all $\boldsymbol{i} \in \mathcal{J}_\ell$ there exist $a_{\boldsymbol{i}} \in \mathbb{C}$ such that

$$f_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathcal{J}_\ell} a_{\boldsymbol{i}} f_{\boldsymbol{i}}.$$

Hence, by virtue of (ii) and equations (3.2-3.3) it follows that

$$b_{\boldsymbol{j}} - \sum_{\boldsymbol{i}\in\mathcal{J}_\ell} a_{\boldsymbol{i}} b_{\boldsymbol{i}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \mathrm{Sp}\left\{ b_{\boldsymbol{i}} \,\middle|\, \boldsymbol{i} \in \bigcup_{m=1}^{\ell-1} I^m \right\}.$$

Using the induction hypothesis for the elements $b_{\boldsymbol{i}}$ in the last summation one obtains (3.12).

It remains to show that the sum in (3.12) is direct. Let $\boldsymbol{j} \in \mathcal{J}$. Then $f_{\boldsymbol{j}}$ is nonzero. Because of the explicit formula (3.1) we have

$$b_{\boldsymbol{j}} - f_{\boldsymbol{j}} \in \mathfrak{n}_X^+ + \mathfrak{h}^\theta + \mathbb{C}\theta(f_{\boldsymbol{j}}) + \mathrm{Sp}\{b_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathcal{J}, \alpha_{\boldsymbol{i}} < \alpha_{\boldsymbol{j}}\}. \tag{3.13}$$

Hence $f_{\boldsymbol{j}} = \pi_{-\alpha_{\boldsymbol{j}}}(b_{\boldsymbol{j}})$ for all $\boldsymbol{j} \in \mathcal{J}$, where π_α is the projection on $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$, see (2.3). Thus the linear independence of $\{f_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathcal{J}}$ together with (2.3) implies that the sum is direct.

(iii) $\implies$ *(iv):* By definition, $\mathfrak{h}^\theta \subseteq \mathfrak{k}\cap\mathfrak{h}$ so it suffices to show that $\mathfrak{k}\cap\mathfrak{h} \subseteq \mathfrak{h}^\theta$. Suppose $h \in \mathfrak{k}\cap\mathfrak{h}^\theta$. Since $\pi_{-\alpha_{\boldsymbol{j}}}(b_{\boldsymbol{j}}) = f_{\boldsymbol{j}}$ and the triangular decomposition (2.3), part (iii) implies $h \in \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \subseteq \mathfrak{g}^\theta$ so $h \in \mathfrak{h}^\theta$.

(iv) $\implies$ *(ii):* We prove the contrapositive. If (3.8) fails then (3.14) and (3.7) imply that either $\gamma_j h_i - \gamma_i h_j \in \mathfrak{k}\cap(\mathfrak{h}\backslash\mathfrak{h}^\theta)$ with $\gamma_i \neq \gamma_j$ or $2h_i + h_j \in \mathfrak{k}\cap(\mathfrak{h}\backslash\mathfrak{h}^\theta)$. In either case (3.10) fails. □

It is convenient to have an explicit description of $\mathfrak{h}^\theta$. Given $i \in I$, by applying θ to $\theta(h_i) - h_i - \theta(h_{\tau(i)}) + h_{\tau(i)} \in \mathfrak{g}_X \cap \mathfrak{h}$ one obtains $\theta(h_i - h_{\tau(i)}) = h_i - h_{\tau(i)}$. From this we straightforwardly deduce

$$\mathfrak{h}^\theta = \bigoplus_{i\in X}\mathbb{C}h_i \oplus \bigoplus_{\substack{i\in I^*\\ i\neq\tau(i)}} \mathbb{C}(h_i - h_{\tau(i)}) \tag{3.14}$$

We denote $\Phi_X = \Phi\cap Q_X$ and note that $|\mathcal{J}| = |\Phi|/2$; from (3.14) we also obtain $\dim(\mathfrak{h}^\theta) = |I| - |I^*|$. Hence, given $(X,\tau) \in \mathrm{GSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$, Theorem 3.2 (iii) implies

$$\dim(\mathfrak{k}) = |\Phi_X|/2 + |I| - |I^*| + |\Phi|/2. \tag{3.15}$$

Corollary 3.4. *Let $(X,\tau) \in \mathrm{GSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. The generating set*

$$\{h_i, e_i\}_{i\in X} \cup \{h_i - h_{\tau(i)}\}_{i\in I^*, i\neq\tau(i)} \cup \{b_i\}_{i\in I},$$

and the relations (3.2-3.6) provide a presentation of $\mathfrak{k}$.

Proof. There are no relations for the b_i other than (3.2), (3.3) and (3.7): otherwise applying $\pi_{-\alpha}$ with $\alpha \in \Phi^+$ maximal produces a relation for the f_i inequivalent to a relation (2.1), (2.2). □

3.1. **Ideal structure of $\mathfrak{k}$.** In this section we assume that A is indecomposable, so that $\mathfrak{g}$ is simple. In order to describe the derived subalgebra of $\mathfrak{k}$ recall the set $I_{\mathrm{diff}} \in I^*$ and define

$$\begin{aligned} I_{\mathrm{ns}} &= \{i \in I \,|\, (\theta(\alpha_i))(h_i) = -2\} = \{i \in I \,|\, i = \tau(i),\, \check{X}(i) = \emptyset\}, \\ I_{\mathrm{nsf}} &= \{j \in I_{\mathrm{ns}} \,|\, \forall i \in I_{\mathrm{ns}}\; a_{ij} \in 2\mathbb{Z}\}. \end{aligned} \tag{3.16}$$

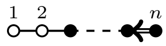
Proposition 3.5. *Let $(X,\tau) \in \mathrm{GSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. As vector spaces we have*

$$\mathfrak{k} = \mathfrak{k}' \oplus \bigoplus_{i\in I_{\mathrm{diff}}} \mathbb{C}(h_i - h_{\tau(i)}) \oplus \bigoplus_{i\in I_{\mathrm{nsf}}} \mathbb{C}b_i.$$

Proof. Fix $(X,\tau) \in \mathrm{GSat}(A)$. Note that neither $h_i - h_{\tau(i)}$ $(i \in I_{\mathrm{diff}})$ nor b_j $(j \in I_{\mathrm{nsf}})$ is a linear combination of Lie brackets in $\mathfrak{k}$. This follows from Corollary 3.4 and (3.2-3.7): these elements do not appear as in the expressions for Lie brackets in the defining relations of $\mathfrak{k}$.

It now suffices to show that the remaining basis elements specified in (3.9) are linear combinations of Lie brackets in $\mathfrak{k}$, for which we argue as follows.

$b_{\boldsymbol{i}}$ *with $\boldsymbol{i} \in \mathcal{J}_\ell$, $\ell > 1$:* This holds by definition.
e_i, f_i, h_i *with $i \in X$:* This follows from (3.2-3.4).
$h_i - h_{\tau(i)}$ *with $i \in I^*\backslash I_{\mathrm{diff}}$ and $i \neq \tau(i)$:* The given condition is equivalent to $w_X(\alpha_i) = \alpha_i$ and $a_{i\,\tau(i)} = 0$. Hence (3.7) implies that $h_i - h_{\tau(i)} = \gamma_i^{-1}[b_i, b_{\tau(i)}]$.
b_j *with $\check{X}(j) \neq \emptyset$:* There exists $i \in X$ such that $a_{ij} \neq 0$. By (3.3) we have $b_j = -a_{ij}^{-1}[h_i, b_j]$.
b_j *with $j \neq \tau(j)$:* Note that $a_{\tau(j)\,j} \leq 0$. By (3.3) we have $b_j = (a_{\tau(j)\,j} - 2)^{-1}[h_j - h_{\tau(j)}, b_j]$.
b_j *with $j \in I_{\mathrm{ns}}\backslash I_{\mathrm{nsf}}$:* By definition of I_{nsf} there exists $i \in I_{\mathrm{ns}}$ such that $a_{ij} \in \{-1,-3\}$. According to (3.7), $b_j = -\gamma_i^{-1}\mathrm{ad}(b_i)^2(b_j)$ if $a_{ij} = -1$ and $b_j = -(2\gamma_i)^{-1}\mathrm{ad}(b_i)^2(b_j) - (6\gamma_i^2)^{-1}\mathrm{ad}(b_i)^4(b_j)$ if $a_{ij} = -3$; in either case $b_j \in \mathfrak{k}'$. □

It follows that the codimension of $\mathfrak{k}'$ in $\mathfrak{k}$ equals $|I_{\rm diff}| + |I_{\rm nsf}|$. For $(X,\tau) \in {\rm Sat}(A)$, in [Let02, Sec. 7, Variation 1] it was noted that $|I_{\rm diff}| \leq 1$ if A is of finite type. In light of the above it is natural to generalize this in two directions: also involve the set $I_{\rm nsf}$ and allow $(X,\tau) \in {\rm GSat}(A)$. It turns out the same upper bound holds true and there are generalized Satake diagrams with $|I_{\rm diff}| + |I_{\rm nsf}| = 1$ unless A is of type E_8, F_4 or G_2. From Table 1 it follows that the only elements of ${\rm GSat}(A)\backslash{\rm Sat}(A)$ for which $|I_{\rm diff}| + |I_{\rm nsf}| = 1$ are of the form  with $n > 2$ in which case $I_{\rm nsf} = \{1\}$ and $\zeta(\alpha_2) = -1$.

For the reasons that will become clear a bit later we introduce a further refinement of generalized Satake diagrams. In particular, we define the set of *weak Satake diagrams* by

${\rm WSat}(A) = \{(X,\tau) \in {\rm GSat}(A)\backslash{\rm Sat}(A) \mid (X,\tau)$ contains no minimal subdiagram of the form $\}$.

As mentioned in Table 1, for elements of ${\rm GSat}(A)\backslash{\rm Sat}(A)$ a case-by-case analysis yields that there can be at most one $i \in I\backslash X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. For $(X,\tau) \in {\rm WSat}(A)$ we will obtain a semidirect product decomposition in terms of a reductive Lie subalgebra and a nilpotent ideal in which this unique $i \in i\backslash X$ plays an important role.

For any $r \in \mathbb{Z}_{\geq 0}$ and any $i \in I$ denote by $\mathfrak{k}(i)_r$ the span of all $b_{\boldsymbol{j}}$ such that the coefficient of α_i in $\alpha_{\boldsymbol{j}}$ is precisely r. We then have the following decomposition

$$\langle b_i \rangle_{i \in I} = \bigoplus_{r=0}^{\infty} \mathfrak{k}(i)_r.$$

Consider the subspace

$$\mathfrak{k}(i) := \bigoplus_{r=1}^{\infty} \mathfrak{k}(i)_r$$

and the subalgebras

$$\mathfrak{k}_{\hat{i}} := \langle \mathfrak{n}_X^+, \mathfrak{h}^\theta, \{b_j\}_{j \in I\backslash\{i\}} \rangle \subseteq \mathfrak{k}, \qquad \mathfrak{g}_{\hat{i}} := \langle \{e_j, f_j, h_j\}_{j \in I\backslash\{i\}} \rangle \subset \mathfrak{g}.$$

Note that $\mathfrak{k} = \mathfrak{k}_{\hat{i}} + \mathfrak{k}(i)$ (not necessarily a direct sum, since e.g. b_i may lie in $\mathfrak{k}_{\hat{i}}$).

Proposition 3.6. *Let $(X,\tau) \in {\rm WSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. Denote by i the unique element of $I\backslash X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. Then $\mathfrak{k}(i)_r = \{0\}$ if $r > 2$ and we have the lower central series*

$$\mathfrak{k}(i) = \mathfrak{k}(i)_1 \oplus \mathfrak{k}(i)_2 \supset \mathfrak{k}(i)_2 \supset \{0\}$$

so that $\mathfrak{k}(i)$ is nilpotent of class 2. Furthermore, both $\mathfrak{k}(i)_1$ and $\mathfrak{k}(i)_2$ are $\mathfrak{k}_{\hat{i}}$-modules under the adjoint action, $\mathfrak{k}(i)$ is an ideal of $\mathfrak{k}$, $\mathfrak{k}_{\hat{i}}$ is the fixed-point subalgebra of $\theta|_{\mathfrak{g}_{\hat{i}}}$ and we have $\mathfrak{k} = \mathfrak{k}_{\hat{i}} \ltimes \mathfrak{k}(i)$.

Proof. Note that (3.7) implies, for all $j \in I\backslash\{i\}$, that

$$\mathrm{ad}(b_i)^{1-a_{ij}}(b_j) = 0 \tag{3.17}$$

$$\mathrm{ad}(b_j)^{1-a_{ji}}(b_i) \in \sum_{r=1}^{-a_{ij}} \mathbb{F}\mathrm{ad}(b_j)^r(b_i) \subseteq \mathfrak{k}(i)_1. \tag{3.18}$$

Since (3.3) and (3.18) are the only relations in $\mathfrak{k}$ with b_i appearing on the right-hand side, it follows that $\mathfrak{k}_{\hat{i}} = \langle \mathfrak{n}_+^X, \mathfrak{h}^\theta, \mathfrak{k}(i)_0 \rangle$ and $\mathfrak{k} = \mathfrak{k}_{\hat{i}} \oplus \mathfrak{k}(i)$ (as vector spaces). Deleting the node i from any diagram in Table 1 one obtains a (possibly disconnected) Satake diagram such that $\theta|_{\mathfrak{g}_{\hat{i}}}$ by virtue of (2.8) is an involution. From Table 1 it also follows that $I^* = I\backslash X$ so that $\mathfrak{k}_{\hat{i}}$ is the fixed-point subalgebra of $\mathfrak{g}_{\hat{i}}$ for the involution $\theta_{\boldsymbol{\gamma}}$, see (2.11).

Combined with (3.2-3.3), (3.18) implies that each summand $\mathfrak{k}(i)_r$ is a $\mathfrak{k}_{\hat{i}}$-module. Hence $\mathfrak{k}(i)$ is a $\mathfrak{k}_{\hat{i}}$-module and by virtue of (3.17) it is a subalgebra of $\mathfrak{k}$. It follows that $\mathfrak{k}(i)$ is an ideal. Automatically we have that $[\oplus_{r=1}^s \mathfrak{k}(i)_r, \mathfrak{k}(i)_1] \subseteq \oplus_{r=1}^{s+1} \mathfrak{k}(i)_r$ for all $s \in \mathbb{Z}_{\geq 1}$. A case-by-case analysis using Table 1 yields that the coefficient in front of α_i in the highest root of Φ is always 2. This implies $\mathfrak{k}(i)_3 = 0$ so that $\mathfrak{k}(i)_2$ is the centre of $\mathfrak{k}(i)$ and we obtain the indicated lower central series. ∎

Regarding the centre $\mathfrak{z}$ of $\mathfrak{k}$ for $(X,\tau) \in \mathrm{WSat}(A)$, recall the notation i for the unique element of $I\backslash X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. Since the centre of the ideal $\mathfrak{k}(i)$ is $\mathfrak{k}(i)_2$, we must have $\mathfrak{z} \subseteq \mathfrak{k}(i)_2$. Define

$$\mathcal{J}_{\mathrm{even}} := \{\boldsymbol{j} \in \mathcal{J} \mid \forall k \in I\backslash X \text{ the coefficient of } \alpha_{\boldsymbol{j}} \text{ in front of } \alpha_k \text{ is even}\}$$

so that

$$\mathfrak{k}(i)_{2,\mathrm{even}} := \bigoplus_{\boldsymbol{j} \in \mathcal{J}_{\mathrm{even}}} \mathbb{C} b_{\boldsymbol{j}} \subset \mathfrak{k}(i)_2.$$

We claim without proof that $\mathfrak{z}$ is generated by a single element of $\mathfrak{k}(i)_{2,\mathrm{even}}$.

Let us now explain the motivation behind the definition of the set $\mathrm{WSat}(A)$. Consider the excluded generalized Satake diagram . By definition, $\mathfrak{k}$ is the subalgebra of $\mathfrak{g} = \mathrm{Lie}(G_2)$ generated by $e_1, h_1, b_1 = f_1$ and $b_2 = f_2 + \gamma_2 \theta(f_2)$ for some $\gamma_2 \in \mathbb{C}^\times$. The relations (3.2-3.7) specialize to

$$[e_1, b_1] = h_1, \qquad [e_1, b_2] = 0, \qquad [h_1, b_1] = -2b_1, \qquad [h_1, b_2] = b_2, \qquad [h_1, e_1] = 2e_1,$$
$$[b_1, [b_1, b_2]] = 0, \qquad [b_2, [b_2, [b_2, [b_2, b_1]]]] = -18\gamma_2^2 e_1.$$

According to (3.15) we have $\dim(\mathfrak{k}) = 8$. A natural basis is given by

$$e_1, \qquad b_1, \qquad h_1, \qquad b_2, \qquad b_{(2,1)}, \qquad b_{(2,2,1)}, \qquad b_{(2,2,2,1)}, \qquad b_{(1,2,2,2,1)}.$$

Using the adjoint action of e_1, b_1 and b_2 on $\mathfrak{k}$ it is easy to verify that an ideal of $\mathfrak{k}$ equals $\mathfrak{k}$ if it contains any of the generators listed above. This together with some straightforward computations shows that $\mathfrak{k}$ is in fact a simple Lie algebra. Since $\dim(\mathfrak{k}) = 8$, it is must be isomorphic to $\mathfrak{sl}_3$. On the other hand, if $(X,\tau) \in \mathrm{WSat}(A)$, since $\mathfrak{k}$ has a nonzero nilpotent ideal by Proposition 3.6, $\mathfrak{k}$ is not a reductive Lie algebra.

Proposition 3.7. *Let $(X,\tau) \in \mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. Then $\mathfrak{k}$ is not the fixed-point subalgebra of any automorphism of $\mathfrak{g}$.*

Proof. We first show this for the case when (X,τ) is . Suppose there exists $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\mathfrak{k} = \mathfrak{g}^\phi$. From $[h_2, b_1] = 3b_1$ and $[h_2, e_1] = -3e_1$ one establishes straightforwardly that $\phi(h_2) \in \mathfrak{h}$ and hence that $\phi(h_2) = \frac{3}{2}(m-1)h_1 + mh_2$ for some $m \in \mathbb{C}$. Next, from $\theta(f_2) = e_{(2,1)}$ it follows that $[h_2, b_2] = -f_2 - b_2$; hence $\phi(f_2) = mf_2 + \frac{1}{2}(1-m)b_2$. Combining this with $[f_2, b_2] = 3e_1$ one obtains $m = 1$. But this means that h_2 and f_2 are also fixed points of ϕ, contrary to assumption. Hence such ϕ does not exist. Now let $(X,\tau) \in \mathrm{WSat}(A)$. In this case $\mathfrak{k}$ is not a reductive Lie algebra and [Ja62, Thm. 1] implies that $\mathfrak{k}$ cannot be the fixed-point subalgebra of any automorphism of $\mathfrak{g}$. □

Nevertheless, in Section 4 we will show that for all $(X,\tau) \in \mathrm{GSat}(A)$, the subalgebra $\mathfrak{k}$ can be quantized resulting in a coideal subalgebra possessing a universal K-matrix.

3.2. **The universal enveloping algebra $U(\mathfrak{k})$.** Let $(X,\tau) \in \mathrm{GSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. We identify $\mathfrak{k}$ with its image in $U(\mathfrak{k})$ under the canonical Lie algebra embedding. The generators of $U(\mathfrak{k})$ corresponding to b_i $(i \in I\backslash X)$ can be modified by scalar terms, which is a straightforward generalization of [Ko14, Cor. 2.9].

Proposition 3.8. *For $(X,\tau) \in \mathrm{GSat}(A)$, $\boldsymbol{\gamma} \in \Gamma$ and $\boldsymbol{s} \in \mathbb{C}^{I\backslash X}$, the universal enveloping algebra $U(\mathfrak{k}_{\boldsymbol{\gamma}})_{\boldsymbol{s}}$ is generated by e_i, f_i $(i \in X)$, $h \in \mathfrak{h}^\theta$ and*

$$b_{i;\boldsymbol{\gamma},\boldsymbol{s}} = f_i + \gamma_i \theta(f_i) + s_i \qquad \text{for all } i \in I\backslash X. \tag{3.19}$$

Again, if there is no cause for confusion, we will suppress $\boldsymbol{\gamma}$ and $\boldsymbol{s}$ from the notation. Because of Corollary 3.4 we immediately obtain the following result, which addresses [Ko14, Rmk. 2.10].

Proposition 3.9. *For $(X,\tau) \in \mathrm{GSat}(A)$, $\boldsymbol{\gamma} \in \Gamma$ and $\boldsymbol{s} \in \mathbb{C}^{I\backslash X}$, the defining relations of the universal enveloping algebra $U(\mathfrak{k})$ are given by (3.2-3.6), with the Lie bracket interpreted as commutator.*

We may view $U(\mathfrak{k})$ as a Hopf subalgebra of $U(\mathfrak{g})$ so that Lie algebra automorphisms of $\mathfrak{g}$ lift to Hopf algebra automorphisms of $U(\mathfrak{g})$. Call two Hopf subalgebras B, B' of $U(\mathfrak{g})$ equivalent if there exists $\phi \in \mathrm{Aut}_{\mathrm{Hopf}}(U(\mathfrak{g}))$ such that $B' = \phi(B)$. Define

$$\widetilde{\Gamma} := \{\boldsymbol{\gamma} \in \Gamma \mid \gamma_i = 1 \text{ unless } i \in I_{\mathrm{diff}}\}, \qquad \mathcal{S} := \{\boldsymbol{s} \in \mathbb{C}^{I\backslash X} \mid s_i = 0 \text{ unless } i \in I_{\mathrm{nsf}}\}. \tag{3.20}$$

Proposition 3.10. *Let $(X,\tau) \in \mathrm{GSat}(A)$, $\boldsymbol{\gamma} \in \Gamma$ and $\boldsymbol{s} \in \mathbb{C}^{I\backslash X}$. There exist $\widetilde{\boldsymbol{\gamma}} \in \widetilde{\Gamma}$ and $\boldsymbol{s}' \in \mathcal{S}$ such that $U(\mathfrak{k}_{\boldsymbol{\gamma}})_{\boldsymbol{s}}$ is equivalent to $U(\mathfrak{k}_{\widetilde{\boldsymbol{\gamma}}})_{\boldsymbol{s}'}$.*

Proof. The existence of $\widetilde{\boldsymbol{\gamma}}$ can be proven in an argument entirely analogous to the proof of [Ko14, Prop. 9.2 (i)]. It follows that $U(\mathfrak{k}_{\boldsymbol{\gamma}})_{\boldsymbol{s}}$ is equivalent to $U(\mathfrak{k}_{\widetilde{\boldsymbol{\gamma}}})_{\widetilde{\boldsymbol{s}}}$ for some $\widetilde{\boldsymbol{s}} \in \mathbb{C}^{I\backslash X}$.

Regarding the existence of $\boldsymbol{s}' \in \mathcal{S}$, note that $b_{i,\widetilde{\boldsymbol{\gamma}}} \in (\mathfrak{k}_{\widetilde{\boldsymbol{\gamma}}})'$ unless $i \in I_{\mathrm{nsf}}$ owing to Prop. 3.5. Hence $U(\mathfrak{k}_{\widetilde{\boldsymbol{\gamma}}})_{\widetilde{\boldsymbol{s}}}$ is already generated by e_i, f_i $(i \in X)$, $h \in \mathfrak{h}^\theta$, $b_{i;\widetilde{\boldsymbol{\gamma}},0}$ for $i \in (I\backslash X)\backslash I_{\mathrm{nsf}}$ and $b_{i;\widetilde{\boldsymbol{\gamma}},\widetilde{\boldsymbol{s}}}$ for $i \in I_{\mathrm{nsf}}$. Hence we may take $s'_i = \widetilde{s}_i$ if $i \in I_{\mathrm{nsf}}$ and $s'_i = 0$ otherwise. □

4. The universal K-matrix revisited

Assume the d_i are dyadic rationals and let $\mathbb{K}$ be a quadratic closure of $\mathbb{C}(q)$ where q is an indeterminate, so that $q_i := q^{d_i} \in \mathbb{K}$ for all $i \in I$. The Drinfeld-Jimbo quantum group $U_q = U_q(\mathfrak{g})$ is an associative unital algebra over $\mathbb{K}$ which quantizes the universal enveloping algebra $U(\mathfrak{g})$. It is generated by $\{E_i, F_i, t_i^{\pm 1}\}$ where $i \in I$, satisfying the relations given in e.g. [Lu94, 3.1.1]. It is a Hopf algebra whose structure is defined by the choice of the coproduct:

$$\Delta(E_i) = E_i \otimes 1 + t_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes t_i^{-1} + 1 \otimes F_i, \qquad \Delta(t_i) = t_i \otimes t_i.$$

For $\alpha = \sum_i n_i \alpha_i \in Q$ with $n_i \in \mathbb{Z}$ we write $t_\alpha = \prod_{i\in I} t_i^{n_i}$. The Hopf subalgebra $U_q^0 = U_q(\mathfrak{h})$ is the subalgebra generated by $t_i^{\pm1}$ for $i \in I$ and spanned by $\{t_\alpha\}_{\alpha \in Q}$. In terms of the quantum root spaces

$$(U_q)_\alpha = \{u \in U_q \,|\, \forall i \in I \; t_i u t_i^{-1} = q_i^{\alpha(h_i)} u\}$$

where $\alpha \in Q$, we have the Q-grading

$$U_q = \bigoplus_{\alpha \in Q} (U_q)_\alpha, \qquad (U_q)_\alpha (U_q)_\beta \subseteq (U_q)_{\alpha+\beta}. \tag{4.1}$$

According to [Tw92, Thm. 2.1] we have $\mathrm{Aut}_{\mathrm{Hopf}}(U_q) = \mathrm{Ad}(\widetilde{H}) \rtimes \mathrm{Aut}(A)$ with $\mathrm{Ad}(\chi)$ for $\chi \in \widetilde{H}$ acting on the root space $(U_q)_\alpha$ for $\alpha \in Q$ by multiplication by $\chi(\alpha)$, and $\mathrm{Aut}(A)$ acting by relabelling. Other relevant algebra automorphisms are Lusztig's automorphisms T_i for $i \in I$ given as $T''_{i,1}$ in [Lu94, 37.1.3] which define a braid group action on U_q restricting to the Weyl group action on U_q^0: $T_i(t_\alpha) = t_{r_i(\alpha)}$ for $i \in I$ and $\alpha \in Q$. For $X \subseteq I$ with $w_X = r_{i_1} \cdots r_{i_\ell}$ a reduced decomposition we define $T_X = T_{i_1} \cdots T_{i_\ell}$. Also, we define a quantum analogue of the Chevalley involution by

$$\omega_q(E_i) = -t_i^{-1} F_i, \qquad \omega_q(F_i) = -E_i t_i, \qquad \omega_q(t_i^{\pm1}) = t_i^{\mp1} \tag{4.2}$$

for $i \in I$. Then ω_q commutes with $\mathrm{Aut}(A)$ and with T_i for $i \in I$, see [BK16, Lemma 7.1]. Assuming $\tau(X) = X$, one straightforwardly checks that τ commutes with T_X.

4.1. **Quantum pair algebras.** We will follow the approach of the papers [Ko14, BK15, BK16] and simply highlight where a definition or formula needs to be changed. The quantum analogon of the map $\theta = \mathrm{Ad}(w_X)\tau\omega$ is the map

$$\theta_q = \theta_q(X,\tau) = T_X \tau \omega_q \in \mathrm{Aut}_{\mathrm{alg}}(U_q).$$

Note the absence of the factor $\mathrm{Ad}(s)$, cf. [Ko14, Def. 4.3] or [BK16, Def. 5.4 and Eqn. (5.4)], which was present in *ibid.* to guarantee that θ_q specializes to the appropriate Lie algebra involution, see [Ko14, Prop. 10.2]. Similar to (3.14) it follows that $U_q(\mathfrak{h})^{\theta_q}$ consists of polynomials in $t_i^{\pm1}$ $(i \in X)$ and $(t_i t_{\tau(i)}^{-1})^{\pm1}$ $(i \in I^*, i \neq \tau(i))$. It is equal to the subalgebra denoted ${U_\Theta^0}'$ in [Ko14].

The quantization of the fixed-point subalgebra in the formalism by [Ko14] relies on the presentation of $\mathfrak{g}^{\theta_\gamma}$ in terms of generators given in [Ko14, Lemma 2.8]. Our $\mathfrak{k}(X,\tau)$ with $(X,\tau) \in \mathrm{GSat}(A)$ by definition can be quantized to a right coideal subalgebra in the same way.

Definition 4.1. Let $(X,\tau) \in \mathrm{GSat}(A)$, $\boldsymbol{\gamma} \in (\mathbb{K}^\times)^{I\backslash X}$ and $\boldsymbol{s} \in \mathbb{K}^{I\backslash X}$. Then $B = B_{\boldsymbol{\gamma},\boldsymbol{s}}(X,\tau)$ is the coideal subalgebra generated by $U_q(\mathfrak{g}_X)$, $U_q(\mathfrak{h})^{\theta_q}$ and the elements

$$B_i = B_{i;\boldsymbol{\gamma},\boldsymbol{s}} = F_i + \gamma_i \theta_q(F_i t_i) t_i^{-1} + s_i t_i^{-1} \qquad \text{for all } i \in I\backslash X. \qquad \varnothing$$

To make a direct match between the Kolb-Balagović formalism based on fixed-point subalgebras and our more general approach one should set, for all $i \in I\backslash X$,

$$\gamma_i = s(\alpha_{\tau(i)}) c_i,$$

see also [BK16, Eqn. (7.7)]. If the tuples $\boldsymbol{\gamma}$, $\boldsymbol{s}$ lie in the sets

$$\begin{aligned} \Gamma_q &= \{\boldsymbol{\gamma} \in (\mathbb{K}^\times)^{I\backslash X} \mid \forall i \in I^* \, \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\mathrm{diff}}\}, \\ \mathcal{S}_q &= \{\boldsymbol{s} \in \mathbb{K}^{I\backslash X} \mid s_i = 0 \text{ unless } i \in I_{\mathrm{nsf}}\} \end{aligned} \tag{4.3}$$

respectively, then according to [Ko14, Sec. 5.3 and Sec. 6] one obtains decompositions of B yielding the quantum analogue of (3.10), namely $B \cap U_q(\mathfrak{h}) = U_q(\mathfrak{h})^{\theta_q}$. The key condition for Satake diagrams, see (2.10), is only used in [Ko14, Proof of Lemma 5.11, Step 1], but it is clear that what is needed is precisely the weaker condition appearing in the definition of a generalized Satake diagram, see Definition 2.2. The rest of [Ko14] is applicable without change in the setting of generalized Satake diagrams; in particular in the specialization ($q \to 1$) one recovers $U(\mathfrak{k})$, see [Ko14, Sec. 10].

In [BK15] the bar involutions for U_q and B are studied, following earlier work by [ES13] and [BW13] in the case of quantum symmetric pairs of $\mathfrak{gl}_N$ type. The proof of [BK15, Prop. 2.3] relies on a case-by-case analysis of Satake diagrams of finite type from Araki's work [Ara62]. We claim here without proof that a similar analysis using Table 1 yields the same result for all generalized Satake diagrams, in other words that [BK15, Prop. 2.5] holds with $\nu_i = 1$ for all $i \in I\backslash X$ (otherwise $\nu_i = -1$). In the remainder of [BK15] the defining condition of Satake diagrams or a case-by-case analysis is not used so that these results remain valid.

The universal K-matrix for the algebra B is constructed in [BK16] in the case $(X,\tau) \in \mathrm{Sat}(A)$. We restate some key conditions in terms of the parameters $\boldsymbol{\gamma}$. Assuming $\nu_i = 1$ for all $i \in I\backslash X$, condition [BK16, Eqn. (5.17)] is equivalent to

$$\gamma_{\tau(i)} = \zeta(\alpha_i) q_i^{(\theta(\alpha_i) - 2\rho_X)(h_i)} \overline{\gamma_i},$$

where ρ_X is the Weyl vector of $\mathfrak{g}_X$ and $\overline{\cdot}$ denotes the bar involution of U_q, which by definition fixes E_i, F_i and inverts $t_i^{\pm 1}$ and q. In [BK16, Proof of Lemma 6.4] the defining condition of Satake diagrams is used, but as before the defining condition of generalized Satake diagrams is what is needed. Then [BK16, Eqn. (7.14)] needs to be replaced by

$$\overline{T_{w_X}(E_{\tau(i)})} = \zeta(\alpha_i) q_i^{-2\rho_X(h_i)} T^{-1}_{w_X}(E_{\tau(i)})$$

so that the scalar ρ_i appearing in [BK16, Lemma 9.3] equals $q_i^{-\theta(\alpha_i)(h_i)} \gamma_{\tau(i)}$ since [BK16, Eqn. (9.8)] is equivalent to

$$\overline{\gamma_i T_{w_X}(E_{\tau(i)})} = q_i^{-\theta(\alpha_i)(h_i)} \gamma_{\tau(i)} T^{-1}_{w_X}(E_{\tau(i)}).$$

Finally, we highlight the paper [DK18] which establishes an elegant factorization property of the quasi K-matrix in terms of the restricted Weyl group of $\mathfrak{g}$. Sections 2.2 and 2.3 in *ibid.* entail an analysis of the restricted Weyl group and restricted root system following [Lu76]. For completeness, in reference to a comment in [DK18, between Eqs. (2.9) and (2.10)] we remark that also for all $(X,\tau) \in \mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ the set X is invariant under the diagram automorphism $\tau_0 = \tau_{I,0}$ corresponding to the longest element of W; this follows from Table 1. The upshot of this in [DK18] is that $\tau_{0,X[i]}$ stabilizes X (for all $i \in I^*$). This is used to derive that the $\widetilde{r}_i = w_X w_{X[i]}$ form a Coxeter system for the group they generate. Alternatively, this result follows from Corollary 2.5 for all generalized Satake diagrams.

A. Deriving modified Serre relations for $\mathfrak{k}$

The following three technical lemmas are used to derive the key equation (3.7). It is convenient to introduce the notation $Q_X = \sum_{i \in X} \mathbb{Z}\alpha_i$ and $Q_X^+ := Q^+ \cap Q_X$.

Lemma A.1. *Let $(X,\tau) \in \mathrm{CDec}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I\backslash X}$. For all $i \in X$, $j \in I$ and $m \in \mathbb{Z}_{\geq 1}$ we have*

$$\mathrm{ad}(b_i)^m(b_j) = \begin{cases} \mathrm{ad}(f_i)^m(f_j) + \gamma_j\, \theta\,(\mathrm{ad}(f_i)^m(f_j)) & \text{if } j \in I\backslash X, \\ \mathrm{ad}(f_i)^m(f_j) & \text{if } j \in X. \end{cases}$$

Proof. This follows immediately from (2.7) and the fact that θ is a Lie algebra automorphism. □

Lemma A.2. *Let $(X,\tau) \in \mathrm{CDec}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I\backslash X}$. For all $i \in I\backslash X$, $j \in X$ and $m \in \mathbb{Z}_{\geq 1}$ we have*

$$\mathrm{ad}(b_i)^m(b_j) = \mathrm{ad}(f_i)^m(f_j) + \gamma_i^m\, \theta\,(\mathrm{ad}(f_i)^m(f_j)) + \mathrm{LO}_{ij}(m)$$

where

$$\mathrm{LO}_{ij}(m) = \begin{cases} (1+\zeta(\alpha_i))\gamma_i\,[\theta(f_i),[f_i,f_j]] \in \mathfrak{n}_X^+ & \text{if } \tau(i)=i,\, w_X(\alpha_i)-\alpha_i-\alpha_j \in \Phi^+,\, m=2, \\ -\gamma_i\,(2h_i - a_{ij}h_j) & \text{if } \tau(i)=i,\, w_X(\alpha_i)=\alpha_i+\alpha_j,\, m=2, \\ -3\,(2+a_{ij})\,\gamma_i\,(f_i-\theta(f_i)) & \text{if } \tau(i)=i,\, w_X(\alpha_i)=\alpha_i+\alpha_j,\, m=3, \\ -6\,a_{ij}(2+a_{ij})\,\gamma_i^2\, e_j & \text{if } \tau(i)=i,\, w_X(\alpha_i)=\alpha_i+\alpha_j,\, m=4, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By induction with respect to m. For $m=1$, (2.7) implies

$$\mathrm{ad}(b_i)^1(b_j) = [f_i + \gamma_i\,\theta(f_i), f_j] = \mathrm{ad}(f_i)^1(f_j) + \gamma_i^1\, \theta\,(\mathrm{ad}(f_i)(f_j)) + \mathrm{LO}_{ij}(1)$$

with $\mathrm{LO}_{ij}(1) = 0$ as required. Now assume $m \in \mathbb{Z}_{>1}$ and suppose the statement holds for all smaller values. Then, by virtue of the induction hypothesis, the fact that θ is a Lie algebra automorphism and (2.7), we find

$$\begin{aligned} \mathrm{ad}(b_i)^m(b_j) &= \big[b_i, \mathrm{ad}(b_i)^{m-1}(b_j)\big] \\ &= \big[f_i + \gamma_i\,\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j) + \gamma_i^{m-1}\theta\left(\mathrm{ad}(f_i)^{m-1}(f_j)\right) + \mathrm{LO}_{ij}(m-1)\big] \\ &= \mathrm{ad}(f_i)^m(f_j) + \gamma_i^m\theta\,(\mathrm{ad}(f_i)^m(f_j)) \\ &\quad + \gamma_i\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right] + \gamma_i^{m-1}\left[f_i, \theta\left(\mathrm{ad}(f_i)^{m-1}(f_j)\right)\right] + \left[b_i, \mathrm{LO}_{ij}(m-1)\right]. \end{aligned}$$

Using (2.8) we have $\theta^2(f_i) = \zeta(\alpha_i) f_i$ so that

$$\mathrm{LO}_{ij}(m) = \gamma_i\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right] + \zeta(\alpha_i)\gamma_i^{m-1}\theta\left(\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right]\right) + [b_i, \mathrm{LO}_{ij}(m-1)]. \tag{A.1}$$

Suppose that $\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right] \neq 0$. Then $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi \cup \{0\}$. Now $\Phi = \Phi^+ \cup \Phi^-$ implies that $\tau(i) = i$ and $j \in \check{X}(i)$.

If $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^+$ we must have $\tau(i) = i$ and $m = 2$; because $w_X(\alpha_{\tau(i)}) - \alpha_i - \alpha_j \in Q_X^+$ it follows that $[\theta(f_i),[f_i,f_j]] \in \mathfrak{n}_X^+$. The claimed expression for $\mathrm{LO}_{ij}(2)$ follows immediately from (A.1); those for $\mathrm{LO}_{ij}(m)$ with $m > 2$ from (3.2).

Now suppose $w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j \in \Phi^- \cup \{0\}$. Then $\tau(i) = i$ and $w_X(\alpha_i) \leq (m-1)\alpha_i + \alpha_j$ so that $\check{X}(i) = \{j\}$ and hence $a_{ji} < 0$. In this case we readily obtain

$$w_X(\alpha_{\tau(i)}) - (m-1)\alpha_i - \alpha_j = (2-m)\alpha_i - (1+a_{ji})\alpha_j.$$

From $\Phi = \Phi^+ \cup \Phi^-$ it follows that $a_{ji} = -1$. Now $\mathbb{Z}\alpha_i \cap \Phi = \{\pm\alpha_i\}$ implies that $m \in \{2,3\}$. We straightforwardly compute

$$\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right] = \begin{cases} a_{ij}h_j - h_i & \text{if } m=2, \\ -2\,(1+a_{ij})f_i & \text{if } m=3, \end{cases}$$

and the claimed expressions for $\mathrm{LO}_{ij}(m)$ readily follow. □

For $i,j\in I$ and $m,r\in\mathbb{Z}$ such that $0\le r\le \lfloor m/2\rfloor$ define $p_{ij}^{(r,m)}\in\mathbb{Z}$ by

$$p_{ij}^{(0,m)}=-1,\qquad p_{ij}^{(\frac{m+1}{2},m)}=0,\qquad p_{ij}^{(r,m+2)}=p_{ij}^{(r,m+1)}-(m+1)(m+a_{ij})p_{ij}^{(r-1,m)}. \tag{A.2}$$

Lemma A.3. *Let $(X,\tau)\in\mathrm{CDec}(A)$ and $\boldsymbol{\gamma}\in(\mathbb{C}^\times)^{I\backslash X}$. For all $i,j\in I\backslash X$ such that $i\neq j$ and $m\in\mathbb{Z}_{\ge0}$ we have*

$$\mathrm{ad}(b_i)^m(b_j)=\mathrm{ad}(f_i)^m(f_j)+\theta\left(\mathrm{ad}(f_i)^m(f_j)\right)+\mathrm{LO}_{ij}(m)$$

where

$$\mathrm{LO}_{ij}(m)=\begin{cases}(\gamma_i+\zeta(\alpha_i)\gamma_j)\,[\theta(f_i),f_j]\in\mathfrak{n}_X^+ & \text{if } \tau(i)=j,\ w_X(\alpha_i)-\alpha_i\in\Phi^+,\ m=1,\\ \gamma_jh_i-\gamma_ih_j & \text{if } \tau(i)=j,\ w_X(\alpha_i)=\alpha_i,\ m=1,\\ 2\left((\gamma_j-a_{ij}\gamma_i)f_i-\gamma_i(\gamma_i-a_{ij}\gamma_j)e_j\right) & \text{if } \tau(i)=j,\ w_X(\alpha_i)=\alpha_i,\ m=2,\\ \displaystyle\sum_{r=1}^{\lfloor m/2\rfloor}p_{ij}^{(r,m)}\gamma_i^r\,\mathrm{ad}(b_i)^{m-2r}(b_j) & \text{if } \tau(i)=i,\ w_X(\alpha_i)=\alpha_i,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. As in the proof of Lemma A.2 we apply induction with respect to m. For $m=0$ we have

$$\mathrm{ad}(b_i)^0(b_j)=b_j=f_j+\gamma_j\,\theta(f_j)=\mathrm{ad}(f_i)^0(f_j)+\gamma_i^0\gamma_j\,\theta\left(\mathrm{ad}(f_i)^0(f_j)\right)+\mathrm{LO}_{ij}(0)$$

with $\mathrm{LO}_{ij}(0)=0$ as required. Now assume $m\in\mathbb{Z}_{>0}$ and suppose the statement holds for all smaller values. Then, by the induction hypothesis,

$$\begin{aligned}\mathrm{ad}(b_i)^m(b_j)&=[b_i,\mathrm{ad}(b_i)^{m-1}(b_j)]\\&=[f_i+\gamma_i\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)+\gamma_i^{m-1}\gamma_j\theta\left(\mathrm{ad}(f_i)^{m-1}(f_j)\right)+\mathrm{LO}_{ij}(m-1)].\end{aligned}$$

Rearranging terms and using that θ is a Lie algebra automorphism we obtain

$$\begin{aligned}\mathrm{ad}(b_i)^m(b_j)&=\mathrm{ad}(f_i)^m(f_j)+\gamma_i^{m-1}\gamma_j\,\theta\left(\mathrm{ad}(f_i)^m(f_j)\right)+\mathrm{LO}_{ij}(m)\qquad\text{where}\\ \mathrm{LO}_{ij}(m)&=\gamma_i\left[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)\right]+\gamma_i^{m-1}\gamma_j\left[f_i,\theta\left(\mathrm{ad}(f_i)^{m-1}(f_j)\right)\right]+[b_i,\mathrm{LO}_{ij}(m-1)].\end{aligned}$$

Using (2.8) we obtain

$$\mathrm{LO}_{ij}(m)=\gamma_i\left[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)\right]+\zeta(\alpha_i)\gamma_i^{m-1}\gamma_j\,\theta\left(\left[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)\right]\right)+[b_i,\mathrm{LO}_{ij}(m-1)]. \tag{A.3}$$

If $\left[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)\right]\neq0$ then $w_X(\alpha_{\tau(i)})-(m-1)\alpha_i-\alpha_j\in\Phi\cup\{0\}$.

If $w_X(\alpha_{\tau(i)})-(m-1)\alpha_i-\alpha_j\in\Phi^+$ we must have $j=\tau(i)$, $\check{X}(i)\neq\emptyset$, $m=1$; since $w_X(\alpha_{\tau(i)})-\alpha_j\in Q_X^+$ it follows that $[\theta(f_i),f_j]\in\mathfrak{n}_X^+$. The expression for $\mathrm{LO}_{ij}(1)$ follows from (A.3); $\mathrm{LO}_{ij}(m)=0$ with $m>1$ is a consequence of (3.2).

Now suppose $w_X(\alpha_{\tau(i)})-(m-1)\alpha_i-\alpha_j\in\Phi^-\cup\{0\}$. It follows that $\check{X}(i)=\emptyset$, so $\zeta(\alpha_i)=1$, and $\tau(i)\in\{i,j\}$. If $\tau(i)=j$ then $\mathbb{Z}\alpha_i\cap\Phi=\{\pm\alpha_i\}$ implies that $m\in\{1,2\}$. Furthermore, $\theta(f_i)=-e_j$ and $a_{ij}=a_{ji}$. Now (A.3) implies, as required, $\mathrm{LO}_{ij}(1)=\gamma_jh_i-\gamma_ih_j$,

$$\begin{aligned}\mathrm{LO}_{ij}(2)&=\gamma_i\gamma_j\,\theta\left([-e_j,[f_i,f_j]]\right)+\gamma_i\,[-e_j,[f_i,f_j]]+[b_i,\mathrm{LO}_{ij}(1)]\\&=\gamma_i\gamma_j\,\theta\left([h_j,f_i]\right)+\gamma_i\,[h_j,f_i]+[\gamma_ih_j-\gamma_jh_i,f_i-\gamma_i\,e_j]\\&=2\left((\gamma_j-a_{ij}\gamma_i)f_i-\gamma_i(\gamma_i-a_{ij}\gamma_j)e_j\right)\end{aligned}$$

and $\mathrm{LO}_{ij}(m)=0$ if $m>2$.

It remains to deal with the case $\check{X}(i)=\emptyset$ and $\tau(i)=i$, in which case $\theta(f_i)=-e_i$. A straightforward computation gives

$$[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)]=(m-1)(m-2+a_{ij})\mathrm{ad}(f_i)^{m-2}(f_j).$$

By virtue of the induction hypothesis, (A.3) simplifies to

$$\mathrm{LO}_{ij}(m)=(m-1)(m-2+a_{ij})\gamma_i\left(\mathrm{ad}(b_i)^{m-2}(b_j)-\mathrm{LO}_{ij}(m-2)\right)+[b_i,\mathrm{LO}_{ij}(m-1)],$$

from which the recursion (A.2) follows straightforwardly. □

References

[Ara62] Sh. Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces.* J. Math. Osaka City Univ. **13** (1962), no. 1, 1–34.

[BB10] P. Baseilhac, S. Belliard, *Generalized q-Onsager algebras and boundary affine Toda field theories.* Lett. Math. Phys. **93** (2010), 213–228. arXiv:0906.1215.

[BK15] M. Balagović, S. Kolb, *The bar involution for quantum symmetric pairs.* Rep. Thy. of the Amer. Math. Soc. **19** (2015), no. 8, 186–210. arXiv:1409.5074.

[BK16] ———, *Universal K-matrix for quantum symmetric pairs.* Journal für die reine und angewandte Mathematik (Crelles Journal). arXiv:1507.06276.

[BW13] H. Bao, W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs.* Astérisque (to appear), arXiv:1310.0103v2.

[BW16] ———, *Canonical bases arising from quantum symmetric pairs.* Preprint at arXiv:1610.09271.

[DK18] L. Dobson, S. Kolb, *Factorisation of quasi K-matrices for quantum symmetric pairs.* Preprint at arXiv:1804.02912.

[Dr87] V.G. Drinfeld, *Quantum groups.* Proceedings ICM 1986, Amer. Math. Soc. (1987), 798–820.

[ES13] M. Ehrig, C. Stroppel, *Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality* Preprint at arXiv:1310.1972v2.

[He84] A. Heck, *Involutive automorphisms of root systems.* J. Math. Soc. Japan **36** (1984), no. 4, 643–658.

[Ja62] N. Jacobson, *A note on automorphisms of Lie algebras.* Pac. J. of Math. **12**, no. 1 (1962), 303–315.

[Ji85] M. Jimbo, *A q-analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation.* Lett. Math. Phys. **11** (1985), 63–69.

[Ka90] M. Kashiwara, *Crystallizing the q-analogue of universal enveloping algebras.* Comm. Math. Phys. **133** no. 2 (1990), 249–260.

[KR90] N. Kirillov, N. Reshetikhin, *q-Weyl group and a multiplicative formula for universal R-matrices.* Comm. Math. Phys. **134** (1990), 421–431.

[Ko14] S. Kolb, *Quantum symmetric Kac-Moody pairs.* Adv. Math. **267** (2014), 395–469. arXiv:1207.6036.

[Ko17] ———, *Braided module categories via quantum symmetric pairs.* Preprint at arXiv:1705.04238.

[Let99] G. Letzter, *Symmetric Pairs for Quantized Enveloping Algebras.* J. Algebra **220** (1999), 729–767.

[Let02] ———, *Coideal Subalgebras and Quantum Symmetric Pairs.* New Directions in Hopf Algebras, MSRI publications 43, Cambridge University Press (2002), 117–166. arXiv:math/0103228.

[Let03] ———, *Quantum Symmetric Pairs and Their Zonal Spherical Functions.* Transformation Groups **8**, no. 3 (2003), 261–292 arXiv:math/0204103.

[LS90] S. Z. Levendorskiĭ, Ya.S. Soibelman, *Some applications of the quantum Weyl groups.* J. Geom. Phys. **7** (1990), 241–254.

[Lu76] G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius.* Inv. Math. **28** (1976), 101–159.

[Lu94] ———, *Introduction to quantum groups.* Birkhäuser, Boston, 1994.

[Lu02] ———, *Hecke algebras with unequal parameters.* CRM Monographs Ser., vol. 18, Amer. Math. Soc., enlarged and updated version at arXiv:math/0208154v2.

[Mu02] A. Mudrov, *Characters of $U_q(\mathfrak{gl}(n))$-reflection equation algebra.* Lett. in Math. Phys. **60**, no. 3 (2002), 283–291.

[NDS95] M. Noumi, M.S. Dijkhuizen, T. Sugitani, *Multivariable Askey-Wilson polynomials and quantum complex Grassmannians*, AMS Fields Inst. Commun. **14** (1997), 167-177. arXiv:q-alg/9603014.

[NS95] M. Noumi, T. Sugitani, *Quantum symmetric spaces and related q-orthogonal polynomials*, in: Group Theoretical Methods in Physics (ICGTMP) (Toyonaka, Japan, 1994), World Scientific Publishing, River Edge, NJ, (1995), 28–40. arXiv:math/9503225.

[Sat71] I. Satake, *Classification theory on semi-simple algebraic groups*, Lecture notes in pure and applied mathematics, vol. 3, Dekker, New York (1971).

[Tw92] E. Twietmeyer, *Real forms of $U_q(\mathfrak{g})$.* Lett. Math. Phys. **24** (1992), 49–58.

Department of Mathematics, University of York, York, YO10 5DD, UK and
Institute of Theoretical Physics and Astronomy, Vilnius University, Saulėtekio av. 3, Vilnius 10257, Lithuania.

E-mail address: vidas.regelskis@gmail.com

Department of Mathematics, University of York, York, YO10 5DD, UK and
Department of Mathematics, Heriot-Watt University, Edinburgh, EH14 4AS, UK

E-mail address: b.vlaar@hw.ac.uk