A MEASURE THEORETIC RESULT FOR APPROXIMATION BY DELONE SETS

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Abstract. With a view to establishing measure theoretic approximation properties of Delone sets, we study a setup which arises naturally in the problem of averaging almost periodic functions along exponential sequences. In this setting, we establish a full converse of the Borel–Cantelli lemma. This provides an analogue of more classical problems in the metric theory of Diophantine approximation, but with the distance to the nearest integer function replaced by distance to an arbitrary Delone set.

1. Introduction

In a previous work [2], motivated by problems emerging in aperiodic order [1], we considered the problem of establishing asymptotic formulas for averages of Bohr almost periodic functions along exponential sequences of the form $(\alpha^n x)_{n\in\mathbb{N}}$, where α is a fixed real number with $|\alpha| > 1$, and $x \in \mathbb{R}$ is arbitrary. The almost everywhere convergence results which were obtained can be viewed as analogues of Birkhoff's ergodic theorem [8]. Their proofs relied on knowing that, for almost every $x \in \mathbb{R}$, the elements of our exponential sequence with $n \leq N$ cannot be approximated too well by elements of a given Delone set. In fact, it was precisely this occurrence of a Delone set that showed how natural the replacement of the integers by such a more general set is, and how well it blends in the realm of metric Diophantine approximation problems. Once again, systems of aperiodic order have thus suggested how one can go meaningfully beyond the standard lattice setting, while remaining firmly within the realm of physical and mathematical significance. In this short note, we return to this problem in order to more fully investigate the Diophantine approximation properties of Delone sets. Our main result is a generalization of Khintchine's theorem (even without monotonicity) to this setting (see [4] or [7] for detailed background on the metric theory of Diophantine approximation). It provides a characterization which tells us precisely when we should expect, for almost every $x \in \mathbb{R}$, to have infinitely many approximations of a certain quality.

In what follows, we suppose that Y is a Delone set in $\mathbb{R}$, with packing radius r and covering radius R. For $x \in \mathbb{R}$ and $\rho > 0$, we use $B(x, \rho)$ to denote the closed ball of radius ρ centered at x. The symbol λ denotes Lebesgue measure on $\mathbb{R}$, and $\mathrm{card}(S)$ denotes the cardinality of a set S. We also use the standard Vinogradov $\ll$ notation, so that if $f, g : D \to \mathbb{R}$ are two functions on some common domain D, then $f \ll g$

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means that there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in D$. Our main result is the following theorem.

Theorem. *Let $\psi : \mathbb{N} \to [0, \infty)$ be any function, and suppose that α is a real number with $|\alpha| > 1$. If*

$$\sum_{n=1}^{\infty} \psi(n) < \infty \tag{1}$$

then, for almost every $x \in \mathbb{R}$, there are only finitely many $n \in \mathbb{N}$ and $y \in Y$ for which

$$|\alpha^n x - y| \leq \psi(n). \tag{2}$$

On the other hand, if

$$\sum_{n=1}^{\infty} \psi(n) = \infty, \tag{3}$$

then, for almost every $x \in \mathbb{R}$, there are infinitely many $n \in \mathbb{N}$ and $y \in Y$ that satisfy the above inequality.

We point out that our result amounts to a full converse of the Borel–Cantelli lemma for special collections of subsets of $\mathbb{R}$. The less trivial direction of our proof is the divergence part, in which we assume that (3) holds, and prove that the associated limsup set (the set of real numbers x for which the relevant inequalities have infinitely many solutions) has full measure. It is quite common in these types of problems to first establish a zero-full lemma (e.g. [3, 6, 7]), which proves that, whether or not (3) holds, the associated limsup set either has zero measure, or its complement does. Then all that is left to show, for the difficult part of the proof, is that the limsup set has *positive* measure. In our proof, we deviate slightly from this approach and, instead of establishing a separate zero-full lemma, we use a simple application of the Lebesgue density theorem [5, Theorem 3.21] to complete our proof. In essence, what we show is that, at all scales, the intersections of the individual intervals which contribute to the limsup set behave in the same way. In other words, roughly the same picture appears after zooming in on any part of the real line. Since the integers in our setup have been replaced by a completely arbitrary Delone set, this is clearly another manifestation of a behavior which should be termed 'aperiodic order'.

2. Proof of main theorem

We will consider only the case when $\alpha > 1$, since the case when $\alpha < -1$ follows as an easy corollary. Let $a < b$ be real numbers and assume, with little loss of generality, that $a \geq 0$ (the cases with $a < 0$ can be dealt with by trivial modifications of our argument). For each $n \in \mathbb{N}$, define a set $\mathcal{A}_n \subseteq [a, b)$ by

$$\mathcal{A}_n = \{x \in [a, b) : |\alpha^n x - y| \leq \psi(n) \text{ for some } y \in Y\}.$$

In what follows, all constants implied by the use of the $\ll$ notation will be universal, not depending on a, b, r, or R, unless otherwise stated. Whenever implied constants do depend on some of these quantities, we will indicate this by attaching the appropriate subscripts to the $\ll$ symbol.

The proof of the first part of the theorem is a straightforward application of the convergence part of the Borel–Cantelli lemma. Write the non-negative elements of Y in increasing order as

$$y_1 < y_2 < \cdots$$

and, for each $n \in \mathbb{N}$, define

$$\mathcal{I}_n = \left\{ i \in \mathbb{N} : [a,b) \cap B\left(\frac{y_i}{\alpha^n}, \frac{\psi(n)}{\alpha^n}\right) \neq \varnothing \right\}.$$

From our hypothesis on Y we have, for all sufficiently large n (depending on a and b), that

$$\frac{(b-a)\,\alpha^n}{R} \ll \operatorname{card}(\mathcal{I}_n) \ll \frac{(b-a)\,\alpha^n}{r},$$

and it follows from this that, for n sufficiently large,

$$\lambda(\mathcal{A}_n) \ll (b-a)\,\psi(n)/r.$$

Since we are assuming that (1) holds, we conclude that almost every $x \in [a,b)$ falls in only finitely many of the sets $\mathcal{A}_n$, which is equivalent to the assertion that there are only finitely many solutions to the inequality (2). This gives the conclusion of the first part of the theorem.

The proof of the second part is slightly more complicated. Ideally, we would like to demonstrate that the sets $\mathcal{A}_n$ above are quasi-independent, or in other words that

$$\lambda(\mathcal{A}_m \cap \mathcal{A}_n) \ll_{r,R} (b-a)^{-1}\lambda(\mathcal{A}_m)\lambda(\mathcal{A}_n) \quad \text{for} \quad m \neq n.$$

Unfortunately, this is not quite true, so we need some technical modifications in our setup. First of all, choose an integer J with the property that

$$\alpha^J \geq \frac{2R(1+\frac{2}{r})}{r} + 1, \tag{4}$$

and then choose a residue class $j \in \{0, 1, \ldots, J-1\}$ modulo J with the property that

$$\sum_{n=1}^{\infty} \psi(nJ+j) = \infty.$$

Since there are only J residue classes to choose from, and since we are assuming that (3) holds, it is clear that this is possible.

Now write $\beta = \alpha^J$ and, for each $n \in \mathbb{N}$, define a set $Y^{(n)} \subseteq Y$ by

$$Y^{(n)} = Y \setminus \bigcup_{m=1}^{n-1} B(\beta^{n-m}Y, 1). \tag{5}$$

The reason for introducing these sets, which will become clearer later in the proof, is to remove the bad overlaps which occur between the sets $\mathcal{A}_n$ which were used in the previous argument. However, we will still need to show that we have not discarded too much from Y so as to make the sum of the measures of our new sets fail to diverge. For each $X \in \mathbb{R}$, we have that

$$\operatorname{card}\{y \in Y : aX \le y < bX\} \ge \left\lfloor \frac{(b-a)X}{R} \right\rfloor$$

and, for each $\ell \in \mathbb{N}$, we also have that

$$\operatorname{card}\{y \in \beta^{\ell} Y : aX \le y < bX\} \le \frac{(b-a)X}{r\beta^{\ell}}.$$

Since the number of points of Y in a ball of radius 1 is bounded above by $1 + \frac{2}{r}$, it follows that, for $n, N \in \mathbb{N}$,

$$\begin{aligned} \operatorname{card}\{y \in Y^{(n)} : aX \le y < bX\} &\ge \left\lfloor \frac{(b-a)X}{R} \right\rfloor - \sum_{\ell=1}^{n-1} \frac{(b-a)X}{r\beta^{\ell}} \cdot (1 + \tfrac{2}{r}) \\ &\ge (b-a)X \left(\frac{1}{R} - \frac{1+\frac{2}{r}}{r(\beta-1)} \right) - 1 \\ &\ge \frac{(b-a)X}{2R} - 1. \qquad (6) \end{aligned}$$

The final inequality here is a result of our choice of J in (4).

For $n \in \mathbb{N}$ we now define $\mathcal{A}_n' \subseteq [0,1)$, our replacement for the set $\mathcal{A}_n$ above, by

$$\mathcal{A}_n' = \{x \in [a,b) : |\alpha^j \beta^n x - y| \le \psi(nJ+j) \text{ for some } y \in Y^{(n)}\}.$$

From (6), and using the same arguments as above, we have that, for all sufficiently large n (depending again only on a and b),

$$\lambda(\mathcal{A}_n') \gg \frac{(b-a)\,\psi(nJ+j)}{R}.$$

For $m \neq n$, we now would like to derive an upper bound for the measure of the intersection of $\mathcal{A}_m'$ with $\mathcal{A}_n'$. With this purpose in mind, let

$$\delta = \delta(m,n) = \min\left\{ \frac{2\psi(mJ+j)}{\alpha^j \beta^m}, \frac{2\psi(nJ+j)}{\alpha^j \beta^n} \right\},$$

and

$$\Delta = \Delta(m,n) = \max\left\{ \frac{2\psi(mJ+j)}{\alpha^j \beta^m}, \frac{2\psi(nJ+j)}{\alpha^j \beta^n} \right\}.$$

Each set $\mathcal{A}_n'$ is a union of connected components, which which we refer to as its component intervals. The component intervals of $\mathcal{A}_n'$ are intervals centered at points of the form $y/\alpha^j\beta^n$. To be fully accurate, at the endpoints of $[a,b)$ it may be the case that there are (at most 2) component intervals which are not of this form, but this fact is negligible in the argument we are about to give. If a component interval from $\mathcal{A}_m'$ intersects a component interval from $\mathcal{A}_n'$, the centers of these intervals must be within

Δ of one another. Furthermore, the intersection of any two such intervals has measure at most δ. This translates into the upper bound

$$\lambda(\mathcal{A}'_m \cap \mathcal{A}'_n) \ll \delta \cdot \mathrm{card}\left\{(y,y') \in Y^{(m)} \times Y^{(n)} \ : \quad a \leq \frac{y}{\alpha^j \beta^m} < b, \ a \leq \frac{y'}{\alpha^j \beta^n} < b, \right.$$

$$\left. \left| \frac{y}{\alpha^j\beta^m} - \frac{y'}{\alpha^j\beta^n} \right| \leq \Delta \right\}, \tag{7}$$

which holds for all m and n sufficiently large. Suppose without loss of generality that $m < n$, and let us derive an estimate for the number of pairs (y, y') which we are counting on the right hand side above. If $y \in Y^{(m)}$ and $y' \in Y^{(n)}$ satisfy

$$|\beta^{n-m}y - y'| \leq \alpha^j \beta^n \Delta,$$

then, from the definition (5) of $Y^{(n)}$, we must also have that

$$1 \leq |\beta^{n-m}y - y'| \leq \alpha^j \beta^n \Delta. \tag{8}$$

It is worth pointing out that this was the reason for the introduction of the sets $Y^{(n)}$. Without the lower bound of a fixed positive constant here, the next part of the argument would not work. From Eqs. (7) and (8) we now derive that, for all m and n sufficiently large (depending on a and b),

$$\begin{aligned}\lambda(\mathcal{A}'_m \cap \mathcal{A}'_n) &\ll_{r,R} \delta \cdot \big((b-a)\,\alpha^j\beta^m\big) \cdot \big(\alpha^j\beta^n\Delta\big) \\ &\ll (b-a)\,\psi(mJ+j)\,\psi(nJ+j).\end{aligned}$$

This implies that there is a constant $K > 0$ (depending only on r and R) with the property that

$$\limsup_{N\to\infty} \left(\sum_{n\leq N} \lambda(\mathcal{A}'_n)\right)^2 \left(\sum_{m,n\leq N} \lambda(\mathcal{A}'_m \cap \mathcal{A}'_n)\right)^{-1} \geq (b-a)K. \tag{9}$$

Since the sum of the measures of the sets $\mathcal{A}'_n$ diverges, by standard arguments from probability theory (see [7, Lemma 2.3]), it follows that the set of x which fall in infinitely many of the sets $\mathcal{A}'_n$ has measure greater than or equal to $(b-a)K$.

Finally, supposing still that the divergence condition (3) holds, let $\mathcal{W} \subseteq \mathbb{R}$ be the set of $x \in \mathbb{R}$ for which the inequality (2) is satisfied by infinitely many $n \in \mathbb{N}$ and $y \in Y$. If it were the case that $\lambda(\mathcal{W}^c) > 0$ then, by the Lebesgue density theorem [5, Theorem 3.21], we could find a point of metric density x_0 of the set $\mathcal{W}^c$. However, this would imply that

$$\lim_{\epsilon\to 0^+} \frac{\lambda(\mathcal{W} \cap B(x_0, \epsilon))}{2\epsilon} = 0,$$

which contradicts (9). Therefore we conclude that $\lambda(\mathcal{W}^c) = 0$, thereby completing the proof of our main result.

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