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CUBIC HYPERSURFACES AND A VERSION OF THE CIRCLE METHOD FOR NUMBER FIELDS

by

T.D. Browning & P. Vishe

Abstract. — A version of the Hardy–Littlewood circle method is developed for number fields $K/\mathbb{Q}$ and is used to show that non-singular projective cubic hypersurfaces over K always have a K-rational point when they have dimension at least 8.

Contents

1. Introduction

Much of analytic number theory has only been extensively developed for problems defined over the rational numbers $\mathbb{Q}$. While many generalisations to finite extensions of $\mathbb{Q}$ are straightforward, there remain a number of areas where substantial technical obstructions persist. One such lacuna may be found in the Hardy–Littlewood circle method, which over $\mathbb{Q}$ begins with a generating function

$$S(z) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i z n}.$$

Typically one is interested in the term $a(0)$, detected via

$$a(0) = \int_0^1 S(z) \mathrm{d}z,$$

the idea being to break $[0,1]$ into subintervals $[a/q - \delta, a/q + \delta]$ over which the integral is more easily estimated. A key innovation, due to Kloosterman [9], involves decomposing $[0,1]$ using a Farey dissection to keep track of the precise endpoints of the intervals. This allows one to introduce non-trivial averaging over the numerators of the approximating fractions a/q, an approach that is usually called the "Kloosterman refinement". This approach is not immediately available to us when passing to finite extensions of $\mathbb{Q}$, since aside from work of Cassels, Ledermann and Mahler [1] particular to the imaginary quadratic fields $\mathbb{Q}(i)$ and $\mathbb{Q}(\rho)$, no suitable generalisation is known of the Farey dissection to the number field analogue of $[0,1]$.

The primary aim of this paper is to provide an alternative route to the Kloosterman refinement over an arbitrary number field, circumventing the need for a Farey dissection. We will illustrate the utility of this new approach by applying it to a long-standing problem in Diophantine geometry. Given any cubic hypersurface $X \subseteq \mathbb{P}_K^{n-1}$ defined over a number field K, a "folklore" conjecture predicts that the set $X(K)$ of K-rational points on X is non-empty as soon as $n \geqslant 10$. The following result establishes this conjecture for generic cubic hypersurfaces.

Theorem 1.1. — *Let K be a number field and let $X \subseteq \mathbb{P}_K^{n-1}$ be a non-singular cubic hypersurface defined over K. If $n \geqslant 10$ then $X(K) \neq \emptyset$.*

In fact, as conjectured by Colliot-Thélène [13, Appendix A], we expect the Hasse principle to hold for non-singular cubic hypersurfaces $X \subseteq \mathbb{P}_K^{n-1}$ with $n \geqslant 5$. Note that work of Lewis [11] ensures that $X(K_v) \neq \emptyset$ for every valuation v of K when $n \geqslant 10$, so that Theorem 1.1 confirms the Hasse principle for the family of hypersurfaces considered here.

The resolution of Theorem 1.1 for the case $K = \mathbb{Q}$ goes back to pioneering work of Heath-Brown [7]. Extending this approach to general number fields K, the best result in the literature is due to Skinner [15], who requires $n \geqslant 13$ variables. The loss of precision is entirely due to the lack of a suitable Kloosterman refinement, a situation that we remedy in the present investigation. When no constraints are placed on the singular locus of X, work of Pleasants [12] shows that $n \geqslant 16$ variables are needed to ensure that $X(K)$ is non-empty. Finally, if the singular locus of X contains a set of three conjugate points then Colliot-Thélène and Salberger [2] have shown that the Hasse principle holds provided only that $n \geqslant 3$.

It is now time to present the main technical tool provided in this work. Let K be a number field of degree $d \geqslant 1$ over $\mathbb{Q}$, with ring of integers $\mathfrak{o}$. The ideal norm will be designated $N\mathfrak{a} = \#\mathfrak{o}/\mathfrak{a}$ for any integral ideal $\mathfrak{a} \subseteq \mathfrak{o}$. In line with our description of the Hardy–Littlewood circle method, we would like to use Fourier analysis to detect when elements of $\mathfrak{o}$ are zero. In fact we will be able to handle the the indicator function

$$\delta_K(\mathfrak{a}) = \begin{cases} 1, & \text{if } \mathfrak{a} = (0), \\ 0, & \text{otherwise,} \end{cases}$$

defined on integral ideals $\mathfrak{a} \subseteq \mathfrak{o}$. When $K = \mathbb{Q}$ an extremely useful formula for δ_K was developed by Duke, Friedlander and Iwaniec [4]. This was later revisited by Heath-Brown [8, Theorem 1] in an effort to relate it to classical versions of the circle method. In this paper we adapt the latter approach to the setting of arbitrary number fields K, as follows.

Theorem 1.2. — *Let $Q \geqslant 1$ and let $\mathfrak{a} \subseteq \mathfrak{o}$ be an ideal. Then there exists a positive constant c_Q and an infinitely differentiable function $h(x, y) : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that*

$$\delta_K(\mathfrak{a}) = \frac{c_Q}{Q^{2d}} \sum_{(0) \neq \mathfrak{b} \subseteq \mathfrak{o}} \sideset{}{^*}\sum_{\sigma \,(\mathrm{mod}\, \mathfrak{b})} \sigma(\mathfrak{a}) h\left(\frac{\mathrm{N}\,\mathfrak{b}}{Q^d}, \frac{\mathrm{N}\,\mathfrak{a}}{Q^{2d}}\right),$$

where the notation $\sum_{\sigma \,(\mathrm{mod}\, \mathfrak{b})}^*$ means that the sum is taken over primitive additive characters modulo $\mathfrak{b}$ extended to ideals. The constant c_Q satisfies

$$c_Q = 1 + O_N(Q^{-N}),$$

for any $N > 0$. Furthermore, we have $h(x, y) \ll x^{-1}$ for all y and $h(x, y) \neq 0$ only if $x \leqslant \max\{1, 2|y|\}$.

This result will be established in §3. The number field K is considered fixed once and for all. Thus all implied constants in our work are allowed to depend implicitly on K. One obtains a formula for the indicator function on $\mathfrak{o}$ by restricting to principal ideals, in which case one writes $\delta_K((\alpha)) = \delta_K(\alpha)$ for any $\alpha \in \mathfrak{o}$.

Given the broad impact that [4] and [8] have had on number-theoretic problems over $\mathbb{Q}$, one might view Theorem 1.2 as foundational in a systematic programme of work to extend our understanding to the setting of general number fields. In §5 we will indicate how Theorem 1.2 can be used to count K-rational points of bounded height on projective hypersurfaces. This will form the basis of our proof of Theorem 1.1.

2. Technical preliminaries

Our work will require a good deal of notation. In this section we collect together the necessary conventions, in addition to some preliminary technical tools, relevant to our number field K of degree d over $\mathbb{Q}$. Let r_1 (resp. $2r_2$) be the number of distinct real (resp. complex) embeddings of K, with $d = r_1 + 2r_2$. Given any $\alpha \in K$ we will denote the norm and trace by $\mathrm{N}_{K/\mathbb{Q}}(\alpha)$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha)$, respectively. Let $\rho_1, \ldots, \rho_{r_1}$ be the r_1 distinct real embeddings and let $\rho_{r_1+1}, \ldots, \rho_{r_1+2r_2}$ be a complete set of $2r_2$ distinct complex embeddings, with ρ_{r_1+i} conjugate to $\rho_{r_1+r_2+i}$ for $1 \leqslant i \leqslant r_2$. Let V denote the d-dimensional commutative $\mathbb{R}$-algebra

$$K \otimes_{\mathbb{Q}} \mathbb{R} \cong \bigoplus_{l=1}^{r_1+r_2} K_l,$$

where K_l is the completion of K with respect to ρ_l, for $1 \leqslant l \leqslant r_1 + r_2$. Thus $K_l = \mathbb{R}$ (resp. $K_l = \mathbb{C}$) for $1 \leqslant l \leqslant r_1$ (resp. for $r_1 < l \leqslant r_2$). There is a canonical embedding of K into V

and we shall identify K with its image in V. Given $v = (v^{(1)}, \ldots, v^{(r_1+r_2)}) \in V$ we define

$$\mathrm{Nm}(v) = v^{(1)} \cdots v^{(r_1)} |v^{(r_1+1)}|^2 \cdots |v^{(r_1+r_2)}|^2,$$
$$\mathrm{Tr}(v) = v^{(1)} + \cdots + v^{(r_1)} + 2\Re(v^{(r_1+1)}) + \cdots + 2\Re(v^{(r_1+r_2)}). \tag{2.1}$$

Furthermore, we define the character $\mathrm{e}(\cdot) = e^{2\pi i \,\mathrm{Tr}(\cdot)}$ on V. We will typically write $v^{(l)} \in K_l$ for the projection of any $v \in V$ onto the lth component, for $1 \leqslant l \leqslant r_1 + r_2$. Thus any $v \in V$ can be written $v = \bigoplus_l v^{(l)}$. Likewise, given a vector $\mathbf{v} \in V^n$, we will usually denote by $\mathbf{v}^{(l)} \in K_l^n$ the projection of the vector onto the lth component.

The canonical embedding of K into V is given by $\alpha \mapsto (\rho_1(\alpha), \ldots, \rho_{r_1+r_2}(\alpha))$. As mentioned previously we shall identify K with its image in V, under which any fractional ideal becomes a lattice in V. Let $\{\omega_1, \ldots, \omega_d\}$ be a $\mathbb{Z}$-basis for $\mathfrak{o}$. Then $\{\omega_1, \ldots, \omega_d\}$ forms an $\mathbb{R}$-basis for V and we may view V as the set $\{x_1\omega_1 + \cdots + x_d\omega_d : x_i \in \mathbb{R}\}$.

We will need to introduce some norms on V and V^n. To begin with let

$$\langle v \rangle = \max_{1 \leqslant l \leqslant r_1+r_2} |v^{(l)}|,$$

for any $v \in V$. We extend this to V^n in the obvious way. Next, let

$$c_l = \begin{cases} 1, & \text{if } 1 \leqslant l \leqslant r_1, \\ 2, & \text{if } r_1 < l \leqslant r_1 + r_2. \end{cases} \tag{2.2}$$

We will also need to introduce a Euclidean norm $\| \cdot \|$ on V, given by

$$\|v\| = \sqrt{\sum_{1 \leqslant l \leqslant r_1+r_2} c_l |v^{(l)}|^2}.$$

We extend this to V^n by setting

$$\|\mathbf{v}\| = \sqrt{\sum_{1 \leqslant i \leqslant n} \|v_i\|^2} = \sqrt{\sum_{1 \leqslant l \leqslant r_1+r_2} c_l |\mathbf{v}^{(l)}|^2},$$

if $\mathbf{v} = (v_1, \ldots, v_n) \in V^n$.

We will make frequent use of the dual form with respect to the trace. For any fractional ideal $\mathfrak{a}$ in K one defines the dual ideal

$$\hat{\mathfrak{a}} = \{\alpha \in K : \mathrm{Tr}_{K/\mathbb{Q}}(\alpha\mathfrak{a}) \subseteq \mathbb{Z}\}.$$

In particular $\hat{\mathfrak{a}} = \mathfrak{a}^{-1}\mathfrak{d}^{-1}$, where

$$\mathfrak{d} = \{\alpha \in K : \alpha\hat{\mathfrak{o}} \subseteq \mathfrak{o}\}$$

denotes the different ideal of K and is itself an integral ideal. One notes that $\hat{\mathfrak{o}} = \mathfrak{d}^{-1}$. Furthermore, we have $\hat{\mathfrak{a}} \subseteq \hat{\mathfrak{b}}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}$. An additional integral ideal featuring in our work is the denominator ideal

$$\mathfrak{a}_\gamma = \{\alpha \in \mathfrak{o} : \alpha\gamma \in \mathfrak{o}\},$$

associated to any $\gamma \in K$.

We let D_K denote the absolute discriminant of K. Finally, we will reserve E_K for denoting the set of units in $\mathfrak{o}$. For an integral ideal $\mathfrak{a}$, the notation $\sum_{\alpha \in \mathfrak{a}/E_K}$ means that the sum is over elements in $\mathfrak{a}$ modulo the action of E_K. Given an element $v \in V$ it will sometimes prove advantageous to use the action of E_K to control the size of each component $v^{(l)}$ of v. This is the object of the following standard result.

Lemma 2.1. — *Let $v = \bigoplus_l v^{(l)} \in V$. Then there exists $u \in E_K$ such that*

$$|\operatorname{Nm}(v)|^{1/d} \ll |(uv)^{(l)}| \ll |\operatorname{Nm}(v)|^{1/d},$$

for $1 \leqslant l \leqslant r_1 + r_2$.

Proof. — Let $\varphi : E_K \to \mathbb{R}^{r_1+r_2}$ be the group homomorphism

$$u \mapsto (c_1 \log |u^{(1)}|, \ldots, c_{r_1+r_2} \log |u^{(r_1+r_2)}|),$$

where c_l is given by (2.2). Then $\varphi(E_K)$ forms a full $r_1 + r_2 - 1$ dimensional lattice in the hyperplane

$$H = \{\mathbf{e} \in \mathbb{R}^{r_1+r_2} : e_1 + \cdots + e_{r_1+r_2} = 0\}.$$

It follows that for any $\mathbf{e} \in H$, there exists a unit $u \in E_K$ such that $|\varphi(u) - \mathbf{e}| \ll 1$.

Now let $a_l = c_l \log |v^{(l)}|$ and $b_l = c_l(\log |\operatorname{Nm}(v)|)/d$, for $1 \leqslant l \leqslant r_1 + r_2$. Then it is clear that $\mathbf{e} = \mathbf{a} - \mathbf{b} \in H$. Hence we can find $w \in E_K$ such that $|\varphi(w) - \mathbf{e}| \ll 1$, which implies that $|\mathbf{a} - \varphi(w) - \mathbf{b}| \ll 1$. The lemma follows on taking $u = w^{-1}$. $\qquad\square$

2.1. The Dedekind zeta function.

— In this section we discuss the Dedekind zeta function associated to K. All of the facts that we record may be found in the work of Landau [10], for example. The Dedekind zeta function is defined to be

$$\zeta_K(s) = \sum_{(0) \neq \mathfrak{a} \subseteq \mathfrak{o}} (\operatorname{N} \mathfrak{a})^{-s},$$

for any $s = \sigma + it \in \mathbb{C}$ with $\sigma = \Re(s) > 1$. The zeta function admits a meromorphic continuation to the entire complex plane with a simple pole at $s = 1$. Let

$$A = \frac{\sqrt{D_K}}{2^{r_2} \pi^{d/2}}.$$

The functional equation for the Dedekind zeta function may be written $\Phi(s) = \Phi(1-s)$, with

$$\Phi(s) = A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s).$$

One recalls that $\Gamma(s)$ has simple poles at each non-positive integer. By the class number formula we have

$$\Delta_K = \operatorname{Residue}_{s=1} \zeta_K(s) = \frac{h_K 2^{r_1} (2\pi)^{r_2} R_K}{w_K \sqrt{D_K}}, \tag{2.3}$$

where h_K is the class number of K, w_K is the number of roots of unity in K and R_K is the regulator. It follows from the functional equation that $\zeta_K(s)$ has a zero of order $r_1 + r_2 - 1$ at $s = 0$ and zeros of order $r_1 + r_2$ at all negative even integers. Noting that $\Gamma(1/2) = \sqrt{\pi}$ it easily follows that

$$\Upsilon_K = \operatorname{Residue}_{s=0} \frac{\zeta_K(s)}{s^{r_1+r_2}} = -\frac{\Delta_K \sqrt{D_K}}{2^{r_1} (2\pi)^{r_2}}. \tag{2.4}$$

We will also require some information about the order of magnitude of $\zeta_K(s)$. The functional equation yields $\zeta_K(s) = \gamma_K(s) \zeta_K(1-s)$, with

$$\gamma_K(s) = A^{1-2s} \left(\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right)^{r_1} \left(\frac{\Gamma(1-s)}{\Gamma(s)}\right)^{r_2}.$$

In any fixed strip $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, an application of Stirling's formula yields the existence of $\Lambda_\sigma \in \mathbb{C}$ and $\Lambda \in \mathbb{R}$, depending on K, such that

$$\gamma_K(s) = \Lambda_\sigma t^{(1/2-\sigma)d} e^{-\pi it(\log t - \Lambda)} \left(1 + O\left(\frac{1}{t}\right)\right), \tag{2.5}$$

as $t \to \infty$. This is established in [**10**, Satz 166]. Here $\Lambda = \log 2\pi + 1 - \frac{1}{d} \log D_K$ is an absolute constant but Λ_σ depends on σ and satisfies $|\Lambda_\sigma| = D_K^{1/2-\sigma} (2\pi)^{d(\sigma-1/2)}$. Let

$$\mu_K(\sigma) = \limsup_{t \to \pm\infty} \frac{\log|\zeta_K(\sigma + it)|}{\log|t|}.$$

We have

$$\mu_K(\sigma) \leqslant \begin{cases} 0, & \text{if } \sigma > 1, \\ (1-\sigma)d/2, & \text{if } 0 \leqslant \sigma \leqslant 1, \\ (1/2 - \sigma)d, & \text{if } \sigma < 0. \end{cases} \tag{2.6}$$

Here the first inequality is obvious and the final inequality follows from the functional equation for $\zeta_K(s)$ and (2.5). The middle inequality is a consequence of the convexity of $\mu_K(\sigma)$.

2.2. Smooth weight functions. — Let K be a number field of degree d, as previously, and let V be the associated $\mathbb{R}$-algebra $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. When dealing with functions on V or V^n it will occasionally be convenient to work with alternative coordinates $v = (\mathbf{v}, \mathbf{y} + i\mathbf{z})$ on V, with

$$\mathbf{v} = (v_1 \ldots, v_{r_1}), \quad \mathbf{y} = (y_1 \ldots, y_{r_2}), \quad \mathbf{z} = (z_1 \ldots, z_{r_2}).$$

We will then employ the volume form

$$\mathrm{d}v = \mathrm{d}v_1 \cdots \mathrm{d}v_{r_1} \mathrm{d}y_1 \mathrm{d}z_1 \cdots \mathrm{d}y_{r_2} \mathrm{d}z_{r_2},$$

on V, where $\mathrm{d}v_i$ is the standard Lebesgue measure on $\mathbb{R}$ for $1 \leqslant i \leqslant r_1$ and $\mathrm{d}y_j, \mathrm{d}z_j$ are, respectively, the standard Lebesgue measures on $\Re(\mathbb{C})$ and $\Im(\mathbb{C})$, for $1 \leqslant j \leqslant r_2$. The corresponding volume form on V^n will be denoted by $\mathrm{d}\mathbf{v}$ or $\mathrm{d}\mathbf{x}$.

Our work will make prevalent use of smooth weight functions on V, and more generally on V^n, for integer $n \geqslant 1$. For us a smooth weight function on V^n is any infinitely differentiable function $w : V^n \to \mathbb{C}$ which has compact support. The latter is equivalent to the existence of $A > 0$ such that w is supported on the hypercube $[-A, A]^{dn}$. Let $n = 1$. For any $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_{\geqslant 0}^d$ and any smooth weight function w on V, we will use the notation

$$\partial^{\boldsymbol{\beta}} w(v) = \prod_{i=1}^{r_1} \partial_{v_i}^{\beta_i} \prod_{j=1}^{r_2} \partial_{y_j}^{\beta_{j+r_1}} \partial_{z_j}^{\beta_{j+r_1+r_2}} w(v).$$

We will denote the "degree" of $\boldsymbol{\beta}$ by $|\boldsymbol{\beta}| = \beta_1 + \cdots + \beta_d$. When $n \geqslant 1$ is arbitrary, an analogous definition of $\partial^{\boldsymbol{\beta}} w(\mathbf{x})$ will be used for any smooth weight w on V^n, for $\boldsymbol{\beta} \in \mathbb{Z}_{\geqslant 0}^{dn}$. For a smooth weight w on V^n, and any $N \geqslant 0$, we let

$$\lambda_w^N = \sup_{\substack{\mathbf{x} \in V^n \\ |\boldsymbol{\beta}| \leqslant N}} \left| \partial^{\boldsymbol{\beta}} w(\mathbf{x}) \right|. \tag{2.7}$$

We will henceforth write $\mathscr{W}_n(V)$ for the set of smooth weight functions w on V^n for which λ_w^N is bounded by an absolute constant $A_N > 0$, for each integer $N \geqslant 0$. We will write $\mathscr{W}_n^+(V)$ for the subset of $w \in \mathscr{W}_n(V)$ which take values on non-negative real numbers only.

In what follows, unless explicitly indicated otherwise, we will allow the implied constant in any estimate involving a weight $w \in \mathscr{W}_n(V)$ to depend implicitly on A and A_N.

2.3. Additive characters over K. — Given any integral ideal $\mathfrak{b}$ of K, an additive character modulo $\mathfrak{b}$ is defined to be a non-zero function σ on $\mathfrak{o}/\mathfrak{b}$ which satisfies

$$\sigma(\alpha_1 + \alpha_2) = \sigma(\alpha_1)\sigma(\alpha_2),$$

for any $\alpha_1, \alpha_2 \in \mathfrak{o}$. Such a character is said to be primitive if it is not a character modulo $\mathfrak{c}$ for any ideal $\mathfrak{c} \mid \mathfrak{b}$, with $\mathfrak{c} \neq \mathfrak{b}$. We will make use of the basic orthogonality relation

$$\sum_{\sigma \, (\mathrm{mod}\, \mathfrak{b})} \sigma(\alpha) = \begin{cases} \mathrm{N}\,\mathfrak{b}, & \text{if } \mathfrak{b} \mid (\alpha), \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

for any $\alpha \in \mathfrak{o}$, where the notation $\sum_{\sigma \, (\mathrm{mod}\, \mathfrak{b})}$ means that the sum is taken over additive characters modulo $\mathfrak{b}$. In fact there is an isomorphism between the additive characters modulo $\mathfrak{b}$ and the residue classes modulo $\mathfrak{b}$ and so the number of distinct characters is $\mathrm{N}\,\mathfrak{b}$. In this isomorphism primitive characters correspond to residue classes that are relatively prime to $\mathfrak{b}$. Hence there are $\varphi(\mathfrak{b})$ distinct primitive characters modulo $\mathfrak{b}$. It is easy to see that if σ_0 is a fixed primitive character modulo $\mathfrak{b}$ then, as β runs through elements of $\mathfrak{o}/\mathfrak{b}$ (respectively, through elements of $(\mathfrak{o}/\mathfrak{b})^*$), the functions $\sigma_0(\beta\cdot)$ give all the characters (respectively, primitive characters) modulo $\mathfrak{b}$ exactly once.

For a given integral ideal $\mathfrak{b}$ we now proceed to construct an explicit non-trivial primitive character modulo $\mathfrak{b}$. For this will need some preliminary algebraic facts to hand. Recall the notation $\mathfrak{d}$ and $\mathfrak{a}_\gamma$, for the different ideal and denominator ideal, respectively.

Lemma 2.2. — *Let $\varepsilon > 0$, let $\mathfrak{b}, \mathfrak{c}$ be integral ideals and let $\gamma \in K^*$. Then we have the following:*

(i) *there exists $\alpha \in \mathfrak{b}$ such that $\mathrm{ord}_\mathfrak{p}(\alpha) = \mathrm{ord}_\mathfrak{p}(\mathfrak{b})$ for every prime ideal $\mathfrak{p} \mid \mathfrak{c}$;*
(ii) *there exists $\alpha \in \mathfrak{b}$ and an unramified prime ideal $\mathfrak{p}$ coprime to $\mathfrak{b}$, with $\mathrm{N}\,\mathfrak{p} \ll (\mathrm{N}\,\mathfrak{b})^\varepsilon$, such that $(\alpha) = \mathfrak{b}\mathfrak{p}$.*

Proof. — Part (i) is due to Skinner [15, Lemma 1]. For part (ii) we note that $\mathfrak{b}$ has $\omega(\mathfrak{b}) = O(\log \mathrm{N}\,\mathfrak{b})$ prime ideal divisors. However the number of distinct prime ideals with norm at most $(\mathrm{N}\,\mathfrak{b})^\varepsilon$ is $\gg (\mathrm{N}\,\mathfrak{b})^{\varepsilon/2}$. Using the equidistribution of prime ideals in ideal classes we therefore deduce that there exists an unramified prime ideal $\mathfrak{p}$ such that $\mathrm{N}\,\mathfrak{p} \ll (\mathrm{N}\,\mathfrak{b})^\varepsilon$, with $\mathfrak{p} \nmid \mathfrak{b}$ and $\mathfrak{b}\mathfrak{p}$ principal. Any generator α of $\mathfrak{b}\mathfrak{p}$ then satisfies the properties claimed. $\square$

The α constructed in part (ii) satisfies $\mathrm{ord}_\mathfrak{q}(\alpha) = \mathrm{ord}_\mathfrak{q}(\mathfrak{b})$ for every prime ideal $\mathfrak{q} \mid \mathfrak{b}$, with $|\mathrm{N}_{K/\mathbb{Q}}(\alpha)| \ll (\mathrm{N}\,\mathfrak{b})^{1+\varepsilon}$. Thus part (ii) is a refinement of part (i) in the special case $\mathfrak{c} = \mathfrak{b}$, with additional control over the size of the norm of α. We now have everything in place to construct our non-trivial character modulo an integral ideal $\mathfrak{b}$. Consider the integral ideal

$$\mathfrak{c} = \mathfrak{b}\mathfrak{d}.$$

By Lemma 2.2 (ii) we can find an integer $\alpha \in \mathfrak{c}$ and an unramified prime ideal $\mathfrak{p}_1$ coprime to $\mathfrak{b}$, with $\mathrm{N}\,\mathfrak{p}_1 \ll (\mathrm{N}\,\mathfrak{b})^\varepsilon$, such that $(\alpha) = \mathfrak{c}\mathfrak{p}_1$. Applying Lemma 2.2 (i) we see that there exists $\nu \in \mathfrak{o}$ such that $\mathfrak{p}_1 \mid (\nu)$ but (ν) and $\mathfrak{c}$ are coprime. It follows that $\mathfrak{c} = \mathfrak{a}_\gamma$ with $\gamma = \nu/\alpha$. Indeed, $\beta \in \mathfrak{a}_\gamma$ if and only if $(\beta\nu) \subseteq (\alpha) = \mathfrak{c}\mathfrak{p}_1$, which is if and only if $\beta \in \mathfrak{c}$. We claim that

$$\sigma_0(\cdot) = \mathrm{e}(\gamma\cdot)$$

defines a non-trivial primitive character modulo $\mathfrak{b}$.

To check the claim we note that σ_0 is a trivial character if and only if $\gamma \in \hat{\mathfrak{o}}$. But this holds if and only if $\mathfrak{a}_\gamma \supseteq \mathfrak{d}$, which is so if and only if $\mathfrak{c} = \mathfrak{a}_\gamma \mid \mathfrak{d}$. This is clearly impossible. Suppose now that $x, z \in \mathfrak{o}$, with $\mathfrak{b} \mid (z)$. Then it is clear that $\mathfrak{a}_\gamma \mid (z)\mathfrak{d}$, whence $\gamma z \in \hat{\mathfrak{o}}$. Thus $\sigma_0(x + z) = \sigma_0(x)$ and so it follows that σ_0 is a non-trivial character modulo $\mathfrak{b}$. Lastly, we need to check the primitivity of the character. Let $\mathfrak{b}_1 \mid \mathfrak{b}$ be any proper ideal divisor. Then the ideal $\mathfrak{b}_1 \gamma$ is not contained in $\hat{\mathfrak{o}}$ and so there exists $z \in \mathfrak{b}_1$ such that $\gamma z \notin \hat{\mathfrak{o}}$. This implies that σ_0 is not a character modulo any ideal $\mathfrak{b}_1 \mid \mathfrak{b}$, with $\mathfrak{b}_1 \neq \mathfrak{b}$, so that it is in fact a primitive character modulo $\mathfrak{b}$.

We may summarise our investigation in the following result.

Lemma 2.3. — *Let $\varepsilon > 0$ and let $\mathfrak{b}$ be an integral ideal. Then there exists $\gamma \in K$, with $\gamma = \nu/\alpha$ for $\alpha \in \mathfrak{b}\mathfrak{d}$ and $\nu \in \mathfrak{o}$ such that (ν) is coprime to $\mathfrak{b}\mathfrak{d}$, together with a prime ideal $\mathfrak{p}_1 \mid (\nu)$ satisfying $\mathrm{N}\,\mathfrak{p}_1 \ll (\mathrm{N}\,\mathfrak{b})^\varepsilon$ and $(\alpha) = \mathfrak{b}\mathfrak{d}\mathfrak{p}_1$, such that $\mathrm{e}(\gamma \cdot)$ defines a non-trivial primitive additive character modulo $\mathfrak{b}$. In particular, we have*

$$\sideset{}{^*}\sum_{\sigma \,(\mathrm{mod}\,\mathfrak{b})} \sigma(x) = \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \mathrm{e}(a\gamma x),$$

for any $x \in \mathfrak{o}$.

Finally, we need to discuss how additive characters modulo $\mathfrak{b}$ can be extended to arbitrary integral ideals $\mathfrak{a} \subseteq \mathfrak{o}$, as in the statement of Theorem 1.2. By part (i) of Lemma 2.2 there exists $\alpha \in \mathfrak{o}$ such that $\mathrm{ord}_\mathfrak{p}(\alpha) = \mathrm{ord}_\mathfrak{p}(\mathfrak{a})$ for all $\mathfrak{p} \mid \mathfrak{a}\mathfrak{b}\mathfrak{d}$. Given any additive character modulo $\mathfrak{b}$ we will extend it to $\mathfrak{a}$ by setting $\sigma(\mathfrak{a}) = \sigma(\alpha)$. On noting that $\mathfrak{b} \mid (\alpha)$ if and only if $\mathfrak{b} \mid \mathfrak{a}$, we conclude from (2.8) that

$$\sum_{\sigma \,(\mathrm{mod}\,\mathfrak{b})} \sigma(\mathfrak{a}) = \begin{cases} \mathrm{N}\,\mathfrak{b}, & \text{if } \mathfrak{b} \mid \mathfrak{a}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

We will need to know that the sum

$$\sideset{}{^*}\sum_{\sigma \,(\mathrm{mod}\,\mathfrak{b})} \sigma(\mathfrak{a}),$$

over primitive characters modulo $\mathfrak{b}$, does not depend on the choice of α. This is achieved in the following result.

Lemma 2.4. — *Let $\mathfrak{a}$ be an integral ideal. Let $\alpha_1, \alpha_2 \in \mathfrak{o}$ such that $\mathrm{ord}_\mathfrak{p}(\alpha_i) = \mathrm{ord}_\mathfrak{p}(\mathfrak{a})$ for $i = 1, 2$ and all $\mathfrak{p} \mid \mathfrak{a}\mathfrak{b}\mathfrak{d}$. Then*

$$\sideset{}{^*}\sum_{\sigma \,(\mathrm{mod}\,\mathfrak{b})} \sigma(\alpha_1) = \sideset{}{^*}\sum_{\sigma \,(\mathrm{mod}\,\mathfrak{b})} \sigma(\alpha_2).$$

Proof. — Let $\mathfrak{a}_1, \mathfrak{a}_2$ be coprime integral ideals, which are also coprime to $\mathfrak{a}\mathfrak{b}\mathfrak{d}$, such that $(\alpha_1^{-1}\alpha_2) = \mathfrak{a}_1^{-1}\mathfrak{a}_2$. Since $\mathfrak{a}_1^{-1}\mathfrak{a}_2$ is principal, the ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are in the same ideal class. Let $\mathfrak{p}_1$ be a prime ideal, coprime to $\mathfrak{a}\mathfrak{b}\mathfrak{d}$, such that $\mathfrak{a}_1\mathfrak{p}_1$ and $\mathfrak{a}_2\mathfrak{p}_1$ are principal ideals. Thus we can choose integers x and y such that $\mathfrak{a}_1\mathfrak{p}_1 = (\beta_1)$ and $\mathfrak{a}_2\mathfrak{p}_1 = (\beta_2)$, with $\beta_1\alpha_2 = \beta_2\alpha_1$. In particular β_1, β_2 are coprime to $\mathfrak{b}\mathfrak{d}$. Denoting by $\overline{\beta_2}$ the multiplicative inverse of β_2 modulo

$\mathfrak{b}\mathfrak{d}$, it follows that

$$\sideset{}{^*}\sum_{\sigma\,(\mathrm{mod}\,\mathfrak{b})} \sigma(\alpha_1) = \sideset{}{^*}\sum_{\sigma\,(\mathrm{mod}\,\mathfrak{b})} \sigma(\overline{\beta_2}\beta_2\alpha_1)$$

$$= \sideset{}{^*}\sum_{\sigma\,(\mathrm{mod}\,\mathfrak{b})} \sigma(\overline{\beta_2}\beta_1\alpha_2)$$

$$= \sideset{}{^*}\sum_{\sigma\,(\mathrm{mod}\,\mathfrak{b})} \sigma(\alpha_2),$$

as required. $\qquad\square$

2.4. Counting rational points on algebraic varieties. — Let $G \in \mathfrak{o}[X_1, \ldots, X_n]$ be an absolutely irreducible homogeneous polynomial, with degree at least 2. We will need an estimate for the number of $\mathbf{x} \in \mathfrak{o}^n$ for which $G(\mathbf{x}) = 0$, subject to certain constraints.

Lemma 2.5. — *Let $B = \bigoplus_l B^{(l)} \in V$, with $B^{(l)} > 0$ and $\mathrm{Nm}(B) \geqslant 2$. Let $\varepsilon > 0$, let $\mathfrak{c}$ be an integral ideal and let $\mathbf{a} \in \mathfrak{o}^n$. Then we have*

$$\#\left\{\mathbf{x} \in \mathfrak{o}^n : \begin{array}{l} |\mathbf{x}^{(l)}| \leqslant B^{(l)}, \\ G(\mathbf{x}) = 0, \ \mathbf{x} \equiv \mathbf{a}\,(\mathrm{mod}\,\mathfrak{c}) \end{array}\right\} \ll (\mathrm{Nm}(B))^\varepsilon \left(1 + \frac{\mathrm{Nm}(B)}{\mathrm{N}\,\mathfrak{c}}\right)^{n-3/2}.$$

The implied constant in this estimate depends at most on G and the choice of ε. Lemma 2.5 is a generalisation of [**7**, Lemma 15] to the number field setting. One expects that one should be able to replace the exponent $3/2$ by 2.

Proof of Lemma 2.5. — We denote by $N(B; \mathfrak{c}, \mathbf{a})$ the quantity that is to be estimated. For the proof we may assume without loss of generality that $\mathrm{N}\,\mathfrak{c} \leqslant \mathrm{Nm}(B)/2$, for otherwise the result follows from the trivial bound $N(B; \mathfrak{c}, \mathbf{a}) \ll (1 + \mathrm{Nm}(B)/\mathrm{N}\,\mathfrak{c})^n$. Let us define $N(B; \mathfrak{c}) = \max_{\mathbf{a}\in\mathfrak{o}^n} N(B; \mathfrak{c}, \mathbf{a})$ and let $\delta > 0$.

Applying Lemma 2.2(ii) we see that there exists $\gamma \in \mathfrak{o}$ and an unramified prime ideal $\mathfrak{p}$ coprime to $\mathfrak{c}$, such that $(\gamma) = \mathfrak{p}\mathfrak{c}$ and $\mathrm{N}\,\mathfrak{p} \ll (\mathrm{N}\,\mathfrak{c})^\delta$. Let us set $\mathfrak{g} = (\gamma)$. Then it follows that

$$N(B; \mathfrak{c}, \mathbf{a}) = \sum_{\mathbf{b}\in(\mathfrak{c}/\mathfrak{g})^n} N(B; \mathfrak{g}, \mathbf{a} + \mathbf{b}) \ll (\mathrm{N}\,\mathfrak{c})^{\delta n} N(B; \mathfrak{g}), \qquad (2.10)$$

since $\#\mathfrak{c}/\mathfrak{g} = [\mathfrak{c} : \mathfrak{g}] = [\mathfrak{o} : \mathfrak{g}]/[\mathfrak{o} : \mathfrak{c}] = \mathrm{N}\,\mathfrak{p}$. For any $u \in E_K$, where E_K denotes the group of units of $\mathfrak{o}$, we let $\tilde{u} = \bigoplus_l |u^{(l)}|$. Then we have

$$N(B; \mathfrak{g}) \leqslant \max_{\mathbf{a}\in\mathfrak{o}} \#\left\{u\mathbf{x} \in \mathfrak{o}^n : \begin{array}{l} |u^{(l)}\mathbf{x}^{(l)}| \leqslant |u^{(l)}|B^{(l)}, \\ G(u\mathbf{x}) = 0, \ u\mathbf{x} \equiv u\mathbf{a}\,(\mathrm{mod}\,\mathfrak{g}) \end{array}\right\}$$

$$\leqslant \max_{\mathbf{a}\in\mathfrak{o}} \#\left\{\mathbf{y} \in \mathfrak{o}^n : \begin{array}{l} |\mathbf{y}^{(l)}| \leqslant (\tilde{u}B)^{(l)}, \\ G(\mathbf{y}) = 0, \ \mathbf{y} \equiv \mathbf{a}\,(\mathrm{mod}\,\mathfrak{g}) \end{array}\right\}$$

$$= N(\tilde{u}B; \mathfrak{g}).$$

According to Lemma 2.1 it therefore suffices to estimate the quantity $N(B; \mathfrak{g})$, for B satisfying $B^{(l)} \ll \mathrm{Nm}(B)^{1/d}$. Likewise, a further application of Lemma 2.1 allows us to assume that $\mathfrak{g} = (\gamma)$, with $(\mathrm{N}\,\mathfrak{g})^{1/d} \ll |\gamma^{(l)}| \ll (\mathrm{N}\,\mathfrak{g})^{1/d}$. Now let $\mathbf{x}$ be any vector in V^n. Then there exists $\mathbf{y} \in \mathfrak{o}^n$ such that $\langle \gamma^{-1}\mathbf{x} - \mathbf{y}\rangle \ll 1$. This implies that $\langle \mathbf{x} - \gamma\mathbf{v}\rangle \ll (\mathrm{N}\,\mathfrak{g})^{1/d}$, whence given any $\mathbf{x} \in V^n$ we can find a vector $\mathbf{y} \in \mathfrak{g}^n = \gamma\mathfrak{o}^n$ such that $\langle \mathbf{x} - \mathbf{y}\rangle \ll (\mathrm{N}\,\mathfrak{g})^{1/d}$. Therefore, in our analysis of $N(B; \mathfrak{g}, \mathbf{a})$ for given $\mathbf{a} \in \mathfrak{o}^n$, it suffices to assume that $\langle \mathbf{a}\rangle \ll (\mathrm{N}\,\mathfrak{g})^{1/d}$.

Let $Y \subseteq \mathbb{A}_K^n$ be the hypersurface $G(\mathbf{a} + \gamma\mathbf{x}) = 0$. This is clearly absolutely irreducible of degree at least 2. Given any $H \geqslant 2$, it follows from an application of the large sieve (see Serre [14, Chapter 13]) that

$$\#\{\mathbf{x} \in \mathfrak{o}^n \cap Y : \langle \mathbf{x} \rangle \leqslant H\} \ll H^{(n-3/2)d} \log H.$$

Furthermore, an inspection of the proof reveals that the implied constant is independent of $\mathbf{a}$ and γ. It now follows

$$\begin{aligned}
N(B; \mathfrak{g}, \mathbf{a}) &= \# \left\{ \mathbf{x} \in \mathfrak{o}^n : \begin{array}{l} |(\mathbf{a} + \gamma\mathbf{x})^{(l)}| \leqslant B^{(l)}, \\ G(\mathbf{a} + \gamma\mathbf{x}) = 0 \end{array} \right\} \\
&\leqslant \# \left\{ \mathbf{x} \in \mathfrak{o}^n \cap Y : |\mathbf{x}^{(l)}| \ll H \right\} \\
&\ll H^{(n-3/2)d} \log \mathrm{Nm}(B),
\end{aligned}$$

where $H = (1 + \mathrm{Nm}(B)/\mathrm{N}\,\mathfrak{g})^{1/d}$. Inserting this into (2.10) and taking any $\delta < \varepsilon/n$ therefore leads to the conclusion of the lemma. $\qquad\square$

2.5. Poisson summation over K. — We will need a version of the Poisson summation formula for number fields. This is provided for us by the work of Friedman and Skoruppa [5], in which we take the base field to be $\mathbb{Q}$ (and so $m = 1$). On $\mathbb{R}_{>0}$ we define the function

$$\begin{aligned}
k_{r_1,r_2}(t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} \left(\frac{\pi^{-z}\Gamma(\frac{z}{2})}{\Gamma(\frac{1-z}{2})} \right)^{r_1} \left(\frac{(2\pi)^{-2z}\Gamma(z)}{\Gamma(1-z)} \right)^{r_2} \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} g(z)^{r_1} h(z)^{r_2} \mathrm{d}z,
\end{aligned}$$

say, for any $0 < c < 1/6$. Then k_{r_1,r_2} is a Mellin convolution $k_{1,0}^{r_1} * k_{0,1}^{r_2}$, with

$$k_{1,0}(t) = \frac{2}{\sqrt{\pi}} \cos(2\pi t), \quad k_{0,1}(t) = J_0(4\pi\sqrt{t}).$$

Let $f \in \mathscr{W}_1(\mathbb{R})$ and let $\mathfrak{a}$ be a fractional ideal of K. Then the version of Poisson summation that we need takes the form

$$\sum_{\substack{\alpha \in \mathfrak{a}/E_K \\ \alpha \neq 0}} f(|\mathrm{N}_{K/\mathbb{Q}}\,\alpha|) = \frac{\Delta_K}{h_K \mathrm{N}\,\mathfrak{a}} \int_0^\infty f(t)\mathrm{d}t + \frac{2^{r_2}\pi^{d/2}}{\sqrt{D_K}\mathrm{N}\,\mathfrak{a}} \sum_{\substack{\beta \in \hat{\mathfrak{a}}/E_K \\ \beta \neq 0}} \tilde{f}(|\mathrm{N}_{K/\mathbb{Q}}\,\beta|), \qquad (2.11)$$

where Δ_K is given by (2.3) and

$$\tilde{f}(y) = \int_0^\infty f(t)k_{r_1,r_2}(ty)\mathrm{d}t,$$

as a function on $\mathbb{R}_{>0}$. In fact we have

$$\tilde{f}(y) \ll_N y^{-N}, \qquad (2.12)$$

for any $N \geqslant 0$. This follows on noting that

$$\begin{aligned}
\tilde{f}(y) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-z} g(z)^{r_1} h(z)^{r_2} \int_0^\infty f(t)t^{-z}\mathrm{d}t\mathrm{d}z \\
&= \frac{1}{2\pi i} \int_{1-c-i\infty}^{1-c+i\infty} F(w)y^{w-1} g(1-w)^{r_1} h(1-w)^{r_2}\mathrm{d}w,
\end{aligned}$$

where F is the Mellin transform of f. Since f is smooth and compactly supported, it follows that F is entire and of rapid decay, enough to counter the polynomial growth of the functions g and h. The estimate (2.12) follows on shifting the contour sufficiently far to the left.

Using (2.11) we may deduce a corresponding version in which the sum over elements is replaced by a sum over ideals. In fact, we will often be led to consider sums of the form

$$\sum_{\mathfrak{b} \neq (0)} f(R \operatorname{N} \mathfrak{b}),$$

for a parameter $R > 0$, with $f \in \mathscr{W}_1(\mathbb{R})$ and the sum being taken over integral ideals. To handle this sum we let $[\mathfrak{c}_1], \ldots, [\mathfrak{c}_{h_K}]$ denote distinct cosets for the class group $C(K)$. For each $1 \leqslant j \leqslant h_K$, we have a bijection between the set S_j of integral ideals in $[\mathfrak{c}_j]$ and the elements of the ideal $\mathfrak{c}_j^{-1}$ modulo the action of the unit group E_K. The explicit bijection is $\psi_j : S_j \to \mathfrak{c}_j^{-1}/E_K$, given by $\psi_j(\mathfrak{b}) = \mathfrak{c}_j^{-1} \mathfrak{b}$. Using our decomposition into ideal classes we may therefore write

$$\sum_{\mathfrak{b} \neq (0)} f(R \operatorname{N} \mathfrak{b}) = \sum_{j=1}^{h_K} \sum_{\substack{\alpha \in \mathfrak{c}_j^{-1}/E_K \\ \alpha \neq 0}} f(R| \operatorname{N}(\alpha \mathfrak{c}_j)|).$$

Let $f_j(t) = f(Rt \operatorname{N} \mathfrak{c}_j)$. We analyse the inner sum using the Poisson summation formula in the form (2.11). This gives

$$\sum_{\substack{\alpha \in \mathfrak{c}_j^{-1}/E_K \\ \alpha \neq 0}} f_j(| \operatorname{N}_{K/\mathbb{Q}}(\alpha)|) = \frac{\Delta_K \operatorname{N} \mathfrak{c}_j}{h_K} \int_0^\infty f_j(t) \mathrm{d}t + \frac{2^{r_2} \pi^{d/2} \operatorname{N} \mathfrak{c}_j}{\sqrt{D_K}} \sum_{\substack{\beta \in \hat{\mathfrak{c}}_j^{-1}/E_K \\ \beta \neq 0}} \tilde{f}_j(| \operatorname{N}_{K/\mathbb{Q}}(\beta)|).$$

An obvious change of variables reveals that

$$\int_0^\infty f_j(t) \mathrm{d}t = \frac{1}{R \operatorname{N} \mathfrak{c}_j} \int_0^\infty f(t) \mathrm{d}t.$$

Moreover, it follows from (2.12) that

$$\begin{aligned}
\tilde{f}_j(y) &= \int_0^\infty f(Rt \operatorname{N} \mathfrak{c}_j) k_{r_1, r_2}(ty) \mathrm{d}t \\
&= \frac{1}{R \operatorname{N} \mathfrak{c}_j} \tilde{f}\left(\frac{y}{R \operatorname{N} \mathfrak{c}_j} \right) \\
&\ll_N R^{-1} \left(\frac{y}{R} \right)^{-N},
\end{aligned}$$

for any $N \geqslant 0$. Reintroducing the sum over j, we have therefore established the following result.

Lemma 2.6. — *Let $f \in \mathscr{W}_1(\mathbb{R})$ and let $R > 0$. Then we have*

$$\sum_{\mathfrak{b} \neq (0)} f(R \operatorname{N} \mathfrak{b}) = \frac{\Delta_K}{R} \int_0^\infty f(t) \mathrm{d}t + O_N\left(R^N \right),$$

for any $N > 0$.

3. The smooth δ-function over K

In this section we establish Theorem 1.2 and some further basic properties of the function $h(x, y)$. Let w be any infinitely differentiable bounded non-negative function on $\mathbb{R}$ which is supported in the interval $[1/2, 1]$ and satisfies

$$\int_{-\infty}^{\infty} w(t)\mathrm{d}t = 1.$$

In particular $w \in \mathscr{W}_1^+(\mathbb{R})$. For $Q \geqslant 1$ define

$$c_Q^{-1} = \Delta_K^{-1} Q^{-d} \sum_{\mathfrak{c}} w(Q^{-d} \, \mathrm{N} \, \mathfrak{c}),$$

where Δ_K is given by (2.3) and the sum is over integral ideals $\mathfrak{c} \subseteq \mathfrak{o}$. For any non-zero ideal $\mathfrak{a} \subseteq \mathfrak{o}$, we have

$$\sum_{\mathfrak{c} | \mathfrak{a}} \left\{ w\left(\frac{\mathrm{N}\,\mathfrak{c}}{Q^d}\right) - w\left(\frac{\mathrm{N}\,\mathfrak{a}}{Q^d \, \mathrm{N}\,\mathfrak{c}}\right) \right\} = 0,$$

where the sum is over integral ideals $\mathfrak{c}$ dividing $\mathfrak{a}$. On the other hand, when $\mathfrak{a} = (0)$, we have

$$\sum_{\mathfrak{c} | \mathfrak{a}} \left\{ w\left(\frac{\mathrm{N}\,\mathfrak{c}}{Q^d}\right) - w\left(\frac{\mathrm{N}\,\mathfrak{a}}{Q^d \, \mathrm{N}\,\mathfrak{c}}\right) \right\} = \sum_{\mathfrak{c}} w(Q^{-d} \, \mathrm{N}\,\mathfrak{c}).$$

In this way we deduce that

$$\delta_K(\mathfrak{a}) = c_Q \Delta_K^{-1} Q^{-d} \sum_{\mathfrak{c} | \mathfrak{a}} \left\{ w\left(\frac{\mathrm{N}\,\mathfrak{c}}{Q^d}\right) - w\left(\frac{\mathrm{N}\,\mathfrak{a}}{Q^d \, \mathrm{N}\,\mathfrak{c}}\right) \right\}.$$

Using (2.9) to detect the divisibility condition, we may therefore write

$$\delta_K(\mathfrak{a}) = c_Q \Delta_K^{-1} Q^{-d} \sum_{\mathfrak{c}} \frac{1}{\mathrm{N}\,\mathfrak{c}} \sum_{\sigma \, (\mathrm{mod}\, \mathfrak{c})} \sigma(\mathfrak{a}) \left\{ w\left(\frac{\mathrm{N}\,\mathfrak{c}}{Q^d}\right) - w\left(\frac{\mathrm{N}\,\mathfrak{a}}{Q^d \, \mathrm{N}\,\mathfrak{c}}\right) \right\}. \tag{3.1}$$

It is convenient to restrict the sum over additive characters to a sum over primitive characters, using the identity

$$\sum_{\sigma \, (\mathrm{mod}\, \mathfrak{c})} \sigma(\mathfrak{a}) = \sum_{\mathfrak{b} | \mathfrak{c}} \sideset{}{^*}\sum_{\sigma \, (\mathrm{mod}\, \mathfrak{b})} \sigma(\mathfrak{a}).$$

This is well-defined by Lemma 2.4. Inserting this into (3.1) and re-ordering the summation we arrive at the expression for $\delta_K(\mathfrak{a})$ in Theorem 1.2, with

$$h(x, y) = \frac{1}{\Delta_K} \sum_{\mathfrak{z}} \frac{1}{x \, \mathrm{N}\,\mathfrak{z}} \left\{ w(x \, \mathrm{N}\,\mathfrak{z}) - w\left(\frac{|y|}{x \, \mathrm{N}\,\mathfrak{z}}\right) \right\}. \tag{3.2}$$

Since w is bounded and supported in $[1/2, 1]$ we see that $h(x, y) \neq 0$ only if $x \leqslant \max\{1, 2|y|\}$ and, furthermore,

$$h(x, y) \ll \frac{1}{x} \left(\sum_{(2x)^{-1} \leqslant \mathrm{N}\,\mathfrak{z} \leqslant x^{-1}} \frac{1}{\mathrm{N}\,\mathfrak{z}} + \sum_{|y|/x \leqslant \mathrm{N}\,\mathfrak{z} \leqslant 2|y|/x} \frac{1}{\mathrm{N}\,\mathfrak{z}} \right) \ll \frac{1}{x}.$$

Finally, $h(x, y)$ is easily seen to be infinitely differentiable on $(0, \infty) \times \mathbb{R}$ since the individual summands are infinitely differentiable.

In order to complete the proof of Theorem 1.2, it remains to show that

$$c_Q = 1 + O_N(Q^{-N}) \tag{3.3}$$

for any $N > 0$. For this we apply Lemma 2.6 with $R = Q^{-d}$ and $f = w$, concluding that

$$\sum_{\mathfrak{c}} w(Q^{-d} \, \mathrm{N} \, \mathfrak{c}) = \Delta_K Q^d \int_0^\infty w(t) \mathrm{d}t + O_N\left(Q^{-N}\right)$$

for any $N > 0$. The first term here is $\Delta_K Q^d$, whence

$$c_Q^{-1} = \Delta_K^{-1} Q^{-d} \sum_{\mathfrak{c}} w(Q^{-d} \, \mathrm{N} \, \mathfrak{c}) = 1 + O_N(Q^{-N}),$$

as claimed.

We now turn to a detailed analysis of the function $h(x, y)$ in (3.2). If a_m are the coefficients appearing in the Dedekind zeta function $\zeta_K(s)$, then it is clear that

$$h(x, y) = \frac{1}{\Delta_K} \sum_{m=1}^\infty \frac{a_m}{xm} \left\{ w(xm) - w\left(\frac{|y|}{xm}\right) \right\}.$$

Our task is to achieve analogues of the corresponding facts established by Heath-Brown [8, §4] concerning $h(x, y)$ in the case $K = \mathbb{Q}$, which corresponds to taking $a_m = 1$ for all m. However, rather than the Euler–Maclaurin formula, which is used extensively by Heath-Brown, we will use the Poisson summation formula in the form Lemma 2.6. The basic structure of the proofs will nonetheless remain similar. We begin with the following result.

Lemma 3.1. — *The function $h(x, y)$ vanishes when $x \geqslant 1$ and $|y| \leqslant x/2$. When $x \leqslant 1$ and $|y| \leqslant x/2$ the function $h(x, y)$ is constant with respect to y, taking the value*

$$\frac{1}{\Delta_K} \sum_{\mathfrak{z}} \frac{1}{x \, \mathrm{N} \, \mathfrak{z}} w(x \, \mathrm{N} \, \mathfrak{z}).$$

Proof. — This is obvious on recalling that w has support $[1/2, 1]$. $\qquad\square$

Our next task is to show that the sum involved in $h(x, y)$ nearly cancels if $x = o(\min(1, |y|))$, for which we will closely follow the argument used in [8, Lemma 4]. We will require some preliminary estimates for $h(x, y)$ and its partial derivatives with respect to x and y. In view of (3.2) we may write $h(x, y) = h_1(x, y) - h_2(x, y)$, with

$$h_k(x, y) = \frac{1}{\Delta_K} \sum_{\mathfrak{z}} f_k(\mathrm{N} \, \mathfrak{z}) \tag{3.4}$$

for $k = 1, 2$, where

$$f_1(t) = \frac{1}{xt} w(xt), \quad f_2(t) = \frac{1}{xt} w\left(\frac{|y|}{xt}\right).$$

Let $i, j \in \mathbb{Z}_{\geqslant 0}$. A simple induction argument reveals that

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} \left(\frac{1}{x} w\left(\frac{y}{xn}\right)\right) = x^{-i-1} y^{-j} \sum_{0 \leqslant t \leqslant i} c_{i,j,t} \left(\frac{y}{xn}\right)^{j+t} w^{(j+t)}\left(\frac{y}{xn}\right) \tag{3.5}$$

and

$$\frac{\partial^i}{\partial x^i} \left(\frac{1}{x} w(xn)\right) = \sum_{0 \leqslant t \leqslant i} d_{i,t} x^{-i+t-1} n^t w^{(t)}(xn), \tag{3.6}$$

for certain constants $c_{i,j,t} = O_{i,j}(1)$ and $d_{i,t} = O_i(1)$, with $c_{i,0,0} = d_{i,0}$.

Suppose that $x \leqslant 1$ and $|y| \leqslant x/2$. In particular $n^t w^{(t)}(xn) \ll x^{-t}$ whenever $1/2 \leqslant xn \leqslant 1$. Thus it follows from Lemma 3.1 and (3.6) that

$$\frac{\partial^i}{\partial x^i} h(x,y) \ll_i x^{-i-1} \sum_{(2x)^{-1} \leqslant \mathrm{N}\, \mathfrak{z} \leqslant x^{-1}} \frac{1}{\mathrm{N}\, \mathfrak{z}} \ll_i x^{-i-1}. \tag{3.7}$$

Next suppose that $|y| \geqslant x/2$. We claim that

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} h(x,y) \ll_{i,j} x^{-i-1}|y|^{-j}. \tag{3.8}$$

Now the terms $(x \mathrm{N}\, \mathfrak{z})^{-1} w(x \mathrm{N}\, \mathfrak{z})$ in the definition (3.4) of $h_1(x,y)$ only contribute to the partial derivative when $x \leqslant 1$ and $j = 0$, in which case they contribute $O_i(x^{-i-1})$ as before, which is satisfactory. For the terms in $h_2(x,y)$ we apply (3.5). Since

$$\left(\frac{y}{xn}\right)^{j+t} w^{(j+t)}\left(\frac{y}{xn}\right) \ll_{j,t} 1,$$

it follows that

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} h_2(x,y) \ll_{i,j} x^{-i-1}|y|^{-j} \sum_{x/|y| \leqslant \mathrm{N}\, \mathfrak{z} \leqslant 2x/|y|} \frac{1}{\mathrm{N}\, \mathfrak{z}} \ll_{i,j} x^{-i-1}|y|^{-j},$$

which is satisfactory.

Lemma 3.2. — *Let $i, j, N \in \mathbb{Z}_{\geqslant 0}$. Then we have*

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} h(x,y) \ll_{i,j,N} x^{-i-j-1}\left(x^N + \min\{1, (x/|y|)^N\}\right).$$

The term x^N on the right can be omitted if $j \neq 0$.

Proof. — If $x \geqslant 2|y|$ then the result is an immediate consequence of Lemma 3.1 and (3.7). If $|y| \leqslant x \leqslant 2|y|$ then it follows from (3.8). Hence we may assume that $x \leqslant |y|$. The case $N = 0$ is trivial and so we suppose that $N \geqslant 1$.

Let $i, j \in \mathbb{Z}_{\geqslant 0}$. Our argument is similar to the proof of (3.3), being based on Lemma 2.6. Writing $h(x,y) = h_1(x,y) - h_2(x,y)$, as in (3.4), we deduce from (3.6) that

$$\frac{\partial^i}{\partial x^i} h_1(x,y) = \frac{1}{\Delta_K} \sum_{\mathfrak{z}} (\mathrm{N}\, \mathfrak{z})^{-1} \frac{\partial^i}{\partial x^i}\left(\frac{1}{x} w(x \mathrm{N}\, \mathfrak{z})\right)$$

$$= \frac{1}{\Delta_K} x^{-i} \sum_{0 \leqslant t \leqslant i} d_{i,t} \sum_{\mathfrak{z}} (x \mathrm{N}\, \mathfrak{z})^{t-1} w^{(t)}(x \mathrm{N}\, \mathfrak{z}).$$

We may assume that $x \leqslant 1$, else $h_1(x,y) = 0$. Let $g_t(z) = z^{t-1} w^{(t)}(z)$. On applying integration by parts repeatedly, we deduce that

$$\int_0^\infty g_t(z) \mathrm{d}z = 0, \tag{3.9}$$

for $t \geqslant 0$. Calling upon Lemma 2.6 with $f = g_t$ and $R = x$, we obtain

$$\frac{\partial^i}{\partial x^i} h_1(x, y) = x^{-i-1} \sum_{0 \leqslant t \leqslant i} d_{i,t} \left(\int_0^\infty g_t(z) \mathrm{d}z + O_N(x^N) \right)$$

$$= x^{-i-1} \left(d_{i,0} \int_0^\infty \frac{w(z)}{z} \mathrm{d}z + O_N(x^N) \right). \tag{3.10}$$

Turning to the term $h_2(x, y)$ in (3.4), we deduce from (3.5) that

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} h_2(x, y) = \frac{1}{\Delta_K} \sum_{\mathfrak{z}} (\mathrm{N}\,\mathfrak{z})^{-1} \frac{\partial^{i+j}}{\partial x^i \partial y^j} \left(\frac{1}{x} w \left(\frac{|y|}{x\,\mathrm{N}\,\mathfrak{z}} \right) \right)$$

$$= \frac{1}{\Delta_K} x^{-i-1} |y|^{-j} \sum_{\mathfrak{z}} (\mathrm{N}\,\mathfrak{z})^{-1} \sum_{0 \leqslant t \leqslant i} c_{i,j,t} \left(\frac{|y|}{x\,\mathrm{N}\,\mathfrak{z}} \right)^{j+t} w^{(j+t)} \left(\frac{|y|}{x\,\mathrm{N}\,\mathfrak{z}} \right)$$

$$= \frac{1}{\Delta_K} x^{-i} |y|^{-j-1} \sum_{0 \leqslant t \leqslant i} c_{i,j,t} \sum_{\mathfrak{z}} \left(\frac{|y|}{x\,\mathrm{N}\,\mathfrak{z}} \right)^{j+t+1} w^{(j+t)} \left(\frac{|y|}{x\,\mathrm{N}\,\mathfrak{z}} \right).$$

Let $f_{j,t}(z) = z^{-j-t-1} w^{(j+t)}(1/z)$ be functions on $(0, \infty)$. By making the change of variables $u = 1/z$, it easily follows from (3.9) that

$$\int_0^\infty f_{j,t}(z) \mathrm{d}z = \int_0^\infty u^{j+t-1} w^{(j+t)}(u) \mathrm{d}u = 0,$$

when $j + t \geqslant 1$. Applying lemma 2.6 with $f = f_{j,t}$ and $R = x/|y|$, we get

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} h_2(x, y) = x^{-i} |y|^{-j-1} \sum_{0 \leqslant t \leqslant i} c_{i,j,t} \left(\frac{|y|}{x} \int_0^\infty f_{j,t}(z) \mathrm{d}z + O_N \left((|y|/x)^{-N} \right) \right)$$

$$= x^{-i-1} |y|^{-j} \left(c_{i,j,0} \int_0^\infty f_{j,0}(z) \mathrm{d}z + O_N \left((|y|/x)^{-N} \right) \right)$$

When $j \neq 0$ the integral vanishes and this estimate is satisfactory for the lemma. On the other hand, when $j = 0$, this becomes

$$\frac{\partial^i}{\partial x^i} h_2(x, y) = x^{-i-1} \left(c_{i,0,0} \int_0^\infty \frac{w(z)}{z} \mathrm{d}z + O_N \left((|y|/x)^{-N} \right) \right).$$

Once combined with (3.10) and the fact that $c_{i,0,0} = d_{i,0}$, this therefore concludes the proof of the lemma. $\qquad \square$

A crucial step in the proof of Lemma 3.2 involved writing $h(x, y) = h_1(x, y) - h_2(x, y)$, where $h_1(x, y)$ satisfies the asymptotic formula (3.10) with $i = 0$. The latter expression is proved under the assumption that $x \leqslant 1$, but it is actually valid even when $x > 1$. It follows that

$$h(x, y) = x^{-1} H(|y|/x) + O_N(x^N), \tag{3.11}$$

for any $N > 0$, where $H : \mathbb{R}_{>0} \to \mathbb{R}$ is given by

$$H(v) = \int_0^\infty \frac{w(t)}{t} \mathrm{d}t - \frac{1}{\Delta_K} \sum_{\mathfrak{z}} \frac{1}{\mathrm{N}\,\mathfrak{z}} w \left(\frac{v}{\mathrm{N}\,\mathfrak{z}} \right).$$

4. Weighted averages of $h(x, y)$

A crucial ingredient in our main term analysis will be a suitable variant of [8, Lemma 9], showing that for small values of x the function $h(x, y)$ acts like a δ-function. In point of fact we shall be interested in the weighted average

$$I(x) = \int_V f(v) h(x, \mathrm{Nm}(v)) \mathrm{d}v,$$

for $0 < x \ll 1$ and $f \in \mathscr{W}_1(V)$. In a departure from our earlier conventions, in this section we will need to keep track of the dependence on f in any implied constant.

When $K = \mathbb{Q}$ one finds that $I(x)$ is approximated by $f(0)$ to within an error of $O_{f,N}(x^N)$, for any $N > 0$. This is achieved through multiple applications of the Euler–Maclaurin summation formula, an approach that is not readily adapted to the setting of general K. Instead, we will argue using Mellin transforms. Recall the definition (2.7) of λ_f^N, for any smooth weight function $f : V \to \mathbb{C}$ with compact support. Our goal in this section is a proof of the following result.

Lemma 4.1. — *Let $f \in \mathscr{W}_1(V)$ and let $N > 0$. Then we have*

$$I(x) = \frac{\sqrt{D_K}}{2^{r_2}} f(0) + O_N\left(\lambda_f^{2d(N+1)} x^N\right).$$

We have not attempted to obtain a dependence on λ_f^M, with M minimal, since all that is required in our application is that M be polynomial in N. When $K = \mathbb{Q}$, so that $D_K = 1$ and $r_2 = 0$, we retrieve [8, Lemma 9]. Since $h(x, y) \ll x^{-1}$, by (3.7), we see that Lemma 4.1 is trivial when $x > 1$. We therefore assume that $x \leqslant 1$ for the remainder of this section.

Since $f \in \mathscr{W}_1(V)$ there exists an absolute constant $0 < A \ll 1$ such that $f(v) = 0$ unless $v = (v^{(1)}, \ldots, v^{(r_1+r_2)})$ satisfies $|v^{(l)}| \leqslant A$ for $1 \leqslant l \leqslant r_1 + r_2$. We introduce a parameter T, to be selected in due course, which satisfies $0 < T \leqslant A$. We may then write

$$I(x) = \sum_{S \subseteq \{1, \ldots, r_1+r_2\}} \int_{V_S} f(v) h(x, \mathrm{Nm}(v)) \mathrm{d}v,$$

where S runs over all subsets of $\{1, \ldots, r_1 + r_2\}$ and V_S is the set of $v \in V$ for which

$$|v^{(l)}| \begin{cases} \leqslant T, & \text{if } l \in S, \\ > T, & \text{if } l \notin S. \end{cases}$$

Let us denote by $I_S(x)$ the integral over V_S. In our proof of Lemma 4.1 we will obtain an asymptotic formula for $I_{\{1, \ldots, r_1+r_2\}}(x)$ and upper bounds for all other $I_S(x)$.

Given a vector $\mathbf{e} \in \mathbb{Z}_{\geqslant 0}^k$ and a vector $\mathbf{t} = (t_1, \ldots t_k)$, we write $\mathbf{t}^{\mathbf{e}} = t_1^{e_1} \cdots t_k^{e_k}$ for the associated monomial of degree

$$|\mathbf{e}| = \sum_{1 \leqslant i \leqslant k} e_i.$$

Let us work with a particular set S. By symmetry we may assume without loss of generality that $S = \{1, \ldots, m\} \cup \{r_1 + 1, \ldots, r_1 + n\}$ for some $m \leqslant r_1$ and $n \leqslant r_2$. Throughout this section it will be convenient to work with the coordinates $v = (\mathbf{v}, \mathbf{y} + i\mathbf{z})$ on V, and to put

$$\mathbf{v} = (\mathbf{v}', \mathbf{v}''), \quad \mathbf{y} = (\mathbf{y}', \mathbf{y}''), \quad \mathbf{z} = (\mathbf{z}', \mathbf{z}''),$$

where $\mathbf{v}' = (v_1, \ldots, v_m)$ and $\mathbf{v}'' = (v_{m+1}, \ldots, v_{r_1})$, and similarly for $\mathbf{y}, \mathbf{z}$. The idea is now to take a power series expansion of f around the point $(\mathbf{0}, \mathbf{v}'', \mathbf{0}, \mathbf{y}'' + i\mathbf{z}'')$ to produce an

approximating polynomial of degree M in the variables $\mathbf{v}', \mathbf{y}', \mathbf{z}'$, with error $O_M(\lambda_f^{M+1} T^{M+1})$. This leads to an expression

$$f(v) = \sum_{\substack{\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^m, \mathbf{b}, \mathbf{c} \in \mathbb{Z}_{\geqslant 0}^n \\ |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| \leqslant M}} \varphi_{\mathbf{a},\mathbf{b},\mathbf{c}} \mathbf{v}'^{\mathbf{a}} \mathbf{y}'^{\mathbf{b}} \mathbf{z}'^{\mathbf{c}} + O_M\left(\lambda_f^{M+1} T^{M+1}\right), \qquad (4.1)$$

for suitable coefficients $\varphi_{\mathbf{a},\mathbf{b},\mathbf{c}} = \varphi_{\mathbf{a},\mathbf{b},\mathbf{c}}(\mathbf{v}'', \mathbf{y}'', \mathbf{z}'')$, with $|\varphi_{\mathbf{a},\mathbf{b},\mathbf{c}}| = O_M(\lambda_f^M)$ and

$$\varphi_{\mathbf{0},\mathbf{0},\mathbf{0}} = f(\mathbf{0}, \mathbf{v}'', \mathbf{0}, \mathbf{y}'' + i\mathbf{z}'').$$

Note that (4.1) holds trivially if $S = \emptyset$, since then $f(v) = \varphi_{\mathbf{0},\mathbf{0},\mathbf{0}}$.

Next we set $B(H) = \{(y,z) \in \mathbb{R}^2 : y^2 + z^2 \leqslant H^2\}$ for the ball of radius H centred on the origin. We may therefore write

$$I_S(x) = \sum_{\substack{\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^m, \mathbf{b}, \mathbf{c} \in \mathbb{Z}_{\geqslant 0}^n \\ |\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| \leqslant M}} \int_{\mathbf{v}'', \mathbf{y}'', \mathbf{z}''} \varphi_{\mathbf{a},\mathbf{b},\mathbf{c}} K \, \mathrm{d}\mathbf{v}'' \mathrm{d}\mathbf{y}'' \mathrm{d}\mathbf{z}'' + O_M\left(\frac{\lambda_f^{M+1} T^{M+1}}{x}\right), \qquad (4.2)$$

the integral being over $\mathbf{v}'' \in ([-A,T] \cup [T,A])^{r_1 - m}$ and $(\mathbf{y}'', \mathbf{z}'') \in (B(A) \setminus B(T))^{r_2 - n}$, and where

$$K = \int_{[-T,T]^m} \int_{B(T)^n} \mathbf{v}'^{\mathbf{a}} \mathbf{y}'^{\mathbf{b}} \mathbf{z}'^{\mathbf{c}} h(x, \mathrm{Nm}(\mathbf{v}, \mathbf{y} + i\mathbf{z})) \mathrm{d}\mathbf{v}' \mathrm{d}\mathbf{y}' \mathrm{d}\mathbf{z}'.$$

Recalling the definition (2.1) of $\mathrm{Nm}(v)$, we see that

$$\mathrm{Nm}(\mathbf{v}, \mathbf{y} + i\mathbf{z}) = v_1 \cdots v_{r_1}(y_1^2 + z_1^2) \cdots (y_{r_2}^2 + z_{r_2}^2).$$

We extend this to $\mathrm{Nm}(\mathbf{v}', \mathbf{y}' + i\mathbf{z}')$ and $\Phi = \mathrm{Nm}(\mathbf{v}'', \mathbf{y}'' + i\mathbf{z}'')$ in the obvious way, so that $\mathrm{Nm}(\mathbf{v}, \mathbf{y} + i\mathbf{z}) = \Phi \, \mathrm{Nm}(\mathbf{v}', \mathbf{y}' + i\mathbf{z}')$. We now make the change of variables

$$(\tilde{\mathbf{v}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}) = |\Phi|^{1/(m+2n)}(\mathbf{v}', \mathbf{y}', \mathbf{z}'),$$

which leads to the conclusion

$$K = |\Phi|^{-(|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}|)/(m+2n) - 1} I_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}(x; T|\Phi|^{1/(m+2n)}), \qquad (4.3)$$

where for any $Y > 0$ we set

$$I_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}(x; Y) = \int_{[-Y,Y]^m} \int_{B(Y)^n} \tilde{\mathbf{v}}^{\mathbf{a}} \tilde{\mathbf{y}}^{\mathbf{b}} \tilde{\mathbf{z}}^{\mathbf{c}} h(x, \mathrm{Nm}(\tilde{\mathbf{v}}, \tilde{\mathbf{y}} + i\tilde{\mathbf{z}})) \mathrm{d}\tilde{\mathbf{v}} \mathrm{d}\tilde{\mathbf{y}} \mathrm{d}\tilde{\mathbf{z}}. \qquad (4.4)$$

Note that $I_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}(x; Y)$ is completely independent of the variables $\mathbf{v}'', \mathbf{y}'', \mathbf{z}''$. The analysis of this integral is rather involved and will lead to the following estimate.

Lemma 4.2. — *Let $N_1, N_2 > 0$, let $0 < Y \ll 1$ and let $m, n \in \mathbb{Z}_{\geqslant 0}$, with $m \leqslant r_1$ and $n \leqslant r_2$. Let $\mathbf{a} \in \mathbb{Z}_{\geqslant 0}^m$ and $\mathbf{b}, \mathbf{c} \in \mathbb{Z}_{\geqslant 0}^n$, with $|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| \leqslant M$. Then we have $I_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}(x; Y) = 0$ unless a_i, b_j, c_j are all even, in which case*

$$I_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}(x; Y) = -\frac{2^{r_1} \pi^{r_2} \Upsilon_K c_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}}{\Delta_K} + O_{M,N_1}\left(\left\{\frac{x}{Y^{m+2n}}\right\}^{N_1}\right) + O_{N_2}(x^{N_2}),$$

with Υ_K given by (2.4) and

$$c_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})} = \begin{cases} 1, & \text{if } (m,n) = (r_1, r_2) \text{ and } (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{0},\mathbf{0},\mathbf{0}), \\ 0, & \text{otherwise.} \end{cases}$$

Let us delay the proof of Lemma 4.2 momentarily, in order to see how it can be used to conclude the proof of Lemma 4.1. It is clear that $|\Phi| \geqslant T^{d-(m+2n)}$ for any $\mathbf{v}'', \mathbf{y}'', \mathbf{z}''$ appearing in (4.2). It therefore follows from inserting Lemma 4.2 into (4.3) that

$$K = |\Phi|^{-(|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|)/(m+2n)-1} \left\{ -\frac{2^{r_1}\pi^{r_2}\Upsilon_K c_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}}{\Delta_K} + O_{M,N_1}\left(\left\{\frac{x}{T^d}\right\}^{N_1}\right) + O_{N_2}(x^{N_2}) \right\}.$$

Here we have $|\Phi|^{-(|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|)/(m+2n)-1} \leqslant T^{-d(M/(m+2n)+1)} \leqslant T^{-d(M+1)}$. Noting that $\Phi = 1$ when $(m,n) = (r_1, r_2)$, and integrating trivially over $\mathbf{v}'', \mathbf{y}'', \mathbf{z}''$, we deduce from (4.2) that

$$I_S(x) = -\frac{2^{r_1}\pi^{r_2}\Upsilon_K c_{m,n}^{(\mathbf{0},\mathbf{0},\mathbf{0})}}{\Delta_K} f(0) + O_M\left(\frac{\lambda_f^{M+1} T^{M+1}}{x}\right)$$
$$+ O_{M,N_1,N_2}\left(\lambda_f^M T^{-d(M+1)}\left(\left\{\frac{x}{T^d}\right\}^{N_1} + x^{N_2}\right)\right).$$

Let $N > 0$ be an integer. We now make the selection $T = x^{1/(2d)}$, which is clearly $O(1)$. We choose M so that $M + 1 = 2d(N+1)$. This ensures that the first error term is satisfactory for Lemma 4.1. The second error term is seen to be satisfactory on choosing N_1 and N_2 sufficiently large in terms of M. It remains to deduce from (2.3) and (2.4) that

$$-\frac{2^{r_1}\pi^{r_2}\Upsilon_K}{\Delta_K} = \frac{\sqrt{D_K}}{2^{r_2}}.$$

Taken together, we may now conclude that

$$I(x) = I_{\{1,\dots,r_1+r_2\}}(x) + \sum_{S \subsetneq \{1,\dots,r_1+r_2\}} I_S(x)$$
$$= \frac{\sqrt{D_K}}{2^{r_2}} \cdot f(0) + O_N(\lambda_f^{2d(N+1)} x^N),$$

as required to complete the proof of Lemma 4.1.

Proof of Lemma 4.2. — For notational convenience let us write $(\tilde{\mathbf{v}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}) = (\mathbf{v}, \mathbf{y}, \mathbf{z})$ in the integral $I_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}(x;Y)$ defined in (4.4). To begin with, (3.11) yields

$$I_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}(x;Y) = x^{-1}\int_{[-Y,Y]^m}\int_{B(Y)^n} \mathbf{v}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}\mathbf{z}^{\mathbf{c}} H\left(\frac{|\mathrm{Nm}(\mathbf{v},\mathbf{y}+i\mathbf{z})|}{x}\right) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} + O_{N_2}(x^{N_2}),$$

for any $N_2 > 0$. Note that the main term vanishes unless all the components of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are even, which we now assume. Now it is clear that

$$\int_{[-Y,Y]^m} \mathbf{v}^{\mathbf{a}}\mathrm{d}\mathbf{y} = \frac{2^m Y^{m+|\mathbf{a}|}}{\prod_{1\leqslant i\leqslant m}(a_i+1)}.$$

Likewise we have

$$\int_{B(Y)^n} \mathbf{y}^{\mathbf{b}}\mathbf{z}^{\mathbf{c}}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} = \prod_{1\leqslant j\leqslant n} F(b_j, c_j),$$

with

$$F(b,c) = \int_{B(Y)} y^b z^c \mathrm{d}y\mathrm{d}z,$$

for even integers b and c. Switching to polar coordinates, we call upon the identities found in Gradshteyn and Ryzhik [6, §2.511], deducing that

$$\int_0^{2\pi} \cos^{2p}(t)\sin^{2q}(t)\mathrm{d}t = \frac{(2p-1)!!(2q-1)!!}{(2p+2q)!!} \cdot 2\pi,$$

for any integers $p, q \geqslant 0$. Hence it follows that

$$F(b,c) = \int_0^Y \int_0^{2\pi} \rho^{b+c+1}\cos^b(\theta)\sin^c(\theta)\mathrm{d}\rho\mathrm{d}\theta$$
$$= \frac{Y^{b+c+2}}{b+c+2} \cdot \frac{(b-1)!!(c-1)!!}{(b+c)!!} \cdot 2\pi,$$

whence

$$\int_{B(Y)^n} \mathbf{y}^{\mathbf{b}}\mathbf{z}^{\mathbf{c}}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} = C_{\mathbf{b},\mathbf{c}}Y^{2n+|\mathbf{b}|+|\mathbf{c}|},$$

with

$$C_{\mathbf{b},\mathbf{c}} = (2\pi)^n \prod_{1\leqslant j\leqslant n} \left(\frac{(b_j-1)!!(c_j-1)!!}{(b_j+c_j)!!} \right).$$

It will be convenient to define

$$\nu_{\mathbf{a},\mathbf{b},\mathbf{c}} = 2^m C_{\mathbf{b},\mathbf{c}} \prod_{1\leqslant i\leqslant m} \left(\frac{1}{a_i+1} \right) \prod_{1\leqslant j\leqslant n} \left(\frac{1}{b_j+c_j+2} \right) \int_0^\infty \frac{w(u)}{u}\mathrm{d}u. \tag{4.5}$$

This allows us to conclude that

$$I_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}(x;Y) = \frac{\nu_{\mathbf{a},\mathbf{b},\mathbf{c}}Y^{|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|+m+2n}}{x} - \frac{2^m C_{\mathbf{b},\mathbf{c}}}{\Delta_K} \sum_{\mathfrak{z}} g(\mathrm{N}\,\mathfrak{z}) + O_{N_2}(x^{N_2}), \tag{4.6}$$

with

$$g(t) = \frac{1}{C_{\mathbf{b},\mathbf{c}}} \int_{[0,Y]^m} \int_{B(Y)^n} \mathbf{v}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}\mathbf{z}^{\mathbf{c}} \cdot \frac{1}{xt} w\left(\frac{\mathrm{Nm}(\mathbf{v},\mathbf{y}+i\mathbf{z})}{xt} \right) \mathrm{d}\mathbf{v}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z},$$

for $t > 0$. Switching to polar coordinates as above we may write

$$g(t) = \int_{[0,Y]^m} \int_{[0,Y]^n} \mathbf{v}^{\mathbf{a}}\boldsymbol{\rho}^{\mathbf{b}+\mathbf{c}} \cdot \frac{1}{xt} w\left(\frac{v_1\cdots v_m \rho_1^2\cdots\rho_n^2}{xt} \right) \mathrm{d}\mathbf{v}\mathrm{d}\boldsymbol{\rho}.$$

We are therefore led to analyse the sum

$$S = \sum_{\mathfrak{z}} g(\mathrm{N}\,\mathfrak{z}),$$

for which we will use properties of the Mellin transform

$$\hat{g}(s) = \int_0^\infty t^{s-1}g(t)\mathrm{d}t.$$

It follows that

$$S = \sum_{n=1}^\infty a_n g(n) = \frac{1}{2\pi i} \int_{(2)} \zeta_K(s)\hat{g}(s)\mathrm{d}s,$$

where a_n are the coefficients appearing in the Dedekind zeta function $\zeta_K(s)$ and the integral is over the line $\sigma = \Re(s) = 2$. In order to move the line of integration further to the left we will need a better understanding of the analyticity of $\hat{g}$.

We will write $u = (xt)^{-1}v_1 \cdots v_m \rho_1^2 \cdots \rho_n^2$ and substitute for one of the variables. If $m \neq 0$ then we substitute for v_1. Alternatively, if $m = 0$, then we substitute for ρ_1 and argue similarly. Assuming without loss of generality that $m \neq 0$, it will be convenient to put $L = v_2 \cdots v_m \rho_1^2 \cdots \rho_n^2$ and to set $\mathbf{v}_1 = (v_2, \ldots, v_m)$ and $\mathbf{a}_1 = (a_2, \ldots, a_m)$. Then for $\Re(s) > 0$ it follows that

$$\hat{g}(s) = \int_0^\infty t^{s-1} \int_{[0,Y]^{m-1}} \int_{[0,Y]^n} \frac{\mathbf{v}_1^{\mathbf{a}_1} \boldsymbol{\rho}^{\mathbf{b}+\mathbf{c}}}{L} \int_0^{\frac{LY}{xt}} \left(\frac{uxt}{L} \right)^{a_1} w(u) \mathrm{d}u \mathrm{d}\mathbf{v}_1 \mathrm{d}\boldsymbol{\rho} \mathrm{d}t$$

$$= x^{a_1} \int_0^\infty u^{a_1} w(u) \int_{[0,Y]^{m-1}} \int_{[0,Y]^n} \frac{\mathbf{v}_1^{\mathbf{a}_1} \boldsymbol{\rho}^{\mathbf{b}+\mathbf{c}}}{L^{a_1+1}} \int_0^{\frac{LY}{xu}} t^{a_1+s-1} \mathrm{d}t \mathrm{d}\mathbf{v}_1 \mathrm{d}\boldsymbol{\rho} \mathrm{d}u.$$

Carrying out the integration over t, we obtain

$$\hat{g}(s) = \frac{Y^{a_1+s}}{(a_1+s)x^s} \int_0^\infty \frac{w(u)}{u^s} \mathrm{d}u \int_{[0,Y]^{m-1}} \int_{[0,Y]^n} \mathbf{v}_1^{\mathbf{a}_1} \boldsymbol{\rho}^{\mathbf{b}+\mathbf{c}} L^{s-1} \mathrm{d}\mathbf{v}_1 \mathrm{d}\boldsymbol{\rho} = F_Y(s),$$

with

$$F_R(s) = \frac{R^{|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|+(m+2n)s}}{x^s \prod_{1 \leqslant i \leqslant m}(a_i+s) \prod_{1 \leqslant j \leqslant n}(b_j+c_j+2s)} \int_0^\infty \frac{w(u)}{u^s} \mathrm{d}u,$$

for $\Re(s) > 0$ and $R > 0$. As a formula this continues to make sense in the half-plane $\Re(s) \leqslant 0$ and so provides a meromorphic continuation of $\hat{g}$ to the whole of $\mathbb{C}$, with at most poles at the non-positive even integers. One sees that the pole at $s = 0$ has order at most $m + n - 1$ when $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \neq (\mathbf{0}, \mathbf{0}, \mathbf{0})$ and order $m + n$ otherwise. Likewise, the poles at the negative even integers have order at most $m + n$.

We take this opportunity to record an upper bound for $|F_R(s)|$. Assume that $s = \sigma + it$ with $|t| \geqslant 1$ and recall that $w \in \mathscr{W}_1^+(\mathbb{R})$. Repeated integration by parts then yields

$$\int_0^\infty \frac{w(u)}{u^s} \mathrm{d}u = \frac{1}{(s-1) \cdots (s-N)} \int_0^\infty u^{N-s} w^{(N)}(u) \mathrm{d}u,$$

for any integer $N \geqslant 0$. In this way we conclude that

$$|F_R(s)| \ll_{M,N,\sigma} \frac{R^{|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|}}{(1+|t|)^N} \left(\frac{R^{m+2n}}{x} \right)^\sigma,$$

if $s \neq 0$, for any $N \geqslant 0$.

Returning to our formula for S we seek to move the line of integration back to the line $\sigma = -N_1$ for some positive constant N_1. This is facilitated by the polynomial decay in $|t|$ that we have observed in $F_R(s)$. Indeed, in view of our convexity estimates (2.6), we are able to shift the line of integration arbitrarily far to the left. In doing so will encounter poles at $s = 1$ and possibly also at the non-positive even integers. We note that if $\hat{g}(s)$ has a pole of order $\leqslant r_1 + r_2 - 1$ at $s = 0$ then it will be compensated for by the presence of $\zeta_K(s)$, which has a zero of order $r_1 + r_2 - 1$ at $s = 0$. Similarly, any poles at the negative even integers will be compensated for by the zeros of $\zeta_K(s)$ of order $r_1 + r_2$ at these places. The residue at $s = 1$ of $\zeta_K(s)\hat{g}(s)$ is

$$\Delta_K \hat{g}(1) = \frac{\Delta_K}{2^m C_{\mathbf{b},\mathbf{c}}} \cdot \frac{\nu_{\mathbf{a},\mathbf{b},\mathbf{c}} Y^{|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|+m+2n}}{x},$$

in the notation of (2.3) and (4.5). When $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ and $(m, n) = (r_1, r_2)$, then $\zeta_K(s)\hat{g}(s)$ has a simple pole at $s = 0$ with residue

$$\frac{\Upsilon_K}{2^{r_2}} \int_0^\infty w(u)\mathrm{d}u = \frac{\Upsilon_K}{2^{r_2}},$$

where Υ_K is given by (2.4). Putting this together we may therefore conclude that

$$S = \frac{\Delta_K}{2^m C_{\mathbf{b},\mathbf{c}}} \cdot \frac{\nu_{\mathbf{a},\mathbf{b},\mathbf{c}} Y^{|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|+m+2n}}{x} + \frac{\Upsilon_K c_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}}{2^{r_2}}$$
$$+ O_{M,N_1}\left(Y^{|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|}\left\{\frac{Y^{m+2n}}{x}\right\}^{-N_1}\right),$$

with $c_{m,n}^{(\mathbf{a},\mathbf{b},\mathbf{c})}$ as in the statement of the lemma. Recalling that $Y \ll 1$ and substituting this into (4.6), this therefore concludes the proof of Lemma 4.2. $\square$

5. Application to hypersurfaces

Suppose that $Y \subseteq \mathbb{P}_K^{n-1}$ is a projective hypersurface defined by a homogeneous polynomial $F \in \mathfrak{o}[X_1, \ldots, X_n]$. In order to gauge whether or not $Y(K)$ is empty it is sometimes fruitful to study the asymptotic behaviour of sums

$$N_W(F; P) = \sum_{\mathbf{x} \in \mathfrak{o}^n} \delta_K(F(\mathbf{x})) W(\mathbf{x}/P),$$

as $P \to \infty$, where $W \in \mathscr{W}_n(V)$. For any $Q \geqslant 1$ we deduce from Theorem 1.2 that

$$N_W(F, P) = \frac{c_Q}{Q^{2d}} \sum_{\mathfrak{b}} \sideset{}{^*}\sum_{\sigma \pmod{\mathfrak{b}}} \sum_{\mathbf{x} \in \mathfrak{o}^n} \sigma(F(\mathbf{x})) W(\mathbf{x}/P) h\left(\frac{\mathrm{N}\,\mathfrak{b}}{Q^d}, \frac{|\mathrm{Nm}(F(\mathbf{x}))|}{Q^{2d}}\right).$$

In view of the fact that $h(x, y) \neq 0$ only if $x \leqslant \max(1, 2|y|)$, it is clear that the sum over $\mathfrak{b}$ is restricted to $\mathrm{N}\,\mathfrak{b} \ll Q^d$, if Q is taken to be of order $P^{(\deg F)/2}$. Breaking the inner sum over $\mathbf{x}$ into residue classes modulo $\mathfrak{b}$, we see that it can be written

$$\sum_{\mathbf{a} \in (\mathfrak{o}/\mathfrak{b})^n} \sigma(F(\mathbf{a})) \sum_{\mathbf{x} \in \mathfrak{b}^n} W\left((\mathbf{x} + \mathbf{a})/P\right) h\left(\frac{\mathrm{N}\,\mathfrak{b}}{Q^d}, \frac{|\mathrm{Nm}(F(\mathbf{x} + \mathbf{a}))|}{Q^{2d}}\right),$$

for any primitive character σ modulo $\mathfrak{b}$. We apply the usual multi-dimensional Poisson summation formula (in the form found in Skinner [15, §5], for example), finding that the inner sum over $\mathbf{x}$ is

$$\frac{2^{r_2 n}}{D_K^{n/2}(\mathrm{N}\,\mathfrak{b})^n} \sum_{\mathbf{m} \in \hat{\mathfrak{b}}^n} \mathrm{e}(\mathbf{m}.\mathbf{a}) \int_{V^n} W(\mathbf{x}/P) h\left(\frac{\mathrm{N}\,\mathfrak{b}}{Q^d}, \frac{|\mathrm{Nm}(F(\mathbf{x}))|}{Q^{2d}}\right) \mathrm{e}(-\mathbf{m}.\mathbf{x}) \mathrm{d}\mathbf{x},$$

where D_K is the absolute discriminant of K and $\hat{\mathfrak{b}}$ is the dual of $\mathfrak{b}$ taken with respect to the trace. Putting everything together we have therefore established the following result.

Theorem 5.1. — *We have*

$$N_W(F, P) = \frac{c_Q 2^{r_2 n}}{D_K^{n/2} Q^{2d}} \sum_{\mathfrak{b}} \sum_{\mathbf{m} \in \hat{\mathfrak{b}}^n} (\mathrm{N}\,\mathfrak{b})^{-n} S_{\mathfrak{b}}(\mathbf{m}) I_{\mathfrak{b}}(\mathbf{m}),$$

where the sum over $\mathfrak{b}$ is over non-zero integral ideals and

$$S_{\mathfrak{b}}(\mathbf{m}) = \sideset{}{^*}\sum_{\sigma \,(\mathrm{mod}\,\mathfrak{b})} \sum_{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{b})} \sigma(F(\mathbf{a}))\mathrm{e}(\mathbf{m}.\mathbf{a}),$$

$$I_{\mathfrak{b}}(\mathbf{m}) = \int_{V^n} W(\mathbf{x}/P)h\left(\frac{\mathrm{N}\,\mathfrak{b}}{Q^d}, \frac{|\,\mathrm{Nm}(F(\mathbf{x}))|}{Q^{2d}}\right)\mathrm{e}\left(-\mathbf{m}.\mathbf{x}\right)\mathrm{d}\mathbf{x}.$$

This result is a number field analogue of [**8**, Theorem 2]. We apply this in the case that $F \in \mathfrak{o}[X_1, \ldots, X_n]$ is a non-singular cubic form in $n \geqslant 10$ variables. Using the embedding of K into V, we may write

$$F = \bigoplus_{l=1}^{r_1+r_2} F^{(l)},$$

where each $F^{(l)}$ is a cubic form over K_l for $1 \leqslant l \leqslant r_1 + r_2$. Recall the definition of the norms $\langle\cdot\rangle, \|\cdot\|$ from §2.

Let $\boldsymbol{\xi} = \bigoplus_l \boldsymbol{\xi}^{(l)} \in V^n$ be a suitable point chosen as in [**15**, Lemma 13(i)]. Let $\delta_0 > 0$ be a small constant such that the inverse function theorem can be used for F in the region $\langle\mathbf{x} - \boldsymbol{\xi}\rangle \leqslant \delta_0$. Let $\mathbf{u} = \mathbf{x} - \boldsymbol{\xi}$. Thus, without loss of generality, we may assume that there exists a smooth function f, such that for each $1 \leqslant l \leqslant r_1 + r_2$ we have

$$u_1^{(l)} = f^{(l)}(F^{(l)}(\mathbf{u}^{(l)} + \boldsymbol{\xi}^{(l)}), u_2^{(l)}, \ldots, u_n^{(l)}), \tag{5.1}$$

whenever $|\mathbf{u}^{(l)}| \leqslant \langle\mathbf{u}\rangle \leqslant \delta_0$. We henceforth view δ_0 as being fixed once and for all.

Next, we take $w_0 \in \mathscr{W}_n^+(V)$ to be a smooth weight function which takes the value 1 on the region $\langle\mathbf{x} - \boldsymbol{\xi}\rangle \leqslant \delta_0/2$ and is zero outside the region $\langle\mathbf{x} - \boldsymbol{\xi}\rangle \leqslant \delta_0$. Let $\omega : V^n \to \mathbb{R}_{\geqslant 0}$ be the smooth weight function

$$\omega(\mathbf{x}) = \exp(-(\log P)^4\|\mathbf{x} - \boldsymbol{\xi}\|^2). \tag{5.2}$$

The function $W : V^n \to \mathbb{R}_{\geqslant 0}$ that we shall work with in $N_W(F; P)$ is

$$W(\mathbf{x}) = w_0(\mathbf{x})\omega(\mathbf{x}).$$

In particular W is supported on the region $\langle\mathbf{x}\rangle \ll 1$. Let $Q = P^{3/2}$. It now follows from Theorem 5.1 that

$$N_W(F, P) = \frac{c_Q 2^{r_2 n}}{D_K^{n/2} Q^{2d}} \sum_{\substack{(0)\neq\mathfrak{b}\subseteq\mathfrak{o} \\ \mathrm{N}\,\mathfrak{b}\ll Q^d}} \sum_{\mathbf{m}\in\hat{\mathfrak{b}}^n} (\mathrm{N}\,\mathfrak{b})^{-n}S_{\mathfrak{b}}(\mathbf{m})I_{\mathfrak{b}}(\mathbf{m}).$$

The definition of h in $I_{\mathfrak{b}}(\mathbf{m})$ means that one can freely replace $|\,\mathrm{Nm}(F(\mathbf{x}))|$ by $\mathrm{Nm}(F(\mathbf{x}))$.

We will use this expression to obtain an asymptotic lower bound for $N_W(F, P)$, as $P \to \infty$. The terms corresponding to $\mathbf{m} = \mathbf{0}$ will give the main contribution. The following section will be concerned with the exponential integrals $I_{\mathfrak{b}}(\mathbf{m})$. Our analysis of the complete exponential sums $S_{\mathfrak{b}}(\mathbf{m})$ will take place in §7. Finally, in §8 we shall handle the terms with $\mathbf{m} = \mathbf{0}$ and draw together the various estimates in order to conclude the proof of Theorem 1.1.

Let $\rho = Q^{-d}\mathrm{N}\,\mathfrak{b} \ll 1$. The first task in §6 will be to prove the following result, which is based on repeated integration by parts.

Lemma 5.2. — *For any non-zero $\mathbf{m} \in V^n$ and any integer $N \geqslant 0$, we have*

$$I_{\mathfrak{b}}(\mathbf{m}) \ll_N \rho^{-1}P^{dn}\left(\frac{(\log P)^2}{\rho P\langle\mathbf{m}\rangle}\right)^N.$$

This result shows that $I_{\mathfrak{b}}(\mathbf{m})$ decays faster than any polynomial decay in $\langle \mathbf{m} \rangle$. The tail of the series involving $\mathbf{m}$ in our expression for $N_W(F, P)$ therefore makes a negligible contribution, leaving us free to truncate the sum over $\mathbf{m}$ by $\langle \mathbf{m} \rangle \leqslant P^A$, for some appropriate absolute constant $A > 0$. Thus we have

$$N_W(F, P) = \frac{c_Q 2^{r_2 n}}{D_K^{n/2} Q^{2d}} \sum_{\substack{(0) \neq \mathfrak{b} \subseteq \mathfrak{o} \\ \mathrm{N}\,\mathfrak{b} \ll Q^d}} \sum_{\substack{\mathbf{m} \in \hat{\mathfrak{b}}^n \\ \langle \mathbf{m} \rangle \leqslant P^A}} (\mathrm{N}\,\mathfrak{b})^{-n} S_{\mathfrak{b}}(\mathbf{m}) I_{\mathfrak{b}}(\mathbf{m}) + O(1). \tag{5.3}$$

We close this section by recording some further notation that will feature in our analysis. For a subset $S = \{s_1, \ldots, s_k\}$ of $\{1, \ldots, r_1 + r_2\}$, we define the restricted norm

$$\mathrm{Nm}_S(v) = \prod_{\substack{l \in S \\ l \leqslant r_1}} v^{(l)} \prod_{\substack{l \in S \\ l > r_1}} |v^{(l)}|^2,$$

on V. We follow the convention that $\mathrm{Nm}_\emptyset(v) = 1$ and define the "degree of S" to mean $d(S) = \sum_{l \in S} c_l$, with c_l given by (2.2). For any $v \in V$, let

$$T(v) = \{1 \leqslant l \leqslant r_1 + r_2 : |v^{(l)}| > 1\}. \tag{5.4}$$

The quantity $|\mathrm{Nm}_{T(v)}(v)|$ will provide a useful measure of the "height" of a point $v \in V$, and we henceforth set

$$\mathscr{H} : V \to \mathbb{R}_{>0}, \quad v \mapsto |\mathrm{Nm}_{T(v)}(v)|. \tag{5.5}$$

We may now establish the following result.

Lemma 5.3. — *Let $A > 0$ and let $\alpha < -1$. Then we have*

$$\int_{\{v \in V : \mathscr{H}(v) \leqslant A\}} \mathscr{H}(v)^\alpha \mathrm{d}v \ll_\alpha 1,$$

uniformly in A.

Proof. — Given any subset S of $\{1, \ldots, r_1 + r_2\}$, let R_S denote the set of $v \in V$ for which $|\mathrm{Nm}_S(v)| \leqslant A$ and $T(v) = S$. In order to establish the lemma it suffices to prove the desired bound for each integral

$$I_S = \int_{R_S} \mathscr{H}(v)^\alpha \mathrm{d}v.$$

Note that for $T(v) = \emptyset$, we have $\mathscr{H}(v) = 1$, whence $I_\emptyset \ll 1$ in this case. We next assume that $S = \{s_1, ..., s_{l+m}\}$ is non-empty, with $s_i \leqslant r_1$ for $1 \leqslant i \leqslant l$ and $s_{l+i} > r_1$ for $1 \leqslant i \leqslant m$. Let us make the polar change of variables $(v^{(s_1)}, \ldots, v^{(s_{l+m})})$ goes to $(u_1, \ldots, u_{l+m}, \theta_1, \ldots, \theta_m)$, with

$$u_i = \begin{cases} |v^{(s_i)}|, & \text{if } 1 \leqslant i \leqslant l, \\ |v^{(s_i)}|^2, & \text{if } l < i \leqslant l + m. \end{cases}$$

In particular $\mathrm{d}v^{(s_{l+i})} = 2^{-1} \mathrm{d}u_{l+i} \mathrm{d}\theta_i$, for $1 \leqslant i \leqslant m$ and $\mathscr{H}(v) = u_1 \cdots u_{l+m}$. It follows that

$$I_S \ll \int_{\substack{u_1, \ldots, u_{l+m} \geqslant 1 \\ u_1 \cdots u_{l+m} \leqslant A}} (u_1 \cdots u_{l+m})^\alpha \mathrm{d}u_1 \cdots \mathrm{d}u_{l+m}.$$

The statement of the lemma is clearly trivial unless $A \geqslant 1$, which we now assume. Write $\sigma = l + m$ for the cardinality of S and denote by J_σ the integral on the right hand side. In order to complete the proof of the lemma it suffices to show that $J_\sigma \ll_\alpha 1$. We do so by

induction on σ, the case $\sigma = 1$ being trivial. For $\sigma > 1$ we integrate over u_σ, finding that $J_\sigma \ll_\alpha J_{\sigma-1} \ll_\alpha 1$, by the induction hypothesis. This completes the proof of the lemma. $\square$

Next, for any $v \in V$ let

$$\mathfrak{I}(v) = \int_{\substack{\mathbf{x} \in V^n \\ F(\mathbf{x}) = v}} W(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{5.6}$$

with W as above. It is easy to see that $\mathfrak{I}(v)$ is an infinitely differentiable function on V with compact support. Furthermore, for any $N \in \mathbb{Z}_{\geqslant 0}$, we have

$$\lambda_{\mathfrak{I}}^N \ll_N (\log P)^{2N}, \tag{5.7}$$

in the notation of (2.7). It turns out that $\mathfrak{I}(0)$ is the "singular integral" for the problem. The "singular series" is given by the infinite sum

$$\mathfrak{S} = \sum_{(0) \neq \mathfrak{b} \subseteq \mathfrak{o}} (\mathrm{N}\,\mathfrak{b})^{-n} S_{\mathfrak{b}}(\mathbf{0}). \tag{5.8}$$

Our main aim is to establish the following result, which clearly suffices to conclude the proof of Theorem 1.1.

Theorem 5.4. — *Assume that $n \geqslant 10$. Then there exists $\Delta > 0$ such that*

$$N_W(F, P) = \frac{2^{r_2(n-1)}}{D_K^{(n-1)/2}} \mathfrak{S}\mathfrak{I}(0) P^{(n-3)d} + O(P^{(n-3)d-\Delta}),$$

with

$$(\log P)^{-2d(n-1)} \ll \mathfrak{S}\mathfrak{I}(0) \ll (\log P)^{-2d(n-1)}.$$

6. Cubic exponential integrals

Let $\mathfrak{b}$ be a non-zero integral ideal with $\mathrm{N}\,\mathfrak{b} \ll Q^d$ and define

$$\rho = Q^{-d}\,\mathrm{N}\,\mathfrak{b}.$$

Thus $\rho \in \mathbb{R}$ satisfies $\rho \ll 1$. In this section we shall produce a number of estimates for the exponential integral $I_{\mathfrak{b}}(\mathbf{m})$ in (5.3), beginning with a proof of Lemma 5.2. By a change of variables we get

$$I_{\mathfrak{b}}(\mathbf{m}) = P^{dn} \int_{V^n} W(\mathbf{x})h(\rho, \mathrm{Nm}(F(\mathbf{x})))\mathrm{e}(-P\mathbf{m}.\mathbf{x})\mathrm{d}\mathbf{x}, \tag{6.1}$$

where W is supported on the region $\langle \mathbf{x} \rangle \ll 1$. Note that

$$\partial^{\boldsymbol{\beta}}\{W(\mathbf{x})h(\rho, \mathrm{Nm}(F(\mathbf{x})))\} = \sum_{\boldsymbol{\beta} = \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2} \partial^{\boldsymbol{\beta}_1}W(\mathbf{x})\partial^{\boldsymbol{\beta}_2}h(\rho, \mathrm{Nm}(F(\mathbf{x}))),$$

for $\boldsymbol{\beta}, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ running over $\mathbb{Z}_{\geqslant 0}^{dn}$. Since F is a polynomial, there exist polynomials $f_{\boldsymbol{\beta}_2, j}$, such that

$$\partial^{\boldsymbol{\beta}_2}h(\rho, \mathrm{Nm}(F(\mathbf{x}))) = \sum_{j=0}^{|\boldsymbol{\beta}_2|} f_{\boldsymbol{\beta}_2, j}(\mathbf{x})h^{(j)}(\rho, \mathrm{Nm}(F(\mathbf{x}))),$$

with $h^{(j)}(x, y) = \frac{\partial^j}{\partial y^j}h(x, y) \ll_j x^{-j-1}$, by Lemma 3.2. Moreover, $|\partial^{\boldsymbol{\beta}_1}W(\mathbf{x})| \ll_{|\boldsymbol{\beta}_1|} (\log P)^{2|\boldsymbol{\beta}_1|}$ for any $\mathbf{x} \in V^n$. Combining these estimates, together with the inequality $\rho \ll 1$, we get

$$\partial^{\boldsymbol{\beta}}\{W(\mathbf{x})h(\rho, \mathrm{Nm}(F(\mathbf{x})))\} \ll_{|\boldsymbol{\beta}_1|} \rho^{-1}(\rho^{-1}(\log P)^2)^{|\boldsymbol{\beta}|}.$$

Using repeated integration by parts in (6.1), we arrive at the statement of Lemma 5.2.

We now turn to a more sophisticated treatment of $I_{\mathfrak{b}}(\mathbf{m})$. Let L be a constant, with $1 \ll L \ll 1$, such that $\langle F(\mathbf{x}) \rangle \leqslant L$ for every $\mathbf{x} \in \operatorname{supp}(W)$. Set $w_2(v) = w_1(v/2L)$, where $w_1 \in W_1^+(V)$ is any weight function which takes the value 1 in the region $\langle v \rangle \leqslant 1$. Then we clearly have $W(\mathbf{x}) = W(\mathbf{x})w_2(F(\mathbf{x}))$ for every $\mathbf{x} \in V^n$. This allows us to write

$$I_{\mathfrak{b}}(\mathbf{m}) = \int_{V^n} W(\mathbf{x}/P) \left\{ w_2(Q^{-2}F(\mathbf{x}))h(\rho, \operatorname{Nm}(Q^{-2}F(\mathbf{x}))) \right\} \mathrm{e}(-\mathbf{m}.\mathbf{x})\mathrm{d}\mathbf{x}.$$

The Fourier inversion formula implies that

$$w_2(Q^{-2}F(\mathbf{x}))h(\rho, \operatorname{Nm}(Q^{-2}F(\mathbf{x}))) = \int_V p_\rho(v)\mathrm{e}(vQ^{-2}F(\mathbf{x}))\mathrm{d}v,$$

where

$$p_\rho(v) = \int_V w_2(x)h(\rho, \operatorname{Nm}(x))\mathrm{e}(-vx)\mathrm{d}x. \tag{6.2}$$

This yields

$$I_{\mathfrak{b}}(\mathbf{m}) = \int_V p_\rho(v)K(Q^{-2}v, \mathbf{m})\mathrm{d}v, \tag{6.3}$$

with

$$K(v, \mathbf{m}) = \int_{V^n} W(\mathbf{x}/P)\mathrm{e}(vF(\mathbf{x}) - \mathbf{m}.\mathbf{x})\mathrm{d}\mathbf{x}.$$

6.1. Weighted exponential integrals. — It is clear from our work above that we will need good estimates for integrals of the form

$$\int_V w(x)h(\rho, \operatorname{Nm}(x))\mathrm{e}(-vx)\mathrm{d}x,$$

for $w \in \mathscr{W}_1^+(V)$. In fact, in the context of (6.3), in order for the integral over v to converge we require an estimate for $p_\rho(v)$ that decays sufficiently fast. It turns out that we will require savings of the form $\mathscr{H}(v)^{-N}$, in the notation of (5.5). The principal means of achieving this will be the use of integration by parts repeatedly. However, it will be crucial to apply this process in multiple directions, with respect to every component $v^{(l)}$ of v such that $l \in T(v)$.

Recall our decomposition $v = (\mathbf{v}, \mathbf{y} + i\mathbf{z})$, from §2.2. Let ∂_l denote ∂_{v_l} for $1 \leqslant l \leqslant r_1$. For $l > r_1$, we let ∂_l denote $\partial_{y_{l-r_1}}^2 + \partial_{z_{l-r_1}}^2$. Given a subset S of $\{1, \ldots, r_1 + r_2\}$, we let

$$\partial_S = \prod_{l \in S} \partial_l,$$

and we let $S^c = \{1, \ldots, r_1 + r_2\} \setminus S$. Recall the notation $h^{(m)}(x, y) = \frac{\partial^m}{\partial y^m}h(x, y)$ for any $m \in \mathbb{Z}_{\geqslant 0}$. We are now ready to establish the following result.

Lemma 6.1. — *Let $S \subseteq \{1, \ldots, r_1 + r_2\}$ and let $w \in \mathscr{W}_1(V)$. Then given any $k_1, k_2 \in \mathbb{Z}_{\geqslant 0}$, there exist weight functions $w_{S,k_1,k_2}, w_{S,k_1,k_2}^{(m)} \in \mathscr{W}_1(V)$, such that*

$$\partial_S\left\{w(v)\operatorname{Nm}(v)^{k_1}h^{(k_2)}(\rho, \operatorname{Nm}(v))\right\}$$

$$= w_{S,k_1,k_2}(v)\operatorname{Nm}(v)^{k_1}h^{(k_2)}(\rho, \operatorname{Nm}(v))$$

$$+ \sum_{m=0}^{d(S)} w_{S,k_1,k_2}^{(m)}(v)\operatorname{Nm}_{S^c}(v)\operatorname{Nm}(v)^{\max\{0, k_1+m-1\}}h^{(k_2+m)}(\rho, \operatorname{Nm}(v)).$$

Proof. — The proof will be given using induction on the cardinality of the set S. We will first prove the lemma when $\#S \leqslant 1$. Let $2 \leqslant M \leqslant r_1 + r_2$. Then, assuming the result to be true for all subsets of $\{1, \ldots, r_1 + r_2\}$, with cardinality at most $M - 1$, we will decompose a given subset $S \subseteq \{1, \ldots, r_1 + r_2\}$ of cardinality M as $S = S_1 \cup \{j\}$, with $\#S_1 = M - 1$. We will then use the fact that $\partial_S = \partial_j \partial_{S_1}$ and use the induction hypothesis accordingly. Notice that for $1 \leqslant j \leqslant r_1$ we have

$$\partial_j \operatorname{Nm}(v) = \operatorname{Nm}_{\{j\}^c}(v),$$

and for $r_1 < j \leqslant r_2$,

$$\frac{\partial}{\partial y_j} \operatorname{Nm}(v) = 2y_j \operatorname{Nm}_{\{j\}^c}(v), \quad \frac{\partial}{\partial z_j} \operatorname{Nm}(v) = 2z_j \operatorname{Nm}_{\{j\}^c}(v).$$

Let $K(v) = w(v) \operatorname{Nm}(v)^{k_1} h^{(k_2)}(\rho, \operatorname{Nm}(v))$.

The case $S = \emptyset$ is trivial. Suppose next that $S = \{j\}$. By symmetry it suffices to deal with cases when $S = \{1\}$ and $S = \{r_1 + 1\}$. Suppose first that $j = 1$. This implies that

$$\begin{aligned}
\partial_S K(v) &= \partial_1 (w(v) \operatorname{Nm}(v)^{k_1} h^{(k_2)}(\rho, \operatorname{Nm}(v))) \\
&= (\partial_1 w(v)) \operatorname{Nm}(v)^{k_1} h^{(k_2)}(\rho, \operatorname{Nm}(v)) \\
&\quad + k_1 w(v) \operatorname{Nm}_{S^c}(v) \operatorname{Nm}(v)^{k_1 - 1} h^{(k_2)}(\rho, \operatorname{Nm}(v)) \\
&\quad + w(v) \operatorname{Nm}_{S^c}(v) \operatorname{Nm}(v)^{k_1} h^{(k_2 + 1)}(\rho, \operatorname{Nm}(v)).
\end{aligned}$$

This is clearly of the required form, with $w_{S, k_1, k_2} = \partial_1 w$. Suppose next that $S = \{r_1 + 1\}$, so that $\partial_S = \partial_{y_1}^2 + \partial_{z_1}^2$. Notice that

$$\begin{aligned}
\partial_{y_1}^2 K(v) &= \partial_{y_1} \left(\partial_{y_1} \left(w(v) \operatorname{Nm}(v)^{k_1} h^{(k_2)}(\rho, \operatorname{Nm}(v)) \right) \right) \\
&= \partial_{y_1} \left((\partial_{y_1} w(v)) \operatorname{Nm}(v)^{k_1} h^{(k_2)}(\rho, \operatorname{Nm}(v)) \right) \\
&\quad + 2k_1 \partial_{y_1} \left(y_1 \operatorname{Nm}_{S^c}(v) w(v) \operatorname{Nm}(v)^{k_1 - 1} h^{(k_2)}(\rho, \operatorname{Nm}(v)) \right) \\
&\quad + \partial_{y_1} \left(2y_1 \operatorname{Nm}_{S^c}(v) w(v) \operatorname{Nm}(v)^{k_1} h^{(k_2 + 1)}(\rho, \operatorname{Nm}(v)) \right).
\end{aligned}$$

We deal here only with the third term in this expression, the remaining two terms being of a similar ilk. Let $f(v) = 2 \operatorname{Nm}_{S^c}(v) w(v) \operatorname{Nm}(v)^{k_1} h^{(k_2 + 1)}(\rho, \operatorname{Nm}(v))$. We find that

$$\begin{aligned}
\partial_{y_1} (y_1 f(v)) &= 2 \operatorname{Nm}_{S^c}(v) \{w(v) + y_1 \partial_{y_1} w(v)\} \operatorname{Nm}(v)^{k_1} h^{(k_2 + 1)}(\rho, \operatorname{Nm}(v)) \\
&\quad + 4k_1 y_1^2 (\operatorname{Nm}_{S^c}(v))^2 w(v) \operatorname{Nm}(v)^{k_1 - 1} h^{(k_2 + 1)}(\rho, \operatorname{Nm}(v)) \\
&\quad + 4y_1^2 (\operatorname{Nm}_{S^c}(v))^2 w(v) \operatorname{Nm}(v)^{k_1} h^{(k_2 + 2)}(\rho, \operatorname{Nm}(v)).
\end{aligned}$$

We carry out the same process for z_1 and get similar expressions. Adding together the expressions corresponding to y_1 and z_1 we get the contribution

$$\partial_{y_1}(y_1 f(v)) + \partial_{z_1}(z_1 f(v))$$
$$= 2\,\mathrm{Nm}_{S^c}(v)\,\{2w(v) + y_1\partial_{y_1}w(v) + z_1\partial_{z_1}w(v)\}\,\mathrm{Nm}(v)^{k_1}h^{(k_2+1)}(\rho,\mathrm{Nm}(v))$$
$$+ 4k_1(y_1^2 + z_1^2)(\mathrm{Nm}_{S^c}(v))^2 w(v)\,\mathrm{Nm}(v)^{k_1-1}h^{(k_2+1)}(\rho,\mathrm{Nm}(v))$$
$$+ 4(y_1^2 + z_1^2)(\mathrm{Nm}_{S^c}(v))^2 w(v)\,\mathrm{Nm}(v)^{k_1}h^{(k_2+2)}(\rho,\mathrm{Nm}(v))$$
$$= E_1(v)\,\mathrm{Nm}_{S^c}(v)\,\mathrm{Nm}(v)^{k_1}h^{(k_2+1)}(\rho,\mathrm{Nm}(v))$$
$$+ E_2(v)\,\mathrm{Nm}_{S^c}(v)\,\mathrm{Nm}(v)^{k_1+1}h^{(k_2+2)}(\rho,\mathrm{Nm}(v)),$$

with $E_1(v) = (4 + 4k_1)w(v) + 2(y_1\partial_{y_1}w(v) + z_1\partial_{z_1}w(v))$ and $E_2(v) = 4w(v)$. On dealing with the other terms in a similar fashion, this concludes the proof of the lemma when $\#S = 1$.

Let $2 \leqslant M \leqslant r_1 + r_2$. For the inductive step let us assume the veracity of the lemma for all subsets of $\{1, \ldots, r_1 + r_2\}$ of cardinality at most $M - 1$. Now let S be any subset of $\{1, \ldots, r_1 + r_2\}$ with M elements. We write $S = S_1 \cup \{j\}$ for some S_1 of cardinality $M - 1$ elements. For simplicity we shall assume that $j \leqslant r_1$, the complex case being handled similarly. The induction hypothesis implies that there exist smooth weights $w_{S_1,k_1,k_2}, w_{S_1,k_1,k_2}^{(m)} \in \mathscr{W}_1(V)$, such that

$$\partial_{S_1}K(v) = w_{S_1,k_1,k_2}(v)\,\mathrm{Nm}(v)^{k_1}h^{(k_2)}(\rho,\mathrm{Nm}(v))$$
$$+ \sum_{m=0}^{d(S_1)} w_{S_1,k_1,k_2}^{(m)}(v)\,\mathrm{Nm}_{S_1^c}(v)\,\mathrm{Nm}(v)^{\max\{0,k_1+m-1\}}h^{(k_2+m)}(\rho,\mathrm{Nm}(v)).$$

Taking the derivative of this with respect to v_j, we obtain

$$\partial_S K(v) = \partial_j\left(w_{S_1,k_1,k_2}(v)\,\mathrm{Nm}(v)^{k_1}h^{(k_2)}(\rho,\mathrm{Nm}(v))\right)$$
$$+ \sum_{m=0}^{d(S_1)} \partial_j\left(w_{S_1,k_1,k_2}^{(m)}(v)\,\mathrm{Nm}_{S_1^c}(v)\,\mathrm{Nm}(v)^{\max\{0,k_1+m-1\}}h^{(k_2+m)}(\rho,\mathrm{Nm}(v))\right).$$

An application of the induction hypothesis shows that the first term here is satisfactory for the lemma. Turning to the mth summand, we see that

$$\partial_j\left(w_{S_1,k_1,k_2}^{(m)}(v)\,\mathrm{Nm}_{S_1^c}(v)\,\mathrm{Nm}(v)^{\max\{0,k_1+m-1\}}h^{(k_2+m)}(\rho,\mathrm{Nm}(v))\right)$$
$$= \partial_j(\mathrm{Nm}_{S_1^c}(v))\left(w_{S_1,k_1,k_2}^{(m)}(v)\,\mathrm{Nm}(v)^{\max\{0,k_1+m-1\}}h^{(k_2+m)}(\rho,\mathrm{Nm}(v))\right)$$
$$+ \mathrm{Nm}_{S_1^c}(v)\partial_j\left(w_{S_1,k_1,k_2}^{(m)}(v)\,\mathrm{Nm}(v)^{\max\{0,k_1+m-1\}}h^{(k_2+m)}(\rho,\mathrm{Nm}(v))\right).$$

Since $j \leqslant r_1$, we have $\partial_j(\mathrm{Nm}_{S_1^c}(v)) = \mathrm{Nm}_{S^c}(v)$ here. Thus the first term is already in the desired form and we need to investigate the second term, $G(v)$, say. Applying the induction hypothesis we obtain weights $w_{S,k_1,k_2}^{(m)}, w_{S,k_1,k_2}^{(m,n)} \in \mathscr{W}_1(V)$, such that

$$G(v) = \mathrm{Nm}_{S_1^c}(v)w_{S,k_1,k_2}^{(m)}(v)(\mathrm{Nm}(v))^{\max\{0,k_1+m-1\}}h^{(k_2+m)}(\rho,\mathrm{Nm}(v))$$
$$+ \mathrm{Nm}_{S_1^c}(v)\sum_{n=0,1}\mathrm{Nm}_{\{j\}^c}(v)w_{S,k_1,k_2}^{(m,n)}(v)(\mathrm{Nm}(v))^{\max\{0,k_1+m+n-2\}}h^{(k_2+m+n)}(\rho,\mathrm{Nm}(v)).$$

But $\mathrm{Nm}_{S_1^c}(v) = v_j \, \mathrm{Nm}_{S^c}(v)$ and $\mathrm{Nm}_{S_1^c}(v) \, \mathrm{Nm}_{\{j\}^c}(v) = \mathrm{Nm}_{S^c}(v) \, \mathrm{Nm}(v)$, whence

$$
\begin{aligned}
G(v) = {} & \mathrm{Nm}_{S^c}(v) v_j w_{S,k_1,k_2}^{(m)}(v)(\mathrm{Nm}(v))^{\max\{0,k_1+m-1\}} h^{(k_2+m)}(\rho, \mathrm{Nm}(v)) \\
& + \sum_{n=0,1} \mathrm{Nm}_{S^c}(v) w_{S,k_1,k_2}^{(m,0)}(v)(\mathrm{Nm}(v))^{\max\{1,k_1+m+n-1\}} h^{(k_2+m+1)}(\rho, \mathrm{Nm}(v)).
\end{aligned}
$$

This is satisfactory for the lemma and so concludes its proof. $\qquad\square$

We shall ultimately be interested in a version of Lemma 6.1 in the special case $k_1 = k_2 = 0$, when ∂_S is replaced by an arbitrary power of ∂_S. This is an easy consequence of our work so far, as the following result attests.

Lemma 6.2. — *Assume the same notation as in Lemma 6.1 and let $N \in \mathbb{Z}_{\geq 0}$. Then there exist weight functions $w_{S,N}, w_{S,N}^{(m,n)} \in \mathscr{W}_1(V)$, such that*

$$
\begin{aligned}
\partial_S^N \{ w(v) h(\rho, \mathrm{Nm}(v)) \} = {} & w_{S,N}(v) h(\rho, \mathrm{Nm}(v)) \\
& + \sum_{n=1}^{N} \sum_{m=0}^{Nd(S)} w_{S,N}^{(m,n)}(v) \, \mathrm{Nm}_{S^c}(v)^n \, \mathrm{Nm}(v)^{\max\{0,m-n\}} h^{(m)}(\rho, \mathrm{Nm}(v)).
\end{aligned}
$$

Proof. — We argue by induction on $N \geq 1$, the case $N = 1$ being Lemma 6.1. The induction hypothesis ensures that

$$
\partial_S^{N+1} \{ w(v) h(\rho, \mathrm{Nm}(v)) \} = \tilde{w}_{S,N}(v) + \sum_{n=1}^{N} \sum_{m=0}^{Nd(S)} \mathrm{Nm}_{S^c}(v)^n \tilde{w}_{S,N}^{(m,n)}(v),
$$

where

$$
\begin{aligned}
\tilde{w}_{S,N}(v) &= \partial_S(w_{S,N}(v) h(\rho, \mathrm{Nm}(v))), \\
\tilde{w}_{S,N}^{(m,n)}(v) &= \partial_S \left(w_{S,N}^{(m,n)}(v) \, \mathrm{Nm}(v)^{\max\{0,m-n\}} h^{(m)}(\rho, \mathrm{Nm}(v)) \right),
\end{aligned}
$$

for $w_{S,N}, w_{S,N}^{(m,n)} \in \mathscr{W}_1(V)$. Invoking Lemma 6.1 to evaluate $\tilde{w}_{S,N}$ and $\tilde{w}_{S,N}^{(m,n)}$, it is a simple matter to check that one arrives at an expression suitable for the conclusion of the lemma. $\quad\square$

6.2. Estimation of $p_\rho(v)$. — We now apply our work in the previous section to the task of estimating $p_\rho(v)$, as given by (6.2). Let $\rho \ll 1$, with $\log \rho$ having order of magnitude $\log P$. The following result shows that $p_\rho(v)$ is essentially supported on the set of $v \in V$ for which $\mathscr{H}(v) \ll \rho^{-1} P^\varepsilon$, in the notation of (5.5)

Lemma 6.3. — *Let $\varepsilon > 0$ and $N \in \mathbb{Z}_{\geq 0}$. Then we have*

$$
p_\rho(v) \ll_N \rho^{-1} \left(\rho^{-1} P^\varepsilon |\mathscr{H}(v)|^{-1} \right)^N.
$$

Proof. — Recall that

$$
p_\rho(v) = \int_V w_2(x) h(\rho, \mathrm{Nm}(x)) \mathrm{e}(-vx) \mathrm{d}x,
$$

where $w_2 \in \mathscr{W}_1(V)$. In particular $\langle x \rangle \ll 1$ for all $x \in \mathrm{supp}(w_2)$. This in turn implies that $\mathrm{Nm}_S(x) \ll 1$ for any $x \in \mathrm{supp}(w_2)$ and any subset S of $\{1, \ldots, r_1 + r_2\}$.

Let $S \subseteq \{1, \ldots, r_1 + r_2\}$ and let $N \in \mathbb{Z}_{\geqslant 0}$. Lemma 6.2 implies that

$$\partial_S^N \{w_2(x) h(\rho, \mathrm{Nm}(x))\} \ll_N \sum_{n=0}^{N} \sum_{m=0}^{Nd(S)} |\mathrm{Nm}(x)|^{\max\{0, m-n\}} |h^{(m)}(\rho, \mathrm{Nm}(x))|. \qquad (6.4)$$

Integration by parts yields

$$p_\rho(v) \ll_N |\mathrm{Nm}_S(v)|^{-N} \int_{\langle x \rangle \ll 1} \left| \partial_S^N \{w_2(x) h(\rho, \mathrm{Nm}(x))\} \right| \mathrm{d}x. \qquad (6.5)$$

Let $\varepsilon > 0$. We split the integration here into two parts $J_1 + J_2$, where J_1 arises from $|\mathrm{Nm}(x)| \leqslant \rho P^{\varepsilon/d}$ and J_2 is the contribution from $|\mathrm{Nm}(x)| > \rho P^{\varepsilon/d}$.

Beginning with the latter, we deduce from taking $\mathrm{Nm}(x) \ll 1$ in (6.4) that

$$J_2 \ll_N \sum_{m=0}^{Nd(S)} \int_{\substack{\langle x \rangle \ll 1 \\ |\mathrm{Nm}(x)| \geqslant \rho P^{\varepsilon/d}}} |h^{(m)}(\rho, \mathrm{Nm}(x))| \mathrm{d}x.$$

On the domain of integration Lemma 3.2 implies that $h^{(m)}(\rho, \mathrm{Nm}(x)) \ll_{m,M} \rho^{-1-m} P^{-M\varepsilon/d}$, for any $M \in \mathbb{Z}_{\geqslant 0}$. Recall here that $\log \rho$ has order $\log P$. Thus, on taking M sufficiently large, we have $J_2 \ll_N P^{-N\varepsilon/d}$. Turning to the estimation of J_1, a further application of Lemma 3.2 yields $h^{(m)}(\rho, \mathrm{Nm}(x)) \ll \rho^{-m-1}$ for any $m \in \mathbb{Z}_{\geqslant 0}$. Hence (6.4) gives

$$\partial_S^N \{w_2(x) h(\rho, \mathrm{Nm}(x))\} \ll_N \sum_{n=0}^{N} \rho^{-1-n} \sum_{m=0}^{Nd(S)} (\rho^{-1} |\mathrm{Nm}(x)|)^{\max\{0, m-n\}}$$

$$\ll_N \rho^{-1-N} \sum_{m=0}^{Nd(S)} |\rho^{-1} \mathrm{Nm}(x)|^m.$$

We therefore obtain

$$J_1 \ll_N \rho^{-1-N} \sum_{m=0}^{Nd(S)} \int_{\substack{\langle x \rangle \ll 1 \\ |\mathrm{Nm}(x)| \leqslant \rho P^{\varepsilon/d}}} |\rho^{-1} \mathrm{Nm}(x)|^m \mathrm{d}x$$

$$\ll_N \rho^{-1-N} P^{N\varepsilon},$$

since $d(S) \leqslant r_1 + r_2 \leqslant d$.

Combining our estimates for J_1, J_2 in (6.5), we therefore obtain

$$p_\rho(v) \ll_N |\mathrm{Nm}_S(v)|^{-N} \left(P^{-N\varepsilon/d} + \rho^{-1-N} P^{N\varepsilon} \right).$$

Since $\rho \ll 1$ the second term here clearly dominates the first and we therefore conclude the proof of the lemma on taking $S = T(v)$, in the notation of (5.4) and (5.5). □

Lemma 6.3 will be effective when $\mathscr{H}(v)$ is large, but we will need a companion "trivial" estimate to deal with the remaining cases. This is provided by the following result.

Lemma 6.4. — *We have $p_\rho(v) \ll |\log \rho|^{r_1+r_2-1}$.*

Proof. — We break the integral over $x = (x^{(1)}, \ldots, x^{(r_1+r_2)}) \in V$ in (6.2) into two parts $I_1 + I_2$, say, where I_1 is the contribution from x such that $|x^{(l)}| < \rho$ for some $1 \leqslant l \leqslant r_1 + r_2$, and I_2 has $|x^{(l)}| \geqslant \rho$ for every $1 \leqslant l \leqslant r_1 + r_2$. Since $\mathrm{supp}(w_2) \ll 1$ we deduce from Lemma 3.2 with $i = j = N = 0$ that $I_1 \ll 1$.

Turning to the second term, we take $i = j = 0$ and $N = 2$ in Lemma 3.2 to deduce that

$$I_2 \ll 1 + \rho^{-1} \int_{\substack{\langle x \rangle \ll 1 \\ \min_l |x^{(l)}| \geqslant \rho}} \min\left\{1, \frac{\rho}{|\operatorname{Nm}(x)|}\right\}^2 \mathrm{d}x$$

$$\ll 1 + \rho^{-1} \int_{\substack{\langle x \rangle \ll 1 \\ \min_l |x^{(l)}| \geqslant \rho \\ \rho > |\operatorname{Nm}(x)|}} \mathrm{d}x + \rho \int_{\substack{\langle x \rangle \ll 1 \\ \min_l |x^{(l)}| \geqslant \rho \\ \rho \leqslant |\operatorname{Nm}(x)|}} \frac{1}{|\operatorname{Nm}(x)|^2} \mathrm{d}x$$

$$\ll 1 + |\log \rho|^{r_1 + r_2 - 1}.$$

This completes the proof of the lemma since $|\log \rho| \gg 1$. $\square$

6.3. Application of Skinner's estimates. — In this section we show how Skinner's treatment of cubic exponential integrals in [**15**, §6] can be recycled here to help deal with the integral $K(v, \mathbf{m})$ that appears in (6.3). Recall the definition (5.2) of the weight function ω and define

$$I(v, \mathbf{m}) = \int_{V^n} \omega(\mathbf{x}/P) \mathrm{e}(vF(\mathbf{x}) - \mathbf{m}.\mathbf{x}) \mathrm{d}\mathbf{x},$$

for $v \in V$ and $\mathbf{m} \in V^n$. Note that this is equal to the integral $I(v, -\mathbf{m})$ introduced in [**15**, Eq. (5.7)]. Then it easily follows that

$$|I(v, \mathbf{m}) - K(v, \mathbf{m})| \ll e^{-(\log P)^2/2}.$$

Returning to (5.3) and (6.3), we shall in this section mainly be concerned with the contribution from non-zero phases

$$\begin{aligned}
M(P) &= \sum_{\substack{(0) \neq \mathfrak{b} \subseteq \mathfrak{o} \\ \mathrm{N}\,\mathfrak{b} \ll Q^d}} \sum_{\substack{\mathbf{0} \neq \mathbf{m} \in \hat{\mathfrak{b}}^n \\ \langle \mathbf{m} \rangle \leqslant P^A}} (\mathrm{N}\,\mathfrak{b})^{-n} S_\mathfrak{b}(\mathbf{m}) I_\mathfrak{b}(\mathbf{m}) \\
&= \sum_{\substack{(0) \neq \mathfrak{b} \subseteq \mathfrak{o} \\ \mathrm{N}\,\mathfrak{b} \ll Q^d}} \sum_{\substack{\mathbf{0} \neq \mathbf{m} \in \hat{\mathfrak{b}}^n \\ \langle \mathbf{m} \rangle \leqslant P^A}} (\mathrm{N}\,\mathfrak{b})^{-n} S_\mathfrak{b}(\mathbf{m}) \int_V p_\rho(v) K(Q^{-2}v, \mathbf{m}) \mathrm{d}v \\
&= \sum_{\substack{(0) \neq \mathfrak{b} \subseteq \mathfrak{o} \\ \mathrm{N}\,\mathfrak{b} \ll Q^d}} \int_V \sum_{\substack{\mathbf{0} \neq \mathbf{m} \in \hat{\mathfrak{b}}^n \\ \langle \mathbf{m} \rangle \leqslant P^A}} (\mathrm{N}\,\mathfrak{b})^{-n} S_\mathfrak{b}(\mathbf{m}) p_\rho(v) I(Q^{-2}v, \mathbf{m}) \mathrm{d}v + O(1)
\end{aligned} \tag{6.6}$$

Let $J^{(l)}(z, \mathbf{m})$ denote the lth component of $I(z, \mathbf{m})$, so that $I(z, \mathbf{m}) = \prod_l J^{(l)}(z, \mathbf{m})$. We are now in a position to apply [**15**, Eq. (6.32) and Lemma 20], which gives the following result.

Lemma 6.5. — *Let $\varepsilon > 0$ and $B^{(l)}(z) = P^\varepsilon(P^{-1} + |z^{(l)}|P^2)$. Then we have*

$$J^{(l)}(z, \mathbf{m}) \ll \begin{cases} P^\varepsilon \min\{P^{nc_l}, |Pz^{(l)}|^{-nc_l/2}\}, & \text{if } |\mathbf{m}^{(l)}| \ll B^{(l)}(z), \\ \exp(-\frac{1}{3}(\log P(2 + |\mathbf{m}^{(l)}|))^2), & \text{otherwise.} \end{cases}$$

In particular, when $|\mathbf{m}^{(l)}| \ll B^{(l)}(z)$ for all l, we have

$$I(z, \mathbf{m}) \ll P^{d\varepsilon} \prod_{l=1}^{r_1 + r_2} \min\{P^{nc_l}, |Pz^{(l)}|^{-nc_l/2}\}.$$

Lemma 6.5 allows us to freely truncate the $\mathbf{m}$ summation in (6.6) to $\mathbf{m}$ satisfying

$$|\mathbf{m}^{(l)}| \ll B^{(l)}(Q^{-2}v) = P^{-1+\varepsilon}(1 + |v^{(l)}|), \tag{6.7}$$

with acceptable error. For such $\mathbf{m}$ we deduce that

$$I(Q^{-2}v, \mathbf{m}) \ll P^{d\varepsilon} \prod_l \min\{P^{nc_l}, |PQ^{-2}v^{(l)}|^{-nc_l/2}\}$$

$$\ll P^{d\varepsilon} \prod_l \min\{P^{nc_l}, |P^{-2}v^{(l)}|^{-nc_l/2}\}$$

$$\ll P^{dn+d\varepsilon}\mathscr{H}(v)^{-n/2},$$

where we recall that $\mathscr{H}(v)$ is given by (5.5). Inserting this into (6.6) yields

$$M(P) \ll 1 + P^{dn+d\varepsilon} \sum_{\substack{(0)\neq\mathfrak{b}\subseteq\mathfrak{o} \\ \mathrm{N}\,\mathfrak{b}\ll Q^d}} \int_V \mathscr{H}(v)^{-n/2} \sum_{\substack{\mathbf{0}\neq\mathbf{m}\in\hat{\mathfrak{b}}^n \\ (6.7)\ \text{holds}}} (\mathrm{N}\,\mathfrak{b})^{-n}|S_\mathfrak{b}(\mathbf{m})||p_\rho(v)|\mathrm{d}v.$$

Let

$$R_\rho = \{v \in V : \mathscr{H}(v) \ll \rho^{-1}P^{2\varepsilon}\}.$$

Using Lemma 6.3 to estimate $p_\rho(v)$, we see that the overall contribution to the above estimate from $v \in V \setminus R_\rho$ is $O(1)$. When $v \in R_\rho$ we will simply invoke Lemma 6.4, which gives $p_\rho(v) \ll |\log\rho|^{r_1+r_2-1} \ll P^\varepsilon$. Hence we may write

$$M(P) \ll 1 + P^{dn+(d+1)\varepsilon} \sum_{\substack{(0)\neq\mathfrak{b}\subseteq\mathfrak{o} \\ \mathrm{N}\,\mathfrak{b}\ll Q^d}} \int_{R_\rho} \mathscr{H}(v)^{-n/2} \sum_{\substack{\mathbf{0}\neq\mathbf{m}\in\hat{\mathfrak{b}}^n \\ (6.7)\ \text{holds}}} (\mathrm{N}\,\mathfrak{b})^{-n}|S_\mathfrak{b}(\mathbf{m})|\mathrm{d}v.$$

Notice that $R_\rho \subseteq \{v \in V : \mathscr{H}(v) \ll P^{3d/2+2\varepsilon}\} = R$, say. Let $\chi_\rho(v)$ be the characteristic function of R_ρ. Then we may write

$$M(P) \ll 1 + P^{dn+(d+1)\varepsilon} \int_R E(v, P)\mathrm{d}v, \tag{6.8}$$

where

$$E(v, P) = \mathscr{H}(v)^{-n/2} \sum_{\substack{(0)\neq\mathfrak{b}\subseteq\mathfrak{o} \\ \mathrm{N}\,\mathfrak{b}\ll Q^d}} \chi_\rho(v) \sum_{\substack{\mathbf{0}\neq\mathbf{m}\in\hat{\mathfrak{b}}^n \\ (6.7)\ \text{holds}}} (\mathrm{N}\,\mathfrak{b})^{-n}|S_\mathfrak{b}(\mathbf{m})|.$$

We proceed to show that the outer sum is actually restricted to $Q_0 \ll \mathrm{N}\,\mathfrak{b} \ll Q_1$, for suitable parameters Q_0 and Q_1.

Notice that for $v \in R$, we have $v \in R_\rho$ if and only if $\mathscr{H}(v) \ll \rho^{-1}P^{2\varepsilon}$. Recalling that $\rho = Q^{-d}\mathrm{N}\,\mathfrak{b}$, we see that $v \in R_\rho$ if and only if

$$\mathrm{N}\,\mathfrak{b} \ll P^{2\varepsilon}Q^d\mathscr{H}(v)^{-1} = P^{3d/2+2\varepsilon}\mathscr{H}(v)^{-1} = Q_1,$$

say. Recall from (6.7) that $|\mathbf{m}^{(l)}| \ll P^{-1+\varepsilon}(1 + |v^{(l)}|) = B^{(l)}$, say, for $\mathbf{m} = (m_1, \ldots, m_n) \in \hat{\mathfrak{b}}^n$. Hence

$$|\mathrm{Nm}(m_i)| \ll |\mathrm{Nm}(B)| \ll P^{-d(1-\varepsilon)}\mathscr{H}(v),$$

for $1 \leqslant i \leqslant n$. Let $\mathbf{m} \in \hat{\mathfrak{b}}^n$ be non-zero and suppose, without loss of generality, that $m_1 \neq 0$. Then $m_1 \in \hat{\mathfrak{b}} = \mathfrak{b}^{-1}\mathfrak{d}^{-1}$, and so $m_1\mathfrak{b}\mathfrak{d}$ is integral. It follows that $\mathrm{Nm}(m_1)\mathrm{Nm}(\mathfrak{b}) \geqslant 1$ and so $\mathrm{Nm}(m_1) \geqslant \mathrm{Nm}(\mathfrak{b})^{-1}$. This in turn yields $(\mathrm{N}\,\mathfrak{b})^{-1} \ll P^{-d(1-\varepsilon)}\mathscr{H}(v)$, which implies that

$$\mathrm{N}\,\mathfrak{b} \gg P^{d(1-\varepsilon)}\mathscr{H}(v)^{-1} = Q_0,$$

say. Bringing this all together in (5.3) and (6.8), and replacing 2ε by ε, we may now record the following result.

Lemma 6.6. — *Let $\varepsilon > 0$ and let $B = \bigoplus_l B^{(l)}$, with $B^{(l)} = P^{-1+\varepsilon}(1 + |v^{(l)}|)$. Let*

$$Q_0 = P^{d(1-\varepsilon)}\mathscr{H}(v)^{-1}, \quad Q_1 = P^{3d/2+\varepsilon}\mathscr{H}(v)^{-1}.$$

Then we have

$$N_W(F, P) = \frac{c_Q 2^{r_2 n}}{D_K^{n/2}Q^{2d}} \sum_{\substack{(0)\neq\mathfrak{b}\subseteq\mathfrak{o} \\ N\,\mathfrak{b}\ll Q^d}} (N\,\mathfrak{b})^{-n}S_\mathfrak{b}(\mathbf{0})I_\mathfrak{b}(\mathbf{0}) + O\left(1 + P^{d(n-3)+(d+1)\varepsilon}\int_R E(v, P)\mathrm{d}v\right),$$

with $R = \{v \in V : \mathscr{H}(v) \ll P^{3d/2+\varepsilon}\}$ and

$$E(v, P) = \mathscr{H}(v)^{-n/2} \sum_{\substack{(0)\neq\mathfrak{b}\subseteq\mathfrak{o} \\ Q_0\ll N\,\mathfrak{b}\ll Q_1}} \sum_{\substack{\mathbf{0}\neq\mathbf{m}\in\hat{\mathfrak{b}}^n \\ |\mathbf{m}^{(l)}|\ll B^{(l)}}} (N\,\mathfrak{b})^{-n}|S_\mathfrak{b}(\mathbf{m})|.$$

In applying this result the goal is to show that $\int_R E(v, P)\mathrm{d}v \ll P^{-\Delta}$, for a suitable constant $\Delta > 0$. When $\mathscr{H}(v)$ is small, one sees that Q_0 is large and one can hope to gain sufficient cancellation in the exponential sums $S_\mathfrak{b}(\mathbf{m})$. On the other hand, when $\mathscr{H}(v)$ is large, then Q_1 is small and the exponential sums involved are small. However, in this case, the factor $\mathscr{H}(v)^{-n/2}$ will produce the necessary saving. This is the main idea behind our application of Lemma 6.6.

7. Cubic exponential sums

The purpose of this section is to make a careful analysis of the exponential sums $S_\mathfrak{b}(\mathbf{m})$ occurring in Theorem 5.1 and Lemma 6.6 when $F \in \mathfrak{o}[X_1, \ldots, X_n]$ is a non-singular cubic form, with $n \geqslant 3$. Here $\mathfrak{b}$ is an arbitrary integral ideal and $\mathbf{m} \in \hat{\mathfrak{b}}^n$. Applying Lemma 2.3, we see that there exists $\gamma = \nu/\alpha$, for $\nu, \alpha \in \mathfrak{o}$ such that (ν) is coprime to $\mathfrak{b}$, which allows us to write

$$S_\mathfrak{b}(\mathbf{m}) = \sum_{a\in(\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{b})} \mathrm{e}\left(a\gamma F(\mathbf{a}) + \mathbf{m}.\mathbf{a}\right).$$

It follows from our work in §2.3 that

$$\mathfrak{a}_\gamma = \mathfrak{b}\mathfrak{d}. \tag{7.1}$$

Moreover, there exists a prime ideal $\mathfrak{p}_1$ coprime to $\mathfrak{b}\mathfrak{d}$, such that $(\alpha) = \mathfrak{b}\mathfrak{d}\mathfrak{p}_1$. The following simple result, established by Skinner [15, Lemma 3], will prove useful in our analysis.

Lemma 7.1. — *Suppose that $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are integral ideals such that $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$. Then we have the following:*

(i) *if $\alpha \in \mathfrak{o}$ satisfies $\mathrm{ord}_\mathfrak{p}(\alpha) = \mathrm{ord}_\mathfrak{p}(\mathfrak{b})$ for all $\mathfrak{p} \mid \mathfrak{a}$, then*

$$\mathfrak{o}/\mathfrak{a} = \{\beta + \mu\alpha : \beta \in \mathfrak{o}/\mathfrak{b}, \ \mu \in \mathfrak{o}/\mathfrak{c}\};$$

(ii) *if, furthermore, $\mathfrak{b}$ and $\mathfrak{c}$ are coprime and if $\alpha, \lambda \in \mathfrak{o}$ satisfy $\mathrm{ord}_\mathfrak{p}(\alpha) = \mathrm{ord}_\mathfrak{p}(\mathfrak{b})$ and $\mathrm{ord}_\mathfrak{p}(\lambda) = \mathrm{ord}_\mathfrak{p}(\mathfrak{c})$ for all $\mathfrak{p} \mid \mathfrak{a}$, then*

$$\mathfrak{o}/\mathfrak{a} = \{\alpha\mu + \lambda\beta : \beta \in \mathfrak{o}/\mathfrak{b}, \ \mu \in \mathfrak{o}/\mathfrak{c}\}.$$

There is an abuse of notation at play in this lemma, in that the sets involved are actually coset representatives for $\mathfrak{o}/\mathfrak{a}, \mathfrak{o}/\mathfrak{b}$ and $\mathfrak{o}/\mathfrak{c}$.

Since we aim to monopolise upon the existing work of Skinner, we will begin by indicating how our expression for $S_{\mathfrak{b}}(\mathbf{m})$ is related to the exponential sums

$$S(\gamma, \mathbf{b}) = \sum_{\mathbf{a} \,(\mathrm{mod}\,\mathfrak{a}_\gamma)} \mathrm{e}\,(\gamma F(\mathbf{a}) - \mathbf{b}.\mathbf{a}),$$

that emerge in Skinner [**15**, Eq. (5.6)], for $\mathbf{b} \in \hat{\mathfrak{a}}_\gamma$. This is the object of the following result.

Lemma 7.2. — *For any integral ideal $\mathfrak{b}$ and $\mathbf{m} \in \hat{\mathfrak{b}}^n$ we have*

$$S_{\mathfrak{b}}(\mathbf{m}) = D_K^{-n} \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} S(a\gamma, -\mathbf{m}).$$

Proof. — Recall from (7.1) that $\mathfrak{a}_\gamma = \mathfrak{b}\mathfrak{d}$. Hence for any $\mathbf{m} \in \hat{\mathfrak{b}}^n$ we have

$$\sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} S(a\gamma, -\mathbf{m}) = \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{a} \,(\mathrm{mod}\,\mathfrak{b}\mathfrak{d})} \mathrm{e}(a\gamma F(\mathbf{a}) + \mathbf{m}.\mathbf{a}).$$

By Lemma 2.2 (i) we may find $\beta \in \mathfrak{b}$ such that $\mathrm{ord}_{\mathfrak{p}}(\beta) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{b})$, for each prime ideal $\mathfrak{p} \mid \mathfrak{b}\mathfrak{d}$. Then Lemma 7.1 allows us to write $\mathbf{a} = \mathbf{b} + \beta\mathbf{c}$, with $\mathbf{b}$ running modulo $\mathfrak{b}$ and $\mathbf{c}$ running modulo $\mathfrak{d}$, giving

$$\sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} S(a\gamma, -\mathbf{m}) = \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{c} \,(\mathrm{mod}\,\mathfrak{d})} \sum_{\mathbf{b} \,(\mathrm{mod}\,\mathfrak{b})} \mathrm{e}(a\gamma F(\mathbf{b} + \beta\mathbf{c}) + \mathbf{m}.(\mathbf{b} + \beta\mathbf{c})).$$

Since $\mathfrak{a}_\gamma = \mathfrak{b}\mathfrak{d}$ and $\mathfrak{b} \mid (\beta)$, it follows that $\gamma\beta \in \hat{\mathfrak{o}}$, whence $\mathrm{e}(a\gamma F(\mathbf{b} + \beta\mathbf{c})) = \mathrm{e}(a\gamma F(\mathbf{b}))$. Moreover, if $\mathbf{m} \in \hat{\mathfrak{b}}^n$ then one has $\mathrm{e}(\beta\mathbf{m}.\mathbf{c}) = 1$, since $\beta\hat{\mathfrak{b}} = \beta\mathfrak{b}^{-1}\hat{\mathfrak{o}} \subseteq \hat{\mathfrak{o}}$. Thus

$$\sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} S(a\gamma, -\mathbf{m}) = (\mathrm{N}\,\mathfrak{d})^n \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{b} \,(\mathrm{mod}\,\mathfrak{b})} \mathrm{e}(a\gamma F(\mathbf{b}) + \mathbf{m}.\mathbf{b})$$
$$= D_K^n S_{\mathfrak{b}}(\mathbf{m}),$$

as required.   □

It will be convenient to pass from exponential sums modulo $\mathfrak{b}$ indexed by $\hat{\mathfrak{b}}^n$, to exponential sums modulo $\mathfrak{b}$ indexed by $\mathfrak{o}^n$. Define

$$\tilde{S}_{\mathfrak{b}}(\mathbf{v}) = \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{a} \,(\mathrm{mod}\,\mathfrak{b})} \mathrm{e}\,(\gamma\{aF(\mathbf{a}) + \mathbf{v}.\mathbf{a}\}), \tag{7.2}$$

for any $\mathbf{v} \in \mathfrak{o}^n$. It follows from our work in §2.3 that this expression is independent of the precise choice of γ. Since (ν) is coprime to $\mathfrak{b}$, we may write

$$S_{\mathfrak{b}}(\mathbf{m}) = \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{a} \,(\mathrm{mod}\,\mathfrak{b})} \mathrm{e}\,(a\gamma F(\nu\mathbf{a}) + \nu\mathbf{m}.\mathbf{a}) = \tilde{S}_{\mathfrak{b}}(\alpha\mathbf{m}), \tag{7.3}$$

on making a suitable change of variables in the a and $\mathbf{a}$ summations. Note here that $\alpha\mathbf{m}$ belongs to $\mathfrak{o}^n$. In fact, following the notation introduced at the start of this section, one easily sees that $\alpha\mathbf{m}$ belongs to $\mathfrak{p}_1^n$.

The exponential sums (7.2) satisfy the basic multiplicativity property

$$\tilde{S}_{\mathfrak{b}_1\mathfrak{b}_2}(\mathbf{v}) = \tilde{S}_{\mathfrak{b}_1}(\mathbf{v})S_{\mathfrak{b}_2}(\mathbf{v}),$$

if $\mathfrak{b}_1, \mathfrak{b}_2$ are coprime integral ideals. We will use this decomposition to analyse $\tilde{S}_{\mathfrak{b}}(\mathbf{v})$ at the square-free and square-full parts of $\mathfrak{b}$ separately. To check this we write $\mathfrak{b} = \mathfrak{b}_1 \mathfrak{b}_2$ and recall that $\mathfrak{a}_\gamma = \mathfrak{b}\mathfrak{d}$. Applying Lemma 2.2(i) we find α and λ such that $\mathrm{ord}_{\mathfrak{p}}(\alpha) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{b}_1)$ and $\mathrm{ord}_{\mathfrak{p}}(\lambda) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{b}_2)$ for all $\mathfrak{p} \mid \mathfrak{b}\mathfrak{d}$. Lemma 7.1(ii) then allows us to write $a = \lambda b + \alpha c$ and $\mathbf{a} = \lambda \mathbf{b} + \alpha \mathbf{c}$, for $b, \mathbf{b} \,(\mathrm{mod}\,\mathfrak{b}_1)$ and $c, \mathbf{c} \,(\mathrm{mod}\,\mathfrak{b}_2)$, to get

$$\tilde{S}_{\mathfrak{b}}(\mathbf{v}) = \sum_{b \in (\mathfrak{o}/\mathfrak{b}_1)^*} \sum_{c \in (\mathfrak{o}/\mathfrak{b}_2)^*} \sum_{\mathbf{b}\,(\mathrm{mod}\,\mathfrak{b}_1)} \sum_{\mathbf{c}\,(\mathrm{mod}\,\mathfrak{b}_2)} \mathrm{e}\left(\gamma\}(\lambda b + \alpha c)F(\lambda \mathbf{b} + \alpha \mathbf{c}) + \mathbf{v}.(\lambda \mathbf{b} + \alpha \mathbf{c})\}\right)$$

$$= \sum_{b \in (\mathfrak{o}/\mathfrak{b}_1)^*} \sum_{\mathbf{b}\,(\mathrm{mod}\,\mathfrak{b}_1)} \mathrm{e}\left(\lambda\gamma\{b\lambda^3 F(\mathbf{b}) + \mathbf{v}.\mathbf{b}\}\right) \sum_{c \in (\mathfrak{o}/\mathfrak{b}_2)^*} \sum_{\mathbf{c}\,(\mathrm{mod}\,\mathfrak{b}_2)} \mathrm{e}\left(\alpha\gamma\{c\alpha^3 F(\mathbf{c}) + \mathbf{v}.\mathbf{c}\}\right)$$

$$= \sum_{b \in (\mathfrak{o}/\mathfrak{b}_1)^*} \sum_{\mathbf{b}\,(\mathrm{mod}\,\mathfrak{b}_1)} \mathrm{e}\left(\lambda\gamma\{bF(\mathbf{b}) + \mathbf{v}.\mathbf{b}\}\right) \sum_{c \in (\mathfrak{o}/\mathfrak{b}_2)^*} \sum_{\mathbf{c}\,(\mathrm{mod}\,\mathfrak{b}_2)} \mathrm{e}\left(\alpha\gamma\{cF(\mathbf{c}) + \mathbf{v}.\mathbf{c}\}\right).$$

The multiplicativity property therefore follows on noting that $\mathfrak{a}_{\lambda\gamma} = \mathfrak{b}_1\mathfrak{d}$ and $\mathfrak{a}_{\alpha\gamma} = \mathfrak{b}_2\mathfrak{d}$.

7.1. Square-free $\mathfrak{b}$. — Let $G \in \mathfrak{o}[X_1, \ldots, X_n]$ be the dual form of F, whose zero locus parameterises the set of hyperplanes whose intersection with the cubic hypersurface $F = 0$ produce a singular variety. It is well-known that G is absolutely irreducible and has degree $3 \cdot 2^{n-2}$. The primary aim of this section is to establish the following result, which is an exact analogue of [7, Lemma 13].

Lemma 7.3. — *Let $\mathfrak{b}$ be a square-free integral ideal and let $\mathbf{v} \in \mathfrak{o}^n$. Then there exists an absolute constant $A > 0$ such that*

$$|\tilde{S}_{\mathfrak{b}}(\mathbf{v})| \leqslant A^{\omega(\mathfrak{b})}(\mathrm{N}\,\mathfrak{b})^{(n+1)/2}(\mathrm{N}\,\mathfrak{h})^{1/2},$$

where $\mathfrak{h}$ is the greatest common divisor of $\mathfrak{b}$ and $(G(\mathbf{v}))$.

Now for any square-free ideal $\mathfrak{b}$ it follows from [15, Lemma 23], Lemma 7.2 and (7.3) that

$$|\tilde{S}_{\mathfrak{b}}(\mathbf{v})| \leqslant A^{\omega(\mathfrak{b})}(\mathrm{N}\,\mathfrak{b})^{n/2+1}.$$

Hence, by multiplicativity, in order to complete the proof of Lemma 7.3 it will suffice to show that

$$\tilde{S}_{\mathfrak{p}}(\mathbf{v}) \ll (\mathrm{N}\,\mathfrak{p})^{(n+1)/2}, \tag{7.4}$$

when $\mathfrak{p}$ is an unramified prime ideal such that $\mathfrak{p} \nmid G(\mathbf{v})$. In particular we may freely assume that F and $F_{\mathbf{v}}$ are both non-singular modulo $\mathfrak{p}$, where $F_{\mathbf{v}}$ is the cubic form in $n-1$ variables obtained by eliminating a variable from the pair of equations $F(\mathbf{X}) = 0$ and $\mathbf{v}.\mathbf{X} = 0$. In order to bound $\tilde{S}_{\mathfrak{p}}(\mathbf{v})$ we introduce a dummy sum over an extra variable t to get

$$\varphi(\mathfrak{p})\tilde{S}_{\mathfrak{p}}(\mathbf{v}) = \sum_{a \in (\mathfrak{o}/\mathfrak{p})^*} \sum_{t \in (\mathfrak{o}/\mathfrak{p})^*} \sum_{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{p})} \mathrm{e}(\gamma\{a\bar{t}^3 F(\mathbf{a}) + \mathbf{v}.\mathbf{a}\})$$

$$= \sum_{a \in (\mathfrak{o}/\mathfrak{p})^*} \sum_{t \in (\mathfrak{o}/\mathfrak{p})^*} \sum_{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{p})} \mathrm{e}(\gamma\{aF(\mathbf{a}) + t\mathbf{v}.\mathbf{a}\})$$

$$= \sum_{a \in (\mathfrak{o}/\mathfrak{p})^*} \left(\sum_{t\,(\mathrm{mod}\,\mathfrak{p})} \sum_{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{p})} \mathrm{e}(\gamma\{aF(\mathbf{a}) + t\mathbf{v}.\mathbf{a}\}) - \sum_{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{p})} \mathrm{e}(a\gamma F(\mathbf{a})) \right),$$

where $\bar{t}$ is the multiplicative inverse of t modulo $\mathfrak{p}$. But clearly

$$\sum_{a \in (\mathfrak{o}/\mathfrak{p})^*} \sum_{t \,(\mathrm{mod}\,\mathfrak{p})} \sum_{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{p})} \mathrm{e}\left(\gamma\{aF(\mathbf{a}) + t\mathbf{v}.\mathbf{a}\}\right) = \mathrm{N}\,\mathfrak{p} \sum_{a \in (\mathfrak{o}/\mathfrak{p})^*} \sum_{\substack{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{p})\\ \mathfrak{p}|\mathbf{v}.\mathbf{a}}} \mathrm{e}(a\gamma F(\mathbf{a})),$$

since $\mathbf{v} \in \mathfrak{o}^n$. Assuming without loss of generality that $v_n \neq 0$, we may eliminate a_n from this exponential sum, leading to the expression

$$\varphi(\mathfrak{p})\tilde{S}_\mathfrak{p}(\mathbf{v}) = \sum_{a \in (\mathfrak{o}/\mathfrak{p})^*} \left(\mathrm{N}\,\mathfrak{p} \sum_{\mathbf{a}'\,(\mathrm{mod}\,\mathfrak{p})} \mathrm{e}(a\gamma F_\mathbf{v}(\mathbf{a}')) - \sum_{\mathbf{a}\,(\mathrm{mod}\,\mathfrak{p})} \mathrm{e}(a\gamma F(\mathbf{a})) \right),$$

where $\mathbf{a}' = (a_1, \ldots, a_{n-1})$. Since F and $F_\mathbf{v}$ are non-singular modulo $\mathfrak{p}$, it follows from Deligne's estimate [3, Theorem 8.4] that the two terms in the brackets are $O((\mathrm{N}\,\mathfrak{p})^{(n+1)/2})$. This therefore establishes (7.4), which concludes the proof of Lemma 7.3.

7.2. Square-full $\mathfrak{b}$. — In this section we examine the exponential sum $\tilde{S}_\mathfrak{b}(\mathbf{v})$ in (7.2) for square-full integral ideals $\mathfrak{b}$ and suitable $\mathbf{v} \in \mathfrak{o}^n$, the main idea being to average over the $\mathbf{v}$. We begin with the following result.

Lemma 7.4. — *Let $B = \bigoplus_l B^{(l)} \in V$, with $B^{(l)} > 0$. Let $\varepsilon > 0$ and let $\mathfrak{b}$ be any square-full integral ideal. Let $\mathfrak{c}$ be an integral ideal which is coprime to $\mathfrak{b}\mathfrak{d}$, with $\mathfrak{b}\mathfrak{d}\mathfrak{c}$ principal. Then we have*

$$\sum_{\substack{\mathbf{v} \in \mathfrak{c}^n\\ |\mathbf{v}^{(l)}| \leqslant B^{(l)}}} |\tilde{S}_\mathfrak{b}(\mathbf{v})| \ll (\mathrm{N}\,\mathfrak{b})^{n/2+1+\varepsilon} \left(\mathrm{Nm}(B)^n + (\mathrm{N}\,\mathfrak{b})^{n/3} \right).$$

The implied constant in this estimate does not depend on $\mathfrak{c}$.

Proof. — By hypothesis there exists $\alpha \in \mathfrak{o}$ such that $\mathfrak{b}\mathfrak{d}\mathfrak{c} = (\alpha)$. Therefore $\alpha\hat{\mathfrak{b}} = \mathfrak{b}\mathfrak{d}\mathfrak{c}(\mathfrak{b}\mathfrak{d})^{-1} = \mathfrak{c}$ and it follows that any $\mathbf{v} \in \mathfrak{c}^n$ can be written $\mathbf{v} = \alpha\mathbf{m}$ for $\mathbf{m} \in \hat{\mathfrak{b}}^n$. Following our conventions, we may view $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(r_1+r_2)})$ as an element of V by setting $\alpha^{(j)} = \rho_j(\alpha)$. Hence (7.3) and Lemma 7.2 yield

$$\sum_{\substack{\mathbf{v} \in \mathfrak{c}^n\\ |\mathbf{v}^{(l)}| \leqslant B^{(l)}}} |\tilde{S}_\mathfrak{b}(\mathbf{v})| = \sum_{\substack{\mathbf{v} \in \mathfrak{c}^n\\ |\mathbf{v}^{(l)}| \leqslant B^{(l)}}} |S_\mathfrak{b}(\alpha^{-1}\mathbf{v})|$$

$$\leqslant \sum_{\substack{\mathbf{m} \in \hat{\mathfrak{b}}^n\\ |\mathbf{m}^{(l)}| \leqslant |\alpha^{(l)}|^{-1}B^{(l)}}} |S_\mathfrak{b}(\mathbf{m})|$$

$$\leqslant \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\substack{\mathbf{m} \in \hat{\mathfrak{b}}^n\\ |\mathbf{m}^{(l)}| \leqslant |\alpha^{(l)}|^{-1}B^{(l)}}} |S(a\gamma, -\mathbf{m})|.$$

Since $\mathfrak{b} \mid \mathfrak{a}_\gamma$ it follows that $\hat{\mathfrak{b}} \subseteq \hat{\mathfrak{a}}_\gamma$. Enlarging the sum over $\mathbf{m}$, [15, Lemma 21] yields

$$\sum_{\substack{\mathbf{v} \in \mathfrak{c}^n\\ |\mathbf{v}^{(l)}| \leqslant B^{(l)}}} |\tilde{S}_\mathfrak{b}(\mathbf{v})| \ll \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} (\mathrm{N}\,\mathfrak{b})^{n/2+\varepsilon} \left(\mathrm{Nm}(\alpha)^{-n}\,\mathrm{Nm}(B)^n(\mathrm{N}\,\mathfrak{b})^n + (\mathrm{N}\,\mathfrak{b})^{n/3} \right),$$

for any $\varepsilon > 0$. The lemma follows on noting that $\mathrm{Nm}(\alpha)^{-1}\,\mathrm{N}\,\mathfrak{b} = (\mathrm{N}\,\mathfrak{d}\,\mathrm{N}\,\mathfrak{c})^{-1} \leqslant 1$. $\qquad\square$

We will need a companion estimate which deals with the problem of averaging over those $\mathbf{v}$ at which the dual form G vanishes. The rest of this section will be devoted to a proof of the following result.

Lemma 7.5. — *Let $B = \bigoplus_l B^{(l)} \in V$, with $B^{(l)} > 0$. Let $\varepsilon > 0$ and let $\mathfrak{b}$ be any square-full integral ideal. Then we have*

$$\sum_{\substack{\mathbf{v} \in \mathfrak{o}^n \\ |\mathbf{v}^{(l)}| \leqslant B^{(l)} \\ G(\mathbf{v}) = 0}} |\tilde{S}_{\mathfrak{b}}(\mathbf{v})| \ll (\mathrm{N}\,\mathfrak{b}\,\mathrm{Nm}(B))^{\varepsilon} \left((\mathrm{N}\,\mathfrak{b})^{n+1/2} + (\mathrm{Nm}(B))^{n-3/2}(\mathrm{N}\,\mathfrak{b})^{n/2+4/3} \right).$$

Lemmas 7.4 and 7.5 are precise analogues of Lemmas 14 and 16, respectively, of Heath-Brown [7]. The proofs for general number fields are very similar to the case $K = \mathbb{Q}$ and we shall attempt to be brief in our demonstration of Lemma 7.5. The rationale behind this result is the need to make up for the loss in Lemma 7.3 when $G(\mathbf{v})$ vanishes. Following Heath-Brown [7, §7] we will get extra savings from two sources: firstly by summing non-trivially over a and secondly by using the relative sparsity of vectors $\mathbf{v}$ such that $G(\mathbf{v}) = 0$.

Since $\mathfrak{b}$ is square-full we may write $\mathfrak{b} = \mathfrak{b}_1^2 \mathfrak{b}_2$, where $\mathfrak{b}_2$ is square-free and $\mathfrak{b}_2 \mid \mathfrak{b}_1$. Let us set $q = \mathrm{N}\,\mathfrak{b}$ and $q_i = \mathrm{N}\,\mathfrak{b}_i$ for $i = 1, 2$. Let γ be as before and recall (7.1). Applying Lemma 2.2(i), let $\beta \in \mathfrak{b}_1$ and $\mu \in \mathfrak{b}_2$ such that $\mathrm{ord}_{\mathfrak{p}}(\beta) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{b}_1)$ and $\mathrm{ord}_{\mathfrak{p}}(\mu) = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{b}_2)$, for all $\mathfrak{p} \mid \mathfrak{b}\mathfrak{d}$. Notice that $\mathfrak{a}_{\gamma\beta^2\mu} = \mathfrak{d}$. Adapting the argument leading to [15, Eq. (8.5)], we obtain

$$\tilde{S}_{\mathfrak{b}}(\mathbf{v}) = \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{a} \,(\mathrm{mod}\,\mathfrak{b})} \mathrm{e}(\gamma\{aF(\mathbf{a}) + \mathbf{v}.\mathbf{a}\})$$

$$= \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{f} \,(\mathrm{mod}\,\mathfrak{b}_1\mathfrak{b}_2)} \sum_{\mathbf{g} \,(\mathrm{mod}\,\mathfrak{b}_1)} \mathrm{e}(\gamma\{aF(\mathbf{f} + \beta\mu\mathbf{g}) + \mathbf{v}.(\mathbf{f} + \beta\mu\mathbf{g})\})$$

$$= \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\mathbf{f} \,(\mathrm{mod}\,\mathfrak{b}_1\mathfrak{b}_2)} \mathrm{e}(\gamma\{aF(\mathbf{f}) + \mathbf{v}.\mathbf{f}\}) \sum_{\mathbf{g} \,(\mathrm{mod}\,\mathfrak{b}_1)} \mathrm{e}(\gamma\beta\mu\{a\mathbf{g}.\nabla F(\mathbf{f}) + \mathbf{v}.\mathbf{g}\}).$$

Here $\mathrm{e}(\gamma\cdot)$ is a primitive character modulo $\mathfrak{b}$ and $\mathrm{e}(\gamma\beta\mu\cdot)$ is a primitive character modulo $\mathfrak{b}_1$, since $\mathfrak{a}_{\gamma\beta\mu} = \mathfrak{b}_1\mathfrak{d}$. This implies that

$$\tilde{S}_{\mathfrak{b}}(\mathbf{v}) = q_1^n \sum_{a \in (\mathfrak{o}/\mathfrak{b})^*} \sum_{\substack{\mathbf{f} \,(\mathrm{mod}\,\mathfrak{b}_1\mathfrak{b}_2) \\ a\nabla F(\mathbf{f}) + \mathbf{v} \in \mathfrak{b}_1^n}} \mathrm{e}(\gamma\{aF(\mathbf{f}) + \mathbf{v}.\mathbf{f}\}).$$

Writing $a = t + u\beta$, we obtain

$$\tilde{S}_{\mathfrak{b}}(\mathbf{v}) = q_1^n \sum_{t \in (\mathfrak{o}/\mathfrak{b}_1)^*} \sum_{\substack{\mathbf{f} \,(\mathrm{mod}\,\mathfrak{b}_1\mathfrak{b}_2) \\ t\nabla F(\mathbf{f}) + \mathbf{v} \in \mathfrak{b}_1^n}} \mathrm{e}(\gamma\{tF(\mathbf{f}) + \mathbf{v}.\mathbf{f}\}) \sum_{u \,(\mathrm{mod}\,\mathfrak{b}_1\mathfrak{b}_2)} \mathrm{e}(\gamma\beta u F(\mathbf{f}))$$

$$= q_1^{n+1} q_2 \sum_{t \in (\mathfrak{o}/\mathfrak{b}_1)^*} \sideset{}{^{(1)}}\sum_{\mathbf{f}} \mathrm{e}(\gamma\{tF(\mathbf{f}) + \mathbf{v}.\mathbf{f}\}),$$

where $\sum^{(1)}$ indicates that $\mathbf{f}$ runs modulo $\mathfrak{b}_1\mathfrak{b}_2$ subject to the constraints $t\nabla F(\mathbf{f}) + \mathbf{v} \in \mathfrak{b}_1^n$ and $F(\mathbf{f}) \in \mathfrak{b}_1\mathfrak{b}_2$. Here we have used the fact that $\mathrm{e}(\gamma\beta\cdot)$ is a primitive character modulo $\mathfrak{b}_1\mathfrak{b}_2$. Since $\beta \in \mathfrak{b}_1$ and $\mathfrak{b}_2 \mid \mathfrak{b}_1$, it follows that $\beta^2 \in \mathfrak{b}_1\mathfrak{b}_2$. We next substitute $\mathbf{f} = \mathbf{h} + \beta\mathbf{j}$.

Then, if $F(\mathbf{h}) = \beta m$, we will have $F(\mathbf{f}) \in \mathfrak{b}_1\mathfrak{b}_2$ precisely when $m + \mathbf{j}.\nabla F(\mathbf{h}) \in \mathfrak{b}_2$. Moreover, if $t\nabla F(\mathbf{h}) = -\mathbf{v} + \beta\mathbf{k}$, then

$$tF(\mathbf{f}) + \mathbf{v}.\mathbf{f} \equiv tF(\mathbf{h}) + \mathbf{v}.\mathbf{h} + \beta^2(\mathbf{k}.\mathbf{j} + t\mathbf{h}.\nabla F(\mathbf{j})) \, (\mathrm{mod}\, \mathfrak{b}).$$

It follows that

$$\left| \sum_{\mathbf{f}}^{(1)} \mathrm{e}(\gamma\{aF(\mathbf{f}) + \mathbf{v}.\mathbf{f}\}) \right| \leqslant \sum_{\mathbf{h}}^{(2)} \max_{m,\mathbf{k}\,(\mathrm{mod}\,\mathfrak{b}_2)} \left| \sum_{\mathbf{j}}^{(3)} \mathrm{e}(\gamma\beta^2\{\mathbf{k}.\mathbf{j} + t\mathbf{h}.\nabla F(\mathbf{j})\}) \right|,$$

where $\sum^{(2)}$ is for $\mathbf{h}$ modulo $\mathfrak{b}_1$ such that $t\nabla F(\mathbf{h}) + \mathbf{v} \in \mathfrak{b}_1^n$ and $F(\mathbf{h}) \in \mathfrak{b}_1$, and $\sum^{(3)}$ is over $\mathbf{j}$ modulo $\mathfrak{b}_2$ for which $m + \mathbf{j}.\nabla F(\mathbf{h}) \in \mathfrak{b}_2$.

Let $\chi = \gamma\beta^2$ and note that $\mathfrak{b}_2\mathfrak{d} = \mathfrak{a}_\chi$. Therefore $\mathrm{e}(\chi\cdot)$ denotes a primitive character modulo $\mathfrak{b}_2$. We proceed by bounding the inner sum over $\mathbf{j}$, which we write as $S^{(3)}$. By orthogonality we have

$$S^{(3)} = q_2^{-1} \sum_{l\,(\mathrm{mod}\,\mathfrak{b}_2)} \sum_{\mathbf{j}\,(\mathrm{mod}\,\mathfrak{b}_2)} \mathrm{e}\left(\chi\{lm + (\mathbf{k} + l\nabla F(\mathbf{h})).\mathbf{j} + t\mathbf{h}.\nabla F(\mathbf{j})\}\right)$$

$$\ll \max_{\mathbf{k}\,(\mathrm{mod}\,\mathfrak{b}_2)} \left| \sum_{\mathbf{j}\,(\mathrm{mod}\,\mathfrak{b}_2)} \mathrm{e}\left(\chi\{\mathbf{k}.\mathbf{j} + t\mathbf{h}.\nabla F(\mathbf{j})\}\right) \right|.$$

We proceed by adapting the argument leading to [**15**, Eq. (8.13)]. Let $S_{\mathbf{k},\mathbf{h}}$ denote the sum over $\mathbf{j}\,(\mathrm{mod}\,\mathfrak{b}_2)$. Then

$$|S_{\mathbf{k},\mathbf{h}}|^2 = \sum_{\mathbf{j}_1\,(\mathrm{mod}\,\mathfrak{b}_2)} \sum_{\mathbf{j}_2\,(\mathrm{mod}\,\mathfrak{b}_2)} \mathrm{e}\left(\chi\{\mathbf{k}.(\mathbf{j}_1 - \mathbf{j}_2) + t\mathbf{h}.(\nabla F(\mathbf{j}_1) - \nabla F(\mathbf{j}_2))\}\right).$$

Write $\mathbf{j}_1 = \mathbf{j}_2 + \mathbf{j}_3$ to get

$$\mathbf{h}.(\nabla F(\mathbf{j}_1) - \nabla F(\mathbf{j}_2)) = \mathbf{h}.\nabla F(\mathbf{j}_3) + 6\mathbf{j}_2.\mathbf{B}(\mathbf{h}, \mathbf{j}_3) \, (\mathrm{mod}\, \mathfrak{b}_2),$$

where $\mathbf{B}(\mathbf{x}, \mathbf{y})$ is the system of n bilinear forms defined in [**15**, §2]. It follows that

$$|S_{\mathbf{k},\mathbf{h}}|^2 = \sum_{\mathbf{j}_3\,(\mathrm{mod}\,\mathfrak{b}_2)} \mathrm{e}\left(\chi\{\mathbf{k}.\mathbf{j}_3 + t\mathbf{h}.\nabla F(\mathbf{j}_3)\}\right) \sum_{\mathbf{j}_2\,(\mathrm{mod}\,\mathfrak{b}_2)} \mathrm{e}\left(6\chi t\mathbf{j}_2.\mathbf{B}(\mathbf{h}, \mathbf{j}_3)\right)$$

$$\ll q_2^n \#\{\mathbf{j}_3\,(\mathrm{mod}\,\mathfrak{b}_2) : \mathbf{B}(\mathbf{h}, \mathbf{j}_3) \in (6^{-1}\mathfrak{b}_2 \cap \mathfrak{o})^n\}.$$

Thus we have shown that $S^{(3)} \ll q_2^{n/2} K(\mathfrak{b}_3; \mathbf{h})^{1/2}$, where $\mathfrak{b}_3 = 6^{-1}\mathfrak{b}_2 \cap \mathfrak{o}$ and

$$K(\mathfrak{b}_3; \mathbf{x}) = \#\left\{\mathbf{y}\,(\mathrm{mod}\,\mathfrak{b}_3) : \mathbf{B}(\mathbf{x}, \mathbf{y}) \in \mathfrak{b}_3^n\right\}.$$

Once inserted into our work so far we therefore obtain

$$\tilde{S}_\mathfrak{b}(\mathbf{v}) \ll q_1^{n+1} q_2^{n/2+1} \sum_{t\in(\mathfrak{o}/\mathfrak{b}_1)^*} \sum_{\mathbf{h}}^{(2)} K(\mathfrak{b}_3; \mathbf{h})^{1/2}.$$

It is now time to introduce the summation over $\mathbf{v}$, defining

$$\mathscr{N} = \max_{\mathbf{r}\,(\mathrm{mod}\,\mathfrak{b}_1)} \#\left\{\mathbf{v} \in \mathfrak{o}^n : |\mathbf{v}^{(l)}| \leqslant B^{(l)},\ G(\mathbf{v}) = 0,\ \mathbf{v} \equiv \mathbf{r}\,(\mathrm{mod}\,\mathfrak{b}_1)\right\}. \tag{7.5}$$

Then we have

$$\sum_{\substack{\mathbf{v}\in\mathfrak{o}^n \\ |\mathbf{v}^{(l)}|\leqslant B^{(l)} \\ G(\mathbf{v})=0}} |\tilde{S}_{\mathfrak{b}}(\mathbf{v})| \ll q_1^{n+2}q_2^{n/2+1}\mathscr{N}\sum_{\substack{\mathbf{h}\,(\mathrm{mod}\,\mathfrak{b}_1) \\ F(\mathbf{h})\in\mathfrak{b}_1}} K(\mathfrak{b}_3;\mathbf{h})^{1/2} = q_1^{n+2}q_2^{n/2+1}\mathscr{N}S(\mathfrak{b}),$$

say. The following result is concerned with an upper bound for $S(\mathfrak{b})$.

Lemma 7.6. — *There exists an absolute constant $A > 0$ such that $S(\mathfrak{b}) \leqslant A^{\omega(\mathfrak{b})}q_1^{n-1}q_2^{1/2}$.*

The proof of Lemma 7.6 is the exact analogue of the treatment of $S(q)$ in [**7**, §7] and we have decided to omit the proof. Note that the analogue of [**7**, Lemma 4] (resp. the estimate $S_0(p^g;\mathbf{0}) \ll p^{(5n/6+1+\varepsilon)g}$) is provided by [**15**, Lemma 10] (resp. by Lemmas 7.3 and 7.4). Applying this result in our work above now yields

$$\sum_{\substack{\mathbf{v}\in\mathfrak{o}^n \\ |\mathbf{v}^{(l)}|\leqslant B^{(l)} \\ G(\mathbf{v})=0}} |\tilde{S}_{\mathfrak{b}}(\mathbf{v})| \ll A^{\omega(\mathfrak{b})}q_1^{2n+1}q_2^{(n+3)/2}\mathscr{N},$$

where $\mathscr{N}$ is given by (7.5). Finally, it follows from Lemma 2.5 that

$$\sum_{\substack{\mathbf{v}\in\mathfrak{o}^n \\ |\mathbf{v}^{(l)}|\leqslant B^{(l)} \\ G(\mathbf{v})=0}} |\tilde{S}_{\mathfrak{b}}(\mathbf{v})| \ll (q\,\mathrm{Nm}(B))^\varepsilon q_1^{2n+1}q_2^{(n+3)/2}\left(1+\frac{\mathrm{Nm}(B)}{q_1}\right)^{n-3/2},$$

for any $\varepsilon > 0$. Since $q = q_1^2 q_2$ and $q_2 \leqslant q^{1/3}$, the above bound is

$$\ll (q\,\mathrm{Nm}(B))^\varepsilon\left\{q^{n+1/2} + (\mathrm{Nm}(B))^{n-3/2}(q_1^2 q_2)^{n/2+5/4}q_2^{1/4}\right\}$$

$$\ll (q\,\mathrm{Nm}(B))^\varepsilon\left\{q^{n+1/2} + (\mathrm{Nm}(B))^{n-3/2}q^{n/2+4/3}\right\}.$$

This therefore concludes the proof of Lemma 7.5.

8. Final deduction of Theorem 1.1

In this section we complete the proofs of Theorems 1.1 and 5.4, by combining our analysis of the exponential sums in §7 with Lemma 6.6. In what follows it will be notationally convenient to follow the convention that the small positive constant ε takes different values at different parts of the argument.

Recall from Lemma 6.6 that

$$E(v,P) = \mathscr{H}(v)^{-n/2}\sum_{\substack{(0)\neq\mathfrak{b}\subseteq\mathfrak{o} \\ Q_0\ll\mathrm{N}\,\mathfrak{b}\ll Q_1}}\sum_{\substack{\mathbf{0}\neq\mathbf{m}\in\hat{\mathfrak{b}}^n \\ |\mathbf{m}^{(l)}|\ll B^{(l)}}} (\mathrm{N}\,\mathfrak{b})^{-n}|S_{\mathfrak{b}}(\mathbf{m})|.$$

where $\mathscr{H}(v)$ is given by (5.5) and $B^{(l)} = P^{-1+\varepsilon}(1+|v^{(l)}|)$, with

$$Q_0 = P^{d(1-\varepsilon)}\mathscr{H}(v)^{-1}, \quad Q_1 = P^{3d/2+\varepsilon}\mathscr{H}(v)^{-1}.$$

We will need to show that there exists an absolute constant $\Delta > 0$, which is independent of ε, such that

$$\int_R E(v,P)\mathrm{d}v \ll P^{-\Delta}, \tag{8.1}$$

where $R = \{v \in V : \mathscr{H}(v) \ll P^{3d/2+\varepsilon}\}$.

We now write $\mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2$, where $\mathfrak{b}_1$ is square-free and $\mathfrak{b}_2$ is square-full. By Lemma 2.2(ii) there exists $\alpha' \in \mathfrak{b}_2\mathfrak{d}$ and an unramified prime ideal $\mathfrak{p}_2$ coprime to $\mathfrak{b}_2$, such that $(\alpha') = \mathfrak{b}_2\mathfrak{d}\mathfrak{p}_2$ and $\mathrm{N}\,\mathfrak{p}_2 \ll (\mathrm{N}\,\mathfrak{b})^\varepsilon$. Likewise, a second application of this result shows that there exists $\alpha \in \mathfrak{b}\mathfrak{d}$ and an unramified prime ideal $\mathfrak{p}_1$ coprime to $\mathfrak{b}\mathfrak{p}_2$, such that $(\alpha) = \mathfrak{b}\mathfrak{d}\mathfrak{p}_1\mathfrak{p}_2$ and $\mathrm{N}\,\mathfrak{p}_1 \ll (\mathrm{N}\,\mathfrak{b})^\varepsilon$. Let us write $\mathfrak{q} = \mathfrak{p}_1\mathfrak{p}_2$ in what follows. In particular we have $\hat{\mathfrak{b}} = \alpha^{-1}\mathfrak{q}$. Let $B_1^{(l)} = B^{(l)}|\alpha^{(l)}|$ and $B_1 = \bigoplus_l B_1^{(l)}$. In order to apply our estimates for exponential sums from the preceding section, we invoke the connection in (7.3) to get

$$E(v,P) = \mathscr{H}(v)^{-n/2} \sum_{Q_0 \ll \mathrm{N}\,\mathfrak{b} \ll Q_1} (\mathrm{N}\,\mathfrak{b})^{-n} \sum_{\substack{0 \neq \mathbf{m} \in (\alpha^{-1}\mathfrak{q})^n \\ |\mathbf{m}^{(l)}| \ll B^{(l)}}} |\tilde{S}_{\mathfrak{b}}(\alpha\mathbf{m})|$$

$$= \mathscr{H}(v)^{-n/2} \sum_{Q_0 \ll \mathrm{N}\,\mathfrak{b} \ll Q_1} (\mathrm{N}\,\mathfrak{b})^{-n} \sum_{\substack{0 \neq \mathbf{m} \in \mathfrak{q}^n \\ |\mathbf{m}^{(l)}| \ll B_1^{(l)}}} |\tilde{S}_{\mathfrak{b}}(\mathbf{m})|.$$

Observe that $\mathrm{Nm}(B) \ll P^{-d+\varepsilon}\mathscr{H}(v)$ and $\mathrm{Nm}(\alpha) \ll \mathrm{N}\,\mathfrak{b}\,\mathrm{N}\,\mathfrak{q}$, whence

$$\mathrm{Nm}(B_1) \ll (\mathrm{N}\,\mathfrak{b})^{1+\varepsilon}P^{-d+\varepsilon}\mathscr{H}(v). \tag{8.2}$$

In all that follows we will use the notation $q = \mathrm{N}\,\mathfrak{b}$, $q_1 = \mathrm{N}\,\mathfrak{b}_1$ and $q_2 = \mathrm{N}\,\mathfrak{b}_2$.

Lemma 7.3 implies that $\tilde{S}_{\mathfrak{b}_1}(\mathbf{m}) \ll q_1^{(n+1)/2+\varepsilon}(\mathrm{N}\,\mathfrak{h})^{1/2}$, where $\mathfrak{h}$ is the greatest common divisor of $\mathfrak{b}_1$ and $(G(\mathbf{m}))$. Hence, since $\mathfrak{q} \subseteq \mathfrak{p}_2$, it follows from multiplicativity that

$$E(v,P) \ll P^\varepsilon \mathscr{H}(v)^{-n/2} \sum_{\substack{Q_0 \ll \mathrm{N}\,\mathfrak{b} \ll Q_1 \\ \mathfrak{b} = \mathfrak{b}_1\mathfrak{b}_2}} q^{-n}q_1^{(n+1)/2} \sum_{\substack{0 \neq \mathbf{m} \in \mathfrak{p}_2^n \\ |\mathbf{m}^{(l)}| \ll B_1^{(l)}}} (\mathrm{N}\,\mathfrak{h})^{1/2}|\tilde{S}_{\mathfrak{b}_2}(\mathbf{m})|.$$

Our goal is to show that (8.1) holds for a suitable absolute constant $\Delta > 0$. On breaking the sum over $\mathfrak{b}_1, \mathfrak{b}_2$ into dyadic intervals for $\mathrm{N}\,\mathfrak{b}_1$ and $\mathrm{N}\,\mathfrak{b}_2$, it will suffice to show that

$$\int_R \max_{\substack{M_1, M_2 \gg 1 \\ Q_0 \ll M_1 M_2 \ll Q_2}} E(v, P; \mathbf{M})\mathrm{d}v \ll P^{-\Delta},$$

where $E(v, P; \mathbf{M})$ denotes the contribution to the right hand side of the above estimate for $E(v, P)$ from $\mathfrak{b}_1, \mathfrak{b}_2$ such that $M_i \leqslant q_i < 2M_i$, for $i = 1, 2$.

It is now time to distinguish between whether $G(\mathbf{m})$ is zero or non-zero in the summation over $\mathbf{m}$, where G is the dual form that we met in §7. Accordingly, we write

$$E(v, P; \mathbf{M}) = P^\varepsilon \mathscr{H}(v)^{-n/2}\left(E_1(v, P; \mathbf{M}) + E_2(v, P; \mathbf{M})\right),$$

where

$$E_1(v, P; \mathbf{M}) = \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ M_i \leqslant q_i < 2M_i}} q^{-n}q_1^{(n+1)/2} \sum_{\substack{\mathbf{m} \in \mathfrak{p}_2^n \\ G(\mathbf{m}) \neq 0 \\ |\mathbf{m}^{(l)}| \ll B_1^{(l)}}} (\mathrm{N}\,\mathfrak{h})^{1/2}|\tilde{S}_{\mathfrak{b}_2}(\mathbf{m})|,$$

$$E_2(v, P; \mathbf{M}) = \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ M_i \leqslant q_i < 2M_i}} q^{-n}q_1^{n/2+1} \sum_{\substack{0 \neq \mathbf{m} \in \mathfrak{p}_2^n \\ G(\mathbf{m}) = 0 \\ |\mathbf{m}^{(l)}| \ll B_1^{(l)}}} |\tilde{S}_{\mathfrak{b}_2}(\mathbf{m})|.$$

It now suffices to show that

$$F_i = \int_R \mathcal{H}(v)^{-n/2} \max_{\substack{M_1, M_2 \gg 1 \\ Q_0 \ll M_1 M_2 \ll Q_2}} E_i(v, P; M_1, M_2) dv \ll P^{-\Delta}, \tag{8.3}$$

for $i = 1, 2$, for a suitable absolute constant $\Delta > 0$. Let us put $M = M_1 M_2$.

8.1. Treatment of F_1. — Let $A \in \mathfrak{o}$ be non-zero and let $(\mathfrak{b}_1, A)$ denote the greatest common divisor of $\mathfrak{b}_1$ and (A). Then there are at most $O(A^\varepsilon)$ ideal divisors of A and it follows that

$$\sum_{\substack{\mathfrak{b}_1 \\ M_1 \leqslant q_1 < 2M_1}} (\mathrm{N}(\mathfrak{b}_1, A))^{1/2} \leqslant \sum_{\mathfrak{c}|(A)} (\mathrm{N}\,\mathfrak{c})^{1/2} \sum_{\substack{\mathfrak{c}|\mathfrak{b}_1 \\ M_1 \leqslant q_1 < 2M_1}} 1$$

$$\ll \sum_{\mathfrak{c}|(A)} (\mathrm{N}\,\mathfrak{c})^{1/2} \left(\frac{M_1}{\mathrm{N}\,\mathfrak{c}} \right)$$

$$\ll M_1 A^\varepsilon.$$

Applying this with $A = G(\mathbf{m})$, we deduce that

$$E_1(v, P; \mathbf{M}) \ll P^\varepsilon M_1^{(3-n)/2} M_2^{-n} \sum_{\substack{\mathfrak{b}_2 \\ M_2 \leqslant q_2 < 2M_2}} \sum_{\substack{\mathbf{0} \neq \mathbf{m} \in \mathfrak{p}_2^n \\ |\mathbf{m}^{(l)}| \ll B_1^{(l)}}} |\tilde{S}_{\mathfrak{b}_2}(\mathbf{m})|.$$

Now it is clear that there are $O(M_2^{1/2+\varepsilon})$ square-full integral ideals $\mathfrak{b}_2$ with norm of order M_2. But then Lemma 7.4 and (8.2) yield

$$E_1(v, P; \mathbf{M}) \ll P^\varepsilon M_1^{(3-n)/2} M_2^{-n} \sum_{\substack{\mathfrak{b}_2 \\ M_2 \leqslant q_2 < 2M_2}} M_2^{n/2+1}(\mathrm{Nm}(B_1)^n + M_2^{n/3})$$

$$\ll P^\varepsilon M_1^{(3-n)/2} M_2^{-n} M_2^{(n+3)/2}(P^{-dn}\mathcal{H}(v)^n M^n + M_2^{n/3})$$

$$\ll P^\varepsilon M^{(3-n)/2}(P^{-dn}\mathcal{H}(v)^n M^n + M^{n/3})$$

$$= P^\varepsilon (P^{-dn}\mathcal{H}(v)^n M^{(n+3)/2} + M^{3/2-n/6}).$$

Notice that the exponent $3/2 - n/6$ is negative for $n \geqslant 10$. Since $Q_0 \ll M \ll Q_1$ and $n \geqslant 10$, we therefore obtain

$$E_1(v, P; \mathbf{M}) \ll P^\varepsilon \left(P^{-dn}\mathcal{H}(v)^n Q_1^{(n+3)/2} + Q_0^{3/2-n/6} \right)$$

$$= P^\varepsilon \left(P^{-d(n-9)/4}\mathcal{H}(v)^{(n-3)/2} + P^{-d(n-9)/6}\mathcal{H}(v)^{n/6-3/2} \right)$$

$$\ll P^{-d/6+\varepsilon} \left(\mathcal{H}(v)^{(n-3)/2} + \mathcal{H}(v)^{n/6-3/2} \right).$$

Inserting this into (8.3) and applying Lemma 5.3, this therefore shows that

$$F_1 \ll P^{-d/6+\varepsilon} \int_R \mathcal{H}(v)^{-3/2} dv \ll P^{-d/6+\varepsilon},$$

which is satisfactory.

8.2. Treatment of F_2. — According to Lemmas 7.4 and 7.5 we have

$$E_2(v, P; \mathbf{M}) \ll \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ M_i \leqslant q_i < 2M_i}} M^{-n} M_1^{n/2+1} \sum_{\substack{0 \neq \mathbf{m} \in \mathfrak{p}_2^n \\ G(\mathbf{m}) = 0 \\ |\mathbf{m}^{(l)}| \ll B_1^{(l)}}} |\tilde{S}_{\mathfrak{b}_2}(\mathbf{m})|$$

$$\ll P^\varepsilon \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ M_i \leqslant q_i < 2M_i}} M^{-n} M_1^{n/2+1} M_2^{n/2+1} \mathscr{M}.$$

Here

$$\mathscr{M} = \min \left\{ \mathrm{Nm}(B_1)^n + M_2^{n/3}, M_2^{(n-1)/2} + \mathrm{Nm}(B_1)^{n-3/2} M_2^{1/3} \right\}$$

$$\leqslant M_2^{n/3} + \mathrm{Nm}(B_1)^{n-3/2} M_2^{1/3} + \min \left\{ \mathrm{Nm}(B_1)^n, M_2^{(n-1)/2} \right\}$$

$$= \mathscr{M}_1 + \mathscr{M}_2 + \mathscr{M}_3,$$

say. We write $E_{2,j}$ for the overall contribution to $E_2(v, P; M_1, M_2)$ from $\mathscr{M}_j$, for $j \in \{1, 2, 3\}$.

Beginning with the contribution from $\mathscr{M}_1$, we sum over $O(M_1)$ ideals $\mathfrak{b}_1$ and $O(M_2^{1/2+\varepsilon})$ square-full ideals $\mathfrak{b}_2$, finding that

$$E_{2,1} \ll P^\varepsilon \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ M_i \leqslant q_i < 2M_i}} M^{-n} M_1^{n/2+1} M_2^{5n/6+1}$$

$$\ll P^\varepsilon M_1^{2-n/2} M_2^{3/2-n/6}$$

$$\ll P^\varepsilon M^{(9-n)/6},$$

since $2 - n/2 < 3/2 - n/6 < 0$ for $n \geqslant 10$. Taking $M \gg Q_0$, inserting the outcome into the left hand side of (8.3) and then applying Lemma 5.3, this therefore shows that this case contributes $O(P^{-d/6+\varepsilon})$ to F_2, which is satisfactory.

Turning to the contribution from $\mathscr{M}_2$, we apply (8.2) and sum over $\mathfrak{b}_1$ and $\mathfrak{b}_2$, as before. In this way we obtain

$$E_{2,2} \ll P^\varepsilon \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ M_i \leqslant q_i < 2M_i}} M^{-n} M_1^{n/2+1} M_2^{n/2+1} \mathrm{Nm}(B_1)^{n-3/2} M_2^{1/3}$$

$$\ll P^{-d(n-3/2)+\varepsilon} \mathscr{H}(v)^{n-3/2} \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ M_i \leqslant q_i < 2M_i}} M^{-3/2} M_1^{n/2+1} M_2^{n/2+4/3}$$

$$\ll P^{-d(n-3/2)+\varepsilon} \mathscr{H}(v)^{n-3/2} M_1^{n/2+1/2} M_2^{n/2+1/3}$$

$$\ll P^{-d(n-3/2)+\varepsilon} \mathscr{H}(v)^{n-3/2} M^{n/2+1/2}.$$

Taking $M \ll Q_1$, inserting the outcome into the left hand side of (8.3) and then applying Lemma 5.3, the $\mathscr{M}_2$ term is therefore seen to contribute

$$\ll P^{-d(n-3/2)+3d(n+1)/4+\varepsilon} \int_R \mathscr{H}(v)^{-2} \mathrm{d}v$$

$$\ll P^{d(-n/4+9/4)+\varepsilon}$$

$$\ll P^{-d/4+\varepsilon}$$

to F_2, since $n \geqslant 10$.

It remains to deal with the term $\mathscr{M}_3$, for which we take $\min\{A, B\} \leqslant A^{(n-2)/(n-1)} B^{1/(n-1)}$. In view of (8.2), this gives

$$\mathscr{M}_3 \ll P^\varepsilon \left(M P^{-d} \mathscr{H}(v) \right)^{n(n-2)/(n-1)} M_2^{1/2}.$$

Summing over $\mathfrak{b}_1, \mathfrak{b}_2$ as before, we obtain

$$E_{2,3} \ll P^\varepsilon \left(P^{-d} \mathscr{H}(v) \right)^{n(n-2)/(n-1)} \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ M_i \leqslant q_i < 2M_i}} M^{-n/(n-1)} M_1^{n/2+1} M_2^{n/2+3/2}$$

$$\ll P^\varepsilon \left(P^{-d} \mathscr{H}(v) \right)^{n(n-2)/(n-1)} M^{n/2+2-n/(n-1)}.$$

Taking $M \ll Q_1$, inserting the outcome into the left hand side of (8.3) and applying Lemma 5.3, the $\mathscr{M}_3$ term therefore contributes

$$\ll P^{-dn(n-2)/(n-1)+\varepsilon} \int_R \mathscr{H}(v)^{-n/2+n(n-2)/(n-1)} Q_1^{n/2+2-n/(n-1)} \mathrm{d}v$$

$$\ll P^{-dn(n-2)/(n-1)+3d(n/4+1-n/(2n-2))+\varepsilon} \int_R \mathscr{H}(v)^{-2} \mathrm{d}v$$

$$\ll P^{-dn(n-2)/(n-1)+3d(n/4+1-n/(2n-2))+\varepsilon}$$

to F_2. This is $O(P^{-d/18+\varepsilon})$, since $n \geqslant 10$, which therefore concludes the proof that (8.3) holds for $i = 1, 2$.

8.3. Conclusion. — Bringing everything together, our work so far has established the following result.

Lemma 8.1. — *Assume that $n \geqslant 10$. Then there exists an absolute constant $\Delta > 0$ such that*

$$N_W(F, P) = \frac{c_Q 2^{r_2 n}}{D_K^{n/2} Q^{2d}} \sum_{\substack{(0) \neq \mathfrak{b} \subseteq \mathfrak{o} \\ \mathrm{N}\,\mathfrak{b} \ll Q^d}} (\mathrm{N}\,\mathfrak{b})^{-n} S_\mathfrak{b}(\mathbf{0}) I_\mathfrak{b}(\mathbf{0}) + O(P^{d(n-3)-\Delta}).$$

Our remaining task is to show that the main term here converges to the main term predicted in Theorem 5.4. Recall from §6 that $\rho = Q^{-d} \mathrm{N}\,\mathfrak{b}$. Lemma 8.1 implies that it is enough to consider the sum

$$\sum_{\substack{(0) \neq \mathfrak{b} \subseteq \mathfrak{o} \\ \mathrm{N}\,\mathfrak{b} \ll Q^d}} (\mathrm{N}\,\mathfrak{b})^{-n} S_\mathfrak{b}(\mathbf{0}) I_\mathfrak{b}(\mathbf{0}).$$

According to (5.6) and (6.1) we have

$$I_\mathfrak{b}(\mathbf{0}) = P^{dn} \int_{V^n} W(\mathbf{x}) h(\rho, \mathrm{Nm}(F(\mathbf{x}))) \mathrm{d}\mathbf{x}$$

$$= P^{dn} \int_V \mathfrak{I}(v) h(\rho, \mathrm{Nm}(v)) \mathrm{d}v.$$

Now it follows from Lemma 4.1 that

$$\int_V f(v) h(\rho, \mathrm{Nm}(v)) \mathrm{d}v = \frac{\sqrt{D_K}}{2^{r_2}} f(0) + O_N(\lambda_f^{2d(N+1)} \rho^N),$$

for any $N > 0$ and any $f \in \mathscr{W}_1(V)$, in the notation of (2.7). Recalling the properties of the weight function $\mathfrak{I}$ discussed in §5, and in particular (5.7), it therefore follows that

$$I_\mathfrak{b}(\mathbf{0}) = \frac{\sqrt{D_K}}{2^{r_2}} P^{dn} \mathfrak{I}(0) + O_N\left(P^{dn+\varepsilon} \rho^N\right). \tag{8.4}$$

The analysis of $\mathfrak{I}(0)$ is the object of the following result.

Lemma 8.2. — *We have $(\log P)^{2d(1-n)} \ll \mathfrak{I}(0) \ll (\log P)^{2d(1-n)}$.*

Proof. — Recall that $W(\mathbf{x}) = w_0(\mathbf{x})\omega(\mathbf{x})$, with w_0 and ω as in §5. It follows from (5.6) that

$$\begin{aligned}
\mathfrak{I}(0) &= \int_{\substack{\mathbf{x} \in V^n \\ F(\mathbf{x})=0}} W(\mathbf{x})\mathrm{d}\mathbf{x} \\
&= \prod_{l=1}^{r_1+r_2} \int_{\substack{\mathbf{x}^{(l)} \in K_l^n \\ F^{(l)}(\mathbf{x}^{(l)})=0}} w_0(\mathbf{x}^{(l)})\omega(\mathbf{x}^{(l)})\mathrm{d}\mathbf{x}^{(l)} \\
&= \prod_{l=1}^{r_1+r_2} \mathfrak{I}_l(0) + O(e^{-(\log P)^2}),
\end{aligned}$$

where

$$\mathfrak{I}_l(0) = \int \omega(\mathbf{x}^{(l)})\mathrm{d}\mathbf{x}^{(l)},$$

the integral being taken over $\mathbf{x}^{(l)} \in K_l^n$ for which $F^{(l)}(\mathbf{x}^{(l)}) = 0$ and $\|\mathbf{x}^{(l)} - \boldsymbol{\xi}^{(l)}\| \leqslant (\log P)^{-1}$. In the last step we have used the fact that outside the set $\|\mathbf{x}^{(l)} - \boldsymbol{\xi}^{(l)}\| \leqslant (\log P)^{-1}$ we have $\omega(\mathbf{x}^{(l)}) = O(e^{-(\log P)^2})$, whereas $w_0(\mathbf{x}^{(l)}) = 1$ within it.

We begin by analysing $\mathfrak{I}_l(0)$ for $l \leqslant r_1$, making the change of variables $\mathbf{u} = \mathbf{x} - \boldsymbol{\xi}$ and then

$$v_1 = F^{(l)}(\mathbf{u}^{(l)} + \boldsymbol{\xi}^{(l)}), \quad v_i = u_i^{(l)} \text{ for } 2 \leqslant i \leqslant n.$$

According to (5.1) we have $u_1^{(l)} = f^{(l)}(\mathbf{v})$, whence $\mathrm{d}\mathbf{x}^{(l)} = \mathrm{d}\mathbf{u}^{(l)} = \frac{\partial f^{(l)}}{\partial v_1}\mathrm{d}\mathbf{v}$. Let

$$g(v) = f^{(l)}(\mathbf{v})^2 + v_2^2 + \cdots + v_n^2.$$

It now follows that

$$\mathfrak{I}_l(0) = I_l(P) + O(e^{-(\log P)^2}),$$

with

$$I_l(P) = \int \exp\left(-(\log P)^4 g(0, v_2, \ldots, v_n)\right) \frac{\partial f^{(l)}(0, v_2, ..., v_n)}{\partial v_1} \mathrm{d}v_2 \cdots \mathrm{d}v_n,$$

where the integral is over $v_2, \ldots, v_n \in \mathbb{R}$ such that $g(0, v_2, \ldots, v_n) \leqslant (\log P)^{-2}$. The term $I_l(P)$ precisely coincides with the term $I_1(0)$ from [7, page 252], the analysis of which shows that $(\log P)^{2(1-n)} \ll \mathfrak{I}_l(0) \ll (\log P)^{2(1-n)}$.

An analogous method can be used to show that when $l > r_1$, one has

$$(\log P)^{4(1-n)} \ll \mathfrak{I}_l(0) \ll (\log P)^{4(1-n)}.$$

In fact one finds that $\mathfrak{I}_l(0)$ is asymptotically equal to $I_2(0, 0)$, in the notation of [15, §12]. Taken together with our work above, this leads to the conclusion of the lemma. □

Taking $N = 1$ in (8.4) we obtain

$$\sum_{N\,\mathfrak{b}\ll Q^d} (N\,\mathfrak{b})^{-n} S_\mathfrak{b}(\mathbf{0}) I_\mathfrak{b}(\mathbf{0}) = \frac{\sqrt{D_K}}{2^{r_2}} P^{dn} \mathfrak{I}(0) \sum_{N\,\mathfrak{b}\ll Q^d} (N\,\mathfrak{b})^{-n} S_\mathfrak{b}(\mathbf{0})$$

$$+ O\left(P^{dn+\varepsilon} \sum_{N\,\mathfrak{b}\ll Q^d} \rho(N\,\mathfrak{b})^{-n} |S_\mathfrak{b}(\mathbf{0})| \right). \tag{8.5}$$

We begin by dealing with the error term in this expression. Thus let

$$\mathfrak{S}_1 = \sum_\mathfrak{b} (N\,\mathfrak{b})^{-n} |S_\mathfrak{b}(\mathbf{0})|,$$

the sum being over all non-zero integral ideals $\mathfrak{b}$. Recalling that $\rho = Q^{-d} N\,\mathfrak{b}$, we see that

$$\sum_{N\,\mathfrak{b}\ll Q^d} \rho(N\,\mathfrak{b})^{-n} |S_\mathfrak{b}(\mathbf{0})| \ll P^{-1} \sum_{N\,\mathfrak{b}\ll P^{-1}Q^d} (N\,\mathfrak{b})^{-n} |S_\mathfrak{b}(\mathbf{0})| + \sum_{N\,\mathfrak{b}\gg P^{-1}Q^d} (N\,\mathfrak{b})^{-n} |S_\mathfrak{b}(\mathbf{0})|$$

$$\ll P^{-1} \mathfrak{S}_1 + \sum_{N\,\mathfrak{b}\gg P^{-1}Q^d} (N\,\mathfrak{b})^{-n} |S_\mathfrak{b}(\mathbf{0})|.$$

Adopting the notation q, q_1, q_2 from above, it follows from Lemmas 7.3 and 7.4 that

$$S_\mathfrak{b}(\mathbf{0}) \ll q_1^{n/2+1} q_2^{5n/6+1}.$$

Hence, on breaking into dyadic intervals for q_1, q_2, we easily deduce that

$$\sum_{N\,\mathfrak{b}\geqslant A} (N\,\mathfrak{b})^{-n} |S_\mathfrak{b}(\mathbf{0})| \ll \sum_{\substack{\mathfrak{b}_1, \mathfrak{b}_2 \\ q\geqslant A}} q_1^{1-n/2} q_2^{1-n/6} \ll A^{3/2-n/6+\varepsilon} \ll A^{-1/6+\varepsilon},$$

for $n \geqslant 10$. In particular this establishes the convergence of $\mathfrak{S}_1$ and, a fortiori, the absolute convergence of $\mathfrak{S}$ in (5.8). We deduce that

$$\sum_{N\,\mathfrak{b}\ll Q^d} (N\,\mathfrak{b})^{-n} S_\mathfrak{b}(\mathbf{0}) = \mathfrak{S} + O(P^{-3/12+\varepsilon})$$

and

$$\sum_{N\,\mathfrak{b}\ll Q^d} \rho(N\,\mathfrak{b})^{-n} |S_\mathfrak{b}(\mathbf{0})| \ll P^{-1} + (P^{-1}Q^d)^{-1/6+\varepsilon} \ll P^{-1/12+\varepsilon},$$

in (8.5). Once inserted into Lemma 8.1, under the assumption that $n \geqslant 10$, we see that there is an absolute constant $\Delta > 0$ such that

$$N_W(F, P) = \frac{c_Q 2^{r_2(n-1)}}{D_K^{(n-1)/2}} \mathfrak{S}\mathfrak{I}(0) P^{d(n-3)} + O(P^{d(n-3)-\Delta}).$$

Here $c_Q = 1 + O_N(Q^{-N})$. Moreover, the lower bound $\mathfrak{S} \gg 1$ is standard and can be established using the argument of Pleasants [12, Lemma 7.4], for example. Hence an application of Lemma 8.2 shows that $\mathfrak{S}\mathfrak{I}(0)$ has the order of magnitude claimed in Theorem 5.4, thereby concluding our argument.

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T.D. Browning, School of Mathematics, University of Bristol, Bristol, BS8 1TW, United Kingdom
    E-mail : `t.d.browning@bristol.ac.uk`

P. Vishe, Department of Mathematics, KTH, Lindstedtsvagen 25, 10044 Stockholm, Sweden
    E-mail : `vishe@kth.se`