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$R = T$ THEOREMS FOR WEIGHT ONE MODULAR FORMS

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ABSTRACT. We prove modularity of certain residually reducible ordinary 2-dimensional p-adic Galois representations with determinant a finite order odd character χ. For certain non-quadratic χ we prove an $R = T$ result for T the weight 1 specialisation of the cuspidal Hida Hecke algebra acting on non-classical weight 1 forms. Under the additional assumption that no two cuspidal Hida families congruent to an Eisenstein series cross in weight 1 we show that T is reduced. For quadratic χ we prove that the quotient of R corresponding to deformations split at p is isomorphic to the Hecke algebra acting on classical CM weight 1 modular forms.

1. INTRODUCTION

Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be an irreducible Galois representation unramified outside a finite set Σ of primes with $p \in \Sigma$ which is residually reducible, p-distinguished and ordinary at p. Suppose that $\det \rho = \chi\epsilon^{k-1}$ where χ is a finite order character, ϵ is the p-adic cyclotomic character and k is a positive integer such that $\chi\epsilon^{k-1}$ is odd and that the associated residual representation has semi-simplification $1 \oplus \overline{\chi\epsilon}^{k-1}$. If $k \geq 2$, the modularity of such representations by cusp forms of weight k was proved by Skinner and Wiles [SW97, SW99] (recently generalised by Lue Pan [Pan21] to the non-ordinary case). The case of $k = 1$ is different because while one can still expect that the Galois representations should arise from weight one cusp forms, in general not all such forms are classical, i.e., there are purely p-adic weight one ordinary modular forms. This phenomenon was first observed by Mazur and Wiles [MW86]. By a classical modular form of weight k (even if $k = 1$) in this article we mean an element of $M_k(\Gamma_1(N), \mathbf{C})$ for some positive integer N (as in e.g. [Miy89]).

In this article we prove the first modularity theorem for residually reducible Galois representations with $k = 1$ where the Galois representations in question are modular but not necessarily by a classical cusp form of weight one. In fact, it was shown by Dummigan and Spencer [DS21] that if χ is not quadratic there are no classical cusp forms of weight 1 whose associated residual Galois representation has semi-simplification $1 \oplus \overline{\chi}$ (see Remarks 3.4 and 4.8). In that case we prove a

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modularity theorem by purely p-adic weight 1 cusp forms. If χ is quadratic we prove a modularity theorem by classical weight 1 cusp forms with complex multiplication.

While we follow a well-established approach of identifying an appropriate deformation ring with a Hecke algebra, if the character is not quadratic we introduce some novel elements into the method which considerably shift the focus of the approach to dealing with some new challenges. In particular, we work with a Hida Hecke algebra $\mathbf{T}$ to obtain the relevant "$R \to T$"-map whose existence seems difficult to prove directly as the weight one specialization of $\mathbf{T}$ need not be reduced. The proof of the principality of the ideal of reducibility of R is also new. The key input on the automorphic side is Wiles' result from [Wil90] on Λ-adic Eisenstein congruences.

On the other hand, if χ is quadratic we directly establish sufficiently many Eisenstein congruences of classical cusp forms of weight 1. To ensure modularity by classical forms we are also forced to work with a stronger deformation condition at p (see Corollary 5.8).

Let us now explain the contents of this paper in more detail. Let E be a finite extension of $\mathbf{Q}_p$ with integer ring $\mathcal{O}$, uniformizer ϖ and residue field $\mathbf{F}$. Let G_Σ be the Galois group of the maximal Galois extension of $\mathbf{Q}$ unramified outside Σ. Let $\chi : G_\Sigma \to \mathcal{O}^\times$ be an odd Galois character associated with a Dirichlet character mod Np whose N-part is primitive and let $\overline{\chi} : G_\Sigma \to \mathbf{F}^\times$ be its mod ϖ reduction. We assume that $\overline{\chi}|_{D_p} \neq 1$. In particular, we allow $\overline{\chi}$ to be unramified at p. Let

$$\rho_0 : G_\Sigma \to \mathrm{GL}_2(\mathbf{F}), \quad \rho_0 = \begin{bmatrix} 1 & * \\ 0 & \overline{\chi} \end{bmatrix} \not\cong 1 \oplus \overline{\chi}$$

be a homomorphism (such a ρ_0 exists and is unique if one assumes that the χ^{-1}-part of the class group of the splitting field of χ is non-zero and cyclic). We study deformations ρ of ρ_0 which are ordinary at p (for precise definition see section 3.3), and such that $\rho|_{I_\ell} = 1 \oplus \chi$ for all $\ell \in \Sigma$ with $\ell \equiv 1 \bmod p$. We furthermore require that $\det \rho = \chi$, i.e., that $k = 1$. Such a deformation problem is representable by a universal deformation ring R. We then also study the deformation problem with the (stronger) assumption that $\rho|_{D_p}$ is split with corresponding universal deformation ring R^{split}. We refer to these two cases as "ordinary" and "split".

We do not use the Taylor-Wiles method. Instead, we prove that there is a surjection from R to a suitable Hecke algebra T, then show that reducible deformations are modular by demonstrating that $R/I \cong T/J$ for I the reducibility ideal and J the Eisenstein ideal. After establishing the principality of I we then use the commutative algebra criterion from [BK11] to deduce $R = T$. For R^{split} we use a similar approach.

However, to implement this general strategy we use very different routes in the ordinary and the split case. This dichotomy reflects the fact that in the first case we need to deal with non-classical, while in the latter one with classical forms.

Let us first discuss the ordinary case. In that case we work with Wiles' Λ-adic cuspidal Hecke algebra $\mathbf{T}$ and consider a certain quotient $\mathbf{T}_1$ of it - its specialisation at weight one. More precisely, we take a localisation $\mathbf{T}_{\mathfrak{M}}$ of $\mathbf{T}$ (respectively $\mathbf{T}_{1,\mathfrak{m}}$ of $\mathbf{T}_1$) at a maximal ideal corresponding to forms congruent to a certain Λ-adic Eisenstein family $\mathcal{E}$ (respectively a weight 1 specialisation of $\mathcal{E}$).

To construct a map from R to $\mathbf{T}_{1,\mathfrak{m}}$ we first establish that there is a surjection $R^{\mathrm{ord}} \twoheadrightarrow \mathbf{T}_{\mathfrak{M}}$, where R^{ord} is the universal ordinary deformation ring without the determinant condition of R. For this we use that $\mathbf{T}$ is reduced under our assumptions and prove that there exists a suitable integral lattice for the Galois representations associated to the Λ-adic newforms congruent to $\mathcal{E}$. We achieve this by refining a result of Bellaïche-Chenevier to only require our assumption that the χ^{-1}-eigenspace $C_F^{\chi^{-1}}$ of the class group of the splitting field F of χ (rather than the full extension group) is at most 1-dimensional. By combining this with an argument of Kisin to show that R^{ord} is generated by traces we obtain a surjection $R^{\mathrm{ord}} \twoheadrightarrow \mathbf{T}_{\mathfrak{M}}$ in Proposition 4.15.

We deduce from this that we have a (surjective) map from R to $\mathbf{T}_{1,\mathfrak{m}}$, and we prove that it descends to an isomorphism $R/I \to \mathbf{T}_{1,\mathfrak{m}}/J_{1,\mathfrak{m}}$ roughly following the method of [BK13], which boils down to bounding the orders of R/I and $\mathbf{T}_{1,\mathfrak{m}}/J_{1,\mathfrak{m}}$. Here $J_{1,\mathfrak{m}}$ is the (localisation at $\mathfrak{m}$ of the weight one specialisation of the) Eisenstein ideal. To bound $\mathbf{T}_{1,\mathfrak{m}}/J_{1,\mathfrak{m}}$ from below we use a theorem of Wiles from his proof of the Main Conjecture in [Wil90] which gives such a bound on the Eisenstein quotient of the Λ-adic Hecke algebra. For the corresponding upper bound on R/I let us only mention that again a crucial ingredient in the proof is the cyclicity of $C_F^{\chi^{-1}}$. A similar condition has been applied in various situations by Skinner-Wiles [SW97], the authors [BK13, BK20], and by Wake–Wang–Erickson [WWE20]. For a full list of assumptions see section 3.1.

To conclude that $R \cong \mathbf{T}_{1,\mathfrak{m}}$ we utilize the commutative algebra criterion from [BK11], but to apply it we need to show that I is a principal ideal. This is a major technical result of the paper that is needed in both the ordinary and the split case and uses different conditions in the two cases (Theorem 3.10). In fact, the condition needed in the ordinary case excludes quadratic characters (see Remark 3.11(i)), so that case concerns exclusively non-classical forms. Hence for us the ordinary and the split case are in fact disjoint.

We want to highlight that our main result $R = \mathbf{T}_{1,\mathfrak{m}}$ (Theorem 4.17) also applies in cases when $\mathbf{T}_{1,\mathfrak{m}}$ is not reduced, as we now discuss. A priori $\mathbf{T}_{1,\mathfrak{m}}$ is just an abstract quotient of the Λ-adic Hecke algebra and we study in section 4.5 when we can identify $\mathbf{T}_{1,\mathfrak{m}}$ as a subalgebra of $\prod_{\mathcal{F}} \mathcal{O}$ where the product runs over all Λ-adic newforms congruent to $\mathcal{E}$, i.e. when $\mathbf{T}_{1,\mathfrak{m}}$ is actually reduced. The obstruction to this injection occurs when two Hida families congruent to $\mathcal{E}$ cross at weight one. Let us note here that under our assumptions this cannot happen if the weight one specialisations are classical by a result of Bellaïche-Dimitrov (indeed, in their terminology we are in the regular case and cannot have real multiplication, so they prove that the eigencurve is étale at the corresponding point). Even in the non-classical case we know of no example when this happens, however, we cannot rule it out. As the second result of this article (Proposition 4.21), we prove that the lack of such crossings (i.e., the étaleness of the eigencurve at weight one) is indeed also a sufficient condition for having an injection $\mathbf{T}_{1,\mathfrak{m}} \hookrightarrow \prod_{\mathcal{F}} \mathcal{O}$ in the general (also non-classical) case. In section 4.7 we discuss examples when the non-crossing condition is satisfied.

In section 4.6 we further prove the irreducibility of the Galois representation $\rho_{\mathcal{F}}^1$ associated with the weight 1 specialisation $\mathcal{F}_1$ of a cuspidal Hida family $\mathcal{F}$ congruent to $\mathcal{E}$. If $\mathcal{F}_1$ is a classical cusp form, this is a theorem of Deligne and Serre [DS74]. However, a priori $\mathcal{F}_1$ could be a classical Eisenstein series. This happens if the

Kubota-Leopoldt p-adic L-function has a trivial zero, but our p-distinguishedness assumption rules out this possibility (see Remark 4.29). The other issue (that does arise in our context) is that $\mathcal{F}_1$ is not classical. As $\mathcal{F}_1$ is a p-adic limit of classical forms in higher weights, and these have irreducible Galois representations, the question becomes that of proving that this irreducibility is preserved in the limit. In general, of course, a limit of irreducible representations may be reducible. Here however, we show that in this context this does not happen (see Theorem 4.28), as a consequence of the finiteness of class groups.

To complete our treatment of the modularity of residually reducible Galois representations with determinant a finite order character χ we prove in Theorem 5.6 that $R^{\rm split} = (\mathbf{T}_1^{\rm class})_{\mathfrak{m}}$ when χ is the character corresponding to an imaginary quadratic extension $F/\mathbf{Q}$ and p is inert in $F/\mathbf{Q}$ and divides the class number of F. Here $(\mathbf{T}_1^{\rm class})_{\mathfrak{m}}$ is the localisation at the Eisenstein maximal ideal of the Hecke algebra acting on weight 1 classical cusp forms of level d_F with complex multiplication. Whilst the usual methods for proving Eisenstein congruences do not apply in this case, it turns out that there is a very direct link here between elements of the Selmer group bounding $R^{\rm split}/I^{\rm split}$ and cusp forms congruent to the corresponding weight 1 Eisenstein series. To establish the required lower bound on the congruence module $(\mathbf{T}_1^{\rm class})_{\mathfrak{m}}/J$ we can therefore count the total depth of Eisenstein congruences provided by CM forms and apply the result of [BKK14]. This, in turn, requires us to know the principality of J which we deduce from the principality of $I^{\rm split}$.

If f is a classical weight one cusp form, then by [DS74] its Galois representation has finite image. However, there is no a priori reason why this should be true of an arbitrary deformation of ρ_0. In particular, if f is classical and ordinary then $\rho_f|_{D_p}$ must be split (as it must be of finite order). Conjecturally this happens only for classical weight 1 cusp forms. Under our assumptions we note that for χ unramified at p our result $R^{\rm split} = (\mathbf{T}_1^{\rm class})_{\mathfrak{m}}$ (Theorem 5.6) establishes the following equivalence (see Corollary 5.8): an ordinary deformation $\rho : G_\Sigma \to \mathrm{GL}_2(\mathcal{O})$ of ρ_0 is modular by a classical weight 1 cusp form if and only if it is unramified at p and χ is quadratic.

The modularity direction of this result is the analogue of that of Buzzard-Taylor [BT99] (see also a recent result of Pan [Pan22]) on the modularity of residually irreducible p-distinguished representations of G_Σ that are unramified at p (and therefore establishes another case of conjecture 5a in Fontaine-Mazur [FM95] that p-adic representations of G_Σ that are unramified at p have finite image). Theorem 5.6 also complements the work of Castella, Wang-Erickson and Hida [CWEH21] in the residually irreducible case on Greenberg's conjecture that ρ_f is split at p if and only if f is CM.

Finally, let us note that the problem considered in [SW97] is a related one even though it assumes that $k \geq 2$. For a comparison of theirs and our result as well as the methods see Remark 4.20.

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2. Selmer groups

Let p be a prime. For Σ a finite set of finite places of $\mathbf{Q}$ containing p we write G_Σ for the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside of Σ and infinity. For any prime ℓ we write $D_\ell \subset G_\Sigma$ for a decomposition group at ℓ and $I_\ell \subset D_\ell$ for the inertia subgroup.

We fix an embedding $\overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$. Let E be a finite extension of $\mathbf{Q}_p$. Write $\mathcal{O}$ for the valuation ring of E, ϖ for a choice of a uniformizer and $\mathbf{F}$ for the residue field.

For $\psi : G_\Sigma \to \mathcal{O}^\times$ a non-trivial character of finite order prime to p we consider the p-adic coefficients $M = E(\psi)$, $E/\mathcal{O}(\psi)$ or $(\mathcal{O}/\varpi^n)(\psi)$ for $n \geq 1$. Here ψ indicates the G_Σ-action on M. By Class Field Theory there is a one-to-one correspondence between such characters and Dirichlet characters. To keep track of which one we have in mind we will use a Greek letter with a tilde to emphasize that the character is a Dirichlet character, while the same Greek letter by itself will denote the corresponding Galois character. We also write $\overline{\psi} : G_\Sigma \to \mathbf{F}^\times$ for the mod ϖ reduction of ψ.

Remark 2.1. Note that if G is a subgroup of G_Σ such that $\psi|_G \neq 1$, then $(E/\mathcal{O})(\psi)^G = 0$. Indeed, as the order of ψ is prime to p the image of ψ is contained in the prime-to-p roots of unity of $\mathcal{O}$ and so ψ is the Teichmüller lift of $\overline{\psi}$. This guarantees that if $\psi|_G \neq 1$ then there exists $\sigma \in G$ such that $\psi(\sigma) \not\equiv 1 \bmod \varpi$.

Let $\Sigma' \subset \Sigma$. For M as above we define the Selmer group $H^1_{\Sigma'}(\mathbf{Q}, M)$ to be the subgroup of $H^1(G_\Sigma, M)$

$$H^1_{\Sigma'}(\mathbf{Q}, M) = \ker(H^1(G_\Sigma, M) \to \prod_{\ell \in \Sigma \setminus \Sigma'} (H^1(\mathbf{Q}_\ell, M)/H^1_{\mathrm{f}}(\mathbf{Q}_\ell, M))),$$

where the local conditions are defined as follows:

For $M = E(\psi)$ we take for all primes ℓ, including p,

$$H^1_{\mathrm{f}}(\mathbf{Q}_\ell, M) = H^1_{\mathrm{ur}}(\mathbf{Q}_\ell, M) = \ker(H^1(\mathbf{Q}_\ell, M) \to H^1(\mathbf{Q}_{\ell,\mathrm{ur}}, M)),$$

where $\mathbf{Q}_{\ell,\mathrm{ur}}$ is the maximal unramified extension of $\mathbf{Q}_\ell$. This induces conditions for $M = (E/\mathcal{O})(\psi)$ and $(\mathcal{O}/\varpi^n)(\psi)$ via

$$H^1_{\mathrm{f}}(\mathbf{Q}_\ell, E/\mathcal{O}(\psi)) = \mathrm{im}(H^1_{\mathrm{ur}}(\mathbf{Q}_\ell, E(\psi)) \to H^1(\mathbf{Q}_\ell, E/\mathcal{O}(\psi)))$$

and

$$H^1_{\mathrm{f}}(\mathbf{Q}_\ell, (\mathcal{O}/\varpi^n)(\psi)) = i_n^{-1} H^1_{\mathrm{f}}(\mathbf{Q}_\ell, E/\mathcal{O}(\psi)) \text{ for } i_n : H^1(\mathbf{Q}_\ell, (\mathcal{O}/\varpi^n)(\psi)) \to H^1(\mathbf{Q}_\ell, E/\mathcal{O}(\psi))$$

the natural map induced by the canonical injection $(\mathcal{O}/\varpi^n)(\psi) \to E/\mathcal{O}(\psi)$.

For $\ell \neq p$ [Rub00] Lemma 1.3.5(iii) tells us that

$$H^1_{\mathrm{f}}(\mathbf{Q}_\ell, E/\mathcal{O}(\psi))) = H^1_{\mathrm{ur}}(\mathbf{Q}_\ell, E/\mathcal{O}(\psi)))$$

since $(E/\mathcal{O})(\psi)^{I_\ell}$ is divisible as ψ has order prime to p. Indeed, if ψ is unramified then the invariants are isomorphic to $E/\mathcal{O}$ as $\mathcal{O}$-modules, hence divisible. If ψ is ramified then the invariants are zero by Remark 2.1. By the same [Rub00] Lemma 1.3.5(iii) $H^1_{\mathrm{f}}(\mathbf{Q}_\ell, \mathcal{O}(\psi)))$ (defined as preimage of $H^1_{\mathrm{f}}(\mathbf{Q}_\ell, E(\psi)))$ agrees with $\mathrm{im}(H^1_{\mathrm{ur}}(\mathbf{Q}_\ell, \mathcal{O}(\psi))$, which by the proof of [Rub00] Lemma 1.3.8 also gives $H^1_{\mathrm{ur}}(\mathbf{Q}_\ell, (\mathcal{O}/\varpi^n)(\psi)) = H^1_{\mathrm{f}}(\mathbf{Q}_\ell, (\mathcal{O}/\varpi^n)(\psi))$.

For $\ell = p$ we also have $H^1_{\mathrm{f}}(\mathbf{Q}_p, E/\mathcal{O}(\psi)) = H^1_{\mathrm{ur}}(\mathbf{Q}_p, E/\mathcal{O}(\psi))$, by the proof of [Rub00] Proposition 1.6.2 as the order of ψ is coprime to p. In addition an easy diagram chase like in the proof of [Rub00] Lemma 1.3.5 for $H^1_{\rm f}(K,T)$ shows that

$$H^1_{\rm ur}(\mathbf{Q}_p, (\mathcal{O}/\varpi^n)(\psi)) \subset H^1_{\rm f}(\mathbf{Q}_p, (\mathcal{O}/\varpi^n)(\psi)). \tag{2.1}$$

By [Rub00] Lemma 1.5.4 and Lemma 1.2.2(i) we have

$$H^1_{\Sigma'}(\mathbf{Q}, (\mathcal{O}/\varpi^n)(\psi)) = H^1_{\Sigma'}(\mathbf{Q}, (E/\mathcal{O})(\psi))[\varpi^n] \tag{2.2}$$

since $(E/\mathcal{O})(\psi)^{G_\Sigma} = 0$ by Remark 2.1.

Proposition 2.2 ([Rub00] Proposition 1.6.2)**.**

$$H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\psi)) \cong \mathrm{Hom}(\mathrm{Cl}(\mathbf{Q}(\psi)), E/\mathcal{O}(\psi))^{\mathrm{Gal}(\mathbf{Q}(\psi)/\mathbf{Q})}$$

Lemma 2.3. *Let $\tilde{\xi}$ be a Dirichlet character. Let $\xi : G_\Sigma \to \mathcal{O}^\times$ be the associated Galois character and write $\overline{\xi}$ for its mod ϖ reduction. Let s be a positive integer. Set $W = E/\mathcal{O}(\xi^{-1})$ and $W_s = W[\varpi^s]$. Suppose $\ell \in \Sigma - \{p\}$ and let $\Sigma' \subset \Sigma$ with $\ell \notin \Sigma'$. Assume that either*

(i) *$\xi_s := \xi$ mod ϖ^s is unramified at ℓ and $\ell\overline{\xi}(\mathrm{Frob}_\ell) \neq 1$;*
or
(ii) *$\overline{\xi}$ is ramified at ℓ.*

Then one has

$$H^1_{\Sigma' \cup \{\ell\}}(\mathbf{Q}, W_s) = H^1_{\Sigma'}(\mathbf{Q}, W_s).$$

Proof. First assume that $\overline{\xi}$ is ramified at ℓ. Then $W_1^{I_\ell} = 0$ and so $W^{I_\ell} = 0$ and we use [BK13] Lemma 5.6 to conclude that

$$H^1_{\Sigma' \cup \{\ell\}}(\mathbf{Q}, W_s) = H^1_{\Sigma'}(\mathbf{Q}, W_s).$$

We note that the definition of the global Selmer group $H^1_\Sigma(\mathbf{Q}, W_s)$ in [BK13] differs from our definition here in that it uses the Fontaine-Laffaille condition at p, rather than assuming that classes are unramified. But on the level of divisible coefficients the definitions agree, so we apply [BK13] Lemma 5.6 to conclude $H^1_{\Sigma' \cup \{\ell\}}(\mathbf{Q}, W) = H^1_{\Sigma'}(\mathbf{Q}, W)$ and then invoke (2.2).

From now on assume that ξ_s and so also W_s is unramified at ℓ. By [Rub00], Theorem 1.7.3 we have an exact sequence

$$0 \to H^1_{\Sigma'}(\mathbf{Q}, W_s) \to H^1_{\Sigma' \cup \{\ell\}}(\mathbf{Q}, W_s) \to \frac{H^1(\mathbf{Q}_\ell, W_s)}{H^1_{\rm ur}(\mathbf{Q}_\ell, W_s)}.$$

Lemma 1.3.8(ii) in [Rub00] tells us that $H^1_{\rm ur}(\mathbf{Q}_\ell, W_s) = H^1_f(\mathbf{Q}_\ell, W_s)$.

To prove the claim it is enough to show that the image of the map $H^1(\mathbf{Q}_\ell, W_s) \to H^1(I_\ell, W_s)$ is zero. To do so consider the inflation-restriction sequence (where we set $G := \mathrm{Gal}(\mathbf{Q}_\ell^{\rm ur}/\mathbf{Q}_\ell)$):

$$H^1(G, W_s) \to H^1(\mathbf{Q}_\ell, W_s) \to H^1(I_\ell, W_s)^G \to H^2(G, W_s).$$

The last group in the above sequence is zero since $G \cong \hat{\mathbf{Z}}$ and $\hat{\mathbf{Z}}$ has cohomological dimension one. This means that the image of the restriction map $H^1(\mathbf{Q}_\ell, W_s) \to H^1(I_\ell, W_s)$ equals $H^1(I_\ell, W_s)^G$. Let us show that the latter module is zero. Indeed,

$$\begin{aligned} H^1(I_\ell, W_s)^G &= \mathrm{Hom}_G(I_\ell, W_s) = \mathrm{Hom}_G(I_\ell^{\rm tame}, W_s) \\ &= \mathrm{Hom}_G(\mathbf{Z}_p(1), \varpi^{-s}\mathcal{O}/\mathcal{O}(\xi^{-1})) = \mathrm{Hom}_G(\mathbf{Z}_p, \varpi^{-s}\mathcal{O}/\mathcal{O}(\xi^{-1}\epsilon^{-1})). \end{aligned} \tag{2.3}$$

So, $\phi \in H^1(I_\ell, W_s)$ lies in

$$H^1(I_\ell, W_s)^G = \mathrm{Hom}_G(\mathbf{Z}_p, \varpi^{-s}\mathcal{O}/\mathcal{O}(\xi^{-1}\epsilon^{-1}))$$

if and only if $\phi(x) = g \cdot \phi(g^{-1} \cdot x) = g \cdot \phi(x) = \xi_s^{-1}\epsilon^{-1}(g)\phi(x)$ for every $x \in I_\ell$ and every $g \in G$, i.e., if and only if

$$(\xi_s^{-1}\epsilon^{-1}(g) - 1)\phi(x) \in \mathcal{O} \quad \text{for every } x \in I_\ell,\ g \in G. \tag{2.4}$$

Since Frob_ℓ topologically generates G, we see that (2.4) holds if and only if it holds for every $x \in I_\ell$ and for $g = \mathrm{Frob}_\ell$. So condition (2.4) becomes

$$(1 - \xi_s^{-1}(\mathrm{Frob}_\ell)\ell^{-1})\phi(x) \in \mathcal{O} \quad \text{for every } x \in I_\ell. \tag{2.5}$$

Since $\overline{\xi}(\mathrm{Frob}_\ell)\ell \neq 1$, the factor $\mathrm{val}_p(1 - \xi_s^{-1}(\mathrm{Frob}_\ell)\ell^{-1}) = 0$, we get that $\phi(x) \in \mathcal{O}$, as claimed. $\square$

3. Deformation theory

3.1. **Assumptions.** Let $p > 2$ be a prime and N a positive integer with $p \nmid N$. Let $\tilde{\chi} : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathbf{C}^\times$ denote a Dirichlet character of order prime to p with $\tilde{\chi}(-1) = -1$. We write $\tilde{\chi} = \tilde{\chi}_N\tilde{\chi}_p$ where $\tilde{\chi}_N$ is a Dirichlet character mod N and $\tilde{\chi}_p$ is a Dirichlet character mod p. We assume that $\tilde{\chi}_N$ is primitive. In particular, we allow but do not require that $\tilde{\chi}$ has p in its conductor.

Write Σ for a finite set of primes containing p and the primes dividing N. Let $\chi : G_\Sigma \to \mathcal{O}^\times$ be the Galois character associated to $\tilde{\chi}$ and write $\overline{\chi} : G_\Sigma \to \mathbf{F}^\times$ for its mod ϖ reduction. We assume that $\overline{\chi}|_{D_p} \neq 1$.

Write F for the splitting field of χ and $\mathrm{Cl}(F)$ for the class group of F. Set $C_F := \mathrm{Cl}(F) \otimes_{\mathbf{Z}} \mathcal{O}$. For any character $\psi : \mathrm{Gal}(F/\mathbf{Q}) \to \mathcal{O}^\times$ we write C_F^ψ for the ψ-eigenspace of C_F under the canonical action of $\mathrm{Gal}(F/\mathbf{Q})$, i.e.

$$C_F^\psi = \{c \in C_F | g \cdot c = \psi(g)c \text{ for all } g \in G_\Sigma\}.$$

In this paper we work under the following assumptions:

(1) $C_F^{\chi^{-1}}$ is a non-zero cyclic $\mathcal{O}$-module, i.e., $\dim_{\mathbf{F}} C_F^{\chi^{-1}} \otimes_{\mathcal{O}} \mathbf{F} = 1$;
(2) if $\ell \in \Sigma$ but $\ell \nmid Np$ then $\tilde{\chi}(\ell)\ell \not\equiv 1 \bmod \varpi$;
(3) if $\ell \in \Sigma$ but $\ell \nmid Np$ then $\tilde{\chi}(\ell) \not\equiv \ell \bmod \varpi$.

Remark 3.1. While Assumption (2) comes in for Propositions 3.3, 3.9, Theorem 3.10, Assumption (3) is only used for Theorem 3.10.

Remark 3.2. We note that $C_F^{\chi^{-1}} \neq 0$ is equivalent to $\mathrm{val}_p(L(0, \tilde{\chi})) > 0$. This is so because under our assumptions on χ, we have that (cf. Theorem 2 in [MW84])

$$\#C_F^{\chi^{-1}} = \#\mathcal{O}/L(0, \tilde{\chi}). \tag{3.1}$$

Let $\rho_0 : G_\Sigma \to \mathrm{GL}_2(\mathbf{F})$ be a continuous homomorphism of the form

$$\rho_0 = \begin{bmatrix} 1 & * \\ 0 & \overline{\chi} \end{bmatrix} \not\cong 1 \oplus \overline{\chi}$$

such that $\rho_0|_{D_p} \cong 1 \oplus \overline{\chi}|_{D_p}$. We show the existence of such a ρ_0 in section 4.3.

3.2. **The residual representation.** We begin by proving the uniqueness of ρ_0 up to isomorphism. Note that for this result we do not need to assume that ρ_0 is split on D_p, but only on I_p.

Proposition 3.3. *Let $\rho' : G_\Sigma \to \mathrm{GL}_2(\mathbf{F})$ be a continuous homomorphism of the form*
$$\rho' = \begin{bmatrix} 1 & * \\ 0 & \overline{\chi} \end{bmatrix} \not\cong 1 \oplus \overline{\chi}$$
such that $\rho'|_{I_p} \cong 1 \oplus \overline{\chi}|_{I_p}$. Then $\rho' \cong \rho_0$.

Proof. Let ρ' be as in the statement of the proposition. Then $*$ gives rise to a non-zero element c in $H^1_\Sigma(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1}))$. Using Lemma 2.3 and Assumption (2) above we conclude that $H^1_\Sigma(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1})) = H^1_{\{p\}}(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1}))$. By the assumption that $\rho'|_{I_p} \cong 1 \oplus \overline{\chi}|_{I_p}$ we see that c is unramified at p, hence in fact $c \in H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1}))$ by (2.1). By Proposition 2.2 we have that

$$H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\chi^{-1})) \cong \mathrm{Hom}(\mathrm{Cl}(F), E/\mathcal{O}(\chi^{-1}))^{\mathrm{Gal}(F/\mathbf{Q})} \cong C_F^{\chi^{-1}}, \tag{3.2}$$

where the last isomorphism is non-canonical. By Assumption (1), the group $C_F^{\chi^{-1}}$ is non-zero and cyclic, hence so is $H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\chi^{-1}))$. By (2.2) we get that $H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1})) \cong H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\chi^{-1}))[\varpi]$, so $H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1}))$ is one-dimensional. Hence the extension given by c is a non-zero scalar multiple of the one given by ρ_0. The claim follows. $\square$

3.3. **The deformation problems.** Set R to be the universal deformation ring for deformations $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ of ρ_0 for A an object in $\mathrm{CNL}(\mathcal{O})$, the category of local complete Noetherian $\mathcal{O}$-algebras with residue field $\mathbf{F}$, such that:

(i) $\det \rho = \chi$

(ii) $\rho|_{D_p} \cong \begin{bmatrix} \psi_1 & * \\ & \psi_2 \end{bmatrix}$ with ψ_2 unramified and $\psi_2 \equiv 1 \mod \mathfrak{m}_A$ (ordinary and p-distinguished)

(iii) If $\ell \in \Sigma$ is such that $\ell \equiv 1 \pmod p$ then $\rho|_{I_\ell} = 1 \oplus \chi$.

Note that this problem is indeed representable due to the fact that ρ_0 has scalar centralizer. Also note that in condition (ii) one must have $* \in \mathfrak{m}_A$ because ρ_0 splits when restricted to D_p. Let $\rho^{\mathrm{univ}} : G_\Sigma \to \mathrm{GL}_2(R)$ be the universal deformation. Write I for the ideal of reducibility of ρ^{univ}.

Let R^{split} be the universal deformation ring for the deformations where (ii) is strengthened to assuming that $\rho|_{D_p}$ is split. We denote the universal deformation for the stronger condition by ρ^{split} and write I^{split} for its ideal of reducibility.

We will refer to deformations satisfying (i)-(iii) as *ordinary deformations* (or simply as *deformations*), whilst calling the ones satisfying the stronger condition *split deformations*. It is clear that every split deformation is a deformation, so we get a natural map $R \to R^{\mathrm{split}}$.

We will not use the notation R^{ord} for the ring R, as this usually denotes the universal ring of deformations satisfying only (ii) and (iii), and we will, in fact, use this ring later in section 4.3.

Remark 3.4. Note that for $\rho : G_\Sigma \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ by Corollary to Theorem 11 in [Sen81] the assumption that $\rho|_{D_p}$ is split corresponds to $\rho|_{D_p}$ being Hodge-Tate (and even de Rham) with Hodge-Tate weights 0. Such representations ρ are expected to correspond to classical cusp forms of weight 1. See Theorem 1.0.5 in [Pan22] for a recent result on this in the residually irreducible case.

If one knows that $\rho(G_{\mathbf{Q}})$ is finite then one can easily prove this special case of Artin's conjecture: From the classification of subgroups of $\mathrm{GL}_2(\mathbf{C})$ one can show (see e.g. section 2 of [DS21]) that the residual reducibility requires the image of ρ to be dihedral. From this one deduces (see e.g. section 7 in [Ser77]) that there exists a quadratic extension $F'/\mathbf{Q}$ for which $\rho \otimes \chi_{F'/\mathbf{Q}} \cong \rho$, where $\chi_{F'/\mathbf{Q}}$ is the unique character of $G_{\mathbf{Q}}$ that factors through the non-trivial character of $\mathrm{Gal}(F'/\mathbf{Q})$. This implies that $\chi = \chi_{F'/\mathbf{Q}}$ (and so $F' = F$), hence F has to be imaginary quadratic as χ is odd. It further follows that ρ is the induction of a finite order Galois character of G_F, i.e. that ρ corresponds to a weight 1 CM form. In section 5 we prove (without the assumption that $\rho(G_{\mathbf{Q}})$ is finite) that split deformations of ρ_0 with $\chi = \chi_{F/\mathbf{Q}}$ indeed correspond to classical weight 1 CM forms.

3.4. **Reducible deformations.** We record the following general lemma regarding pseudocharacters that helps us study reducible deformations.

Lemma 3.5. *Let A be a Henselian local ring with a maximal ideal $\mathfrak{m}$ and let G be a group. Let $\tau_1, \tau_2 : G \to (A/\mathfrak{m})^\times$ be two distinct characters which we can regard as homomorphisms from $A[G]$ to $A/\mathfrak{m}$. Let $T : A[G] \to A$ be a pseudocharacter of dimension 2 such that there exist characters T_1, T_2 with $T = T_1 + T_2$ with the property that $T_i \otimes_A A/\mathfrak{m} = \tau_i$ for $i = 1, 2$. Then T_1 and T_2 are uniquely determined.*

Proof. This is the last assertion of Proposition 1.5.1 in [BC09] where we take $J = 0$ and $R = A[G]$ and then $\mathrm{dec}_{\mathcal{P}}$ is satisfied for $\mathcal{P} = \{\{1\}, \{2\}\}$ with $I_{\mathcal{P}} = 0$. □

Proposition 3.6. *There do not exist any non-trivial upper-triangular deformations of ρ_0 to $\mathrm{GL}_2(\mathbf{F}[X]/X^2)$.*

Proof. Suppose that $\rho : G_\Sigma \to \mathrm{GL}_2(\mathbf{F}[X]/X^2)$ is such a deformation. Write

$$\rho = \begin{bmatrix} 1 + aX & b \\ & \overline{\chi}(1 + dX) \end{bmatrix}$$

with $a, d \in \mathrm{Hom}(G_\Sigma, \mathbf{F})$ and $b : G_\Sigma \to \mathbf{F}[X]/X^2$. Our deformation conditions guarantee that a and d are unramified at all primes. Indeed, a and d are at most tamely ramified at all primes $\ell \neq p$, but if $\ell \in \Sigma - \{p\}$ and $\ell \not\equiv 1 \bmod p$, then there is no abelian p-extension of $\mathbf{Q}$ that is tamely ramified at ℓ. On the other hand if $\ell \equiv 1 \bmod p$ then the deformation condition (iii) guarantees that $a|_{I_\ell} = 0$. So, a can only be ramified at p.

By condition (ii) we have an isomorphism of $\mathbf{F}[X]/X^2[D_p]$-modules

$$\begin{bmatrix} 1 + aX & b \\ & \overline{\chi} + dX \end{bmatrix} \cong \begin{bmatrix} \psi_1 & * \\ & \psi_2 \end{bmatrix}, \tag{3.3}$$

where each of the entries is considered to be restricted to D_p and $\psi_2 \equiv 1 \bmod X$. Using Lemma 3.5 we see that we therefore must have $1 + aX = \psi_2$ as $\overline{\chi}|_{D_p} \neq 1$. As ψ_2 is unramified, we conclude that a is unramified (at p). Hence a is unramified everywhere, so $a = 0$. By condition (i) we must also have $d = 0$.

Now consider the entry $b = b_0 + b_1 X$ with $b_0, b_1 : G_\Sigma \to \mathbf{F}$. Using the basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} X \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ X \end{bmatrix}$ we can write ρ as a 4-dimensional representation over $\mathbf{F}$:

$$\rho = \begin{bmatrix} 1 & b_0 & & \\ & \overline{\chi} & & \\ & b_1 & 1 & b_0 \\ & & & \overline{\chi} \end{bmatrix}$$

which clearly has a subquotient isomorphic to $\begin{bmatrix} 1 & b_1 \\ & \overline{\chi} \end{bmatrix}$. If this subquotient is split we are done. Otherwise it must be isomorphic to ρ_0 by Proposition 3.3. From this it is easy to see that $\rho' \cong \rho_0$ as desired (cf. Proof of Proposition 7.2 in [BK13] for details). $\square$

Corollary 3.7. *The structure maps $\mathcal{O} \to R/I$ and $\mathcal{O} \to R^{\rm split}/I^{\rm split}$ are surjective.*

Proof. Using Proposition 3.6 this is proved like Proposition 7.10 in [BK13]. $\square$

As a consequence of Corollary 3.7 one gets as in Proposition 7.13 in [BK13] the following proposition.

Proposition 3.8. *The ring R is topologically generated as an $\mathcal{O}$-algebra by the set $\{\mathrm{tr}\, \rho^{\rm univ}(\mathrm{Frob}_\ell) \mid \ell \not\in \Sigma\}$ and $R^{\rm split}$ is topologically generated by $\{\mathrm{tr}\, \rho^{\rm split}(\mathrm{Frob}_\ell) \mid \ell \not\in \Sigma\}$.*

Proposition 3.9. *One has $\# R^{\rm split}/I^{\rm split} \leq \# R/I \leq \# C_F^{\chi^{-1}}$.*

Proof. Note that Proposition 3.8 implies that the map $R \to R^{\rm split}$ is a surjection. This implies (see e.g. [BK13] Lemma 7.11) that the map $R/I \to R^{\rm split}/I^{\rm split}$ is also surjective, so we only need to prove the second inequality.

By Corollary 3.7 we get $R/I = \mathcal{O}/\varpi^r$ (allowing for $r = \infty$). Using Corollary 7.8 in [BK13] we know that any deformation to $\mathrm{GL}_2(R/I)$ is equivalent to one of the form $\begin{bmatrix} \Psi_1' & b' \\ & \Psi_2' \end{bmatrix}$ with Ψ_1' reducing to 1 mod ϖ and Ψ_2' reducing to $\overline{\chi}$ mod ϖ. (Note that the corollary assumes that the ring R/I is Artinian. However, its proof uses Theorem 7.7 which allows for the ring to be Hausdorff and complete, so the case of $r = \infty$ is also covered.) Let $s \leq r$ be a (finite) positive integer. Let $\rho : G_\Sigma \to \mathrm{GL}_2(\mathcal{O}/\varpi^s)$ be the composition of the deformation $\begin{bmatrix} \Psi_1' & b' \\ & \Psi_2' \end{bmatrix}$ with the canonical projection $R/I \twoheadrightarrow \mathcal{O}/\varpi^s$. Then $\rho = \begin{bmatrix} \Psi_1 & b \\ & \Psi_2 \end{bmatrix}$, where the non-primed entries are simply the reductions of the primed entries modulo ϖ^s. Write $\Psi_1 = 1 + \alpha\varpi$ for some group homomorphism $\alpha : G_\Sigma \to \mathcal{O}/\varpi^{s-1}$. Hence Ψ_1 cuts out an abelian extension K of $\mathbf{Q}$ that is of p-power degree. Let $\ell \in \Sigma$ be a prime different from p. Then K can be at most tamely ramified at ℓ, so it must be unramified unless $\ell \equiv 1$ mod p. So the deformation condition (iii) guarantees that K can only be ramified at p.

By condition (ii) we get that $\rho|_{D_p} \cong \begin{bmatrix} \psi_1 & * \\ & \psi_2 \end{bmatrix}$ with ψ_2 unramified at p and reducing to the trivial character mod ϖ. Using Lemma 3.5 we must therefore have that $\Psi_1|_{D_p} = \psi_2$ as $\overline{\chi}|_{D_p} \neq 1$. Hence α must be unramified at p. Thus we have shown that Ψ_1 is unramified everywhere and hence $\Psi_1 = 1$. Then condition (i) implies that $\Psi_2 = \chi$.

We thus get that b gives rise to a cohomology class in $H^1_\Sigma(\mathbf{Q}, \mathcal{O}/\varpi^s(\chi^{-1})) = H^1_\Sigma(\mathbf{Q}, W_s)$, where $W = E/\mathcal{O}(\chi^{-1})$. Using Lemma 2.3 and Assumption (2) we see that this group equals $H^1_{\{p\}}(\mathbf{Q}, W_s)$. Condition (ii) now again forces b to be unramified at p as well, so in fact the class of b lies in $H^1_\emptyset(\mathbf{Q}, W_s)$.

By (2.2) we have $H^1_\emptyset(\mathbf{Q}, W_s) = H^1_\emptyset(\mathbf{Q}, W)[\varpi^s]$ and by (3.2) we have a non-canonical isomorphism $H^1_\emptyset(\mathbf{Q}, W) \cong C_F^{\chi^{-1}}$.

Define k by $\#C_F^{\chi^{-1}} \cong \#\mathcal{O}/\varpi^k$. Then we conclude that ϖ^k annihilates the class in $H^1_\emptyset(\mathbf{Q}, W)$ arising from b. As b is not a coboundary mod ϖ by the assumption that ρ_0 is not split, we get that the image of b in $\mathcal{O}/\varpi^s$ generates $\mathcal{O}/\varpi^s$ over $\mathcal{O}$. So, the class of b generates an $\mathcal{O}$-submodule of $H^1_\emptyset(\mathbf{Q}, W)$ isomorphic to $\mathcal{O}/\varpi^s$. Hence $s \leq k$. If $r < \infty$, we can always take $s = r$, so this forces also $r \leq k$. If $r = \infty$, we could take $s = k+1$, which would lead to a contradiction, so r cannot be infinite. $\square$

3.5. **Principality of ideals of reducibility.** Let $\tilde{\omega} : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{C}^\times$ be the Teichmüller character. We denote by $\omega : G_\Sigma \to \mathbf{Z}_p^\times$ the corresponding p-adic Galois character.

In this section we will prove the principality of the reducibility ideals $I^{\text{split}} \subset R^{\text{split}}$ and $I \subset R$ (the latter case for $\chi^2 \neq 1$). In both cases one needs to control the number of generators of C_F^{χ} (note that Assumption (1) in section 3.3 concerned $C_F^{\chi^{-1}}$) as an $\mathcal{O}$-module.

Theorem 3.10.

(1) *Suppose C_F^{χ} is a cyclic $\mathcal{O}$-module, then the ideal I^{split} is principal.*
(2) *Suppose that $C_F^{\chi} = 0$. Assume further that at least one of the following conditions is satisfied:*
  (i) *$e < p-1$ where e is the ramification index of p in F or*
  (ii) *$\chi = \omega^s$ for some integer s or*
  (iii) *$\tilde{\chi}_N(p) \neq 1$.*
  Then I is principal.

Remark 3.11.

(i) Note that χ in part (2) of the Theorem is automatically non-quadratic as we assume that $C_F^{\chi} = 0$ while we have $C_F^{\chi^{-1}} \neq 0$.
(ii) Let χ_N (resp. χ_p for later usage) be the Galois character associated with $\tilde{\chi}_N$ (resp. $\tilde{\chi}_p$). Part (2) of Theorem 3.10 does not cover the case where $\chi = \omega^s \chi_N$ where $(s, p-1) = 1$, so $e = p-1$, but χ_N is a non-trivial character with $\chi_N(p) = 1$, which means that F is an extension of $\mathbf{Q}(\zeta_p)$ where all primes of $\mathbf{Q}(\zeta_p)$ lying over p split completely in $F/\mathbf{Q}(\zeta_p)$.

Proof of Theorem 3.10. The universal deformations give rise to R-algebra homomorphisms $\rho := \rho^{\text{univ}} : R[G_\Sigma] \to M_2(R)$ and $\rho^{\text{split}} : R^{\text{split}}[G_\Sigma] \to M_2(R^{\text{split}})$. Fix $? \in \{\emptyset, \text{split}\}$. The image of $\rho^?$ is a Generalized Matrix Algebra (GMA) in the sense of [BC09] of the form

$$\begin{bmatrix} R^? & B^? \\ C^? & R^? \end{bmatrix},$$

where $B^?$ is the ideal of $R^?$ generated by $b^?(x)$ as x runs over $R^?[G_\Sigma]$ and similarly for $C^?$. As the residual representation is non-split we get $B^? = R^?$, so we get $I^? = B^? C^? = C^?$. Arguing as in the proof of Theorem 1.5.5. in [BC09], and using the fact that $I^? \subset \mathfrak{m}^?$ (where $\mathfrak{m}^?$ is the maximal ideal of $R^?$) we get an injection:

$$\iota^? : \mathrm{Hom}_{R^?}(C^?, R^?/\mathfrak{m}^?) = \mathrm{Hom}_{R^?}(C^?, \mathbf{F}) \hookrightarrow H^1(\mathbf{Q}, \mathbf{F}(\overline{\chi})).$$

For the convenience of the reader we split the rest of the proof into several lemmas.

Lemma 3.12. *The image of $\iota^?$ lies in $H^1_{\{p\}}(\mathbf{Q}, \mathbf{F}(\overline{\chi}))$, and if it is at most one-dimensional then I is principal.*

Proof. We need to first show that the image of $\iota^?$ consists only of classes that are unramified outside p. First note that the map $\iota^?$ is given by (cf. Proof of Theorem 1.5.5 in [BC09])

$$f \mapsto \left(x \mapsto \begin{bmatrix} a^?(x) \pmod{\mathfrak{m}^?} & 0 \\ f(c^?(x)) & d^?(x) \pmod{\mathfrak{m}^?} \end{bmatrix}\right).$$

Then it is clear that the image of $\iota^?$ is contained in $H^1_\Sigma(\mathbf{Q}, \mathbf{F}(\overline{\chi}))$, i.e., is unramified outside Σ.

Now suppose the image of $\iota^?$ is n-dimensional for $n \leq 1$. Then so is $\mathrm{Hom}_{R^?}(C^?, R^?/\mathfrak{m}^?)$. An application of the complete version of Nakayama's Lemma gives us that $C^?/\mathfrak{m}C^?$ is an n-dimensional $\mathbf{F}$-vector space, from which we conclude that $I^?$ is principal. □

So, it remains to prove that the image of $\iota^?$ is of dimension ≤ 1. By Lemma 2.3 applied with $\tilde{\xi} = \tilde{\chi}^{-1}$ and $s = 1$ we see that $H^1_\Sigma(\mathbf{Q}, \mathbf{F}(\overline{\chi})) = H^1_{\Sigma'}(\mathbf{Q}, \mathbf{F}(\overline{\chi}))$ where $\Sigma' \subset \Sigma$ consists only of p and those primes ℓ such that χ is unramified at ℓ and $\tilde{\chi}(\ell) \equiv \ell \bmod p$. If ℓ is one of the latter primes then by our assumption (3) $\rho^?$ is unramified at ℓ. This implies that the image of $\iota^?$ is contained in $H^1_{\{p\}}(\mathbf{Q}, \mathbf{F}(\overline{\chi}))$.

Now suppose we are in the case (1) of the Theorem. Then $? =$ split and the image of ι^{split} is in fact contained in $H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}))$. Arguing as in the proof of Proposition 3.3 we see that the assertion that this Selmer group is at most one-dimensional is equivalent to the $\mathcal{O}$-cyclicity of C_F^χ. This proves part (1) of the Theorem.

From now on we study case (2) when $? = \emptyset$ and we will show that

$$\dim_{\mathbf{F}} H^1_{\{p\}}(\mathbf{Q}, \mathbf{F}(\overline{\chi})) \leq 1.$$

As by Remark 2.1, the character χ is the Teichmüller lift of $\overline{\chi}$, to ease notation below we will not distinguish between χ and $\overline{\chi}$ and always write χ. Write S for the set of primes of F (the splitting field of χ) lying over p.

Lemma 3.13. *Let $H = \mathrm{Gal}(L/F)$, where L is the maximal abelian extension of F unramified away from S. Set $V = H/H^p$. If $\dim_{\overline{\mathbf{F}}_p} V^\chi \leq 1$, then $\dim_{\mathbf{F}} H^1_{\{p\}}(\mathbf{Q}, \mathbf{F}(\overline{\chi})) \leq 1$.*

Proof. Write G for $\mathrm{Gal}(F/\mathbf{Q})$. The restriction map gives us an isomorphism

$$\mathrm{res} : H^1(G_\Sigma, \mathbf{F}(\chi)) \cong H^1(\ker\chi, \mathbf{F}(\chi))^G = \mathrm{Hom}_G((\ker\chi)^{\mathrm{ab}}, \mathbf{F}(\chi)) \tag{3.4}$$

which carries the subgroup $H^1_{\{p\}}(\mathbf{Q}, \mathbf{F}(\chi))$ into the subgroup of $\mathrm{Hom}_G((\ker\chi)^{\mathrm{ab}}, \mathbf{F}(\chi))$ consisting of all the homomorphisms which vanish on I_ℓ for all $\ell \neq p$. Each of these homomorphisms factors through H, thus they land in the group $\mathrm{Hom}_G(H, \mathbf{F}(\chi)) \cong \mathrm{Hom}_G(V, \mathbf{F}(\chi))$ (which injects into $\mathrm{Hom}_G((\ker\chi)^{\mathrm{ab}}, \mathbf{F}(\chi))$ by left exactness of the Hom-functor).

As one has $\dim_{\mathbf{F}} \mathrm{Hom}_G(V, \mathbf{F}(\chi)) = \dim_{\overline{\mathbf{F}}_p} \mathrm{Hom}_{\overline{\mathbf{F}}_p[G]}(V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p, \overline{\mathbf{F}}_p(\chi))$ it suffices to prove that the dimension of the latter space is no greater than one.

One has

$$V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p = \bigoplus_{\varphi \in \mathrm{Hom}(G, \overline{\mathbf{F}}^\times)} V^\varphi,$$

where

$$V^\varphi = \{v \in V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p \mid g \cdot v = \varphi(g)v \text{ for every } g \in G\}.$$

It is clear that

$$\mathrm{Hom}_{\overline{\mathbf{F}}_p[G]}(V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p, \overline{\mathbf{F}}_p(\chi)) \cong \mathrm{Hom}_{\overline{\mathbf{F}}_p[G]}(V^\chi, \overline{\mathbf{F}}_p(\chi)).$$

Hence the claim of the lemma follows. □

Set $\mathcal{O}$ to be the ring of integers in F. For $\mathfrak{p} \in S$ let $\mathcal{O}_\mathfrak{p}$ denote the completion of $\mathcal{O}$ at $\mathfrak{p}$ and let $M = \prod_{\mathfrak{p} \in S}(1 + \mathfrak{p}\mathcal{O}_\mathfrak{p})$. Let T denote the torsion submodule of M. Note that there is a natural G-action on both M and T (see e.g. [Neu99] p. 374).

Lemma 3.14. *If*

$$\dim_{\overline{\mathbf{F}}_p}(M/T \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi = 1, \tag{3.5}$$

then

$$\dim_{\overline{\mathbf{F}}_p} V^\chi \leq 1.$$

Proof. Let $\mathcal{E}$ denote the image of the global units of F in $\prod_{\mathfrak{p}\in S} \mathcal{O}_\mathfrak{p}^\times$ and $\overline{\mathcal{E}}$ the topological closure of $\mathcal{E}$ in this product. Using the assumption that $C_F^\chi = 0$ it can easily be deduced from Corollary 13.6 in [Was97] that $M/\overline{\mathcal{E}}_1 \cong \mathrm{Gal}(K/H)$, where K denotes the maximal abelian pro-p extension of F unramified outside of S and $\overline{\mathcal{E}}_1 = \overline{\mathcal{E}} \cap M$.

Note that $K \subset L$ and the resulting canonical surjection $H = \mathrm{Gal}(L/F) \to \mathrm{Gal}(K/F)$ descends to an isomorphism $H \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p \xrightarrow{\sim} \mathrm{Gal}(K/F) \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$. However, one also clearly has $H \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p \cong V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$.

Hence we get an exact sequence

$$M/\overline{\mathcal{E}}_1 \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p \to V \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p \to \mathrm{Cl}(F) \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p \to 0.$$

Using the assumption that $C_F^\chi = 0$, we see that $(M/\overline{\mathcal{E}}_1 \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi$ surjects onto V^χ, which implies that V^χ is a quotient of $(M \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi$.

To prove that $\dim_{\overline{\mathbf{F}}_p} V^\chi \leq 1$ it suffices to show that $\dim_{\overline{\mathbf{F}}_p}(M/\overline{\mathcal{E}}_1 \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi \leq 1$. This would clearly follow if we had that $\dim_{\overline{\mathbf{F}}_p}(M \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi \leq 1$, but instead we will show that it also follows under the weaker assumption (3.5). Clearly, if $e < p-1$, then $T = 0$, hence (3.5) implies that $\dim_{\overline{\mathbf{F}}_p}(M \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi = 1$. In the case when $\tilde{\chi}_N(p) \neq 1$, we show that χ does not occur in $T \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$, hence again (3.5) gives us $\dim_{\overline{\mathbf{F}}_p}(M \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi = 1$. In the case when $\chi = \omega^s$, it can occur in T, but we will then show that $T \subset \overline{\mathcal{E}}_1$, so proving $\dim_{\overline{\mathbf{F}}_p}(M/T \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi = 1$ again suffices for demonstrating that $\dim_{\overline{\mathbf{F}}_p}(M/\overline{\mathcal{E}}_1 \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\chi \leq 1$.

Note that M/T is a free $\mathbf{Z}_p$-module of rank $|G|$. We now analyze the action of G on T. For $\mathfrak{p} \in S$ we write $G_\mathfrak{p}$ for the stabilizer of $\mathfrak{p}$ in G. As $\chi = \chi_p\chi_N$ and χ_N is unramified at p, we get that $e \leq p-1$. In fact, for the splitting field $\mathbf{Q}(\chi_p)$ of χ_p we have $\mathbf{Q}(\chi_p) \subset \mathbf{Q}(\mu_p)$, so $e = p-1$ if and only if $\mathbf{Q}(\chi_p) = \mathbf{Q}(\mu_p)$. Hence we conclude that $T \neq 0$ if and only if $\mathbf{Q}(\chi_p) = \mathbf{Q}(\mu_p)$. So, assume $\mathbf{Q}(\chi_p) = \mathbf{Q}(\mu_p)$. Using the group structure of $1 + \mathfrak{p}\mathcal{O}_\mathfrak{p}$ (cf. e.g., Proposition (5.7) on page 140 of [Neu99]) we conclude

$$T \cong (\mu_{p^a})^{\#S} \text{ as } G\text{-modules for some } a \in \mathbf{Z}_{>0}$$

and $T \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p = (\mu_p)^{\#S} \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$. As $\mu_p \not\subset \mathbf{Q}_p$, the group $G_\mathfrak{p}$ acts on the corresponding copy of μ_p via ω.

Suppose ξ is a character occurring in the G-action on $T \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$. Then $\xi|_{G_\mathfrak{p}} = \omega$ and $G/G_\mathfrak{p}$ acts faithfully by permuting the elements of S, so $\xi \bmod G_\mathfrak{p}$ (which we denote by ψ) is of order $\#S$ (because $G/G_\mathfrak{p}$ is cyclic of order $\#S$).

Consider the case that χ occurs in $T \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$, i.e., $\chi = \xi$. We have $\chi = \chi_p \cdot \chi_N : G = \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q}) \times \mathrm{Gal}(\mathbf{Q}(\chi_N)/\mathbf{Q}) \to \overline{\mathbf{F}}_p^\times$. So, we can treat $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ as a subgroup of G and then $\chi|_{\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})} = \chi_p$. As the elements of $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ stabilize $\mathfrak{p}$ we see that $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})) \subset G_\mathfrak{p}$. Using the injectivity of χ one can easily see that this inclusion has to be an equality. As $\mathrm{Gal}(\mathbf{Q}(\chi_N)/\mathbf{Q})$ is the quotient of G by $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ we thus get that $\psi = \chi_N$.

Since the order of ψ is $\#S$, we get that $\mathbf{Q}(\chi_N)/\mathbf{Q}$ has degree $\#S$. As p splits into $\#S$ primes in $\mathbf{Q}(\chi_N)$ we conclude that p splits completely in $\mathbf{Q}(\chi_N)$ which is equivalent to saying that $\tilde{\chi}_N(p) = 1$. This shows that if χ appears in $T \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p$ then $\tilde{\chi}_N(p) = 1$.

If $\chi = \omega^s$ there is only one prime above p and either $e < p-1$ (which implies $T = 0$) or $T = \mu_p$. However, in the latter case also $\mathcal{E}$ (and hence $\overline{\mathcal{E}}_1$) contains a copy of μ_p, so in the quotient $M/\overline{\mathcal{E}}_1 \otimes \overline{\mathbf{F}}_p$, the torsion part T gets annihilated. □

We are now ready to complete the proof of Theorem 3.10. By Lemmas 3.12, 3.13 and 3.14 it now suffices to show (3.5). Note that to decompose $(M/T) \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$ it is enough to decompose $\prod_{\mathfrak{p}\in S} \mathfrak{p}\mathcal{O}_\mathfrak{p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$, since $(1+\mathfrak{p}\mathcal{O}_\mathfrak{p})/(\text{torsion}) \cong \mathfrak{p}\mathcal{O}_\mathfrak{p}$ as $\mathbf{Z}_p[G_\mathfrak{p}]$-modules. One has $\dim_{\overline{\mathbf{F}}_p} \mathfrak{p}\mathcal{O}_\mathfrak{p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p = t$, where $t = |G_\mathfrak{p}|$ and $\dim_{\overline{\mathbf{F}}_p} \prod_{\mathfrak{p}\in S} \mathfrak{p}\mathcal{O}_\mathfrak{p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p = r$, where $r = |G|$.

As $|\operatorname{Hom}(G, \overline{\mathbf{F}}_p^\times)| = r$, it suffices to show that $(M/T \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p)^\varphi \neq 0$ for all $\varphi \in \operatorname{Hom}(G, \overline{\mathbf{F}}_p^\times)$. Note that G being isomorphic to the image of χ, hence to a subgroup of $\mathbf{F}^\times$, is a cyclic group. If we denote by ζ a primitive rth root of unity in $\overline{\mathbf{F}}_p$ then the characters $g \mapsto \zeta^i$ for $i = 0, 1, \dots, r-1$ exhaust all the characters in $\operatorname{Hom}(G, \overline{\mathbf{F}}_p^\times)$.

Let $\alpha \in \mathfrak{p}\mathcal{O}_\mathfrak{p}$ be such that $\{g^i\alpha \mid i = 0, 1, ..., t-1\}$ is linearly independent where g is a generator of $G_\mathfrak{p}$. This is possible because the extension $F/\mathbf{Q}$ has degree prime to p, so is at most tamely ramified at p, hence the ideal $\mathfrak{p}$ possesses a normal integral basis - cf. Theorem 1 in [Ull70].

We now claim that the set $\{\gamma\alpha \mid \gamma \in G\}$ is a linearly independent set in $\prod_{\mathfrak{p}\in S} \mathfrak{p} \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$. Indeed, if $x \in \mathfrak{p}'\mathcal{O}_{\mathfrak{p}'}$ for some prime $\mathfrak{p}'$ of F over p, and $\delta \in G$ is an element such that $\delta x \in \mathfrak{p}\mathcal{O}_\mathfrak{p}$, then by the above we can write

$$\delta x = a_0\alpha + a_1 g\alpha + \cdots + a_{t-1} g^{t-1}\alpha \quad \text{for some } a_0, a_1, \dots, a_{t-1} \in \mathbf{Z}_p.$$

Hence we conclude that

$$x = a_0\delta^{-1}\alpha + a_1\delta^{-1}g\alpha + \cdots + a_{t-1}\delta^{-1}g^{t-1}.$$

This shows that there exist $\gamma_1, \gamma_2, \dots, \gamma_t \in G$ such that $\{\gamma_1\alpha, \dots, \gamma_t\alpha\}$ is a $\overline{\mathbf{F}}_p$-basis of $\mathfrak{p}' \otimes_{\mathbf{Z}_p} \overline{\mathbf{F}}_p$, hence our claim is proved.

With this we fix a generator g of G and observe that for each $i \in \{0, 1, \dots, r-1\}$ the vector

$$v_i = \alpha + \zeta^{-i} g\alpha + \zeta^{-2i} g^2\alpha + \cdots + \zeta^{-(r-1)i} g^{r-1}\alpha$$

is an eigenvector for the action of G on which G acts via the character $g \mapsto \zeta^i$. □

4. $R = T$ Theorem in the ordinary case

Our methods for proving an $R = T$ theorem in the split and the ordinary case are different. The ordinary case will be treated in this section using Wiles' Theorem on the Λ-adic Eisenstein congruences and T will be a Hecke algebra acting on non-classical weight one cusp forms. In the split case (treated in section 5) we will construct Eisenstein congruences with classical weight one cusp forms directly (i.e., without using Wiles' result) - see also Remark 4.8.

4.1. **Λ-adic Eisenstein congruences.** Let $\theta : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathbf{C}^\times$ be a primitive even Dirichlet character. Let $\mathcal{O}' \subset \overline{\mathbf{Q}}_p$ be the valuation ring of any finite extension of $\mathbf{Q}_p$ containing the values of θ. Put $\Lambda = \mathcal{O}'[[T]]$ and let $\mathfrak{X} = \{(k, \zeta) \mid k \in \mathbf{Z}, k \geq 1, \zeta \in \mu_{p^\infty}\}$. For every $(k, \zeta) \in \mathfrak{X}$ we have an $\mathcal{O}'$-algebra homomorphism $\nu_{k,\zeta} : \Lambda \to \mathcal{O}'[\zeta]$ induced by $\nu_{k,\zeta}(1 + T) = \zeta u^{k-2}$ where $u = \epsilon(\gamma)$ for γ a topological generator of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ and $\epsilon : \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \xrightarrow{\sim} 1 + p\mathbf{Z}_p$ the p-adic cyclotomic character. Here $\mathbf{Q}_\infty$ is the unique $\mathbf{Z}_p$-extension of $\mathbf{Q}$. Note that under our assumption on the conductor of θ we have $\mathbf{Q}(\theta) \cap \mathbf{Q}_\infty = \mathbf{Q}$.

We fix an algebraic closure $\overline{F}_\Lambda$ of F_Λ, the fraction field of Λ, and regard all finite extensions of F_Λ as embedded in that algebraic closure. For $L \subset \overline{F}_\Lambda$ a finite extension of F_Λ, and $\mathcal{O}_L$ the integral closure of Λ in L put $\mathfrak{X}_L = \{\varphi : \mathcal{O}_L \to \overline{\mathbf{Q}}_p$ extending some $\nu_{k,\zeta}\}$.

We define an $\mathcal{O}_L$-adic modular form of tame level N and character θ to be a collection of Fourier coefficients $c(n, \mathcal{F})$, $n \in \mathbf{Z}_{\geq 0}$ with the property that for all but finitely many $\varphi \in \mathfrak{X}_L$ extending $\nu_{k,\zeta}$ with $(k, \zeta) \in \mathfrak{X}$ and ζ of exact order p^{r-1} for $r \geq 1$ there is an element $\mathcal{F}_\varphi \in M_k(Np^r, \theta\tilde{\omega}^{2-k}\chi_\zeta, \varphi(\mathcal{O}_L))$ whose nth Fourier coefficient equals $\nu_{k,\zeta}(c(n, \mathcal{F}))$. Here

$$\chi_\zeta : (\mathbf{Z}/p^r\mathbf{Z})^\times \to \mu_{p^{r-1}} \subset \mathbf{C}^\times$$

is the unique Dirichlet character mapping the image of $1 + p$ in $(\mathbf{Z}/p^r\mathbf{Z})^\times$ to ζ. A form $\mathcal{F}$ is called a *cusp form* if all the $\mathcal{F}_\varphi$ are cusp forms. We denote by $\mathcal{M}_{\mathcal{O}_L}(N, \theta)$ the $\mathcal{O}_L$-torsion free module consisting of $\mathcal{O}_L$-adic modular forms having character θ and we set $\mathcal{M}_L(N, \theta) = \mathcal{M}_{\mathcal{O}_L}(N, \theta) \otimes_{\mathcal{O}_L} L$ and similarly for $\mathcal{S}_L(N, \theta)$ (cf. [Wil88], p. 545). This module has a natural action of Hecke operators and we denote by $\mathcal{M}^0_{\mathcal{O}_L}(N, \theta)$ the submodule $e\mathcal{M}_{\mathcal{O}_L}(N, \theta)$ cut out by applying the Hida ordinary projector e. The corresponding subspace of cusp forms will be denoted by $\mathcal{S}^0_{\mathcal{O}_L}(N, \theta)$.

Let $\mathbf{T}$ denote the Λ-algebra generated by all the Hecke operators T_n, $n \in \mathbf{Z}_+$ acting on $\mathcal{S}^0_\Lambda(N, \theta)$. By a result of Hida (Theorem 3.1 in [Hid86]) $\mathbf{T}$ is finitely generated and free as a Λ-module. Newforms are then defined in an obvious way, see [Wil88] section 1.5.

Fix $L \subset \overline{F}_\Lambda$ to be a finite extension of F_Λ over which all newforms in $\mathcal{S}^0_\Lambda(N, \theta) \otimes_\Lambda \overline{F}_\Lambda$ are defined. Let $\mathcal{N}'$ be the set of all newforms in $\mathcal{S}^0_L(N, \theta)$ and fix a complete set $\mathcal{S}' \subset \mathcal{N}'$ of representatives of the Galois conjugacy classes (over F_Λ) of all the elements of $\mathcal{N}'$. For $\mathcal{F} \in \mathcal{N}'$, if we denote by $L_\mathcal{F}$ the extension of F_Λ generated by the Fourier coefficients of $\mathcal{F} \in \mathcal{S}^0_\Lambda(N, \theta) \otimes_\Lambda \overline{F}_\Lambda$ then - using the assumption that θ is primitive - $\mathbf{T}$ can be naturally viewed (by mapping an operator t to the tuple $(c(1, t\mathcal{F}))_\mathcal{F}$) as a subring of the F_Λ-algebra $\prod_{\mathcal{F} \in \mathcal{S}'} L_\mathcal{F}$ and one has $\mathbf{T} \otimes_\Lambda F_\Lambda = \prod_{\mathcal{F} \in \mathcal{S}'} L_\mathcal{F}$ (cf. [Wil90], eq. (4.1)). In fact, we have $\mathbf{T} \subset \prod_{\mathcal{F} \in \mathcal{S}'} \mathcal{O}_{L_\mathcal{F}}$ as $c(1, t\mathcal{F})$ are integral over Λ, see e.g. [Wil88] p. 546.

Definition 4.1. For each prime $\ell \neq p$ put $c_\ell := 1 + \theta(\ell)\ell(1+T)^{a_\ell}$, where $a_\ell \in \mathbf{Z}_p$ is defined by $\ell = \tilde{\omega}(\ell)(1+p)^{a_\ell}$. Put $c_p := 1$.

These are the Hecke eigenvalues of a Λ-adic Eisenstein series with constant term given by $L_p(s, \theta\tilde{\omega}^2)/2$. Here the Kubota-Leopoldt p-adic L-function $L_p(s,\theta)$ is an analytic function for $s \in \mathbf{Z}_p - \{1\}$ (and even at $s=1$ if $\theta \neq 1$), which satisfies the interpolation property

$$L_p(1-k,\theta) = (1-\theta\tilde{\omega}^{-k}(p)p^{k-1})L(1-k,\theta\tilde{\omega}^{-k}) \tag{4.1}$$

for $k \in \mathbf{Z}_{\geq 1}$. Iwasawa showed (see e.g., [Wil90], equation (1.3)) that there exists a unique power series $G_\theta(T) \in \Lambda$ such that

$$L_p(1-s,\theta) = G_\theta(u^s - 1).$$

Note that in general there is a denominator H_θ but for us it is identically 1 since θ is of type S (in the sense that $\mathbf{Q}(\theta) \cap \mathbf{Q}_\infty = \mathbf{Q}$).

Put $\hat{G}_\theta(T) = G_{\theta\tilde{\omega}^2}(u^2(1+T)-1)$ and $\hat{G}^0_\theta = \pi^{-\mu}\prod_{\zeta\in\mu_{p^\infty}}(1+T-\zeta u^{-1})^{-s_\zeta}\hat{G}_\theta(T)$, where π is a uniformizer of $\mathbf{Z}_p[\theta]$. Here π^μ (respectively $(1+T-\zeta u^{-1})^{s_\zeta}$) is the highest power of π (respectively $(1+T-\zeta u^{-1})$) common to all coefficients of $\hat{G}_\theta$.

Definition 4.2. Define the Eisenstein ideal $J \subset \mathbf{T}$ to be the ideal generated by $T_\ell - c_\ell$ for all primes ℓ and by $\hat{G}^0_\theta(T)$.

We have the following result due to Wiles (who, in fact, proves this for totally real fields, generalizing earlier joint work with Mazur for $\mathbf{Q}$).

Theorem 4.3 (Wiles, [Wil90], Theorem 4.1)**.** *If $\theta \neq \tilde{\omega}^{-2}$ then one has*

$$\mathbf{T}/J \cong \Lambda/\hat{G}^0_\theta(T).$$

Let $\theta = \tilde{\chi}\tilde{\omega}^{-1}$ and put $\mathcal{O}' = \mathcal{O}$. (Note that the values of ω are already contained in $\mathbf{Z}_p$.)

Remark 4.4. Note that the theorem rules out $\chi = \omega^{-1}$ while Lemma 4.5 below rules out $\chi = \omega$. However, in both cases $C_F^{\chi^{-1}} = 0$ by [Was97] Proposition 6.16 and Theorem 6.17, so these cases are not relevant for our deformation problem as we assume in section 3.1 that $C_F^{\chi^{-1}} \neq 0$. Our assumption that $\tilde{\chi}_N$ is primitive is needed for Theorem 4.3. Note that our assumption that $\tilde{\chi}$ is of order prime to p implies that $\tilde{\chi}$ is of type S as considered in [Wil90].

Lemma 4.5. *Let $\theta = \tilde{\chi}\tilde{\omega}^{-1}$. Assume $\tilde{\chi}\tilde{\omega}^{-1}(p) \neq 1$. Then one has $\mu = s_\zeta = 0$ for all $\zeta \in \mu_{p^\infty}$ in $\hat{G}^0_\theta$.*

Proof. The μ-invariant is zero by [FW79]. Let $\zeta \in \mu_{p^\infty}$. We need to show that $\hat{G}_\theta$ does not have a zero at $T = \zeta u^{-1} - 1$. We calculate $\hat{G}_\theta(\zeta u^{-1}-1) = G_{\theta\tilde{\omega}^2}(u^2\zeta u^{-1} - 1)$. By (1.4) in [Wil90] and (4.1) this equals $L_p(0, \tilde{\chi}\tilde{\omega}\chi_\zeta) = L(0,\tilde{\chi}\chi_\zeta)(1-\tilde{\chi}\tilde{\omega}^{-1}\chi_\zeta(p))$.

Since $\tilde{\chi}\chi_\zeta$ is odd (as $\chi_\zeta(-1) = +1$ since -1 is not congruent to 1 modulo p) we have $L(0,\tilde{\chi}\chi_\zeta) \neq 0$ by the class number formula. Since $\tilde{\chi}$ is of order prime to p the Euler factor could only vanish for $\zeta = 1$ and if $(\tilde{\chi}\tilde{\omega}^{-1})(p) = 1$. ☐

Theorem 4.3 and Lemma 4.5 imply the following corollary (note that by Remark 4.4 we have $\chi \neq \omega^{\pm 1}$).

Corollary 4.6. *One has*

$$\mathbf{T}/J \cong \Lambda/\hat{G}_{\tilde{\chi}\tilde{\omega}^{-1}}(T).$$

Remark 4.7. We thank one of the referees for informing us that this Corollary was also proven in [Oht05] (3.4.2) and [Laf15] Section 3.3.2.

4.2. **The weight one specialisations.** We set $\mathbf{T}_k := \mathbf{T}/(\ker \nu_{k,1})\mathbf{T}$. It is a well-known result of Hida that for $k \geq 2$ the algebra $\mathbf{T}_k$ coincides with the Hecke algebra acting on the space of classical cusp forms $S_k^0(Np, \tilde{\chi})$, and that all the specialisations are classical, but this is not the case in weight 1. We write J_k for the image of J in $\mathbf{T}_k$ under the map $\mathbf{T} \to \mathbf{T}/(\ker \nu_{k,1})\mathbf{T}$ which we will also denote by $\nu_{k,1}$.

Remark 4.8. A classical specialisation in weight 1 corresponds to Galois representations with finite image, which can only happen if $\chi = \chi_{F/\mathbf{Q}}$ for an imaginary quadratic field F, as explained in Remark 3.4. In the ordinary case such characters are excluded by Remark 3.11(i). In section 5 we prove that split deformations of ρ_0 with $\chi = \chi_{F/\mathbf{Q}}$ are modular by classical CM-forms using a different method.

The following lemma will allow us to later relate J_1 to the reducibility ideal.

Lemma 4.9. *The ideal J_k is generated by the set*

$$S = \{T_\ell - 1 - \tilde{\chi}(\ell)\tilde{\omega}^{1-k}(\ell)\ell^{k-1} \mid \ell \neq p\} \cup \{T_p - 1\}.$$

Proof. If A is a set of generators for J as an ideal of $\mathbf{T}$ (i.e., as a $\mathbf{T}$-module), then the images under $\mathbf{T} \to \mathbf{T}_k$ of the elements of A generate J_k as a $\mathbf{T}_k$-module. We have $A = \{T_\ell - c_\ell \mid \ell \in \operatorname{Spec} \mathbf{Z}\}$ with $c_p = 1$ and $c_\ell = 1 + \tilde{\omega}^{-1}\tilde{\chi}(\ell)\ell(1+T)^{a_\ell}$ with $\ell = \tilde{\omega}(\ell)(1+p)^{a_\ell}$ if $\ell \neq p$ (see Definition 4.1).

The lemma follows as we have

$$\nu_{k,1}((1+T)^{a_\ell}) = (1+p)^{(k-2)a_\ell} = \ell^{k-2}\tilde{\omega}^{2-k}(\ell).$$

□

Corollary 4.10. *We have a surjection*

$$\mathbf{T}_1/J_1 \twoheadrightarrow \mathcal{O}/L(0, \tilde{\chi}).$$

Proof. We note that for $k = 1$ and $\zeta = 1$ we get

$$\begin{aligned} \nu_{k,\zeta} \circ \hat{G}_\theta(T) &= \nu_{k,\zeta} \circ \hat{G}_{\tilde{\chi}\tilde{\omega}^{-1}}(T) = \nu_{k,\zeta} \circ G_{\tilde{\chi}\tilde{\omega}}(u^2(1+T) - 1) \\ &= G_{\tilde{\chi}\tilde{\omega}}(u-1) = L_p(0, \tilde{\chi}\tilde{\omega}) = (1 - \tilde{\chi}(p))L(0, \tilde{\chi}). \end{aligned} \tag{4.2}$$

The Euler factor $(1 - \tilde{\chi}(p))$ is a p-unit by our assumption $\overline{\chi}|_{D_p} \neq 1$.

We thus have the following commutative diagram whose vertical arrows are surjective and whose top row is exact by Corollary 4.6. In the top row Ψ is the inclusion map and Φ is the canonical surjection. The maps in the bottom row are defined in the following way: ψ is the natural injection, and $\phi(t) = \nu_{1,1}(\Phi(\tilde{t}))$, where $\tilde{t}$ is any lift of t to $\mathbf{T}$. This is well-defined as $\Phi(\ker \nu_{1,1}\mathbf{T}) = \ker \nu_{1,1} + \hat{G}_\theta\Lambda$ as Φ is a Λ-algebra map.

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r, "\Psi"] \arrow[d] & \mathbf{T} \arrow[r, "\Phi"] \arrow[d] & \Lambda/\hat{G}_\theta \arrow[r] \arrow[d, "\nu_{1,1}"] & 0 \\ 0 \arrow[r] & J_1 \arrow[r, "\psi"] & \mathbf{T}_1 \arrow[r, "\phi"] & \mathcal{O}/L_p(0, \tilde{\chi}\tilde{\omega}) \arrow[r] & 0 \end{tikzcd} \tag{4.3}$$

Note also that the bottom row is clearly exact except possibly at $\mathbf{T}_1$. We do not need exactness at $\mathbf{T}_1$, only that ϕ factors through $\mathbf{T}_1/J_1$, which follows from $\Phi \circ \Psi = 0$. □

4.3. **Modularity of reducible deformations.** For each Λ-adic newform $\mathcal{F}$ we have an associated Galois representation.

Theorem 4.11 (Hida, Wiles, Carayol). *Let $\mathcal{F} \in S^0_{\mathcal{O}_L}(N,\theta)$ be a newform. Then there exists a continuous irreducible odd Galois representation*

$$\rho_{\mathcal{F}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(L)$$

unramified outside Np such that

$$\mathrm{Tr}(\rho_{\mathcal{F}}(\mathrm{Frob}_\ell)) = c(\ell, \mathcal{F})$$

for all primes $\ell \nmid Np$ and

$$\det(\rho_{\mathcal{F}}(\mathrm{Frob}_\ell)) = \theta(\ell)\ell(1+T)^{a_\ell},$$

where $\ell = \tilde{\omega}(\ell)(1+p)^{a_\ell}$ for $a_\ell \in \mathbf{Z}_p$.

(1) *We have $\rho_{\mathcal{F}}|_{D_p} \cong \begin{bmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{bmatrix}$ with ϵ_2 unramified and $\epsilon_2(\mathrm{Frob}_p) = c(p, \mathcal{F})$.*

(2) *For $\ell \mid N$ we have $\rho_\lambda|_{D_\ell} = \begin{bmatrix} \psi & 0 \\ 0 & \delta_\ell \end{bmatrix}$ with $\psi|_{I_\ell} = \chi|_{I_\ell}$, δ_ℓ unramified and $\delta_\ell(\mathrm{Frob}_\ell) = c(\ell, \mathcal{F})$ since $\theta = \tilde{\chi}\tilde{\omega}^{-1}$ is assumed to be primitive of conductor Np.*

We prove a generalization of [BC06] Corollary 2 (who assume $\mathrm{Ext}^1{}_{\mathbf{F}[G_\Sigma]}(\overline{\chi}, 1) = \mathbf{F}$ instead of our $H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1})) \cong \mathbf{F}$ but do not consider ordinary representations):

Proposition 4.12. *Let A be a local reduced Noetherian Henselian ring. Consider $\rho : G_\Sigma \to \mathrm{GL}_2(K)$ for $K = \mathrm{Frac}(A) = \prod K_i$ such that $\mathrm{tr}\,\rho \subset A$, $\mathrm{tr}\,\rho \equiv 1 + \overline{\chi} \mod \mathfrak{m}_A$, $\rho_i : G_\Sigma \to \mathrm{GL}_2(K_i)$ is irreducible, and ordinary (in the sense that $\rho|_{D_p} \cong \begin{bmatrix} \psi_1 & * \\ & \psi_2 \end{bmatrix}$ with ψ_2 unramified and $\psi_2 \equiv 1 \mod \mathfrak{m}_A$). If $H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1})) \cong \mathbf{F}$ then there exists a lattice L in K^2 for which we have $\rho_L : G_\Sigma \to \mathrm{GL}_2(A)$ and*

$$\rho_L \mod \mathfrak{m}_A = \begin{bmatrix} 1 & * \\ 0 & \overline{\chi} \end{bmatrix} \ncong 1 \oplus \overline{\chi}$$

with $\rho_L \mod \mathfrak{m}_A|_{D_p} \cong 1 \oplus \overline{\chi}|_{D_p}$.

Proof. Following [BC06] for $g \in D_p$ we will call a basis $\{e_1, e_2\}$ of ρ *adapted to* g if $g.e_1 = \lambda_1 e_1$ and $ge_2 = \lambda_2 e_2$ for $\lambda_1, \lambda_2 \in A$ lifting 1 and $\overline{\chi}(g)$, respectively. As $\overline{\chi}|_{D_p} \neq 1$, such an adapted basis exists for some $g \in D_p$ and we fix it from now on and we assume that the matrix form of ρ is written with respect to this basis. Let B be the A-submodule of K generated by the top-right entries of $\rho(G_\Sigma)$.

By an argument as in [loc.cit.], it follows that there is a canonical embedding of $\mathrm{Hom}_A(B, \mathfrak{m})$ into $\mathrm{Ext}^1{}_{\mathbf{F}[G_\Sigma]}(\overline{\chi}, 1)$. We claim that the image of this embedding lies in $H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1})) \cong \mathbf{F}$. To see this it is enough to show that the top-right entry of $\rho(h)$ equals 0 for all $h \in I_p$.

Note that $\rho(g) = \begin{bmatrix} \psi_2(g) & 0 \\ 0 & \psi_1(g) \end{bmatrix}$ by adaptedness and ordinarity implies that there exists $M \in \mathrm{GL}_2(K)$ such that $M\rho(D_p)M^{-1}$ is lower-triangular. We can then find a lower-triangular $N \in \mathrm{GL}_2(K)$ such that $NM\rho(g)M^{-1}N^{-1} = \begin{bmatrix} \psi_2(g) & 0 \\ 0 & \psi_1(g) \end{bmatrix}$ and $NM\rho(D_p)M^{-1}N^{-1}$ is lower-triangular.

Since $NM\rho(g)M^{-1}N^{-1} = \rho(g)$ this forces NM to be diagonal, which shows that already with respect to the basis $\{e_1, e_2\}$ $\rho(D_p)$ is lower triangular.

We now conclude the proof as in [BC06]: By Nakayama's lemma B is a cyclic A-module. Since the image of B in K_i is non-zero for all i we can rescale the second basis vector to achieve $B = A$. □

Let $\mathcal{S} \subset \mathcal{S}'$ be the subset of $\mathcal{F} \in \mathcal{S}'$ whose Hecke eigenvalue at ℓ is congruent to c_ℓ modulo the maximal ideal $\mathfrak{m}_{\mathcal{O}_{L_\mathcal{F}}}$ of $\mathcal{O}_{L_\mathcal{F}}$ for all primes ℓ. Recall also that we assume that $C_F^{\chi^{-1}} \otimes_\mathcal{O} \mathbf{F}$ has dimension one. As $\#C_F^{\chi^{-1}} = \#\mathcal{O}/L(0,\chi)$ (see (3.1)) we conclude that $\mathrm{val}_\varpi(L(0,\chi)) > 0$. By Corollary 4.10 we get that $J_1 \neq \mathbf{T}_1$, so $J \neq \mathbf{T}$, hence $\mathcal{S}$ is not empty. For $\mathcal{F} \in \mathcal{S}$ the semi-simplification of the mod $\mathfrak{m}_{\mathcal{O}_{L_\mathcal{F}}}$ reduction $\overline{\rho}_\mathcal{F}$ of $\rho_\mathcal{F}$ has the form $1 \oplus \overline{\chi}$.

Applying Proposition 4.12 (together with the ordinarity result from Theorem 4.11 (1)) for $A = \mathcal{O}_{L_\mathcal{F}}$ and any $\rho_\mathcal{F} : G_\Sigma \to \mathrm{GL}_2(L_\mathcal{F})$ for $\mathcal{F} \in \mathcal{S}$, i.e. $\overline{\rho}_\mathcal{F}^{\mathrm{ss}} = 1 \oplus \overline{\chi}$, we obtain the existence of non-split residual extensions $\overline{\rho}_{\mathcal{F},\Lambda_\mathcal{F}}$, which are split when restricted to D_p. Hence it follows from Proposition 3.3 that for any $\mathcal{F}, \mathcal{F}'$ as above we can choose lattices $\Lambda_\mathcal{F}, \Lambda_{\mathcal{F}'}$ such that $\overline{\rho}_{\mathcal{F},\Lambda_\mathcal{F}} = \overline{\rho}_{\mathcal{F}',\Lambda_{\mathcal{F}'}}$. For each $\mathcal{F} \in \mathcal{S}$ we make such a choice and set

$$\rho_0 = \overline{\rho}_{\mathcal{F},\Lambda_\mathcal{F}}.$$

For a construction of ρ_0 that makes use of irreducibility of weight one specializations of $\mathcal{F}$ rather than Proposition 4.12, see Remark 4.30.

Let R^{ord} be the universal deformation ring of deformations $\rho : G_\Sigma \to \mathrm{GL}_2(A)$ of ρ_0 for A an object in $\mathrm{CNL}(\mathcal{O})$, the category of local complete Noetherian $\mathcal{O}$-algebras with residue field $\mathbf{F}$, such that:

(ii) $\rho|_{D_p} \cong \begin{bmatrix} \psi_1 & * \\ & \psi_2 \end{bmatrix}$ with ψ_2 unramified and $\psi_2 \equiv 1 \mod \mathfrak{m}_A$ (ordinary and p-distinguished)

(iii) If $\ell \in \Sigma$ such that $\ell \equiv 1 \pmod p$ then $\rho|_{I_\ell} = 1 \oplus \chi$.

We denote the corresponding universal ordinary deformation by ρ^{ord}.

Lemma 4.13 (Kisin)**.** *The ring R^{ord} is topologically generated by the set*

$$\{\mathrm{tr}\,\rho^{\mathrm{ord}}(\mathrm{Frob}_\ell) | \ell \notin \Sigma\}.$$

Proof. [Kis09] Corollary 1.4.4 (2) proves this under the assumption that $\mathrm{Ext}^1{}_{\mathbf{F}[G_\Sigma]}(\overline{\chi}, 1) = \mathbf{F}$. Kisin shows that the universal pseudodeformation ring R^{ps} of pseudodeformations of $1 + \overline{\chi}$ surjects onto the universal deformation ring R_{ρ_0} (which in turn surjects onto R^{ord}). This is based on Lemma 1.4.3 in [loc. cit.], in which he proves that the tangent space of the universal pseudodeformation ring equals the tangent space of R_{ρ_0}.

We work instead under the assumption that $C_F^{\chi^{-1}}$ is cyclic. As explained in the proof of Proposition 3.2 this corresponds to $H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1})) \cong \mathbf{F}$. We claim that this implies that R^{ps} surjects onto R^{ord}. In the proof of Lemma 1.4.3 Kisin considers the b-entry (i.e. the top right entry) of infinitesimal deformations of $\rho_0 = \begin{bmatrix} 1 & * \\ 0 & \overline{\chi} \end{bmatrix}$. As in the proof of Proposition 3.2 we know that under our assumption this gives rise to elements in $H^1_{\{p\}}(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1}))$. For ordinary deformations we further know that $b|_{D_p} = 0$ since $\overline{\chi}|_{D_p} \neq 1$, so b lies, in fact, in $H^1_\emptyset(\mathbf{Q}, \mathbf{F}(\overline{\chi}^{-1})) \cong \mathbf{F}$. With this single change the proof of Kisin's Lemma 1.4.3 then carries over. □

Remark 4.14. Proposition 4.13 gives an alternative proof of the fact that R (which is a quotient of R^{ord}) is generated by traces (cf. Proposition 3.8).

Let $\mathfrak{M}$ be the maximal ideal of $\mathbf{T}$ containing J. We write $\mathbf{T}_{\mathfrak{M}}$ for the localisation of $\mathbf{T}$ at $\mathfrak{M}$. By a standard argument one can view $\mathbf{T}_{\mathfrak{M}}$ as a direct summand of $\mathbf{T}$ and so $\mathbf{T}_{\mathfrak{M}}$ can be naturally viewed as a subring of $\prod_{\mathcal{F}\in\mathcal{S}} L_{\mathcal{F}}$, where $\mathcal{S}\subset\mathcal{S}'$ is defined as above.

Recall that $\mathbf{T}$ and hence also $\mathbf{T}_{\mathfrak{M}}$ is a finitely generated Λ-module. This implies that $\mathbf{T}$ is a semi-local ring hence it is the direct product of its localisations. As $\mathbf{T}$ is also a free Λ-module we get that $\mathbf{T}_{\mathfrak{M}}$, being a direct summand of $\mathbf{T}$, is a projective, and so also flat, Λ-module. Since finite flat modules over local rings are free, we conclude that $\mathbf{T}_{\mathfrak{M}}$ is free over Λ.

Proposition 4.15. *We have an $\mathcal{O}$-algebra surjection $R^{\mathrm{ord}}\to\mathbf{T}_{\mathfrak{M}}$ given by* $\mathrm{tr}\,\rho^{\mathrm{ord}}(\mathrm{Frob}_\ell)\mapsto T_\ell$ *for all $\ell\notin\Sigma$.*

Proof. By Proposition 4.12 we know that for each $\rho_{\mathcal{F}}: G_\Sigma\to\mathrm{GL}_2(L_{\mathcal{F}})$ for $\mathcal{F}\in\mathcal{S}$ there exists a lattice $\Lambda_{\mathcal{F}}\subset L^2_{\mathcal{F}}$ such that $\rho_{\mathcal{F},\Lambda_{\mathcal{F}}}: G_\Sigma\to\mathrm{GL}_2(\mathcal{O}_{L_{\mathcal{F}}})$ is a deformation of ρ_0.

Theorem 4.11(1) tells us that $\rho_{\mathcal{F}}$ is ordinary, i.e. satisfies deformation condition (ii) (recall that $c(p,\mathcal{F})\equiv 1 \mod \mathfrak{m}_{\mathcal{O}_{\mathcal{F}}}$).

Since N is the conductor of $\tilde{\chi}$ and the tame level of $\mathcal{F}$ we see that condition (iii) of our deformation conditions is satisfied by Theorem 4.11(ii). If $\ell\in\Sigma$, but $\ell\nmid N$, then $\rho_{\mathcal{F},\Lambda}$ is unramified at ℓ. Hence $\rho_{\mathcal{F},\Lambda}$ is a deformation of ρ_0.

Due to the universality of R^{ord} we deduce the existence of a map

$$\Phi: R^{\mathrm{ord}}\to\prod_{\mathcal{F}\in\mathcal{S}}\mathcal{O}_{L_{\mathcal{F}}}.$$

We claim that $\mathrm{im}(\Phi)\supset\mathbf{T}_{\mathfrak{M}}$. Clearly all operators T_ℓ for primes $\ell\nmid Np$ are in the image of Φ. For T_p and T_ℓ with $\ell\mid N$ we adapt an argument from the proof of [WWE21] Proposition A.2.3 (see also [Wil95] Remark 2.11 and page 507): Since we assume that $\overline{\chi}|_{D_p}\neq 1$ there exists $\sigma\in D_p$ lifting $\mathrm{Frob}_p\in D_p/I_p\cong G_{\mathbf{F}_p}$ such that $\overline{\chi}(\sigma)\neq 1$. It is clear that it can be done if $\overline{\chi}|_{I_p}=1$. Otherwise take σ', any lift of Frobenius. If it happens to satisfy $\overline{\chi}(\sigma')=1$, multiply σ' by an element in inertia for which $\overline{\chi}$ is non-trivial.

For $\ell\mid N$ we argue similarly (using that we assume $\overline{\chi}|_{I_\ell}\neq 1$) to get a lift σ of Frob_ℓ with $\overline{\chi}(\sigma)\neq 1$.

In both cases we know that the characteristic polynomial for $\rho^{\mathrm{ord}}(\sigma)$ reduces modulo $\mathfrak{m}_R$ to

$$(x-1)(x-\overline{\chi}(\sigma)).$$

Let $U\in R^{\mathrm{ord}}$ be the root of the characteristic polynomial of $\rho^{\mathrm{univ}}(\sigma)$ such that $U\equiv 1 \mod \mathfrak{m}_{R^{\mathrm{ord}}}$ (which exists and is unique by Hensel's lemma). We claim that $\Phi(U)=T_p$ (or T_ℓ if σ is a lift of Frob_ℓ for $\ell\mid N$). It suffices to check for each $\mathcal{F}\in\mathcal{S}$ that $\Phi(U)$ maps to $c(p,\mathcal{F})$ (respectively $c(\ell,\mathcal{F})$) in $\mathcal{O}_{L_{\mathcal{F}}}$. Since $\rho_{\mathcal{F}}$ is a deformation of ρ_0 we know that U maps to a root of the characteristic polynomial of $\rho_{\mathcal{F}}(\sigma)$. If σ is a lift of Frob_ℓ for $\ell\mid N$ we know by Theorem 4.11(2) that this characteristic polynomial equals

$$(x-c(\ell,\mathcal{F}))(x-\psi(\sigma))$$

with $\psi(\sigma) \equiv \overline{\chi}(\sigma) \mod \mathfrak{m}_{\mathcal{O}_{L_\mathcal{F}}}$, so U must map to $c(\ell, \mathcal{F}) \equiv c_\ell \equiv 1 \pmod{\mathfrak{m}_{\mathcal{O}_{L_\mathcal{F}}}}$ - note that the second congruence holds as χ is ramified at $\ell \mid N$. If σ is a lift of Frob_p we argue similarly with Theorem 4.11(1).

The arguments above prove that $\mathrm{im}(\Phi) \supset \mathbf{T}_\mathfrak{M}$. On the other hand, Lemma 4.13 implies that this image is contained in $\mathbf{T}_\mathfrak{M}$, so we conclude that it equals $\mathbf{T}_\mathfrak{M}$. □

Let $\mathfrak{m}$ be the maximal ideal of $\mathbf{T}_1$ containing J_1. Write $\mathbf{T}_{1,\mathfrak{m}}$ for the localization of $\mathbf{T}_1$ at $\mathfrak{m}$. Note that $\mathbf{T}_{1,\mathfrak{m}} = \mathbf{T}_\mathfrak{M}/(\ker \nu_{1,1}\mathbf{T}_\mathfrak{M})$.

Corollary 4.16. *There exists a surjective $\mathcal{O}$-algebra map $\Phi : R \to \mathbf{T}_{1,\mathfrak{m}}$ given by $\mathrm{tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_\ell) \mapsto T_\ell + (\ker \nu_{1,1})\mathbf{T}_\mathfrak{M}$ for all $\ell \notin \Sigma$ which induces an isomorphism $R/I \xrightarrow{\sim} \mathbf{T}_{1,\mathfrak{m}}/J_{1,\mathfrak{m}}$.*

Proof. We compose $R^{\mathrm{ord}} \to \mathbf{T}_\mathfrak{M}$ with the surjection to $\mathbf{T}_{1,\mathfrak{m}}$. This gives us a deformation of ρ_0 to $\mathbf{T}_{1,\mathfrak{m}}$ satisfying all our deformation conditions (i)-(iii) in section 3.3 (for the determinant condition we note that $\det \rho_\mathcal{F}$ as in Theorem 4.11 specializes to χ by a calculation similar to the one at the end of the proof of Lemma 4.9). By the universality of R this gives us that the map $R^{\mathrm{ord}} \to \mathbf{T}_{1,\mathfrak{m}}$ (which we will call Φ) factors through the surjection $R^{\mathrm{ord}} \to R$.

Composing Φ with ρ^{univ} we get a representation into $\mathrm{GL}_2(\mathbf{T}_{1,\mathfrak{m}})$ whose trace is reducible modulo $J_{1,\mathfrak{m}}$. So we get $\Phi(I) \subseteq J_{1,\mathfrak{m}}$. Hence using Corollary 4.10 we obtain a surjection

$$R/I \twoheadrightarrow \mathbf{T}_{1,\mathfrak{m}}/J_{1,\mathfrak{m}} \cong \mathbf{T}_1/J_1 \twoheadrightarrow \mathcal{O}/L(0,\chi).$$

By Proposition 3.9 we get that $\#R/I \leq \#C_F^{\chi^{-1}}$. The Corollary now follows from the fact that $\#C_F^{\chi^{-1}} = \#\mathcal{O}/L(0,\chi)$ (see (3.1)). □

4.4. **The main result.** We keep the assumptions of section 3.1.

Theorem 4.17. *Assume $\dim_\mathbf{F} C_F^{\chi^{-1}} \otimes_\mathcal{O} \mathbf{F} = 1$ and $C_F^\chi = 0$. Assume further that at least one of the following conditions is satisfied:*

(i) *$e < p-1$ where e is the ramification index of p in F or*
(ii) *$\chi = \omega^s$ for some integer s or*
(iii) *$\chi_N(p) \neq 1$.*

Then the $\mathcal{O}$-algebra map $\Phi : R \to \mathbf{T}_{1,\mathfrak{m}}$ given by $\mathrm{tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_\ell) \mapsto T_\ell + (\ker \nu_{1,1})\mathbf{T}_\mathfrak{M}$ for all $\ell \notin \Sigma$ is an isomorphism. Here R is the ordinary universal deformation ring of ρ_0 defined in section 3.3, which is generated by $\{\mathrm{tr}\,\rho^{\mathrm{univ}}(\mathrm{Frob}_\ell) \mid \ell \notin \Sigma\}$.

Proof of Theorem 4.17. The existence of the map Φ was proved in Corollary 4.16.

Note that $\mathbf{T}_{1,\mathfrak{m}} = \mathbf{T}_\mathfrak{M} \otimes_\Lambda \Lambda/\ker \nu_{1,1}$. As $\Lambda/\ker_{\nu_{1,1}} \cong \mathcal{O}$ and $\mathbf{T}_\mathfrak{M} \cong \Lambda^{r(\mathbf{T}_\mathfrak{M})}$ as Λ-modules, we get that $\mathbf{T}_{1,\mathfrak{m}}$ is a finitely generated free $\mathcal{O}$-module of rank $r(\mathbf{T}_\mathfrak{M})$, and hence has no $\mathbf{Z}$-torsion.

We can therefore apply Proposition 6.9 in [BK11] to the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\Phi} & \mathbf{T}_{1,\mathfrak{m}} \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & \mathbf{T}_{1,\mathfrak{m}}/J_{1,\mathfrak{m}} \end{array}$$

noting that the top arrow is surjective and the bottom arrow is an isomorphism by Corollary 4.16 and I is principal by Theorem 3.10. □

Remark 4.18. In contrast to many results in the literature this result manages to prove $R = T$, without knowing that these rings are reduced.

Let us record a consequence of Theorem 4.17.

Corollary 4.19. *The Eisenstein ideal $J_{1,\mathfrak{m}}$ is principal.*

Remark 4.20. Let us note that Skinner and Wiles in [SW97] considered a related deformation problem. While [SW97] assume throughout their article that $k \geq 2$, it appears to us that it would be possible to infer a certain modularity result in the weight one case from their isomorphism of the ordinary universal deformation ring with the ordinary Hecke algebra. Even given that, however, there are significant differences in the setup and method. The non-semisimple residual representation considered in [SW97] has the form

$$\begin{bmatrix} \chi & * \\ 0 & 1 \end{bmatrix},$$

so has the opposite order of the characters on the diagonal to our ρ_0. Using this, the authors construct a certain reducible characteristic zero deformation of it, but its uniqueness (necessary for their method) requires that χ be ramified at p. In our setup such a deformation does not exist, and consequently our approach is different (not only because [SW97] work in weight 2 and we work directly in weight 1). Also, while both [SW97] and we need a cyclicity assumption for the uniqueness of the residual representation, the fact that our approach requires the principality of the ideal of reducibility forces us to make the additional assumption that $C_F^{\chi} = 0$. However, as a benefit we do not need to assume that χ is ramified at p. Such unramified characters can occur in this context as we demonstrate in section 4.7.

4.5. **Non-crossing at weight one.** In this section we will show that under an additional assumption that no two Hida families cross at weight one, one can identify the Hecke algebra $\mathbf{T}_{1,\mathfrak{m}}$ with a subring of $\prod_{\mathcal{F}\in\mathcal{S}} \overline{\mathbf{Q}}_p$ (see Proposition 4.21). In particular this will show that in that case $\mathbf{T}_{1,\mathfrak{m}}$ and hence also the universal deformation ring R is reduced. This is not the case if the non-crossing assumptions is not satisfied.

Recall that $\mathbf{T}_{\mathfrak{M}}$ is a finitely generated free Λ-module. We will write $r(\mathbf{T}_{\mathfrak{M}})$ for the Λ-rank of $\mathbf{T}_{\mathfrak{M}}$. As already discussed there is a Λ-algebra map $\mathbf{T} \hookrightarrow \prod_{\mathcal{F}\in\mathcal{S}'} \mathcal{O}_{L_{\mathcal{F}}}$, so we can also view $\mathbf{T}_{\mathfrak{M}}$ as a Λ-subalgebra of $\prod_{\mathcal{F}\in\mathcal{S}} \mathcal{O}_{L_{\mathcal{F}}}$. To be more precise, for each newform $\mathcal{F}$ let $\lambda_{\mathcal{F}} : \mathbf{T} \to \mathcal{O}_{L_{\mathcal{F}}}$ be the map sending a Hecke operator to its eigenvalue corresponding to $\mathcal{F}$. We denote the Λ-algebra map $\mathbf{T}_{\mathfrak{M}} \hookrightarrow \prod_{\mathcal{F}\in\mathcal{S}} \mathcal{O}_{L_{\mathcal{F}}}$ by ι. Note that $\iota = \oplus_{\mathcal{F}\in\mathcal{S}}\lambda_{\mathcal{F}}$.

Recall that we denote by $\mathcal{N}'$ the set of all newforms in $\mathcal{S}_L^0(N,\theta)$. In analogy with $\mathcal{S} \subset \mathcal{S}'$ define $\mathcal{N} \subset \mathcal{N}'$ to consist of all the newforms whose Fourier coefficients are congruent to c_ℓ modulo the maximal ideal of $L_{\mathcal{F}}$ for all primes ℓ. Let $\varphi : \mathcal{O}_L \to \overline{\mathbf{Q}}_p$ be an extension of $\nu_{1,1}$. Here $\mathcal{O}_L$ is the ring of integers of L. For each $\mathcal{F} \in \mathcal{N}$ we define $\varphi_{\mathcal{F}} : \mathcal{O}_{L_{\mathcal{F}}} \to \overline{\mathbf{Q}}_p$ to be the restriction of φ to $\mathcal{O}_{L_{\mathcal{F}}}$. Then the map $\oplus_{\mathcal{F}\in\mathcal{S}}\varphi_{\mathcal{F}} \circ \iota : \mathbf{T}_{\mathfrak{M}} \to \prod_{\mathcal{F}\in\mathcal{S}} \varphi_{\mathcal{F}}(\mathcal{O}_{L_{\mathcal{F}}})$ factors through $\mathbf{T}_1$. Even more, it factors through $\mathbf{T}_{1,\mathfrak{m}}$ where $\mathfrak{m}$ is the maximal ideal of $\mathbf{T}_1$ containing J_1. Note that the canonical map from $\mathbf{T} \to \mathbf{T}/(\ker \nu_{1,1})\mathbf{T} = \mathbf{T}_1$ restricts to a surjective map $\nu : \mathbf{T}_{\mathfrak{M}} \to \mathbf{T}_{\mathfrak{M}}/(\ker \nu_{1,1})\mathbf{T}_{\mathfrak{M}} = \mathbf{T}_{1,\mathfrak{m}}$. In other words there exists an $\mathcal{O}$-algebra map ϕ such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{T}_{\mathfrak{M}} \arrow[r, "\iota"] \arrow[d, "\nu"] & \prod_{\mathcal{F}\in\mathcal{S}} \mathcal{O}_{L_{\mathcal{F}}} \arrow[d, "\oplus_{\mathcal{F}\in\mathcal{S}}\varphi_{\mathcal{F}}"] \\ \mathbf{T}_{1,\mathfrak{m}} \arrow[r, "\phi"] & \prod_{\mathcal{F}\in\mathcal{S}} \varphi_{\mathcal{F}}(\mathcal{O}_{L_{\mathcal{F}}}) \end{tikzcd} \tag{4.4}$$

Note that $\prod_{\mathcal{F}\in\mathcal{S}} \varphi_{\mathcal{F}}(\mathcal{O}_{L_{\mathcal{F}}})$ is a free $\mathcal{O}$-module of finite rank.

Proposition 4.21. *Assume that there exists an extension $\varphi : \mathcal{O}_L \to \overline{\mathbf{Q}}_p$ of $\nu_{1,1}$ such that $\varphi_{\mathcal{F}} \circ \lambda_{\mathcal{F}} \neq \varphi_{\mathcal{F}'} \circ \lambda_{\mathcal{F}'}$ for all $\mathcal{F}, \mathcal{F}' \in \mathcal{N}$ with $\mathcal{F}' \neq \mathcal{F}$. Then ϕ is injective and so $\mathbf{T}_{1,\mathfrak{m}}$ is reduced.*

Remark 4.22. We note that the assumption in Proposition 4.21 cannot be weakened. The problem is that two Λ-adic families may cross at weight one (a phenomenon that does not occur in higher weights) - cf. [DG12]. To this end let us illustrate this issue with a commutative algebra example.

Consider the case where $\mathcal{N}$ consists of only two forms $\mathcal{F}$ and $\mathcal{G}$ and suppose for simplicity that $\mathcal{O}_{L_{\mathcal{F}}} = \mathcal{O}_{L_{\mathcal{G}}} = \Lambda$. In particular, this implies that $\mathcal{F}$ is not a Galois conjugate of $\mathcal{G}$, so $\mathcal{S} = \mathcal{N}$ and $\varphi_{\mathcal{F}} = \varphi_{\mathcal{G}} = \varphi$. Then $\iota = \lambda_{\mathcal{F}} \oplus \lambda_{\mathcal{G}}$. Assume that the families $\mathcal{F}$, $\mathcal{G}$ cross at weight one, i.e., that $\varphi \circ \lambda_{\mathcal{F}} = \varphi \circ \lambda_{\mathcal{G}}$.

Then while the image of $\varphi \oplus \varphi$ is clearly $\mathcal{O} \times \mathcal{O}$, the image of $(\varphi \oplus \varphi) \circ \iota$ equals the diagonally embedded copy of $\mathcal{O}$ inside $\mathcal{O} \times \mathcal{O}$. However, the $\mathcal{O}$-rank of $\mathbf{T}_{1,\mathfrak{m}}$ is still 2 (see proof of Lemma 4.24 below), so the map ϕ cannot be injective.

It is perhaps worth noting that this potential mismatch between the rank of $\mathbf{T}_{1,\mathfrak{m}}$ and the rank of the image of $(\varphi \oplus \varphi) \circ \iota$ means that in general $\mathbf{T}_{1,\mathfrak{m}}$ does not act faithfully on weight one specialisations (even the non-classical ones) of Λ-adic newforms which are congruent to an Eisenstein series.

Remark 4.23. We note that in the presence of Galois conjugate forms, it is not enough to assume that $\varphi_{\mathcal{F}} \circ \lambda_{\mathcal{F}} \neq \varphi_{\mathcal{F}'} \circ \lambda_{\mathcal{F}'}$ for all $\mathcal{F}, \mathcal{F}' \in \mathcal{S}$ with $\mathcal{F}' \neq \mathcal{F}$. This is so because it is a priori possible for two elements of a Λ-adic Galois conjugacy class to specialize to the same weight one form.

Proof of Proposition 4.21. We begin by proving two lemmas.

Lemma 4.24. *The $\mathcal{O}$-module $\mathbf{T}_{1,\mathfrak{m}}$ is finitely generated and free of rank $r(\mathbf{T}_{\mathfrak{M}})$. If the $\mathcal{O}$-rank of $\phi(\mathbf{T}_{1,\mathfrak{m}})$ equals $r(\mathbf{T}_{\mathfrak{M}})$, then ϕ is injective.*

Proof. That $\mathbf{T}_{1,\mathfrak{m}}$ is finitely generated and free of rank $r(\mathbf{T}_{\mathfrak{M}})$ was shown in the proof of Theorem 4.17. If the $\mathcal{O}$-rank of $\phi(\mathbf{T}_{1,\mathfrak{m}})$ equals $r(\mathbf{T}_{\mathfrak{M}})$ we conclude that $\ker \phi$ is a torsion submodule of $\mathbf{T}_{1,\mathfrak{m}}$, so must be zero. □

Lemma 4.25. *The Λ-rank of $\prod_{\mathcal{F}\in\mathcal{S}} \mathcal{O}_{L_{\mathcal{F}}}$ equals $r(\mathbf{T}_{\mathfrak{M}})$. If the $\mathcal{O}$-rank of the image of $\oplus_{\mathcal{F}\in\mathcal{S}}\varphi_{\mathcal{F}} \circ \iota$ equals $r(\mathbf{T}_{\mathfrak{M}})$, then ϕ is injective.*

Proof. First note that as $\mathbf{T}_{\mathfrak{M}} \otimes_{\Lambda} F_{\Lambda} \cong \prod_{\mathcal{F}\in\mathcal{S}} L_{\mathcal{F}}$ we get that the image of the embedding ι is a Λ-submodule of $\prod_{\mathcal{F}\in\mathcal{S}} \mathcal{O}_{L_{\mathcal{F}}}$ of full rank. So, we conclude that the Λ-rank of $\prod_{\mathcal{F}\in\mathcal{S}} \mathcal{O}_{L_{\mathcal{F}}}$ equals $r(\mathbf{T}_{\mathfrak{M}})$. The lemma then follows from the commutativity of (4.4) and Lemma 4.24. □

In light of Lemma 4.25 it is enough to prove that the $\mathcal{O}$-rank s of the image $\mathcal{I}$ of $\oplus_{\mathcal{F}\in\mathcal{S}}\varphi_{\mathcal{F}} \circ \iota$ equals $r(\mathbf{T}_{\mathfrak{M}})$. As the map $\oplus_{\mathcal{F}\in\mathcal{S}}\varphi_{\mathcal{F}}$ is surjective, we get that $s \leq r(\mathbf{T}_{\mathfrak{M}})$. For the reverse inequality first note that $r(\mathbf{T}_{\mathfrak{M}}) = \#\mathcal{N}$. Indeed, we have that $\mathbf{T} \otimes_\Lambda L \cong \prod_{\mathcal{F}\in\mathcal{N}'} L$ by [Wil90], p. 507 from which it follows that $\mathbf{T}_\mathfrak{M} \otimes_\Lambda L \cong \prod_{\mathcal{F}\in\mathcal{N}} L$.

Thus it suffices to prove that $s \geq \#\mathcal{N}$. Note that the map

$$f \mapsto (i \otimes 1 \mapsto f(i))$$

gives rise to an injective map

$$\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathcal{I}, \overline{\mathbf{Q}}_p) \hookrightarrow \mathrm{Hom}_{\overline{\mathbf{Q}}_p-\mathrm{alg}}(\mathcal{I} \otimes_\mathcal{O} \overline{\mathbf{Q}}_p, \overline{\mathbf{Q}}_p).$$

As the $\mathcal{O}$-rank of $\mathcal{I}$ equals s, we get

$$\mathcal{I} \otimes_\mathcal{O} \overline{\mathbf{Q}}_p \cong \overline{\mathbf{Q}}_p^s \quad \text{as } \overline{\mathbf{Q}}_p\text{-algebras.}$$

From this we conclude that $\#\,\mathrm{Hom}_{\overline{\mathbf{Q}}_p-\mathrm{alg}}(\mathcal{I} \otimes_\mathcal{O} \overline{\mathbf{Q}}_p, \overline{\mathbf{Q}}_p) = s$. It follows that $\#\,\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathcal{I}, \overline{\mathbf{Q}}_p) \leq s$. Thus by Lemma 4.26 below we get $s \geq \#\mathcal{N}$, as desired. □

Lemma 4.26. *Under the assumptions of Proposition 4.21 one has*

$$\#\,\mathrm{Hom}_{\mathcal{O}-\mathrm{alg}}(\mathcal{I}, \overline{\mathbf{Q}}_p) \geq \#\mathcal{N}.$$

Proof. By our non-crossing assumption we know that we have $\#\mathcal{N}$ distinct maps $\varphi_\mathcal{F} \circ \lambda_\mathcal{F} : \mathbf{T}_\mathfrak{M} \to \overline{\mathbf{Q}}_p$. It suffices to show that each of them factors through $\mathcal{I}$. For each $\mathcal{F} \in \mathcal{S}$ this follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbf{T}_\mathfrak{M} \arrow[r, "\iota"] \arrow[dr, "\lambda_\mathcal{F}"'] & \prod_{\mathcal{F}\in\mathcal{S}} \mathcal{O}_{L_\mathcal{F}} \arrow[r, "\oplus_{\mathcal{F}\in\mathcal{S}}\varphi_\mathcal{F}"] \arrow[d, "\pi_\mathcal{F}"] & \prod_{\mathcal{F}\in\mathcal{S}} \varphi_\mathcal{F}(\mathcal{O}_{L_\mathcal{F}}) \arrow[d, "\pi_\mathcal{F}"] \\
& \mathcal{O}_{L_\mathcal{F}} \arrow[r, "\varphi_\mathcal{F}"] & \varphi(\mathcal{O}_{L_\mathcal{F}})
\end{tikzcd} \tag{4.5}$$

(the triangle commutes by the definition of $\lambda_\mathcal{F}$ and the square commutes also by the definition of the maps involved).

Now, let $\mathcal{F}'$ be a Galois conjugate of $\mathcal{F}$. Then there exists a Galois element $\sigma = \sigma(\mathcal{F}') \in \mathrm{Gal}(L/F_\Lambda)$ such that $\mathcal{F}' = \sigma\mathcal{F}$, and so $\mathcal{O}_{L_{\mathcal{F}'}} = \sigma\mathcal{O}_{L_\mathcal{F}}$. Furthermore, if $\mathfrak{p}_\mathcal{F}$ denotes the kernel of $\varphi_\mathcal{F}$, then $\sigma\mathfrak{p}_\mathcal{F}$ is the kernel of $\varphi_{\mathcal{F}'}$. Note that $\lambda_{\mathcal{F}'} = \sigma \circ \lambda_\mathcal{F}$.

With this we amend the diagram (4.5) and obtain a new commutative diagram.

$$\begin{tikzcd}
\mathbf{T}_\mathfrak{M} \arrow[r, "\iota"] \arrow[dr, "\lambda_\mathcal{F}"'] \arrow[ddr, bend right, "\lambda_{\mathcal{F}'}"'] & \prod_{\mathcal{F}\in\mathcal{S}} \mathcal{O}_{L_\mathcal{F}} \arrow[r, "\oplus_{\mathcal{F}\in\mathcal{S}}\varphi_\mathcal{F}"] \arrow[d, "\pi_\mathcal{F}"] & \prod_{\mathcal{F}\in\mathcal{S}} \varphi_\mathcal{F}(\mathcal{O}_{L_\mathcal{F}}) \arrow[d, "\pi_\mathcal{F}"] \\
& \mathcal{O}_{L_\mathcal{F}} \arrow[r, "\varphi_\mathcal{F}"] \arrow[d, "\sigma"] & \varphi(\mathcal{O}_{L_\mathcal{F}}) \arrow[d, "\tilde{\sigma}"] \\
& \mathcal{O}_{L_{\mathcal{F}'}} \arrow[r, "\varphi_{\mathcal{F}'}"'] & \varphi(\mathcal{O}_{L_{\mathcal{F}'}})
\end{tikzcd} \tag{4.6}$$

Here we define the map $\tilde{\sigma} : \varphi_\mathcal{F}(\mathcal{O}_{L_\mathcal{F}}) \to \varphi_\mathcal{F}(\mathcal{O}_{L_{\mathcal{F}'}})$ by taking the pre-image under $\varphi_\mathcal{F}$ and then applying $\varphi_{\mathcal{F}'} \circ \sigma$. This is well-defined because if $s, t \in \mathcal{O}_{L_\mathcal{F}}$ are pre-images of the same element in $\varphi_\mathcal{F}(\mathcal{O}_{L_\mathcal{F}})$, then $t - s \in \mathfrak{p}_\mathcal{F}$, so $\sigma t - \sigma s \in \sigma\mathfrak{p}_\mathcal{F}$.

From this diagram we see (as σ is an $\mathcal{O}$-algebra map) that $\varphi_{\mathcal{F}'} \circ \lambda_{\mathcal{F}'} : \mathbf{T}_\mathfrak{M} \to \overline{\mathbf{Q}}_p$ also factors through $\mathcal{I}$, as desired. □

4.6. **Irreducibility of Galois representations in weight 1.**

Definition 4.27. We will write $\rho^1_{\mathcal{F}} : G_\Sigma \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ for the semi-simple Galois representation associated with $\varphi_{\mathcal{F}} \circ \mathrm{tr}\, \rho_{\mathcal{F}}$.

Note that $\det \rho^1_{\mathcal{F}} = \chi$ using that $\theta = \tilde{\chi}\tilde{\omega}^{-1}$ as in the proof of Lemma 4.9. In this section we will prove irreducibility of $\rho^1_{\mathcal{F}}$.

Theorem 4.28. *For $\mathcal{F} \in \mathcal{S}$ the representation $\rho^1_{\mathcal{F}} : G_\Sigma \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is irreducible.*

Proof. Suppose $\mathrm{tr}\, \rho^1_{\mathcal{F}}$ is a sum of two characters ψ_1 and ψ_2 such that ψ_1 reduces to 1 mod ϖ and ψ_2 reduces to $\overline{\chi}$. By ordinarity we can assume that ψ_1 is unramified at p. Furthermore, by Theorem 4.11 (2) we see that for $\ell \in \Sigma - \{p\}$ we have

$$(\rho^1_{\mathcal{F}})^{\mathrm{ss}}|_{I_\ell} = 1 \oplus \chi|_{I_\ell}.$$

By Remark 2.1 this forces $\psi_1 = 1$ and thus $\psi_2 = \chi$ since $\det \rho^1_{\mathcal{F}} = \chi$.

Let k be a positive integer such that $k \equiv 1 \mod (p-1)$. Let $\varphi_{\mathcal{F},k}$ be an extension of $\nu_{k,1}$. Let E_k be a finite extension of the compositum of $\varphi_{\mathcal{F},k}(L)\varphi_{\mathcal{F}}(L)$ and write $\mathcal{O}_k$ for its ring of integers. This is a finite extension of $\mathbf{Q}_p$. Let ϖ_k be a uniformizer of $\mathcal{O}_k$. As $\mathcal{F}$ is a cusp form we know that $\varphi_{\mathcal{F},k}(\mathcal{F})$ is a cusp form for an infinite subset $\mathcal{W}$ of ks as above. Assume that $k \in \mathcal{W}$. Similarly to the case of weight 1, composing $\rho_{\mathcal{F}}$ with $\varphi_{\mathcal{F},k}$ gives rise to a Galois representation $\rho^k_{\mathcal{F}} : G_\Sigma \to \mathrm{GL}_2(\varphi_{\mathcal{F},k}(L))$. By cuspidality of $\varphi_{\mathcal{F},k}(\mathcal{F})$, this Galois representation is irreducible. Then by Proposition 2.1 in [Rib76] there exists a lattice inside the representation space of $\rho^k_{\mathcal{F}}$ such that with respect to that lattice the mod ϖ_k reduction $\overline{\rho}^k_{\mathcal{F}}$ of $\rho^k_{\mathcal{F}}$ is of the form

$$\overline{\rho}^k_{\mathcal{F}} = \begin{bmatrix} 1 & * \\ & \overline{\chi} \end{bmatrix} \not\cong 1 \oplus \overline{\chi}. \tag{4.7}$$

Let m_k be the largest positive integer m such that $\varphi_{\mathcal{F}}(c(1, t\mathcal{F})) \equiv \varphi_{\mathcal{F},k}(c(1, t\mathcal{F}))$ mod ϖ^m for all $t \in \mathbf{T}$. Note that this makes sense as $\varpi \in \mathcal{O}_k$. Using congruence (4.7) and Theorem 1.1 in [Urb01] we conclude that there is a lattice Λ in the space of $\rho^k_{\mathcal{F}}$ such that with respect to a certain basis $\{e_1, e_2\}$ one has

$$\rho^k_{\mathcal{F}} = \begin{bmatrix} 1 & * \\ & \chi \end{bmatrix} \pmod{\varpi^{m_k}} \tag{4.8}$$

with $*$ still not split mod ϖ_k.

So by Theorem 4.11(1) we get

$$\rho^k_{\mathcal{F}}|_{D_p} \cong_{E_k} \begin{bmatrix} \chi\beta^{-1}\omega^{1-k}\epsilon^{k-1} & * \\ & \beta \end{bmatrix},$$

where β is unramified and maps Frob_p to $\varphi_{\mathcal{F},k}(c(p, \mathcal{F}))$.

We claim that it is possible to change the basis of Λ such that in that new basis

$$\rho^k_{\mathcal{F}}|_{D_p} = \begin{bmatrix} \chi\beta^{-1}\omega^{1-k}\epsilon^{k-1} & * \\ & \beta \end{bmatrix}.$$

By ordinarity there exists a vector $v = ae_1 + be_2$, with $a, b \in E_k$, on which D_p acts by $\chi\beta^{-1}\omega^{1-k}\epsilon^{k-1}$. Multiply this by a power of ϖ_k such that $\varpi_k^s a, \varpi_k^s b \in \mathcal{O}_k$ and $v' := \varpi_k^s v \not\equiv 0 \mod \varpi_k$.

Assume that $\varpi_k^s a$ is a ϖ_k-unit. Then we have $\Lambda = \mathcal{O}_k v' + \mathcal{O}_k e_2$. (If $\varpi_k^s b \in \mathcal{O}_k^\times$ then $\Lambda = \mathcal{O}_k v' + \mathcal{O}_k e_1$.) As $\det(\rho_{\mathcal{F}}^k) = \chi\omega^{1-k}\epsilon^{k-1}$ we see that in the basis $\mathcal{B}' = \{v', e_2\}$ (respectively $\mathcal{B}' = \{v', e_1\}$) we have

$$\rho_{\mathcal{F}}^k|_{D_p} = \begin{bmatrix} \chi\beta^{-1}\omega^{1-k}\epsilon^{k-1} & * \\ & \beta \end{bmatrix}. \tag{4.9}$$

We know that $\beta \equiv 1 \mod \varpi_k$ as $\mathcal{F} \in \mathcal{S}$ and $\chi|_{D_p} \not\equiv 1 \mod \varpi_k$ by assumption, hence $\beta \not\equiv \chi|_{D_p} \mod \varpi_k$. So Lemma 3.5 tells us that $\beta \equiv 1 \bmod \varpi^{m_k}$ by comparing the above to (4.8).

Thus we get that in the basis $\mathcal{B}'$ of Λ we have

$$\rho_{\mathcal{F}}^k|_{D_p} = \begin{bmatrix} \chi & *' \\ & 1 \end{bmatrix} \quad (\text{mod } \varpi^{m_k} = \varpi_k^{e_k m_k}),$$

where e_k is the ramification index of $\mathcal{O}_k$ over $\mathcal{O}$. Comparing this with (4.8) we conclude that there exists a matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}_2(\mathcal{O}_k/\varpi^{m_k})$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & * \\ & \chi \end{bmatrix} = \begin{bmatrix} \chi & *' \\ & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.10}$$

where χ and $*$ are considered after restrictions to D_p. We first note that $\overline{C} \neq 0$ mod ϖ_k. Indeed, if the reduction $\overline{C}$ of C mod ϖ_k were 0 then reducing the equation (4.10) mod ϖ_k and comparing the top-left entries we would get $\overline{A} = \overline{\chi}\overline{A}$. Note that if $\overline{C} = 0$ then we must have $\overline{A} \neq 0$ as the matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is invertible. This contradicts the assumption that $\overline{\chi}|_{D_p} \neq 1$. Hence we must have that C is a unit in $\mathcal{O}_k/\varpi^{m_k}$.

Now compare the top-left entries of (4.10) to get that $A = A\chi + *'C$, from which we get that $*' = (A/C)(1-\chi)$. Hence the cocycle induced by $*'$ in $H^1(D_p, \mathcal{O}/\varpi^{m_k}(\chi))$ is a coboundary. In other words,

$$\rho_{\mathcal{F}}^k|_{D_p} \cong \chi \oplus 1 \pmod{\varpi^{m_k}}$$

Hence the $*$ in (4.8) gives rise to an element $c_k \in H^1_{\Sigma-\{p\}}(\mathbf{Q}, \mathcal{O}_k/\varpi^{m_k}(\chi^{-1}))$ which is not annihilated by ϖ^{m_k-1}. We now use Lemma 2.3 for primes $\ell \mid N$ or such that $\tilde{\chi}(\ell)\ell \not\equiv 1 \bmod p$ to deduce that $c_k \in H^1_\emptyset(\mathbf{Q}, \mathcal{O}_k/\varpi^{m_k}(\chi^{-1}))$ using the fact that by assumption (2) in section 3.1 this covers all the primes in $\Sigma - \{p\}$.

We now claim that there exists an element of $H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\chi^{-1}))$ which is not annihilated by ϖ^{m_k-1}. For this first note that for every positive integer r one has $\mathcal{O}_k/\varpi^r = (\mathcal{O}/\varpi^r)^s$ where $s = [E_k : E]$. As the formation of Selmer groups commutes with direct sums we get

$$H^1_{\mathrm{f}}(\mathbf{Q}, \mathcal{O}_k/\varpi^{m_k}) \cong (H^1_{\mathrm{f}}(\mathbf{Q}, \mathcal{O}/\varpi^{m_k}(\chi^{-1})))^s.$$

Since $\varpi^{m_k-1}c_k \neq 0$ we conclude that there must exist an element $c_k' \in H^1_\emptyset(\mathbf{Q}, \mathcal{O}/\varpi^{m_k}(\chi^{-1}))$ such that $\varpi^{m_k-1}c_k' \neq 0$.

By (2.2) we have that $\iota : H^1_\emptyset(\mathbf{Q}, \mathcal{O}/\varpi^r(\chi^{-1})) \to H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\chi^{-1}))[\varpi^r]$ is an isomorphism. Therefore the elements c_k give rise to an infinite sequence of elements $c_k' \in H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\chi^{-1}))$ for $k \in \mathcal{W}$ with the property that $\varpi^{m_k-1}c_k' \neq 0$.

As $m_k \to \infty$ when k approaches 1 p-adically this forces $H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\chi^{-1}))$ to be infinite. However, one has by Proposition 2.2 that

$$H^1_\emptyset(\mathbf{Q}, E/\mathcal{O}(\chi^{-1})) \cong \mathrm{Hom}(\mathrm{Cl}(F), E/\mathcal{O}(\chi^{-1}))^{\mathrm{Gal}(F/\mathbf{Q})},$$

so we get a contradiction to the finiteness of class groups. □

Remark 4.29. We thank the referee for alerting us to the following alternative argument: If $\rho^1_{\mathcal{F}}$ is reducible then it is given by $\psi_1 \oplus \psi_2$ with ψ_1 unramified and $\psi_1 \equiv 1 \mod \varpi$. This forces ψ_1 to be of finite order (and therefore also ψ_2 since $\det(\rho^1_{\mathcal{F}}) = \chi$ has finite order). (As in the proof of Theorem 4.28 we can, in fact, show that $(\psi_1, \psi_2) = (1, \chi)$.) This means that $\mathcal{F}_1$ is a p-stabilization of the weight 1 Eisenstein series associated to (ψ_1, ψ_2) (see [BDP22] Proposition 4.4 for the local-global compatibility that relates the T_ℓ eigenvalues for such Eisenstein series for $\ell \mid N$ to the action of Frob_ℓ on $(\rho^1_{\mathcal{F}})^{I_\ell}$). Since this Eisenstein series is p-adically cuspidal one must have $\psi_1/\psi_2(D_p) = 1$ (see [DLR15] Proposition 1.3), which contradicts our assumption that $\overline{\chi}|_{D_p} \neq 1$. This assumption also rules out so-called trivial zeros of the Kubota-Leopoldt p-adic L-function, which govern the occurrence of irregular weight one Eisenstein series as part of a cuspidal Hida family (see [BDP22] for more details).

Remark 4.30. For every $\mathcal{F} \in \mathcal{S}$, we have that $\rho^1_{\mathcal{F}} : G_\Sigma \to \mathrm{GL}_2(\varphi_{\mathcal{F}}(L_{\mathcal{F}}))$ is irreducible by Theorem 4.28. Then by Proposition 2.1 in [Rib76] there exists a G_Σ-stable lattice Λ in the space of $\rho^1_{\mathcal{F},\Lambda}$ such that with respect to that lattice we have

$$\overline{\rho}^1_{\mathcal{F},\Lambda} = \begin{bmatrix} 1 & * \\ & \overline{\chi} \end{bmatrix} \not\cong 1 \oplus \overline{\chi}. \tag{4.11}$$

Furthermore by (1) in Theorem 4.11 we know that $\rho^1_{\mathcal{F}}|_{D_p}$ has a D_p-stable $\varphi_{\mathcal{F}}(L_{\mathcal{F}})$-line L on which D_p acts via $\chi\beta^{-1}$ and a quotient on which D_p acts by β with β an unramified $\varphi_{\mathcal{F}}(\mathcal{O}_{L_{\mathcal{F}}})$-valued character which reduces to the identity mod ϖ (because $\mathcal{F} \in \mathcal{S}$ and $\overline{\chi}|_{D_p} \neq 1$ by our assumption).

Arguing as in the proof of Theorem 4.28 we conclude that this combined with (4.11) shows that $\overline{\rho}^1_{\mathcal{F},\Lambda}$ splits when restricted to D_p. In particular, it splits when restricted to I_p. Hence it follows from Proposition 3.3 that for any $\mathcal{F}, \mathcal{F}'$ as above we can choose lattices $\Lambda_{\mathcal{F}}, \Lambda_{\mathcal{F}'}$ such that $\overline{\rho}^1_{\mathcal{F},\Lambda_{\mathcal{F}}} = \overline{\rho}^1_{\mathcal{F}',\Lambda_{\mathcal{F}'}}$ (note that $\mathcal{S} \neq \emptyset$). Note that this provides an alternative construction of the residual representation ρ_0 defined in section 4.3.

4.7. **Examples.** In this section we will demonstrate that our non-crossing assumption for the Hida families in Proposition 4.21 is often satisfied. We will also present an example of a character χ that is unramified at p and satisfies all the assumptions of Theorem 4.17. Such unramified characters cannot be handled by the methods of [SW97].

As mentioned in the Introduction the related question of the geometry of the eigencurve at classical weight one points has been studied extensively. In particular, Bellaïche and Dimitrov [BD16] prove that the eigencurve is smooth at such points if they are *regular*, i.e. have distinct roots of the Hecke polynomial at p. This translates to the form lying in a unique Hida family up to Galois conjugacy. Our p-distinguishedness assumption ensures that our forms are regular. Bellaïche and Dimitrov further prove that the eigencurve is étale at such points if there does not exist a real quadratic field K in which p splits and such that the corresponding Galois representation becomes reducible over K. Note that our assumption that $\overline{\rho}^{\mathrm{ss}} = 1 \oplus \chi$ rules out such a real multiplication case, as χ has to be odd. Our condition of not having Hida families (even Galois conjugate ones) cross at non-classical weight 1 specialisations corresponds to étaleness at all weight 1 specialisations of the connected components of the eigenvariety containing the Hida families $\mathcal{F} \in \mathcal{S}$.

As far as we know, the geometry at non-classical points has not been studied. We can nevertheless exhibit many cases in which there is a unique Hida family, so that our non-crossing assumption is satisfied. In particular, [BP19] calculate that for many irregular pairs (p, k) (irregular in the sense that p divides the Bernoulli number B_k) one indeed has

$$\dim_\Lambda \mathbf{T}_{\mathfrak{M}} = 1, \quad \text{(unique Hida family)}$$

where $\mathbf{T}$ is the universal ordinary Hecke algebra of tame level $N = 1$ and $\mathfrak{M}$ is a maximal ideal containing the Eisenstein ideal J corresponding to the Λ-adic Eisenstein series which specializes at the particular weight k to

$$E_k = -\frac{B_k}{2} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

This corresponds to the Hecke algebra we considered for $\chi = \omega^{k-1}$. Furthermore, by Corollary 5.15 in [Was97] we know that p-divisibility of B_k implies that of $B_1(\omega^{k-1}) = -L(0, \omega^{k-1})$.

Set $N = 1$ and fix a finite set Σ satisfying assumptions (2) and (3). Since $p \mid B_k$ there exists a representation $\rho_0 : G_\Sigma \to \mathrm{GL}_2(\mathbf{F})$ of the form

$$\rho_0 = \begin{bmatrix} 1 & * \\ 0 & \overline{\chi} \end{bmatrix},$$

which does not split, but splits when restricted to D_p (see section 4.4). In this case (as χ is a power of ω), the existence of ρ_0 can also be deduced from Theorem 1.3 in [Rib76].

We discuss the case $(p, k) = (37, 32)$ in detail to demonstrate that all our assumptions are satisfied for $\chi = \omega^{k-1}$. Indeed, since the class number of the splitting field $F = \mathbf{Q}(\zeta_{37})$ of χ is 37, we know that $p \| B_1(\chi)$, $C_F^{\chi^{-1}}$ is a cyclic $\mathcal{O}$-module and $C_F^{\chi} = 0$. We also have $\overline{\chi}|_{I_p} \neq 1$ as χ has order 36 and is ramified at p.

Theorem 4.17 therefore shows that $R = \mathbf{T}_{1,\mathfrak{m}} = \mathcal{O}$, where R is the universal deformation ring of ρ_0 defined in section 3.3 and $\mathbf{T}_{1,\mathfrak{m}}$ is as in section 4.4. This implies that there is a unique characteristic zero ordinary deformation of ρ_0 with determinant χ, and this corresponds to a non-classical p-adic cuspform of weight 1, as χ is not quadratic (see Remark 3.4).

An example of a character χ unramified at p that satisfies the assumptions in Theorem 4.17 is the following: there is an odd order 4 character of conductor 157, which is identified by its Conrey number of 28 (see [LMF22b]). Using Sage [The18] one can check that $L(0, \chi)$ is divisible by a prime above 5 in $\mathbf{Q}(i)$, whereas $L(0, \chi^{-1})$ is a 5-unit. This example (and others) can be found by using [LMF22a] to search for totally imaginary cyclic extensions with class number 5.

We could not check in this example whether the non-crossing assumption for Proposition 4.21 is satisfied (by checking for uniqueness of the Hida family) as the coefficient field of the specialisation in weight 5 has degree 102 over $\mathbf{Q}$, so we could not confirm whether there is a unique Galois conjugate of this cuspform of weight 5 and level 157 congruent to $1 + \chi$ for a fixed prime above 5.

5. $R = T$ theorem in the split case

In this section we will treat the split case of the deformation problem for odd quadratic characters. We keep all the assumptions of section 3.1.

Let $\chi = \chi_{F/\mathbf{Q}} : \mathrm{Gal}(F/\mathbf{Q}) \to \mathbf{Z}_p^\times$ be the quadratic character associated to an imaginary quadratic extension $F/\mathbf{Q}$ (so $N = \mathrm{cond}(\chi) = d_F$). We assume $p > 2$ is inert in $F/\mathbf{Q}$ (to have $\overline{\chi}|_{D_p} \neq 1$). We note that assumption (1) in the case of a quadratic character implies that C_F^χ is a cyclic $\mathcal{O}$-module (so the assumption of Theorem 3.10(1) is satisfied). This is actually equivalent to assuming that C_F is a cyclic $\mathcal{O}$-module, as $\mathrm{Gal}(F/\mathbf{Q})$ acts on $\mathrm{Cl}(F)$ via χ.

As before we will write $\tilde{\chi} : (\mathbf{Z}/d_F\mathbf{Z})^\times \to \mathbf{C}^\times$ for the Dirichlet character associated with χ. We will also denote by $\overline{\chi}$ the mod ϖ reduction of χ. The Dirichlet Class Number Formula and the functional equation imply that $L(0, \tilde{\chi}) = 2\frac{h_F}{w_F}$, where h_F is the class number of F and $w_F := \#\mathcal{O}_F^\times$. If $p \mid h_F$ then there exists a non-split representation $\rho_0 : G_\Sigma \to \mathrm{GL}_2(\mathbf{F})$ of the form

$$\rho_0 = \begin{bmatrix} 1 & * \\ 0 & \overline{\chi} \end{bmatrix},$$

which is split on I_ℓ for all primes, and also split on D_p since $p\mathcal{O}_F$ is a principal ideal and therefore splits completely in the Hilbert class field. By Proposition 3.3 this representation is unique up to isomorphism.

Write $S_1(d_F, \tilde{\chi})^{\mathrm{CM}}$ for the space of weight 1 classical cusp forms of level d_F and character $\tilde{\chi}$ spanned by the set $\mathcal{N}'$ of newforms with complex multiplication, i.e. such that for $f \in \mathcal{N}'$ one has $a_\ell(f)\tilde{\chi}(\ell) = a_\ell(f)$ for all primes ℓ, where $a_\ell(f)$ denotes the T_ℓ-eigenvalue corresponding to f. Suppose that all the forms $f \in \mathcal{N}'$ are defined over the extension $E/\mathbf{Q}_p$.

We define $\mathbf{T}_1^{\mathrm{class}}$ as the $\mathcal{O}-$subalgebra of $\prod_{f \in \mathcal{N}'} \mathcal{O}$ generated by $(a_\ell(f))_f$ for all primes $\ell \notin \Sigma$. By [Ser77] section 7.3 (see also [DS21] Proposition 2.4) each $f \in \mathcal{N}'$ has the form $f = f_\varphi$, where f_φ is induced from a non-trivial non-quadratic character $\varphi : \mathrm{Cl}(F) \cong \mathrm{Gal}(H/F) \to \mathbf{C}^\times$ of finite order (for H the Hilbert class field of F), with associated Galois representation

$$\rho_{f_\varphi} = \mathrm{ind}_F^{\mathbf{Q}}(\varphi)$$

and $\det \rho_{f_\varphi} = \chi$. We recall that $a_\ell(f_\varphi) = 0$ if ℓ is inert in $F/\mathbf{Q}$, and

$$a_\ell(f_\varphi) = \varphi(\mathfrak{l}) + \varphi(\mathfrak{l}^c) \text{ if } (\ell) = \mathfrak{l}\mathfrak{l}^c. \tag{5.1}$$

We define the Eisenstein ideal $J \subset \mathbf{T}_1^{\mathrm{class}}$ as the ideal generated by $T_\ell - 1 - \chi(\ell)$ for $\ell \notin \Sigma$ and $\mathfrak{m}$ the maximal ideal containing J. We also note (see section 4.7 in [Miy89]) that the space of classical Eisenstein series of weight 1, level d_F and character $\tilde{\chi}$ is spanned by the Eisenstein series $E_1(\tilde{\chi})$ with constant term $\frac{L(0,\tilde{\chi})}{2}$ at infinity and Hecke eigenvalues $1 + \tilde{\chi}(\ell)$ for $\ell \nmid d_F$. Let $\mathcal{N}$ be the subset of $\mathcal{N}'$ of newforms f congruent to $E_1(\tilde{\chi})$. We note that $(\mathbf{T}_1^{\mathrm{class}})_\mathfrak{m}$ is naturally a subring of $\prod_{f \in \mathcal{N}} \mathcal{O}$, which is of full rank as an $\mathcal{O}$-module.

Let R^{split} be the deformation ring defined in section 3.3.

Proposition 5.1. *We have an $\mathcal{O}$-algebra surjection $\Phi : R^{\mathrm{split}} \twoheadrightarrow (\mathbf{T}_1^{\mathrm{class}})_\mathfrak{m}$ mapping $\mathrm{tr}(\rho^{\mathrm{univ}}(\mathrm{Frob}_\ell))$ to T_ℓ for all primes $\ell \notin \Sigma$.*

Proof. We need to check that ρ_{f_φ} with $\overline{\rho}^{\rm ss}_{f_\varphi} \cong 1 \oplus \overline{\chi}$ satisfies the deformation conditions. By the universality of $R^{\rm split}$ we then get surjections $R^{\rm split} \to \mathcal{O}$, which will induce $R^{\rm split} \twoheadrightarrow (\mathbf{T}_1^{\rm class})_{\mathfrak{m}}$ as $R^{\rm split}$ is generated by traces (by Proposition 3.8).

Condition (i) is clear. For $\ell \mid d_F$ one can check that deformation condition (iii) is satisfied by Mackey theory. Indeed, for $\rho_{f_\varphi} = {\rm ind}_F^{\mathbf{Q}}(\varphi)$ this gives ${\rm ind}_F^{\mathbf{Q}}(\varphi)|_{I_\ell} = {\rm ind}_{I_\lambda}^{I_\ell}(\varphi|_{I_\lambda})$ for $(\ell) = \lambda^2$, and therefore ${\rm ind}_F^{\mathbf{Q}}(\varphi)|_{I_\ell} = {\rm ind}_{I_\lambda}^{I_\ell}(1) = 1 \oplus \chi$.

For (ii) we note that $a_p(f_\varphi) = 0$ as p is inert in $F/\mathbf{Q}$. This means that the Hecke polynomial of an eigenform f_φ is $x^2 + \tilde{\chi}(p) = (x-1)(x-\tilde{\chi}(p)) = (x-1)(x+1)$, which implies that $\rho_{f_\varphi}|_{D_p}$ is split. □

Corollary 5.2. *The Eisenstein ideal $J \subset (\mathbf{T}_1^{\rm class})_{\mathfrak{m}}$ is principal.*

Proof. As the reducibility ideal $I^{\rm split} \subset R^{\rm split}$ is the smallest ideal I of $R^{\rm split}$ such that ${\rm tr}(\rho^{\rm split}) \mod I$ is the sum of two characters, this means that $I^{\rm split}$ is contained in the ideal I_0 generated by ${\rm tr}(\rho^{\rm split}({\rm Frob}_\ell)) - 1 - \chi({\rm Frob}_\ell)$ for $\ell \notin \Sigma$. It follows from the proof of Proposition 3.9 that ${\rm tr}(\rho^{\rm split}) = 1 + \chi \mod I^{\rm split}$, so $I^{\rm split} = I_0$.

Under $\Phi : R^{\rm split} \twoheadrightarrow (\mathbf{T}_1^{\rm class})_{\mathfrak{m}}$ the generators of $I^{\rm split}$ map to the generators of J, so $\Phi(I^{\rm split}) = J$, and the principality of J follows from that of $I^{\rm split}$ (cf. part (1) of Theorem 3.10). □

5.1. **Proving Eisenstein congruences in weight 1.**

Theorem 5.3. *We have $\#((\mathbf{T}_1^{\rm class})_{\mathfrak{m}}/J) \geq \#C_F$.*

Proof. We put $\mathbf{T} := (\mathbf{T}_1^{\rm class})_{\mathfrak{m}}$. Note that all $f \in \mathcal{N}$ are defined over the completion E' of $\mathbf{Q}(\mu_{p^n})^+$ at the prime above p, where $p^n \| h_F$. As both sides of the inequality in the statement increase by the same factor if we extend the field E we can and will assume that $E = E'$. The ramification index e of E over $\mathbf{Q}_p$ is $\frac{1}{2}\phi(p^n) = \frac{1}{2}p^{n-1}(p-1)$.

From [BKK14] Proposition 4.3 and the principality of J we deduce the following (noting the correction made in [BK19] Remark 5.13 about the missing factor of $[E:\mathbf{Q}_p]$).

Proposition 5.4. *For every $\mathcal{O}$-algebra morphism $\lambda : \mathbf{T} \to \mathcal{O}$ write m_λ for the largest integer such that $1 + \chi(\ell) \equiv \lambda(T_\ell) \mod \varpi^{m_\lambda}$ for all $\ell \notin \Sigma$. Then*

$$\frac{[E:\mathbf{Q}_p]}{e} \cdot \sum_\lambda m_\lambda = {\rm val}_p(\#\mathbf{T}/J). \tag{5.2}$$

We will show that

$$\frac{1}{e} \cdot \sum_\lambda m_\lambda \geq {\rm val}_p(h_F), \tag{5.3}$$

which together with Proposition 5.4 implies the theorem, as we will now explain:

Indeed these would give us

$$\frac{1}{[E:\mathbf{Q}_p]} {\rm val}_p(\#\mathbf{T}/J) = \frac{1}{e} \cdot \sum m_\lambda \geq {\rm val}_p(h_F) = n.$$

Hence one gets from this:

$${\rm val}_p(\#\mathbf{T}/J) \geq [E:\mathbf{Q}_p]n = {\rm val}_p(\#C_F),$$

as desired.

Thus it remains to prove (5.3). Consider a character $\varphi : \mathrm{Cl}(F) \cong \mathrm{Gal}(H/F) \to \overline{\mathbf{Q}}_p^\times$ of exact order p^m for $1 \le m \le n$. We note (as in the proof of [DS21] Theorem 2.8) that since the values of φ are p^m-th roots of unity we have $\varphi(\mathfrak{q}) \equiv 1 \mod \varpi_m$ for ϖ_m the prime in $\mathbf{Q}(\mu_{p^m})$ above p and $\mathfrak{q}$ any ideal of $\mathcal{O}_F$. Note that $\varpi^{p^{n-m}}$ is a uniformizer in the completion of $\mathbf{Q}(\mu_{p^m})^+$ at the prime ideal above p.

We deduce that $(\varphi + \varphi^c)(\mathfrak{q}) \equiv 2 = 1 + \chi(\mathfrak{q}) \mod \varpi^{p^{n-m}}$ for any prime q of $\mathbf{Z}$ which splits in $\mathcal{O}_F$ as $\mathfrak{q}\overline{\mathfrak{q}}$. By (5.1) this tells us that $m_\lambda \ge p^{n-m}$ for λ corresponding to f_φ.

It remains to count how many such cusp forms f_φ congruent to $E_1(\tilde{\chi})$ we have. Since φ and $\varphi^{-1} = \varphi^c$ induce to the same cusp form we need to count how many (unordered) pairs $\{\varphi, \varphi^{-1}\}$ with φ exact order p^m exist for each $1 \le m \le n$. Since we assume that $C_F = \mathrm{Cl}(F) \otimes_{\mathbf{Z}} \mathcal{O}$ is cyclic, the p-part of $\mathrm{Cl}(F)$ is a cyclic abelian group G.

The order p^m characters lie in a unique subgroup of the character group of G (which is isomorphic to G) of order p^m, which has $\phi(p^m)$ generators. We therefore have $\frac{1}{2}\phi(p^m)$ pairs $\{\varphi, \varphi^{-1}\}$ with φ exact order p^m.

Hence

$$\frac{1}{e} \cdot \sum_\lambda m_\lambda \ge \frac{1}{e} \sum_{m=1}^n \frac{1}{2}\phi(p^m) \cdot p^{n-m} = \frac{2}{\phi(p^n)} \sum_{m=1}^n \frac{1}{2}\phi(p^n) = n.$$

This gives (5.3) and thus concludes the proof of the theorem. □

Remark 5.5. This bound on the congruence module T/J cannot be proved by the usual methods: For the method used e.g. in [BK19] Proposition 5.2 one needs a modular form with constant term a p-unit. However, the Eisenstein part of $M_1(d_K, \tilde{\chi})$ is spanned by $E_1(\tilde{\chi})$, which has $\frac{1}{2}L(0, \tilde{\chi})$ as constant term. Deducing the bound from Wiles's result Theorem 4.3 is also difficult, as we only know $\mathbf{T} \twoheadrightarrow \mathbf{T}_1 \twoheadrightarrow \mathbf{T}_1^{\mathrm{class}}$ and would need to establish classicality of the specialisation in weight 1. In addition, one would need to show (for the splitting of the associated Galois representation at p) that the specialisation is a p-stabilisation of a form of level d_F.

We obtain the following $R = T$ theorem in the split case:

Theorem 5.6. *Consider $F/\mathbf{Q}$ an imaginary quadratic field and $p > 2$ inert in $F/\mathbf{Q}$ dividing the class number of F. Assume that C_F is a cyclic $\mathcal{O}$-module (and assumptions (2) and (3) in section 3.1). Then the map $\Phi : R^{\mathrm{split}} \to (\mathbf{T}_1^{\mathrm{class}})_{\mathfrak{m}}$ in Proposition 5.1 is an isomorphism.*

Proof. By Proposition 5.1 and the fact that $\Phi(I^{\mathrm{split}}) \subset J$ we get the following commutative diagram

$$\begin{tikzcd}
R^{\mathrm{split}} \arrow[r, "\Phi"] \arrow[d] & (\mathbf{T}_1^{\mathrm{class}})_{\mathfrak{m}} \arrow[d] \\
R^{\mathrm{split}}/I^{\mathrm{split}} \arrow[r] & (\mathbf{T}_1^{\mathrm{class}})_{\mathfrak{m}}/J
\end{tikzcd}$$

where the top map is surjective. By combining Proposition 3.9 and Theorem 5.3 we see that the bottom arrow is an isomorphism. We can then apply Theorem 6.9 in [BK11] to conclude that Φ is also an isomorphism noting that I^{split} is principal by part (1) of Proposition 3.10. □

Remark 5.7. This result complements the work of Castella and Wang-Erickson [CWEH21] on Greenberg's conjecture for ordinary cuspidal eigenforms f, who prove in the residually irreducible case that ρ_f is split at p if and only if f is CM.

Our theorem implies the following equivalence:

Corollary 5.8. *Under the assumptions of section 3.1 let $\rho : G_\Sigma \to \mathrm{GL}_2(\mathcal{O})$ be an ordinary deformation of ρ_0, and assume that χ is unramified at p. Then ρ is modular by a classical weight 1 cusp form if and only if ρ is unramified at p and χ is quadratic.*

Proof. First note that ρ unramified at p is equivalent to $\rho|_{D_p}$ being split under our assumptions. Indeed, if ρ is unramified then p-distinguishedness forces $\overline{\rho}(\mathrm{Frob}_p)$ to have distinct eigenvalues and hence $\rho|_{D_p}$ is split.

We now assume that $\rho|_{D_p}$ is split and χ is quadratic. As we assume that χ is unramified at p and $\chi|_{D_p} \neq 1$ we deduce that p is inert in the imaginary quadratic extension, which is the splitting field of χ. By Theorem 5.6 we conclude that ρ is modular by a classical weight 1 cusp form.

Conversely, if ρ is modular by a classical weight 1 cusp form then by Remark 4.8 we know that χ is quadratic and that $\rho(G_\Sigma)$ is finite hence, in particular, ρ is split on D_p. $\square$

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