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Colour Equality Patterns on Rubik's Cubes

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Introduction

Most readers will be familiar with Rubik's Cube, among the world's most famous puzzles. The original 3×3×3 design is now also available in sizes ranging from 2×2×2 up to a truly astonishing 21×21×21. Interests range from speed contests to various pretty patterns that can be constructed. From a more mathematical perspective, one important result is that any position can be solved in 20 moves or fewer [1]. For the general $n \times n \times n$ cube this 'group diameter' scales as $n^2/\log n$ [2]. The entire group theoretical structure is very interesting and complex, and indeed has been used for teaching group theory on several courses [3,4].

After many hours of playing around with cubes of various sizes, I became interested in certain special patterns, but ones that can be more precisely defined than just being 'nice'. In particular, suppose we define a *Colour Equality Pattern* to be one where all six colours are distributed in exactly the same way. That is, if you had two identically arranged cubes, you could rotate them with respect to one another so that any one colour on the first cube and any other colour on the second cube would coincide in all of their positions. Figure 1 shows three very simple examples. The first is a chessboard, where all the side pieces are exchanged with their diagonal opposites. The second pattern has all the side pieces flipped, but remaining in their original positions. The third pattern combines both previous actions. All three of these configurations clearly satisfy the colour equality definition. The question then is, what other such patterns might there be, to what extent can different actions be combined as in Figure 1c, how systematically can we classify them all, etc. My hope in writing this is that the results might not only be interesting in their own right, but might also be useful for visualising and teaching about some patterns and their underlying symmetries in three-dimensional space.

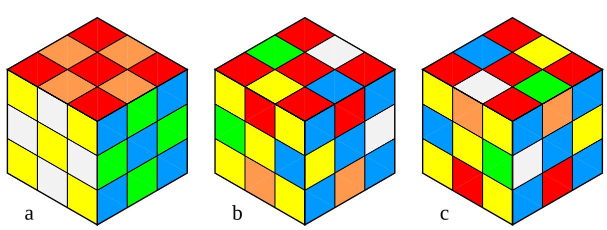

Figure 1: Three very simple colour equality patterns. (See also the colour chart at the end.)

Corner Patterns

The place to start is clearly with the $2 \times 2 \times 2$ cube, especially since the corners of all larger ones obey exactly the same rules as the $2 \times 2 \times 2$ cube consisting only of corners. The number of possible configurations, $7! \cdot 3^6 \approx 4 \cdot 10^6$, is sufficiently small to apply the brute force approach of simply scanning through them all with a computer and picking out the ones that satisfy the colour equality criterion. Including also the original configuration, there turn out to be twelve distinct colour equality patterns, shown in Figure 2. The representation here shows three faces that constitute the 'front', then imagine flipping the cube over to show the three faces on the 'back'. Note though that to really appreciate the various patterns and their symmetries, it is best to construct actual 3D cubes. To facilitate this, there is *Supplementary Material* available at the end with pre-printed templates to cut out and tape together to form cubes.

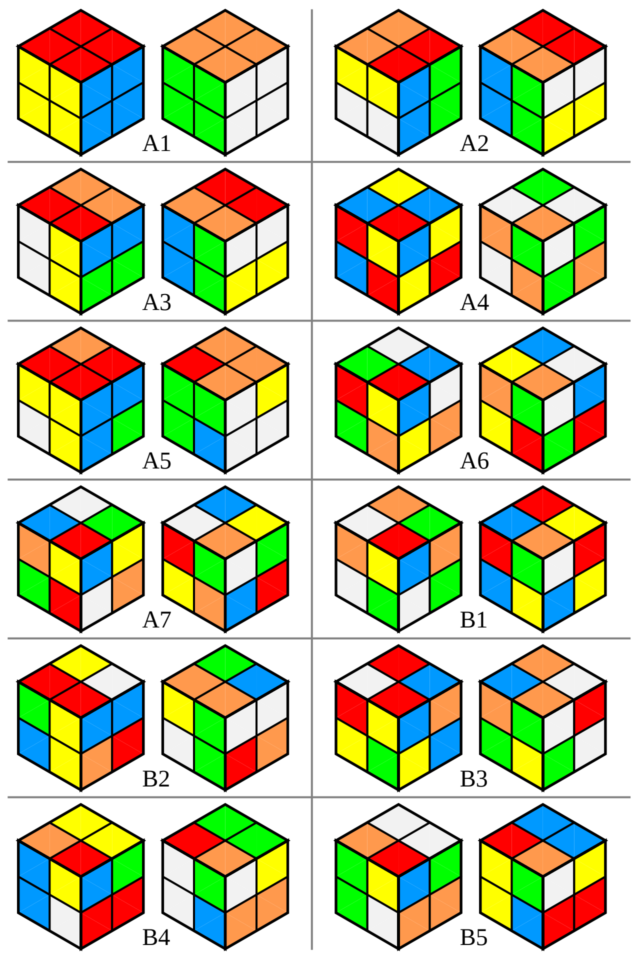

Figure 2: The twelve possible arrangements of the corners.

Besides the original pattern A1, the easiest to understand is A2, where all the pieces remain in their original positions, and are alternately rotated clockwise or counter-clockwise one twist. One could argue that there are two such patterns, depending on which half of the pieces are rotated clockwise and which counter-clockwise, but these are ultimately the same pattern, so there is only one listing. At least in terms of how the pieces move, the next easiest is A6, since here again all the pieces remain in their original positions. The way they now rotate is that two opposing corners, here chosen as red/yellow/blue (r/y/b) and orange/white/green (o/w/g), do not rotate at all, and all others rotate clockwise one twist. This again means that really there are two such patterns, since the rotation could also have been counter-clockwise. This produces a pattern having the opposite handedness to the one here, but is otherwise basically the same, and is therefore again not listed as a separate entry. Also, this pattern then obviously has a preferred 'axis' going from the r/y/b corner to the o/w/g one. Since we could have chosen any of four possible such axes, there are really eight possible such patterns, but again all fundamentally the same.

All of the remaining patterns also have such a preferred axis, here consistently chosen to be r/y/b for the 'front' and o/w/g for the 'back' to facilitate comparison. All except B1 are also handed, and thus have a mirror-symmetric version not included here. The front (r/y/b) and back (o/w/g) pieces remain in their original locations for all patterns; the other six pieces then move in a variety of ways that are left for readers to explore. There are also several other cases, for example A3 and A4, and also B2 and B3, where all the pieces are in the same positions, and only their orientations are different. (The reason for labelling some of the patterns as A and others as B will become apparent later on, when we add in the side pieces.)

Other comparisons that can be made are that A2 and A3 look vaguely similar, also A6 and A7, and B4 and B5. A6 and A7 have the interesting property that the three front faces are pairwise identical to the three back faces. This is already visible in Figure 2, but is even clearer when an actual 3D cube is viewed from any desired angle. Note also the clear difference between the front and back views of A5; both views exhibit the same three-fold symmetry seen in all the patterns, but the details of front versus back are different. Nonetheless, by constructing a 3D cube, one can easily verify that A5 is indeed a colour equality pattern, just like all the others. The vast majority of these patterns are sufficiently non-obvious though that I for one would not have discovered them without the computer search!

Combining Corners and Centres

The next item to consider is how these corner patterns in Figure 2 can fit together with the centres in 3×3×3 cubes. Since these centres form a fixed lattice, there are only 24 ways that lattice can be oriented with respect to the Figure 2 patterns. The options can therefore be examined directly, without needing a computer. The results nevertheless reveal some surprises. While all twelve patterns have at least one orientation of the centres that still constitutes a colour equality pattern, some of them yield up to three possible orientations. The details, as well as understanding what symmetries of the patterns in question cause this, are left as an exercise to the reader.

Side Piece Patterns

The final ingredient are the side pieces. There are $12! \cdot 2^{10} \approx 5 \cdot 10^{11}$ possible configurations, so certainly more challenging than the corners were, but still manageable by the brute force computational approach. Figure 3 shows three simple patterns that I had already worked out before. The first takes the three side pieces around the r/y/b corner, and similarly the three side pieces around the o/w/g corner, and rotates the entire sets either clockwise (as shown here) or counter-clockwise. The second pattern takes these same 'three plus three' side pieces and flips them all in place. The third one takes the other six side pieces, constituting a 'ring' separating the front and back halves, and flips all those in place. (So Figures 3b and 3c together are the same as Figure 1b.)

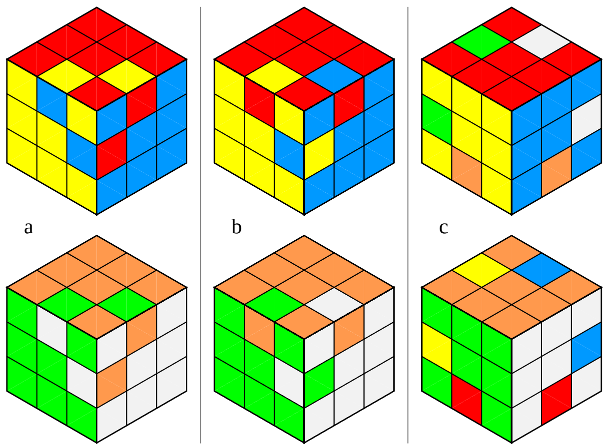

Figure 3: Three simple arrangements of the side pieces.

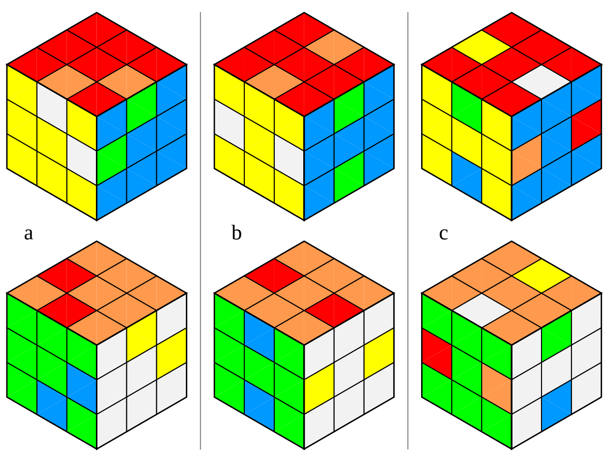

Figure 4: Three not so simple arrangements of the sides.

Figure 4 shows another three far less obvious patterns, which I was not aware of until the computer scan revealed them to be colour equality patterns. Figure 4a is especially interesting, as it has a clear difference between front and back views, similar to what we previously saw in the A5 corner pattern. Combining patterns like this, exhibiting front/back differences, with patterns as in Figure 3 that do not have such differences, will then typically not yield new colour equality patterns. That is, some patterns can be combined, but others can't. Understanding how the different symmetries of various patterns do or don't match up correctly is part of the charm of the whole topic though.

Combining Corners and Sides

All these results in Figures 3 and 4 have had the corners in their original A1 configuration. What about combining corner and side piece results? First though, Figure 5 shows A4 together with the side pieces (and centres) in their original positions. This same arrangement works for all of A1 through A7. However, if we try to construct the equivalent for B1, it can't be done. That is, the resulting pattern would be a colour equality pattern, but when you try to actually construct it on a cube, the last two side pieces always end up in the wrong positions. This is a well-known property of the 3×3×3 cube, that two side pieces can't be switched while leaving everything else in place.

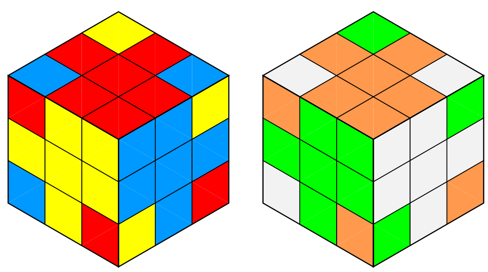

Figure 5: Pattern A4 with its 'standard' side arrangement.

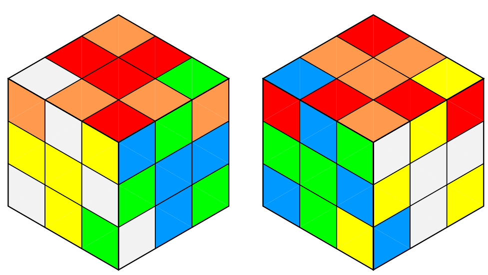

Figure 6: Pattern B1 with its 'standard' side arrangement.

Fortunately, the situation can be remedied. In particular, suppose we go back to Figure 3b, where the three plus three side pieces were flipped in place. If instead we had wanted to switch them with their diagonally opposite partners front to back, that would still constitute a colour equality pattern. The reason it wasn't listed in Figure 3 is that it would suffer from this same problem, that side pieces can't be switched as odd numbers of pairs. Well, a moment's thought convinces us that in this case two wrongs do make a right, and the two problems will conveniently cancel each other. Figure 6 then shows B1 together with its suitable arrangement of side pieces and centres. This configuration is not only a colour equality pattern, it can actually be achieved on the $3\times3\times3$ cube. This is also the distinction between the A and B corner patterns; A have their 'standard' side piece configuration as in Figure 5, whereas B have their 'standard' side piece configuration as in Figure 6.

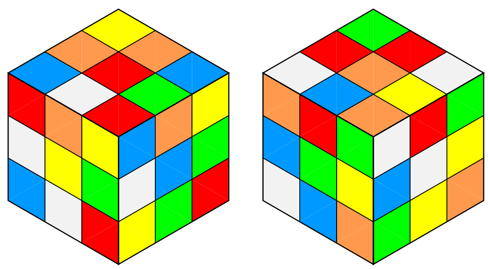

Figure 7: A4 with a more complicated side arrangement.

Suppose we next take the Figure 5 pattern, add the chessboard from Figure 1a, and also the 'three plus three flip in place' from Figure 3b. The result is shown in Figure 7. This is not only yet another colour equality pattern, it also has the colours so uniformly distributed around the cube that every face contains all six colours. Readers are invited to explore for themselves which other Figure 2 corner patterns can be combined with which side piece patterns to yield the same effect.

4×4×4 Cubes

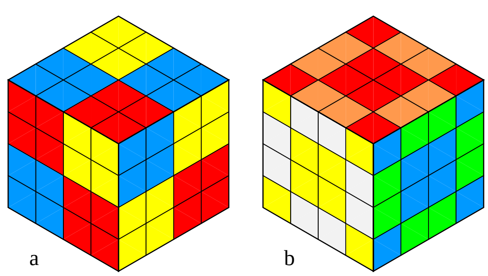

Figure 8: Two previous patterns repeated on a $4\times4\times4$ cube.

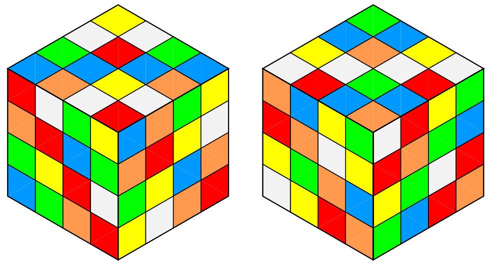

Figure 9: A very complicated new pattern on a 4×4×4 cube.

Finally, what about the next size up, the 4×4×4 cube? A complete analysis of its colour equality patterns is far beyond the scope of this article. Nevertheless, there is a simple trick we can apply to yield a surprising variety of new patterns there as well. We begin by noting that we can treat a $4 \times 4 \times 4$ cube either as an 'effective' $2 \times 2 \times 2$ cube, or an 'effective' $3 \times 3 \times 3$ cube. That is, by only allowing certain moves on the cube, any patterns on the smaller cubes can also be reproduced on the $4 \times 4 \times 4$ cube, as in Figure 8. By itself this is obviously trivial, and we haven't generated any new patterns at all. However, suppose we first arrange the cube as in Figure 8a, and then apply the chessboard effect as in Figure 8b. The result is shown in Figure 9, and is an impressively complicated new 4×4×4 colour equality pattern. Readers are left to work out for themselves what the result would be of first applying 8b and then 8a, as well as many other patterns that can be created in this way.

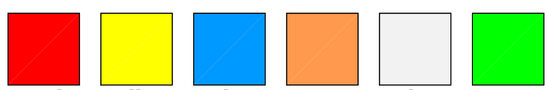

Figure 10: A colour chart labelling the six colours used throughout the article.

References

[1] Rokicki, T., Kociemba, H., Davidson, M. and Dethridge, J. (2013) The diameter of the Rubik's cube group is twenty, *SIAM J. Discrete Math.*, vol. 27, pp. 1082-1105.

[2] Demaine, E.D., Demaine, M.L., Eisenstat, S., Lubiw, A. and Winslow, A. (2011) Algorithms for solving Rubik's cubes, *Algorithms ESA 2011*, pp. 689-700.

[3] Cornock, C. (2015) Teaching group theory using Rubik's cubes, *Int. J. Math. Ed. Sci. Tech.*, vol. 46, pp. 957-967.

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Supplementary Material

All of the patterns presented here really are best studied by constructing actual 3D cubes. To make this as convenient as possible, the following pages consist of ready-made templates that can be cut out and folded up into cubes, with the colours all correctly filled in already. First, there are:

(a) Sixteen pages with templates for all the patterns presented in Figures 1-9 in the main text.

(b) Six pages with the twelve possible corner patterns already filled in on $3\times3\times3$ cubes, but the other slots left blank. These templates could therefore be useful if readers want to try out for themselves how particular arrangements of centres and/or sides might or might not fit together with a given corner pattern.

Spoiler Alert!

The article left various items as challenges to readers:

1. It was claimed that for most corner patterns there is only one way that the centres can be oriented to still form a colour equality pattern, but that for some corner patterns up to three orientations were possible.

2. Readers were invited to play around with some of the other corner patterns to try to achieve the same effect as in Figure 7, where every face contains all six colours.

The final five pages contain templates presenting some (but still not all) of these possibilities. So any readers who want to tackle these challenges themselves first should avoid looking at any of the templates beyond item (b) above.

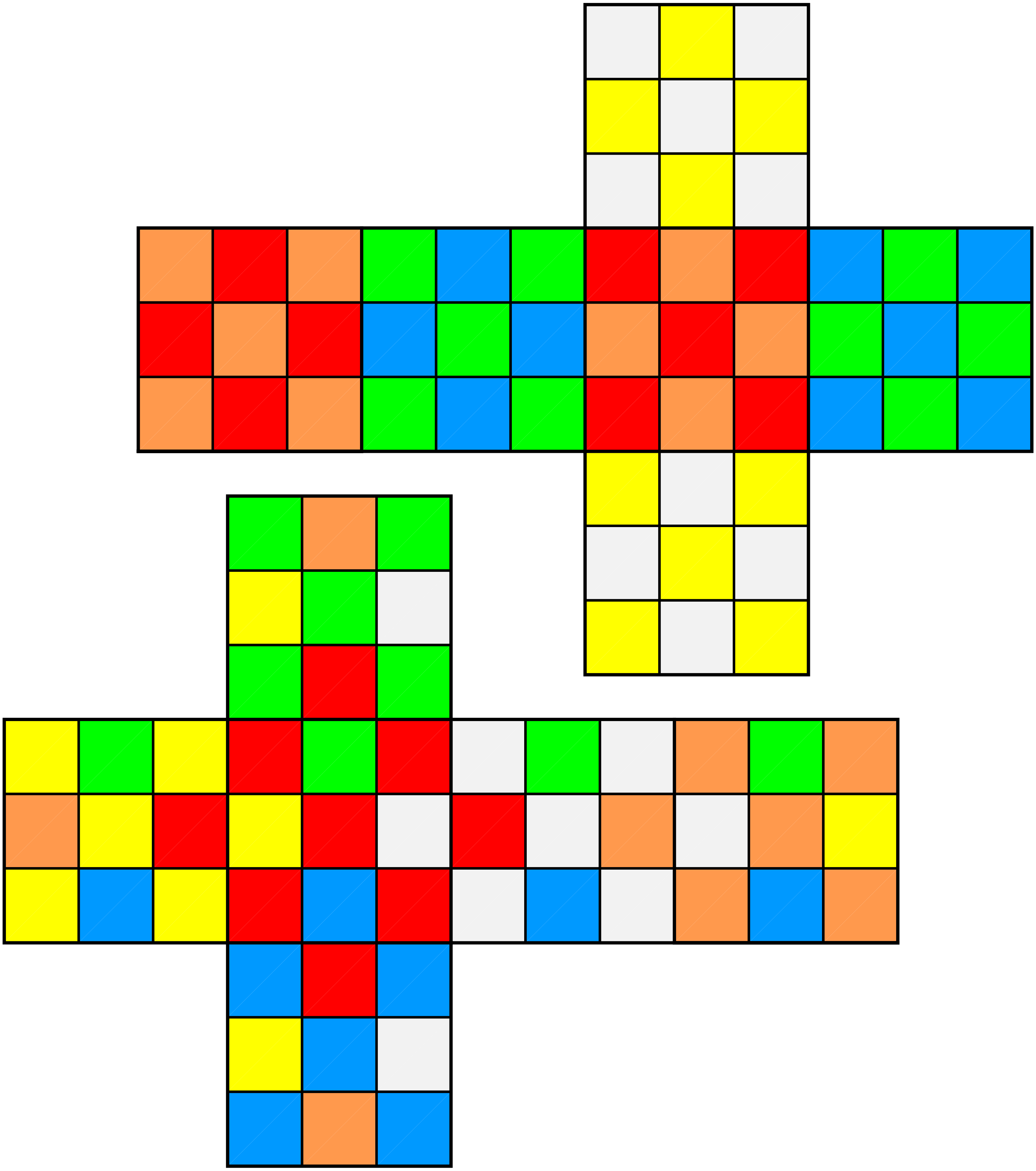


Figure 11: Templates for Figures 1a and 1b.

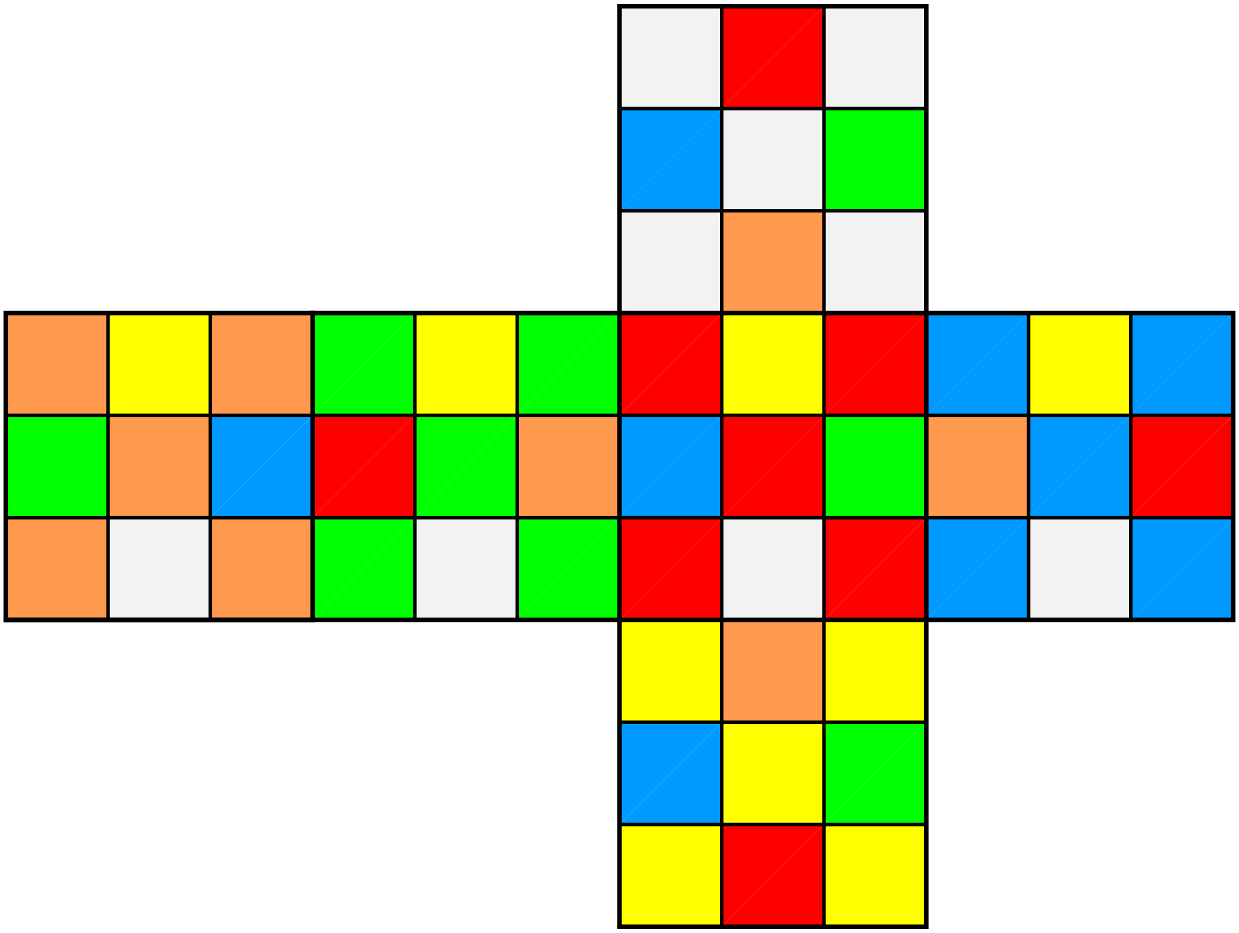


Figure 12: Template for Figure 1c.

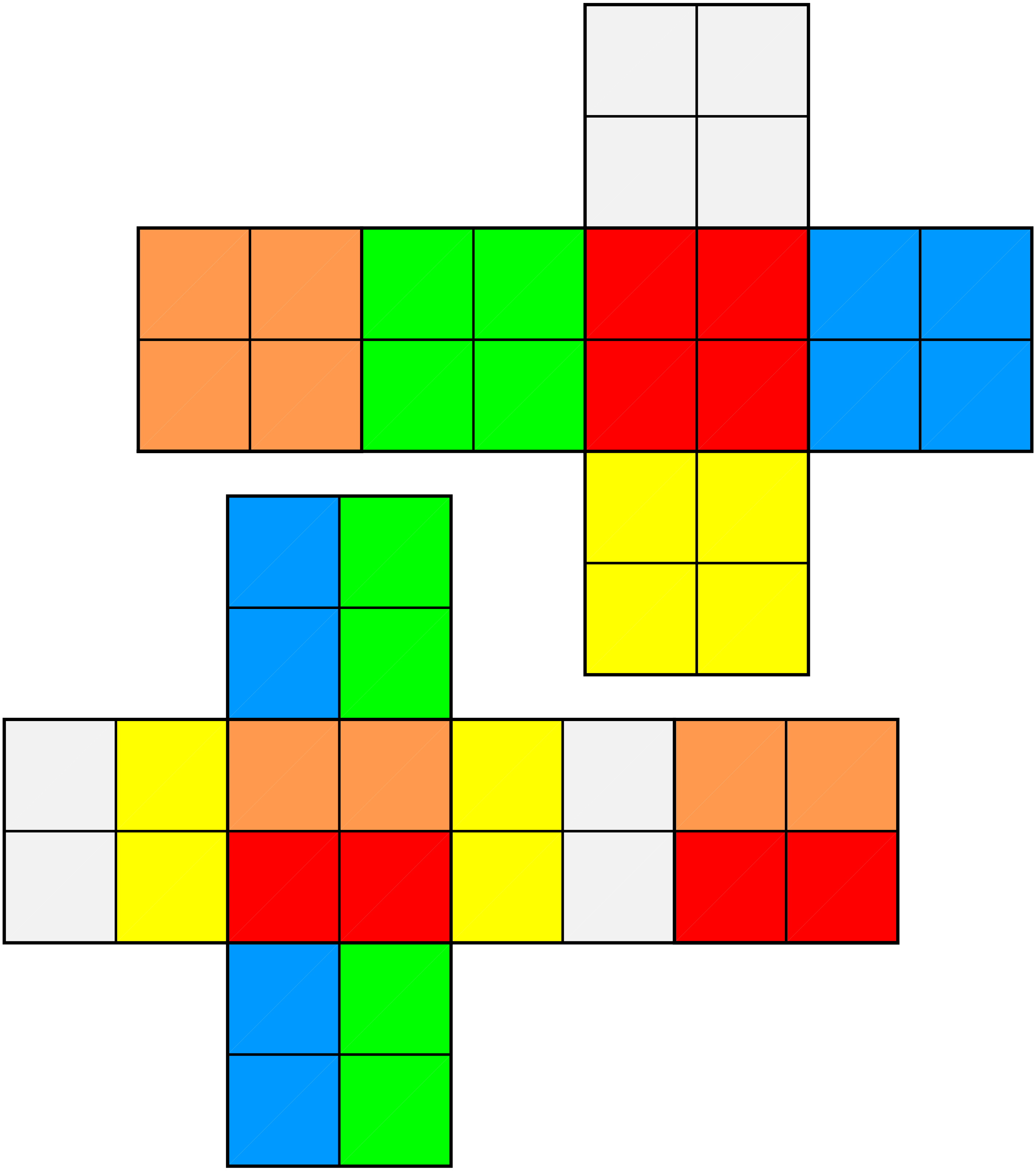


Figure 13: Templates for corner patterns A1 and A2.

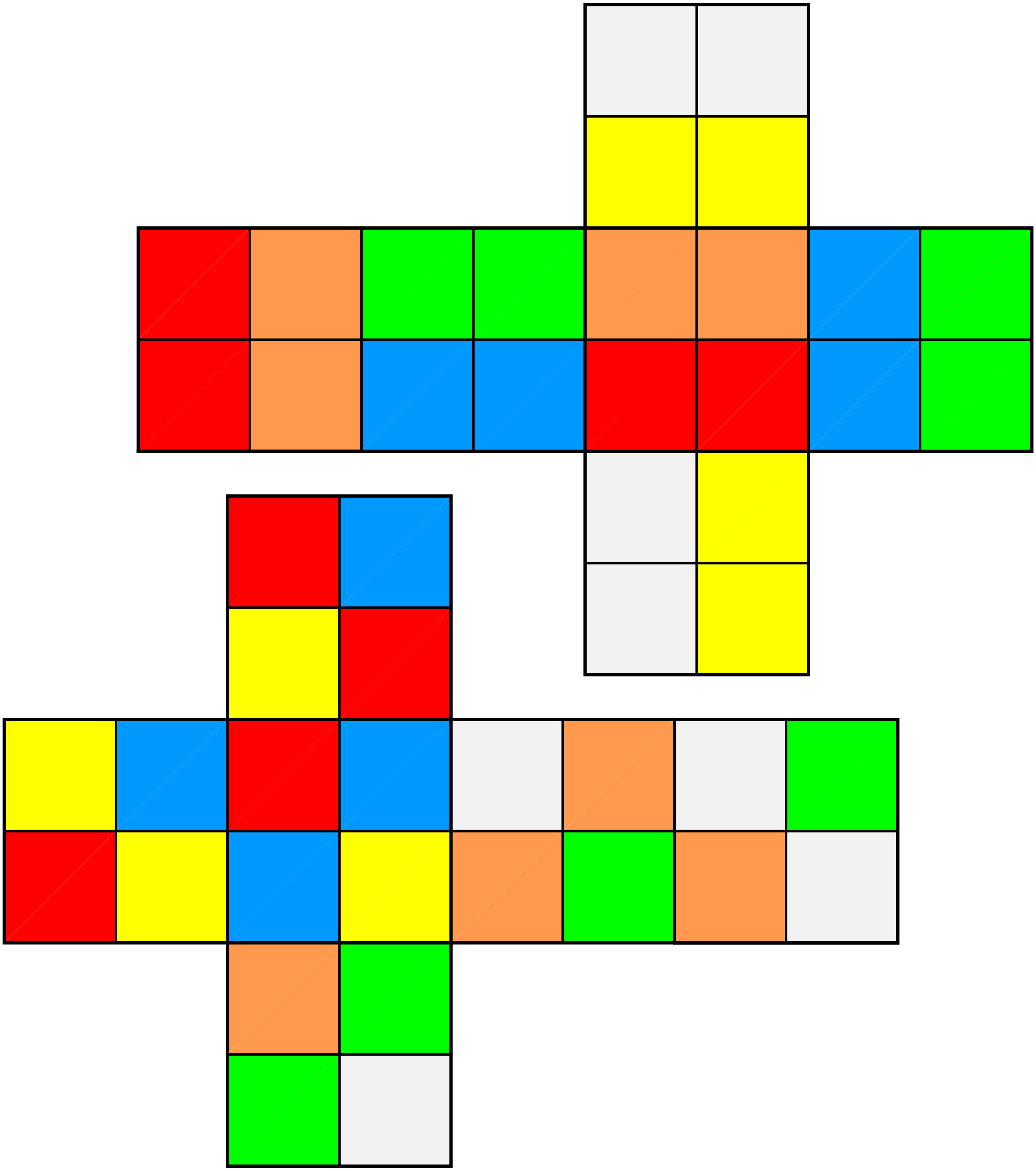


Figure 14: Templates for corner patterns A3 and A4.

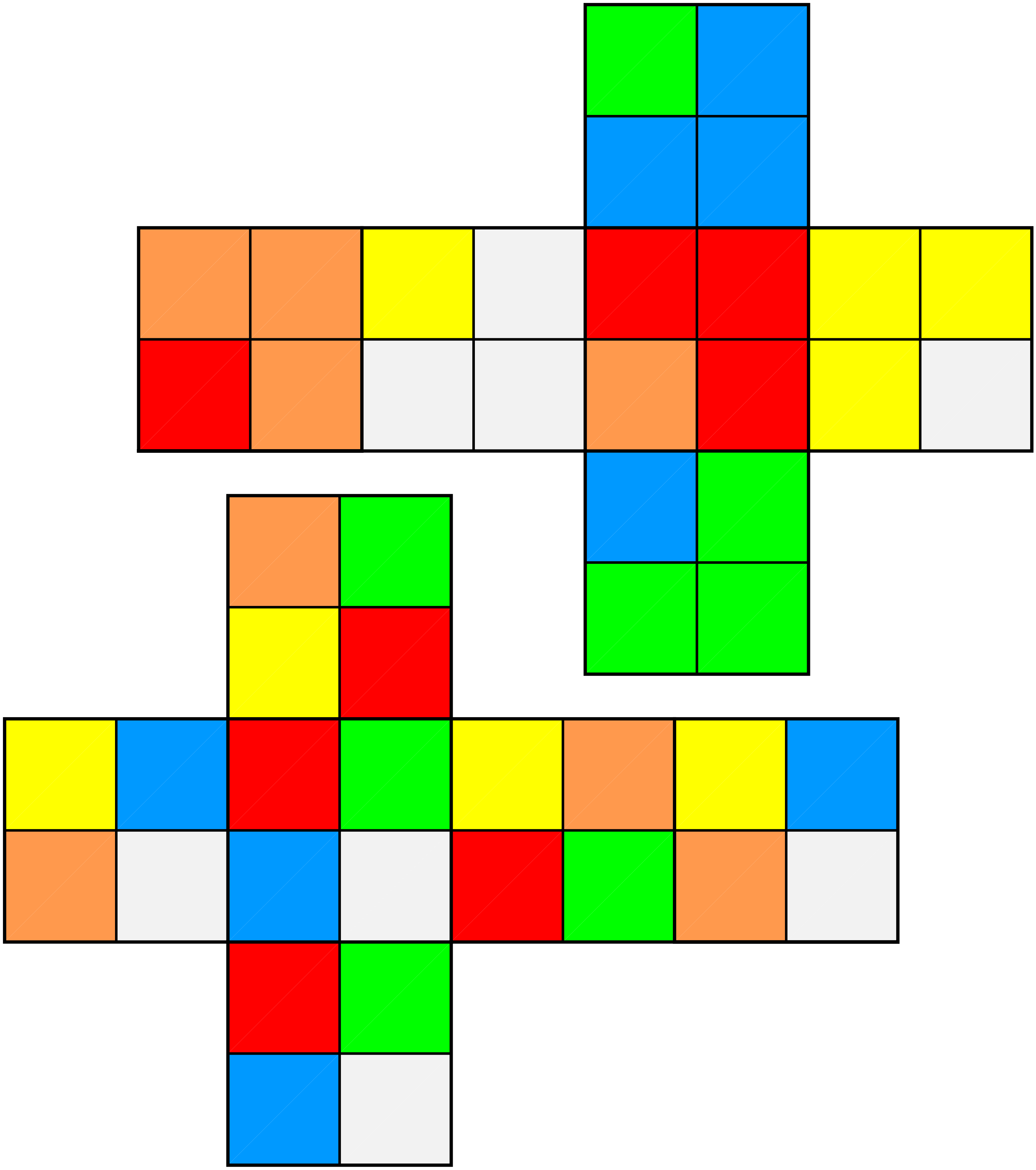


Figure 15: Templates for corner patterns A5 and A6.

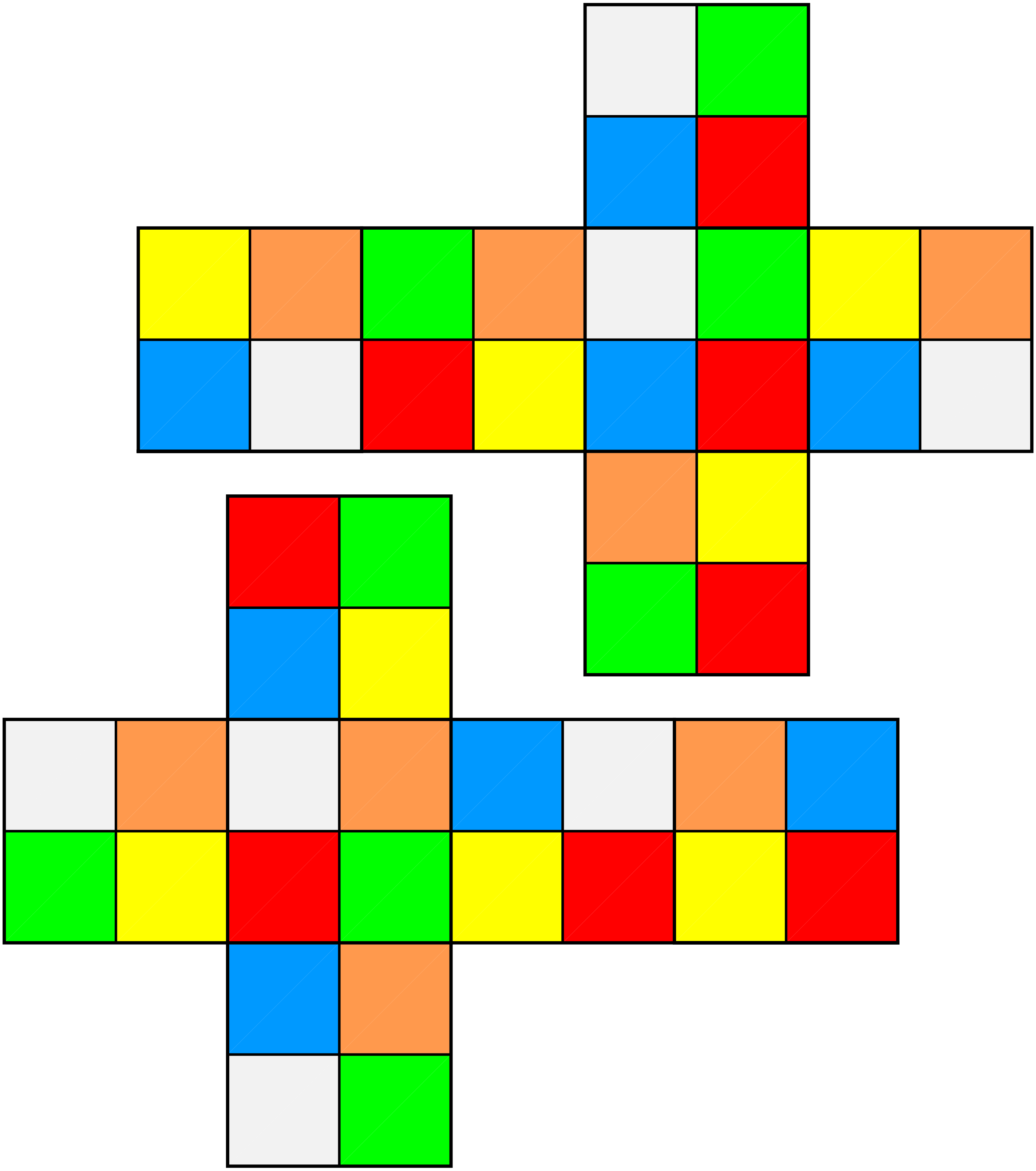


Figure 16: Templates for corner patterns A7 and B1.

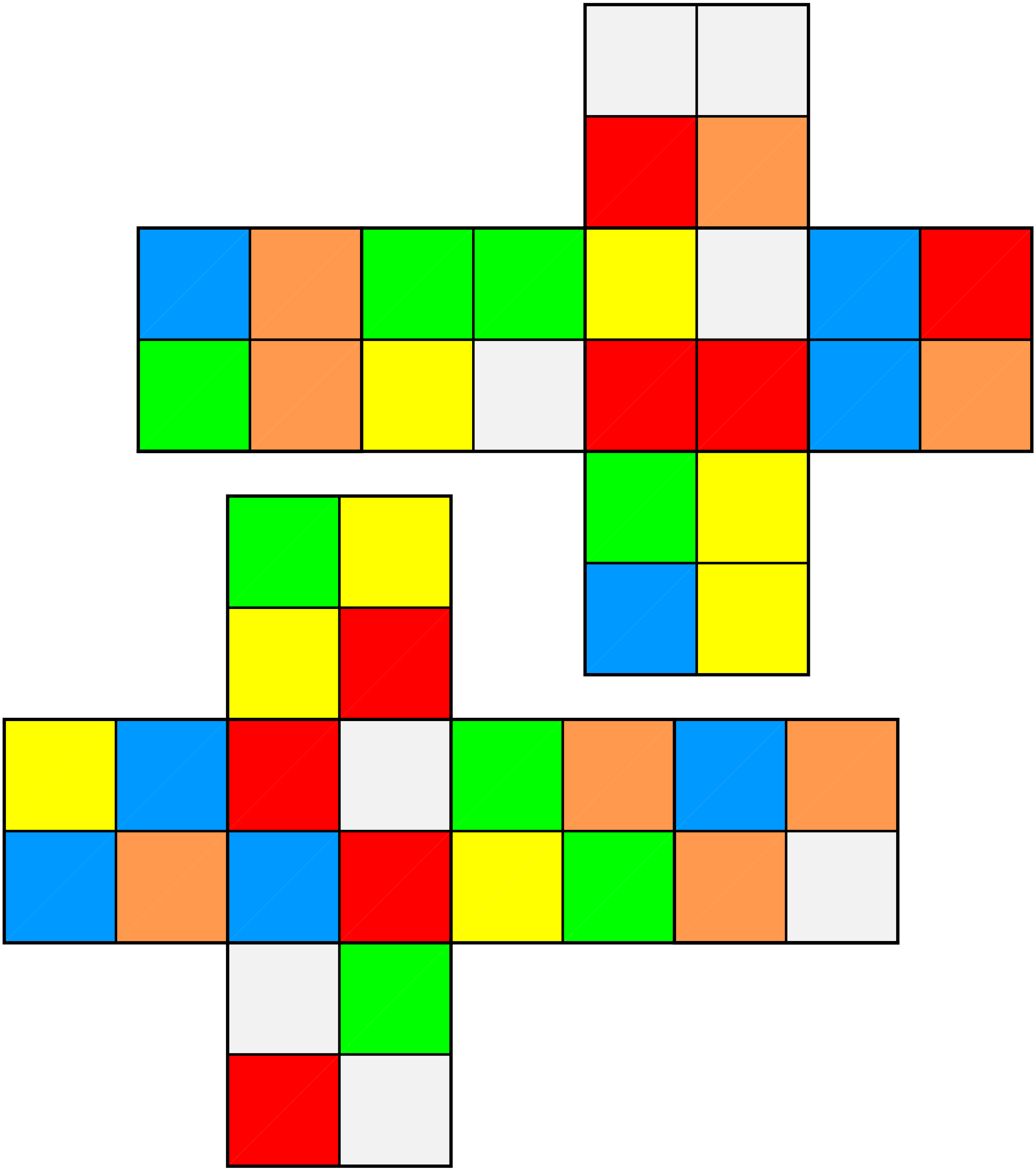


Figure 17: Templates for corner patterns B2 and B3.

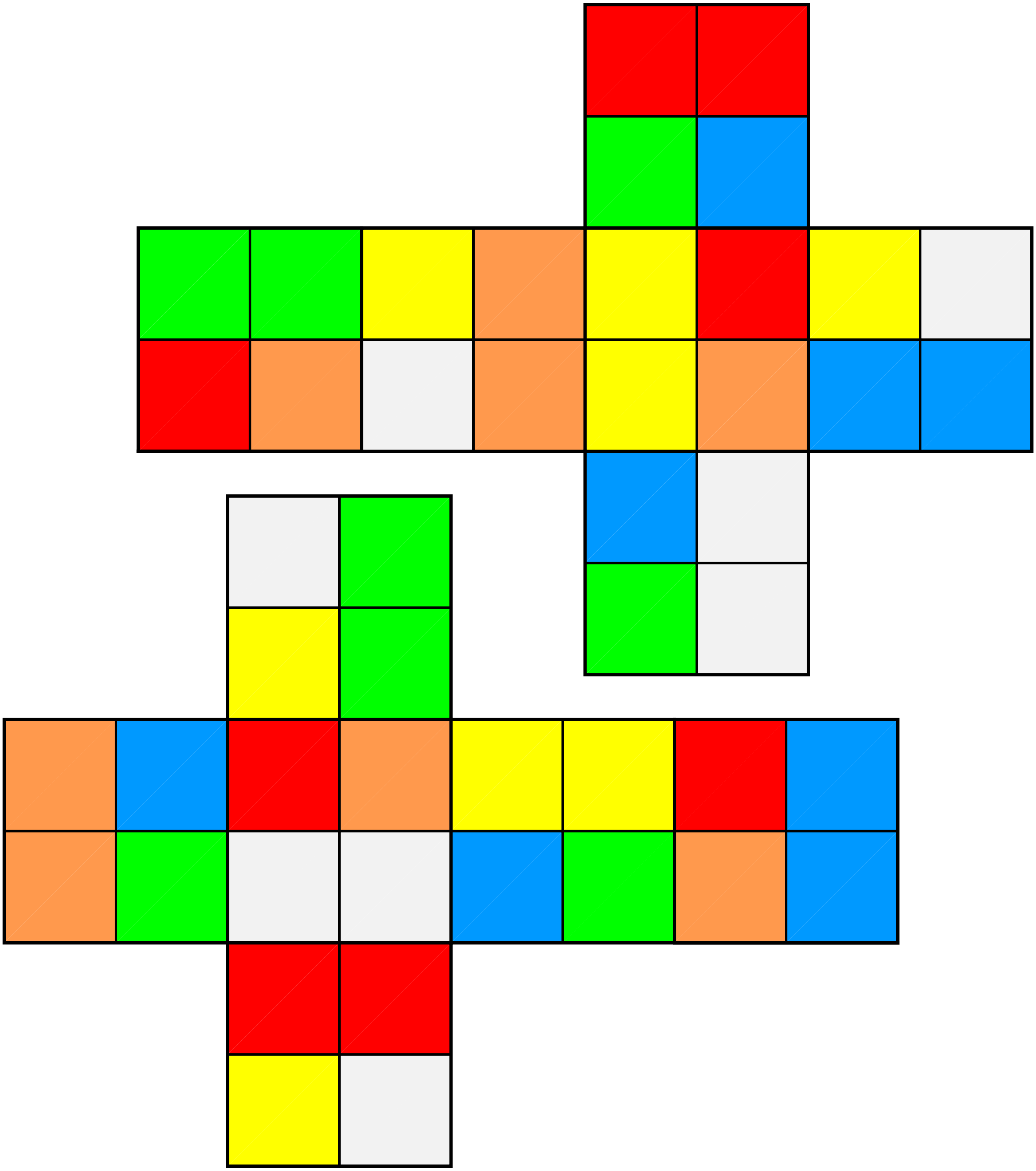


Figure 18: Templates for corner patterns B4 and B5.

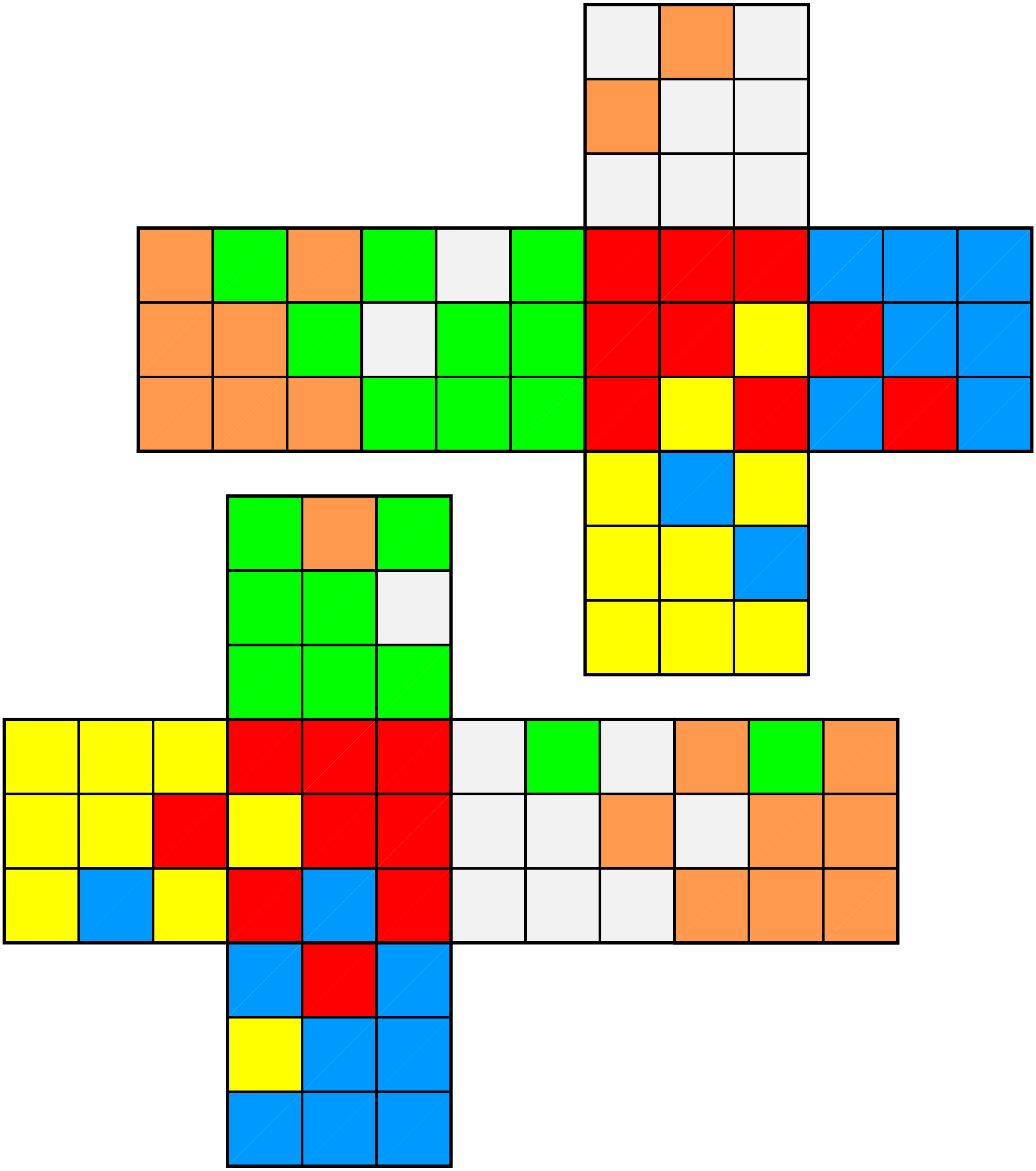


Figure 19: Templates for Figures 3a and 3b.

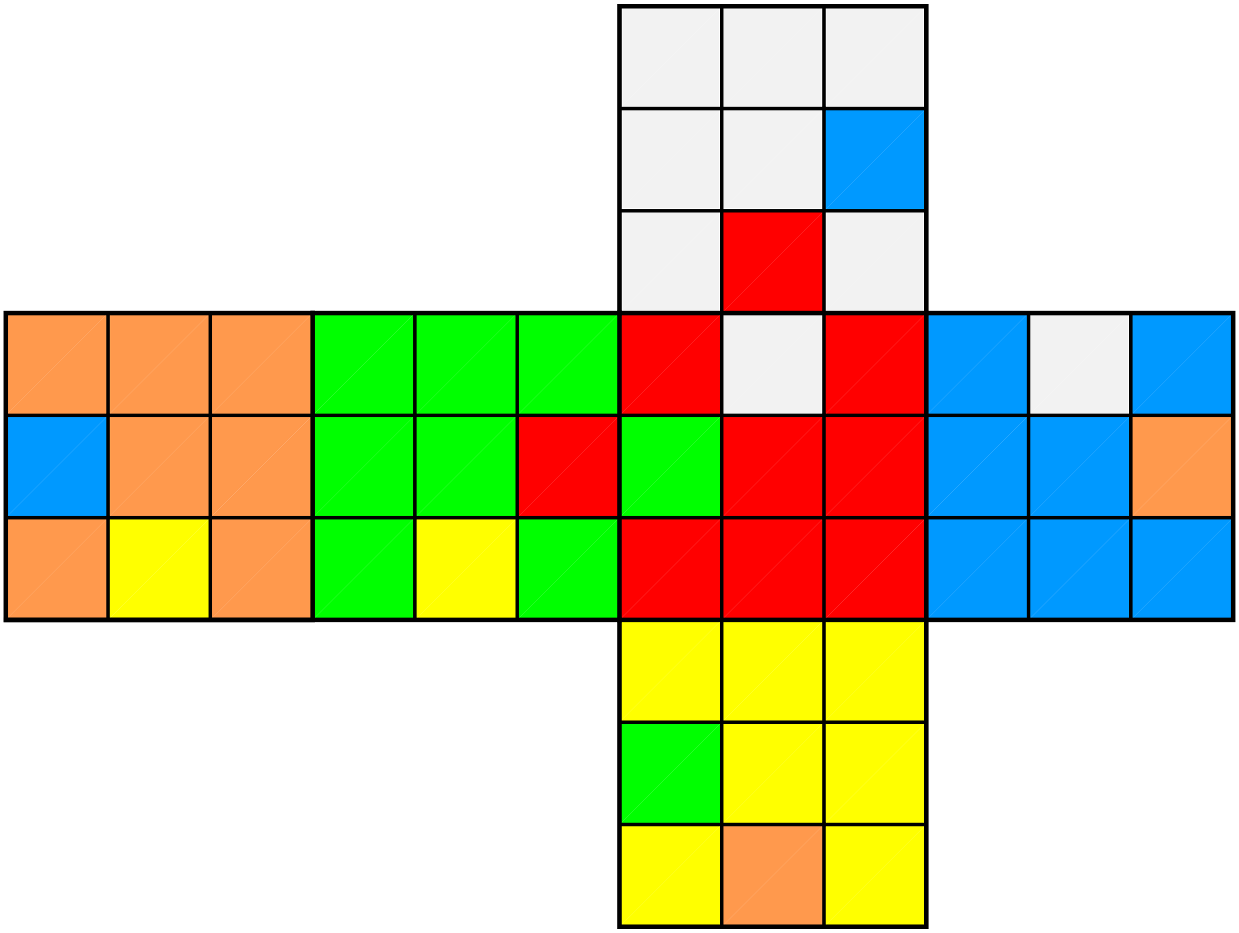


Figure 20: Template for Figure 3c.

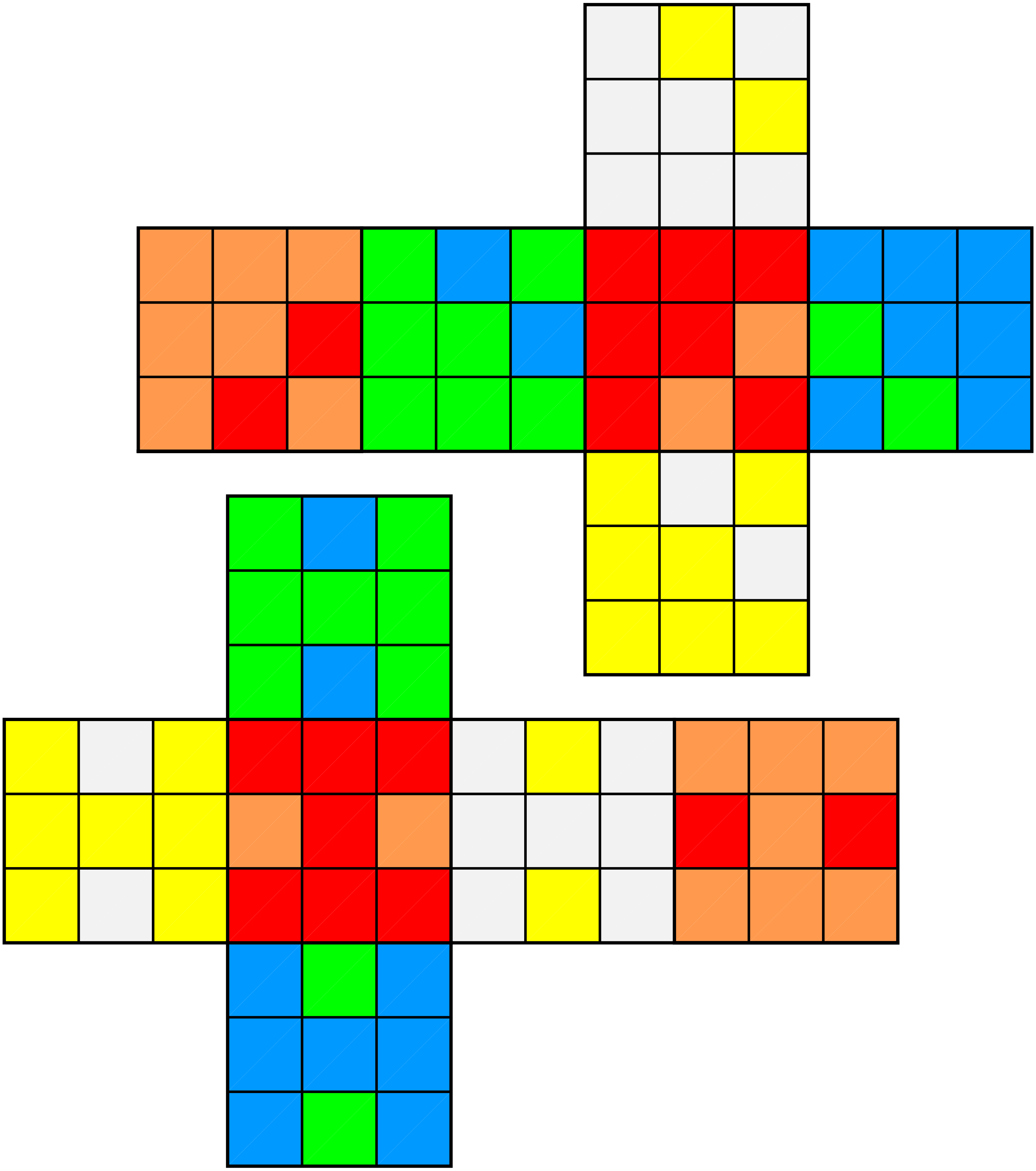


Figure 21: Templates for Figures 4a and 4b.

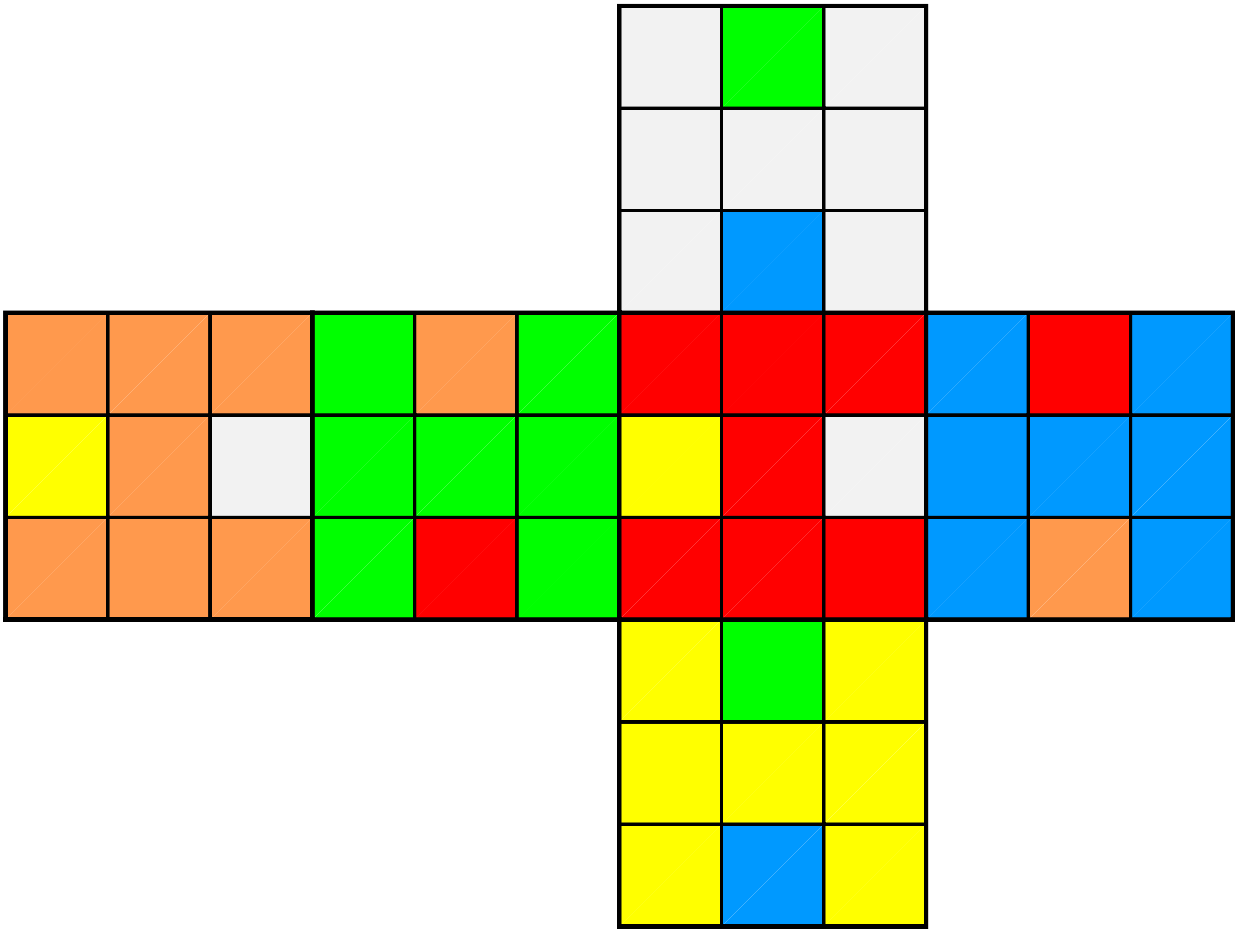


Figure 22: Template for Figure 4c.

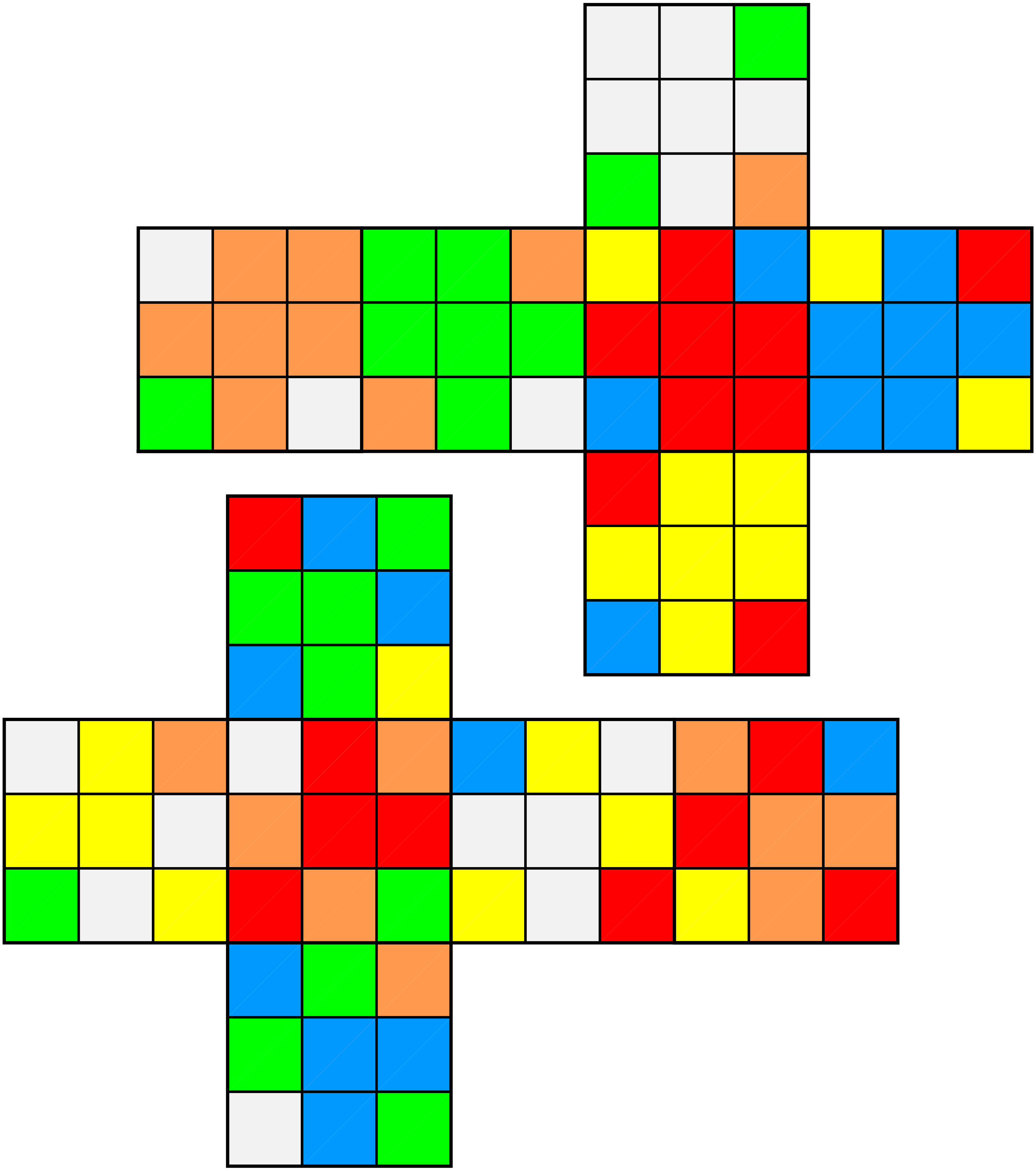


Figure 23: Templates for Figures 5 and 6.

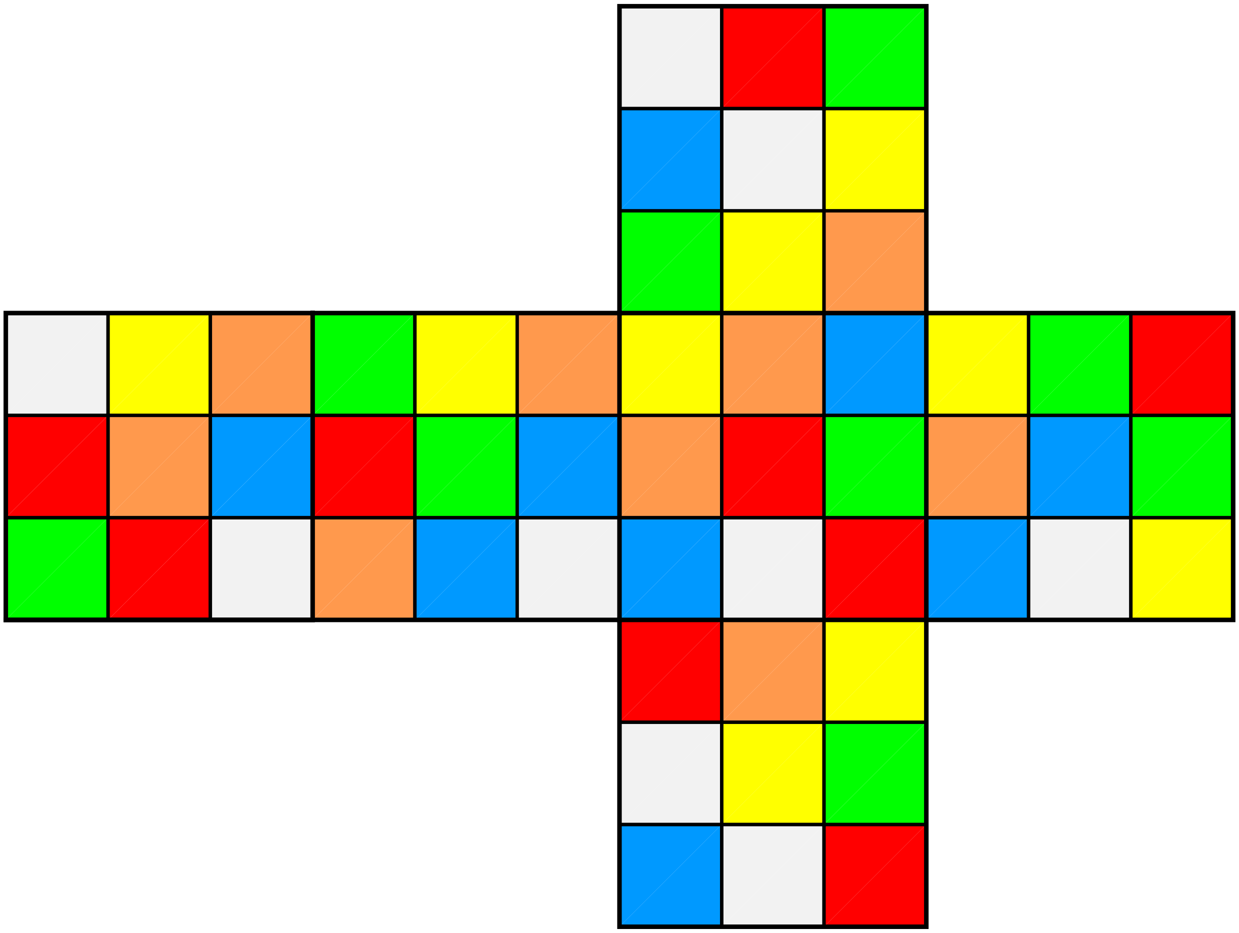


Figure 24: Template for Figure 7.

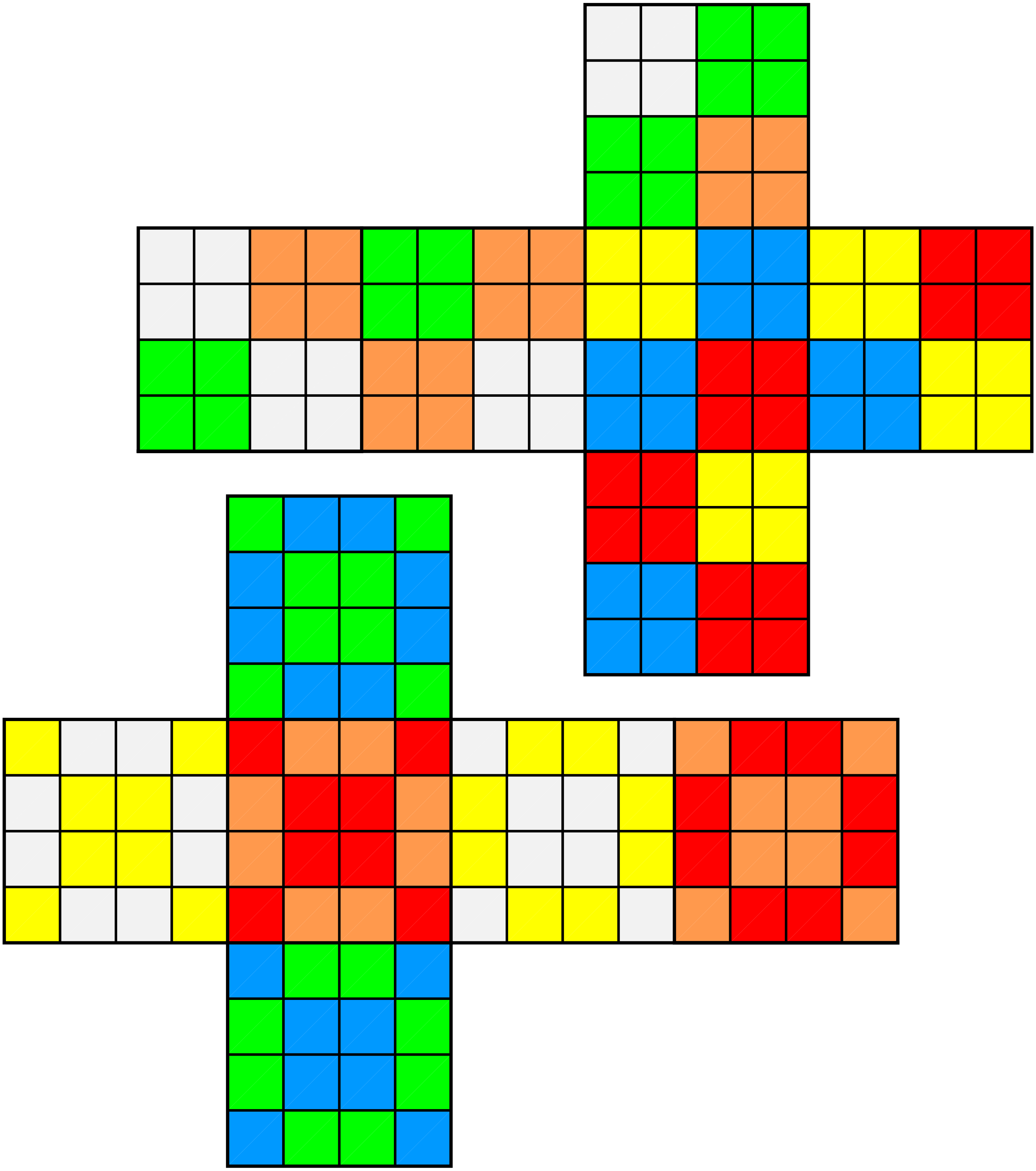


Figure 25: Templates for Figure 8.

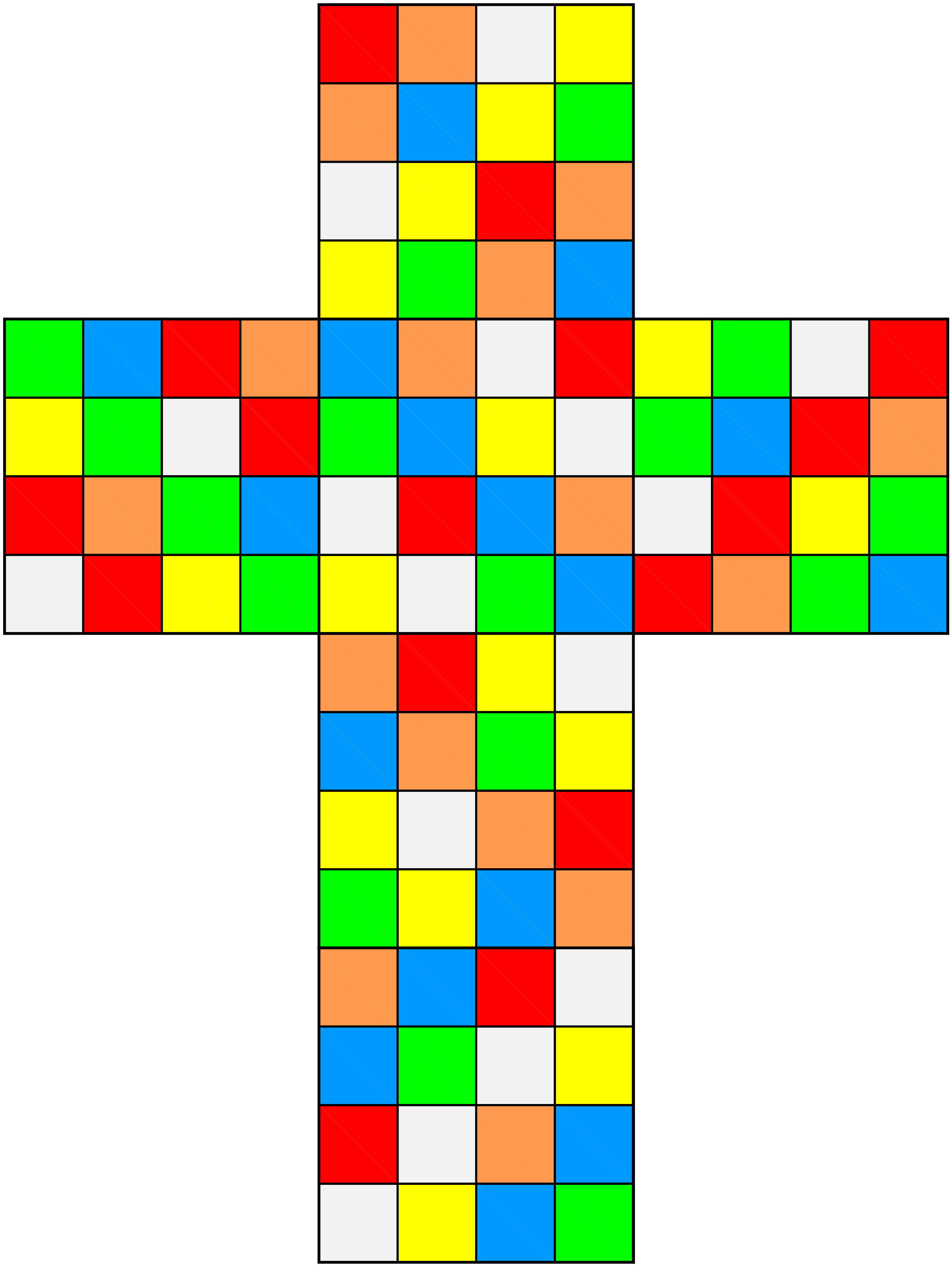


Figure 26: Template for Figure 9.

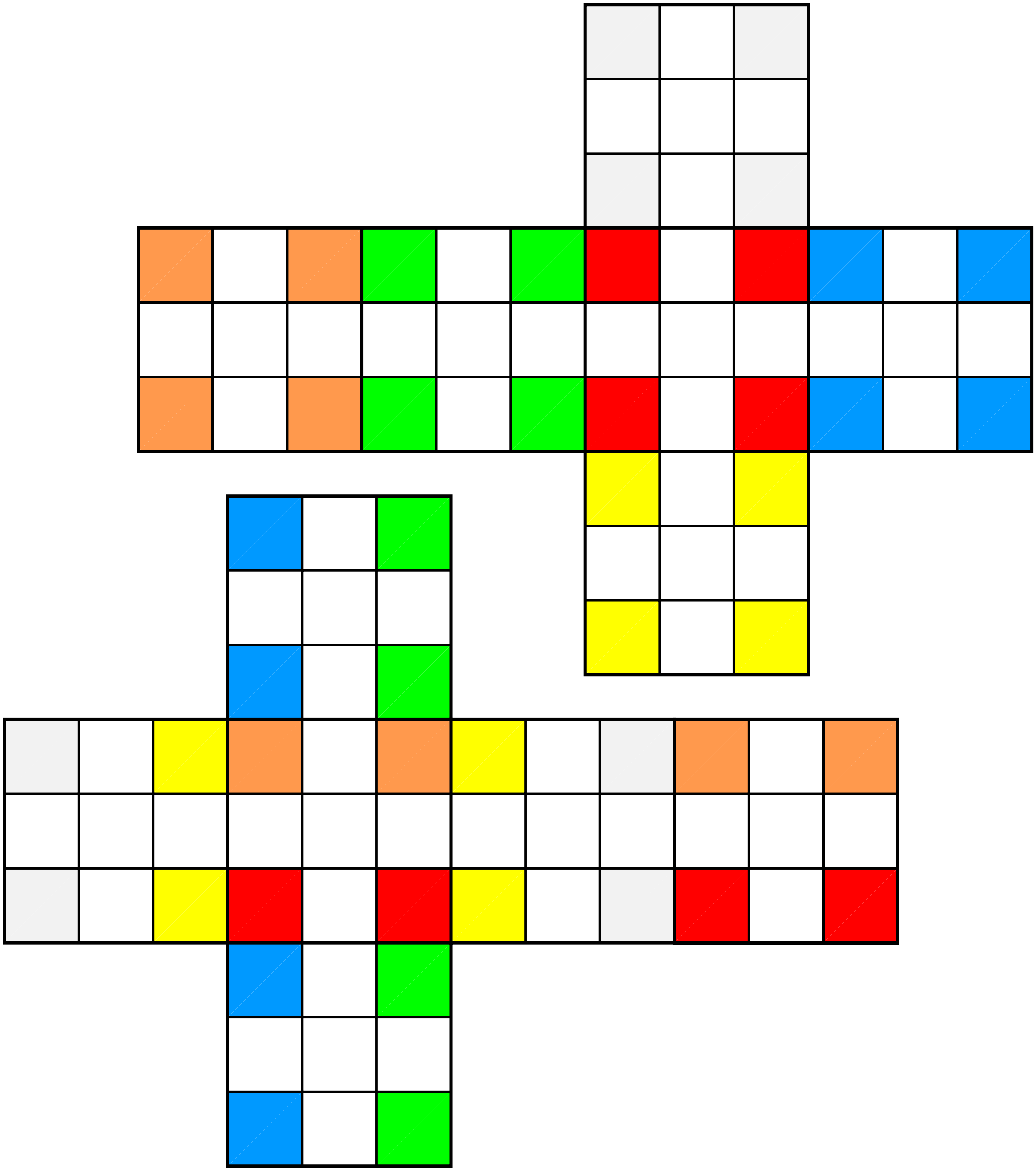


Figure 27: Corner patterns A1 and A2 on 3×3×3 cubes.

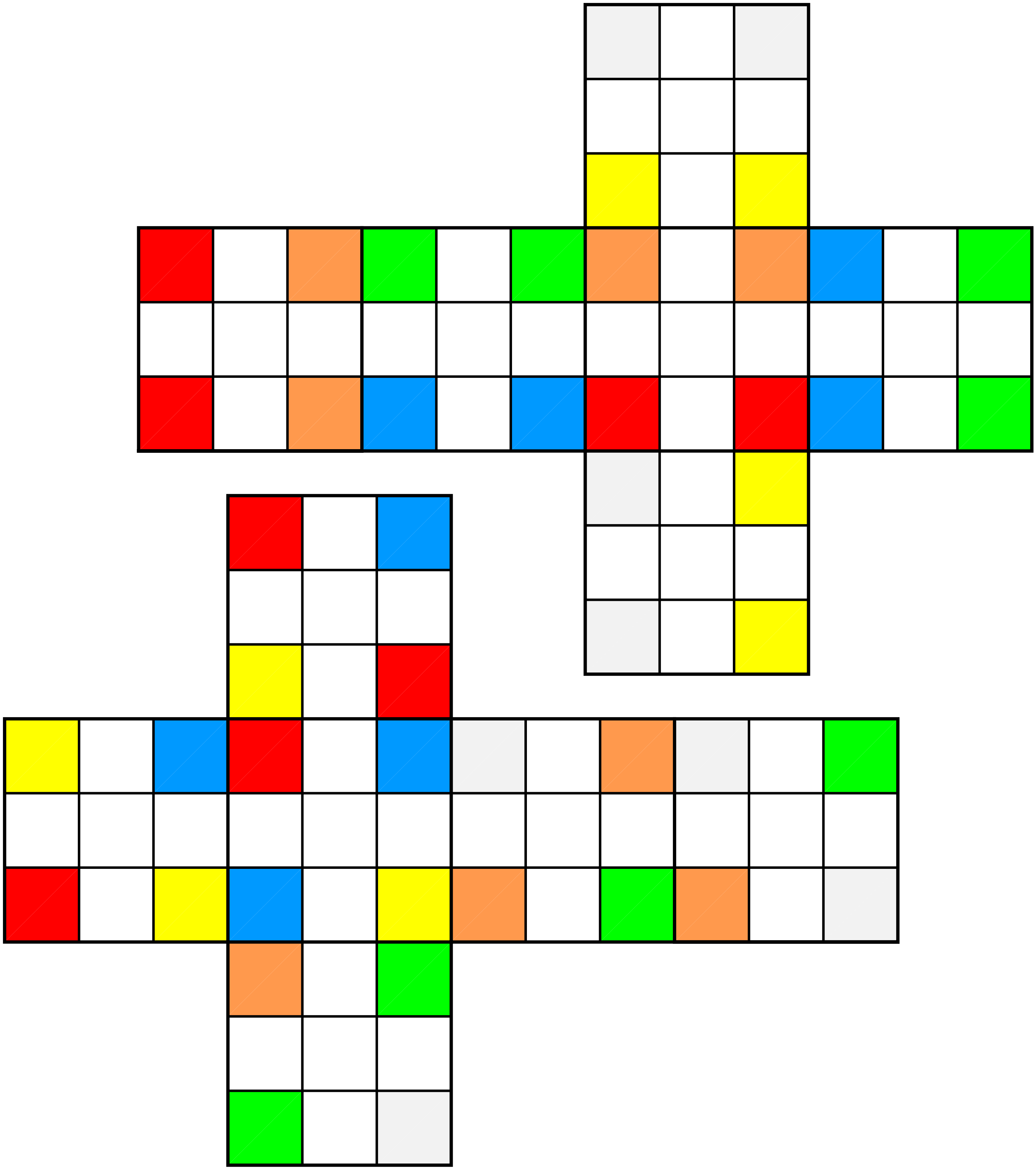


Figure 28: Corner patterns A3 and A4 on 3×3×3 cubes.

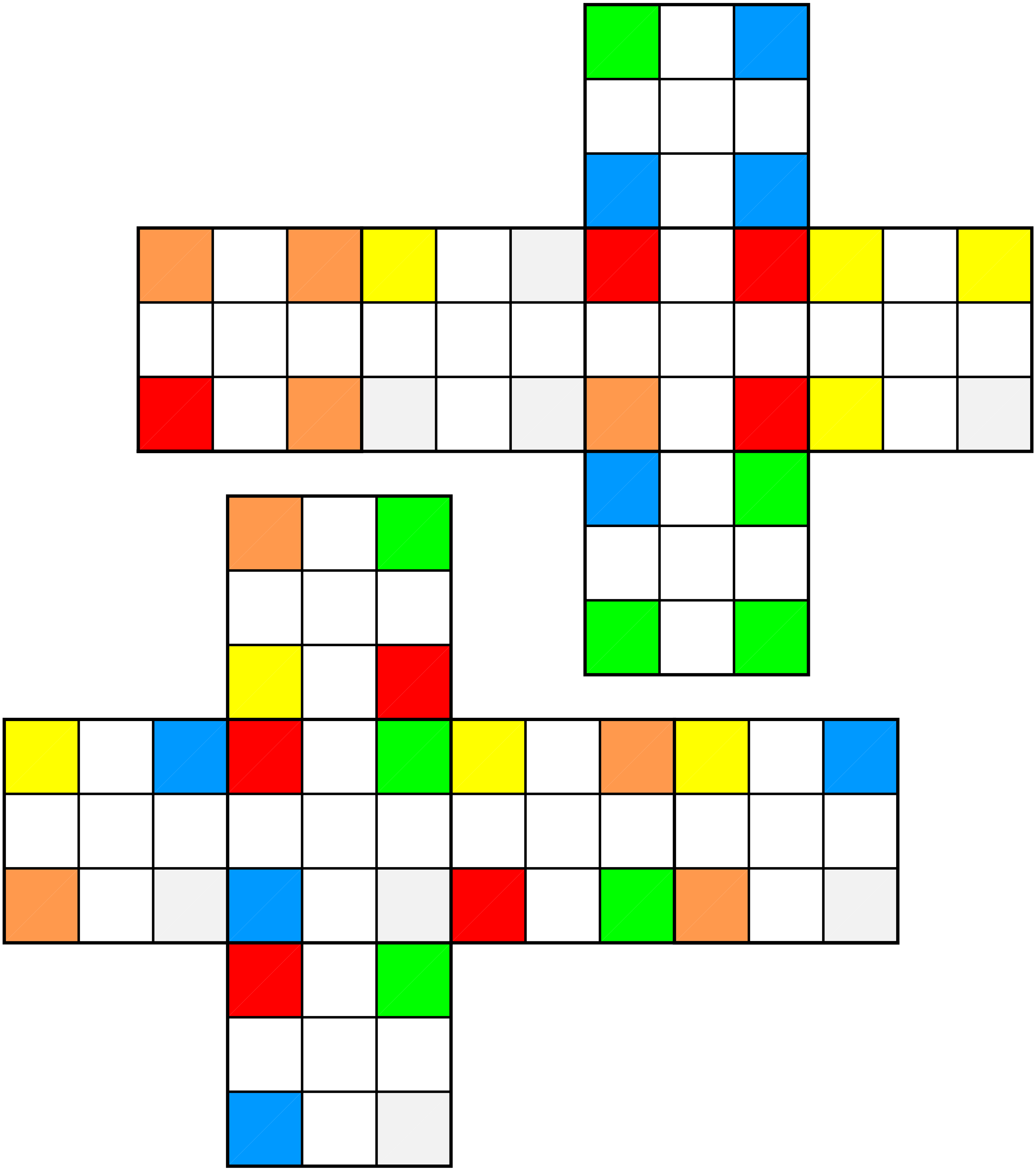
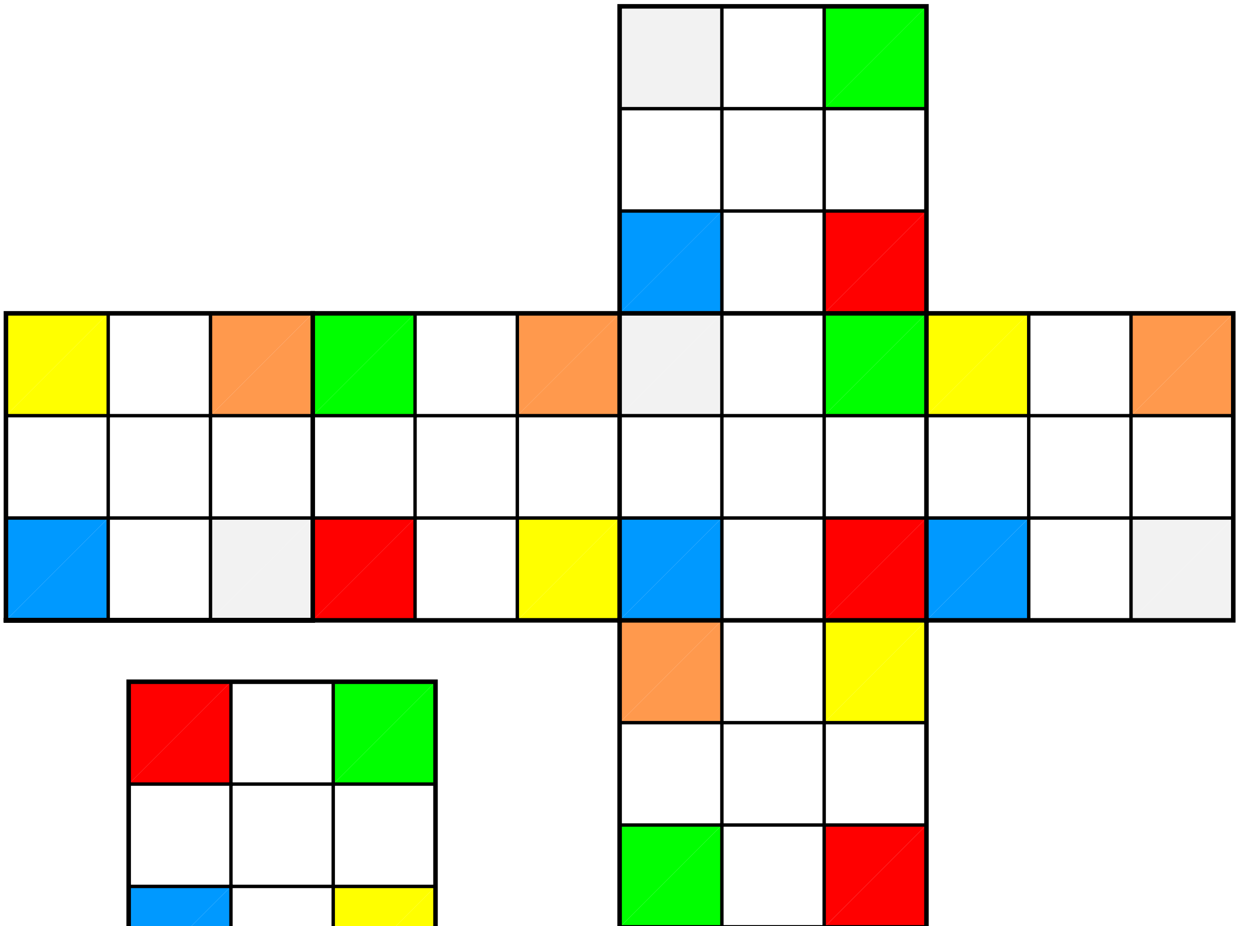


Figure 29: Corner patterns A5 and A6 on 3×3×3 cubes.



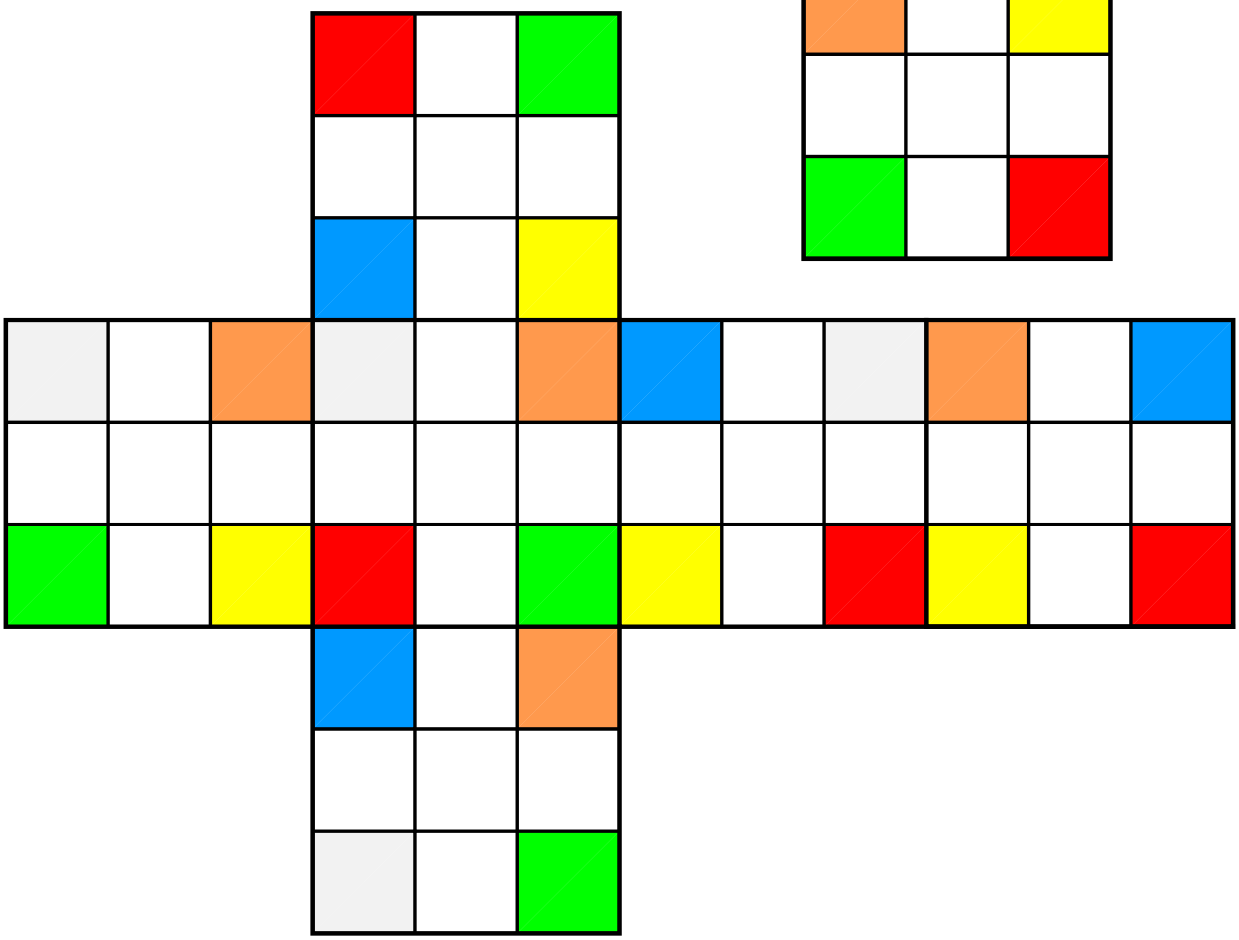


Figure 30: Corner patterns A7 and B1 on 3×3×3 cubes.

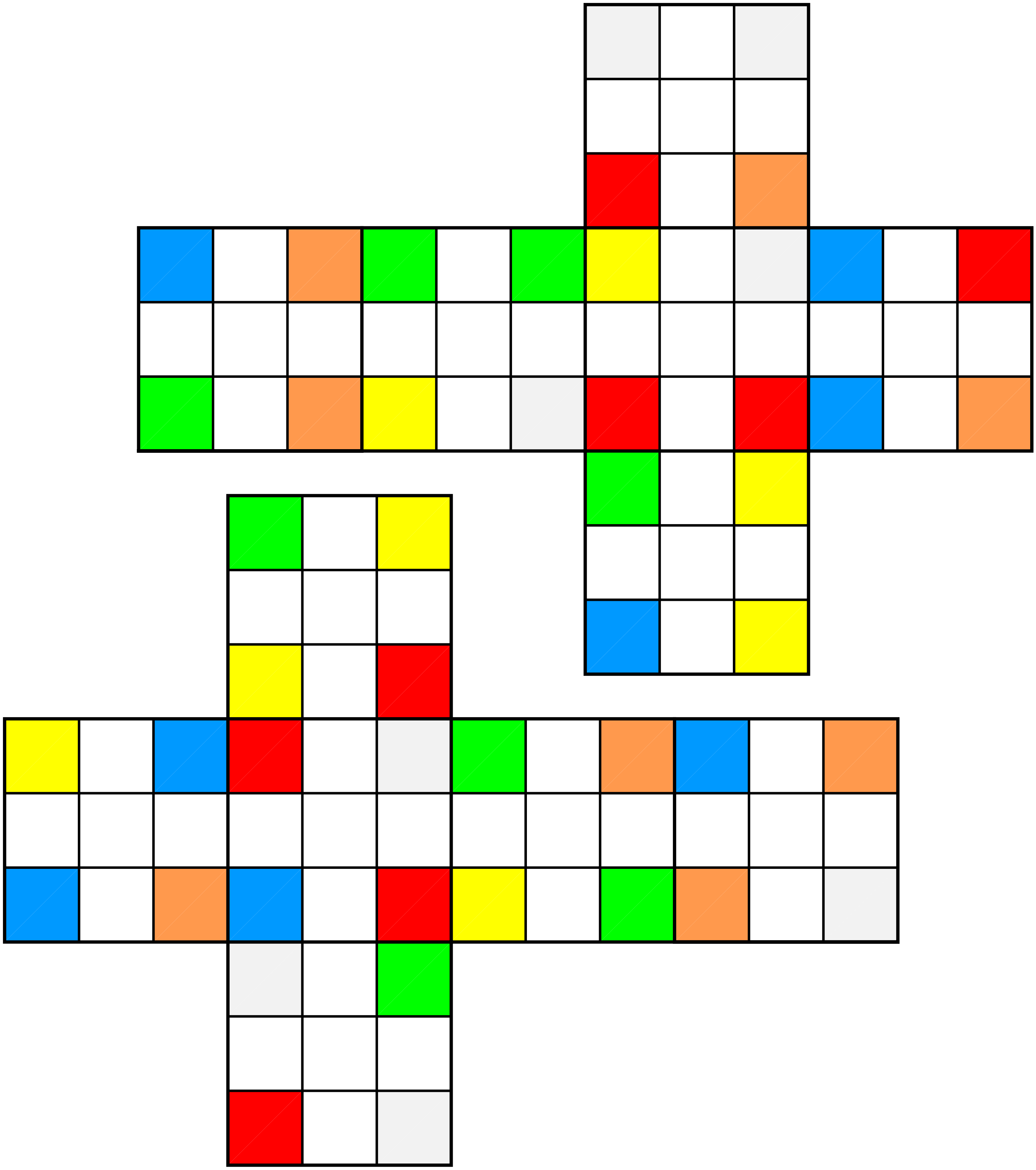


Figure 31: Corner patterns B2 and B3 on 3×3×3 cubes.

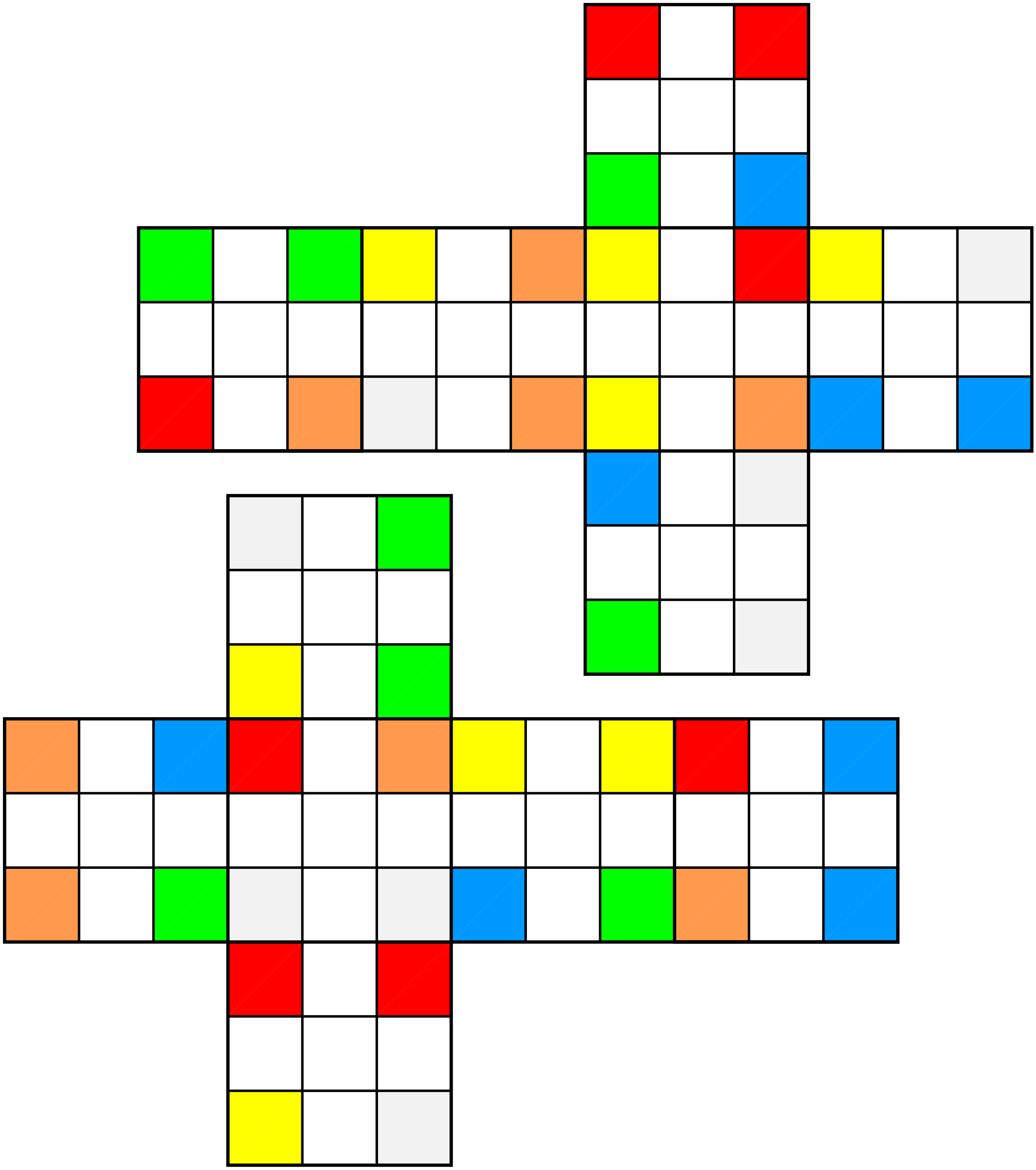


Figure 32: Corner patterns B4 and B5 on 3×3×3 cubes.

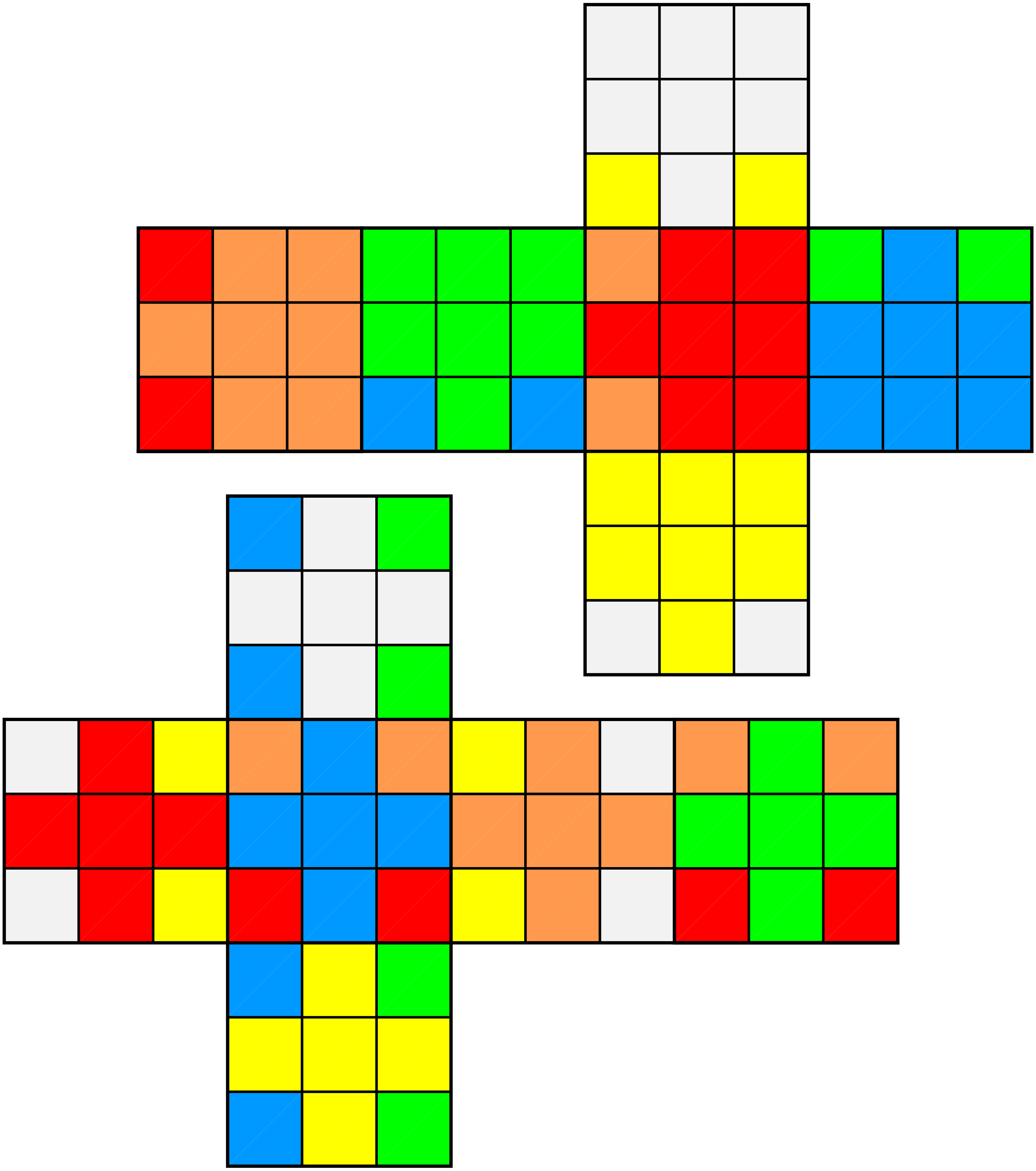


Figure 33: Two centre arrangements for corner pattern A2.

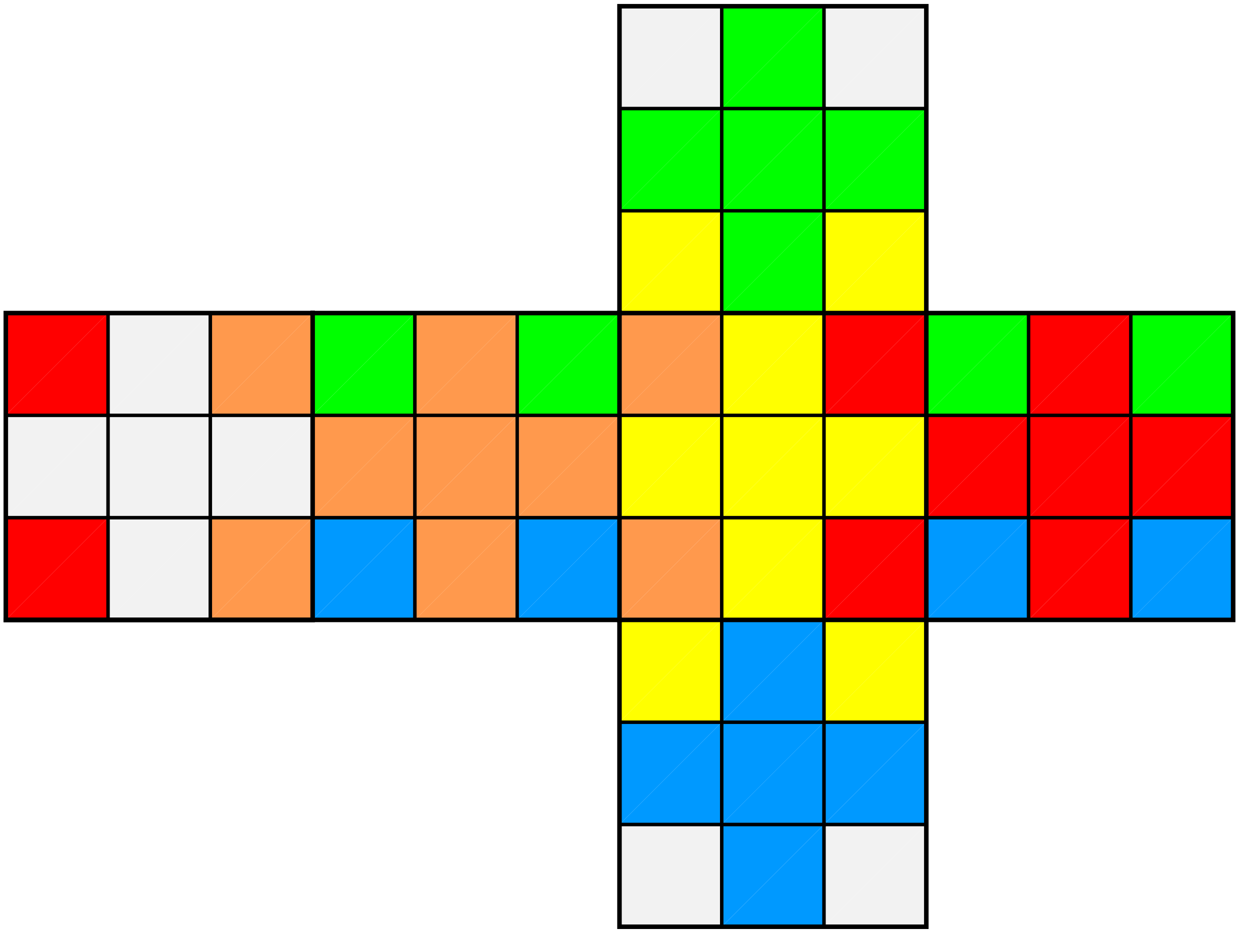


Figure 34: And even a third centre arrangement for A2.

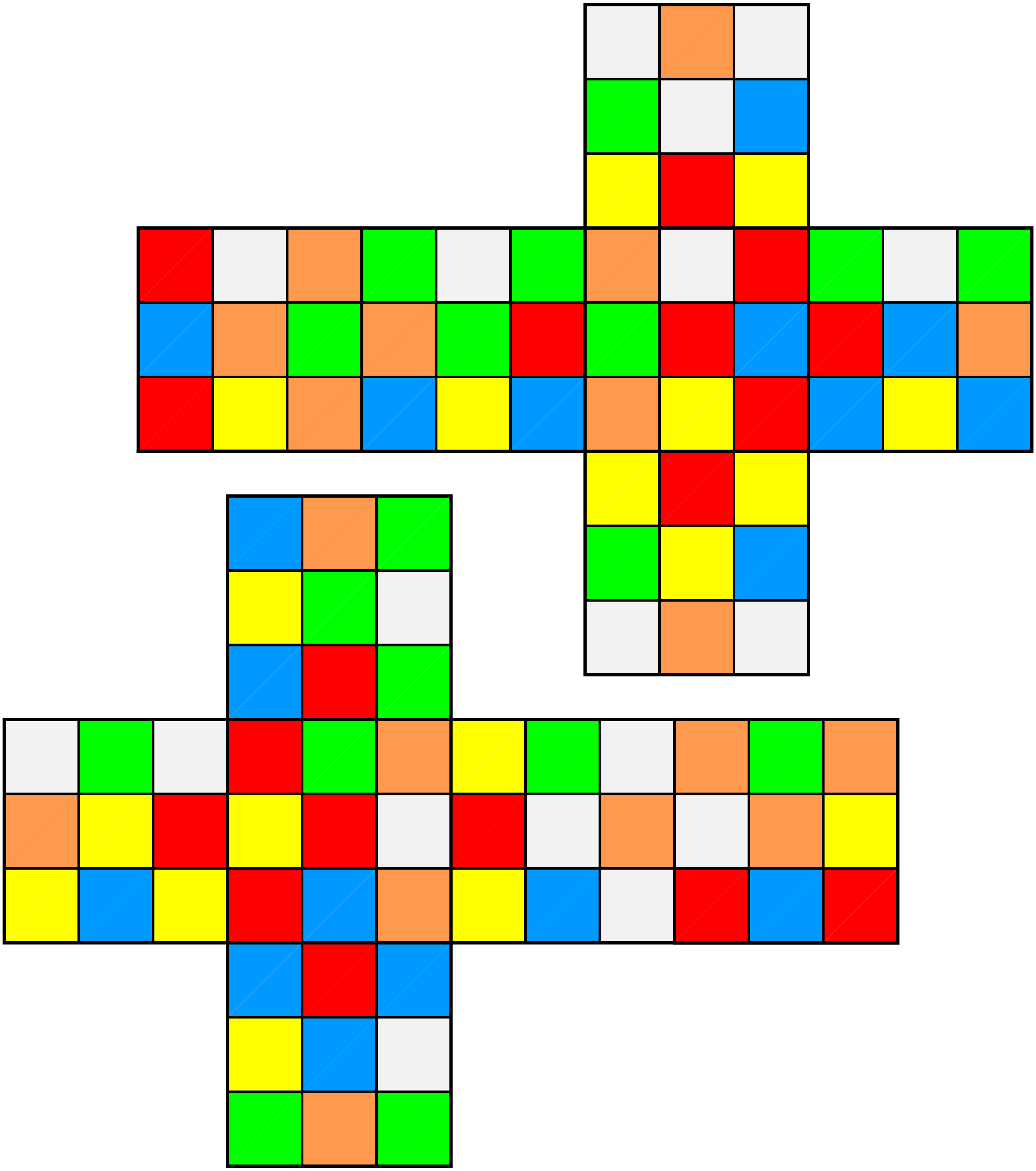


Figure 35: Two examples as in Figure 7 in the main text.

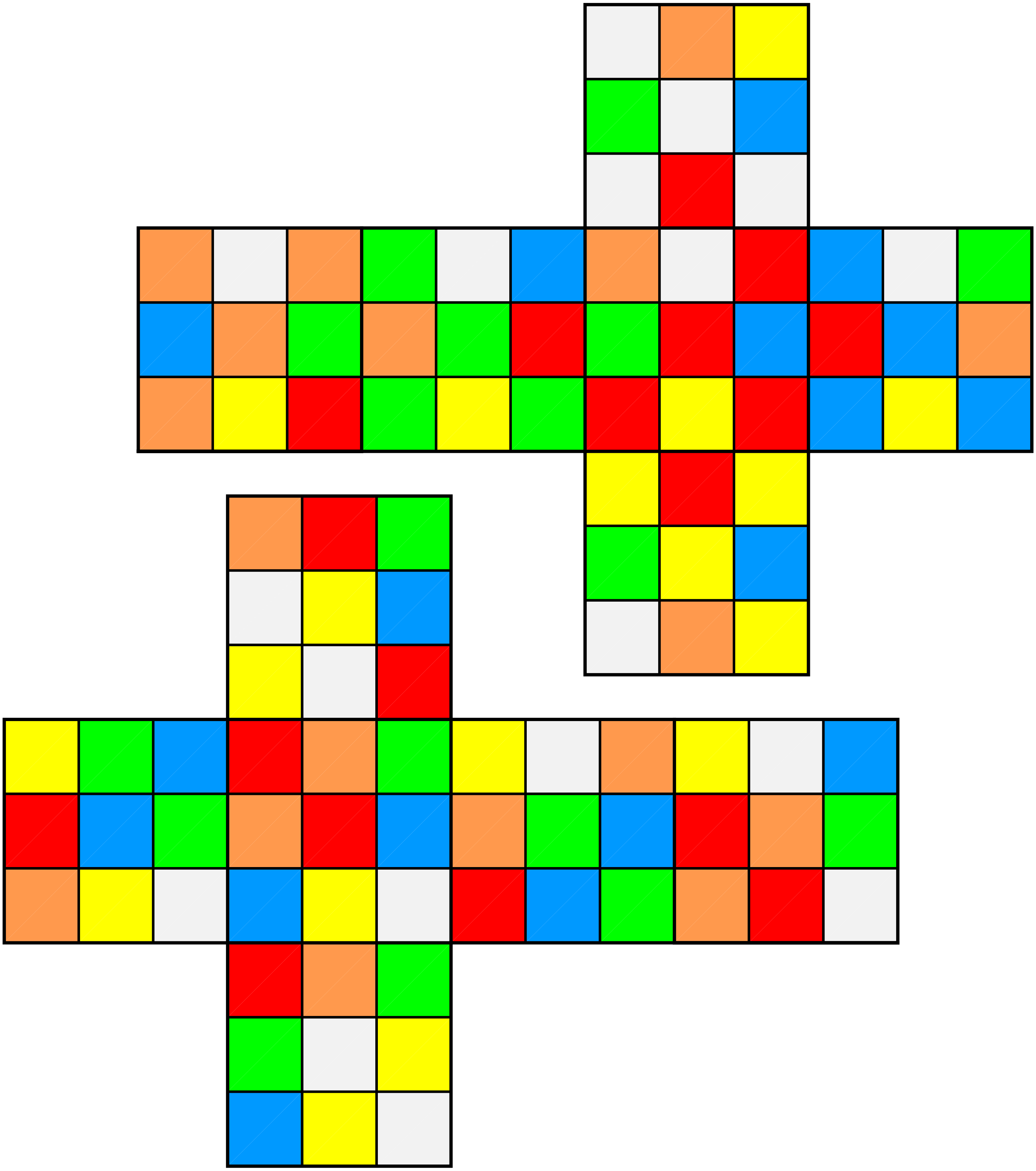


Figure 36: Another two examples as in Figure 7 in the main text.

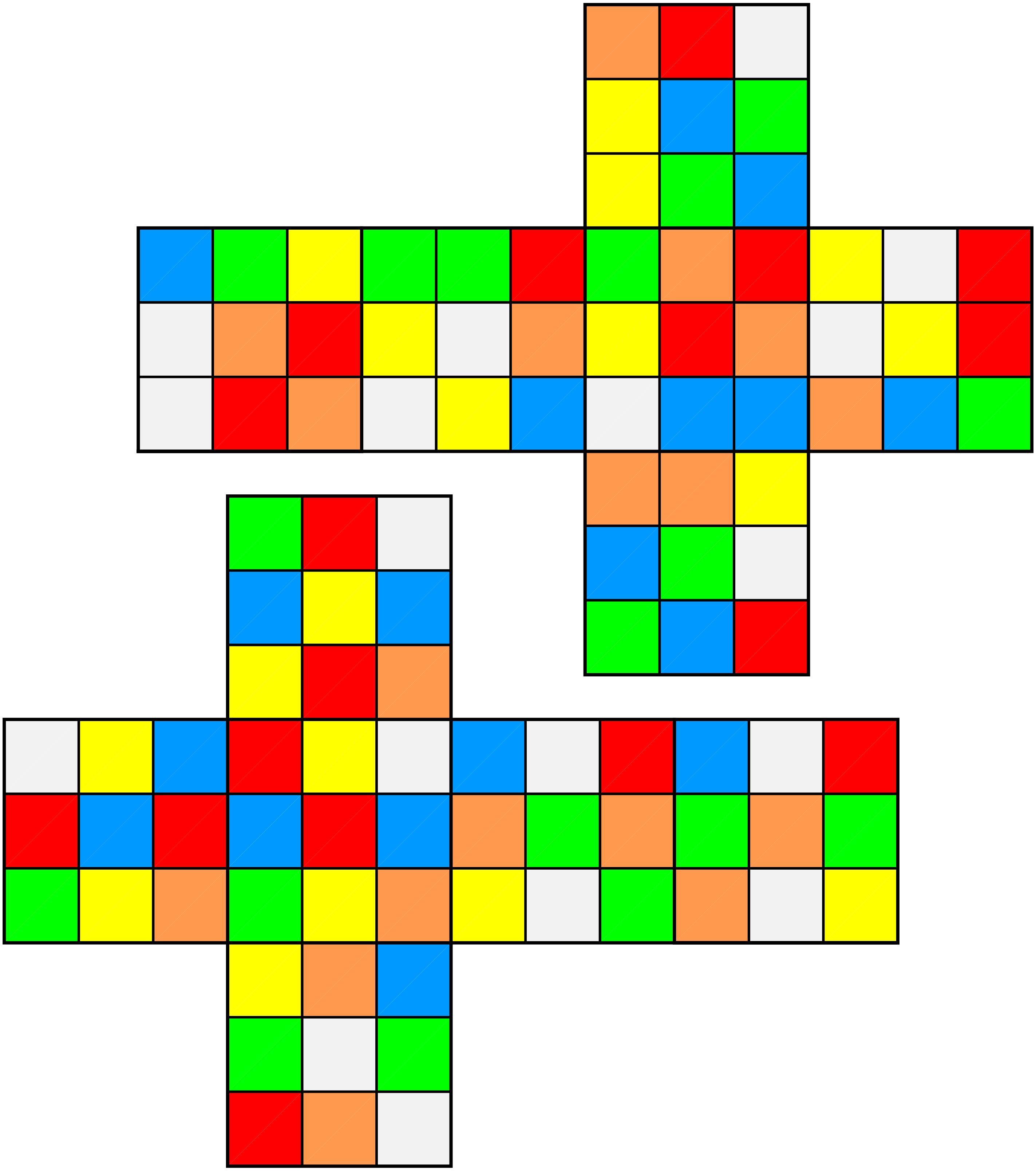


Figure 37: And another two as in Figure 7 in the main text.