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Divide and choose: An informationally robust strategic approach to bankruptcy problems[☆]

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ABSTRACT

This paper proposes a simple and general strategic approach for analyzing bankruptcy problems. We construct three strategic bargaining games and show that they yield unique subgame perfect equilibrium (SPE) outcomes that coincide with the allocations given by the three prominent solution concepts, the constrained equal awards rule, the constrained equal losses rule and the Talmud rule, respectively. We also discuss the robustness of the result in the presence of certain incomplete information. The approach can be readily extended to study alternative solutions for bankruptcy problems or other settings such as surplus sharing problems, and is further enriched by considering a voting stage. Central to all these bargaining protocols is an extended and context-fitting "divide-and-choose" mechanism.

1. Introduction

The seminal paper of Aumann and Maschler (1985) solves a long-standing problem that has puzzled researchers for 2000 years. It discovers the logic behind the Talmud rule for bankruptcy problems in relation to the Nucleolus solution (Schmeidler, 1969) for TU (transferable utility) games. This has substantially developed our understanding about how to resolve problems of conflicting claims.

In a bankruptcy problem (or claim problem), every creditor has a certain claim over a perfectly divisible endowment which is insufficient to grant all the claims.[1] The bankruptcy problem captures the essence of most rationing situations, such as the execution of a will to disburse the insufficient property to the beneficiaries, the distribution of the liquidation of a firm among its creditors, the collection of a certain amount of tax among tax payers with different responsibilities, etc. Such problems, arising from the Talmud, are first studied by O'Neill (1982) and Aumann and Maschler (1985). Since then the literature has flourished along an axiomatic perspective to study the *fair* allocation rules including the constrained equal awards rule (CEA), the constrained equal losses rule (CEL), as well as the Talmud rule (T). A comprehensive survey is provided by Thomson (2003) and further updated in Thomson (2015).

Such problems and the related rules have general and wider implications, even nowadays for contemporary issues. In 2020, as a measure to support firms and workers during the pandemic, the UK government introduced a scheme that pays 80% of salary for staff who are kept on by their employers, covering wages of *up to* £2500 a month. For the current energy crisis, many countries introduced measures to provide subsidies to families according to certain income levels. These are all applications or in the same spirit of, e.g., the constrained equal awards rule.

In this paper, we provide a general non-cooperative approach to bankruptcy problems, with the aim to better understand the strategic elements underlying those axiomatically justified allocation rules. It offers a way to address the problem where centralized methods are not applicable, e.g., when the social planner is lack of certain information. Hence, we provide the arbitrator of a bankruptcy situation decentralized and easy-to-implement procedures such that the players (or claimants) can resolve their disputes on the sharing of the underlying endowment through bargaining among themselves. We first propose

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1 We follow Thomson (2015) and use *endowment* to describe the total resource to be divided.

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two bargaining games whose unique subgame perfect equilibrium (SPE) outcomes coincide with the CEA and CEL allocations, respectively. Built upon these two games, we construct a third one that yields the Talmud allocation as the unique SPE outcome. We also discuss other variations that lead to alternative bankruptcy rules.

The strategic analysis of bankruptcy problems is relatively less explored, although a non-cooperative formulation was proposed as early as in the seminal article by O'Neill (1982). Chun (1989) defined a game where players propose order preservative rules and the limit point of a process of adjustment coincides with the CEA allocation. Herrero (2003) studied the dual game of the above mechanism that reaches the CEL allocation as the unique Nash equilibrium outcome. García-Jurado et al. (2006) implemented a range of bankruptcy rules in Nash equilibrium. Different from the above approaches that focus on Nash implementation, the current paper designs strategic games consisting of multi-stages and obtains the allocations prescribed by the respective rules by subgame perfect implementation. Sonn (1992) studied a game of demand, in a modified alternating offer bargaining game style, where the SPE outcome converges to the CEA allocation when the discount factor goes to 1. Different from the above approach, the games proposed in the current paper reach the CEA and CEL in SPE, respectively, rather than the limit point of the procedures. It allows the players to achieve the desirable allocation in finite steps.

As for the non-cooperative mechanisms for the Talmud allocation, important contributions were made by Serrano (1995) and Dagan et al. (1997). Hu et al. (2012, 2018) provide strategic justifications of the constrained equal benefit rule and the nucleolus in the nested-cost sharing problem, underlining the important role that the consistency properties play in strategically justifying the rules of related problems. This is also a major difference from the strategic games introduced in our paper, where obtaining the Talmud allocation as the SPE outcome does not resort to the properties of consistency.[2] A recent study in this literature is due to Tsay and Yeh (2019).

The games proposed in this paper are inspired by the well-known divide-and-choose mechanism in fair division (Brams and Taylor, 1996). When two people want to share a cake fairly, one is randomly chosen to cut the cake, and the other one has the right to choose first which piece he wants. Apparently, the optimal strategy for the player who cuts the cake is to cut it exactly in halves. The unique SPE outcome is an equal division of the cake. We extend this idea to the context of n-player bankruptcy problems by restricting that no player can get a payoff higher than his claim – a standard requirement in the literature (O'Neill, 1982).

For expositional purpose, we first present and analyze the mechanisms in an environment where the endowment and claims are assumed to be common knowledge to players and the social planner, which is in line with Serrano (1995) and Dagan et al. (1997). Such an informational structure was also discussed in Thomson (2003). Generally, comparing centralized solutions with decentralized bargaining protocols is interesting and helps better understand the properties of the solutions.

We then relax the assumption of complete information, but allow the social planner to have only limited information. This makes the current research more relevant and better justified, as the social planner will be unable to implement a desired outcome directly. In addition, we further make part of the information private for players and show all our major results hold with such incomplete information.

We first introduce a game where the player with the highest claim acts as the executor and makes a proposal which is an efficient allocation of the endowment. To be precise, please note that a proposal is a *multiset* that may contain multiple instances of its elements and the order of its elements is irrelevant. In mathematics, a multiset is a modification of the concept of a set that, unlike a set, allows for multiple instances for each of its elements. For example, consider a 5-player problem with the endowment $E = 100$. A proposal can be $\{6, 12, 12, 30, 40\}$ where the element 12 occurs twice, which means one player can take 12 and another player can take 12, too.

Then, players, following a reverse order according to their claims, sequentially choose an element from the proposal as their payoffs. That is, players with lower claims are given priority to choose early on in the game, while the player who chooses last is given the power to make the division. The privilege of dividing the endowment and choosing early is spread out among the players. If at any point, a player's choice is higher than his claim, he only gets the amount equal to his claim and the executor makes a new proposal with respect to the total remaining endowment to be chosen by the players who have not yet got their payoffs. The game ends when the executor takes (passively chooses) the remaining endowment after all other players have received their payoffs. It can be shown that the unique SPE outcome of this game is the CEA allocation of the corresponding bankruptcy problem. A similar game, where the deficit between the endowment and total claim is distributed by the executor, yields the CEL allocation as the unique outcome in all SPE. A game that suitably combines these two games yields the Talmud allocation as the unique SPE outcome. In each of the above games, the outcome is reached in finite steps.

We then discuss the construction of alternative games to implement various bankruptcy rules. Apart from the simplicity and generality of the divide-and-choose approach, another innovative aspect lies in that the results are robust in an environment of incomplete information.

The rest of the paper is organized as follows. Section 2 introduces the bankruptcy problems and the allocation rules. In Section 3, we present three strategic bargaining games and the main results. In Section 4 we introduce settings of incomplete information and show the robustness of the results obtained in Section 3. In Section 5 we discuss the extensions of the mechanism and, in particular, we consider a model that allows players to vote for games to adopt. Section 6 concludes.

2. Bankruptcy problems and allocation rules

Let $N = \{1, 2, \ldots, n\}$ be the finite set of players. For each $i \in N$, let $c_i \in \mathbb{R}_+$ denote player i's claim and $c = (c_i)_{i\in N}$ the vector of claims. $E \in \mathbb{R}_+$ is the perfectly divisible endowment to be divided among all players. A bankruptcy problem is a pair (E, c), such that $0 < E < \sum c_i$.[3] Without loss of generality, players are assumed to be ordered according to their claims, that is, for players $1, 2, \ldots, n$, we have $0 \le c_1 \le c_2 \cdots \le c_n$. The order is randomly decided in case of equal claims. An allocation in a bankruptcy problem is an n-tuple $x(E, c) = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$, with $\sum x_i = E$ and $0 \le x_i \le c_i$. An allocation rule is a function that assigns a unique allocation to each bankruptcy problem. The three prominent rules for this problem are the so-called the

[2] The consistency property also plays a crucial role in Serrano (1995)'s mechanism: if a creditor rejects a proposal, he will enter a bilateral negotiation with the proposer, where he will receive an amount according to the Contested Garment (CG) consistent rule (à la Aumann and Maschler, 1985). This amount, which could be different from the creditor's claim, needs to be enforced by the social planner (or a "CG bilateral court"). By contrast, our game obtains the Talmud allocation without resorting to the CG consistency property. The social planner in our game only needs to monitor and ensure that no player gets more than her claim.

[3] For simplicity of notation, $\sum$ stands for $\sum_{i=1}^n$ unless specified otherwise.

constrained equal awards rule, the constrained equal losses rule and the Talmud rule.

The constrained equal awards rule (CEA) divides the endowment equally among all players, subjected to no player gets more than his claim.

$$CEA_i(E,c) = \min\{\alpha, c_i\}, \text{ with } \alpha \in \mathbb{R}_+ \text{ solves } \sum \min\{\alpha, c_i\} = E.$$

The constrained equal losses rule (CEL) assigns equal loss to each player, subjected to no player receives negative payoff.

$$CEL_i(E,c) = \max\{0, c_i - \beta\}, \text{ with } \beta \in \mathbb{R}_+ \text{ solves}$$
$$\sum \max\{0, c_i - \beta\} = E.$$

The Talmud rule (T) combines CEA and CEL. The Talmud allocation is obtained by applying the CEA and CEL to the bankruptcy problem sequentially, with half-claims instead of claims being used as the switchpoint.

$$T_i(E,c) = \begin{cases} CEA_i\left(E, \frac{c_i}{2}\right) & \text{when } \sum c_i \geq 2E; \\ \frac{c_i}{2} + CEL_i\left(E - \sum \frac{c_i}{2}, \frac{c_i}{2}\right) & \text{when } \sum c_i < 2E. \end{cases}$$

For the desirable axiomatic properties underlying the above three rules, please refer to Thomson (2015).

3. The extended divide-and-choose mechanism

We propose an extended divide-and-choose mechanism that consists of variants of multi-step extensive form games, and show that they all have unique SPE outcomes, respectively coinciding with the CEA, CEL and Talmud allocations.

Consider a bankruptcy problem (E, c). We study the following three games.

3.1. Game Γ^{cea} for the constrained equal awards rule

Game $\Gamma^{cea}(E, c)$ is a divide-and-choose game with respect to the bankruptcy problem (E, c). Denote the payoff for player i in game Γ^{cea} as π_i^{cea}. With respect to the number of players n in the associated bankruptcy problem, the game consists of n stages. The basic idea is as follows. At stage 1, player n makes a proposal for a division of the endowment. Then, player 1, who has the lowest claim, picks one from the proposed amounts. If the chosen amount is not higher than her claim c_1, then she gets that chosen amount. Otherwise, she gets her claim. At stage 2, player n, forever as the proposer, makes a new proposal as a division of the residual endowment (the original endowment minus what player 1 got at stage 1) for the remaining players. Then, player 2 chooses from the proposal at this stage. If the chosen amount is not higher than her claim c_2, then she gets that chosen amount. Otherwise, she gets her claim. The game continues in this pattern until stage $n-1$ where player n proposes again on how to divide the remaining endowment and then player $n-1$ makes a choice. If the chosen amount is not higher than her claim, then player $n-1$ gets the amount of her choice. Otherwise, she gets her claim. After that, the game enters the final stage n where player n simply receives the residual amount, and the entire game ends. Such a simple mechanism will lead to the CEA allocation in SPE.

Note that in this game, other than player n who is the proposer, all players are not allowed to get more than their claims, while no restriction is placed on player n about how much he could obtain. However, as one can see in the following analysis of the game, in equilibrium player n will not be able to receive an amount that is more than his claim because any other player will not choose an amount that is lower than her claim when there exists an amount that is higher than her claim in the proposal.

The game Γ^{cea} is formally described below. It has n stages and starts at stage 1.

At any stage s, where $s = 1, \ldots, n$, the game proceeds as follows. Player n makes a proposal that is a multiset, $A^s = \left\{x_s^s, x_{s+1}^s, \ldots, x_n^s\right\}$, such that $x_i^s \in \mathbb{R}_+$ for all $i = s, \ldots, n$ with $\sum_s^n x_i^s = E$ when $s = 1$ and $\sum_s^n x_i^s = E - \sum_{j=1}^{s-1} \pi_j^{cea}$ when $s > 1$. After observing A^s, player s chooses an element, denoted by θ_s, from the proposal A^s. For any $s < n$, if $\theta_s \leq c_s$, player s leaves the game with payoff $\pi_s^{cea} = \theta_s$, and if $\theta_s > c_s$, player s' payoff is restricted to her claim and she leaves the game with $\pi_s^{cea} = c_s$. Then, the game proceeds to stage $s+1$ if $s < n$, or stops when $s = n$, where player n receives $E - \sum_{j=1}^{n-1} \pi_j^{cea}$.

Note that one can consider an alternative mechanism such that after the proposer made the first proposal, all other players will sequentially make a choice, following a reverse order according to their claims, while the proposer would only be called for taking action again when some player i chooses an amount that is higher than her claim c_i, in which case the proposer is to make a new proposal for the rest. This will also implement the constrained equal awards rule. The reason we would focus on the current mechanism that allows the proposer to make a proposal at every stage is twofold. One is for the convenience and clarity of the proof of the main result, as this specification helps to establish the same structure in every subgame. The other is due to its generality, as the current rule essentially admits and practically supersedes the alternative one: the proposer can effectively maintain the same proposal, albeit taking away the elements chosen by the previous players, so long as they are no higher than their claims, which is strategically equivalent to making such a "new" proposal.

Below we will show that game Γ^{cea} has a unique SPE outcome that coincides with the allocation prescribed by CEA.

Define the remaining endowment at stage 1 as $E_1 = E$ and at stage $s = 2, \ldots, n-1$ as $E_s = E - \sum_{i=1}^{s-1} \pi_i^{cea}$.

The following lemma implies that at any stage, if the remaining endowment is not sufficient to award every remaining player the amount of the lowest claim among the remaining players, the proposal at this stage must be an equal division in equilibrium. By reverse induction on the number of the remaining players, we show that if the proposal is not an equal division, the proposer would end up with a lower payoff. Thus, any strategy of the proposer that leads to a non-equal division at such a stage will not be in SPE.

Lemma 3.1. *If $E_s < (n-s+1)c_s$ at any stage s, $s = 1, 2, \ldots, n-1$, the unique SPE outcome of the subgame starting from stage s with respect to the bankruptcy problem $(E_s, (c_s, c_{s+1}, \ldots, c_n))$ is $\pi_i^{cea} = \frac{E_s}{n-s+1}$, for all $i = s, \ldots, n$.*

Prior to the proof, we like to note that, given any $s, s = 1, 2, \ldots, n-1$, if $E_s < (n-s+1)c_s$, then $E_s - \frac{E_s}{n-s+1} < (n-s)c_{s+1}$ because $c_s < c_{s+1}$. This means that at any stage s, if player s takes an equal share of the estate in that stage, then the remaining estate (i.e., for stage $s+1$) will not be sufficient to award every remaining player the amount of the lowest claim (i.e., c_{s+1}) among the remaining players. One can readily see that this will have a knock-on effect till the end of the game. That is, if at any stage $s, s = 1, 2, \ldots, n-1$, $E_s < (n-s+1)c_s$, then for any stage t, where $t = s+1$, it holds that $E_t < (n-t+1)c_t$, where $E_t = E_s - \frac{E_s}{n-s+1}$. Hence, if at any stage s, there is $E_s < (n-s+1)c_s$, provided that from now on at each stage the player with the lowest claim at the corresponding stage will get the average of the remaining estate of that stage, then the remaining estate at any of the following stages will not be sufficient to award every remaining player the amount of the lowest claim among the remaining players.

Proof (*of Lemma 3.1*).

The proof is done by reverse induction on s.

We first show that the lemma holds for $s = n-1$.

When $s = n-1$, the lemma says if $E_{n-1} < 2c_{n-1}$, then the unique SPE outcome is that both player $n-1$ and player n have the same payoff $\frac{E_{n-1}}{2}$. Consider the following strategies adopted by the two players. Player n makes proposal $A^{n-1} = \left\{\frac{E_{n-1}}{2}, \frac{E_{n-1}}{2}\right\}$ and player $n-1$ chooses $\theta_{n-1} \in \max\left\{x_{n-1}^{n-1}, x_n^{n-1}\right\}$. It is easy to see that both choices are best responses and therefore constitute an SPE that leads to the outcome $\left\{\frac{E_{n-1}}{2}, \frac{E_{n-1}}{2}\right\}$. One can also readily verify that there does not exist any other SPE. Choosing $\theta_{n-1} \in \max\left\{x_{n-1}^{n-1}, x_n^{n-1}\right\}$ is the best response of player $n-1$. Player n would have no incentive to make a different proposal $(A^{n-1})' = \left\{\frac{E_{n-1}}{2} - \Delta, \frac{E_{n-1}}{2} + \Delta\right\}$, where $\Delta \in \left(0, \frac{E_{n-1}}{2}\right]$. Otherwise, player $n-1$ is at best response to choose $\frac{E_{n-1}}{2} + \Delta$. Since $(\pi_{n-1}^{cea})' = \min\left\{\frac{E_{n-1}}{2} + \Delta, c_{n-1}\right\} > \frac{E_{n-1}}{2}$, the game ends up with player n having payoff $(\pi_n^{cea})' = E_{n-1} - (\pi_{n-1}^{cea})' < \frac{E_{n-1}}{2} = \pi_n^{cea}$. Hence $(A^{n-1})'$ cannot be part of SPE.

Now assume the lemma holds for $s+1$, where $s < n-1$. We show that the lemma holds for s.

At stage s, if $E_s < (n-s+1)c_s$, consider the following strategies. Player s chooses $\theta_s \in \{x_i^s | x_i^s \in A^s, x_i^s \geq c_s\} \cup \max A^s$. That is, player s is indifferent about all the elements in the proposal that are greater than or equal to her claim; if there does not exist an element that is greater than or equal to her claim, then player s chooses the biggest element in the proposal. Player n makes a proposal $A^s = \left\{x_s^s, x_{s+1}^s, \ldots, x_n^s\right\}$ where $x_i^s = \frac{E_s}{n-s+1}, i = s, s+1, \ldots, n$. Here, the payoff for player s is $\pi_s^{cea} = \frac{E_s}{n-s+1}$. Then, at stage $s+1$, the remaining endowment will be $E_{s+1} = E_s - \frac{E_s}{n-s+1} < (n-s)c_{s+1}$. By the induction hypothesis, the unique SPE outcome for the subgame starting from stage $s+1$ is $\pi_i^{cea} = \frac{E_{s+1}}{n-s} = \frac{E_s}{n-s+1}$, for all $i = s+1, s+2, \ldots, n$. Combining with the payoff for player s, the SPE outcome for the subgame starting from stage s is $\pi_i^{cea} = \frac{E_s}{n-s+1}, i = s, s+1, \ldots, n$.

Next we show there does not exist any other SPE. It is easy to see that in any SPE player s chooses $\theta_s \in \{x_i^s | x_i^s \in A^s, x_i^s \geq c_s\} \cup \max A^s$. Player n would have no incentive to make a different proposal $(A^s)' = \left\{(x_s^s)', (x_{s+1}^s)', \ldots, (x_n^s)'\right\}$, where $(x_s^s)' \leq (x_{s+1}^s)' \leq \cdots \leq (x_n^s)'$ with at least one strict inequality and $\sum_{i=s}^n (x_i^s)' = E_s$. Otherwise, such a proposal will yield that $(x_s^s)' < \frac{E_s}{n-s+1}$ and $(x_n^s)' > \frac{E_s}{n-s+1}$. Player s will be at best response to choose $(x_n^s)'$ and obtain payoff $(\pi_s^{cea})' = \min\left\{(x_n^s)', c_s\right\} > \frac{E_s}{n-s+1}$. Then at stage $s+1$, $(E_{s+1})' = E_s - \min\left\{(x_n^s)', c_s\right\} < \frac{(n-s)E_s}{n-s+1} < (n-s)c_s \leq (n-s)c_{s+1}$. By the induction hypothesis, the unique SPE outcome for the subgame starting from stage $s+1$ is $\left(\pi_i^{cea}\right)' = \frac{(E_{s+1})'}{n-s} < \frac{E_s}{n-s+1} = \pi_i^{cea}$, for all $i = s+1, s+2, \ldots, n$.[4] Hence, any deviation from A^s would make player n strictly worse off. □

The next theorem shows that game Γ^{cea} has a unique SPE outcome which coincides with the allocation prescribed by CEA in the corresponding bankruptcy problem.

Theorem 3.2. *For any bankruptcy problem (E, c), the associated game $\Gamma^{cea}(E, c)$ has a unique SPE outcome that is $\pi^{cea} = CEA(E, c)$.*

Proof. The proof is done by induction on the number of players.

We first show that the theorem holds for $|N| = 2$. There are two cases.

Case 1, $E < 2c_1$. $CEA_i(E, c) = \frac{E}{2}, i = 1, 2$. By Lemma 3.1, the unique SPE outcome for the game with respect to the bankruptcy problem (E, c) is $\pi_i^{cea} = \frac{E}{2} = CEA_i(E, c), i = 1, 2$.

Case 2, $E \geq 2c_1$. $CEA(E, c) = (c_1, E - c_1)$. In any SPE, player 1's choice is $\theta_1 = \max\left\{x_1^1, x_2^1\right\}$ if $\min\left\{x_1^1, x_2^1\right\} < c_1$; otherwise, if $\min\left\{x_1^1, x_2^1\right\} \geq c_1$, player 1 is indifferent between the two elements. Player 1's payoff is $\pi_1^{cea} = c_1$. A different choice would lead to a lower payoff for player 1. At stage 1, any efficient proposal made by player 2 constitutes an SPE and results in the payoff $\pi_2^{cea} = E - c_1$ for player 2.

Assuming the theorem holds for $|N| = n-1$, we then show it also holds for $|N| = n$. Consider the following two cases.

Case 1, if $E < nc_1$, $CEA_i(E, c) = \frac{E}{n}$ for all $i = 1, 2, \ldots, n$. By Lemma 3.1, the unique SPE outcome for the game with respect to the bankruptcy problem (E, c) is $\pi_i^{cea} = \frac{E}{n} = CEA_i(E, c)$ for all $i = 1, 2, \ldots, n$.

Case 2, if $nc_1 \leq E < \sum c_i$, $CEA_i(E, c) = \min\{c_i, \alpha\}$, where $\sum \min\{c_i, \alpha\} = E$.

Consider the following strategy profile. Player n, at stage 1, proposes $A^1 = \left\{x_1^1, x_2^1, \ldots, x_n^1\right\}$, where $x_i^1 = c_i$ for $i = 1, \ldots, q, q < n$ and $x_i^1 = \alpha$ for $i = q+1, q+2, \ldots, n$, such that $\sum x_i^1 = E$ and $c_q \leq \alpha < c_{q+1}$.

At any subsequent stage s, player n makes the proposal $A^s = \left\{x_s^s, x_{s+1}^s, \ldots, x_n^s\right\}$, where $x_i^s = c_i$ if $i \leq q$ and $x_i^s = \frac{E - \sum_{k=1}^s \pi_k^{cea} - \sum_{l=s+1}^q c_l}{n-s}$, if $i > q$. At stage s, where $s \leq q$, player s chooses $\theta_s = x_s^s \in A^s$ if $x_s^s \geq c_s$; if $x_s^s < c_s$, player s chooses $\theta_s \in \max\left\{x_s^s, x_{s+1}^s, \ldots, x_n^s\right\}$. At stage s, where $s > q$, player s chooses $\theta_s \in \max\left\{x_s^s, x_{s+1}^s, \ldots, x_n^s\right\}$.

The payoffs from the above strategies are $\pi_i^{cea} = c_i$ for $i = 1, \ldots, q$ and $\pi_i^{cea} = \frac{E - \sum_{i=1}^q c_i}{n-q}$ for $i = q+1, q+2, \ldots, n$, which coincides with the CEA allocation.

To verify that the above strategies constitute an SPE, one can readily see that for each player $i = 1, \ldots, q$, he has no incentive to deviate from the current strategy as the payoff from the current choice θ_i has already been the maximum he could achieve, which equals to his claim. Player n's proposals at stages 1 to q are also the best responses. Any other proposal would still lead to a payoff c_i for player i, $i = 1, \ldots, q$ given their choosing strategies, leaving $E_{q+1} = E - \sum_{i=1}^q c_i < (n-q)c_{q+1}$. By Lemma 3.1, the unique SPE outcome for the subgame starting from stage $q+1$ is $\pi_i^{cea} = \frac{E - \sum_{i=1}^q c_i}{n-q}$ for $i = q+1, q+2, \ldots, n$. Hence, any deviation would not change the payoff to player n.

We now show that all SPE yield the same outcome which is the CEA allocation. In any SPE, given that $E \geq nc_1$ and the proposal $A^1 = \left\{x_1^1, x_2^1, \ldots, x_n^1\right\}$ must be efficient, there necessarily exists an x_j^1 from the proposal such that $x_j^1 \geq c_1$. Player 1's best response at stage 1 is $\theta_1 = x_j^1 \geq c_1$ and gets payoff $\pi_1^{cea} = c_1$. Therefore, for stage 2, with the remaining endowment $E_2 = E - c_1$ and the $n-1$ players, by the induction hypothesis, this subgame has a unique SPE outcome which is $\pi_i^{cea} = \min\{c_i, \alpha\}$, for all $i = 2, \ldots, n$ such that $\sum_{i=2}^n \pi_i^{cea} = E - c_1$. Since $\pi_1^{cea} = c_1$ and $c_1 < \alpha$, then $\pi_1^{cea} = \min\{c_1, \alpha\}$. Thus, the SPE outcome of the entire game with n players could be expressed as $\pi_i^{cea} = \min\{c_i, \alpha\} = CEA_i(E, c), i = 1, 2, \ldots, n$ and $\sum \pi_i^{cea} = E$. □

As we have shown in the proof of Theorem 3.2, there may exist multiple SPE, but all SPE yield the same outcome that is the CEA allocation. To see the multiplicity of SPE, one can consider the following case. If at any stage s, the remaining estate is sufficiently large such that the equal split of the remaining estate is an amount that is greater than the claim of player s, then any proposal from player n constitutes an SPE. Player s will choose an amount greater than or equal to her claim but leave with the amount equal to her claim. So, in this situation, how player n makes a proposal does not matter, so long as it is efficient,

4 $(\pi_n^{cea})' - \pi_n^{cea} = \frac{(E_{s+1})'}{n-s} - \frac{E_s}{n-s+1} = \frac{(n-s+1)(E_{s+1})' - (n-s)E_s}{(n-s)(n-s+1)} = \frac{E_s - (n-s+1)\cdot\min\left\{(x_n^s)', c_s\right\}}{(n-s+1)(n-s)}$. Since $E_s < (n-s+1)(x_n^s)'$, $E_s < (n-s+1)c_s$, $E_s - (n-s+1)\cdot\min\left\{(x_n^s)', c_s\right\} < 0$, which means $(\pi_n^{cea})' < \pi_n^{cea}$.

because it has no impact on the payoff of player s and will not change the size of the remaining estate to be divided in the next stage, either.

It is worth noting that in the above game (and also the following variants) the ordering of the players in making choices matters. That is, if the order is not increasing from the lowest claim to the highest claim (even if requiring player n to be the last one to choose), then player n could manipulate the proposal to his advantage, as shown by the following example.

Example 1. Consider a bankruptcy problem $(330, (100, 200, 300))$. The CEA allocation for this problem is $(100, 115, 115)$. Suppose the order to choose is: player 2, player 1, player 3. The optimal proposal for player 3 would be $\{110, 110, 110\}$. The SPE outcome payoff is $(100, 110, 120)$.

Corollary 3.3. *Given a random choosing order for players $i = 1, 2, \ldots, n-1$, player n maximize his expected payoff by proposing equal division of the endowment.*

Corollary 3.4. *If $E > nc_1$, there exists a choosing order such that $\pi_n^{cea} \geq CEA_n(E, c)$.*

The proofs of the above the corollaries are omitted as they can be readily constructed.

In the above game Γ^{cea} (and the following variants of the game), it is important for the player with the highest claim to act as the executor, so that the unique SPE outcome coincides with the CEA allocation. It is easy to see that if the executor is not the last to choose from the proposal, he would have the chance to manipulate the proposal to get a higher payoff. We further use an example to illustrate the case when some player other than player n is the executor, and even if the executor is the last one to make choice from the proposal, there exists an SPE outcome that differs from the CEA allocation.

Example 2. Consider the bankruptcy problem in Example 1. Suppose player 1 is the proposer. Players make choices by a sequential order of players 2, 3 and 1. Consider a proposal of $(100, 101, 129)$ made by player 1. It is easy to verify that this strategy would be part of an SPE. The SPE payoffs would be 100, 129 and 101 for players 1, 2 and 3, respectively, which are different from the CEA allocation. Indeed, any proposal of the form $\{100, a, 230 - a\}$, with $100 < a < 115$, and the strategies that players always choose the highest amount available in the proposal, would constitute an SPE. All such SPE have the same outcome $(100, 230 - a, a)$ that differs from the CEA allocation.

The intuition behind the example is that if any of the players other than player n is the proposer, he would not care about the shares for players with higher claims, as long as he can get his highest possible payoff.

3.2. Game Γ^{cel} for the constrained equal losses rule

Similar to Γ^{cea}, the game Γ^{cel} is a divide-and-choose game where players share the deficit of the endowment with respect to the sum of claims, such that every player's payoff is his claim minus his share of the deficit.

Define the deficit or the total loss for all players N as $L = \sum c_i - E$. Denote the payoff for player i in game Γ^{cel} as π_i^{cel}. The game Γ^{cel} also starts at stage 1 and has n stages.

At any stage s, where $s = 1, \ldots, n$, the game proceeds as follows. Player n makes a proposal $B^s = \left\{x_s^s, x_{s+1}^s, \ldots, x_n^s\right\}$, such that $x_i^s \in \mathbb{R}_+$ for all $i = s, \ldots, n$ while $\sum_s^n x_i^s = L$ when $s = 1$ and $\sum_s^n x_i^s = L - \sum_{j=1}^{s-1} c_j + \sum_{j=1}^{s-1} \pi_j^{cel}$ when $s > 1$. After observing B^s, player s chooses an element, denoted by θ_s, from the proposal B^s. If $\theta_s \leq c_s$, player s leaves the game with payoff $\pi_s^{cel} = c_s - \theta_s$. If $\theta_s > c_s$, player s' loss is restricted to her claim and she leaves the game with $\pi_s^{cel} = 0$. Then, the game proceeds to stage $s + 1$ if $s < n$, or stops when $s = n$.

Without proof, we note that the CEL rule satisfies the *endowment monotonicity* property (Curiel et al., 1987), which says that if the endowment increases, each claimant should be awarded at least as much as initially (Thomson, 2015).[5]

Theorem 3.5. *For any bankruptcy problem (E, c), the unique SPE outcome of game $\Gamma^{cel}(E, c)$ is $\pi_i^{cel} = CEL_i(E, c)$.*

Proof.

The proof is done by induction on the number of players.

We first show that the theorem holds for $|N| = 2$.

Case 1, $c_2 - c_1 < E$, i.e. $L = c_1 + c_2 - E < 2c_1$. First we show that there exist an SPE of the game $\Gamma^{cel}(E, c)$ which yields the outcome $\pi_i^{cel} = CEL_i(E, c) = c_i - \frac{L}{2}$.

Consider the following strategy profile. Player 1's choice is $\theta_1 = \min\{x_1^1, x_2^1\}$ for any proposal. Player 2 makes the proposal $B^1 = \left\{\frac{L}{2}, \frac{L}{2}\right\}$. The outcome of this strategy profile is $\pi_i^{cel} = c_i - \frac{L}{2} = CEL_i(E, c)$. Player 1 is best responding given the proposal B^1. It is easily shown that any proposal different from B^1 would result in a bigger loss for player 2. Thus the above strategy profile constitutes an SPE.

Next, we point out that there exists only one SPE where player 2's proposal is $B^1 = \left\{\frac{L}{2}, \frac{L}{2}\right\}$ and player 1 chooses $\theta_1 = \min\left\{\frac{L}{2}, \frac{L}{2}\right\} = \frac{L}{2}$. Any other proposal would give player 2 a lower payoff, so cannot be part of the SPE. The outcome of the above SPE is $\pi_i^{cel} = CEL_i(E, c) = c_i - \frac{L}{2}$.

Case 2, $E \leq c_2 - c_1$, i.e. $L \geq 2c_1$. In this case, CEL assigns the whole endowment to player 2, and 0 to player 1, that is $CEL_1(E, c) = 0$ and $CEL_2(E, c) = E$.

Consider the following strategy profile. Player 1's choice is $\theta_1 = \min\left\{x_1^1, x_2^1\right\}$ for any proposal. Player 2 makes the proposal $B^1 = \{c_1, L - c_1\}$. The outcome of this strategy profile is $\pi_1^{cel} = 0$ and $\pi_2^{cel} = E$. Player 1 is best responding given the proposal B^1. Player 2 cannot improve his payoff of E. Thus the above strategy profile is an SPE.

Next, we point out that in any SPE, player 2's proposal must be $B^1 = \left\{x_1^1, x_2^1\right\}$ such that $x_i^1 \geq c_1$, $i = 1, 2$ and player 1's choice is $\theta_1 = \min\left\{x_1^1, x_2^1\right\}$ for any proposal. Player 1 is best responding given the proposal B^1 and receiving a payoff of 0. Player 2's payoff is E. Any other proposal would result in player 1 choosing the share of loss smaller than his claim, which gives player 1 a positive payoff and a lower payoff to player 2, so cannot be part of SPE. All the SPE lead to the same outcome which is $\pi_i^{cel} = CEL_i(E, c)$.

Assuming that the theorem holds for $|N| = n - 1$, we show that it also holds for $|N| = n$.

Case 1, if $\sum c_i - nc_1 < E$, i.e., $L = \sum c_i - E < nc_1$, $CEL_i(E, c) = c_i - \frac{L}{n}$ for $i = 1, 2, \ldots, n$.

First, we show that there exist an SPE of the game $\Gamma^{cel}(E, c)$ which yields the outcome of $\pi_i^{cel} = CEL_i(E, c) = c_i - \frac{L}{n}$. Consider the following strategy profile. In every stage s, $s = 1, 2, \ldots, n$, player n makes the proposal $B^s = \left\{x_s^s, x_{s+1}^s, \ldots, x_n^s\right\}$, where $x_i^s = \frac{L}{n}$, for $i = s, s+1, \ldots, n$. Player s' choice is $\theta_s = \min\left\{x_s^s, \ldots, x_n^s\right\}$ for any proposal. The outcome of this strategy profile is $\pi_i^{cel} = c_i - \frac{L}{n} = CEL_i(E, c)$, for $i = 1, 2, \ldots, n$.

5 For two bankruptcy problems (E, c) and (E', c), $CEL_i(E, c) = \max\{0, c_i - \beta\}$, with $\beta \in \mathbb{R}_+$ solves $\sum \max\{0, c_i - \beta\} = E$ and $CEL_i(E', c) = \max\left\{0, c_i - \beta'\right\}$, with $\beta' \in \mathbb{R}_+$ solves $\sum \max\left\{0, c_i - \beta'\right\} = E'$. If $E > E'$, by *endowment monotonicity*, $CEL_i(E, c) \geq CEL_i(E', c)$. It follows that $\beta < \beta'$ and $CEL_n(E, c) > CEL_n(E', c)$.

Now we show that the above strategy profile is an SPE of the game $\Gamma^{cel}(E, c)$. Define the remaining endowment at stage 1 as $E_1 = E$ and at stage $s = 2, \ldots, n-1$ as $E_s = E - \sum_{i=1}^{s-1} \pi_i^{cel}$. In stage 1, player n makes the proposal $B^1 = \{x_1^1, x_2^1, \ldots, x_n^1\}$, where $x_i^1 = \frac{L}{n}$, for $i = 1, 2, \ldots, n$. Player 1's choosing strategy $\theta_1 = \min\{x_1^1, \ldots, x_n^1\}$ is a best response which leads to the payoff $\pi_1^{cel} = c_1 - \frac{L}{n}$. In stage 2, the remaining endowment becomes $E_2 = E - c_1 + \frac{L_1}{n}$. The subgame from stage 2 is a game with $n-1$ players with respect to the bankruptcy problem $(E_2, (c_2, \ldots, c_n))$.[6] By the induction hypothesis, the SPE outcome from stage 2 is $\pi_i^{cel} = CEL_i(E_2, (c_2, \ldots, c_n)) = c_i - \frac{L}{n}$, $i = 2, 3, \ldots, n$. In every stage s, $s = 2, \ldots, n$, player n's proposal $B^s = \{x_s^s, x_{s+1}^s \ldots, x_n^s\}$, where $x_i^s = \frac{L}{n}$, for $i = s, s+1, \ldots, n$ and player s' choice $\theta_s = \min\{x_s^s, \ldots, x_n^s\}$ for any proposal constitute the SPE for the subgame from stage 2. Consider a different proposal made by player n, $(B^1)' = \{(x_1^1)', (x_2^1)', \ldots, (x_n^1)'\}$, where $(x_1^1)' \leq (x_2^1)' \leq \cdots \leq (x_n^1)'$ (at least one strict inequality holds). It must be the case that $(x_1^1)' < \frac{L}{n}$. Player 1's choosing strategy $\theta_1 = \min\{(x_1^1)', (x_2^1)', \ldots, (x_n^1)'\}$ is a best response which leads to the payoff $\pi_1^{cel} = c_1 - (x_1^1)'$. In stage 2, the remaining endowment becomes $(E_2)' = E - c_1 + (x_1^1)' < E - c_1 + \frac{L_1}{n} = E_2$. It is easy to verify that $((E_2)', (c_2, \ldots, c_n))$ is a bankruptcy problem with $n-1$ players.[7] By the induction hypothesis, the SPE outcome of the subgame from stage 2 is $(\pi_i^{cel})' = CEL_i((E_2)', (c_2, \ldots, c_n))$. Since $E_2 > (E_2)'$, by *endowment monotonicity* of CEL, $\pi_n^{cel} > (\pi_n^{cel})'$. So any proposal different from B^1 would result in a lower payoff for player n. Thus the proposal B^1 must be part of the SPE.

Next we show that all SPE result in the same outcome of $\pi_i^{cel} = CEL_i(E, c) = c_i - \frac{L}{n}$. First note that in all SPE, the proposal in stage 1 must be $B^1 = \{x_1^1, x_2^1, \ldots, x_n^1\}$, where $x_i^1 = \frac{L}{n}$, for $i = 1, 2, \ldots, n$ as shown above. Player 1's choice $\theta_1 = \min\{x_1^1, \ldots, x_n^1\}$ leads to the payoff $\pi_1^{cel} = c_1 - \frac{L}{n}$. By the induction hypothesis, from stage 2, for the subgame with $n-1$ players and the remaining endowment $E_2 = E - \frac{L}{n}$, there is a unique SPE outcome $\pi_i^{cel} = CEL_i\left(E - \frac{L}{n}, (c_2, \ldots, c_n)\right) = c_i - \frac{L}{n}$, for $i = 2, \ldots, n$. Combining with player 1's payoff $\pi_1^{cel} = c_1 - \frac{L}{n}$, the unique SPE outcome for the game $\Gamma^{cel}(E, c)$ is $\pi_i^{cel} = c_i - \frac{L}{n} = CEL_i(E, c)$, for $i = 1, \ldots, n$.

Case 2, if $\sum c_i - nc_1 \geq E$, i.e. $L = \sum c_i - E \geq nc_1$, $CEL_i(E, c) = \max\{0, c_i - \beta\} = c_i - \min\{c_i, \beta\}$, where $\sum \min\{c_i, \beta\} = L$. In particular, $CEL_1(E, c) = 0$ and $CEL_n(E, c) = c_n - \beta$.

We first show that there exists an SPE that leads to the outcome of $\pi_i^{cel} = CEL_i(E, c)$. Consider the following strategy profile. In any stage s, $s = 1, 2, \ldots, n$, player n makes the proposal $B^s = \{x_s^s, x_{s+1}^s, \ldots, x_n^s\}$, where $x_i^s = c_i$ for $i = s, s+1, \ldots, k$, $k < n$ and $x_i^s = \beta$ for $i = k+1, k+2, \ldots, n$, such that $\sum_{i=s}^n x_i^s = L - \sum_{j=1}^k c_j$ and $c_k \leq \beta < c_{k+1}$. Player s' choice is $\theta_s = \min\{x_s^s, x_{s+1}^s, \ldots, x_n^s\}$ for any proposal, which is the best response given proposal B^s. Player s is best responding given the proposal B^s. The outcome of the above strategy profile is $\pi_i^{cel} = c_i - \min\{c_i, \beta\} = CEL_i(E, c)$, where $\sum \min\{c_i, \beta\} = L$. In particular, in stage 1, player n makes the proposal $B^1 = \{x_1^1, x_2^1, \ldots, x_n^1\}$, where $x_i^1 = c_i$ for $i = 1, 2, \ldots, k$, $k < n$ and $x_i^1 = \beta$ for $i = k+1, k+2, \ldots, n$, such that $\sum x_i^1 = L$ and $c_k \leq \beta < c_{k+1}$. Player 1 makes choice of $\theta_1 = \min B^1 = x_1^1 = c_1$ which is his best response. Player 1 leaves with the payoff of $\pi_1^{cel} = 0$. In stage 2, the remaining endowment is still E. It is easily shown that the subgame from stage 2 is a game with $n-1$ players with respect to the bankruptcy problem $(E, (c_2, \ldots, c_n))$. By the induction hypothesis, the SPE outcome from stage 2 is $\pi_i^{cel} = CEL(E, (c_2, \ldots, c_n)) = c_i - \min\{c_i, \beta\}$, $i = 2, 3, \ldots, n$, where $\sum_{i=2}^n \min\{c_i, \beta\} = L - c_1$. To see that B^1 is part of an SPE, consider any proposal made by player n. For any proposal, player 1's minimum payoff is $\pi_1^{cel} = 0$ and the maximum endowment to be divided in the subgame of stage 2 is E. By *endowment monotonicity* of CEL, player n cannot improve upon his payoff by deviating from B^1. Thus, B^1 is part of the SPE.

Next, we show that all SPE lead to a unique outcome that is $\pi_i^{cel} = CEL_i(E, c)$. First, player n's proposal in stage 1 must be $B^1 = \{x_1^1, x_2^1, \ldots, x_n^1\}$, where $x_i^1 \geq c_1$ for all $i = 1, 2, \ldots, n$. Player 1's choice $\theta_1 = \min B^1$ is the best response. Player 1 gets payoff $\pi_1^{cel} = 0 = CEL_1(E, c)$. In stage 2, the remaining endowment is still E. By the induction hypothesis, the SPE outcome for the subgame with $n-1$ players from stage 2 is $\pi_i^{cel} = CEL(E, (c_2, \ldots, c_n)) = \max\{0, c_i - \beta\}$, such that $\sum_{i=2}^n \max\{0, c_i - \beta\} = E$. Consider a different proposal by player n, $(B^1)' = \{(x_i^1)', (x_2^1)', \ldots, (x_n^1)'\}$, where $(x_j^1)' = \min(B^1)' < c_1$ for some j. Player 1's best response is to choose $\theta_1 = (x_j^1)'$ and his payoff is $\pi_1^{cel} = c_1 - (x_j^1)'$. From stage 2, the remaining endowment to be divided becomes $E' = E - c_1 + (x_j^1)' < E$. It is easy to verify that $\sum_{i=2}^n c_i > E'$, i.e. from stage 2, $(E', (c_2, \ldots, c_n))$ is a bankruptcy problem with $n-1$ players. By the induction hypothesis, the equilibrium outcome for the subgame from stage 2 is $CEL_i(E', (c_2, \ldots, c_n)) = \max\{0, c_i - \beta'\}$ for $i = 2, \ldots, n$, where $\sum_{i=2}^n \max\{0, c_i - \beta'\} = E'$. Since $E > E'$, by *endowment monotonicity*, $(\pi_n^{cel})' = CEL_i(E', (c_2, \ldots, c_n)) < CEL_n(E, (c_2, \ldots, c_n))$. Any proposal different from B^1 would result in a lower payoff for player n, which, therefore, does not constitute an SPE. So in all SPE, player 1 gets payoff $\pi_1^{cel} = CEL_1(E, c) = 0$. Combining with the SPE outcome payoffs for players 2 to n following proposal B^1, the unique outcome in all SPE is $\pi_i^{cel} = \max\{0, c_i - \beta\} = CEL_i(E, c)$. □

3.3. Game Γ^t for the Talmud rule

The game Γ^{cea} and game Γ^{cel} approach the claim problems from two opposite perspectives: dividing and choosing from either the endowment or the deficits of the endowment compared to the total claims. One may consider combining the two views into one and obtain a game Γ^t that is a hybrid of game Γ^{cea} and game Γ^{cel}. That is, in game Γ^t, players will firstly follow a procedure of Γ^{cea} based on half of their claims and then play the game Γ^{cel} based on the other half of their claims and the remaining endowment.

Consider any bankruptcy problem (E, c).

Step 1 Game $\Gamma^{cea}\left(E, \frac{c}{2}\right)$ is played with respect to $\left(E, \frac{c}{2}\right)$. The allocation for player i (where $i < n$) in this step is his payoff π_i^{cea} in game $\Gamma^{cea}\left(E, \frac{c}{2}\right)$, and the allocation for player n in this step is $\pi_n^{cea} = \min\{E - \sum_{i=1}^{n-1} \pi_i^{cea}, \frac{c_n}{2}\}$.[8] The game continues to Step 2.

Step 2 Game $\Gamma^{cel}\left(E - \sum \pi_i^{cea}, \frac{c}{2}\right)$ is played where the associated bankruptcy problem is $\left(E - \sum \pi_i^{cea}, \frac{c}{2}\right)$. The allocation for player i in Step 2 is his payoff π_i^{cel} in game $\Gamma^{cel}\left(E - \sum \pi_i^{cea}, \frac{c}{2}\right)$.[9]

Step 3 Player i takes the final payoff $\pi_i = \pi_i^{cea} + \pi_i^{cel}$.

The next theorem shows that game $\Gamma^t(E, c)$ has a unique SPE outcome which coincides with the Talmud solution of the

6 $E - c_1 + \frac{L_1}{n} = E_2 < \sum_{i=2}^n c_i$.

7 $(E_2)' < E - c_1 + \frac{L_1}{n} = E_2 < \sum_{i=2}^n c_i$.

8 Here in Step 1, $E < \sum \frac{c_i}{2}$ is not required. However, the proposals should still be efficient. It is also important to note that in game $\Gamma^{cea}\left(E, \frac{c}{2}\right)$, no player can get more than her half claim, including player n. It implies that if $E > \sum \frac{c_i}{2}$, in SPE, the endowment E may not be exhausted in Step 1. One can modify Step 1 to be $\Gamma^{cea}\left(\min\{E, \sum \frac{c}{2}\}, \frac{c}{2}\right)$, which would yield the same SPE outcome. However, we adopt the current specification of the game as it is less restrictive, e.g., the game would allow for the cases where an off-equilibrium strategy profile may lead to $\sum \pi_i^{cea} < \min\{E, \sum \frac{c_i}{2}\}$.

9 The restriction that Γ^{cel} is played with respect to a positive endowment is relaxed. $\pi_i^{cel} = 0$ if $E - \sum \pi_i^{cea} \leq 0$.

bankruptcy problem (E, c). Firstly, it is shown that if the endowment is not enough to grant every player half of her claim, the endowment will be fully distributed by the end of Step 1. Next, if the endowment is more than half of the total claims, in Step 1, every player's payoff is half of his claim. Finally, the payoff for every player is the allocation assigned by the Talmud solution.

Theorem 3.6. *For any bankruptcy problem (E, c), the unique SPE outcome of game $\Gamma^t(E, c)$ is the allocation assigned by the Talmud rule for the problem (E, c).*

Proof. The proof is done by a series of claims.

Claim 1. *if $E \le \sum \frac{c_i}{2}$, in all SPE, $E - \sum \pi_i^{cea} = 0$ in Step 2. Suppose there are two SPE strategy profiles $\mathcal{G}$ and $\mathcal{G}'$. By following $\mathcal{G}$, $E - \sum \pi_i^{cea} = 0$ in Step 2 and the payoff for any player i is $\pi_i = \pi_i^{cea}$, while by following $\mathcal{G}'$, $E - \sum(\pi_i^{cea})' > 0$ and player i's payoff is $(\pi_i)' = (\pi_i^{cea})' + (\pi_i^{cel})'$.*

In $\mathcal{G}'$, at the last stage of Step 1, it must be the case that $\left(\pi_n^{cea}\right)' = \frac{c_n}{2}$.[10] So $\pi_n^{cea} \le (\pi_n^{cea})'$. It implies that $\sum_{i=1}^{n-1} \pi_i^{cea} > \sum_{i=1}^{n-1}(\pi_i^{cea})'$.[11] There must be at least one player $j, j \neq n$, whose allocation in Step 1 is lower in $\mathcal{G}'$ than in $\mathcal{G}$, i.e., $\frac{c_j}{2} \ge \pi_j^{cea} > (\pi_j^{cea})'$. Suppose player j is the only such player. At Step 2, game Γ^{cel} is played with respect to $\left((\pi_j^{cea} - (\pi_j^{cea})') - (\frac{c_n}{2} - \pi_n^{cea}), \frac{c}{2}\right)$. Player j's payoff is $\left(\pi_j^{cel}\right)' < \left(\pi_j^{cea} - (\pi_j^{cea})'\right) - \left(\frac{c_n}{2} - \pi_n^{cea}\right) \le \pi_j^{cea} - (\pi_j^{cea})'$. His final payoff is $(\pi_j)' = (\pi_j^{cea})' + (\pi_j^{cel})' < \pi_j^{cea} = \pi_j$. So player j has an incentive to deviate from $\mathcal{G}'$. $\mathcal{G}'$ cannot be an SPE.

Claim 2. *if $E > \sum \frac{c_i}{2}$, in all SPE, the allocation in Step 1 for every player is $\pi_i^{cea} = \frac{c_i}{2}$. Suppose there are two SPE strategy profiles $\mathcal{G}$ and $\mathcal{G}'$. By following $\mathcal{G}$, $\pi_i^{cea} = \frac{c_i}{2}$ for any i, while by following $\mathcal{G}'$, there exist at least one player j such that $(\pi_j^{cea})' < \frac{c_j}{2}$. Because the proposal at any stage in Step 1 must be efficient, we have $j \neq n$. Without loss of generality, assume player j is the only such player. In $\mathcal{G}$, the game in Step 2 is game Γ^{cel} with respect to $\left(E - \sum \frac{c_i}{2}, \frac{c}{2}\right)$. Player j's allocation in Step 2 is $\pi_j^{cel} = \max\left\{\frac{c_j}{2} - \beta, 0\right\}$, where β solves $\sum \max\left\{\frac{c_i}{2} - \beta, 0\right\} = E - \sum \frac{c_i}{2}$. Player j's final payoff is $\pi_j = \frac{c_j}{2} + \pi_j^{cel}$. In $\mathcal{G}'$, the game in Step 2 is game Γ^{cel} with respect to $(E - \sum \frac{c_i}{2} + (\frac{c_j}{2} - (\pi_j^{cea})'), \frac{c}{2})$. His allocation in Step 2 is $(\pi_j^{cel})' = \max\left\{\frac{c_j}{2} - \beta', 0\right\}$, where β' solves $\sum \max\left\{\frac{c_i}{2} - \beta', 0\right\} = E - \sum \frac{c_i}{2} + \left(\frac{c_j}{2} - (\pi_j^{cea})'\right)$. His final payoff is $\pi_j' = (\pi_j^{cea})' + (\pi_j^{cel})'$. It can be easily shown that $(\pi_j^{cel})' - \pi_j^{cel} < \frac{c_j}{2} - (\pi_j^{cea})'$.[12] So player j must have an incentive to deviate from $\mathcal{G}'$ which is not an SPE.*

Claim 3. *in all SPE, the final payoff for every player in game Γ^t coincides with his payoff assigned by Talmud rule in bankruptcy problem (E, c). By Claim 1 and Theorem 3.2, for all bankruptcy game (E, c) such that $E \le \sum \frac{c_i}{2}$, the final payoff for any player i is $\pi_i = \pi_i^{cea} = CEA_i\left(E, \frac{c}{2}\right) = T_i(E, c)$.*

By Claim 2, for all bankruptcy game (E, c) such that $E > \sum \frac{c_i}{2}$, the allocation in Step 1 is $\pi_i^{cea} = \frac{c_i}{2}$. By Theorem 3.5, in Step 2 the allocation is $\pi_i^{cel} = CEL_i\left(E - \sum \frac{c_i}{2}, \frac{c}{2}\right)$. The final payoff for any player i is $\pi_i = \pi_i^{cea} + \pi_i^{cel} = \frac{c_i}{2} + CEL_i\left(E - \sum \frac{c_i}{2}, \frac{c}{2}\right) = T_i(E, c)$. □

4. Informational robustness

As pointed out by Dagan et al. (1999), implementing a solution becomes harder if there is more private information in the problem. They have also shown the negative result of implementation with fully private information. This holds with our mechanism, too. However, it is interesting to see that with our approach the implementation results can be largely maintained despite a substantial relaxation of complete information. This shows that the extended divide-and-choose mechanism is robust to certain degree of incomplete information.

So far in this paper, we have assumed complete information, that is, the social planner of the mechanism and all players know the total endowment to be divided (E) and the claim of each player (c). Below we will consider relaxing this requirement in two aspects: reducing the knowledge of the social planner and making part of information private for players.

We first consider the following modification of the informational structure. Here, the social planner still knows the claims of all the players and could monitor and enforce the actions taken by the players and the consequent outcome of the game, but does not know the size of the endowment (or it is too costly to know it), which happens often in real life, e.g., the court may not know the value of an asset but the relevant claimants know it; as heirs, the family members have not yet revealed to the adjudicator the exact value of the wealth to be inherited, although all the heirs know it, while they may resort to the adjudicator for a fair procedure of allocating the wealth rather than asking for advice on a vector of payoffs. We call this assumption (A1). Then, apparently, the social planner cannot directly distribute the endowment by any rule on itself because of the incapability of calculating the outcome. Thus, this specification alone renders sufficient validity for considering decentralized implementations via strategic mechanisms. One can readily see that the games presented in Section 3 well work in this scenario. The proposer would have no incentive to make a proposal that may lead to a deficit or surplus on the endowment, as either way would result in a lower payoff for the proposer, given what he could take away from the game will be monitored by the social planner. Such an enforceability is actually a weak condition, as implicitly adopted in almost all implementation literature.

In addition to (A1) that serves as a relaxation on the planner's information, we can further allow players to have a certain degree of private information, which is specified by the following two assumptions.

- (A2) Every player only knows his own claim and his own position in the ordering of all the claims.
- (A3) Player n knows that she herself is the highest claimant and how many claimants are in the game as well as the size of the (remaining) endowment at every stage.

That is, every player's claim will be private information and can only be known to the social planner but not to the other players. We can show that for game Γ^{cea} with incomplete information as described above an equal division proposal from player n at each stage of the game is a weakly dominant strategy. Thus, the CEA rule can be implemented as the unique SPE outcome even with incomplete information, as stated by the following theorem.

Theorem 4.1. *For any bankruptcy problem (E, c), the associated game $\Gamma^{cea}(E, c)$ satisfying (A1), (A2) and (A3) has a unique SPE outcome that is $\pi^{cea} = CEA(E, c)$.*

10 $E - \sum(\pi_i^{cea})' > 0 \Leftrightarrow E - \sum_{i=1}^{n-1}(\pi_i^{cea})' - (\pi_n^{cea})' > 0 \Leftrightarrow E - \sum_{i=1}^{n-1}(\pi_i^{cea})' > (\pi_n^{cea})' > 0$. Because $(\pi_n^{cea})' = \max\{0, \min\{\frac{c_n}{2}, E - \sum_{i=1}^{n-1}(\pi_i^{cea})'\}\}$, we must have $(\pi_n^{cea})' = \frac{c_n}{2}$.

11 At $\mathcal{G}$, if $\pi_n^{cea} = \frac{c_n}{2}$, we must have $E - \sum_{i=1}^{n-1} \pi_i^{cea} = \frac{c_n}{2}$. Since $E - \sum_{i=1}^{n-1}(\pi_i^{cea})' > \frac{c_n}{2}$, which means $E - \sum_{i=1}^{n-1}(\pi_i^{cea})' > E - \sum_{i=1}^{n-1} \pi_i^{cea}$.

12 Note that β solves $\sum \max\left\{\frac{c_i}{2} - \beta, 0\right\} = E - \sum \frac{c_i}{2}$ and β' solves $\sum \max\left\{\frac{c_i}{2} - \beta', 0\right\} = E - \sum \frac{c_i}{2} + \left(\frac{c_j}{2} - (\pi_j^{cea})'\right)$. By *endowment monotonicity*, we have $\beta > \beta'$. $\sum \max\left\{\frac{c_i}{2} - \beta', 0\right\} - \sum \max\left\{\frac{c_i}{2} - \beta, 0\right\} = \left(\frac{c_j}{2} - (\pi_j^{cea})'\right) = (\pi_j^{cel})' - \pi_j^{cel} + (\sum_{i \neq j} \max\left\{\frac{c_i}{2} - \beta', 0\right\} - \sum_{i \neq j} \max\left\{\frac{c_i}{2} - \beta, 0\right\})$. Since $j \neq n$, we must have $(\sum_{i \neq j} \max\left\{\frac{c_i}{2} - \beta', 0\right\} - \sum_{i \neq j} \max\left\{\frac{c_i}{2} - \beta, 0\right\}) > 0$. So $(\pi_j^{cel})' - \pi_j^{cel} < \frac{c_j}{2} - (\pi_j^{cea})'$.

The proof can be constructed along the same lines as in Lemma 3.1 and Theorem 3.2, and therefore is omitted. However, we provide the following remarks to highlight the main intuition behind the result.

If the endowment is small (i.e., $E < nc_1$), equal division of the whole endowment is the dominant strategy for player n. Every player's award is the equal share of the whole endowment, which is the CEA allocation of the problem. The proof is analogous to that of Lemma 3.1.

If the endowment is large enough to grant some players their claims under the CEA allocation (i.e., $E > nc_1$), it is easy to see that there must exist a stage, where according to Lemma 3.1 the equal division is the dominant proposal for player n. Let stage s be the first stage that satisfies the conditions of Lemma 3.1. In the previous $s-1$ stages, any proposal, including equal division of the available endowment, would award players $1, \ldots, s-1$ their claims. However, under incomplete information, where the claim of each player is unknown to player n, she cannot determine when stage s is reached. Any unequal division of the remaining endowment would risk resulting in a lower payoff to the proposer. Thus, the weakly dominant strategy for player n is to propose the equal division of the remaining endowment at every stage.

Similarly, the CEL rule and the Talmud rule can be implemented in the presence of incomplete information as well.

By contrast, most existing mechanisms in the literature usually require players to evaluate the proposals of other players and engage in bilateral negotiation in the case of rejection of the proposal, which requires the claims to be common knowledge, and hence, they do not possess the robustness to the incomplete information as discussed here.

The above informational structure is not the only one that could sustain the implementation results of the CEA, CEL and Talmud rules using our extended divide-and-choose mechanisms. An alternative informational structure that also works is as follows.[13] Consider the following the assumptions:

- (B1): The social planner knows the claims of all the players and could monitor and enforce the actions taken by the players and the consequent outcome of the game, but does not know the size of the endowment E. The social planner also appoints the player with the highest claim as the proposer and specifies the order of choosing from the proposals to be ascending on claims.
- (B2): The proposer knows the number of remaining players and the size of the remaining estate to be divided at every stage.

The main difference from the previous informational structure is (A2) being omitted, i.e., here no player knows their claims. Given that the social planner knows each player's claim, it is reasonable to let the social planner restrict each player receiving no more than her claim and let the social planner decide the order of choosing from the proposal. In the game of Γ^{cea}, when a player is going to choose, her weakly dominant strategy is to simply choose the largest amount in the proposal. Similarly, in Γ^{cel}, her weakly dominant strategy is to choose the smallest amount in the proposal. Hence, (A2) is no longer necessary. One can then readily see that all the implementation results hold in this alternative informational structure.

13 We thank an anonymous referee for generously sharing his/her idea and suggesting this informational structure.

5. Extension

5.1. Alternative rules

The divide-and-choose mechanism can be modified to fit for other rules and alternative settings.

Piniles' rule (see Thomson, 2015) results from applying the CEA twice using half-claims as the award constraints. Replacing Step 2 of game Γ^t with game Γ^{cea}, the Piniles' rule allocation can be achieved as the unique SPE outcome.

Chun et al. (2001) proposed a reverse Talmud rule which, using half claims, applies the CEL first and if there is excess after the initial distribution then the CEA is applied.[14] That is,

$$RT_i(E, c) = \begin{cases} CEL_i\left(E, \frac{c_i}{2}\right) & \text{when } \sum c_i \geq 2E; \\ \frac{c_i}{2} + CEA_i\left(E - \sum \frac{c_i}{2}, \frac{c_i}{2}\right) & \text{when } \sum c_i < 2E. \end{cases}$$

By switching the order of the two games in steps 1 and 2 in Γ^t, we obtain game Γ^{rt} as follows, where players will firstly follow a procedure of Γ^{cel} based on half of their claims and then play the game Γ^{cea} with the other half of their claims and the remaining endowment.

Consider any bankruptcy problem (E, c).

Step 1 Game $\Gamma^{cel}\left(E, \frac{c}{2}\right)$ is played with respect to $\left(E, \frac{c}{2}\right)$. The allocation for player i (where $i < n$), in this step is his payoff π_i^{cel} in game $\Gamma^{cel}\left(E, \frac{c}{2}\right)$. The allocation for player n in this step is $\pi_n^{cel} = \min\{E - \sum_{i=1}^{n-1} \pi_i^{cel}, \frac{c_n}{2}\}$.[15] The game continues to Step 2.

Step 2 Game $\Gamma^{cea}\left(E - \sum \pi_i^{cel}, \frac{c}{2}\right)$ is played where the associated bankruptcy problem is $\left(E - \sum \pi_i^{cel}, \frac{c}{2}\right)$.[16]

Step 3 Player i takes the final payoff $\pi_i^{rt} = \pi_i^{cel} + \pi_i^{cea}$.

Game $\Gamma^{rt}(E, c)$ has a unique SPE outcome which coincides with the reverse Talmud solution of the bankruptcy problem (E, c). The proof is analogous to that of game Γ^t and therefore omitted.

It is well-known that the allocation rule of the convex combination of CEA and CEL is not consistent, which means non-cooperative games that rely on consistency are not suitable to achieve such rules.[17] By introducing nature's random choice of game Γ^{cea} and Γ^{cel}, we could achieve such allocations as the expected outcome of SPE.

Furthermore, the model is not restricted to bankruptcy problems. We can apply the approach to handle the surplus sharing problems, where the endowment to be distributed is bigger than the sum of claims. The mechanism that yields a CEA-like solution for the surplus sharing problem (E, c), where $E > \sum c_i$, can be constructed as a multi-step game. If game Γ^{cea} is repeated k times until the whole endowment is distributed, then the SPE outcome for the game is $\pi_i = (k-1)c_i + CEA_i(E - (k-1)\sum c_i, c)$. That is, the endowment is shared proportional to the players' claims up to a point where the remaining endowment is not enough to cover the sum of all claims. Then the remaining endowment is distributed according to the CEA rule. In the same spirit, the divide-and-choose mechanism can shed light on cost sharing problems.

14 Here, $E < \sum \frac{c_i}{2}$ is not required. If $E \geq \sum \frac{c_i}{2}$, each player is assigned a loss of 0 and receive a payoff of $\frac{c_i}{2}$ in this step.

15 Here, $E < \sum \frac{c_i}{2}$ is not required. However, the proposals should still be efficient.

16 The restriction that Γ^{cea} is played with respect to a positive endowment is relaxed. $\pi_i^{cea} = 0$ if $E - \sum \pi_i^{cel} \leq 0$.

17 The convex combination of CEA and CEL can be described as $A_i(E, c) = \alpha CEA_i(E, c) + (1-\alpha) CEL_i(E, c)$, where $0 \leq \alpha \leq 1$.

5.2. A vote-divide-choose mechanism

Given there are different games (e.g., Γ^{cea} and Γ^{cel}) to play, it seems natural to consider introducing a voting stage[18] to the current divide-and-choose mechanism. Indeed, players with lower claims would prefer the CEA allocation whereas those with higher claims would prefer the CEL allocation. Allowing players to vote for playing either Γ^{cea} or Γ^{cel} can serve as an endogenous way of deciding the order of playing the two games Γ^{cea} and Γ^{cel}.

To be specific, the idea is to delegate the choice of which game (Γ^{cea} or Γ^{cel}) to play to players. That is, if the players could agree unanimously on Γ^{cea} or Γ^{cel}, then the chosen game is played to divide the estate; but if the players cannot reach unanimity, then the players will first play the game chosen by the majority votes with half of their claims, and then play the other game by the minority votes with the remaining estate and the other half of claims. Formally, the vote-divide-choose game Γ^{v} is composed of two stages and proceeds as follows.

Consider any bankruptcy problem (E, c).

Stage 1 All players sequentially[19] vote for either Γ^{cea} or Γ^{cel}. Denote the number of votes for Γ^{cea} and Γ^{cel} by n^{cea} and n^{cel}, respectively.

Stage 2 It has the following five cases.

(i) If $n^{cea} = n$, then game Γ^{cea} is played with respect to the bankruptcy problem (E, c).
(ii) If $n^{cel} = n$, then game Γ^{cel} is played with respect to the bankruptcy problem (E, c).
(iii) If $n^{cea} > n^{cel} > 0$, game Γ^{t} is played. That is, game Γ^{cea} is played with respect to $(E, \frac{c}{2})$, followed by Γ^{cel} with the remaining estate and half-claims.
(iv) If $n^{cel} > n^{cea} > 0$, game Γ^{rt} is played. That is, game Γ^{cel} is played with respect to $(E, \frac{c}{2})$, followed by Γ^{cea} with the remaining estate and half-claims.
(v) If $n^{cea} = n^{cel}$, the game to be played is randomly decided between Γ^{t} and Γ^{rt}.

Denote the number of players whose allocations according to the Talmud solution are greater, smaller than, or equal to those according to the reverse Talmud solutions as n^{t}, n^{rt} and n^{e}, respectively.

Theorem 5.1. *For any bankruptcy problem (E, c) with $E \le \frac{\sum c_i}{2}$, game $\Gamma^{v}(E, c)$ has SPE, and*

(i) if $n^{t} > \frac{n}{2}$, the unique SPE outcome of game $\Gamma^{v}(E, c)$ is the allocation assigned by the Talmud rule for the problem (E, c);
(ii) if $n^{rt} > \frac{n}{2}$, the unique SPE outcome of game $\Gamma^{v}(E, c)$ is the allocation assigned by the reverse Talmud rule for the problem (E, c);
(iii) if neither $n^{t} > \frac{n}{2}$ nor $n^{rt} > \frac{n}{2}$, game $\Gamma^{v}(E, c)$ has (ex post) two SPE outcomes that are the allocations assigned by the Talmud rule or by the reverse Talmud rule for the problem (E, c).

In proving the above theorem, we shall omit the description of players' strategies in stage 2 as these subgames have been analyzed in Section 3 and it would be unnecessarily cumbersome to repeat here. The essential analysis is on stage 1.

Proof. First of all, it is easy to see that with $E \le \frac{\sum c_i}{2}$ both games Γ^{t} and Γ^{rt} end after stage 1 as there will be no remaining estate to be divided in stage 2 in either game Γ^{t} or Γ^{rt}.

For case (i), since $n^{t} > \frac{n}{2}$, we know that $n^{t} > n^{rt} + n^{e}$. Consider the following strategies in stage 1: the players who prefer the Talmud allocation vote for game Γ^{cea}, the players who prefer the reverse Talmud allocation vote for game Γ^{cel}, and the players who are indifferent between the Talmud and the reverse Talmud allocations vote either Γ^{cea} or Γ^{cel}. Since n^{t} is the majority of n, the game to be played is Γ^{t} and the SPE outcome is the Talmud allocation. To verify these strategies are in SPE, one can see that players have no incentive to change the vote as it would either not affect their payoffs (as it will not change the game to be played) or lead to a lower payoff. One can see that these SPE strategies are independent of any specific ordering and, therefore, any ordering can be adopted here.

To be more specific, when $E \le \frac{\sum c_i}{2}$, one can see that the payoff for player 1 who has the lowest claim satisfies $CEA_1 \ge T_1 \ge RT_1 \ge CEL_1$,[20] and the payoff for player n who has the highest claim satisfies $CEL_n \ge RT_n \ge T_n \ge CEA_n$.[21]

Note that there may exist a special class of bankruptcy problems (E, c) such that the CEA allocation coincides with the Talmud allocation, e.g., all players have high claims relative to the estate to be divided and hence both the CEA and the Talmud rule yield the equal allocation of the estate among all players. For this special occasion, all players voting for Γ^{cea} also constitutes an equilibrium, following which game $\Gamma^{cea}(E, c)$ will be played and generates the CEA allocation. No player would have any incentive to change her vote as it would lead to Γ^{t} which yields the same payoff. However, in this case, the equilibrium outcome of CEA allocation is the same as the Talmud allocation for the bankruptcy problem (E, c). So the statement in the theorem remains valid.

When $CEA(E, c) \ne T(E, c)$, it is a dominant strategy for player 1 to vote for game Γ^{cea}, so unanimous voting on Γ^{cel} would not occur. It is a dominant strategy for player n to vote for game Γ^{cel}. Similarly, unanimous voting on Γ^{cea} would not occur, either, as player n would have the incentive to veto the CEA allocations being implemented by voting for Γ^{cel}. Therefore, only the Talmud allocation or the reverse Talmud allocation would be implemented in SPE in this mechanism. Hence, for case (i), it is a (weakly) dominant strategy for every player to vote in accordance with his or her preference of the Talmud and the reverse Talmud allocations, which, given $n^{t} > \frac{n}{2}$, yields the Talmud allocation in SPE.

There exists alternative SPE but they all have the same outcome that is the Talmud allocation. Other SPE are those strategy combinations that would yield the same outcome as above. For instance, more than $\frac{n}{2}$ players from those who prefer the Talmud solution over the reverse Talmud solution vote for Γ^{cea} while the rest vote for Γ^{cel}, which would be in SPE as well for it still guarantees the game to be played is Γ^{t}. The reverse Talmud allocation cannot be sustained as an SPE outcome in this case because with the sequential voting, if this were to happen, those who prefer the Talmud solution over the reverse Talmud solution will deviate from voting for Γ^{cel} and make sure the game to be played is Γ^{t}.

Case (ii) is the opposite of case (i) and its proof can be constructed analogously.

Case (iii) has two subcases. Subcase 1 is that $n^{t} = n^{rt}$ and $n^{e} = 0$. Then, it is easy to see that it is a hybrid of case (i) and case (ii). Its outcome depends on which game (Γ^{t} or Γ^{rt}) to be chosen by the randomized tie-breaking device in stage 2. Hence the ex post SPE outcome can be the allocations by either the Talmud

18 Once again, we are grateful to the anonymous referee for sharing this interesting idea and advising us to make an investigation.

19 Simultaneous voting may cause problems in implementation in this setting because it cannot rule out the "bad" equilibria, e.g., when two or more players simultaneously voted for Γ^{cel} while they should have actually voted for Γ^{cea}.

20 Note than $T_1 > RT_1$ when $E < \frac{\sum c_i}{2}$, and $T_1 = RT_1$ only when $E = \frac{\sum c_i}{2}$. Moreover, $CEA_1 > CEL_1$ whenever $E \le \frac{\sum c_i}{2}$.

21 Note than $RT_n > T_n$ when $E < \frac{\sum c_i}{2}$, and $RT_n = T_n$ only when $E = \frac{\sum c_i}{2}$. Moreover, $CEL_n > CEA_n$ whenever $E \le \frac{\sum c_i}{2}$.

rule or the reverse Talmud rule. Subcase 2 is that $n^e > 0$. Then, those players whose Talmud allocations equal the reverse Talmud allocations will affect the SPE outcome of the entire game. Since they themselves are indifferent between voting for Γ^{cea} and Γ^{cel}, their votes will affect the comparison between the number of players voting for Γ^{cea} and the number of players voting for Γ^{cel}, and consequently, lead to the corresponding SPE outcomes. The proof of this case can be constructed along the same way as in the above cases and therefore omitted. □

For a bankruptcy problem (E, c) with $E > \frac{\sum c_i}{2}$, the vote-divide-choose mechanism is more complicated and may not always have SPE. The reason is that with certain ordering of players to vote there may always exist players who have incentives to deviate in the loop of the CEA, CEL, Talumd and reverse Talmud allocations. However, we can introduce additional conditions to ensure the existence of SPE for the vote-divide-choose mechanism in this case.

Theorem 5.2. *For any bankruptcy problem (E, c) with $E > \frac{\sum c_i}{2}$,*

(i) if $n^t > \frac{n}{2}+1$, game $\Gamma^v(E, c)$ has a unique SPE outcome that is the allocation assigned by the Talmud rule for the bankruptcy problem (E, c);
(ii) if $n^{rt} > \frac{n}{2}+1$, game $\Gamma^v(E, c)$ has a unique SPE outcome that is the allocation assigned by the reverse Talmud rule for the bankruptcy problem (E, c).

Before presenting the proof, we provide some intuition of the result. Firstly, note that the threshold $\frac{n}{2}+1$ implies that when there are less than five players the game $\Gamma^v(E, c)$ with $E > \frac{\sum c_i}{2}$ may not have SPE. A sufficient condition for the existence of SPE would be that players who strictly prefer the Talmud (or reverse Talmud) allocation need to be able to be in control of the voting outcome irrespective of how the other players would vote. Hence, $n^t > \frac{n}{2}+1$ or $n^{rt} > \frac{n}{2}+1$ can guarantee such a full control with respect to the Talmud allocation or the reverse Talmud allocation. To be more specific, please note that when $E > \frac{\sum c_i}{2}$, game Γ^t (game Γ^{rt}) results in allocating the half claim to each player and then dividing the remaining estate according to game Γ^{cel} (game Γ^{cea}) with respect to the other half claims. For example, to have game Γ^{rt} being played requires a majority vote on game Γ^{cel}. However, this means if all such players cast their votes based on their preferences on the reverse Talmud allocation, there may exist a scenario where the other players strictly prefer CEL over the reverse Talmud allocations and, therefore, will have an incentive to also vote for game Γ^{cel} in order to invoke the unanimous vote on game Γ^{cel}, which would make the former players worse off. So voting purely based on the preference between the Talmud and reverse Talmud allocations will not be in SPE as happened in Theorem 5.1. In order to implement the allocation that the majority players prefer and also prevent the situation of a unanimous vote on the less preferred allocation, at least one of these majority players needs to vote against their preference between the Talmud and reverse Talmud allocation. However, this would risk the majority being flipped and the less preferred allocation being implemented. Thus, the number of players who prefer the reverse Talmud solution needs to be at least one larger than the required majority to implement the reverse Talmud solution. This is to allow one of those who prefer the reverse Talmud allocation to vote for Γ^{cea}, while the others (still form the majority of all players) of those who prefer the reverse Talmud solution will simply vote for Γ^{cel}, which will prevent a unanimous voting outcome on Γ^{cel} and guarantee the final game to be played is Γ^{rt}.

Proof. First of all, it is easy to see that when $E > \frac{\sum c_i}{2}$, the payoff for player 1 satisfies $CEA_1 \geq RT_1 > T_1 \geq CEL_1$ and the payoff for player n satisfies $CEL_n \geq T_n > RT_n \geq CEA_n$. Thus, unanimous voting on either Γ^{cea} or Γ^{cel} would not occur as both player 1 and player n would have the incentive to veto, respectively, the CEL and CEA allocations being implemented. Therefore, only the Talmud allocation or the reverse Talmud allocation would be implemented in SPE in this mechanism.

For case (i), consider the following strategies in stage 1: all but one of those players who prefer the Talmud solution vote for Γ^{cea} and the remaining players all vote for Γ^{cel}, which yields the Talmud allocation. This is in SPE because no player would have an incentive to change his or her vote as otherwise it would either not affect their payoffs (as it will not change the game to be played) or lead to a lower payoff. There exists alternative SPE but they all have the same outcome which is the Talmud allocation. One such SPE is that more than $\frac{n}{2}$ of the players who prefer the Talmud solution vote for Γ^{cea} and at least one of the players who prefer the Talmud solution vote for Γ^{cel}, while some of the remaining players vote for Γ^{cea} and some vote for Γ^{cel}. One can see that these voting strategies guarantee the Talmud allocation being implemented by majority voting on Γ^{cea} but avoid the unanimous voting on Γ^{cea} that may lead to the CEA allocation.

Case (ii) is the opposite of (i) and its proof can be constructed analogously. □

5.3. A modified voting game

In order to rule out the situations where SPE may not exist, we consider a modified mechanism Γ^{vm} as follows.

Stage 1 All players sequentially vote for either Γ^{cea} or Γ^{cel}. Denote the number of votes for Γ^{cea} and Γ^{cel} by n^{cea} and n^{cel}, respectively.

Stage 2 It has the following three cases.

(i) If $n^{cea} > n^{cel}$, game Γ^t is played. That is, game Γ^{cea} is played with respect to $(E, \frac{c}{2})$, followed by Γ^{cel} with the remaining estate and half-claims.
(ii) If $n^{cel} > n^{cea}$, game Γ^{rt} is played. That is, game Γ^{cel} is played with respect to $(E, \frac{c}{2})$, followed by Γ^{cea} with the remaining estate and half-claims.
(iii) If $n^{cea} = n^{cel}$, the game to be played is randomly decided between Γ^t and Γ^{rt}.

That is, compared to game Γ^v, here in Γ^{vm} we change the voting rule such that the game being played in stage 2 is decided by the majority of the votes but not by the possible unanimous votes. Thus, even all players unanimously vote for Γ^{cea}, it just implies that game Γ^{cea} will be played first, with respect to $(E, \frac{c}{2})$, while Γ^{cel} will still be played, albeit later, with the remaining estate and half-claims. Thus, such a modification excludes the possibility of CEA and CEL allocations to be the final outcomes, and therefore, it breaks the possible loop that appears in the game Γ^v with (E, c) and $E > \frac{\sum c_i}{2}$.

It can be easily shown that for the above game, SPE always exists. And the SPE outcome depends on the number of players who prefer Talmud or reverse Talmud allocations. If $n^t > \frac{n}{2}$ (if $n^{rt} > \frac{n}{2}$), in SPE, we always have $n^{cea} > n^{cel}$ ($n^{cel} > n^{cea}$) in the voting stage and the SPE outcome is the allocation according to the Talmud (reverse Talmud) solution. If $n^t \leq \frac{n}{2}$ and $n^{rt} \leq \frac{n}{2}$, the game has a unique SPE outcome that is either the Talmud or the reverse Talmud allocation.

6. Concluding remark

This paper proposes and advocates the extended divide-and-choose mechanism for it has three desirable features. Firstly, it is simple. It provides a clear and intuitive explanation of the strategic elements of the allocation rules for bankruptcy problems, which are sometimes complex from conceptual or axiomatic perspectives. Secondly, it is general. It seems that all these major bankruptcy rules can be accommodated within the same divide-and-choose framework. The basic divide-and-choose idea can be modified into variant mechanisms to obtain alternative allocation rules and applied to related settings like surplus sharing problems. The comparison of the variants also helps clearly pin down the essential strategic differences between the rules. Finally, it is robust to certain incomplete information. In particular, claims are no longer needed to be common knowledge. The only essential requirement we need, a rather weak assumption though, is that the proposer knows the number of claimants. Then, all the results hold.

We like to conclude by highlighting three topics that should be of interest for future research. Firstly, one can extend the divide-and-choose idea to study alternative bankruptcy rules and it also seems promising to construct new rules along this strategic perspective. Secondly, it is worth exploring alternative informational structures that may still preserve the results presented in the paper. Generally, implementation with incomplete information for bankruptcy problems is under-developed and much can be done in this area. Finally, one can further investigate other ways of (modeling) strategic interaction among the claimants when they are allowed to play different games. For example, one can study the design of a mechanism such that the right to choose between sharing the endowment directly and sharing the deficits can be endogenously determined. As in Section 5.2, we offered an attempt by introducing a voting stage. More analysis can be done along this line and other ideas can be explored as well.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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