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Can reference governors outperform tube-based predictive control?

Mohammad A.M. Sarbini[1], John A. Rossiter[2], Paul A. Trodden[3]

***Abstract*— This paper makes comparisons between a reference governor scheme and a tube-based MPC algorithm for a simple tracking problem with output disturbances. The key aim is to consider the extent to which a tube based approach is merited in this scenario, or whether the same feasibility assurances and good performance can be obtained with a far simpler approach. The main features, properties, and general setup of both methods are described and compared. The tracking performance of both algorithms are analysed.**

I. INTRODUCTION

Model predictive control (MPC) strategies have been widely and successfully used in various industrial applications [20] due to the ability to handle multivariable processes and constrained control problems. The theoretical foundation has been developed to guarantee asymptotic stability, constraint satisfaction, and robustness for linear and non-linear systems, based on the solution of tractable, online optimisation problems. Most authors have recognised the usefulness of the dual-mode paradigm [26], in which Lyapunov theory is used provide a general framework for stability guarantees and recursive feasibility by employing suitable terminal ingredients, that is, a terminal cost and terminal equality or inequality constraint [15].

The inclusion of terminal conditions however, implicitly imposes restrictions on the states in the presence of disturbances and/or during target changes. The shifted terminal set might not be an admissible invariant set and/or could be unreachable within the available prediction horizon, resulting in loss of feasibility. Moreover, computing the terminal sets online for every new target or to increase the prediction horizon for feasibility gains can be computationally expensive.

To deal with these issues, significant attention has been given to develop stabilising MPC that deals with disturbed and uncertain systems (e.g. [3], [14], [16], [18]). The main objective can be described as defining an appropriate invariant set for: (i) the disturbed system or (ii) the *difference* between the nominal and disturbed system. The latter is particularly well known in the literature as tube-based robust MPC (TMPC) [3], [14]. The core idea is to *contain* the uncertain closed-loop trajectories within the prescribed tubes to ensure feasibility.

An alternative approach to regaining feasibility is to consider, artificially and/or temporarily, changing the reference target [12], [13], [24]. In contrast to standard tracking MPC, the reference or command governors (R/CG) strategies employ an artificial reference target as a decision variable

[1,2,3] Department of Automatic Control and Systems Engineering, The University of Sheffield, Mappin Street, S1 3JD, UK. Email: [1]mamsarbini1, [2]j.a.rossiter, [3]p.a.trodden@sheffield.ac.uk

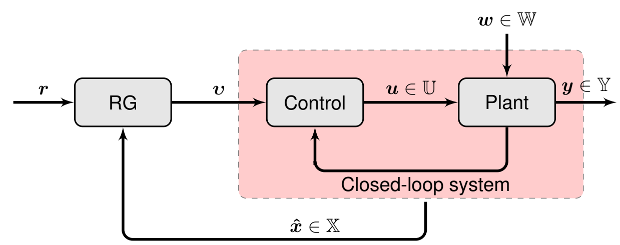

Fig. 1. Reference governor applied to constrained closed-loop system

(see Fig. 1). The main idea is to augment with a non-linear *device* cascaded to a primary closed-loop system to modify, whenever necessary, the desired target r so that constraints are satisfied. The tacit assumption within an R/CG schemes is that infeasibility is often caused by a fast target changes and thus, feasibility is retained by introducing an artificial target. Classical RG schemes, such as presented in [5], [6], [8] focused on simple computations to for the determination of $\boldsymbol{v}$, therefore the tracking performance is suboptimal. For optimality, command governors [1], [2] use similar ideas to MPC where the artificial target trajectory is predicted online at each time step. Interested readers are referred to [4], which provides a recent survey on R/CG schemes.

In some respects, both TMPC and R/CG schemes attempt to tackle infeasibility issues caused by different sources; tubes provide feasibility guarantees against model uncertainties such as disturbances whereas originally RG focused on infeasibility due to fast/large target changes. Infeasibility within an optimisation can also occur due to a fast/large output disturbance acting on the closed-loop system and it appears this issue has primarily being considered under tube-based schemes. However, this paper proposes that in fact a RG scheme may be equally adept at handling this scenario and ensuring feasibility is retained, but with the advantage of being far simpler to code and implement. The main purpose of this paper is to draw attention to the potential of relatively simple predictive approaches and thus pose the question of, to what extent can a RG type of approach be equally effective and ensure recursive feasibility results for a wider range of uncertainty?

The paper is organised as follows. The general control problem, model, assumptions, and control strategy is setup in Section II. A TMPC formulation and R/CG schemes are presented in Sections III and IV. Section V contrasts the two schemes and discusses the implementation issues. Illustration via numerical simulations are presented in Section VI. Section VII gives the conclusion and future work.

Notation: $\mathbb{R}$ and $\mathbb{I}$ denote real and non-negative integer numbers, respectively. $\ominus$ denotes Pontryagin set difference, $\oplus$ denotes Minkowski set sum, $\|x\|_P = \sqrt{x^T P x}$ for $x \in \mathbb{X}^{n_x}$. $\underset{\rightarrow k}{\mathbf{x}}$ represents future states of $\boldsymbol{x}$. $\mathbb{X}^\circ$ represents int($\mathbb{X}$). (a, b) denotes the stacked vector $\left[a^T b^T\right]^T$.

II. PROBLEM SETUP AND PRELIMINARIES

A. Actual system, nominal model, and main assumptions

Consider the linear, discrete-time invariant system,

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + B\boldsymbol{u}_k + \boldsymbol{w}_k \tag{1a}$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{v}_k \tag{1b}$$
$$\boldsymbol{z}_k = F\boldsymbol{x}_k + G\boldsymbol{u}_k \tag{1c}$$

where $\boldsymbol{x}_k \in \mathbb{R}^{n_x}$, $\boldsymbol{u}_k \in \mathbb{R}^{n_u}$, $\boldsymbol{y}_k \in \mathbb{R}^{n_y}$, and $\boldsymbol{z}_k \in \mathbb{R}^{n_z}$ are the states, control inputs, tracking outputs, and constrained outputs, respectively, $\boldsymbol{w}_k \in \mathbb{R}^{n_w}$ is the state disturbance, $\boldsymbol{v}_k \in \mathbb{R}^{n_v}$ is the output disturbance. Disturbances are unknown but bounded and satisfy $\boldsymbol{w}_k \in \mathbb{W} \subset \mathbb{R}^{n_x}$ and $\boldsymbol{v}_k \in \mathbb{V} \subset \mathbb{R}^{n_y}$.

A-1: The pair (A, B) is stabilisable, the pair (C, A) is detectable, and the following condition holds:

$$\operatorname{rank}\begin{bmatrix} A - I_{n_x} & B \\ C & \mathbf{0} \end{bmatrix} = n_x + n_y \tag{2}$$

Remark 1: Condition (2) is necessary for the output $\boldsymbol{y}$ to track an arbitrary set-point target $\boldsymbol{r}_k$, and implies that $n_y \leq n_u$.

The system (1) is subject to the state and input constraints for all $k \in \mathbb{N}$, or more compactly,

$$\boldsymbol{x}_k \in \mathbb{X}, \boldsymbol{u}_k \in \mathbb{U}, \boldsymbol{z}_k \in \mathbb{Z} \tag{3}$$

where $\mathbb{X}$, $\mathbb{U}$, and $\mathbb{Z}$ are specified sets of constraints. Matrices F and G in (1c) are suitable half-space representations of hyperreactangle with upper and lower bounds.

A-2: The set $\mathbb{Z}$ is a convex, compact polyhedron represented by $\mathbb{Z} = \{\, \boldsymbol{z} \mid S\boldsymbol{z} \leq s \,\}$ which satisfies $0 \in \mathbb{Z}^\circ$.

To ensure offset-free set-point tracking, we employ a disturbance model [17] to estimate the states, disturbance, and steady-state values

A-3: Under the specified constraints, all steady-state values are reachable,

$$\begin{bmatrix} A - I_{n_x} & B \\ C & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_\mathrm{s} \\ \boldsymbol{u}_\mathrm{s} \end{bmatrix} = \begin{bmatrix} B_d \boldsymbol{d}_k \\ \boldsymbol{r}_k - D_d \boldsymbol{d}_k \end{bmatrix} \tag{4}$$

More specifically, this paper only considers output disturbances (i.e. $B_d = 0$ and $D_d = 1$). If (4) has a unique solution, we can define the deviation variables based on the steady-states as $\tilde{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_k - \boldsymbol{x}_\mathrm{s}$ and $\tilde{\boldsymbol{u}}_k = \hat{\boldsymbol{u}}_k - \boldsymbol{u}_\mathrm{s}$. A possible control law for tracking the steady-states is:

$$\kappa_\mathrm{s}(\hat{\boldsymbol{x}}) = \boldsymbol{u}_\mathrm{s} - K(\hat{\boldsymbol{x}}_k - \boldsymbol{x}_\mathrm{s}) \tag{5}$$

where K is the state feedback matrix chosen such that the closed-loop system is stable, i.e., $|\lambda(A - BK)| < 1$, and

$$\begin{bmatrix} \boldsymbol{x}_\mathrm{s} \\ \boldsymbol{u}_\mathrm{s} \end{bmatrix} = \begin{bmatrix} M_x \\ M_u \end{bmatrix} (\boldsymbol{r}_k - \boldsymbol{d}_k) \tag{6}$$

Remark 2: The control law (5) can equivalently be written as $\kappa_\mathrm{s}(\hat{\boldsymbol{x}}, \boldsymbol{r}, \boldsymbol{d}) = -K\hat{\boldsymbol{x}}_k + L(\boldsymbol{r}_k - \boldsymbol{d}_k)$ where $L = KM_x + M_u$.

B. The MPC strategy for tracking

It is now well known that the dual-mode MPC schemes in general provide good stability guarantees and recursive feasibility [15]. A standard dual-mode MPC formulation for tracking uses deviation variables:

$$\min_{\underset{\rightarrow k}{\mathbf{u}}} \quad \left\|\tilde{\boldsymbol{x}}_{k+N|k}\right\|^2_{Q_\mathrm{f}} + \sum_{i=0}^{N-1} \left\|\tilde{\boldsymbol{x}}_{k+i|k}\right\|^2_Q + \left\|\tilde{\boldsymbol{u}}_{k+i|k}\right\|^2_R \tag{7a}$$
$$\text{s.t.} \quad \tilde{\boldsymbol{x}}_{k|k} = \tilde{\boldsymbol{x}}_k, \tag{7b}$$
$$\tilde{\boldsymbol{x}}_{k+i+1|k} = A\tilde{\boldsymbol{x}}_{k+i|k} + B\tilde{\boldsymbol{u}}_{k+i|k}, \quad i \in \mathbb{I}_0^{N-1}, \tag{7c}$$
$$\boldsymbol{z}_{k+i|k} \in \mathbb{Z} \quad i \in \mathbb{I}_0^{N-1}, \tag{7d}$$
$$\tilde{\boldsymbol{x}}_{k+N|k} \in \mathbb{X}_\mathrm{f}(\boldsymbol{x}_\mathrm{s}) \tag{7e}$$

where N is the prediction horizon, $\underset{\rightarrow k}{\mathbf{u}} = (\boldsymbol{u}_0, \ldots, \boldsymbol{u}_{N-1})$ are the optimal solution for the input sequence, $(Q, R) \succ 0$ are states and input weighting matrices, Q_f is the terminal cost matrix which satisfies Lyapunov equation, and $\mathbb{X}_\mathrm{f}$ is the terminal set. Readers will note the dependency of the terminal set on the steady-states in (7d). Consequently if the steady-state target to close the constraint boundary, part of the the translated terminal set may lie outside the states constraint set, i.e., $\mathbb{X}_\mathrm{f}(\boldsymbol{x}_\mathrm{s}) \subsetneq \mathbb{X}$, thus invalidating any claims of positive invariance. The set of feasible initial states can be expressed as:

$$\mathbb{X}_N = \left\{\, \boldsymbol{x}_k \;\middle|\; \exists \underset{\rightarrow k}{\mathbf{u}} : \mathbf{F}\boldsymbol{x}_k + \mathbf{G}\underset{\rightarrow k}{\mathbf{u}} \leq \mathbf{h} \,\right\} \tag{8}$$

where $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{h}$ are obtained from A, B, F, G, and h via prediction equations over N. Details on how to compute these are omitted as these are standard in the MPC literature.

C. Closed-loop prediction

Let the predicted control law be [25]:

$$\tilde{\boldsymbol{u}}_{k+i} = \begin{cases} -K\tilde{\boldsymbol{x}}_{k+i} + \boldsymbol{c}_{k+i} & i \in \mathbb{I}_0^{N-1} \\ -K\tilde{\boldsymbol{x}}_{k+i} & i \geq N \end{cases} \tag{9}$$

where the perturbations $\boldsymbol{c}_{k+i}$ are the d.o.f. (or control moves) available for constraint handling. It is well known that one can re-parameterise (7) as

$$J = \min_{\underset{\rightarrow k}{\mathbf{c}}} \quad \left\|\underset{\rightarrow k}{\mathbf{c}}\right\|^2_{S_c} \tag{10a}$$
$$\text{s.t.} \quad \tilde{\boldsymbol{z}}_{k+i|k} \in \mathbb{Z}_N(\boldsymbol{x}_\mathrm{s}) \tag{10b}$$

where $\underset{\rightarrow k}{\mathbf{c}} = (\boldsymbol{c}^T_{k|k}, \ldots, \boldsymbol{c}^T_{k+N-1|k})$, $S_c = \operatorname{diag}(B^T Q_\mathrm{f} B + R)$ and $\mathbb{Z}_N$ is the maximal controlled admissible set (MCAS),

$$\mathbb{Z}_N = \left\{\, \boldsymbol{x}_k \;\middle|\; \exists \underset{\rightarrow k}{\mathbf{c}} : \mathbf{F}\boldsymbol{x}_k + \mathbf{G}\underset{\rightarrow k}{\mathbf{c}} \leq \mathbf{h} \,\right\} \tag{11}$$

where $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{h}$ are computed using admissible set algorithm such as [19] and therefore different from (8).

III. TUBE MPC

A. Tube of trajectories

Central to the notion of tubes is the disturbance invariant sets [11], which is the set of states reachable from the origin in the presence of a bounded disturbance. The parameterisations of states, inputs, and local trajectories and their respective "tubes" can be described in the following. Due to the model uncertainty in (1), the predicted states are described by a tube comprising a sequence of sets, $\{\mathcal{X}_0, \mathcal{X}_1, \dots\}$. Each of these sets contains the possible states at a given future time instant for all realisations of future uncertainty:

$$\boldsymbol{x}_k = \bar{\boldsymbol{x}}_k + \boldsymbol{e}_k, \implies \mathcal{X}_k = \bar{\boldsymbol{x}}_k \oplus \mathcal{E}_k \tag{12}$$

where $\boldsymbol{e}_k \in \mathcal{E}_k$ is the prediction error. To ensure $\boldsymbol{x}_k \in \mathcal{X}_k$ for all $k \geq 0$, the proposed tube control law comprises of two terms: (i) nominal control $\bar{\boldsymbol{u}}_k$, and (ii) correction to account for the prediction error $\boldsymbol{e}_k$,

$$\kappa_\tau(\hat{\boldsymbol{x}}) = \bar{\boldsymbol{u}}_k - K_\tau \boldsymbol{e}_k, \implies \mathcal{U}_k = \bar{\boldsymbol{u}}_k \oplus -K_\tau \mathcal{E}_k \tag{13}$$

where the feedback matrix K_τ is chosen such that $A_\tau = A - BK_\tau$ is Hurwitz and ensures the evolution of $\boldsymbol{e}_k$ is bounded. The actual trajectory thus lies close to the nominal trajectory inside $\mathcal{X}_k$, which can be visualised as a tube of trajectories with $\bar{\boldsymbol{x}}_k$ at the centre. The sets that form the tube $\{\mathcal{E}_0, \mathcal{E}_1, \dots\}$ evolve according to the closed-loop trajectory of (1a), i.e.,

$$\boldsymbol{e}_{k+1} = A_\tau \boldsymbol{e}_k + \boldsymbol{w}_k, \implies \mathcal{E}_{k+1} = A_\tau \mathcal{E}_k \oplus \mathbb{W} \tag{14}$$

Each set generated by Eq. (14) implies that $\mathcal{E}_k \subset \mathcal{E}_{k+1}$ for all $k \geq 0$, and therefore is a robust positive invariant (RPI) set. If $\mathcal{E}_0 = \{0\}$, then $\mathcal{E}_k$ is defined as

$$\mathcal{E}_k = \mathbb{W} \oplus A_\tau \mathbb{W} \oplus \dots \oplus A_\tau^{j-1}\mathbb{W} = \bigoplus_{j=0}^{k-1} A_\tau^j \mathbb{W} \tag{15}$$

B. Robust constraint handling

Robust constraint satisfaction is guaranteed if (1c) holds. By substituting Eq. (13) into Eq. (1c), we get

$$F\boldsymbol{x}_k + G\boldsymbol{u}_k + F_K \boldsymbol{e}_k \leq h, \quad \forall \boldsymbol{e}_k \in \mathcal{E}_k \tag{16}$$

where $F_\tau = F - GK_\tau$ and the term $F_\tau \boldsymbol{e}_k$ represents the additional constraint tightening that is computed by maximising the prediction error $\boldsymbol{e}$; this can done by considering $k \to \infty$ in (15). Since A_τ is stable, then the minimum Robust Invariant Set (mRPI) set exists [11] which is given by

$$\mathbb{X}_\tau^{\mathrm{m}} = \bigoplus_{j=0}^{\infty} A_\tau^j \mathbb{W} \tag{17}$$

However unless A_τ is nilpotent, computing the exact representation of (17) is impossible; instead its close outer approximation is employed, for example, in [23], [27]. Robust

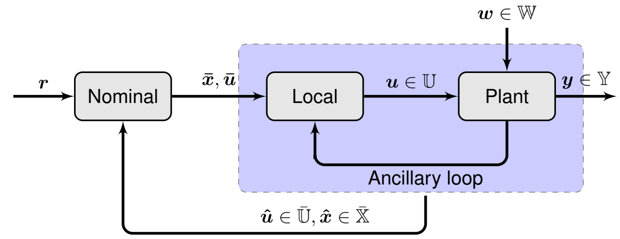

Fig. 2. Tube MPC scheme.

constraint handling can be ensured by tightening the original state and input constraints (3) via Pontryagin difference,

$$\bar{\mathbb{X}} = \mathbb{X} \ominus \mathbb{X}_\tau^{\mathrm{m}} \tag{18a}$$
$$\bar{\mathbb{U}} = \mathbb{U} \ominus K_\tau \mathbb{X}_\tau^{\mathrm{m}} \tag{18b}$$
$$\bar{\mathbb{Z}} = \mathbb{Z} \ominus (\mathbb{X}_\tau^{\mathrm{m}} \times K_\tau \mathbb{X}_\tau^{\mathrm{m}}) \tag{18c}$$

Eq. (18) implies that by forcing a suitable tighter set of constraints for the nominal system, the evolution of the uncertain system controlled by (14) is robustly admissible.

Remark 3: The use of mRPI to characterise the tubes is rigid and conservative. Other works in the literature such as homothetic [21] and parameterised [22] tubes are further extensions but the implementation is non-trivial and therefore are not discussed here.

C. Implementation

The tube-based MPC is therefore similar to conventional MPC with systematic constraint tightening procedure and additional ancillary inner loop. The nominal problem is solved online and the nominal trajectory is used as the new target trajectory for the ancillary tube controller. Fig. 2 illustrates the block diagram of tube MPC scheme. Readers will notice a similarity with cascade control. Tube MPC is implemented according to Algorithm 1.

Algorithm 1 Tube MPC Algorithm

1: At each sampling instant, perform the optimisation (7) or (10) with tightened constraints (18)
2: Apply the first value of the optimal solution for the input sequence, $\boldsymbol{u}_0$
3: Determine the nominal state and input trajectory $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$
4: Apply the control law (13)

IV. REFERENCE/COMMAND GOVERNORS

This section summarises the main concepts of RG/CG schemes [8]. For a detailed exposition readers are referred to the recent survey found in [4].

A. Closed-loop form

In this paper we setup the RG/CG scheme according to the standard form in the RG/CG literature, see e.g. [4], [6], [7], [9]. To that end, let the control law be:

$$\kappa_\upsilon(\hat{\boldsymbol{x}}, \boldsymbol{\upsilon}) = -K\hat{\boldsymbol{x}}_k + L\boldsymbol{\upsilon}_k \tag{19}$$

where $\boldsymbol{v}_k \in \mathbb{R}^{n_v}$ is the modified reference. Applying (19) in (1) yields

$$\boldsymbol{x}_{k+1} = A_K \boldsymbol{x}_k + B_L \boldsymbol{v}_k + B_w \boldsymbol{w}_k \quad (20a)$$
$$\boldsymbol{z}_k = F_K \boldsymbol{x}_k + G_L \boldsymbol{v}_k + D_w \boldsymbol{w}_k \quad (20b)$$

where $A_K = A - BK$, $B_L = BL$, $F_K = F - GK$, $G_L = GL$, and the matrices B_w and D_w are known with appropriate sizes. The following assumptions [7], [9] are made:

A-4: The input $\boldsymbol{v}$ is assumed to be constant for all future instants, i.e. $\boldsymbol{v}_{k+1} = \boldsymbol{v}_k$.

Remark 4: If $\boldsymbol{v}_k = \boldsymbol{r}_k$, Eqs. (5) and (19) are equivalent. Eq. (19) can be re-written as $\kappa_v(\hat{\boldsymbol{x}}, \boldsymbol{v}) = \boldsymbol{u}_v - K(\hat{\boldsymbol{x}} - \boldsymbol{x}_v)$ where $\boldsymbol{x}_v = M_x \boldsymbol{v}_k$, and $\boldsymbol{u}_v = M_u \boldsymbol{v}_k$.

B. Robust constraint handling in RG

The RG relies on the model (20) to predict future constraint violation using the maximal output admissible set (MOAS) [1] [7]. Using Assumption (A-4), the constrained output prediction is expressed as a function of the initial states and inputs,

$$\underset{\rightarrow i}{\mathbf{z}} = \mathbf{F}_i \boldsymbol{x}_0 + \mathbf{G}_i \boldsymbol{v}_0 + \mathbf{d}_i \quad (21a)$$
$$\mathbf{d}_i = F_K \sum_{j=0}^{i-1} A_K^{i-j-1} B_w \boldsymbol{w}_j + D_w \boldsymbol{w}_i \quad (21b)$$

where $\mathbf{F}_i = F_K A_K^i$, $\mathbf{G}_i = H_0 - F_K A_K^i \Gamma$, $H_0 = F_K \Gamma + G_K$, $\Gamma = (I - A_K)^{-1} B_L$ is the map from the constant reference $\boldsymbol{v}$ to the steady-state, and Eq. (21b) computes the possible effects of $\boldsymbol{w}$ on $\boldsymbol{z}$. Consequently, the MOAS is given by

$$\mathbb{O}_\infty = \{\, (\boldsymbol{x}, \boldsymbol{v}) \mid H_x \boldsymbol{x} + H_v \boldsymbol{v} \le h \,\} \quad (22)$$

The algorithm to compute H_x, H_v, and h is well known [7] therefore not elaborated. A key point of note is a close inner approximation of $\mathbb{Z}$, i.e. $\mathbb{Z}^\epsilon = (1-\epsilon)\mathbb{Z}$ is usually required to ensure the MOAS can be represented by a finite number of linear inequalities. Moreover, the effects of $\boldsymbol{w}$ are captured in vector h, using a LP.

C. Computation of $\boldsymbol{v}_k$

The goal of the RG is to enforce constraints by computing $\boldsymbol{v}_k$ online such that it is close to the original reference $\boldsymbol{r}_k$. More specifically, the original RG scheme solves the following LP problem:

$$\begin{aligned} \max_{\kappa \in [0,1]} \quad & \kappa(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{x}) && (23a)\\ \text{s.t.} \quad & \boldsymbol{v}_k = \boldsymbol{v}_{k-1} + \kappa(\boldsymbol{r}_k - \boldsymbol{v}_{k-1}), && (23b)\\ & (\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{O}_\infty && (23c) \end{aligned}$$

The parameter κ can be solved compactly from (22) and (23b) [8]:

$$\kappa(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{x}) = \min\left\{ \min_{j \in J} \{\mathcal{H}(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{x})\}, 1 \right\} \quad (24)$$

where

$$\mathcal{H}(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{x}) = \frac{h^j - H_x^j \boldsymbol{x}_k - H_v^j \boldsymbol{v}_{k-1}}{H_v^j (\boldsymbol{r}_k - \boldsymbol{v}_{k-1})}$$

and the index j represents the row of H and h.

The command governor (CG) schemes [1], [2] involve the solution of Euclidean-norm projection problems, which can be transformed into parametric QP problem similar to solving standard MPC problems,

$$\begin{aligned} \min_{\boldsymbol{v}} \quad & \|\boldsymbol{v}_k - \boldsymbol{r}_k\|_R^2 \\ \text{s.t.} \quad & (\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{O}_\infty \end{aligned} \quad (25)$$

D. Implementation

The R/CG schemes exploits the closed-loop dynamics and the maximal admissible sets theory to predict future constraint violations. These sets are computed offline and finite determination are guaranteed if the A_K is stable, (provided by (A-1)), constraints are convex and compact (provided by (A-2)), and by tightening the original constraint by a small number (recall (A-3)). R/CG is implemented according to Algorithm 2.

Algorithm 2 Reference/Command Governor

1: Determine (22) offline using Algorithm [7].
2: At each sample time, compute $\boldsymbol{v}_k$ by solving (23)(25).
3: Apply (19) in (20).

V. TMPC-R/CG COMPARISON

A. Control laws and d.o.f.

Readers will notice Eqs. (5), (13), and (19) have a similar structure. Eq. (5) is essentially the terminal (or mode-2) control law for the dual-mode MPC. In Eq. (13), the nominal trajectories $(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{u}}_k)$ are in essence the d.o.f of the tube controller, as it was defined in [3] that $\boldsymbol{c}_k = \bar{\boldsymbol{u}}_k + K\bar{\boldsymbol{x}}_k$. In Eq. (19) the d.o.f is *transferred* to the state trajectories $(\boldsymbol{u}_v, \boldsymbol{x}_v)$ (Remark 4) via the extra state $\boldsymbol{v}_k$. Therefore, the "terminal region" for the RG is expanded for every admissible $\boldsymbol{v}_k$.

B. Feasible and terminal regions

In general, $\mathbb{X}_N$ in MPC is used to compute a maximal invariant terminal set that helps to enforce stability of the closed-loop system. In RG, $\mathbb{O}_\infty$ is used for constraint enforcement only and closed-loop stability is enforced by noticing from (23b) that for a constant $\boldsymbol{r}_k$, $\boldsymbol{v}_k$ forms a monotonic sequence over a compact set, which implies that $\boldsymbol{v}_k$ must converge to a constant. Clearly, the shape and size of $\mathbb{O}_\infty$ depends on the choice the state feedback controller K and thus, Q and R. This is illustrated in Fig. 3 using the double integrator example in Section VI-A. The projection of the MOAS on the states, $\mathbb{O}_{t^*}^R(\boldsymbol{x}, \boldsymbol{v})$ at slices of $\boldsymbol{v} \in \{\, -1.9, 0, 1.9 \,\}$ are also illustrated. The terminal set (white) and feasible sets for tube MPC with increasing horizon $N \in [2, 14]$ are represented by darker shades. For small N, which is typically desirable for applications with limited computation, MPC schemes obviously may not be able to

[1]Often referred to as Maximal Positive Invariant (MPI) Set in MPC literature

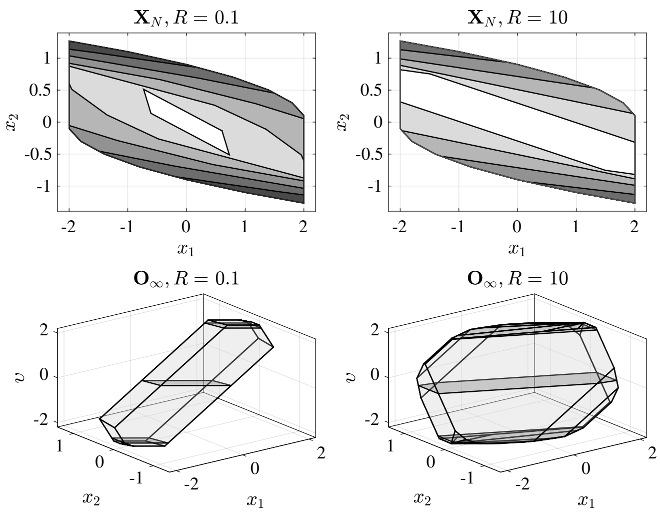

Fig. 3. The effect of input penalty R on the size $\mathbb{X}_0$ (white), feasible sets $\mathbb{X}_N$ with $N = 2, 4, 6, 10, 12, 14$, and $\mathbb{O}_\infty$.

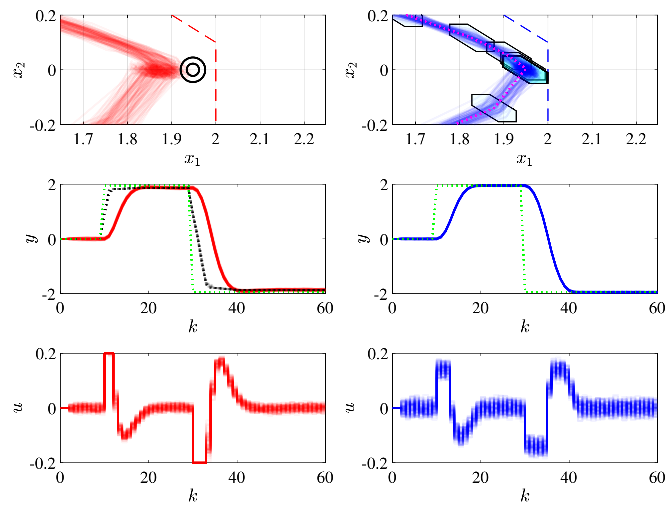

Fig. 4. Comparison of 100 realisations of RG (left) and tube MPC (right) for double integrator example. Top: State trajectory around $\boldsymbol{x}_\mathrm{s} = [1.948, 0]^T$. Middle: Output tracking trajectory. Bottom: Control signals.

reach targets close to the constraint boundary. However, the RG scheme simply *spans* its feasible region due to the d.o.f. provided by the extra state $\boldsymbol{v}$. In the context of predictive control, it is desired to modify the terminal control law (Eqs. (5), (9)) so that the associated terminal regions can be enlarged without increasing N.

VI. NUMERICAL EXAMPLES

A. Double Integrator

Consider a double integrator example

$$\boldsymbol{x}_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \boldsymbol{u}_k + \boldsymbol{w}_k,$$
$$\boldsymbol{y}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{v}_k$$
$$\boldsymbol{z}_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}_k + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{u}_k$$

with state constraints $|\boldsymbol{x}_k| \leq 2$, input constraints $|\boldsymbol{u}_k| \leq 0.2$, state disturbance $|\boldsymbol{w}_k| \leq 0.02$, and output disturbance $\boldsymbol{v}_k = 0$. State feedback controller is employed with $Q = I$ and $R = 10$. For the local tube controller, $Q_\tau = 10I$ and $R = 0.02$ is used and the prediction horizon for the tube MPC is 6. An approximate mRPI set was computed using the algorithm in [23] with $\epsilon = 5 \times 10^{-5}$. We set $B_w = I$ and $D_w = 0$. Set calculations are performed using MPT3 toolbox [10].

B. Results and discussion

The performance of the RG and tube-based MPC for the double integrator system are compared in Fig. 4 for 100 random disturbance samples tracking target at $(\boldsymbol{r}, \boldsymbol{v}) \in \begin{bmatrix} -1.948 & 1.948 \end{bmatrix}$. The simulations show the capability of RG scheme to satisfy constraints by simply moving the target during target changes. The actual state trajectory for the tube MPC has less spread as more aggressive control is employed to contain the effects of random disturbances. However, this comes at the cost of larger spread in the control inputs, due to tighter tube control as implied by Eq. (18b), compared to the RG scheme. It can also be seen that while RG maintains the state trajectory further away from the constraint boundary compared to tube MPC, it is able to operate its input trajectory at the input constraint boundary during target changes. For this particular scenario, the tracking cost for the RG clearly outperforms tube MPC, even though the tracking is suboptimal (i.e. $\|\boldsymbol{r} - \boldsymbol{v}\| \neq 0$, see Table I) and tube MPC is able to reach the true (nominal) steady-state target. Moreover, RG scheme is able to handle large output disturbances effectively by manipulating $\boldsymbol{v}$ without losing feasibility.

Table I compares the sizes of the feasible regions, tracking cost, and the difference between the original and modified reference for varying R/Q. The key observation here is that under certain conditions, the RG-based scheme can outperform tube MPC in terms of the tracking cost. However, increasing R further degrades the overall performance where the modified target deviates further away from the original target. The tube MPC however provides a more consistent performance regardless of the choice of the tracking controller. Moreover, the admissible steady-state target is influenced only by the size of the mRPI set, which depends only on the tube controller.

For tube MPC, constraint tightening explicitly depends on the size of the mRPI set, which depends on the choice of the tube controller gain K_τ. Choosing higher penalty of the states implies smaller tubes; the spread of the state trajectory is smaller and hence the original constraint set is tightened by the size of the mPRI set (see Eq. (18)). The RG scheme has relatively conservative constraint handling due to Eq. (21b), especially if relatively high input penalty is chosen, in contrast to the feasible sets of tube MPC. This is implied by the increase in $\|\boldsymbol{r} - \boldsymbol{v}\|$ as R/Q increases. This is expected as for the RG scheme, the effects of random disturbance

TABLE I

COMPARISON OF SIZES OF $\mathbb{X}_N$, $\mathbb{O}_\infty$, AVERAGE TRACKING COST FOR RG AND TUBE MPC

R/Q	Volume		Cost		$\|\boldsymbol{r}-\boldsymbol{v}\|$
	$\mathbb{O}_\infty$	$\mathbb{X}_N$	RG	TMPC	
0.01	3.3256	5.5281	1.1339	0.8962	0.0000
0.1	3.3912	5.5281	1.0734	0.8887	0.0010
1	3.7546	5.5281	0.8322	0.8465	0.0186
10	5.0858	5.5281	0.7276	0.8620	0.0843
100	5.8189	5.5281	0.9065	1.0299	0.2740

depends on the closed-loop dynamics only. It is possible to alleviate this issue by employing a tube controller within the RG scheme.

VII. CONCLUSION

In this paper, the comparative performance of a tube MPC and RG scheme are presented for the scenario of output disturbances. Tube MPC is a popular robust predictive control technique to ensure recursive feasibility with disturbance uncertainty. The setup relies on the use of RPI sets which, in order to guarantee recursive feasibility, lead to tightened state and input constraints. These restricted constraints are known to cause relatively conservative performance and also, reduce the operating range as one can no longer get as near to the actual constraints. It is perhaps less well investigated and publicised that implicity therefore, tube MPC could lose feasibility more easily during large target changes. This disadvantage is on top of the other challenge, which is that the computation of the tubes themselves is far from straightforward in practice, with the exception of low dimension and simple examples.

An alternative approach for handling infeasibility during target changes is a RG scheme. These are known to be very simple in principle to understand and code. This paper has brought attention to the fact that a simple RG approach can also ensure recursive feasibility in the presence of output disturbances and thus is a potentially a much simpler approach for this scenario. Moreover, it does so without restricting the potential targets that can be tracked and thus may also outperform a tube approach.

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