A Sum-Difference Expansion Scheme for Sparse Array Construction Based on the Fourth-Order Difference Co-Array

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***Abstract*—A generalized sum-difference expansion scheme is proposed to construct sparse arrays based on the fourth-order difference co-array with increased degrees of freedom (DOFs). Different from existing structures, both the second-order sum and difference co-arrays are exploited in array construction under this scheme, leading to a large consecutive fourth-order difference co-array with its number of uniform DOFs (uDOFs) derived. To optimize the provided uDOFs, required design properties of the initial prototype arrays are discussed. Three examples are then provided to demonstrate its superior performance over existing structures in both resolution capacity and estimation accuracy.**

***Index Terms*—Sparse array, fourth-order, difference co-array, sum co-array, DOA estimation.**

I. Introduction

Sparse array is one typical solution to the underdetermined direction finding problem [1]–[5], and has been widely studied in recent years. By calculating the fourth-order cumulants of the received data from an N-sensor array, the generated virtual array (fourth-order difference co-array) can resolve up to $\mathcal{O}(N^4)$ uncorrelated non-Gaussian sources [6]–[8], while only $\mathcal{O}(N^2)$ sources can be resolved via exploiting the second-order statistics [1], [9].

Achieving remarkable resolution capability requires specifically designed sparse structures. The length of the longest uniform linear array (ULA) segment contained in a virtual array is referred to as number of uniform degrees of freedom (uDOFs) [10], [11], which is directly related to the estimation performance and primarily considered as a criterion in sparse array design [6], [10]. A series of nested-array-like (NA-like) structures exploiting the fourth-order difference co-array have been proposed, including four-level NA (FL-NA) [6], NA under expanding and shift scheme (EAS-NA-NA) [12], enhanced FL-NA (E-FL-NA) [7], and improved EAS-NA-NA with larger spacing (EAS-NA-NA$_{\text{LS}}$) [13], etc. These structures consist of four sub-arrays, and $\mathcal{O}(N^4)$ uDOFs can be provided by an N-sensor array.

On the other hand, some array configurations focus on the property of hole-free fourth-order difference co-array, and notable structures include two-level nested sparse array (2L-FO-NA) [14], compressed NA (CNA) [15], extended Cantor array (E-FO-Cantor) [16], and half inverted NA (HINA) [17]. Although the hole-free property guarantees the entire co-array information can be exploited by easy-to-implement subspace-based algorithms [18], [19], the number of uDOFs provided is limited especially for a large sensor number. Among these structures, only the HINA [17] is capable of offering $\mathcal{O}(N^4)$ uDOFs. However, HINA offers less uDOFs than EAS-NA-NA$_{\text{LS}}$ for $N > 8$.

Most existing structures only consider the relationship between the fourth-order and the second-order difference co-arrays in construction, while others focus on a specific prototype array (such as NA) and then design its extension based on the fourth-order difference co-array. In this paper, we propose a generalized sum-difference expansion (SDE) scheme, where both the second-order sum and difference co-arrays are exploited in array construction. For sparse arrays constructed under this scheme, the inter-element spacings of the high-level sub-array can be increased by exploiting the union of the sum and difference co-arrays of the low-level sub-array, leading to a large consecutive fourth-order difference co-array. After deriving the number of uDOFs, the required design properties of the low-level and high-level prototype arrays are introduced to optimize the uDOFs, and three examples are presented to demonstrate the superior performance of the proposed SDE scheme.

II. Signal Model

Consider I far-field narrowband uncorrelated sources impinging on a linear array located at $\mathbb{A} = \{p_0, \cdots, p_{N-1}\}d_u$ with d_u being the unit spacing. Without loss of generality, the leftmost sensor is marked as the zeroth sensor and $\{p_n\}_{n=0}^{N-1} \geq 0$. The signal received by this array is

$$\mathbf{y} = \mathbf{A}(\Theta)\mathbf{s} + \mathbf{n}, \tag{1}$$

where $\mathbf{A}(\Theta) = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_I)]$ is the $N \times I$ steering matrix, and its i-th column vector $\mathbf{a}(\theta_i) = [e^{-j2\pi p_0 d_u \sin\theta_i/\lambda}, \cdots, e^{-j2\pi p_{N-1} d_u \sin\theta_i/\lambda}]^T$ is the steering vector corresponding to the i-th source direction. $\mathbf{s} = [s_1, \cdots, s_I]^T$ is the source signal vector, while $\mathbf{n}$ is the Gaussian white noise vector.

According to [6], [7], [16], the fourth-order circular cumulant matrix of the received signal vector $\mathbf{y}$ is calculated by

$$\mathbf{F_y} = \sum\nolimits_{i=1}^{I} c_{s_i}\big[\mathbf{a}(\theta_i) \otimes \mathbf{a}(\theta_i)^*\big] \times \big[\mathbf{a}(\theta_i) \otimes \mathbf{a}(\theta_i)^*\big]^H. \tag{2}$$

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Here $\{\cdot\}^*$ and $\{\cdot\}^H$ return the conjugate and conjugate transpose of the input, respectively, and $\otimes$ represents Kronecker product. $c_{s_i} = \mathrm{cum}\{s_i, s_i, s_i^*, s_i^*\}$ is the fourth-order cumulant [7], [16] of the i-th source signal s_i $(1 \le i \le I)$.

To construct an equivalent virtual array based on the fourth-order difference co-array, $\mathbf{F_y}$ is vectorized as [6]

$$\mathbf{y}_v = \mathrm{vec}\{\mathbf{F_y}\} = \mathbf{A}_v(\Theta)\mathbf{s}_v, \tag{3}$$
$$\mathbf{A}_v(\Theta) = [\mathbf{a}_v(\theta_1), \cdots, \mathbf{a}_v(\theta_I)], \quad \mathbf{s}_v = [c_{s_1}, \cdots, c_{s_I}]^T.$$

In this model, $\mathbf{A}_v(\Theta)$ is the virtual steering matrix consisting of sensors in the fourth-order difference co-array (see *Definition 1*), while the steering vector $\mathbf{a}_v(\theta_i) = [\mathbf{a}(\theta_i) \otimes \mathbf{a}(\theta_i)^*]^* \otimes [\mathbf{a}(\theta_i) \otimes \mathbf{a}(\theta_i)^*]$ [7], [16], [20].

Definition 1: For an arbitrary linear array $\mathbb{A}$, its fourth-order difference co-array is $\mathbb{D}_{4(\mathbb{A})} = \{\vartheta_1 - \vartheta_2 + \vartheta_3 - \vartheta_4 \mid \vartheta_k \in \mathbb{A}\}$ [16], [21].

$\mathbb{D}_{4(\mathbb{A})}$ is closely related to second-order difference and sum co-arrays ($\mathbb{D}_{2(\mathbb{A})}$ and $\mathbb{S}_{2(\mathbb{A})}$) [7], [22]. We have

$$\mathbb{D}_{4(\mathbb{A})} = \{\vartheta_5 + \vartheta_6 \mid \vartheta_5, \vartheta_6 \in \mathbb{D}_{2(\mathbb{A})}\}, \tag{4}$$
$$= \{\vartheta_7 - \vartheta_8 \mid \vartheta_7, \vartheta_8 \in \mathbb{S}_{2(\mathbb{A})}\}, \tag{5}$$

where $\mathbb{D}_{2(\mathbb{A})} = \{n_1 - n_2 \mid n_1, n_2 \in \mathbb{A}\}$ and $\mathbb{S}_{2(\mathbb{A})} = \{n_3 + n_4 \mid n_3, n_4 \in \mathbb{A}\}$.

III. Sum-Difference Expansion Scheme Based on the Fourth-Order Difference Co-Array

Most representative structures based on the fourth-order difference co-array have been designed by exploiting the relationship between $\mathbb{D}_{2(\mathbb{A})}$ and $\mathbb{D}_{4(\mathbb{A})}$ given in (4), such as FL-NA [6], EAS scheme [12], and E-FL-NA [7], etc. In this section, we consider both (4) and (5), and a generalized sum-difference expansion scheme is proposed.

A. Sum-Difference Expansion Scheme

Definition 2: Consider two prototype arrays $\mathbb{A}_1$ and $\mathbb{A}_2$, and the set of sensor positions $\mathbb{A}_{\text{SDE}}$ for a structure (based on the fourth-order difference co-array) constructed by the SDE scheme is

$$\mathbb{A}_{\text{SDE}} = \{a_1 \mid a_1 \in \mathbb{A}_1\} \cup \{a_2 \cdot D_E \mid a_2 \in \mathbb{A}_2\}. \tag{6}$$

Here D_E is the cardinality of the longest consecutive segment ($\mathbb{U}_E$) in the union of the second-order difference co-array ($\mathbb{D}_{2(A1)}$) and the sum co-array ($\mathbb{S}_{2(A1)}$) of $\mathbb{A}_1$, that is,

$$D_E \triangleq |\mathbb{U}_E|, \ \mathbb{U}_E = \arg \max_{\mathbb{U}_{lr} \subseteq \mathbb{D}_{2(A1)} \cup \mathbb{S}_{2(A1)}} |\mathbb{U}_{lr}|, \tag{7}$$
$$\mathbb{U}_{lr} = \{-l, -l+1, \cdots, r-1, r\}, \ l, r \in \mathbb{N}^+, \tag{8}$$

where $|\cdot|$ returns the cardinality of the input set and $\mathbb{N}^+$ is the set of positive integers.

The consecutive segment in the fourth-order difference co-array of the proposed SDE structure is derived in the following proposition.

Proposition 1: The fourth-order difference co-array (denoted by $\mathbb{D}_{4(\text{SDE})}$) of $\mathbb{A}_{\text{SDE}}$ satisfies

$$\mathbb{D}_{4(\text{SDE})} \supseteq [-\alpha, \alpha], \tag{9}$$
$$\alpha = \begin{cases} (\beta+1)D_E - \min(\mathbb{S}_{c(A1)}), & \text{if } \textit{Assumption 1} \text{ holds}, \\ \beta D_E + \max(\mathbb{D}_{c(A1)}), & \text{otherwise}, \end{cases}$$

where $\min(\cdot)$ and $\max(\cdot)$ return the minimum and maximum values of the input set, respectively. $\beta = \max(\mathbb{U}_{c(A2)})$ with $\mathbb{U}_{c(A2)}$ defined as

$$\mathbb{U}_{c(A2)} = \arg \max_{\mathbb{U}_m \subseteq \mathbb{D}_{2(A2)} \cap \mathbb{S}_{2(A2)}} |\mathbb{U}_m|, \tag{10}$$
$$\mathbb{U}_m = \{0, 1, \cdots, m\}, \ m \in \mathbb{N}^+. \tag{11}$$

$\mathbb{U}_{c(A2)}$ is the longest consecutive segment (starting from 0) in the intersection of the second-order difference and sum co-arrays ($\mathbb{D}_{2(A2)} \cap \mathbb{S}_{2(A2)}$) of $\mathbb{A}_2$. $\mathbb{S}_{c(A1)}$ and $\mathbb{D}_{c(A1)}$ denote the longest consecutive segment in $\mathbb{S}_{2(A1)}$ and $\mathbb{D}_{2(A1)}$, respectively, with

$$\mathbb{S}_{c(A1)} = \arg \max_{\mathbb{U}_{pq} \subseteq \mathbb{S}_{2(A1)}} |\mathbb{U}_{pq}|, \ \mathbb{D}_{c(A1)} = \arg \max_{\mathbb{U}_n \subseteq \mathbb{D}_{2(A1)}} |\mathbb{U}_n|,$$
$$\mathbb{U}_{pq} = \{p, p+1, \cdots, q-1, q\}, \ p, q \in \mathbb{N}, \tag{12}$$
$$\mathbb{U}_n = \{-n, -n+1, \cdots, n-1, n\}, \ n \in \mathbb{N}^+. \tag{13}$$

Assumption 1: $D_E \le \max(\mathbb{S}_{c(A1)}) + \max(\mathbb{D}_{c(A1)}) + 1$ and $\beta + 1 \in \mathbb{S}_{2(A2)}$.

Proof: We focus on the following three sets of cross-subarray co-arrays in $\mathbb{D}_{4(\text{SDE})}$:

$$\phi_1 = \{(\nu_1 + \nu_2)D_E - (\xi_1 + \xi_2) \mid \nu_m \in \mathbb{A}_2, \ \xi_n \in \mathbb{A}_1\},$$
$$\phi_2 = \{(\xi_1 + \xi_2) - (\nu_1 + \nu_2)D_E \mid \xi_n \in \mathbb{A}_1, \ \nu_m \in \mathbb{A}_2\},$$
$$\phi_3 = \{(\nu_1 - \nu_2)D_E + (\xi_1 - \xi_2) \mid \nu_m \in \mathbb{A}_2, \ \xi_n \in \mathbb{A}_1\}.$$

Denote $\mathbb{D}_{2(A1)}$ ($\mathbb{D}_{2(A2)}$) and $\mathbb{S}_{2(A1)}$ ($\mathbb{S}_{2(A2)}$) as the second-order difference and sum co-arrays of $\mathbb{A}_1$ ($\mathbb{A}_2$). According to the definitions of $\mathbb{D}_{2(A1)}$, $\mathbb{S}_{2(A1)}$, $\mathbb{D}_{2(A2)}$, and $\mathbb{S}_{2(A2)}$, the sets ϕ_1, ϕ_2, and ϕ_3 can be rewritten as

$$\phi_1 = \{s_o D_E - s \mid s_o \in \mathbb{S}_{2(A2)}, s \in \mathbb{S}_{2(A1)}\} = -\phi_2, \tag{14}$$
$$\phi_3 = \{d_o D_E + d \mid d_o \in \mathbb{D}_{2(A2)}, d \in \mathbb{D}_{2(A1)}\}. \tag{15}$$

By extracting the common consecutive parts of $\mathbb{S}_{2(A2)}$ and $\mathbb{D}_{2(A2)}$, subsets ϕ_4, ϕ_5, ϕ_6 can be constructed, expressed as

$$\phi_1 \supseteq \phi_4 = \{s_c D_E - s \mid s_c \in [0, \beta], s \in \mathbb{S}_{2(A1)}\}, \tag{16}$$
$$\phi_2 \supseteq \phi_5 = \{s - s_c D_E \mid s \in \mathbb{S}_{2(A1)}, -s_c \in [-\beta, 0]\}, \tag{17}$$
$$\phi_3 \supseteq \phi_6 = \{d_c D_E + d \mid d_c \in [-\beta, \beta], d \in \mathbb{D}_{2(A1)}\}, \tag{18}$$

Note that the integer range $[0, \beta]$ is shared by s_c in ϕ_4 and d_c in ϕ_6. Similarly, variables d_c in ϕ_6 and $-s_c$ in ϕ_5 can both take values in $[-\beta, 0]$. Since $\mathbb{D}_{4(\text{SDE})}$ is symmetric about 0, we only analyze the non-negative part for simplification.

For every $\kappa \in [0, \beta]$, the set of associated co-array lags in ϕ_4 is $\eta(\kappa) = \{\kappa D_E - s \mid s \in \mathbb{S}_{2(A1)}\}$, while in ϕ_6, it becomes $\zeta(\kappa) = \{\kappa D_E + d \mid d \in \mathbb{D}_{2(A1)}\}$. Since $\mathbb{D}_{2(A1)}$ is symmetric about 0, $\eta(\kappa) \cup \zeta(\kappa) = \{\kappa D_E - s' \mid s' \in \mathbb{S}_{2(A1)} \cup \mathbb{D}_{2(A1)}\}$. As a result, $\eta(\kappa) \cup \zeta(\kappa)$ provides a consecutive segment with the same length as that of $\mathbb{U}_E$ (the longest consecutive part of $\mathbb{S}_{2(A1)} \cup \mathbb{D}_{2(A1)}$ defined in (7)). In order to concatenate the elements in $\eta(\kappa) \cup \zeta(\kappa)$ and $\eta(\kappa+1) \cup \zeta(\kappa+1)$ to form a lager consecutive segment, one yields $D_E \le |\mathbb{U}_E|$, and we set $D_E = |\mathbb{U}_E|$ to achieve the largest consecutive segment. Considering all $\kappa \in [0, \beta]$, we derive

$$\bigcup_{\kappa \in [0, \beta]} \big(\eta(\kappa) \cup \zeta(\kappa)\big) \supseteq [0, \beta \cdot D_E + \max(\mathbb{D}_{c(A1)})]. \tag{19}$$

Furthermore, if *Assumption 1* is satisfied by selecting specifically designed prototype arrays $\mathbb{A}_1$ and $\mathbb{A}_2$, additional consecutive co-arrays can be provided. By setting $s_c = \beta + 1$ in (16), a consecutive co-array set is obtained, given as

$$[(\beta+1)\cdot D_E - \max(\mathbb{S}_{c(A1)}), (\beta+1)\cdot D_E - \min(\mathbb{S}_{c(A1)})].$$

Due to the assumption $D_E \leq \max(\mathbb{S}_{c(A1)}) + \max(\mathbb{D}_{c(A1)}) + 1$, we have

$$[0, (\beta+1)\cdot D_E - \min(\mathbb{S}_{c(A1)})] \subseteq \mathbb{D}_{4(\text{SDE})}. \quad (20)$$

According to the symmetrical property of the fourth-order difference co-array, one concludes that $\mathbb{D}_{4(\text{SDE})} \supseteq [-\alpha, \alpha]$, where $\alpha = (\beta+1)\cdot D_E - \min(\mathbb{S}_{c(A1)})$ if *Assumption 1* holds, while $\alpha = \beta \cdot D_E + \max(\mathbb{D}_{c(A1)})$ for other cases. ∎

From *Definition 2* and *Proposition 1*, the expansion factor D_E equals the length of the longest continuous segment in $\mathbb{D}_{2(A1)} \cup \mathbb{S}_{2(A1)}$, and D_E is linearly related to the number of uDOFs guaranteed by $\mathbb{A}_{\text{SDE}}$. For the EAS scheme [12], the expansion factor D_{EAS} is chosen as the length of the longest continuous segment in $\mathbb{D}_{2(A1)}$ with only the difference co-array considered. For the same prototype arrays $\mathbb{A}_1$ and $\mathbb{A}_2$ (where $\mathbb{A}_2$ is a nested array to ensure high DOFs is provided by the EAS scheme), D_E in SDE scheme is always larger than or equal to D_{EAS} in the EAS scheme due to the contribution of $\mathbb{S}_{2(A1)}$, leading to increased uDOFs as derived in (9).

As shown in *Proposition 1*, several properties are required to achieve significantly increased uDOFs via the SDE scheme: 1) a large D_E is preferred, and thus the consecutive segment in $\mathbb{D}_{2(A1)} \cup \mathbb{S}_{2(A1)}$ is expected to be as long as possible; 2) a large number of consecutive elements in $\mathbb{D}_{2(A2)} \cap \mathbb{S}_{2(A2)}$ leads to a large β, which is finally translated to a large number of uDOFs; 3) additional uDOFs can be provided if *Assumption 1* holds.

B. *Examples of Specifically Designed Structures*

Based on the above analysis, we then focus on how to design the prototype arrays, and the following three structures under the proposed SDE scheme are given as examples.

Structure 1: Transformed nested array (TNA) under the sum-difference expansion scheme (TNA-SDE). In TNA-SDE with its configuration given in (6), the prototype array $\mathbb{A}_1$ is chosen as an (N_1+N_2)-sensor TNA [23], defined as

$$\begin{aligned}\mathbb{A}_1 =& \{t_1(N_2+1) \mid t_1 \in [0, N_1-1]\} \\ &\cup \{t_2 + (N_1-1)(N_2+1) \mid t_2 \in [0, N_2]\}, \end{aligned} \quad (21)$$

while another prototype array $\mathbb{A}_2$ is configured as a classic nested array [1] with (N_3+N_4) sensors, i.e.,

$$\begin{aligned}\mathbb{A}_2 =& \{t_3 \mid t_3 \in [0, N_3]\} \\ &\cup \{t_4(N_3+1) - 1 \mid t_4 \in [2, N_4]\}. \end{aligned} \quad (22)$$

The second-order difference co-array of $\mathbb{A}_1$ (TNA) is

$$\mathbb{D}_{2(A1)} = [-(N_2+1)(N_1-1) - N_2, (N_2+1)(N_1-1) + N_2],$$

which is hole-free so that the longest central ULA segment $\mathbb{D}_{c(A1)} = \mathbb{D}_{2(A1)}$. The second-order sum co-array of TNA is

$$\mathbb{S}_{2(A1)} = \{s(N_2+1) \mid s \in [0, 2(N_1-1)]\} \cup \mathbb{S}_{c(A1)},$$

with $\mathbb{S}_{c(A1)} = [(N_1-1)(N_2+1), 2(N_1-1)(N_2+1) + 2N_2]$ [23], [24]. According to the definition of D_E in (7),

$$\begin{aligned} D_E &= 3N_1(N_2+1) - 2 && (23)\\ &\leq \max(\mathbb{S}_{c(A1)}) + \max(\mathbb{D}_{c(A1)}) + 1. && (24)\end{aligned}$$

For the nested array $\mathbb{A}_2$, the intersection of its second-order difference and sum co-arrays is [22]

$$\mathbb{D}_{2(A2)} \cap \mathbb{S}_{2(A2)} = [0, N_4(N_3+1) - 1]. \quad (25)$$

Therefore, $\beta = N_4(N_3+1) - 1$. The *Assumption 1* holds due to $N_4(N_3+1) \in \mathbb{S}_{2(A2)}$ and (24). According to *Proposition 1*, the number of uDOFs offered by TNA-SDE is

$$\begin{aligned} M_{\text{TNA}} = 2\alpha + 1 =& 2N_4(N_3+1)[3N_1(N_2+1) - 2] \\ &- 2(N_1-1)(N_2+1) + 1. \end{aligned} \quad (26)$$

Structure 2: Improved transformed nested array under the sum-difference expansion scheme (ITNA-SDE). For ITNA-SDE, TNA-II (with N_1+N_2 physical sensors) [23] is utilized as $\mathbb{A}_1$, and $\mathbb{A}_2$ is a nested array shown in (22). Similarly, relevant parameters in the configuration are shown as follows:

$$D_E = l_1 + l_3 + 1,\ \beta = N_4(N_3+1) - 1, \quad (27)$$

$$\mathbb{D}_{c(A1)} = [-l_1, l_1],\ \mathbb{S}_{c(A1)} = [l_2, l_3], \quad (28)$$

$$M_{\text{ITN}} = 2\alpha + 1 = 2(\beta+1)D_E - 2l_2 + 1, \quad (29)$$

$$l_1 = N_1N_2 + N_1 - 1,\ l_2 = N_1N_2 + N_1 - N_2 + \lceil N_2/2 \rceil - 1,$$

$$l_3 = \begin{cases} 2N_1N_2 + 2N_1, & \lceil N_2/2 \rceil = 2, \\ 2N_1N_2 + 2N_1 + 2\lceil N_2/2 \rceil - 2, & \lceil N_2/2 \rceil \geq 3. \end{cases}$$

Note that TNA-II exists only when N_2 is greater than 2 [23].

Structure 3: Two-level extended transformed nested array (TwETNA) under the sum-difference expansion scheme (TETNA-SDE). In TETNA-SDE, the prototype arrays $\mathbb{A}_1$ and $\mathbb{A}_2$ are TwETNA and nested array, respectively. The definition and properties of an (N_1+N_2)-sensor TwETNA can be found in [24], while $\mathbb{A}_2$ is shown in (22). The associated parameters of TETNA-SDE can be derived, given by

$$D_E = l_4 + l_6 + 1,\ \beta = N_4(N_3+1) - 1, \quad (30)$$

$$\mathbb{D}_{c(A1)} = [-l_4, l_4],\ \mathbb{S}_{c(A1)} = [l_5, l_6], \quad (31)$$

$$M_{\text{TET}} = 2\alpha + 1 = 2(\beta+1)D_E - 2l_5 + 1, \quad (32)$$

$$l_4 = N_1N_2 + N_1 - 1,\ l_5 = N_1N_2 + N_1,$$

$$l_6 = \begin{cases} 2N_1N_2 + 2N_1 + N_2 + \lceil N_2/2 \rceil - 2, & 7 \leq N_2 \leq 9, \\ 2N_1N_2 + 2N_1 + 2N_2 - 2, & N_2 \geq 10. \end{cases}$$

Note that TwETNA exists only when N_2 is greater than 6 [24].

IV. Comparisons and Simulation Results

A. *Comparisons of Different Array Structures*

The number of physical sensors and uDOFs (related to the consecutive co-arrays) of different array structures based on the fourth-order difference co-array are listed in Table I, where the optimal sensor allocation strategy is adopted to achieve the maximum number of uDOFs for each structure. Obviously, for the same number of physical sensors, the proposed structures, including TNA-SDE, ITNA-SDE and TETNA-SDE, offer

TABLE I
COMPARISONS OF SPARSE ARRAYS BASED ON THE FOURTH-ORDER DIFFERENCE CO-ARRAY

Arrays ($N = 11$)	Sensor Allocation Strategy	Number of uDOFs	Arrays ($N = 30$)	Sensor Allocation Strategy	Number of uDOFs
2L-FO-NA	$(5, 6)$	441	2L-FO-NA	$(15, 15)$	3481
FL-NA	$(4, 4, 3, 3)$	383	FL-NA	$(9, 8, 8, 8)$	10367
CNA	$(5, 6)$	521	CNA	$(15, 15)$	5937
EAS-NA-NA	$(4, 4, 3, 3)$	557	EAS-NA-NA	$(9, 8, 8, 8)$	18303
E-FL-NA	$(4, 4, 3, 3)$	617	E-FL-NA	$(9, 8, 8, 8)$	19297
HINA	$(3, 3, 3, 2)$	721	HINA	$(8, 8, 7, 7)$	20481
EAS-NA-NA$_{\text{LS}}$	$(3, 4, 2, 3)$	797	EAS-NA-NA$_{\text{LS}}$	$(8, 8, 7, 8)$	27249
TNA-SDE	$(4, 3, 2, 3)$	805	TNA-SDE	$(8, 8, 7, 8)$	27267
ITNA-SDE	$(4, 3, 2, 3)$	837	ITNA-SDE	$(8, 8, 7, 8)$	28283
–	–	–	TETNA-SDE	$(8, 8, 7, 8)$	28785

more uDOFs than existing ones, with TETNA-SDE being the best.

In the following *Corollary 1*, we theoretically prove that under a similar sensor allocation strategy, more uDOFs can be achieved by the proposed TNA-SDE, ITNA-SDE, and TETNA-SDE compared with EAS-NA-NA$_{\text{LS}}$ [13], which was shown to be superior to other existing structures.

Corollary 1: Given the same number of physical sensors N_t and a similar sensor allocation strategy, TNA-SDE, ITNA-SDE, and TETNA-SDE always provide more uDOFs than EAS-NA-NA$_{\text{LS}}$.

Proof: Consider $N_t = \sum_{i=1}^{4} N_i - 1$ physical sensors. The sensor allocation strategy is (N_2, N_1, N_3, N_4) for EAS-NA-NA$_{\text{LS}}$, while (N_1, N_2, N_3, N_4) for others. The number of uDOFs provided by EAS-NA-NA$_{\text{LS}}$ [13] is $M_{\text{LS}} = (2N_4N_3 + 2N_4 - 2)(3N_1N_2 + 3N_1 - 2) + 4(N_1N_2 + N_1) - 3$, and those offered by TNA-SDE, ITNA-SDE, and TETNA-SDE are listed in (26), (29), and (32), respectively.

We simply derive that $M_{\text{TNA}} - M_{\text{LS}} = 2(N_2 + 1)$, and

$$M_{\text{ITN}} - M_{\text{LS}} = \begin{cases} 2(2N_4N_3 + 2N_4 + N_2 - \lceil N_2/2 \rceil + 1), & \lceil N_2/2 \rceil = 2, \\ 2[(2N_4(N_3+1) - 1)\lceil N_2/2 \rceil + N_2 + 1], & \lceil N_2/2 \rceil \geq 3, \end{cases}$$

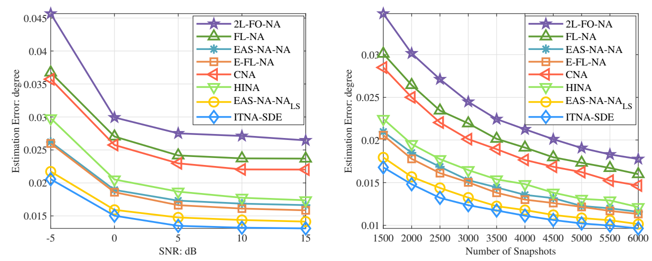
$$M_{\text{TET}} - M_{\text{LS}} = \begin{cases} 2N_4(N_3+1)(N_2 + \lceil N_2/2 \rceil), & 7 \leq N_2 \leq 9, \\ 4N_4(N_3+1)N_2, & N_2 \geq 10. \end{cases}$$

Clearly, $M_{\text{TNA}} - M_{\text{LS}} > 0$, $M_{\text{ITN}} - M_{\text{LS}} > 0$, and $M_{\text{TET}} - M_{\text{LS}} > 0$ for $N_2 \geq 7$. ■

B. DOA Estimation Results

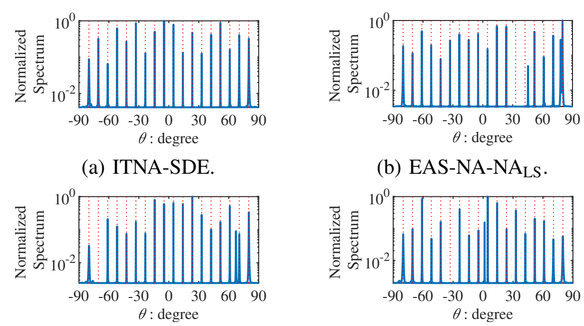
The ITNA-SDE structure with $N = 11$ physical sensors, i.e., $\mathbb{A}_{\text{SDE}} = \{0, 4, 8, 12, 14, 15, 17, 48, 96, 240, 384\}$, is taken as an example for performance comparison with other existing structures shown in Table I, and the SS-MUSIC [6] method is employed to evaluate the root mean square errors (RMSE) via Monte Carlo simulations of 500 independent trials.

First, consider $I = 12$ sources uniformly spaced from $-60°$ to $60°$. The RMSE results versus input SNR for different sparse structures are shown in Fig. 1(a). It is clear that our proposed structure outperforms other structures consistently for a wide range of input SNR with 2500 snapshots. As also shown in Fig. 1(b) with a fixed SNR of 10 dB, the performance of the proposed ITNA-SDE is always the best for different number of snapshots, indicating the superiority of the proposed structure.



(a) RMSE versus the input SNR. (b) RMSE versus the number of snapshots.

Fig. 1. RMSE results of different sparse structures.



(a) ITNA-SDE. (b) EAS-NA-NA$_{\text{LS}}$. (c) HINA. (d) E-FL-NA.

Fig. 2. DOA estimation results of ITNA-SDE, EAS-NA-NA$_{\text{LS}}$, HINA, and E-FL-NA.

Then, we focus on the comparison of resolution capacity with $I = 18$ uncorrelated sources (uniformly distributed between $-80°$ and $80°$). The DOA estimation results under 10 dB SNR and 2000 snapshots are given in Fig. 2, where it can be seen clearly that only the proposed ITNA-SDE structure is capable of resolving all 18 sources.

V. CONCLUSION

A generalized sum-difference expansion scheme was proposed to construct sparse arrays based on the fourth-order difference co-array. Under this scheme, the length of consecutive segment in the fourth-order difference co-array can be expanded by simultaneously exploiting both the sum and difference co-arrays of the prototype arrays. The number of uDOFs achieved by this scheme was derived, and optimal design criteria were given. Three structures under the SDE scheme were presented as examples, all offering more uDOFs than existing ones. Simulation results have shown that superior performance in terms of resolution capacity and estimation accuracy can be achieved through the proposed SDE scheme.

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