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Lagrangian 3-form structure for the Darboux system and the KP hierarchy

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Abstract

A Lagrangian multiform structure is established for a generalisation of the Darboux system describing orthogonal curvilinear coordinate systems. It has been shown in the past that this system of coupled PDEs is in fact an encoding of the entire Kadomtsev–Petviashvili (KP) hierarchy in terms so-called Miwa variables. Thus, in providing a Lagrangian description of this multidimensionally consistent system amounts to a new Lagrangian 3-form structure for the continuous KP system. A generalisation to the matrix (also known as non-Abelian) KP system is discussed.

Keywords Integrable system · Multi-dimensional consistency · Lagrangian multiforms · KP hierarchy · Darboux systems

Mathematics Subject Classification 35A15 · 35Q58 · 37K05 · 37K15 · 39A99 · 49S05 · 53A40 · 70H30 · 81Q99

1 Introduction

The notion of Lagrangian multiforms was introduced in [23] to provide a variational formalism for systems integrable in the sense of multidimensional consistency (MDC). This novel variational approach to integrable systems allows for the derivation of an entire system (called a *hierarchy*) of simultaneous compatible equations from a single variational framework, in which the conventional Lagrange function is replaced by a Lagrangian d-form integrated over arbitrary hypersurfaces in a space of independent variables of arbitrary dimension. Lagrangian multiform theory has undergone a significant development in the last decade (cf. e.g. [42, 46], or [19] and references therein). It has become evident that Lagrangian multiform (or, in its variant formula-

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tion, *pluri-Lagrangian systems*, cf. e.g. [1, 5, 6, 43–45]) forms a universal variational aspect of integrability. It distinguished itself from the conventional least-action principle in that, where the latter produces through the standard Euler–Lagrange (EL) equations only one equation per component of the field variable, the multiform EL equations comprise a multitude of compatible equations for every component of the fields. Furthermore, the Lagrangian components themselves have to be very special (they have to be 'admissible', which implies 'integrable'), and in a precise sense the Lagrangians themselves can be considered as solutions of the systems of generalised EL equations.

In this note, I will focus on the Darboux system of equations, [10], which in the original notation of Darboux reads

$$\frac{\partial \beta_{kk'}}{\partial \rho_{k''}} = \beta_{kk''}\beta_{k''k'} \ , \tag{1.1}$$

where the indices k, k', k'' run over a set of integers, and the quantities $\beta_{kk'}$, etc., are functions of a set of coordinates $\rho_1, \cdots, \rho_n$. These equations describe conjugate nets for a system of curvilinear orthogonal coordinates, following on from earlier work by Lamé [22]. It is well known that the set of Eq. (1.1), or generalisations thereof, are closely related to integrable three-dimensional equations, cf. e.g. [13, 15, 50], in particular the N-component wave equation. In fact, in [27] it was shown that they form a realisation of the KP hierarchy in terms of so-called *Miwa variables*, [28], which are variables depending on a continuous parameter associated with an underlying lattice structure. Here, I will show that this set of equations possesses a Lagrangian 3-form structure, in the sense of [24, 39], cf. also [6]. Whereas our previous treatment of the Lagrange multiform structure of the continuous KP hierarchy used a representation in terms of pseudo-differential operators, going back to [11, 12], the multiform structure of the Darboux system is more compact and can be viewed as a generating system for the KP hierarchy, encoding the latter in a more covariant way.

In the next section, I will present this 3-form structure and demonstrate the salient multiform features, while in the ensuing section I will discuss the connection to the KP hierarchy, and further generalisations in the remainder. Some speculative applications are discussed in the Conclusion section.

2 Lagrangian 3-form structure for the generalised Darboux system

The generalised Darboux system reads

$$\frac{\partial B_{qr}}{\partial \xi_p} = B_{qp}B_{pr} \ , \quad \frac{\partial B_{rq}}{\partial \xi_p} = B_{rp}B_{pq} \ , \tag{2.1a}$$

$$\frac{\partial B_{pr}}{\partial \xi_q} = B_{pq}B_{qr} \ , \quad \frac{\partial B_{rp}}{\partial \xi_q} = B_{rq}B_{qp} \ , \tag{2.1b}$$

$$\frac{\partial B_{pq}}{\partial \xi_r} = B_{pr}B_{rq} \ , \quad \frac{\partial B_{qp}}{\partial \xi_r} = B_{qr}B_{rp} \ , \tag{2.1c}$$

where B_{pq}, etc., are scalar functions (but can be readily generalised to matrices) of the independent variables ξ_p, ξ_q and ξ_r, which are continuous variables labelled by parameters p, q and r, respectively, which themselves are in principle continuous variables taking values in a continuous subset of the real or complex numbers (hence, the term 'generalised'). We assume that these parameters are distinct, and we will not consider for now quantities B for which they coincide (quantities of the type B_{pp}). A main property of the system (2.1) is that it can be extended in a consistent way to an arbitrarily large set of copies of these equations in terms of additional variables ξ_s, etc., similarly labelled by values of the parameters. This compatibility is expressed as follows.

Theorem 2.1 *The PDE system (2.1) for the quantities $B_{..}$ is multidimensionally consistent.*

Proof The proof is by direct computation, introducing a fourth variable ξ_s and associated lattice direction with parameter s, such that the system of independent variables is extended to include B_{ps}, B_{qs}, B_{rs} and B_{sp}, B_{sq}, B_{sr} obeying relations of the form

$$\frac{\partial B_{ps}}{\partial \xi_q} = B_{pq} B_{qs} \ ,$$

etc., and where the other variables depend also on ξ_s such that

$$\frac{\partial B_{pq}}{\partial \xi_s} = B_{ps} B_{sq} \ ,$$

etc. We then establish by direct computation from the extended system of equations comprising (2.1) and the PDEs w.r.t. ξ_s, the relation

$$\frac{\partial}{\partial \xi_s}\left(\frac{\partial}{\partial \xi_p} B_{qr}\right) = \frac{\partial}{\partial \xi_p}\left(\frac{\partial}{\partial \xi_s} B_{qr}\right) \ ,$$

by direct computation. Similarly, all relations obtained from cross-differentiation hold by the same token. □

The system (2.1) possesses a *Lax multiplet*, cf. e.g. [13], in the following sense.

Proposition 2.1 *If the system (2.1) is satisfied, each of the following linear overdetermined systems (one system for a parameter-labelled family of functions $\Phi_.$ and another system for the parameter family of functions $\Psi_.$)*

$$\frac{\partial \Phi_q}{\partial \xi_p} = B_{qp} \Phi_p, \quad \text{and} \quad \frac{\partial \Psi_r}{\partial \xi_p} = \Psi_p B_{pr} \ , \tag{2.2}$$

respectively (and similar relations for all variables ξ_q and ξ_r), is consistent in the sense of possessing a common general solution.

Proof Again, this is by direct computation. Cross-differentiation of two copies of the first equation of (2.2) we get the equality

$$\frac{\partial}{\partial\xi_r}\left(\frac{\partial\Phi_q}{\partial\xi_p}\right)=\left(\frac{\partial}{\partial\xi_r}B_{qp}\right)\Phi_p+B_{qp}B_{pr}\Phi_r$$
$$=\frac{\partial}{\partial\xi_p}\left(\frac{\partial\Phi_q}{\partial\xi_r}\right)=\left(\frac{\partial}{\partial\xi_p}B_{qr}\right)\Phi_r+B_{qr}B_{rp}\Phi_p\ ,$$

and hence the equality of the coefficients of Φ_r and Φ_p give us the desired differential equations for B_{qp} and B_{qr}, respectively. The same hold true for the 'adjoint' Lax multiplet in terms of the functions Ψ.. □

We note that the Lax multiplets (2.2) can be obtained from the Darboux system itself, relying on the multidimensional consistency, by identifying the Lax wave functions Φ and Ψ by fixing two, possibly separate, directions in the space of independent variables, ξ_k and ξ_l, say (where k and l play the role of spectral parameters), such that $\Phi_p=B_{pk}$ and $\Psi_p=B_{lp}$. Furthermore, the quantities Φ and Ψ obey a linear homogeneous set of equations of the form

$$\partial_p\partial_q\Phi_r=(\partial_p\ln\Phi_q)\partial_q\Phi_r+(\partial_q\ln\Phi_p)\partial_p\Phi_r\ , \tag{2.3a}$$
$$\partial_p\partial_q\Psi_r=(\partial_p\ln\Psi_q)\partial_q\Psi_r+(\partial_q\ln\Psi_p)\partial_p\Psi_r\ , \tag{2.3b}$$

thus obeying both an identical equation.

We now introduce the Lagrangian structure. Let us consider the following Lagrangian components:

$$\begin{aligned}\mathcal{L}_{pqr}&=\tfrac{1}{2}\left(B_{rq}\partial_{\xi_p}B_{qr}-B_{qr}\partial_{\xi_p}B_{rq}\right)+\tfrac{1}{2}\left(B_{qp}\partial_{\xi_r}B_{pq}-B_{pq}\partial_{\xi_r}B_{qp}\right)\\&\quad+\tfrac{1}{2}\left(B_{pr}\partial_{\xi_q}B_{rp}-B_{rp}\partial_{\xi_q}B_{pr}\right)+B_{rp}B_{pq}B_{qr}-B_{rq}B_{qp}B_{pr}\ .\end{aligned}\tag{2.4}$$

Then, we have the following main statement.

Theorem 2.2 *The differential of the Lagrangian 3-form*

$$\begin{aligned}L:=\mathcal{L}_{pqr}\,\mathrm{d}\xi_p\wedge\mathrm{d}\xi_q\wedge\mathrm{d}\xi_r+\mathcal{L}_{qrs}\,\mathrm{d}\xi_q\wedge\mathrm{d}\xi_r\wedge\mathrm{d}\xi_s+\\+\mathcal{L}_{rsp}\,\mathrm{d}\xi_r\wedge\mathrm{d}\xi_s\wedge\mathrm{d}\xi_p+\mathcal{L}_{spq}\,\mathrm{d}\xi_s\wedge\mathrm{d}\xi_p\wedge\mathrm{d}\xi_q\ ,\end{aligned}\tag{2.5}$$

has a 'double zero' on the solutions of the set of generalised Darboux equations (2.1), i.e. dL *can be written as*

$$\mathrm{d}L=\mathcal{A}_{pqrs}\,\mathrm{d}\xi_p\wedge\mathrm{d}\xi_q\wedge\mathrm{d}\xi_r\wedge\mathrm{d}\xi_s \tag{2.6}$$

with the coefficient $\mathcal{A}_{pqrs}$ being a sum of products of factors which vanish on solutions of the EL equations.

Proof Computing the components of the differential dL, we obtain

$$
\begin{aligned}
&\partial_{\xi_s}\mathcal{L}_{pqr} - \partial_{\xi_p}\mathcal{L}_{qrs} + \partial_{\xi_q}\mathcal{L}_{rsp} - \partial_{\xi_r}\mathcal{L}_{spq} \\
&\quad = \Gamma_{s;rq}\Gamma_{p;qr} - \Gamma_{p;rq}\Gamma_{s;qr} + \Gamma_{s;qp}\Gamma_{r;pq} - \Gamma_{r;qp}\Gamma_{s;pq} \\
&\qquad + \Gamma_{s;pr}\Gamma_{q;rp} - \Gamma_{q;pr}\Gamma_{s;rp} + \Gamma_{q;sr}\Gamma_{p;rs} - \Gamma_{p;sr}\Gamma_{q;rs} \\
&\qquad + \Gamma_{p;sq}\Gamma_{r;qs} - \Gamma_{r;sq}\Gamma_{p;qs} + \Gamma_{q;ps}\Gamma_{r;sp} - \Gamma_{r;ps}\Gamma_{q;sp} \ ,
\end{aligned}
$$

where

$$
\Gamma_{p;qs} = \partial_{\xi_p} B_{qs} - B_{qp} B_{ps} \ ,
$$

and similarly for the other indices. The set of generalised EL equations in this case are obtained from $\delta\mathcal{A}_{pqrs} = 0$, repeating the general argument, cf. e.g. [41, 42, 45], for deriving the EL equations from the differential of the Lagrangian multiform. Thus, since all the variations δB_{pq}, etc., and their first derivatives, are independent, the coefficients are precisely all the combinations $\Gamma_{r;pq}$, etc., which will have to vanish at the critical point for the action

$$
S[\mathbf{B}(\boldsymbol{\xi}); \mathcal{V}] = \int_{\mathcal{V}} L = \int_{\mathcal{W}} \mathrm{d}L \ , \tag{2.7}
$$

integrated over any arbitrary three-dimensional closed hypersurfaces $\mathcal{V}$ in the multi-variable space of all the ξ_p's, and where the enclosed volume $\mathcal{W}$ is such that $\mathcal{V} = \partial\mathcal{W}$. □

As a corollary, the statement of Theorem 2.2 holds more generally for Lagrangian 3-forms embedded in a higher-dimensional space of independent variables, namely

Corollary 2.1 *In a space of variables $\{\boldsymbol{p} = (p_j)_{j\in I}\}$ where the p_i denote complex-valued continuous variables labelled by an index set I, the Lagrangian 3-form*

$$
\mathsf{L} = \sum_{i,j,k\in I} \mathcal{L}_{p_i,p_j,p_k}\, \mathrm{d}\xi_{p_i} \wedge \mathrm{d}\xi_{p_j} \wedge \mathrm{d}\xi_{p_k} \ , \tag{2.8}
$$

with $\mathcal{L}_{p_i,p_j,p_k}$ as given in (2.4), we have that

$$
\mathrm{d}\mathsf{L} = \sum_{i,j,k,l\in I} \mathcal{A}_{p_i,p_j,p_k,p_l}\, \mathrm{d}\xi_{p_i} \wedge \mathrm{d}\xi_{p_j} \wedge \mathrm{d}\xi_{p_k} \wedge \mathrm{d}\xi_{p_l} \ ,
$$

has a double zero on solutions of the system (2.1) written in the relevant variables labelled by p_i, p_j, p_k, p_l.

The proof is an obvious extension of the one for Theorem 2.2, assuming that all labels of the p_{i_ν} are distinct.

The variational equations obtained from $\delta\mathrm{d}L = 0$ for the Lagrangian multiform (2.5) constitute the set of multiform Euler–Lagrange equations in the language of the variational bicomplex, cf. e.g. [12]. In fact, considering all the fields $B_{\cdot\cdot}$ independent for different labels, this compact form of the variational equations, through the double-zero form, implies the vanishing of all the factors Γ in the above computation, which amounts to the set of generalised Darboux equations. Furthermore, as a direct consequence of the double-zero form for $\mathrm{d}L$ the Lagrangian multiform L is closed on solutions of the Darboux system (2.1), i.e. $\mathrm{d}L|_{\mathrm{EL}} = 0$ (but not trivially so, only 'on-shell'), which implies that for the critical fields which obey the Darboux system the action is invariant under smooth deformations of the hypersurface $\mathcal{V}$. This is precisely the phenomenon of multidimensional consistency: the Darboux system is compatible on any hypersurface in the multidimensional space of Miwa variables.

It is important to mention in this context that multi-time Euler–Lagrange equations were derived in various papers, notably [43, 48] in the case of Lagrangian 1-forms, [44, 45], continuous 2-forms (and in [5, 25] in the discrete case) and, more generally, in [42, 46] in the general continuous case. These derivations follow different approaches, but they have in common that the conventional EL equations associated with a specific choice (meaning in the present context fixing $\mathcal{V}$ in (2.7)) of integration manifold (in the multi-time space of independent variables) must hold simultaneously for all possible choices of $\mathcal{V}$ and are supplemented by a set of additional constraints linking the Lagrangian components arising from 'alien derivatives' (i.e. derivatives w.r.t. independent variables that are not integrated over in the action functional $S[\mathbf{B}(\boldsymbol{\xi}); \mathcal{V}]$ along the direction of the components in question of the Lagrangian multiform L). I don't intend to write down the general formulae here, as they require a lot of additional notations, cf. e.g. [42, 45, 46] for a detailed presentation. In fact, in [42] the double-zero condition is shown to be a sufficient condition for the multi-time EL equations to be satisfied, and this is all we need for the purpose of the present paper.

It may also need pointing out that in the case that the dimension of the embedding space of independent variables coincides with the number of variables in the system (2.1), i.e. when the cardinality of the index set $|I| = 3$ in (2.8), we recover the conventional case of a Lagrangian volume-form with Lagrangian density $\mathcal{L}_{pqr} = \mathcal{L}_{p_i,p_j,p_k}$, in which case the standard Euler–Lagrange equations yield

$$\frac{\delta\mathcal{L}_{pqr}}{\delta B_{pq}} = -\frac{\partial B_{qp}}{\partial\xi_r} + B_{qr}B_{rp}\ , \quad \frac{\delta\mathcal{L}_{pqr}}{\delta B_{qp}} = \frac{\partial B_{pq}}{\partial\xi_r} + B_{pr}B_{rq}\ , \tag{2.9}$$

and similarly for the components B_{qr}, B_{rq}, B_{pr} and B_{rp}. Thus, we obtain the Darboux system (2.1), but the integrability in the sense of MDC is not evident from the conventional Lagrangian formalism.

Another corollary of the multiform structure is that we also have a variational description of the Lax system (2.2). To see this, note that we can extend the set of Miwa variables to include variables ξ_k associated with a 'spectral parameter' k. Thus, we are led to the following statement.

Corollary 2.2 *The Lagrangian 3-form*

$$\begin{aligned} L_{(k)} := \mathcal{L}_{pq(k)}\,\mathrm{d}\xi_p\wedge\mathrm{d}\xi_q\wedge\mathrm{d}\xi_k + \mathcal{L}_{qr(k)}\,\mathrm{d}\xi_q\wedge\mathrm{d}\xi_r\wedge\mathrm{d}\xi_k + \\ + \mathcal{L}_{rp(k)}\,\mathrm{d}\xi_r\wedge\mathrm{d}\xi_p\wedge\mathrm{d}\xi_k + \mathcal{L}_{pqr}\,\mathrm{d}\xi_p\wedge\mathrm{d}\xi_q\wedge\mathrm{d}\xi_r\ , \end{aligned} \tag{2.10}$$

with components

$$\mathcal{L}_{pq(k)} = \tfrac{1}{2}\left(\Psi_q\partial_{\xi_p}\Phi_q - (\partial_{\xi_p}\Psi_q)\Phi_q\right) - \tfrac{1}{2}\left(\Psi_p\partial_{\xi_q}\Phi_p - (\partial_{\xi_q}\Psi_p)\Phi_p\right) \\ + \tfrac{1}{2}\left(B_{qp}\partial_{\xi_k}B_{pq} - B_{pq}\partial_{\xi_k}B_{qp}\right) + \Psi_p B_{pq}\Phi_q - \Psi_q B_{qp}\Phi_p\ , \tag{2.11}$$

through the EL equations $\delta \mathrm{d}\mathsf{L}_{(k)} = 0$ *constitutes a variational description of the Lax multiplet (2.2).*

A similar variational description of the Lax system was obtained in [40] for the 1+1-dimensional Lax system associated with the so-called Zakharov–Mikhailov action. We note that the Lagrangian 3-form (2.10) should really be considered as a Lagrangian 2-form when integrating out the direction ξ_k associated with the (fixed) spectral variable.

3 Discrete Darboux system

A discrete analogue of the Darboux system of orthogonal coordinate systems, i.e. was found in [2, 14] where its integrability was inserted. In fact, an interesting connection with integrable quadrilateral lattices was discovered, as well as with multidimensional circular lattices, cf. [9, 26]. The corresponding discrete analogue of the generalised Darboux system (2.1) reads

$$\Delta_p B_{qr} = B_{qp}T_pB_{pr}\ , \quad \Delta_p B_{rq} = B_{rp}T_pB_{pq}\ , \tag{3.1a}$$
$$\Delta_q B_{rp} = B_{rq}T_qB_{qp}\ , \quad \Delta_q B_{pr} = B_{pq}T_qB_{qr}\ , \tag{3.1b}$$
$$\Delta_r B_{pq} = B_{pr}T_rB_{rq}\ , \quad \Delta_r B_{qp} = B_{qr}T_rB_{rp}\ , \tag{3.1c}$$

where Δ_p denotes the difference operator $\Delta_p = T_p - \mathrm{id}$. This system is related to other multidimensional lattice systems that were formulated in [29].

Theorem 3.1 *The system of difference equations (3.1) is multidimensionally consistent, and furthermore, it is consistent with the differential system (2.1).*

Proof The consistency of the set of difference equations is by direct computation. For instance, rewriting the difference equation as follows:

$$\begin{aligned} B_{qr} &= T_pB_{qr} - B_{qp}T_pB_{pr} \\ &= T_p\left(T_sB_{qr} - B_{qs}T_sB_{sr}\right) - B_{qp}T_p\left(T_sB_{pr} - B_{ps}T_sB_{sr}\right) \\ &= T_pT_sB_{qr} - (T_pB_{qs})T_pT_sB_{sr} - B_{qp}T_pT_sB_{pr} + B_{qp}(T_pB_{ps})T_pT_sB_{sr} \end{aligned}$$

which is equal to the same expression with the labels p and s interchanged. Thus, the latter is equal to

$$= T_sT_pB_{qr} - (T_sB_{qp})T_sT_pB_{pr} - B_{qs}T_sT_pB_{sr} + B_{qs}(T_sB_{sp})T_sT_pB_{pr}\ .$$

Assuming that the shifts T_p and T_s commute, and collecting the factors with $T_pT_sB_{pr}$ and the ones with $T_pT_sB_{pr}$, regarding the latter as independent, we obtain the relations

$$T_pB_{qs} - B_{qs} - B_{qp}T_pB_{ps} = 0\ , \quad \text{and} \quad T_sB_{qp} - B_{qp} - B_{qs}T_sB_{sp} = 0\ ,$$

which are two of the discrete Darboux equations. Thus, the relations are consistent under mutual shifts. The compatibility with the continuous Darboux system (2.1) follows from a similar computation. Abbreviating $\partial/\partial\xi_p$ by ∂_p, we get

$$\partial_p(T_sB_{qr}) = \partial_p\left(B_{qr} + B_{qs}T_sB_{sr}\right) = \partial_pB_{qr} + (\partial_pB_{qs})T_sB_{sr} + B_{qs}\partial_pT_sB_{sr}$$

whereas

$$\begin{aligned} &T_s\partial_pB_{qr} = T_s(B_{qp}B_{pr}) = (T_sB_{qp})T_sB_{pr} = (B_{qp} + B_{qs}T_sB_{sp})T_sB_{pr} \\ &\Rightarrow \quad B_{qp}B_{pr} + B_{qp}B_{ps}T_sB_{sr} + \cancel{B_{qs}T_s\left(B_{sp}B_{pr}\right)} \\ &\qquad = B_{qp}T_sB_{pr} + \cancel{B_{qs}(T_sB_{sp})T_sB_{pr}}\ , \end{aligned}$$

and the remaining terms cancel as well due to the discrete Darboux relation. □

Similarly to the continuous case we have a Lax system, and its adjoint, given by

$$\Delta_p\Phi_q = B_{qp}T_p\Phi_p\ , \quad \Delta_p\Psi_q = \Psi_pT_pB_{pq}\ , \tag{3.2}$$

and the homogeneous linear difference system for an eigenfunctions Φ_r, Ψ_r, respectively,

$$\Delta_p\Delta_q\Phi_r = \frac{\Delta_p(T_q\Phi_q)}{T_q\Phi_q}\,\Delta_q\Phi_r + \frac{\Delta_q(T_p\Phi_p)}{T_p\Phi_p}\,\Delta_p\Phi_r\ , \tag{3.3}$$

$$\Delta_p\Delta_q\Psi_r = \frac{\Delta_p\Psi_q}{T_p\Psi_q}\,\Delta_q(T_p\Psi_r) + \frac{\Delta_q\Psi_p}{T_q\Psi_p}\,\Delta_p(T_q\Psi_r)\ . \tag{3.4}$$

Note that in the discrete case the equations for the eigenfunction and its adjoint are no longer the same. It is natural to assume that the discrete Darboux system (3.1), like its continuous counterpart (2.1), admits a Lagrangian 3-form structure. I intend to settle this question in a future publication [32].

4 Connection with the (scalar) KP system

The KP system of equations is often introduced as the set of Lax equations arising from a Lax operator in a ring of pseudo-differential operators with respect to a singled-out variable x, cf. [38]. This has a disadvantage that the inherent covariant structure of the KP system is broken, and not all independent variables (the higher time variables) appear on the same footing as the variable x. A more covariant approach is provided by the 'direct linearisation' set-up, cf. e.g. [18] and references therein, where there is no need to single out a particular variable to describe the KP hierarchy. It can be argued that the generalised Darboux system (2.1) provides also a covariant description but in the sense of encoding the hierarchy through Miwa variables, cf. [27]. In that sense, the generalised Darboux system is similar in spirit as the 'hierarchy generating PDEs' of [35, 47], but for a three-dimensional system of PDEs instead of the KdV or Boussinesq hierarchies, respectively.

Solutions of the discrete KP system were considered in [17, 33, 34] using the *direct linearisation* (DL) approach, cf. also [37]. The dynamics is governed by *plane-wave factors* which take the form

$$\rho_k = \left[\prod_\nu (p_\nu - k)^{n_\nu}\right] \exp\left\{k\xi - \sum_\nu \frac{\xi_{p_\nu}}{p_\nu - k}\right\} , \tag{4.1a}$$

$$\sigma_{k'} = \left[\prod_\nu (p_\nu - k')^{-n_\nu}\right] \exp\left\{-k'\xi + \sum_\nu \frac{\xi_{p_\nu}}{p_\nu - k'}\right\} . \tag{4.1b}$$

Here, the ξ_{p_ν} are the independent variables of the generalised Darboux system, and the n_ν are associated discrete variables, in terms of which the KP τ function obeys the compatible set Hirota bilinear equations[1]

$$(p-q)(T_pT_q\tau)T_r\tau + (q-r)(T_qT_r\tau)T_p\tau + (r-p)(T_rT_p\tau)T_q\tau = 0 , \tag{4.2}$$

where T_{p_ν} (p, q, r being any three of the p_ν) denotes the elementary shift in the variable n_ν associated with p_ν (which in this context) has the interpretation of a *lattice parameter* measuring the grid width in the discrete direction labelled by n_ν.

The interplay between discrete and continuous variables turns out to be an essential feature of the structure. In fact, the τ-function obeys the relations

$$\frac{\partial\tau}{\partial\xi_p} = -\left(T_p^{-1}\frac{\mathrm{d}}{\mathrm{d}p}T_p\right)\tau := \lim_{\varepsilon\to 0}\frac{T_p^{-1}T_{p-\varepsilon}\tau - \tau}{\varepsilon\tau} , \tag{4.3}$$

for any of the parameters $p_\nu = p$, where we should think of the $p-\varepsilon$ as the lattice parameter associated with lattice directions with elementary lattice shift $T_{p-\varepsilon}$ for all arbitrary small ε. Using the identification between lattice shifts and derivatives as in (4.3), we can perform a limit $r \to p$ on (4.2) and thus obtain the following differential-difference equation for τ:

$$(p-q)\left(\tau\, T_q\frac{\partial\tau}{\partial\xi_p} - (T_q\tau)\,\frac{\partial\tau}{\partial\xi_p}\right) = \tau\, T_q\tau - (T_p\tau)\, T_qT_p^{-1}\tau . \tag{4.4}$$

[1] In the literature, cf. e.g. [15, 27], the dependence on discrete shifts in what is essentially the *scalar* KP system, is confusingly often referred to as the 'multicomponent KP hierarchy' (because lattice-shifted variables are considered as components). This should not be confused with the *matrix KP system*, cf. e.g. [20, 29], which in my opinion more rightfully deserves the name 'multicomponent', and which is related to the system in section 5. The difference between the two resides in that the scalar KP is governed essentially by a scalar integral measure in the underlying DL framework, while the matrix KP is governed by a matrix measure and hence has a much richer solution structure.

Furthermore, the τ-function also obeys the differential-difference equation

$$1+(p-q)^2\frac{\partial^2 \ln\tau}{\partial\xi_p\,\partial\xi_q}=\frac{(T_pT_q^{-1}\tau)T_qT_p^{-1}\tau}{\tau^2}\ , \tag{4.5}$$

which can be readily cast into bilinear form. In fact, Eq. (4.5) is the bilinear form of the 2D Toda equation (with the discrete variable along the skew-diagonal lattice direction in the lattice generated by the T_p and T_q shifts).

It turns out that the Darboux variables of the system (2.1) can be expressed in terms of the KP τ-function exploiting the underlying discrete structure[2]. To do so, consider the quantities

$$S_{a,b}=\frac{T_a^{-1}T_b\tau}{\tau}\ , \tag{4.6}$$

as a consequence (4.2) and (4.4) obey the following relations:

$$(p-b)T_pS_{a,b}-(p-a)S_{a,b}=(a-b)S_{a,p}T_pS_{p,b}\ , \tag{4.7a}$$

$$(p-a)(p-b)\frac{\partial S_{a,b}}{\partial\xi_p}=(a-b)\left(S_{a,p}S_{p,b}-S_{a,b}\right)\ . \tag{4.7b}$$

Similar relations appeared in [15, 27] derived from a different perspective from [17]. These relations are compatible for all parameters p and corresponding shifts and derivatives w.r.t. the corresponding Miwa variables ξ_p. They form the basis for the generalised Darboux and a discrete analogue of the Darboux system, where the latter can be obtained by a gauge transformation with factors $\rho_{-a}\sigma_b$ of the form (4.1). Furthermore, the quantity $S=S_{a,b}$ obeys the following three-dimensional partial difference equation: [33],

$$\begin{aligned}&\frac{\left[(p-b)T_pT_qS-(p-a)T_qS\right]\left[(q-b)T_qT_rS-(q-a)T_rS\right]}{\left[(p-b)T_pT_rS-(p-a)T_rS\right]\left[(q-b)T_pT_qS-(q-a)T_pS\right]}\\ &\qquad\times\frac{\left[(r-b)T_pT_rS-(r-a)T_pS\right]}{\left[(r-b)T_qT_rS-(r-a)T_qS\right]}=1\end{aligned} \tag{4.8}$$

which is essentially the lattice Schwarzian KP equation, first given in its well-known pure form in [16].

The KP hierarchy can be obtained by the expansions

$$\begin{aligned}&t_j=\delta_{j,1}\xi+\sum_\nu\left(\frac{\xi_{p_\nu}}{p_\nu^{j+1}}+\frac{1}{j}\,\frac{n_\nu}{p_\nu^j}\right)\\ &\Rightarrow\quad T_{p_\nu}\tau=\tau\left(\{t_j+\frac{1}{jp_\nu^j}\}\right)\quad\text{and}\quad\frac{\partial\tau}{\partial p_\nu}=\sum_{j=1}^\infty\frac{1}{p_\nu^{j+1}}\frac{\partial\tau}{\partial t_j}\ ,\end{aligned} \tag{4.9}$$

[2] In [15, 27], a similar connection was exhibited, which differs from the present one in that my presentation is based on results from the DL approach, whereas the Sato or fermionic-type approach seems to cover a more restricted solution sector of the theory.

where the t_j are the usual independent time variables in the hierarchy.

From (4.7b), it follows that the Darboux quantities can be identified as

$$B_{pq} = \frac{\sigma_p \rho_q S_{p,q}}{q-p} = \sigma_p \rho_q \frac{T_p^{-1} T_q \tau}{(q-p)\tau} \ , \quad q \neq p \ , \tag{4.10}$$

from which, together with (4.1), we get the relations (2.1) whenever $q \neq p$. When $q = p$, we have $B_{pp} = \mathcal{C}\partial_{\xi_p}(\ln \tau)$, where $\mathcal{C}$ is some constant normalisation factor.

The eigenfunctions of the Lax multiplet are obtained from

$$\phi_a(k) = \frac{S_{a,k}\rho_k}{a-k} \ , \quad \psi_b(k') = \frac{S_{k',b}\sigma_{k'}}{b-k'} \ , \tag{4.11}$$

which obey the set of relations

$$(p-a)\phi_a(k) = T_p\phi_a(k) - S_{a,p}T_p\phi_p(k) \ , \tag{4.12a}$$

$$(p-b)T_p\psi_b(k') = \psi_b(k') - \psi_p(k')T_pS_{p,b} \ , \tag{4.12b}$$

$$(a-p)\frac{\partial}{\partial \xi_p}\phi_a(k) = \phi_a(k) - S_{a,p}\phi_p(k) \ , \tag{4.12c}$$

$$-(b-p)\frac{\partial}{\partial \xi_p}\psi_b(k') = \psi_b(k') - \psi_p(k')S_{p,b} \ , \tag{4.12d}$$

the compatibility conditions of which reproduce Eq. (4.7).

Within the setting of the DL approach, the following combination of the quantities S, for arbitrary values of c, possesses a quadratic eigenfunction expansion of the form

$$S_{a,b} - S_{a,c}S_{c,b} = \iint_D d\zeta(l,l')(l-l')\frac{(c-a)(c-b)}{(c-l)(c-l')}\,\phi_a(l)\psi_b(l') \ . \tag{4.13}$$

Here, the integration is over an arbitrary measure in a region $D \subset \mathcal{C} \times \mathcal{C}$ of values of l, l' in a spectral space with measure $\mathrm{d}\zeta(l,l')$. Under special conditions, these integrals correspond to the generalised Cauchy integrals arising in the $\bar{\partial}$ problem or nonlocal Riemann–Hilbert problems for the KP-type spectral problems, cf. [21, 49]. (The choices of c must be such that singularities in the integrals are avoided, and this requires some conditions on the integrations, which play a role when we consider special solutions. We will not address these issues of analysis here.) Note that when $c = p_\nu$, i.e. coincides with any of the parameters associated with the Miwa variables ξ_p , then the left-hand side of (4.13) coincides with the expression on the right-hand side of (4.7). I.o.w. the right-hand side of (4.13) provides a quadratic eigenfunction expansion for the derivative of $S_{a,b}$ w.r.t. ξ_p (modulo a constant factor). However, (4.13) is independent of the choice of Miwa variables and holds for any c. In particular, in the limit $c \to \infty$ we obtain the following fundamental bilinear identity for the solution of the τ function associated with the choice of measure and integration region D:

$$\iint_D d\zeta(l,l')\frac{(l-l')\rho_l\sigma_{l'}}{(a-l')(a'-l)}(T_{a'}^{-1}T_l\tau)(T_{l'}^{-1}T_a\tau) = \\ = \tau\,(T_{a'}^{-1}T_a\tau) - (T_{a'}^{-1}\tau)(T_a\tau)\ , \tag{4.14}$$

which can be considered as a *bilinear integro-difference equation* for the τ function. The relation (4.14) is reminiscent of the fundamental bilinear identity that plays a central role in the Sato approach to the KP hierarchy, cf. also [3], which is, however, not the approach taken here to derive this relation. It is maybe useful to mention at this juncture that, while all definitions of a τ-function are in a sense non-universal and depend on the solution class under consideration, what may be the most general definition of a τ-function was formulated in [30], namely in terms of a fermionic path integral associated with the direct linearising transform structure.

5 Generalisation to the matrix case

In a talk at the June 1987 NEEDS meeting, I presented a 2+1-dimensional Lagrangian matrix KP system which effectively amounts to a matrix generalisation of the Darboux system, that became a focus of attention in the mid 1990s. We proposed the following Lagrangian, cf. [36],

$$\mathcal{L}_{ijk} = \tfrac{1}{2}\mathrm{tr}\left\{G_{ij}J_i(\partial_k G_{ji})J_j - (\partial_k G_{ij})J_i G_{ji}J_j + \text{cycl. (ijk)}\right\} \\ - \mathrm{tr}\left\{G_{ij}J_iG_{ki}J_kG_{jk}J_j - G_{ji}J_jG_{kj}J_kG_{ik}J_i\right\}\ , \tag{5.1}$$

which is a matrix generalisation of (2.4). In fact, the G_{ij} are $N\times N$ matrix functions of dynamical variables $x_i = \xi_{l_i}^{J_i}, x_j = \xi_{l_j}^{J_j}, x_k = \xi_{l_k}^{J_k}, \ldots$, which are labelled not only by a continuous parameter $l.$ (like the p, q, r in the scalar case), but also by a matrix $J.$ which in a sense 'tunes' a hierarchy of associated KP type equations. while the J_i, J_j, J_k are constant $N\times N$ matrices, which commute among themselves[3], i.e. $[J_i, J_j] = [J_j, J_k] = [J_k, J_i] = 0$. In (5.1) we have denoted $\partial/\partial\xi_{l_j} =: \partial_j$, etc., for the sake of brevity. Like (2.4) the Lagrangian (5.1) can be viewed as a component of a Lagrangian 3-form

$$\mathsf{L} = \sum_{i<j<k}\mathcal{L}_{ijk}\,\mathrm{d}x_i\wedge\mathrm{d}x_j\wedge\mathrm{d}x_k\ , \tag{5.2}$$

which is closed on solutions of the Euler–Lagrange equations

$$\partial_i G_{jk} = G_{ik}J_iG_{ji}\ ,\quad i\neq j\neq k\neq i\ . \tag{5.3}$$

The main statement is that these Lagrangians form the components of a Lagrangian 3-form. Thus, we have

[3] In fact, one can also consider the non-commutative case $[J_i, J_j] = \Gamma_{ij}^k J_k$, in which case we get non-commuting flows on a loop group, for which a Lagrangian description was proposed recently in [7] for (1+1)-dimensional systems.

Theorem 5.1 *The Lagrangian 3-form* (5.2) *has a double zero on solutions of the fundamental set of equations* (5.3).

Proof The proof is again computational, and in essence similar to the one of Theorem 2.2, with the main difference occurring in the matrix ordering within the trace. Computing the differential of L, we get in the matrix case

$$\mathrm{dL} = \sum_{i,j,k,l} \mathcal{A}_{ijkl}\, \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_l \ ,$$

with

$$\begin{aligned}\mathcal{A}_{ijkl} = \tfrac{1}{2}\mathrm{tr}\, \big\{ & \Gamma_{l;i,j} J_i \Gamma_{k;j,i} J_j - \Gamma_{k;i,j} J_i \Gamma_{l;j,i} J_j \\ & + \Gamma_{l;k,i} J_k \Gamma_{j;i,k} J_i - \Gamma_{j;k,i} J_k \Gamma_{l;i,k} J_i \\ & \Gamma_{l;j,k} J_j \Gamma_{i;k,j} J_k - \Gamma_{i;j,k} J_j \Gamma_{l;k,j} J_k \pm \text{cycl} \ \ (ijkl) \big\} \ ,\end{aligned}$$

where the cyclic permutation over the indices (i, j, k, l) is done with alternating signs of the six terms inside the bracket, resulting in 24 terms in total. Here, the quantities Γ are given by

$$\Gamma_{i;j.k} = \partial_i G_{jk} - G_{ik} J_i G_{ji} \ ,$$

and hence we have a double-zero expansion of dL implying that the generalised Euler–Lagrange equations arising from $\delta \mathrm{dL} = 0$ for all G_{ij} varied independently (for different indices) gives rise to the entire system of matrix Darboux equations to yield the critical point of the action

$$S[G_{\cdot,\cdot}(\boldsymbol{x}); \mathcal{V}] = \int_{\mathcal{V}} \mathsf{L} \ ,$$

as a functional of all the matrix fields $G_{\cdot,\cdot}$ as well as of the hypersurfaces $\mathcal{V}$ in the space of independent variables. As a consequence of the double-zero expansion form we have $\mathrm{dL} = 0$ for the fields G obeying the set of EL equations, and hence the action is independent of the choice of hypersurface for those critical fields. □

More or less simultaneously to our paper [36], and independently, Bogdanov and Manakov investigated a (2+1)-dimensional Lagrangian matrix system, cf. [4]. In retrospect both systems are very similar and originate from the consideration of nonlocal inverse problems, either through Direct Linearisation in the case of [36], in a framework also exploited for the case of three-dimensional matrix lattice equations, cf. [29, 34], or using nonlocal $\bar{\partial}$ in the case of [4]. Like in the scalar case of section 2, these Lagrangians can be seen as components of a Lagrangian 3-form, and in a precise sense they generate the entire hierarchy of matrix KP equations (I will not dwell on that aspect in the present note).

To be more precise let us first, in the notation of [36], specify the matrix (or, in the parlance of the last decade, the non-Abelian or non-commutative) KP structure. Note that one of the first papers that addressed the matrix KP system, from an inverse scattering point of view, was [20]. The main set of equations, in fact the matrix generalisation of (4.7), is the family of relations given by

$$\partial_k^J H_{ab} = \frac{J}{k-a} H_{ab} - H_{ab} \frac{J}{k-b} + H_{ak} J H_{kb} \ , \tag{5.4}$$

where $k \neq a \neq b \neq k$ are complex-valued parameters and the derivatives ∂_k^J is with respect to some Miwa-type variables ξ_k^J characterised by the constant matrix J as well as the label k which here is complex parameter. The family of Eqs. (5.4) is multidimensionally consistent for different values of k and commuting sets of matrices J, as can be readily verified.

A Lagrangian for the set of equations (5.4) is given by

$$\begin{aligned}\mathcal{L}_{klm} = \tfrac{1}{2}\mathrm{tr}\Big\{ & H_{ml}\widetilde{J}(\partial_k^J H_{lm})\widehat{J} - (\partial_k^J H_{ml})\widetilde{J} H_{lm}\widehat{J} \\ & + H_{km}\widehat{J}(\partial_l^{\widetilde{J}} H_{mk}) J - (\partial_l^{\widetilde{J}} H_{km})\widehat{J} H_{mk} J \\ & + H_{lk} J (\partial_m^{\widehat{J}} H_{kl})\widetilde{J} - (\partial_m^{\widehat{J}} H_{lk}) J H_{kl}\widetilde{J}\Big\} \\ & + \mathrm{tr}\Big\{ H_{ml}\widetilde{J} H_{lm}\frac{J\widehat{J}}{k-m} - H_{ml}\frac{\widetilde{J}J}{k-l} H_{lm}\widehat{J} \\ & + H_{km}\widehat{J} H_{mk}\frac{\widetilde{J}J}{l-k} - H_{km}\frac{\widehat{J}\widetilde{J}}{l-m} H_{mk} J \\ & + H_{lk} J H_{kl}\frac{\widehat{J}\widetilde{J}}{m-l} - H_{lk}\frac{\widehat{J}J}{m-k} H_{kl}\widetilde{J}\Big\} \\ & + \mathrm{tr}\left\{ H_{lm}\widehat{J} H_{mk} J H_{kl}\widetilde{J} - H_{ml}\widetilde{J} H_{lk} J H_{km}\widehat{J}\right\} \ ,\end{aligned} \tag{5.5}$$

which essentially is equivalent to the Lagrangian of [4]. The variational equations

$$\frac{\delta \mathcal{L}_{klm}}{\delta H_{ml}^T} = 0 \quad \Rightarrow \quad \partial_k^J H_{lm} = \frac{J}{k-l} H_{lm} - H_{lm}\frac{J}{k-m} + H_{lk} J H_{km} \ ,$$

and similarly the other equations with k, l, m and $J, \widetilde{J}, \widehat{J}$, respectively, all permuted, follow from this Lagrangian. By expanding the Miwa variables we can derive Lagrangians for the matrix KP hierarchy (examples of matrix KP hierarchy equations arising from the analogous Lagrange structure were provided in [4]). The main new insight provided here, and which is a direct consequence, in fact a specification, of Theorem 5.1, is that this Lagrangian structure can be extended to a Lagrangian 3-form structure for the matrix KP hierarchy in (matrix) Miwa variables[4], provided by

[4] Since integrable matrix hierarchies comprise not a single sequence of higher time flows, but several families, each generated by a zeroth-order time flow associated with a constant matrix J, this matrix serves as the label for the corresponding sequence of higher times t_j^J, cf. e.g. [12], and associated Miwa-type variables ξ_p^J can be defined by 'compounding' those hierarchies in the sense of [31], i.e. constructing weighted sums of higher time derivatives as in (4.9).

$$\begin{aligned}\mathsf{L} = \mathcal{L}_{klm}\mathrm{d}\xi_k^J \wedge \mathrm{d}\xi_l^{\widetilde{J}} \wedge \mathrm{d}\xi_m^{\widehat{J}} + \mathcal{L}_{lmn}\mathrm{d}\xi_l^{\widetilde{J}} \wedge \mathrm{d}\xi_m^{\widehat{J}} \wedge \mathrm{d}\xi_n^{\overline{J}} \\ + \mathcal{L}_{mnk}\mathrm{d}\xi_m^{\widehat{J}} \wedge \mathrm{d}\xi_n^{\overline{J}} \wedge \mathrm{d}\xi_k^J + \mathcal{L}_{klm}\mathrm{d}\xi_k^J \wedge \mathrm{d}\xi_l^{\widetilde{J}} \wedge \mathrm{d}\xi_m^{\widehat{J}}\end{aligned}$$

(which can be readily extended to a multi-sum involving more variables of the type x_k^J with different labels and different matrices J). As a conclusion, this provides the proper variational structure of the matrix KP hierarchy in its generating form. This is direct consequence of Theorem 5.1, where the correspondence between the matrices G_{kl} and the matrices H_{kl} is obtained by introducing matrix analogues of the plane-wave factors ρ_k and $\sigma_{k'}$, given by nonsingular $N \times N$ matrices φ_k^0 and ${}^t\varphi_l^0$ obeying

$$\partial_m^J \varphi_k^0 = \frac{J}{m-k}\varphi^0 \ , \quad \partial_m^J {}^t\varphi_l^0 = -{}^t\varphi_l^0 \frac{J}{m-l} \ ,$$

(where the superscript ^{0} denotes the aspect that these are 'free' solutions of the underlying linear system), and setting

$$G_{kl} = {}^t\varphi_l^0 H_{lk} \varphi_k^0 \ , \quad \text{and} \quad J_m = ({}^t\varphi_m^0)^{-1} J (\varphi_m^0)^{-1} \ .$$

As a consequence, relying on Theorem 5.1, the Lagrangian (5.5) form the components of a Lagrangian 3-form whose generalised EL equations provide the system of Eq. (5.4). This is essentially the generating set of equations for the matrix KP hierarchy.

6 Discussion

The results in this paper generalise in an essential way those of [39] where the multiform structure of the KP hierarchy was established in the conventional presentation in terms of pseudo-differential operators. In this paper, we consider the KP hierarchy from the point of view of *generating PDEs*, namely through their representation in terms of Miwa variables. This has the advantage that the structure becomes much more covariant. Thus the KP hierarchy is being treated as multi-parameter family of equations in the sense of what we called a *generating PDE*, i.e. a PDE in terms of Miwa-type variables, which by expansion in powers of the parameters lead to the conventional hierarchy of KP equations in the multi-time form (in the cases of (1+1)-dimensional hierarchies these generating PDEs are obtained from the conventional enumerative hierarchies, by a process of 'compounding', cf [31]). A connection between Lagrangian multiforms in this parameter-family representation and the classical r-matrix was recently put forward in [8]. Lifting those results to the case of the (2+1)-dimensional KP hierarchy could provide a novel route to the quantisation of the KP system. Identifying a classical (and possible in due course a quantum) R-matrix for the KP system would form a major step towards both a canonical as well as path integral route towards its quantisation.

In this context it is worth mentioning another connection. In the Direct Linearising Transform (DLT) approach to the KP system (discrete as well as continuous), cf. [17, 29, 33, 34] the invariance under integral transforms with a kernel $G_{kk'}$ is the key element of the construction. This kernel is a path-independent line-integral in the space of independent variables of the system, constructed from of a closed (on solutions of the equation of motion) 1-form constructed from the Lax multiplet eigenfunctions. The kernel $G_{kk'}$ solves the following class of generalised Darboux systems

$$\partial_i G_{kk'} = \iint_{D_i} G_{lk'}\, \mathrm{d}\zeta_i(l, l')\, G_{kl'}\ , \quad i \in I, \tag{6.1}$$

where the integration is over a set (labelled by I) of domains $D_i \in \mathbb{C} \times \mathbb{C}$ in some spectral-type variables l and l' over a set of matrix-valued measures $\mathrm{d}\zeta_i(l, l')$ in that domain. The independent variables are assumed to be characterised by the integration data: $x_i = x(\zeta_i, D_i)$ and $\partial_i = \partial/\partial x_i$. Notable is the dual role played by $G_{kk'}$, on the one hand as the integral kernel of an integral transform, on the other hand as a solution of a parameter family of nonlinear equations of Darboux type, which can be reconstructed from the quantities $H_{k,k'}$ of the previous section. Most important in the present context is the observation that this general system can be endowed with a Lagrangian 3-form structure very similar to the ones described in the previous section, namely given by Lagrangian components

$$\begin{aligned} \mathcal{L}_{ijk} = \tfrac{1}{2} \iint_{D_i} \iint_{D_j} & \mathrm{tr}\, \big\{ G_{l,k'}\, \mathrm{d}\zeta_i(l, l')\, (\partial_k G_{k,l'})\, \mathrm{d}\zeta_j(k, k') \\ & - (\partial_k G_{l,k'})\, \mathrm{d}\zeta_i(l, l')\, G_{k,l'}\, \mathrm{d}\zeta_j(k, k') + \mathrm{cycl}(ijk) \big\} \\ & + \iint_{D_i} \iint_{D_j} \iint_{D_k} \mathrm{tr}\, \big\{ G_{l,k'}\, \mathrm{d}\zeta_i(l, l')\, G_{m,l'}\, \mathrm{d}\zeta_j(m, m')\, G_{k,m'}\, \mathrm{d}\zeta_k(k, k') \\ & \qquad - G_{l,k'}\, \mathrm{d}\zeta_i(l, l')\, G_{m,l'}\, \mathrm{d}\zeta_k(m, m')\, G_{k,m'}\, \mathrm{d}\zeta_j(k, k') \big\}\ . \end{aligned} \tag{6.2}$$

It can be proven by similar computations, and under some generous assumptions on the integrations in the formula, that analogous statements to the ones in the previous sections, that the Lagrangian 3-form with components given by (6.2) possesses a Lagrangian multiform structure. This forms arguably the most general multiform structure so far considered in the theory. Note also that the corresponding action functional $S[G_{\cdot,\cdot}(\boldsymbol{x}); \mathcal{V}]$ where as before $\mathcal{V}$ is an arbitrary three-dimensional hypersurface in the space of independent variables $\boldsymbol{x} = (\{x_i, i \in I\})$, shows some resemblance some action functionals associated with the Chern–Simons theory in topological field theory, but this connection still remains to be explored.

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Declarations

Conflict of interest The corresponding author states that there is no conflict of interest.

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