DEFORMED CALOGERO–MOSER OPERATORS AND IDEALS OF RATIONAL CHEREDNIK ALGEBRAS

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Abstract. We consider a class of hyperplane arrangements $\mathcal{A}$ in $\mathbb{C}^n$ that generalise the locus configurations of [CFV2]. To such an arrangement we associate a second order partial differential operator of Calogero–Moser type, and prove that this operator is completely integrable (in the sense that its centraliser in $\mathcal{D}(\mathbb{C}^n \setminus \mathcal{A})$ contains a commutative subalgebra of Krull dimension n). The proof is based on the study of certain ideals of (the spherical subalgebra of) the rational Cherednik algebra that may be of independent interest. Our examples include the examples of deformed Calogero–Moser systems constructed by A. Sergeev and A. Veselov in [SV1], M. Feigin in [F] as well as new examples recently proposed by D. Gaiotto and M. Rapčák in [GR]. Our approach describes these and other examples in a general framework of rational Cherednik algebras close in spirit to [BEG] and [BC].

1. Introduction

Let V be the Euclidean space $\mathbb{R}^n$ with standard inner product $(\cdot,\cdot)$. Consider a collection $\mathcal{A}_+ = \{\alpha\}$ of nonparallel vectors in V with prescribed 'multiplicities' k_α, which are assumed (for the moment) to be arbitrary real numbers. Set $\mathcal{A} := \mathcal{A}_+ \cup (-\mathcal{A}_+)$ and define $k_{-\alpha} := k_\alpha$ for $\alpha \in \mathcal{A}_+$. We will refer to the pair $(\mathcal{A}, k_\alpha)$ as a *configuration* in $\mathbb{R}^n$, abbreviating it often as $\mathcal{A}$ for simplicity. With such a configuration we associate a *generalised Calogero–Moser operator* of the form

$$L_{\mathcal{A}} := \Delta - \sum_{\alpha\in\mathcal{A}_+} \frac{k_\alpha(k_\alpha+1)(\alpha,\alpha)}{(\alpha,x)^2}\,, \tag{1.1}$$

where Δ is the standard Laplacian on $\mathbb{R}^n$. The usual (rational) Calogero–Moser operator corresponds to the root system of type A_{n-1} with all $k_\alpha = k$:

$$L = \Delta - \sum_{i<j}^{n} \frac{2\,k(k+1)}{(x_i-x_j)^2}\,. \tag{1.2}$$

The operator (1.2) can be viewed as a quantum Hamiltonian of a system of n interacting particles on the line. This is a celebrated example of a quantum completely integrable system: there exist n algebraically independent partial differential operators $L_1, L_2, \dots, L_n$, including L, such that $[L_i, L_j] = 0$ for all $i, j = 1, \dots, n$. In contrast, the quantum Hamiltonian (1.1) is not expected to be integrable for an arbitrary configuration. For instance, we have

Theorem 1.1. *Let* (1.1) *be a completely integrable quantum Hamiltonian such that its quantum integrals* $L_1, \dots, L_n$ *have algebraically independent constant principal symbols* $p_1, \dots, p_n \in \mathbb{R}[V^*]$. *Assume furthermore that* $k_\alpha \notin \mathbb{Z}$ *for all* $\alpha \in \mathcal{A}$. *Then the polynomials* p_i *are invariant under a finite Coxeter group* $W \subset \mathtt{GL}(V)$, *and* $\alpha \in \mathcal{A}$ *form a subset of the root system* R *of* W.

This theorem is a simple consequence of the main result of [T1]. Indeed, if s_α is the orthogonal reflection corresponding to $\alpha \in \mathcal{A}$, then, as shown in [T1], each p_i must be invariant under s_α. Now take $\alpha, \beta \in \mathcal{A}$, and assume that $s_\alpha s_\beta$ is of infinite order. Then p_i must be invariant under an arbitrary rotation in the two-dimensional plane spanned by α, β. However, the ring of polynomials invariant under such rotations has Krull dimension $< n$, which implies that $p_1, \dots, p_n$ cannot be algebraically independent. By contradiction, we conclude that $s_\alpha s_\beta$ is of finite order for any α, β, therefore the reflections $\{s_\alpha\}_{\alpha \in \mathcal{A}}$ generate a finite Coxeter group W, and so $\mathcal{A}$ is a subset of the root system R of W.

Theorem 1.1 tells us that for non-integral parameters k_α, the completely integrable operators of the form (1.1) are closely related to Coxeter groups. Indeed, by a theorem of Heckman [H1], the Calogero–Moser operator (introduced in [OP])

$$L_W := \Delta - \sum_{\alpha \in R_+} \frac{k_\alpha (k_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}, \tag{1.3}$$

is completely integrable for the root system R of an arbitrary finite Coxeter group W and an arbitrary W-invariant function $k : R \to \mathbb{R}$. (For all crystallographic groups W this was already shown in [O].)

On the other hand, in the case when *all* k_α's are integers, there are examples of completely integrable operators of the form (1.1) where $\mathcal{A}$ is not contained in any root system (see [CFV2]). Instead, such configurations satisfy certain algebraic conditions, called the *locus relations*. The purpose of the present work is to study the intermediate case: namely, we are interested in completely integrable operators of the form (1.1) where only *some* of the k_α's are integers. Examples of such operators related to Lie superalgebras were constructed in [SV1]. A common feature of these examples is that the vectors in $\mathcal{A}$ with non-integral multiplicities form a root system R of a finite Coxeter group W, while those with integral multiplicities form a W-invariant set. To ensure integrability, the vectors with integral multiplicities should satisfy certain compatibility conditions analogous to the locus relations of [CFV2]. We call such configurations the *generalised locus configurations*. Our main result, Theorem 3.5, states that for any generalised locus configuration, the operator (1.1) is completely integrable. More precisely, we show that, in this case, there is a commutative algebra of quantum integrals containing the Hamiltonian (1.1), that is isomorphic to the ring of *generalised quasi-invariants* $Q_{\mathcal{A}}^W$ (see Definition 3.4). The complete integrability immediately follows from that. Note that the constructed commutative ring of quantum integrals is, in general, of rank > 1 (see Remark 5.7). Also, we prove that there exists a linear differential operator S intertwining $L_{\mathcal{A}}$ and L_W, i.e.

$$L_{\mathcal{A}}\, S = S\, L_W\,.$$

In various special cases the results of this kind can be found in [CFV1, SV1, F, SV2]; our approach unifies them and applies to a wider class of operators. We also extend some of our results to the case of the Calogero–Moser operators with an additional quadratic oscillatory term.

We now explain how the generalised quasi-invariants $Q_{\mathcal{A},W}$ are related to rational Cherednik algebras. Our approach is inspired by [BEG] and [BC]; however, the Cherednik algebras play a different role in our construction.

If X is an affine algebraic variety, we write $\mathcal{D}(X)$ for the ring of (global) algebraic differential operators on X. It is well known that when X is singular, the ring $\mathcal{D}(X)$ has a complicated structure. A natural way to approach $\mathcal{D}(X)$ geometrically is to relate it to the ring of differential operators on a non-singular variety Y, which is a resolution of X. Specifically (cf. [SS]), assuming that the variety X is irreducible, one can choose a finite birational map $\pi : Y \to X$ with Y smooth and consider the space of differential operators from Y to X:

$$\mathcal{D}(Y, X) := \{D \in \mathcal{D}(\mathbb{K}) \,:\, D[\mathcal{O}(Y)] \subseteq \mathcal{O}(X)\}\,, \qquad (1.4)$$

where $\mathbb{K}$ is the field of rational functions on X. This space is naturally a right module over $\mathcal{D}(Y)$ and a left module over $\mathcal{D}(X)$, and the two module structures are compatible: in other words, $\mathcal{D}(Y, X)$ is an $\mathcal{D}(X)$-$\mathcal{D}(Y)$-bimodule. Taking the endomorphism ring of $\mathcal{D}(Y, X)$ over $\mathcal{D}(Y)$ and mapping the differential operators in $\mathcal{D}(X)$ to (left) multiplication operators on $\mathcal{D}(Y, X)$ gives an algebra homomorphism: $\mathcal{D}(X) \to \mathrm{End}_{\mathcal{D}(Y)}\mathcal{D}(Y, X)$, which — under good circumstances — turns out to be an isomorphism. In [BEG], this construction was used for the varieties of classical quasi-invariants, $X = \mathtt{Spec}\, Q_m$, in which case the resolution $\pi : Y \to X$ is given by the normalization map, with $Y = \tilde{X} \cong V$.

In the present paper, we modify ('deform') the bimodule $\mathcal{D}(Y, X)$ replacing the ring $\mathcal{D}(Y)$ of differential operators on a smooth resolution of X by a (spherical) Cherednik algebra. To be precise, given a generalised locus configuration $(\mathcal{A}, k, W)$, we consider the variety $X := \mathtt{Spec}\, Q_{\mathcal{A}}^W$ together with a natural map $\pi : V/\!/W \to X$ corresponding to the inclusion $Q_{\mathcal{A}}^W \subset \mathbb{C}[V]^W$ (see Definition 4.2). Instead of applying (1.4) directly to π, we first restrict to the subspace $V_{\mathrm{reg}}/\!/W$ of regular W-orbits in $V/\!/W$ (obtained by removing from V the reflection hyperplanes of W), and define the ring $Q_{\mathrm{reg}} \subseteq \mathbb{C}[V_{\mathrm{reg}}]^W$, using the same algebraic conditions as for $Q = Q_{\mathcal{A}}^W$ (see (4.1)) but with $\mathbb{C}[V]^W$ replaced by $\mathbb{C}[V_{\mathrm{reg}}]^W$. Taking $X_{\mathrm{reg}} := \mathtt{Spec}\, Q_{\mathrm{reg}}$, we then consider the bimodule $\mathcal{D}(V_{\rm reg}/\!/W, X_{\rm reg})$ associated to the natural map $\pi_{\rm reg} : V_{\rm reg}/\!/W \to X_{\rm reg}$. Since W acts freely on $V_{\rm reg}$, we have $\mathcal{D}(V_{\rm reg}/\!/W) \cong \mathcal{D}(V_{\rm reg})^W$, and therefore $\mathcal{D}(V_{\rm reg}/\!/W, X_{\rm reg}) \subseteq \mathcal{D}(V_{\rm reg})^W[\delta^{-1}]$, where $\delta := \prod_{\alpha\in\mathcal{A}_+\setminus R}(\alpha, x)^{k_\alpha}$. Now, the spherical subalgebra B_k of the rational Cherednik algebra $H_k(W)$ with $k = \{k_\alpha\}_{\alpha\in R}$ embeds naturally into $\mathcal{D}(V_{\rm reg})^W$ via the Dunkl representation (see (2.5)); thus, we can define

$$\mathcal{M}_{\mathcal{A},W} := \mathcal{D}(V_{\rm reg}/\!/W, X_{\rm reg}) \cap B_k\,.$$

This is a right B_k-module – in fact, an ideal of B_k – that we associate to our generalised locus configuration.[1]

Main results.

Organisation of the paper. In Section 2 we recall a well-known relation between Calogero–Moser systems and Dunkl operators, which goes back to [D, H1, EG1]. In Section 3 we introduce locus relations and locus configurations, following [CFV2], and recall their link to quantum integrability. In Section 4 we define generalised locus configurations and state our main result, Theorem 3.5. The proofs are given in Section 5; the main idea is to define and study, for each generalised locus configuration $\mathcal{A}$, an ideal $\mathcal{M}_{\mathcal{A}}$ (as briefly described above) in the spherical Cherednik algebra. This is an extension of the earlier arguments in the theory of locus configurations and quasi-invariants [B, C1, CFV2, BC], with ideals of Cherednik algebras replacing the Baker–Akhiezer function, which was central in those earlier works. Note that the idea that the Baker–Akhiezer function for the Calogero–Moser system can be re-interpreted in terms of special ideals of the n-th Weyl algebra goes back to [BEG]. The ideals $\mathcal{M}_{\mathcal{A}}$ have some interesting algebraic properties which we discuss in Section 6. In Sections 7 and 8 we describe all generalised locus configurations currently known. In Section 9 we extend our results to the generalised Calogero–Moser operators in the presence of a harmonic oscillatory confinement. Finally, Section 10 briefly discusses the case of affine configurations.

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[1]For technical reasons, it is more convenient for us to work with a twisted ideal which is obtained by replacing $Q_{\rm reg} = \mathcal{O}(X_{\rm reg})$ in the above construction by a rank one torsion-free $\mathcal{O}(X_{\rm reg})$-module $U_{\mathcal{A}}$, see (5.2).

2. Cherednik algebras and Calogero–Moser systems

In this section we recall a well-known relation between rational Cherednik algebras and Calogero–Moser systems. For more details and references, we refer the reader to [EG1].

Let W be a finite Coxeter group with reflection representation V. Throughout the paper we will work over $\mathbb{C}$, so V is a complex vector space with a W-invariant bilinear form $(\cdot,\cdot)$. Each reflection $s \in W$ acts on V by the formula

$$s(x) = x - 2\,\frac{(\alpha, x)}{(\alpha,\alpha)}\,\alpha\ , \tag{2.1}$$

where $\alpha \in V$ is a normal vector to the reflection hyperplane. Denote by R_+ the set of all these normals and put $R = R_+ \cup -R_+$. Only the direction of each normal α is important, so we may assume that they are chosen in such a way that the set R is W-invariant (it is also customary to choose R_+ to be contained in some prescribed half-space). Let us choose a W-invariant function $k : R \to \mathbb{C}$. The elements $\alpha \in R$ are called the roots of W, and $k_\alpha := k(\alpha)$ is called the *multiplicity* of α. Note that we do not assume that W is irreducible, and R may not span the whole V.

We set $V_{\rm reg} := \{x \in V \,|\, (\alpha, x) \neq 0\ \forall \alpha \in R\}$ and denote by $\mathbb{C}[V_{\rm reg}]$ and $\mathcal{D}(V_{\rm reg})$ the rings of regular functions and regular differential operators on $V_{\rm reg}$, respectively. The action of W on V restricts to $V_{\rm reg}$, so W acts naturally on $\mathbb{C}[V_{\rm reg}]$ and $\mathcal{D}(V_{\rm reg})$ by algebra automorphisms. We form the crossed products $\mathbb{C}[V_{\rm reg}] * W$ and $\mathcal{D}W := \mathcal{D}(V_{\rm reg}) * W$. As an algebra, $\mathcal{D}W$ is generated by its two subalgebras, $\mathbb{C}W$ and $\mathcal{D}(V_{\rm reg})$.

The Calogero–Moser operator associated to W and $k = \{k_\alpha\}$ is a differential operator $L_{W,k} \in \mathcal{D}(V_{\rm reg})^W$ defined by

$$L_W := \Delta - u_W\ , \qquad u_W = \sum_{\alpha\in R_+} \frac{k_\alpha(k_\alpha+1)(\alpha,\alpha)}{(\alpha,x)^2}\ , \tag{2.2}$$

where Δ is the Laplacian on V associated with the W-invariant form $(\cdot,\cdot)$.

To describe the link between L_W and Cherednik algebra, we first define the *Dunkl operators* $T_\xi \in \mathcal{D}W$ as

$$T_\xi := \partial_\xi + \sum_{\alpha\in R_+} \frac{(\alpha,\xi)}{(\alpha,x)} k_\alpha s_\alpha\ , \quad \xi \in V\ . \tag{2.3}$$

Note that the operators (2.3) depend on $k = \{k_\alpha\}$, and we sometimes write $T_{\xi,k}$ to emphasize this dependence. The basic properties of Dunkl operators are listed in the following lemma.

Lemma 2.1 ([D]). *For all $\xi, \eta \in V$ and $w \in W$, we have*

(1) commutativity: $T_\xi\, T_\eta - T_\eta\, T_\xi = 0$,

(2) W-equivariance: $w\, T_\xi = T_{w(\xi)}\, w$,

(3) homogeneity: T_ξ *is an operator of degree* -1 *with respect to the natural homogeneous grading on* $\mathcal{D}W$.

In view of Lemma 2.1, the assignment $\xi \mapsto T_\xi$ extends to an (injective) algebra homomorphism

$$\mathbb{C}[V^*] \hookrightarrow \mathcal{D}W\ , \quad q \mapsto T_q\ . \tag{2.4}$$

Identifying $\mathbb{C}[V^*]$ with its image in $\mathcal{D}W$ under (2.4), we now define the *rational Cherednik algebra* $H_k = H_k(W)$ as the subalgebra of $\mathcal{D}W$ generated by $\mathbb{C}[V]$, $\mathbb{C}[V^*]$ and $\mathbb{C}W$. The family $\{H_k\}$ can be viewed as a deformation (in fact, universal deformation) of the crossed product $H_0 = \mathcal{D}(V) * W$ (see [EG1], Theorem 2.16). The above realization of H_k inside $\mathcal{D}W$ is referred to as the *Dunkl representation* of H_k.

The algebra $\mathcal{D}W = \mathcal{D}(V_{\text{reg}}) * W$ carries a natural *differential* filtration, defined by taking $\deg(x) = 0$, $\deg(\xi) = 1$ and $\deg(w) = 0$ for all $x \in V^*$, $\xi \in V$ and $w \in W$. Through the Dunkl representation, this induces a filtration on H_k for all k, and the associated graded ring $\mathtt{gr}\, H_k$ is isomorphic to $\mathbb{C}[V \times V^*] * W$; in particular, it is independent of k. This implies the PBW property for H_k, i.e. a vector space isomorphism

$$H_k \xrightarrow{\sim} \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]\,. \tag{2.5}$$

By definition, the *spherical subalgebra* of H_k is given by $\boldsymbol{e}\, H_k\, \boldsymbol{e}$, where $\boldsymbol{e} = |W|^{-1} \sum_{w\in W} w$. For $k = 0$, we have $H_0 = \mathcal{D}(V) * W$ and $\boldsymbol{e}H_0\boldsymbol{e} \cong \mathcal{D}(V)^W$; thus, the family $\boldsymbol{e}H_k\boldsymbol{e}$ is a deformation (in fact, universal deformation) of the ring of invariant differential operators on V.

The Dunkl representation restricts to the embedding $\boldsymbol{e}H_k\boldsymbol{e} \hookrightarrow \boldsymbol{e}\mathcal{D}W\boldsymbol{e}$. If we combine this with (the inverse of) the isomorphism $\mathcal{D}(V_{\text{reg}})^W \xrightarrow{\sim} \boldsymbol{e}\,\mathcal{D}W\boldsymbol{e}$, $u \mapsto \boldsymbol{e}u\boldsymbol{e} = \boldsymbol{e}u = u\boldsymbol{e}$, we get an algebra map (cf. [H1])

$$\mathtt{Res}:\ \boldsymbol{e}H_k\boldsymbol{e} \hookrightarrow \mathcal{D}(V_{\text{reg}})^W\ , \tag{2.6}$$

representing the spherical subalgebra $\boldsymbol{e}H_k\boldsymbol{e}$ by invariant differential operators. We will refer to (2.6) as the *spherical Dunkl representation* and denote

$$B_k := \mathtt{Res}(\boldsymbol{e}H_k\boldsymbol{e}) \subset \mathcal{D}(V_{\text{reg}})^W\,. \tag{2.7}$$

The differential filtration on H_k induces filtrations on $\boldsymbol{e}H_k\boldsymbol{e}$ and B_k, with

$$\mathtt{gr}\, B_k = \mathbb{C}[V \times V^*]^W\,.$$

Theorem 2.2 ([H1])**.** *Let $\xi_1 \ldots, \xi_n$ be an orthonormal basis of V, and $q = \xi_1^2 + \cdots + \xi_n^2 \in \mathbb{C}[V^*]^W$. Then $\mathtt{Res}(\boldsymbol{e}\,T_q\,\boldsymbol{e}) = L_W$ is the Calogero–Moser operator* (2.2)*. Furthermore, the image of $\boldsymbol{e}\,\mathbb{C}[V^*]^W$ under the spherical Dunkl representation* (2.6) *forms a commutative subalgebra in $\mathcal{D}(V_{\text{reg}})^W$, and the operator L_W, thus, defines a quantum completely integrable system.*

Theorem 1.1 stated in the Introduction implies that if $k_\alpha \notin \mathbb{Z}$ for all α, then the commutative algebra constructed in Theorem 2.2 is a maximal (i.e. coincides with its centralizer) in $\mathcal{D}(V_{\text{reg}})^W$. On the other hand, when k_α's are integers, this algebra can be extended to a larger commutative algebra. This stronger property is known as *algebraic integrability* [CV1, VSC]. To state the result, let us make the following definition, cf. [CV1, VSC, FV1].

Definition 2.3. Let $\{\mathcal{A}, k\}$ be a configuration with $k_\alpha \in \mathbb{Z}_+$ for all $\alpha \in \mathcal{A}$. A polynomial $q \in \mathbb{C}[V]$ is called *quasi-invariant* if

$$q(x) - q(s_\alpha x) \text{ is divisible by } (\alpha, x)^{2k_\alpha} \quad \forall \alpha \in \mathcal{A}_+ . \tag{2.8}$$

The set of all quasi-invariant polynomials in $\mathbb{C}[V]$ is denoted by $Q_\mathcal{A}$. It is easy to check that $Q_\mathcal{A}$ is a subalgebra in $\mathbb{C}[V]$.

In the case when $\mathcal{A} = R$ is a root system of a Coxeter group W, we have $\mathbb{C}[V]^W \subset Q_\mathcal{A} \subset \mathbb{C}[V]$, so the algebra of quasi-invariants $Q_k(W) := Q_R$ interpolates between the invariants and $\mathbb{C}[V]$.

Remark 2.4. In the definition of $Q_\mathcal{A}$ one can replace $2k_\alpha$ by $2k_\alpha + 1$ in (2.8), because $q(x) - q(s_\alpha x)$ is skew-symmetric under s_α.

Consider the Calogero–Moser operator (2.2) with W-invariant multiplicities $k_\alpha \in \mathbb{Z}_+$ and write $L = L_W$, $L_0 = \Delta$.

Theorem 2.5. (1) *There exists a nonzero linear differential operator* $S \in \mathcal{D}(V_{\mathrm{reg}})$ *such that*

$$L\,S \;=\; S\,L_0\,.$$

Equivalently, $L\psi = (\lambda, \lambda)\psi$, *where* $\psi(\lambda, x) := Se^{(\lambda, x)}$, $\lambda \in V$.

(2) *For any quasi-invariant polynomial* $q \in Q_k(W)$ *there exists* $L_q \in \mathcal{D}(V_{\mathrm{reg}})$ *such that* $L_q\psi = q(\lambda)\psi$. *The operators* L_q, $q \in Q_k(W)$, *pairwise commute and the map* $q \mapsto L_q$ *defines an algebra embedding* $\theta : Q_k(W) \hookrightarrow \mathcal{D}(V_{\mathrm{reg}})$.

The first statement follows from the existence of the so-called shift operators, constructed explicitly (in terms of the Dunkl operators) in [H2]. Part (2) is the result of [VSC]. □

The above theorem admits a generalisation where root systems of Coxeter groups are replaced by more general systems of vectors called *locus configurations* [CFV2]. Their definition will be recalled in the next section, where we will also introduce a more general class of configurations for which a result analogous to Theorem 2.5 exists.

3. Generalised locus configurations

Let $(\mathcal{A}, k_\alpha)$ be a configuration of vectors with complex multiplicities in a (complex) Euclidean space V. We assume that the vectors of $\mathcal{A}$ are non-isotropic, i.e. $(\alpha, \alpha) \neq 0$ for all $\alpha \in \mathcal{A}$; the corresponding orthogonal reflections s_α can then be defined by the same formula as in the real case, see (2.1). We write $H_\mathcal{A} := \{H_\alpha\} \subset V$ for the collection of hyperplanes $H_\alpha := \mathtt{Ker}(1 - s_\alpha)$ with $\alpha \in \mathcal{A}$. As in the Introduction, we associate to $(\mathcal{A}, k_\alpha)$ the second order differential operator in $\mathcal{D}(V \backslash H_\mathcal{A})$:

$$L_\mathcal{A} = \Delta - u_\mathcal{A}\ , \qquad u_\mathcal{A} := \sum_{\alpha \in \mathcal{A}_+} \frac{k_\alpha(k_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}\,. \tag{3.1}$$

and recall from [CFV2] the following definition.

Definition 3.1 ([CFV2])**.** Assume that $k_\alpha \in \mathbb{Z}_+$ for all $\alpha \in \mathcal{A}$. The configuration $(\mathcal{A}, k_\alpha)$ is then called a *locus configuration* if for each $\alpha \in \mathcal{A}_+$, the function $u_\mathcal{A}$ in (3.1) satisfies the condition

$$u_\mathcal{A}(x) - u_\mathcal{A}(s_\alpha x) \quad \text{is divisible by} \quad (\alpha, x)^{2k_\alpha}\,. \tag{3.2}$$

Here, we say that a rational function f on V is divisible by $(\alpha, x)^{2k}$ if $(\alpha, x)^{-2k} f$ is regular at a generic point of the hyperplane H_α.

Explicitly, (3.2) can be described by the following set of equations [C1, CFV2]:

$$\sum_{\beta \in \mathcal{A}_+ \setminus \{\alpha\}} \frac{k_\beta(k_\beta+1)(\beta,\beta)(\alpha,\beta)^{2j-1}}{(\beta,x)^{2j+1}} = 0 \quad \text{for } (\alpha, x) = 0 \text{ and } j = 1, \ldots, k_\alpha. \tag{3.3}$$

Remark 3.2. An important feature of (3.3) is that it is sufficient for them to hold for any two-dimensional subconfiguration of $\mathcal{A}$. See [CFV2] for details.

Note that the root system of any Coxeter group W with W-invariant integral k_α obviously satisfies the condition (3.2): these are basic examples of locus configurations. There exist also many examples of locus configurations which do not arise from Coxeter groups (the so-called 'deformed root systems'); a complete classification of all such configurations is still an open problem: the list of all known examples can be found in [C2].

In this paper, we generalise the notion of locus configurations by allowing *some* of the multiplicities k_α to take non-integral values. The locus conditions (3.2) for the α's in $\mathcal{A}$ with non-integer k_α are simply replaced by the symmetry condition under s_α. More precisely,

Definition 3.3. Let $R \subset V$ be the root system of a finite Coxeter group W acting on V by reflections. A configuration $\mathcal{A}$ is called a *generalised locus configuration* of type W if

(1) $\mathcal{A}$ contains R, and both $\mathcal{A}$ and $k : \mathcal{A} \to \mathbb{C}$ are invariant under W;

(2) For any $\alpha \in \mathcal{A} \setminus R$ one has $k_\alpha \in \mathbb{Z}_+$ and the locus condition (3.2) holds.

Next, we introduce generalised quasi-invariant polynomials associated with generalised locus configurations.

Definition 3.4. Let $\mathcal{A}$ be a generalised locus configuration of type W. A polynomial $q \in \mathbb{C}[V]^W$ is called a *generalised quasi-invariant* if

$$q(x) - q(s_\alpha x) \text{ is divisible by } (\alpha, x)^{2k_\alpha} \quad \forall \alpha \in \mathcal{A}_+ \setminus R\,. \tag{3.4}$$

Write $Q_\mathcal{A}^W$ for the space of generalised quasi-invariants. It is easy to check that $Q_\mathcal{A}^W$ is a graded subalgebra of $\mathbb{C}[V]^W$.

In the trivial case $W = \{e\}$ and $R = \emptyset$ the above definitions reduce to the usual locus configurations and quasi-invariants $Q_\mathcal{A}$ as defined in the previous Section. Below we will always identify V and V^* using the bilinear form $(\cdot,\cdot)$, thus making no distinction between $\mathbb{C}[V]$ and $\mathbb{C}[V^*]$ and regarding $Q_{\mathcal{A}}^W$ interchangeably as a subalgebra in $\mathbb{C}[V]^W$ or $\mathbb{C}[V^*]^W$.

Before proceeding further, let us compare our class of configurations with those introduced by Sergeev and Veselov in [SV1]. One difference is that in [SV1] the authors assume that $k_\alpha = 1$ for $\alpha \in \mathcal{A} \setminus R$, while we allow arbitrary $k_\alpha \in \mathbb{Z}_+$. Furthermore, instead of the locus conditions they impose a condition, which they call "the main identity", see [SV1, (12)]. In the rational case this identity takes the following form:

$$\sum_{\alpha\neq\beta,\alpha,\beta\in\mathcal{A}_+} \frac{k_\alpha k_\beta(\alpha,\beta)}{(\alpha,x)(\beta,x)} = 0\,. \tag{3.5}$$

Going through the list of configurations in [SV1, Section 2], one can check that they all satisfy our definition (cf. a remark at the end of [SV1, Section 2]). However, as will become clear from the examples in Sections 7 and 8, there exist generalised locus configurations that do *not* fit in the axiomatics of [SV1]. Therefore, our approach covers a larger class of configurations.

With a generalised locus configuration $\mathcal{A}$ of type W we associate two quantum Hamiltonians, $L_0 := L_W \in \mathcal{D}(V_{\mathrm{reg}})^W$ and $L := L_{\mathcal{A}} \in \mathcal{D}(V_{\mathrm{reg}} \setminus H_{\mathcal{A}})^W$. By Theorem 2.2, L_0 is a member of a commutative family of higher-order Hamitonians $L_{q,0} := \mathtt{Res}(\boldsymbol{e}T_q\boldsymbol{e})$, $q \in \mathbb{C}[V^*]^W$. Our goal is to prove the following theorem that extends the main results of [CFV2] to generalised locus configurations.

Theorem 3.5. *Let $\mathcal{A}$ be a generalised locus configuration of type W.*

(1) *There exists a nonzero linear differential operator ('shift operator') $S \in \mathcal{D}(V_{\mathrm{reg}} \setminus H_{\mathcal{A}})^W$ such that $LS = SL_0$.*

(2) *Furthermore, for any homogeneous generalised quasi-invariant $q \in Q_{\mathcal{A}}^W$ there exists a differential operator L_q such that $L_qS = SL_{q,0}$ where $L_{q,0} = \mathtt{Res}(\boldsymbol{e}T_q\boldsymbol{e})$. The operators L_q pairwise commute and the map $q \mapsto L_q$ defines an algebra embedding $\theta : Q_{\mathcal{A}}^W \hookrightarrow \mathcal{D}(V_{\mathrm{reg}} \setminus H_{\mathcal{A}})^W$.*

(3) *The algebra $Q_{\mathcal{A}}^W$ has Krull dimension $n = \dim V$, i.e. it has n algebraically independent elements and so the quantum Hamiltonian $L_{\mathcal{A}}$ is completely integrable.*

(4) *The algebra $\theta(Q_{\mathcal{A}}^W)$ is a maximal commutative subalgebra in $\mathcal{D}(V_{\mathrm{reg}} \setminus H_{\mathcal{A}})^W$.*

We will prove Theorem 3.5 in Section 5 after we develop an appropriate algebraic framework in the next section.

4. Shift Operators

In this section, we develop an abstract algebraic approach to the problem of constructing differential 'shift' operators. Our main result — Theorem 4.3 — provides necessary and sufficient conditions for the existence of such operators under very general assumptions. In the next section, we will verify these conditions for generalised Calogero-Moser operators, thus deducing our main Theorem 3.5 from Theorem 4.3. Our approach originates from an attempt to understand examples and unify various *ad hoc* constructions of shift operators known in higher dimension ($n > 1$). Some of the ideas go back to old observations of the authors in [B] and [C1]; our main innovation is in clarifying the role of localisation and ad-nilpotency (see Lemma 4.1) as well as the use of a canonical ad-nilpotent filtration (Lemma 4.2) that allows us to state and prove Theorem 4.3 in abstract 'coordinate-free' terms .

4.1. **Existence of shift operators.** Throughout this section, k will denote a fixed field of characteristic zero, and all rings will be k-algebras with 1. If A and B are two rings and M is an A-B-bimodule, we consider M as a left $(A \otimes B^{\circ})$-module, with an element $a \otimes b \in A \otimes B^{\circ}$ acting on $m \in M$ by $(a \otimes b) \cdot m := amb$. Then, for $a \in A$ and $b \in B$, we say that the pair (a, b) acts on M *locally ad-nilpotently* if $a \otimes 1 - 1 \otimes b \in A \otimes B^{\circ}$ acts locally nilpotently: i.e., for every $m \in M$, there is $n > 0$ such that $(a \otimes 1 - 1 \otimes b)^n \cdot m = 0$. We will use the following notation for this action:

$$\mathtt{ad}_{a,b}(m) := (a \otimes 1 - 1 \otimes b) \cdot m = am - mb$$

Note that, using the binomial formula, we can write the elements $\mathtt{ad}^n_{a,b}(m) := (a \otimes 1 - 1 \otimes b)^n \cdot m$ explicitly for all $n \geq 0$:

$$\mathtt{ad}^n_{a,b}(m) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^k m\, b^{n-k} \tag{4.1}$$

When $A = B$ and $a = b$, this becomes the adjoint action, in which case we use the standard notation $\mathtt{ad}_a$ instead of $\mathtt{ad}_{a,a}$; we say that a acts locally ad-nilpotently on M if so does (a, a). We call an element of a ring *locally ad-nilpotent* if it acts locally ad-nilpotently on the ring viewed as a bimodule.

We begin with a general lemma from noncommutative algebra.

Lemma 4.1. *Let B be a noncommutative integral domain, $S \subset B$ a two-sided Ore subset in B, and $A := B[S^{-1}]$ the corresponding ring of fractions. Let $L_0 \in B$ be a locally ad-nilpotent element in B. Then, for $L \in A$, the following conditions are equivalent:*

- (a) *there exists a nonzero $D \in A$ such that $L\, D = D\, L_0$ in A ,*
- (b) *there exists a nonzero $D^* \in A$ such that $D^* L = L_0\, D^*$ in A ,*
- (c) *there exists $\delta \in S$ such that $\mathtt{ad}^{N+1}_{L,L_0}(\delta) = 0$ in A for some $N \geq 0$.*

Proof. The implication $(c) \Rightarrow (a)$ is immediate: if (c) holds, choose the smallest $N \geq 0$ such that $\mathtt{ad}^{N+1}_{L,L_0}(\delta) = 0$, then $D := \mathtt{ad}^{N}_{L,L_0}(\delta) \neq 0$ satisfies (a).

Now, assume that (b) holds. Since S is a right Ore subset in B, for $D^* \in B[S^{-1}]$, there is $\delta \in S$ such that $D^*\delta \in B$. Then, by (4.1), we have

$$\begin{aligned} D^*\,\mathtt{ad}^n_{L,L_0}(\delta) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\,D^*L^k\delta\,L_0^{n-k} \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\,L_0^kD^*\delta\,L_0^{n-k} = \mathtt{ad}^n_{L_0}(D^*\delta)\ , \quad \forall\, n\geq 0\,. \end{aligned}$$

Since L_0 acts locally ad-nilpotently on B, there is $N \geq 0$ such that $\mathtt{ad}^{N+1}_{L_0}(D^*\delta) = 0$. Since $A = B[S^{-1}]$ is a domain and $D^* \neq 0$, the above formula implies $\mathtt{ad}^{N+1}_{L,L_0}(\delta) = 0$. This proves $(b) \Rightarrow (c)$.

Finally, assume that (a) holds. Since S is a left Ore subset in B, for $D \in B[S^{-1}]$, there is $\delta^* \in S$ such that $\delta^*D \in B$. Then, by (4.1), we have $\mathtt{ad}^n_{L_0,L}(\delta^*)\,D = \mathtt{ad}^n_{L_0}(\delta^*D)$ for all $n \geq 0$, which implies that $\mathtt{ad}^n_{L_0,L}(\delta^*) = 0$ for $n \gg 0$. Taking the smallest $N \geq 0$ such that $\mathtt{ad}^{N+1}_{L_0,L}(\delta^*) = 0$, we put $D^* := \mathtt{ad}^N_{L_0,L}(\delta^*) \neq 0$. This satisfies $L_0\,D^* = D^*L$, proving the last implication $(a) \Rightarrow (b)$. □

In this paper, we are concerned with algebraic differential operators. To proceed further we therefore make the following general assumption.

(A) *An algebra B contains a commutative Noetherian domain R with a multiplicative closed subset S and the quotient field $\mathbb{K}$ such that*

$$S \subset R \subset B \subset \mathcal{D}(\mathbb{K})\ ,$$

where $\mathcal{D}(\mathbb{K})$ is the ring of k-linear algebraic differential operators on $\mathbb{K}$, with $R \subset \mathcal{D}(\mathbb{K})$ being the natural inclusion.

By assumption (A), we may think of elements of the algebra B as usual 'partial differential operators with rational coefficients'. More precisely, since R is Noetherian, by Noether's Normalization Lemma, we can choose finitely many algebraically independent elements in R, say $x_1, \ldots, x_n$, so that the quotient field $\mathbb{K}$ of R is a finite extension of $k(x_1, \ldots, x_n)$. The module $\mathtt{Der}_k(\mathbb{K})$ of k-linear derivations of $\mathbb{K}$ is then freely generated (as a $\mathbb{K}$-module) by the 'partial derivatives' $\partial/\partial x_i : \mathbb{K} \to \mathbb{K}$, and the ring $\mathcal{D}(\mathbb{K})$ can be identified as

$$\mathcal{D}(\mathbb{K}) \cong \mathbb{K}\,[\partial/\partial x_1, \ldots, \partial/\partial x_n]\ .$$

Note that (A) formally implies the assumptions of Lemma 4.1. Indeed, $\mathcal{D}(\mathbb{K})$ is a noncommutative domain (see, e.g., [MR, Theorem 15.5.5]); hence, being a subalgebra of $\mathcal{D}(\mathbb{K})$, B must be a domain as well. Furthermore, since $S \subset \mathbb{K}$, the elements of S are represented by zero order differential operators on $\mathbb{K}$ which act, by definition, locally ad-nilpotently on $\mathcal{D}(\mathbb{K})$ (and hence *a fortiori* on B). It follows that S is a two-sided Ore subset. Note that the elements of S are actually units in $\mathcal{D}(\mathbb{K})$, hence, by the universal property of Ore localisation, the inclusion $B \hookrightarrow \mathcal{D}(\mathbb{K})$ extends to $A := B[S^{-1}]$: thus, if (A) holds, we have

$$S \subset R \subset B \subset A \subset \mathcal{D}(\mathbb{K}) \ .$$

Let L_0 be a locally ad-nilpotent element in an algebra B. Following [BW], we associate to L_0 a (positive increasing) filtration on B:

$$F_0 B \subseteq F_1 B \subseteq \ldots \subseteq F_n B \subseteq F_{n+1} B \subseteq \ldots \subseteq B$$

which is defined by induction:

$$F_{-1}B := \{0\} \ , \quad F_{n+1}B := \{b \in B \ : \ \mathtt{ad}_{L_0}(b) \in F_n B\} \ , \tag{4.2}$$

or equivalently,

$$F_n B := \{b \in B \ : \ \mathtt{ad}_{L_0}^{n+1}(b) = 0\} \quad \text{for all } n \ .$$

Since $\mathtt{ad}_{L_0}$ is a locally nilpotent derivation, $\{F_* B\}$ is an exhaustive filtration on B satisfying $(F_n B)\cdot(F_m B) \subseteq F_{n+m}B$ for all $n, m \geq 0$. Note that $F_0 B = C_B(L_0)$ is the centralizer of L_0, which is a (not necessarily commutative) subalgebra of B.

Associated to (4.2) is the degree (valuation) function $\deg_{L_0} : B\backslash\{0\} \to \mathbb{N}$ defined by

$$\deg_{L_0}(b) \ := \ n \quad \text{iff} \quad b \in F_n B \setminus F_{n-1}B \ , \ n \geq 0 \, . \tag{4.3}$$

Note that $\deg_{L_0}(b) = n$ whenever $\mathtt{ad}_{L_0}^{n+1}(b) = 0$ while $\mathtt{ad}_{L_0}^{n}(b) \neq 0$ in B. It is convenient to extend $\deg_{L_0}$ to the whole B by setting $\deg_{L_0}(0) := -\infty$, so that $F_n B = \{b \in B \ : \ \deg_{L_0}(b) \leq n\}$ for all n.

The next lemma shows that, under assumption (A), the above filtration and the associated degree function on B extend to the localised algebra $B[S^{-1}]$.

Lemma 4.2. *Assume that* (A) *holds. Let L_0 be a locally ad-nilpotent element in B with degree function* $\deg_{L_0} : B \to \mathbb{N} \cup \{-\infty\}$*. Then, there is a unique function* $\deg : A \to \mathbb{Z} \cup \{-\infty\}$ *on $A = B[S^{-1}]$ with the following properties*[2]*:*

(0) $\deg(b) = \deg_{L_0}(b), \ \forall\, b \in B$,
(1) $\deg\,(a_1 a_2) = \deg(a_1) + \deg(a_2), \ \forall\, a_1, a_2 \in A$,
(2) $\deg\,(a_1 + a_2) = \max\{\deg(a_1), \deg(a_2)\}, \ \forall\, a_1, a_2 \in A$,
(3) $\deg\,[\mathtt{ad}_{L_0}(a)] \leq \deg(a) - 1, \ \forall\, a \in A$.

Proof. First, observe that the properties (1), (2), (3) hold for the function $\deg_{L_0}$ on B. Indeed, for $\deg_{L_0}$, property (2) is immediate from the definition (4.3), while (3) follows from the inductive construction of the filtration (4.2).

[2]Just as the function $\deg_{L_0}$ on B, its extension to A depends on the ad-nilpotent element L_0. To distinguish between these two degree functions we suppress the dependence of 'deg' on L_0 in our notation.

To verify (1) take two elements $b_1, b_2 \in B$ with $\deg_{L_0}(b_1) = n_1 \geq 0$ and $\deg_{L_0}(b_2) = n_2 \geq 0$. Then, by (4.3),

$$\mathtt{ad}_{L_0}^{n_1+1}(b_1) = \mathtt{ad}_{L_0}^{n_2+1}(b_2) = 0 \ , \tag{4.4}$$

while $\mathtt{ad}_{L_0}^{n_1}(b_1) \neq 0$ and $\mathtt{ad}_{L_0}^{n_2}(b_2) \neq 0$. Since $\mathtt{ad}_{L_0}$ is a derivation on B, by Leibniz rule, we have

$$\mathtt{ad}_{L_0}^{n}(b_1 b_2) \ = \ \sum_{k=0}^{n} \binom{n}{k} \mathtt{ad}_{L_0}^{k}(b_1)\, \mathtt{ad}_{L_0}^{n-k}(b_2) \ .$$

for all $n \geq 0$. In view of (4.4), for $n = n_1 + n_2 + 1$, the above formula implies $\mathtt{ad}_{L_0}^{n_1+n_2+1}(b_1 b_2) = 0$, while, for $n = n_1 + n_2$,

$$\mathtt{ad}_{L_0}^{n_1+n_2}(b_1 b_2) \ = \ \frac{(n_1+n_2)!}{n_1!\, n_2!}\, \mathtt{ad}_{L_0}^{n_1}(b_1)\, \mathtt{ad}_{L_0}^{n_2}(b_2) \ .$$

Since B is a domain, the last equation shows that $\mathtt{ad}_{L_0}^{n_1+n_2}(b_1 b_2) \neq 0$, which means that $\deg_{L_0}(b_1 b_2) = n_1 + n_2$, or equivalently,

$$\deg_{L_0}(b_1 b_2) = \deg_{L_0}(b_1) + \deg_{L_0}(b_2) \ . \tag{4.5}$$

Now, we define the function $\deg : A \backslash \{0\} \to \mathbb{Z}$ by

$$\deg(s^{-1} b) := \deg_{L_0}(b) - \deg_{L_0}(s) \tag{4.6}$$

where $s^{-1}b \in A$ with $s \in S$ and $b \in B$. To see that this definition makes sense consider two different presentations of an element in A by (left) fractions: say $a = s_1^{-1} b_1 = s_2^{-1} b_2$ with $s_1, s_2 \in S$ and $b_1, b_2 \in B$. Since S is commutative (by assumption (A)), we have $s_2 b_1 = s_1 b_2$ in B, which, by (4.5), implies

$$\deg_{L_0}(s_2) + \deg_{L_0}(b_1) = \deg_{L_0}(s_1) + \deg_{L_0}(b_2) \ .$$

Whence

$$\deg(s_1^{-1} b_1) := \deg_{L_0}(b_1) - \deg_{L_0}(s_1) = \deg_{L_0}(b_2) - \deg_{L_0}(s_2) = \deg(s_2^{-1} b_2) \ ,$$

as required. Note that the same argument shows that

$$\deg(b s^{-1}) \ = \ \deg_{L_0}(b) - \deg_{L_0}(s) \ = \ \deg(s^{-1} b) \tag{4.7}$$

for all $b \in B$ and $s \in S$.

Now, with definition (4.6), the property (0) of Lemma 4.2 is obvious. To prove (1) write elements a_1 and a_2 in A as left and right fractions: $a_1 = s_1^{-1} b_1$ and $a_2 = b_2 s_2^{-1}$, and use (4.7) to conclude:

$$\begin{aligned}
\deg(a_1 a_2) \ &= \ \deg(s_1^{-1} b_1 b_2 s_2^{-1}) \\
&= \ \deg_{L_0}(b_1 b_2) - \deg_{L_0}(s_1) - \deg_{L_0}(s_2) \\
&= \ \deg_{L_0}(b_1) + \deg_{L_0}(b_2) - \deg_{L_0}(s_1) - \deg_{L_0}(s_2) \\
&= \ [\,\deg_{L_0}(b_1) - \deg_{L_0}(s_1)\,] + [\,\deg_{L_0}(b_2) - \deg_{L_0}(s_2)\,] \\
&= \ \deg(a_1) + \deg(a_2) \ .
\end{aligned}$$

Note that property (1) implies formally that $\deg(s^{-1}) = -\deg(s)$ for all $s \in S$; together with (0), it entails (4.6), and hence the uniqueness of the function 'deg'.

To prove (2) take $a_1 = s_1^{-1}b_1,\ a_2 = s_2^{-1}b_2$ in A and assume (without loss of generality) that $\deg(a_1) \ge \deg(a_2)$. Note that, by (4.5) and (4.6), this last condition is equivalent to

$$\deg_{L_0}(s_2 b_1) \ge \deg_{L_0}(s_1 b_2) \tag{4.8}$$

Now, using the fact that (2) holds for the degree function $\deg_{L_0}$, we check

$$\begin{aligned}
\deg(a_1 + a_2) &= \deg[(s_1 s_2)^{-1}(s_2 b_1 + s_1 b_2)] \\
&= \deg_{L_0}(s_2 b_1 + s_1 b_2) - \deg_{L_0}(s_1) - \deg_{L_0}(s_2) \\
&\le \max\{\deg_{L_0}(s_2 b_1), \deg_{L_0}(s_1 b_2)\} - \deg_{L_0}(s_1) - \deg_{L_0}(s_2) \\
&= \deg_{L_0}(s_2 b_1) - \deg_{L_0}(s_1) - \deg_{L_0}(s_2) \qquad [\text{by } (4.8)\,] \\
&= \deg_{L_0}(b_1) - \deg_{L_0}(s_1) = \deg(a_1) \\
&= \max\{\deg(a_1),\ \deg(a_2)\}\ .
\end{aligned}$$

Finally, to prove (3) we take $a = s^{-1}b \in A$ and write $\mathtt{ad}_{L_0}(a)$ in the form

$$\mathtt{ad}_{L_0}(s^{-1}b) = s^{-1}\mathtt{ad}_{L_0}(b) - s^{-1}\mathtt{ad}_{L_0}(s)\, s^{-1}b$$

Since

$$\deg[s^{-1}\mathtt{ad}_{L_0}(b)] = \deg_{L_0}[\mathtt{ad}_{L_0}(b)] - \deg_{L_0}(s) \le \deg_{L_0}(b) - \deg_{L_0}(s) - 1$$

and similarly

$$\deg[s^{-1}\mathtt{ad}_{L_0}(s)\, s^{-1}b] \le \deg_{L_0}(b) - \deg_{L_0}(s) - 1\ ,$$

by property (2) we conclude

$$\deg[\mathtt{ad}_{L_0}(a)] \le \deg_{L_0}(b) - \deg_{L_0}(s) - 1 = \deg(a) - 1\ .$$

This completes the proof of the lemma. □

Using the degree function of Lemma 4.2, we can extend the filtration (4.2) on the algebra B to a $\mathbb{Z}$-filtration on the algebra A:

$$F_n A := \{a \in A\ :\ \deg(a) \le n\}\ , \quad \forall\, n \in \mathbb{Z}\,.$$

We write $\mathtt{gr}(A) := \oplus_{n\in\mathbb{Z}} F_n A/F_{n-1}A$ for the associated graded ring, and for each $n \in \mathbb{Z}$, denote by $\sigma_n : F_n A \twoheadrightarrow F_n A/F_{n-1}A \hookrightarrow \mathtt{gr}(A)$ the *symbol map of degree n*. By definition, for $a \in F_n A$, the symbol $\sigma_n(a) = a + F_{n-1}A$ is nonzero if and only if $\deg(a) = n$. For example, we have $\sigma_0(L_0) = L_0 + F_{-1}A$, since $\deg(L_0) = 0$.

We can now state the main result of this section.

Theorem 4.3. *Assume that an algebra B satisfies condition* (A). *In addition, assume that there is an R-submodule $U_0 \subseteq \mathbb{K}$ such that $B = \{a \in A : a[U_0] \subseteq U_0\}$, where $A := B[S^{-1}]$. Let L_0 be a locally ad-nilpotent operator in B. Then, for an operator $L \in A$, there is a nonzero operator $D \in A$ such that*

$$L\,D = D\,L_0 \tag{4.9}$$

if and only if the following conditions hold:

(1) *there is a* k*-linear subspace* $U \subseteq \mathbb{K}$ *such that*
   a) U *is stable under* L*, i.e.* $L[U] \subset U$,
   b) $s\,U_0 \subseteq U \subseteq s^{-1}U_0$ *for some* $s \in S$,

(2) $\sigma_0(L) = \sigma_0(L_0)$ (*in particular,* $\deg(L) = \deg(L_0) = 0$).

Given a subspace $U \subseteq \mathbb{K}$ *satisfying condition* (1b)*, there is at most one operator* $L \in A$ *satisfying* (1a) *and* (2) (*and hence the identity* (4.9)).

Proof. First, we prove that conditions (1) and (2) are sufficient for the existence of D. To this end, we consider the space of all operators in A mapping U_0 to U:

$$\mathcal{M} := \{a \in A \ : \ a[U_0] \subseteq U\} \ .$$

Note that $\mathcal{M}$ is a right B-module which, by (1b), contains the ideal sB and is contained in $s^{-1}B$:

$$sB \subseteq \mathcal{M} \subseteq s^{-1}B \ . \tag{4.10}$$

On the other hand, by (1a), $\mathcal{M}$ is closed under the action of the operator $\mathtt{ad}_{L,L_0}$. We claim that this last operator acts on $\mathcal{M}$ locally nilpotently. Indeed, by Lemma 4.2, it follows from the inclusion $\mathcal{M} \subseteq s^{-1}B$ in (4.10) that

$$\deg(a) \geq -\deg(s) \quad \text{for all } \ a \in \mathcal{M}\backslash\{0\} \ . \tag{4.11}$$

On the other hand, letting $P := L - L_0 \in A$, we can write

$$\mathtt{ad}_{L,L_0}(a) = \mathtt{ad}_{L_0}(a) + Pa \ .$$

By condition (2), we have $\deg(P) \leq -1$, and hence

$$\deg(Pa) = \deg(P) + \deg(a) \leq \deg(a) - 1$$

for all $a \in A$. Then, by Lemma 4.2,

$$\deg[\mathtt{ad}_{L,L_0}(a)] \leq \max\{\deg[\mathtt{ad}_{L_0}(a)], \deg(Pa)\} \leq \deg(a) - 1 \tag{4.12}$$

Now, by (4.10), any element $a \in \mathcal{M}$ can be written in the form $a = s^{-1}b$ with $b \in B$. If we take $N = \deg(b)$, then, for $a = s^{-1}b$, (4.12) implies by induction

$$\deg[\mathtt{ad}^{N+1}_{L,L_0}(a)] \leq \deg(a) - N - 1 = \deg(b) - \deg(s) - N - 1 = -\deg(s) - 1 \ .$$

In view of (4.11), for $a \in \mathcal{M}$, this means that $\deg[\mathtt{ad}^{N+1}_{L,L_0}(a)] = -\infty$, i.e. $\mathtt{ad}^{N+1}_{L,L_0}(a) = 0$. Thus $\mathtt{ad}_{L,L_0}$ acts on $\mathcal{M}$ locally nilpotently. Now, since $s \in \mathcal{M}$ by (1b), we have $\mathtt{ad}^{N+1}_{L,L_0}(s) = 0$ with $N = 2\deg(s)$. This implies the existence of D by Lemma 4.1.

Conversely, suppose that there is $D \neq 0$ in A such that $L\,D = L_0\,D$. This last equation can be rewritten in the form $\mathtt{ad}_{L_0}(D) = -PD$, where $P := L - L_0$. Hence, by Lemma 4.2,

$$\deg(P) + \deg(D) = \deg(PD) = \deg[\mathtt{ad}_{L_0}(D)] \leq \deg(D) - 1 \ ,$$

which implies $\deg(P) \leq -1$. Thus (2) holds.

To construct a subspace $U \subseteq \mathbb{K}$ satisfying condition (1) we apply Lemma 4.1. According to this lemma, there is an element $\delta \in S$ such that $\mathtt{ad}_{L,L_0}^{N+1}(\delta) = 0$ for some $N \geq 0$. We put $D_k := \mathtt{ad}_{L,L_0}^{k}(\delta)$ for $k = 0, 1, 2, \ldots, N+1$, with $D_{N+1} = 0$, and define U to be the smallest subspace of $\mathbb{K}$ that contains the images of U_0 under the D_k's for all k: i.e.,

$$U := \sum_{k=0}^{N} D_k[U_0] \subseteq \mathbb{K} .$$

Since $D_{k+1} = LD_k - D_k L_0$, we have

$$LD_k[U_0] = D_{k+1}[U_0] + D_k L_0[U_0] \subseteq D_{k+1}[U_0] + D_k[U_0] \subseteq U$$

for $k = 0, 1, \ldots, N$. Hence $L[U] \subseteq U$, which is condition (1a). To prove (1b) note that, by construction, all the D_k's are in A, hence there are elements $\delta_k \in S$ such that $\delta_k D_k \in B$ for all k. Put $s := \delta\, \delta_1 \ldots \delta_N \in S$. Then

$$s\, U = \sum_{k=0}^{N} s\, D_k[U_0] \subseteq \sum_{k=0}^{N} B[U_0] = U_0 .$$

On the other hand, since U_0 is an R-module and $S \subset R$, we have

$$s\, U_0 = \delta\, (\delta_1 \ldots \delta_N U_0) \subseteq \delta\, U_0 = D_0[U_0] \subseteq U .$$

Thus, $s\, U_0 \subseteq U \subseteq s^{-1} U_0$ for $s \in S$, which is the required condition (1b).

To prove the last claim of the theorem consider two operators L_1 and L_2 in A satisfying (1a) and (2) for a given subspace U which satisfies (1b). Put $P := L_1 - L_2$. Then, by (1a), $P[U] \subseteq U$, while by (1b) and Lemma 4.2,

$$\deg(P) = \deg(L_1 - L_0 + L_0 - L_2) \leq \max\{\deg(L_1 - L_0), \deg(L_2 - L_0)\} < 0 .$$

The first condition implies that $P \in \mathrm{End}_B(\mathcal{M})$ so that $Pa \in \mathcal{M}$ for all $a \in \mathcal{M}$, while by the second, $\deg(Pa) < \deg(a)$. Taking $a \neq 0$ to be of minimal degree in $\mathcal{M}$, we conclude $Pa = 0$ which means that $P = 0$ or equivalently $L_1 = L_2$. This finishes the proof of the theorem. □

Remark 4.4. Note that an operator $L \in A$ satisfies condition (2) of Theorem 4.3 if and only if $L = L_0 + P$ with $\deg(P) < 0$. By Lemma 4.2, the last inequality holds for $P \in A$ iff there is an $s \in S$ and $n \geq 0$ such that $sP \in B$ and $\mathtt{ad}_{L_0}^{n}(sP) = 0$, while $\mathtt{ad}_{L_0}^{n}(s) \neq 0$. In practice, these conditions are easily verifiable. In applying Theorem 4.3 the main problem is to verify condition (1).

Remark 4.5. Under the assumptions of Theorem 4.3, for an operator L in B, the identity $LD = DL_0$ may hold (with nonzero $D \in \mathcal{D}(\mathbb{K})$) if and only if $L = L_0$. This follows from the last claim of the theorem.

Remark 4.6. Theorem 4.3 extends naturally to the case when a single ad-nilpotent operator $L_0 \in B$ is replaced by an abelian ad-nilpotent family $\mathcal{C}_0 \subset B$ (in the sense of [BW]). The filtration $F_* B$ is defined in this case by $F_{n+1}B := \{b \in B \,:\, \mathtt{ad}_{L_0}(b) \in F_n B \text{ for all } L_0 \in \mathcal{C}_0\}$, and the associated degree function on B determines — under the assumption (A) — a degree function 'deg' on $A = B[S^{-1}]$ with the same properties as in Lemma 4.2. The generalisation of Theorem 4.3 says that, for a family of operators $\mathcal{C} \subset A$, there is a nonzero $D \in A$ such that $\mathcal{C}\,D = D\,\mathcal{C}_0$ if and only if conditions (1) and (2) hold for all $L \in \mathcal{C}$. The family $\mathcal{C}$ is then necessarily abelian, and the algebra generated by $\mathcal{C}$ in A is a commutative ad-nilpotent subalgebra of $\mathrm{End}_B(\mathcal{M})$. We will construct examples of such subalgebras in Section 4.3 below.

We give a few basic examples of algebras of differential operators satisfying the assumptions of Theorem 4.3.

Example 4.7. Let V be a finite-dimensional vector space over $\mathbb{C}$. Take $R = \mathbb{C}[V]$ to be the algebra of polynomial functions on V, and $B = \mathcal{D}(V)$ the ring of differential operators on $U_0 = R = \mathbb{C}[V]$. Then $B \cong A_n(\mathbb{C})$, where $A_n(\mathbb{C}) = \mathbb{C}[x_1, \dots, x_n; \partial_1, \dots, \partial_n]$ is the n-th Weyl algebra with $n = \dim(V)$. The algebra $A_n(\mathbb{C})$ contains the commutative subalgebra $\mathbb{C}[\partial_1, \dots, \partial_n]$ of constant coefficient differential operators $L_0 = P(\partial_1, \dots, \partial_n)$ which act locally ad-nilpotently on B. In the one-dimensional case ($n = 1$), there is a well-known inductive construction of shift operators, using the classical Darboux transformations, that works for an arbitrary L_0. This elementary construction does not extend to higher dimensions: for $n > 1$, only some ad hoc constructions and a few explicit examples are known (see, e.g., [B], [BK], [BCM], [C1], [CFV1], [CFV2] and references therein).

Example 4.8. Let $B = B_k(W)$ be the spherical Cherednik algebra associated to a finite Coxeter group W acting in its reflection representation V. Take $R = \mathbb{C}[V]^W$ and $U_0 = \mathbb{C}[V_{\mathrm{reg}}]^W$, where V_{reg} is the (open) subvariety of V (obtained by removing the reflection hyperplanes of W) on which W acts freely (see Section 2) . It is well-known that B contains a maximal commutative subalgebra of W-invariant differential operators $L_{q,0} = \mathrm{Res}(e\,T_q\,e)$ associated to $q \in \mathbb{C}[V^*]^W$ that act locally ad-nilpotently on B (see, e.g., [BEG]). The Calogero-Moser operator L_W defined by (1.3) is a special example of the $L_{q,0}$ corresponding to the quadratic polynomial $q = |\xi|^2$ (*cf.* Theorem 2.2). The generalised Calogero-Moser operators $L_{\mathcal{A}}$ given by (1.1) are examples of the operators L related to $L_0 = L_W$ by a shift operator in a properly localised Cherednik algebra; in the next section, we will describe the subspaces $U = U_{\mathcal{A}}$ associated to these operators explicitly in terms of locus conditions. This is the main example of the present paper.

Example 4.9. Let G be a complex connected reductive algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Take $B = \mathcal{D}(\mathfrak{g})^G$ to be the ring of invariant polynomial differential operators on $\mathfrak{g}$ with the respect to the natural (adjoint) action of G on $\mathfrak{g}$. The algebra B contains the subalgebra $R = \mathbb{C}[\mathfrak{g}]^G$ of invariant polynomial functions on $\mathfrak{g}$ and acts naturally on $U_0 = \mathbb{C}[\mathfrak{g}_{\mathrm{reg}}]^G$, where $\mathfrak{g}_{\mathrm{reg}} \subset \mathfrak{g}$ is the (open) subvariety of regular semisimple elements of $\mathfrak{g}$ on which G acts freely. Moreover, B contains the commutative subalgebra

$\mathbb{C}[\mathfrak{g}^*]^G$ of constant coefficient invariant differential operators L_0 which act locally ad-nilpotently on B. A special example of such an L_0 is the second order Laplace operator $\Delta_{\mathfrak{g}}$ defined for a G-invariant metric on $\mathfrak{g}$. Applications of Theorem 4.3 to this example seems to deserve a separate study. Of particular interest is a relation to the previous example: specifically, the question whether the generalised Calogero-Moser operators constructed in this paper can be obtained via the (properly localised) deformed Harish-Chandra map $\Phi_k : \mathcal{D}(\mathfrak{g})^G \to B_k(W)$ constructed in [EG1]?

4.2. **Morita context.** We return to the situation of Theorem 4.3. We take an operator L satisfying conditions (1) and (2) of the theorem, fix a subspace $U \subseteq \mathbb{K}$ corresponding to L and consider the module $\mathcal{M}$ of all operators in A mapping U_0 to U (as defined in the proof of Theorem 4.3). This last module has some interesting algebraic properties that we will describe next.

First, we remark that the subspace U satisfying condition (1) of Theorem 4.3 is not uniquely determined by L. However, given two such subspaces, say U_1 and U_2, their sum $U_1 + U_2$ also satisfies (1). Indeed, if $s_1U_0 \subseteq U_1 \subseteq s_1^{-1}U_0$ and $s_2U_0 \subseteq U_2 \subseteq s_2^{-1}U_0$, then $sU_0 \subseteq U_1 + U_2 \subseteq s^{-1}U_0$ for $s = s_1s_2 \in S$, while obviously $L[U_1 + U_2] \subseteq U_1 + U_2$ whenever $L[U_1] \subseteq U_1$ and $L[U_2] \subseteq U_2$. This implies that the poset of all subspaces $U \subseteq \mathbb{K}$ satisfying (1) has *at most* one maximal element — the largest subspace $U_{\max}$. We will see that in our basic example — for the operator $L = L_{\mathcal{A}}$ associated to a generalised locus configuration — such a subspace always exists (Lemma 5.1). In what follows, we will therefore study the module $\mathcal{M}$ for the maximal subspace $U = U_{\max}$, assuming that the latter exists.

Next, we recall a few basic definitions from noncommutative algebra. For a right B-module $\mathcal{M}$, we will denote by $\mathcal{M}^* := \mathrm{Hom}_B(\mathcal{M}, B)$ its dual, which is naturally a left B-module (via left multiplication of B on itself). Applying the Hom-functor twice, we get the double dual $\mathcal{M}^{**} := \mathrm{Hom}_B(\mathrm{Hom}_B(\mathcal{M}, B), B)$, which is a right B-module equipped with a canonical map $\mathcal{M} \to \mathcal{M}^{**}$. A B-module $\mathcal{M}$ is called *reflexive* if the canonical map $\mathcal{M} \to \mathcal{M}^{**}$ is an isomorphism. It is easy to see that every f.g. projective (in particular, free) module is reflexive but, in general, a reflexive module need not be projective. If B is a Noetherian domain, we write $\mathcal{Q} = \mathcal{Q}(B)$ for the quotient skew-field[3] of B, and call $\mathcal{M}$ a *fractional ideal* if $\mathcal{M}$ is a right submodule of $\mathcal{Q}$ such that $pB \subseteq \mathcal{M} \subseteq qB$ for some nonzero $p, q \in \mathcal{Q}$. Furthermore, if B is a Noetherian domain satisfying our condi tion (A), then $B \subseteq \mathcal{D}(\mathbb{K}) \subseteq \mathcal{Q}$; in this case, we call a fractional ideal $\mathcal{M}$ of B *fat* if $\mathcal{M} \subseteq \mathcal{D}(\mathbb{K})$ and $\mathcal{M} \cap \mathbb{K} \neq \{0\}$. Finally, we recall the definition of the B-module $\mathcal{M}$ from Theorem 4.3:

$$\mathcal{M} := \{a \in A \ : \ a[U_0] \subseteq U\} \ , \tag{4.13}$$

[3]Recall, for a (left and/or right) Noetherian domain B, the set $S = B\backslash\{0\}$ of all nonzero elements of B satisfies a (left and/or right) Ore condition (Goldie's Theorem), and the quotient skew-field $\mathcal{Q}(B)$ is obtained in this case by Ore localisation $B[S^{-1}]$.

and in a similar fashion, we define the ring

$$\mathcal{D} := \{a \in A \ : \ a[U] \subseteq U\} \ . \tag{4.14}$$

Proposition 4.10. *Assume that the algebra B, the operators $L_0 \in B$ and $L \in A$ satisfy the assumptions of Theorem 4.3. In addition, assume that B is Noetherian and the subspace $U \subseteq \mathbb{K}$ associated to L by Theorem 4.3 is maximal. Then*

(*a*) *$\mathcal{M}$ is a fat reflexive fractional ideal of B;*

(*b*) *$\mathcal{D} \cong \mathrm{End}_B(\mathcal{M})$, where $\mathrm{End}_B(\mathcal{M})$ is the endomorphism ring of $\mathcal{M}$.*

Proof. (*a*) Note that $\mathcal{M}$ being a fractional right ideal of B follows immediately from condition (1b) of Theorem 4.3: see (4.10). We need only to prove that $\mathcal{M}$ is reflexive. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two fractional (right) ideals of B, we can identify (see [MR, 3.1.15]):

$$\mathrm{Hom}_B(\mathcal{M}_1, \mathcal{M}_2) \cong \{q \in \mathcal{Q} \ : \ q\mathcal{M}_1 \subseteq \mathcal{M}_2\} \ . \tag{4.15}$$

In particular,

$$\mathcal{M}^* \cong \{q \in \mathcal{Q} \ : \ q\mathcal{M} \subseteq B\} \tag{4.16}$$

Now, in addition to the right B-module $\mathcal{M}$, we introduce the left B-module

$$\mathcal{N} := \{a \in A \ : \ a[U] \subseteq U_0\} \ .$$

By condition (1b) of Theorem 4.3,

$$Bs \subseteq \mathcal{N} \subseteq Bs^{-1} \,, \tag{4.17}$$

which shows that $\mathcal{N}$ is a fractional left ideal. Since $B = \{a \in A \ : \ a[U_0] \subseteq U_0\}$, we have $\mathcal{N}\mathcal{M} \subseteq B$. With identification (4.16), this implies $\mathcal{N} \subseteq \mathcal{M}^*$. Dualizing the last inclusion yields $\mathcal{M}^{**} \subseteq \mathcal{N}^*$. On the other hand, for any fractional ideal, we have $\mathcal{M} \subseteq \mathcal{M}^{**}$. Hence, to prove that $\mathcal{M}$ is reflexive it suffices to show

$$\mathcal{N}^* \subseteq \mathcal{M} \,. \tag{4.18}$$

We prove (4.18) in two steps. First, we define $\mathcal{M}^\circ := \{a \in A : \mathtt{ad}_{L,L_0}^N(a) = 0 \text{ for some } N \geq 0\}$ and show that

$$\mathcal{N}^* \subseteq \mathcal{M}^\circ. \tag{4.19}$$

Then, we will prove

$$\mathcal{M}^\circ \subseteq \mathcal{M} \,. \tag{4.20}$$

To see (4.19) we identify $\mathcal{N}^* \cong \{q \in \mathcal{Q} : \mathcal{N}q \in B\}$ similar to (4.16). Since $s \in \mathcal{N}$, for any $q \in \mathcal{N}^*$, we have $sq \in B$, which implies $q \in A$. Hence $\mathcal{N}^* \subseteq A$. On the other hand, the inclusion $\mathcal{N} \subseteq Bs^{-1}$ in (4.17) implies $\deg(a) \geq -\deg(s)$ for all $a \in \mathcal{N}$. Then, the same argument as in the proof of Theorem 4.3 shows that $\mathtt{ad}_{L_0,L}$ acts on $\mathcal{N}$ locally nilpotently. In particular, for $s \in \mathcal{N}$, there is $N = N_s \geq 0$ such that $\mathtt{ad}_{L_0,L}^{N+1}(s) = 0$, while $\mathtt{ad}_{L_0,L}^N(s) \neq 0$. Set $S^* := \frac{1}{N!}\,\mathtt{ad}_{L_0,L}^N(s) \in \mathcal{N}$, so that $L_0 S^* = S^* L$. Now, for any $q \in \mathcal{N}^*$, we have $S^*q \in \mathcal{N}\mathcal{N}^* \subseteq B$. Since L_0 acts on B locally ad-nilpotently, there is $n \geq 0$ such that

$$\mathtt{ad}^n_{L_0}(S^*q) = S^*\,\mathtt{ad}^n_{L,L_0}(q) = 0\,.$$

This implies $\mathtt{ad}^n_{L,L_0}(q) = 0$ since $S^* \neq 0$. Thus $q \in \mathcal{M}^\circ$ for any $q \in \mathcal{N}^*$, which proves (4.19).

To prove (4.20) it suffices to show (by induction) that for $a \in A$,

$$\mathtt{ad}_{L,L_0}(a) \in \mathcal{M} \;\Rightarrow\; a \in \mathcal{M}\,.$$

Note that, if $\mathtt{ad}_{L,L_0}(a) \in \mathcal{M}$, then

$$La[U_0] = \mathtt{ad}_{L,L_0}(a)[U_0] + aL_0[U_0] \subseteq U + a[U_0]$$

Hence, if we set $\tilde{U} := U + a[U_0] \subseteq \mathbb{K}$, then $L[\tilde{U}] \subseteq \tilde{U}$, i.e. $\tilde{U}$ satisfies condition (1a) of Theorem 4.3. On the other hand, since $a \in A$, we can find $s' \in S$ such that $s'a \in B$. Taking $\tilde{s} := ss' \in S$, with $s \in S$ as in (1b) of Theorem 4.3, we have $\tilde{s}\,U_0 \subseteq s\,U_0 \subseteq U \subseteq \tilde{U}$ and

$$\tilde{s}\,\tilde{U} = \tilde{s}\,U + \tilde{s}a[U_0] = s'(s\,U) + s(s'a[U_0]) \subseteq s'\,U_0 + sB[U_0] \subseteq U_0$$

Thus, $\tilde{s}\,U_0 \subseteq \tilde{U} \subseteq \tilde{s}^{-1}\,U_0$ for $\tilde{s} \in S$, i.e. the $\tilde{U}$ also satisfies condition (1b) of Theorem 4.3. Since $U \subseteq \tilde{U}$, by maximality of U, we conclude that $\tilde{U} = U$ which implies that $a[U_0] \subseteq U$, i.e. $a \in \mathcal{M}$. This proves (4.20).

Summing up, we have shown that

$$\mathcal{M} \subseteq \mathcal{M}^{**} \subseteq \mathcal{N}^* \subseteq \mathcal{M}^\circ \subseteq \mathcal{M}\,.$$

Thus, all these subspaces in $\mathcal{Q}$ are equal. In particular, we have $\mathcal{M} = \mathcal{M}^{**}$, which proves the reflexivity of $\mathcal{M}$.

(b) By (4.15), we can identify $\mathrm{End}_B(\mathcal{M}) \cong \{q \in \mathcal{Q} \;:\; q\mathcal{M} \subseteq \mathcal{M}\}$. Since $\mathcal{M}$ is naturally a left $\mathcal{D}$-module, we have $\mathcal{D} \subseteq \mathrm{End}_B(\mathcal{M})$ via left multiplication in $\mathcal{Q}$. We need only to show the opposite inclusion

$$\mathrm{End}_B(\mathcal{M}) \subseteq \mathcal{D} \tag{4.21}$$

This can be proved in the same way as (4.18) in part (a): first, one defines the ring $\mathcal{D}^\circ := \{a \in A \,:\, \mathtt{ad}^N_L(a) = 0$ for some $N \geq 0\}$ and shows that $\mathrm{End}_B(\mathcal{M}) \subseteq \mathcal{D}^\circ$, then one proves the inclusion $\mathcal{D}^\circ \subseteq \mathcal{D}$ arguing by induction (downwards) in N. Note that, just as in part (a), the maximality of U is needed only for the last inclusion. Thus we get the chain of subalgebras in $\mathcal{Q}$:

$$\mathcal{D} \subseteq \mathrm{End}_B(\mathcal{M}) \subseteq \mathcal{D}^\circ \subseteq \mathcal{D}\,,$$

proving that all three are equal. This finishes the proof of the proposition. □

Remark 4.11. The proof of Proposition 4.10 shows that

$$\begin{aligned} \mathcal{M} &= \{a \in A \,:\, \mathtt{ad}^N_{L,L_0}(a) = 0 \text{ for some } N \geq 0\}\,, \\ \mathcal{D} &= \{a \in A \,:\, \mathtt{ad}^N_L(a) = 0 \text{ for some } N \geq 0\}\,, \end{aligned}$$

which gives an intrinsic characterisation of (4.13) and (4.14) for the maximal U. Dually, if we assume the maximality of U_0, i.e. that the U_0 is maximal among all subspaces $\tilde{U}_0 \subseteq U_0[S^{-1}]$ such that $L_0[\tilde{U}_0] \subseteq \tilde{U}_0$ and $U_0 \subseteq \tilde{U}_0 \subseteq s^{-1}U_0$ with $s \in S$, then we get $\mathcal{N} = \mathcal{N}^{**} = \mathcal{M}^*$ and

$$\begin{aligned} \mathcal{N} &= \{a \in A \,:\, \mathtt{ad}^N_{L_0,L}(a) = 0 \text{ for some } N \geq 0\}\,, \\ B &= \{a \in A \,:\, \mathtt{ad}^N_{L_0}(a) = 0 \text{ for some } N \geq 0\}\,. \end{aligned}$$

Proposition 4.10 shows that the quadruple $(\mathcal{M},\, \mathcal{M}^*,\, B,\, \mathcal{D})$ forms a *Morita context* in the sense of [MR, 1.1.5]. It is natural to ask when this context gives an actual *Morita equivalence* between the algebras B and $\mathcal{D}$: i.e., when do these algebras have equivalent module categories? Standard ring theory provides necessary and sufficient conditions for this in the form (see [MR, Cor. 3.5.4]):

$$\mathcal{M}^*\mathcal{M} = B \quad \text{and} \quad \mathcal{M}\,\mathcal{M}^* = \mathcal{D}\ .$$

In general, these conditions are not easy to verify; however, in our situation, they hold automatically under additional homological assumptions on B:

Corollary 4.12. *Assume that B is a simple Noetherian ring of global dimension* $\mathrm{gldim(B)} \leq 2$ *. Then $\mathcal{D}$ is Morita equivalent to B; in particular, $\mathcal{D}$ is a simple Noetherian ring of global dimension* $\mathrm{gldim}(\mathcal{D}) = \mathrm{gldim(B)}$ *. Moreover, if U_0 is a simple B-module, then U is a simple $\mathcal{D}$-module.*

Proof. It is a standard fact of homological algebra that every nonzero reflexive module over a Noetherian ring of global dimension ≤ 2 is f.g. projective (see, e.g., [Bass]). Hence, by part (a) of Proposition 4.10, the B-module $\mathcal{M}$ is f.g. projective; then part (b) — together with Dual Basis Lemma [MR, 3.5.2] — implies $\mathcal{M}\,\mathcal{M}^* = \mathcal{D}$. On the other hand, if B is a simple domain, we have automatically $\mathcal{M}^*\mathcal{M} = B$, since $\mathcal{M}^*\mathcal{M}$ is a (nonzero) two-sided ideal in B. Thus, by [MR, Cor. 3.5.4], B and $\mathcal{D}$ are Morita equivalent algebras. Being Noetherian, simple and having global dimension n are known to be Morita invariant properties of rings, hence $\mathcal{D}$ shares these properties with B.

To prove the last statement consider the map of left B-modules

$$f : \mathcal{M} \otimes_B U_0 \to U$$

given by the action of operators in $\mathcal{M}$ on U_0. The cokernel of this map, $\mathtt{Coker}(f) = U/\mathcal{M}[U_0]$, has a nonzero annihilator in $\mathcal{D}$: indeed, for $s \in S$ as in (1b) of Theorem 4.3, we have $s^2U = s(sU) \subseteq sU_0 \subseteq sB[U_0] \subseteq \mathcal{M}[U_0])$ by (4.10). Hence $\mathtt{Coker}(f) = 0$, since $\mathcal{D}$ is simple. On the other hand, since $\mathcal{M}$ is a progenerator in $\mathtt{Mod}(B)$, the $\mathcal{D}$-module $\mathcal{M} \otimes_B U_0$ is simple, whenever U_0 is simple. Hence $\mathtt{Ker}(f) = 0$. It follows that f is an isomorphism and U is a simple $\mathcal{D}$-module. □

Remark 4.13. In the last statement of Corollary 4.12, we can replace the assumption that U_0 is a simple B-module by U_0 being a finite R-module. The latter implies the former by an argument of [BW, Proposition 8.9].

4.3. **Commutative subalgebras.** The results of the previous section show that the algebras B and $\mathcal{D}$ containing the operators L_0 and L share many common properties, provided L_0 and L are related by the 'shift' identity (4.9). In this section, we will construct two commuting families of operators (including L_0 and L) that generate two isomorphic commutative subalgebras in B and $\mathcal{D}$ intertwined by a common shift operator S. It is interesting to note that the operator S may differ from the operator D that appears in (4.9): in general, there seems to be no simple relation between these two shift operators.

We will keep the assumptions of Theorem 4.3 and keep using the notation from the previous section. In addition, we will introduce a new notation: for a "multiplicative" version of the operator $\mathtt{ad}_{a,b}$ defined in the beginning of Section 4.1. Specifically, for an algebra A and a pair of elements $a, b \in A$, we define a linear map $\mathrm{Ad}_{a,b} : A \to A[[t]]$ with values in the ring of formal power series over A, by

$$\mathrm{Ad}_{a,b}(x) := \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \mathtt{ad}^n_{a,b}(x) \ . \tag{4.22}$$

(As in the case of '$\mathtt{ad}$', we will simply write Ad_a instead of $\mathrm{Ad}_{a,a}$ when $a = b$.)

Note that (a, b) acts locally ad-nilpotently on $x \in A$ if and only if $\mathrm{Ad}_{a,b}(x) \in A[t]$, where $A[t] \subset A[[t]]$ is the subring of polynomials in t with coefficients in A. Moreover, (4.22) has the following useful 'multiplicative' property.

Lemma 4.14. *For all $x, y \in A$, the following identity holds in $A[[t]]$:*

$$\mathrm{Ad}_{a,c}(xy) = \mathrm{Ad}_{a,b}(x)\, \mathrm{Ad}_{b,c}(y) \tag{4.23}$$

Proof. The coefficient under t^n in the left-hand side of (4.23) is $\frac{1}{n!}\, \mathtt{ad}^n_{a,c}(xy)$, while in the right-hand side,

$$\sum_{n_1+n_2=n} \frac{1}{n_1!\, n_2!}\ \mathtt{ad}^{n_1}_{a,b}(x)\ \mathtt{ad}^{n_2}_{b,c}(y)$$

Thus (4.23) is equivalent to the sequence of identities in A:

$$\mathtt{ad}^n_{a,c}(xy) = \sum_{k=0}^{n} \binom{n}{k} \mathtt{ad}^k_{a,b}(x)\ \mathtt{ad}^{n-k}_{b,c}(y)\ , \quad \forall\, n \geq 0\ ,$$

which can be easily verified by induction using the following 'twisted' version of the Leibniz rule

$$\mathtt{ad}_{a,c}(xy) = \mathtt{ad}_{a,b}(x)\, y + x\, \mathtt{ad}_{b,c}(y)\ .$$

An alternative way to prove (4.23) is to use the identity

$$\mathrm{Ad}_{a,b}(x) = e^{ta} x\, e^{-tb} \tag{4.24}$$

that formally holds in $A[[t]]$. To see (4.24) it suffices to notice that the both sides of (4.24) agree at $t = 0$, while satisfy the same differential equation $dF(t)/dt = \mathtt{ad}_{a,b}[F(t)]$ for $F(t) \in A[[t]]$. □

Now, let $L_0 \in B$ and $L \in A$ be as in Theorem 4.3, and let $U \subseteq \mathbb{K}$ be a subspace (not necessarily maximal) associated to L. Recall the fractional ideal $\mathcal{M}$, see (4.13), and the algebra $\mathcal{D}$, see (4.14), attached to U. As shown in the proof of Theorem 4.3, $\mathtt{ad}_{L,L_0}$ acts locally nilpotently on $\mathcal{M}$; in particular, if we take $s \in \mathcal{M}$ as in (1b), then $\mathtt{ad}_{L,L_0}^{N+1}(s) = 0$ for some $N \le 2\deg(s)$. We take the smallest $N \in \mathbb{N}$ with this property and put

$$S := \frac{1}{N!}\,\mathtt{ad}_{L,L_0}^{N}(s) \in \mathcal{M} \tag{4.25}$$

so that $S \neq 0$ while $LS = SL_0$. Using (4.25), it is easy to show that L is locally ad-nilpotent in $\mathcal{D}$. Indeed, as $\mathcal{D}\mathcal{M} \subseteq \mathcal{M}$, we have $aS \in \mathcal{M}$ for any $a \in \mathcal{D}$, and therefore $\mathtt{ad}_L^n(a)\, S = \mathtt{ad}_{L,L_0}^n(aS) = 0$ for $n \gg 0$, which implies $\mathtt{ad}_L^n(a) = 0$ since $S \neq 0$.

Now, we define

$$Q := \mathcal{D} \cap R = \{q \in R \;:\; qU \subseteq U\} \tag{4.26}$$

which is a commutative subring in $\mathcal{D}$. Note that Q is nontrivial: i.e. $Q \neq \{0\}$, since at least $s^2 \in Q$ by condition (1b). Note also that $Q \subseteq R \subseteq B$, i.e. Q is a common commutative subring of B and $\mathcal{D}$. Using the fact that L_0 is locally ad-nilpotent in B and L is locally ad-nilpotent in $\mathcal{D}$, we define for every $q \in Q$:

$$L_{q,0} := \frac{1}{N_{q,0}!}\,\mathtt{ad}_{L_0}^{N_{q,0}}(q)\,, \tag{4.27}$$

$$L_q := \frac{1}{N_q!}\,\mathtt{ad}_{L}^{N_q}(q)\,, \tag{4.28}$$

where $N_{q,0} \ge 0$ and $N_q \ge 0$ are chosen to be the smallest numbers such that $\mathtt{ad}_{L_0}^{N_{q,0}+1}(q) = 0$ and $\mathtt{ad}_{L}^{N_q+1}(q) = 0$. Thus, by definition, $L_{q,0} \in B$ and $L_q \in \mathcal{D}$ are nonzero operators satisfying $[L_{q,0},\, L_0] = 0$ and $[L_q,\, L] = 0$. In addition, we have

Proposition 4.15. *The operators* (4.27) *and* (4.28) *commute in* B *and* $\mathcal{D}$: *i.e.,*

$$[L_{q,0},\, L_{q',0}] = 0 \quad , \quad [L_q,\, L_{q'}] = 0\ , \quad \forall\, q,\, q' \in Q\ . \tag{4.29}$$

Moreover, for all $q \in Q$, *we have*

$$L_q\, S = S\, L_{q,0} \tag{4.30}$$

where S *is the operator defined by* (4.25).

Proof. The commutation relations (4.29) and (4.30) are proved in a similar way, using the identity (4.23) of Lemma 4.14. For example, to prove (4.30) we apply (4.23) to $x = q \in Q$ and $y = s$ as in (1b):

$$\mathrm{Ad}_L(q)\,\mathrm{Ad}_{L,L_0}(s) = \mathrm{Ad}_{L,L_0}(qs) = \mathrm{Ad}_{L,L_0}(sq) = \mathrm{Ad}_{L,L_0}(s)\,\mathrm{Ad}_{L_0}(q) \tag{4.31}$$

Notice that, by ad-nilpotency, all the Ad's in equation (4.31) take values in the polynomial ring $A[t]$. Then, comparing the leading coefficients of polynomials in both sides of (4.31) gives precisely the identity (4.30). Also, comparing the degrees (in t) of these polynomials shows that $N_q = N_{q,0}$ in (4.27) and (4.28). □

It follows from Proposition 4.15 that the operators $\{L_0,\, L_{q,0}\} \subset B$ and $\{L,\, L_q\} \subset \mathcal{D}$ generate two commutative subalgebras in B and $\mathcal{D}$ that are isomorphic to each other, with isomorphism given by $L_0 \mapsto L$ and $L_{q,0} \mapsto L_q$ for all $q \in Q$.

5. Proof of Theorem 3.5

Given a generalised locus configuration $\mathcal{A}$, consider the polynomial $\delta \in \mathbb{C}[V]^W$ defined by[4]

$$\delta := \prod_{\alpha \in \mathcal{A}_+ \setminus R} (\alpha, x)^{k_\alpha} \,. \tag{5.1}$$

[The fact that δ is W-invariant follows from the W-invariance of $\mathcal{A}$ and k_α: indeed, we must have $\delta(s_\alpha x) = \pm\delta(x)$ for any $\alpha \in R$, but $\delta(s_\alpha x) = -\delta(x)$ is impossible since δ does not vanish along $(\alpha, x) = 0$ for $\alpha \in R$.]

The set $S = \{1,\, \delta,\, \delta^2,\, \ldots\}$ is a two-sided Ore subset in the Cherednik algebra H_k, and we write $H_k[\delta^{-1}]$ and $B_k[\delta^{-1}]$ for H_k and B_k localised at S. By (2.7) $B_k \subset \mathcal{D}(V_{\mathrm{reg}})^W$, thus $B := B_k$, $R := \mathbb{C}[V]^W$ and the above S satisfy the assumption (A) of Section 4. Note that the quotient filed $\mathbb{K}$ of R is $\mathbb{C}(V)^W$ where $\mathbb{C}(V)$ denotes the field of rational functions.

The operator L_0 acts on B_k locally ad-nilpotently (**precise reference?**), so by Lemma 4.2 we can associate to it a degree function on $B = B_k$ and $A = B_k[\delta^{-1}]$. An elementary calculation shows that for any $f \in \mathbb{C}[V]^W$, $\deg_{L_0} f$ coincides with the usual homogeneous degree of f. It follows that $\deg(L - L_0) = -2$, which verifies the condition (2) of Theorem 4.9.

Next, let us take $U_0 = \mathbb{C}[V_{\mathrm{reg}}]^W$; clearly, $B_k(U_0) \subset U_0$. Moreover, any $a \in B_k[\delta^{-1}]$ that preserves U_0 must be regular away from the reflection hyperplanes of W, hence, a must necessarily lie in B_k. This proves that

$$B_k = \{a \in B_k[\delta^{-1}] \mid a(U_0) \subset U_0\} \,. \tag{5.2}$$

Our next and most important ingredient is the subspace $U_{\mathcal{A}} \subset \delta^{-1}\mathbb{C}[V_{\mathrm{reg}}]^W$ consisting of functions f satisfying

$$f(s_\alpha x) - (-1)^{k_\alpha} f(x) \text{ is divisible by } (\alpha, x)^{k_\alpha} \quad \forall \alpha \in \mathcal{A}_+ \setminus R \,. \tag{5.3}$$

It is immediate from the definitions that

$$\delta\, \mathbb{C}[V_{\mathrm{reg}}]^W \subseteq U_{\mathcal{A}} \subseteq \delta^{-1}\mathbb{C}[V_{\mathrm{reg}}]^W \,, \qquad Q_{\mathcal{A}}^W\, U_{\mathcal{A}} \subset U_{\mathcal{A}} \,. \tag{5.4}$$

The following properties of $U_{\mathcal{A}}$ are crucial. These are, in fact, the only properties where the locus relations (3.2) play a role.

[4]The polynomial (5.1) should not be confused with the discriminant of the Coxeter group W, i.e. $\prod_{\alpha \in R_+} (\alpha, x)$, which is also denoted frequently by δ in the literature on Cherednik algebras.

Lemma 5.1 (cf.[C1, CEO])**.** *The space $U_{\mathcal{A}}$ is invariant under the action of $L_{\mathcal{A}}$. Moreover, $U_{\mathcal{A}}$ is maximal among all subspaces U with the properties that $U \subseteq \delta^{-r}\mathbb{C}[V_{\mathrm{reg}}]^W$ for some $r > 0$ and $L_{\mathcal{A}}(U) \subseteq U$.*

Proof. The first claim in the case $W = \{e\}$ goes back to [C1] while the second is a slight reformulation of [CEO, Proposition 5.1], and the same arguments work for the general W. Namely, one works 'locally' by considering, for each $\alpha \in \mathcal{A}_+ \setminus R$, Laurent expansions in direction α. Functions $f \in U_{\mathcal{A}}$ are then characterised precisely by the property that their Laurent expansions do not contain terms $(\alpha, x)^j$ with $j \notin \{-k_\alpha + 2\mathbb{Z}_{\geq 0}\} \cup \{k_\alpha + 1 + 2\mathbb{Z}_{\geq 0}\}$. On the other hand, the locus relations (3.2) mean that in a similar Laurent expansion of u there will be no terms of degree $1, 3, \dots, 2k_\alpha - 1$. The invariance of $U_{\mathcal{A}}$ under $L_{\mathcal{A}}$ immediately follows from that. Moreover, if $f \notin U_{\mathcal{A}}$, then one can repeatedly apply $L_{\mathcal{A}}$ to f and obtain a function with a pole of an arbitrarily large order. This, in its turn, would violate the condition $U \subseteq \delta^{-r}\mathbb{C}[V_{\mathrm{reg}}]^W$, thus proving that $U_{\mathcal{A}}$ is maximal. See the proof of [CEO, Proposition 5.1] for the details. □

Remark 5.2. For $L_0 = L_W$ a similar fact is true, namely, $\mathbb{C}[V_{\mathrm{reg}}]^W$ is maximal among all subspaces $U_0 \subset \delta^{-r}\mathbb{C}[V_{\mathrm{reg}}]^W$ for some $r > 0$ such that $L_0(U_0) \subset U_0$. The proof is similar (formally, it corresponds to setting $k_\alpha = 0$ in the above arguments).

Proof of Theorem 3.5. Part (1). This follows from Theorem 4.3 applied to $B = B_k$, $A = B_k[\delta^{-1}]$, $L_0 = L_W$, $L = L_A$ and $U = U_{\mathcal{A}}$. All assumptions of Theorem 4.3 are satisfied, hence there exists $D \in B_k[\delta^{-1}]$ such that $LD = DL_0$. This proves part (1) of Theorem 3.5. Furthermore, the proof of Lemma 4.1 shows that D can be taken to be

$$D = \mathtt{ad}^N_{L,L_0}(\delta)\,, \qquad N = \deg\delta\,.$$

□

With the space $U_{\mathcal{A}}$ we associate the following subspaces:

$$\mathcal{M}_{\mathcal{A}} = \{a \in B_k[\delta^{-1}] \mid a(\mathbb{C}[V_{\mathrm{reg}}]^W) \subset U_{\mathcal{A}}\}\,, \tag{5.5}$$

$$\mathcal{M}^*_{\mathcal{A}} = \{a \in B_k[\delta^{-1}] \mid a(U_{\mathcal{A}}) \subset \mathbb{C}[V_{\mathrm{reg}}]^W\}\,, \tag{5.6}$$

$$\mathcal{D}_{\mathcal{A}} = \{a \in B_k[\delta^{-1}] \mid a(U_{\mathcal{A}}) \subset U_{\mathcal{A}}\}\,. \tag{5.7}$$

We may also consider the subspace consisting of all $a \in B_k[\delta^{-1}]$ that preserve $\mathbb{C}[V_{\mathrm{reg}}]^W$. However, any such a must be regular away from the reflection hyperplanes of W, hence, $a \in B_k$. This proves that

$$B_k = \{a \in B_k[\delta^{-1}] \mid a(\mathbb{C}[V_{\mathrm{reg}}]^W) \subset \mathbb{C}[V_{\mathrm{reg}}]^W\}\,. \tag{5.8}$$

It is clear that $\mathcal{D}_{\mathcal{A}}$ is a ring which acts on $\mathcal{M}_{\mathcal{A}}$ ($\mathcal{M}^*_{\mathcal{A}}$, resp.) by left (right, resp.) multiplication. In fact, $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}^*_{\mathcal{A}}$ can be viewed as $\mathcal{D}_{\mathcal{A}}$ - B_k and B_k - $\mathcal{D}_{\mathcal{A}}$ bimodules in the obvious way. From (5.4), (5.8) we get that

$$\delta B_k \subset \mathcal{M}_{\mathcal{A}} \subset \delta^{-1}B_k\,, \qquad B_k\delta \subset \mathcal{M}^*_{\mathcal{A}} \subset B_k\delta^{-1}\,, \qquad \delta\mathcal{D}_{\mathcal{A}}\delta \subset B_k\,. \tag{5.9}$$

In particular, δ belongs to both $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}^*_{\mathcal{A}}$.

Recall the Calogero–Moser operators $L_0 := L_W$ and $L := L_{\mathcal{A}}$. Since $L_0 \in B_k$ and $L \in \mathcal{D}_{\mathcal{A}}$, the ad-operators $\mathtt{ad}_L$ and $\mathtt{ad}_{L_0}$ act on $\mathcal{D}_{\mathcal{A}}$ and B_k, respectively. We also have $\mathtt{ad}_{L,L_0}$ (resp. $\mathtt{ad}_{L_0,L}$) acting on the bimodule $\mathcal{M}_{\mathcal{A}}$ (resp. $\mathcal{M}^*_{\mathcal{A}}$).

Lemma 5.3. (1) *The operators* $\mathtt{ad}_L$ *and* $\mathtt{ad}_{L_0}$ *act locally nilpotently on* $\mathcal{D}_{\mathcal{A}}$ *and* B_k*, respectively.*

(2) *The operators* $\mathtt{ad}_{L,L_0}$ *and* $\mathtt{ad}_{L_0,L}$ *act locally nilpotently on* $\mathcal{M}_{\mathcal{A}}$ *and* $\mathcal{M}^*_{\mathcal{A}}$*, respectively.*

(3) *We have* $\mathtt{gr}(\mathcal{M}_{\mathcal{A}}), \mathtt{gr}(\mathcal{M}^*_{\mathcal{A}}), \mathtt{gr}(\mathcal{D}_{\mathcal{A}}) \subset \mathbb{C}[V \times V^*]^W$*, where '*$\mathtt{gr}$*' is taken with respect to the differential filtration on* $B_k[\delta^{-1}]$*.*

Proof. By (5.9), we must have

$$\mathtt{gr}(\mathcal{M}_{\mathcal{A}}), \mathtt{gr}(\mathcal{M}^*_{\mathcal{A}}) \subset \delta^{-1}\mathbb{C}[V \times V^*]^W , \qquad \mathtt{gr}(\mathcal{D}_{\mathcal{A}}) \subset \delta^{-2}\mathbb{C}[V \times V^*]^W .$$

Below we will only check the properties of $\mathtt{ad}_{L,L_0}$ and $\mathtt{gr}(\mathcal{M}_{\mathcal{A}})$, all others are proved in the same way.

Consider a positive increasing filtration on the ring $\mathcal{D}(V)$, defined by $\deg(x) = 1$, $\deg(\partial) = 0$. This filtration naturally extends to a $\mathbb{Z}$-filtration on $\mathcal{D}(V_{\mathrm{reg}})[\delta^{-1}]$. It is easy to see that each application of $\mathtt{ad}_{L,L_0}$ lowers the x-degree. On the other hand, considering the differential filtration, the fact that $\mathtt{gr}(\mathcal{M}_{\mathcal{A}}) \subset \delta^{-1}\mathbb{C}[V \times V^*]^W$ tells us that the x-degree of elements in $\mathcal{M}_{\mathcal{A}}$ are bounded below by $-\deg\delta$. It follows that for any $a \in \mathcal{M}_{\mathcal{A}}$ $\mathtt{ad}^N_{L,L_0}(a) = 0$ for N sufficiently large.

To prove that $\mathtt{gr}(\mathcal{M}_{\mathcal{A}}) \subset \mathbb{C}[V \times V^*]^W$, take an arbitrary $a \in \mathcal{M}_{\mathcal{A}}$ and suppose it has order r as a differential operator. Then $\mathtt{ad}_{L,L_0}(a)$ is of order at most $r + 1$, with the terms of order $r + 1$ given by

$$\sum_{i=1}^{n} 2\partial_i(a_0)\partial_i \,, \tag{5.10}$$

where a_0 denotes the leading order terms of a. Suppose a_0 has a pole of order $p > 0$ along a hyperplane $(\alpha, x) = 0$. Then it is easy to see that the expression (5.10) has pole of order $p + 1$ along this hyperplane. As a result, each application of $\mathtt{ad}_{L,L_0}$ increases the order of the pole, and so $\mathtt{ad}^r_{L,L_0}(a)$ remains nonzero for all $r > 0$. This contradicts part (2) of the lemma. We conclude that the leading order terms of a are regular and $\mathtt{gr}(\mathcal{M}_{\mathcal{A}}) \subset \mathbb{C}[V \times V^*]^W$, as needed. □

Corollary 5.4. *With respect to the x-filtration, all elements of* B_k*,* $\mathcal{D}_{\mathcal{A}}$*,* $\mathcal{M}_{\mathcal{A}}$*,* $\mathcal{M}^*_{\mathcal{A}}$ *have non-negative degree.*

Indeed, this follows immediately from the fact that their leading terms w.r.t. the differential filtration are contained in $\mathbb{C}[V \times V^*]$. □

We can now give an intrinsic characterisation of $\mathcal{M}_{\mathcal{A}}$, $\mathcal{M}^*_{\mathcal{A}}$, $\mathcal{D}_{\mathcal{A}}$ and B_k as subspaces in $B_k[\delta^{-1}]$.

Theorem 5.5. *Let $A = B_k[\delta^{-1}]$, $L = L_{\mathcal{A}}$, $L_0 = L_W$. Then for the spaces defined by (??)–(??) we have: $\mathcal{D} = \mathcal{D}_A$, $\mathcal{M} = \mathcal{M}_{\mathcal{A}}$, $\mathcal{M}^* = \mathcal{M}^*_{\mathcal{A}}$, $\mathcal{D}_0 = B_k$. In other words, $\mathcal{M}_{\mathcal{A}}$ is the largest subspace in $B_k[\delta^{-1}]$ on which $\mathtt{ad}_{L,L_0}$ acts locally nilpotently, and similarly for the other cases.*

Proof. Let us prove that $\mathcal{M} = \mathcal{M}_{\mathcal{A}}$, other cases are similar. By Lemma 5.3(2), $\mathcal{M}_{\mathcal{A}} \subset \mathcal{M}$. It remains to prove that if for some $a \in B_k[\delta^{-1}]$ and $n \geq 1$, we have $\mathtt{ad}^n_{L,L_0}(a) = 0$, then $a \in \mathcal{M}_{\mathcal{A}}$. By induction in n, it is enough to show that $\mathtt{ad}_{L,L_0}(a) \in \mathcal{M}_{\mathcal{A}}$ implies $a \in \mathcal{M}_{\mathcal{A}}$. So let $a \in B_k[\delta^{-1}]$ and $b \in \mathcal{M}_{\mathcal{A}}$ are such that

$$La - aL_0 = b \ . \tag{5.11}$$

The differential operators a, b, L, L_0 act on $\mathbb{C}[V_{\mathrm{reg}}]^W[\delta^{-1}]$. Denote $U_0 = \mathbb{C}[V_{\mathrm{reg}}]^W$; our goal is to show that $a(U_0) \subset U_{\mathcal{A}}$. We have $L_0(U_0) \subset U_0$, $b(U_0) \subset U_{\mathcal{A}}$, and $L(U_{\mathcal{A}}) \subset U_{\mathcal{A}}$. Now set $U := a(U_0) + U_{\mathcal{A}}$, then (5.11) implies that $L(U) \subset U$. Since $\delta^r a \in B_k$ for some $r > 0$, we have $a(U_0) \subset \delta^{-r} U_0$. Also, $U_{\mathcal{A}} \subset \delta^{-1} U_0$, and therefore $U \subset \delta^{-r} U_0$. We now use Lemma 5.1 to conclude that $U = U_{\mathcal{A}}$, and thus $a(U_0) \subset U_{\mathcal{A}}$. This proves that $a \in \mathcal{M}_{\mathcal{A}}$, and so $\mathcal{M} = \mathcal{M}_{\mathcal{A}}$, as needed.

For the remaining cases, proofs are essentially the same. For example, for the proof of $\mathcal{D}_0 = B_k$, one replaces (5.11) with $L_0 a - aL_0 = b$ and set $U := a(U_0) + U_0$. In this case instead of Lemma 5.1 one uses Remark 5.2. ☐

5.1. **Proof of Theorem 3.5.** Take any homogeneous polynomial $f \in \mathbb{C}[V]^W$ of degree r. Using (5.10) repeatedly, we have

$$\mathtt{ad}^r_{L_0}(f) = 2^r r! f(\partial) + \dots ,$$

up to lower order terms w.r.t. differential filtration. Since each application of $\mathtt{ad}_{L_0}$ lowers the x-degree by one, the x-degree of $\mathtt{ad}^r_{L_0}(f)$ is zero. Using the previous corollary, we obtain that $\mathtt{ad}^{r+1}_{L_0}(f) = 0$. Hence, the least r such that $\mathtt{ad}^{r+1}_{L_0}(f) = 0$ is given by $r = \deg f$. The same argument works for the other cases, e.g. for $f \in \mathcal{M}_{\mathcal{A}}$ and repeated application of $\mathtt{ad}_{L,L_0}$. For instance, taking $\delta \in \mathcal{M}_{\mathcal{A}}$ and denoting $N := \deg \delta$, we construct an operator

$$S = \frac{1}{2^N N!} \mathtt{ad}^N_{L,L_0}(\delta) \, .$$

By construction, S belongs to $\mathcal{M}_{\mathcal{A}}$ and it has the form

$$S = \prod_{\alpha \in \mathcal{A}_+ \setminus R} (\alpha, \partial)^{k_\alpha} \ + \ \dots \tag{5.12}$$

with lower order terms "..." being of negative x-degree. As a result, $ad^{N+1}_{L,L_0} \delta = 0$ and S intertwines L and L_0. This proves part (1) of Theorem 3.5.

Applying the general construction of Theorem (??), we alos obtain the following result.

Proposition 5.6. *For any homogeneous generalised quasi-invariant $q \in Q_{\mathcal{A}}^W$ of degree $r > 0$, define*

$$L_q = \frac{1}{2^r r!}\mathtt{ad}_L^r(q)\,, \qquad L_{q,0} = \frac{1}{2^r r!}\mathtt{ad}_{L_0}^r(q)\,. \tag{5.13}$$

Then the assignment $q \mapsto L_q$ ($q \mapsto L_{q,0}$, resp.) extends to an injective algebra homomorphism $Q_{\mathcal{A}}^W \hookrightarrow \mathcal{D}_{\mathcal{A}}$ ($Q_{\mathcal{A}}^W \hookrightarrow B_k$, resp.). Moreover, the operators L_q, $L_{q,0}$ have their leading order terms of the form $q(\partial)$ and satisfy

$$L_q\, S = S\, L_{q,0}\,,$$

where S is the shift operator (5.12).

Proof. In the setting of Theorem **??**, consider $B := B_k$, $A := B_k[\delta^{-1}]$, $L := L_{\mathcal{A}}$, $L_0 := L_W$, $\mathcal{M} := \mathcal{M}_A$. We then have $\mathcal{D} = \mathcal{D}_{\mathcal{A}}$, $\mathcal{D}_0 = B_k$. Choose the commutative subalgebras $C = C_0 = Q_{\mathcal{A}}^W \subset \mathcal{D} \cap \mathcal{D}_0$. We have $\delta \in \mathcal{M}$ and, obviously, $C\delta = \delta C_0$. By the previous results, all the assumptions of Theorem **??** are satisfied. As explained above, for any homogeneous polynomial $f \in C = C_0$ the smallest r such that $\mathtt{ad}_L^{r+1} f = 0$ or $\mathtt{ad}_{L_0}^{r+1} f = 0$ equals $r = \deg f$. Thus, we obtain two commutative subalgebras $C^\flat, C_0^\flat$ generated by the operators L_q and $L_{q,0}$, respectively, and the shift operator (5.12) intertwines these subalgebras. □

To finish the proof of part (2) of Theorem 3.5, we need to explain why $L_{q,0}$ is the same as $\mathtt{Res}(\boldsymbol{e}T_q\boldsymbol{e})$. To see that, we notice that both operators belong to B_k and have the same leading order terms $q(\partial)$. From the way these operators are constructed, it is clear that their lower order terms have negative x-degree. Corollary 5.4 then tells us that $L_{q,0} = \mathtt{Res}(\boldsymbol{e}T_q\boldsymbol{e})$.

Finally, part (3) of Theorem 3.5 follows easily from the inclusion $\delta^2\mathbb{C}[V]^W \subset Q_{\mathcal{A}}^W$.

□

Remark 5.7. For the commutative ring $\{L_q \,|\, q \in Q_{\mathcal{A}}^W\}$ we can consider the eigenvalue problem

$$L_q\psi = q(\lambda)\psi \quad \forall\, q \in Q_{\mathcal{A}}^W\,, \tag{5.14}$$

where $\psi = \psi(x, \lambda)$ is a function of x and the spectral variable $\lambda \in V$. The dimension of the solution space to (5.14) for generic λ is usually referred to as the *rank* of the commutative ring (cf. [BrEtGa]). By arguments close to those in [C2, Section 3], one can show that the solution space to (5.14) has dimension $|W|$, the cardinality of W. For the locus configurations, the group W is trivial, and the commutative ring has rank one (see [C2, Theorem 3.11]).

6. Fat ideals of the spherical subalgebra

Recall the $\mathcal{D}_{\mathcal{A}}$ - B_k bimodules $\mathcal{M}_{\mathcal{A}}$ (5.5). When viewed as right B_k-modules, these modules have some very interesting properties. Recall that the spherical subalgebra B_k contains a commutative subalgebra $\mathbb{C}[V]^W$ of polynomials. Following [BCM], we call a right B_k-module $\mathcal{M}$ *fat* if it is isomorphic to an ideal $\mathcal{M}_x \subset B_k$ with $\mathcal{M}_x \cap \mathbb{C}[V]^W \neq \{0\}$. The algebra B_k contains another commutative subalgebra, denoted as $\mathbb{C}[V^*]^W$, consisting of the elements $\mathtt{Res}(\boldsymbol{e}T_q\boldsymbol{e})$ with $q \in \mathbb{C}[V^*]^W$. A fat B_k-module $\mathcal{M}$ is called *very fat* if, in addition, $\mathcal{M}$ is isomorphic to an ideal $\mathcal{M}_y \subset B_k$ with $\mathcal{M}_y \cap \mathbb{C}[V^*]^W \neq \{0\}$. We should remark that not every fat module is very fat when $\dim V > 1$ (for counterexamples, see [BCM]).

Proposition 6.1. *For any generalised locus configuration $\mathcal{A}$, the right B_k-module $\mathcal{M}_\mathcal{A}$ is very fat.*

Proof. Replacing $\mathcal{M}_\mathcal{A}$ by an isomorphic right B_k-module $\mathcal{M}_x := \delta\mathcal{M}_\mathcal{A}$, we get $\mathcal{M}_x \subset B_k$ by (5.9). Also, it is clear that $\mathcal{M}_x \cap \mathbb{C}[V]^W \neq \{0\}$; indeed, $\delta^2 \in \mathcal{M}_x$. Thus, $\mathcal{M}_\mathcal{A}$ is a fat B_k-module.

Recall a B_k-$\mathcal{D}_\mathcal{A}$ bimodule $\mathcal{M}^*_\mathcal{A}$ (5.6). Since δ belongs to $\mathcal{M}^*_\mathcal{A}$, we may consider

$$S^* = \frac{1}{2^N N!}\mathtt{ad}^N_{L_0,L}(\delta)\,, \qquad N = \deg\delta\,. \tag{6.1}$$

It is clear that S^* belongs to $\mathcal{M}^*_\mathcal{A}$. Now let $\mathcal{M}_y = S^*\mathcal{M}_\mathcal{A}$; then

$$\mathcal{M}_y \subset M^*_\mathcal{A}\mathcal{M}_\mathcal{A} \subset B_k\,.$$

By Lemma ??, we have

$$S^*S = \frac{1}{(2^N N!)^2}\mathtt{ad}^{2N}_{L_0}(\delta^2).$$

Up to a constant factor, this is one of the operators $L_{q,0}$ appearing in Theorem 3.5 and Proposition 5.6, namely, $L_{\delta^2,0} = \mathtt{Res}(T_{\delta^2}) \in \mathbb{C}[V^*]^W$. We conclude that $\mathcal{M}_y \cap \mathbb{C}[V^*]^W \neq \{0\}$, so $\mathcal{M}_\mathcal{A}$ is very fat.

□

Remark 6.2. The above results remain true for the locus configurations $\mathcal{A} \subset \mathbb{C}^n$ (when $R = \emptyset$), with the same proofs. The Cherednik algebra B_k in that case is simply replaced by the Weyl algebra A_n. Thus, the locus configurations in $\mathbb{C}^n$ produce very fat right ideals of $\mathcal{D}(V) \cong A_n$.

Remark 6.3. The most well studied case is that of Coxeter locus configurations, when $\mathcal{A} \subset \mathbb{C}^n$ is the root system of a Coxeter group W and k_α's are integers. In this case it is known that the ring $Q_\mathcal{A}$ is Cohen–Macaulay and Gorenstein [FV1, EG2]. Furthermore, by results of [BEG], the ideal $\mathcal{M}_\mathcal{A}$ is projective, and the ring $\mathcal{D}_\mathcal{A}$ is a simple ring, Morita equivalent to A_n. It is an interesting question which of these properties remain true in the general case. For instance, for any locus configuration $\mathcal{A} \subset \mathbb{C}^n$, the right A_n-module $\mathcal{M}_\mathcal{A}$ is reflexive. (For a proof, as well as for further interesting properties of the class of fat/very fat reflexive ideals of A_n, see [BCM].) We expect that the right B_k-modules $\mathcal{M}_\mathcal{A}$ are reflexive for all generalised locus configuration. See also [BCES, ER] where the rings $Q_\mathcal{A}$ are shown to be Cohen–Macaulay for some non-Coxeter cases.

6.1. **Non-twisted configurations.** Many of the known examples of the generalised locus configurations satisfy the additional property (3.5) or, equivalently, the equation $L_{\mathcal{A}}(\delta^{-1}) = 0$. The identity (3.5) can be reformulated as the following system of equations for each $\alpha \in \mathcal{A}_+$:

$$\sum_{\beta\in\mathcal{A}_+\setminus\{\alpha\}} \frac{k_\beta(\alpha,\beta)}{(\beta,x)} = 0 \quad \text{for } (\alpha,x)=0\,. \tag{6.2}$$

Note that these conditions hold automatically for each $\alpha \in R_+$, thanks to the W-invariance. For $\alpha \in \mathcal{A}_+ \setminus R$, let us complement (6.2) by the stronger conditions:

$$(\alpha,\partial)^{2j-1} \prod_{\beta\in\mathcal{A}_+\setminus\{\alpha\}} (\beta,x)^{k_\beta} = 0 \quad \text{for } (\alpha,x)=0 \quad \text{and } j=1,\dots,k_\alpha\,. \tag{6.3}$$

In many examples, we have $k_\alpha = 1$, in which case (6.2) and (6.3) coincide. Also, (6.3) is automatic whenever $\mathcal{A}$ is invariant under the action of s_α. Let us call a configuration $\mathcal{A}$ *non-twisted* if it satisfies the conditions (6.3) for all $\alpha \in \mathcal{A}_+ \setminus R$, and *twisted* otherwise. For the rest of this subsection, assume that $\mathcal{A}$ is non-twisted. In this case we can replace $L_{\mathcal{A}}$ with

$$\widetilde{L}_{\mathcal{A}} = \delta_{\mathcal{A}} L_{\mathcal{A}} \delta_{\mathcal{A}}^{-1} = \Delta - \sum_{\alpha\in\mathcal{A}_+} \frac{2k_\alpha}{(\alpha,x)}(\alpha,\partial)\,.$$

Likewise, we define

$$\widetilde{L}_W = \Delta - \sum_{\alpha\in R_+} \frac{2k_\alpha}{(\alpha,x)}(\alpha,\partial)\,.$$

Crucially, we have

$$\widetilde{L}_W(\mathbb{C}[V]^W) \subset \mathbb{C}[V]^W\,, \qquad \widetilde{L}_{\mathcal{A}}(Q_{\mathcal{A}}^W) \subset Q_{\mathcal{A}}^W\,.$$

Let us modify the Cherednik algebra accordingly, namely, consider the Dunkl operators

$$\widetilde{T}_\xi := \partial_\xi - \sum_{\alpha\in R_+} \frac{(\alpha,\xi)}{(\alpha,x)} k_\alpha(1-s_\alpha)\,, \quad \xi \in V\,, \tag{6.4}$$

and set $\widetilde{H}_k$ be the subalgebra of $\mathcal{D}W$ generated by $\mathbb{C}W$, $\mathbb{C}[V]$ and all $\widetilde{T}_\xi$. As before, the spherical subalgebra is $\widetilde{B}_k := \mathtt{Res}(\boldsymbol{e}\widetilde{H}_k\boldsymbol{e})$. Note that one has $\widetilde{L}_W \in \widetilde{B}_k$ and $\widetilde{B}_k(\mathbb{C}[V]^W) \subset \mathbb{C}[V]^W$. We can also modify the definitions of $\mathcal{M}_{\mathcal{A}}, \mathcal{D}_{\mathcal{A}}$ appropriately:

$$\widetilde{\mathcal{M}}_{\mathcal{A}} = \{a \in \widetilde{B}_k[\delta^{-1}] \mid a(\mathbb{C}[V]^W) \subset Q_{\mathcal{A}}^W\}\,,$$
$$\widetilde{\mathcal{D}}_{\mathcal{A}} = \{a \in \widetilde{B}_k[\delta^{-1}] \mid a(Q_{\mathcal{A}}^W) \subset Q_{\mathcal{A}}^W\}\,.$$

Note that every $a \in \widetilde{\mathcal{M}}_{\mathcal{A}}$ maps $\mathbb{C}[V]^W$ to itself, thus $\widetilde{\mathcal{M}}_{\mathcal{A}} \subset \widetilde{B}_k$.

Let $L = \widetilde{L}_{\mathcal{A}}$ and $L_0 = \widetilde{L}_W$. We then have the following analogues of the previous results, with analogous proofs.

Theorem 6.4. (1) *The action of* $\mathtt{ad}_{L,L_0}$ *on* $\widetilde{\mathcal{M}}_{\mathcal{A}}$ *is locally nilpotent.*

(2) *With respect to the differential filtration, we have* $\mathtt{gr}(\widetilde{\mathcal{M}}_{\mathcal{A}}) \subset \delta\,\mathbb{C}[V \times V^*]^W$ *and* $\mathtt{gr}(\widetilde{\mathcal{D}}_{\mathcal{A}}) \subset \mathbb{C}[V \times V^*]^W$*.*

(3) *Define* $S = \frac{1}{2^N N!}\mathtt{ad}^N_{L,L_0}(\delta^2)$*,* $N = \deg\delta$*. Then* $LS = SL_0$*. Moreover,* S *belongs to* $\widetilde{\mathcal{M}}_{\mathcal{A}}$ *(hence,* $S \in \widetilde{B}_k$*) and has* $\delta(x)\delta(\partial)$ *as its leading term.*

(4) *For any* $q \in Q^W_{\mathcal{A}}$ *with* $\deg q = r$*, define* $L_q = \frac{1}{2^r r!}\mathtt{ad}^r_L(q)$*. Then* $L_q(Q^W_{\mathcal{A}}) \subset Q^W_{\mathcal{A}}$*, and the map* $q \mapsto L_q$ *defines an algebra embedding of* $Q^W_{\mathcal{A}} \hookrightarrow \widetilde{\mathcal{D}}_{\mathcal{A}}$*.*

(5) *The right ideal* $\widetilde{\mathcal{M}}_{\mathcal{A}} \subset \widetilde{B}_k$ *is very fat.*

Assuming the algebra $Q^W_{\mathcal{A}}$ is finitely generated, it can be viewed as the algebra of functions on a (singular) affine variety $X = \mathtt{Spec}\, Q^W_{\mathcal{A}}$, and so the elements of $\widetilde{\mathcal{D}}_{\mathcal{A}}$ are regular differential operators on X. For comparison, if $\mathcal{A}$ is twisted, we define the ring $Q_{\mathrm{reg}} \subset \mathbb{C}[V_{\mathrm{reg}}]^W$, using the same algebraic conditions as for $Q = Q_{\mathcal{A},W}$, see (4.1). Then we have a projective, rank-one Q_{reg}-module $U_{\mathcal{A},W}$ (i.e. a line bundle over $X_{\mathrm{reg}} = \mathtt{Spec}\, Q_{\mathrm{reg}}$), and so the elements of $\mathcal{D}_{\mathcal{A}}$ can be viewed as *twisted* differential operators.

7. Examples of generalised locus configurations

In this section we describe known examples of generalised locus configurations in dimension > 2. The two-dimensional configurations will be discussed in Section 8.

7.1. Coxeter configurations.

The simplest examples can be obtained by considering a pair $W \subset W'$ of finite reflection groups acting on V. Let $R \subset R'$ be the corresponding root systems with a W'-invariant *integral* multiplicities $k : R' \to \mathbb{Z}_+$. Then we can view $\mathcal{A} = R'$ as a generalised locus configuration of type W. In the special case when R coincides with one of the W'-orbits of roots in R', we may allow non-integer multiplicities k_α for $\alpha \in R$.

7.2. Examples related to Lie superalgebras.

These examples were discovered in [SV1]; there are two infinite series and three exceptional cases.

1. *Type* $A(n,m)$

Here $V = \mathbb{C}^{n+m}$ with the standard scalar product, and the group $W = S_n \times S_m$ acts by permuting the first n and the last m of the coordinates. We set

$$I_1 = \{1, \dots, n\}, \quad I_2 = \{n+1, \dots, n+m\}. \tag{7.1}$$

The configuration depends on one parameter $k \neq 0$. It consists of the vectors $\alpha = e_i - e_j$, $i, j \in I_1$, $i \neq j$ with $k_\alpha = k$, the vectors $\alpha = e_i - e_j$, $i, j \in I_2$, $i \neq j$ with $k_\alpha = k^{-1}$, and the vectors $\pm(e_i - \sqrt{k}e_j)$, $i \in I_1$, $j \in I_2$ with $k_\alpha = 1$.

2. *Type* $B(n,m)$

We keep the notation of the previous case, so $V=\mathbb{C}^{n+m}$ with the standard Euclidean product. The configuration depends on parameters $k\neq 0$ and a,b related by

$$2a+1=k(2b+1)\,.$$

It consists of the vectors $\alpha=\pm e_i\pm e_j$, $i,j\in I_1$, $i\neq j$ with $k_\alpha=k$, the vectors $\alpha=\pm e_i\pm e_j$, $i,j\in I_2$, $i\neq j$ with $k_\alpha=k^{-1}$, the vectors $\alpha=\pm e_i$, $i\in I_1$ with $k_\alpha=a$, the vectors $\alpha=\pm e_i$, $i\in I_2$ with $k_\alpha=b$, and the vectors $\pm e_i\pm\sqrt{k}e_j$, $i\in I_1$, $j\in I_2$ with $k_\alpha=1$.

Let us write down the corresponding Calogero–Moser operator. Using Cartesian coordinates $x_1,\dots,x_n$, $y_1,\dots,y_m$ on V, we obtain:

$$\begin{aligned}L_{BC(n,m)}=\Delta &-2k(k+1)\sum_{i<j}^{n}(x_i\pm x_j)^{-2}\\ &-2k^{-1}(k^{-1}+1)\sum_{i<j}^{m}(y_i\pm y_j)^{-2}\\ &-a(a+1)\sum_{i=1}^{n}x_i^{-2}-b(b+1)\sum_{i=1}^{m}y_i^{-2}\\ &-2(k+1)\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i\pm\sqrt{k}y_j)^{-2}\,.\end{aligned}$$

The first three lines of this expression describe the Calogero–Moser operator for the root system $R=B_n\times B_m$; the remaining sum is invariant under the action of the Coxeter group $W=W(B_n)\times W(B_m)$.

Remark 7.1. In [SV1], the operator $L_{BC(n,m)}$ is written in the trigonometric case, and in non-Cartesian coordinates. Our parameters a,b,k are related to the parameters p,q,r,s,k in [SV1] as $a=p+q$ and $b=r+s$.

3. *Type* $AB(1,3)$

In this case $V=\mathbb{C}^4$, and the corresponding Calogero–Moser operator in Cartesian coordinates (x_1,x_2,x_3,y) is given by

$$\begin{aligned}L_{AB(1,3)}=\Delta-\sum_{i=1}^{3}a(a+1)x_i^{-2}-b(b+1)y^{-2}-2c(c+1)\sum_{i<j}^{3}(x_i\pm x_j)^{-2}\\ -2(3k+3)\sum_{\pm}(\sqrt{3k}y\pm x_1\pm x_2\pm x_3)^{-2}\,.\end{aligned}$$

Here the last sum is over all 8 possible combinations of the signs. The parameters a,b,c,k are related by

$$a=\frac{3k+1}{2}\,,\quad b=\frac{1}{2}(k^{-1}-1)\,,\quad c=\frac{3k-1}{4}\,.$$

The configuration of type $AB(1,3)$ contains a Coxeter configuration of type $R = B_3 \times A_1$. The remaining vectors $\alpha = (\pm 1, \pm 1, \pm 1, \pm\sqrt{3k})$ have multiplicities $k_\alpha = 1$.

Remark 7.2. In [SV1], the formula for $L_{AB(1,3)}$ contains a misprint: the numerical factor in front of the last sum in [SV1, (14)] should be $\frac{1}{2}$, not $\frac{1}{4}$.

4. *Type* $G(1,2)$

In this case $V = \mathbb{C}^4$, and the corresponding Calogero–Moser operator in Cartesian coordinates (x_1, x_2, x_3, y) is given by

$$L_{G(1,2)} = \Delta - 2p(p+1)\sum_{i<j}^{3}(x_i - x_j)^{-2}$$
$$- 3q(q+1)\sum_{i\neq j\neq l}^{3}(x_i + x_j - 2x_l)^{-2}$$
$$- r(r+1)y^{-2} - 4(k+1)\sum_{i\neq j}^{3}(\sqrt{2k}y - x_i + x_j)^{-2}\,.$$

The parameters p, q, r, k are related by

$$p = 2k+1\,, \quad q = \frac{2k-1}{3}\,, \quad r = \frac{3}{2}(k^{-1}+1)\,.$$

The configuration of type $G(1,2)$ contains a Coxeter configuration of type $R = G_2 \times A_1$. The remaining vectors $\alpha = \pm(-e_i + e_j + \sqrt{2k}e_4)$ have multiplicities $k_\alpha = 1$.

Remark 7.3. The configuration of type $G(1,2)$ is contained in the hyperplane $x_1 + x_2 + x_3 = 0$ in $\mathbb{C}^4$. In [SV1], the operator $L_{G(1,2)}$ is restricted onto this hyperplane (in non-Cartesian coordinates). Our parameters p, q, r, k are related to the parameters a, b, c, d, k in [SV1] as $p = a$, $q = b$ and $r = c + d$.

5. *Type* $D(2,1,\lambda)$

In this case $V = \mathbb{C}^3$. Let $\lambda_1, \lambda_2, \lambda_3$ be arbitrary non-zero parameters. Introduce

$$m_i = \frac{\lambda_1 + \lambda_2 + \lambda_3}{2\lambda_i} - 1 \quad (i = 1, 2, 3)\,.$$

The configuration $D(2,1,\lambda)$ consists of the vectors $\alpha = \pm e_i$ with $k_\alpha = m_i$ and eight additional vectors $\alpha = \pm\sqrt{\lambda_1}e_1 \pm \sqrt{\lambda_2}e_2 \pm \sqrt{\lambda_3}e_3$ with $k_\alpha = 1$.

The corresponding Calogero–Moser operator is given by

$$L_{D(2,1,\lambda)} = \Delta - \sum_{i=1}^{3} m_i(m_i+1)x_i^{-2}$$
$$- 2(\lambda_1 + \lambda_2 + \lambda_3)\sum_{\pm}(\sqrt{\lambda_1}x_1 \pm \sqrt{\lambda_2}x_2 \pm \sqrt{\lambda_3}x_3)^{-2}\,.$$

7.3. **Type $A_{n-1,2}$ configuration.** It consists of the following vectors in $\mathbb{C}^{n+2}$:

$$\begin{cases} e_i - e_j, & 1 \leq i < j \leq n, \quad \text{with } k_\alpha = k\,, \\ e_i - \sqrt{k} e_{n+1}, & i = 1, \dots, n\,, \quad \text{with } k_\alpha = 1\,, \\ e_i - \sqrt{k^*} e_{n+2}, & i = 1, \dots, n\,, \quad \text{with } k_\alpha = 1\,, \\ \sqrt{k} e_{n+1} - \sqrt{k^*} e_{n+2} & \quad \text{with } k_\alpha = 1\,. \end{cases}$$

Here k is an arbitrary parameter, $k^* = -1 - k$, and $W = S_n$. For $k \in \mathbb{Z}$ this is a locus configuration from [CV2].

7.4. **Restricted Coxeter configurations.** Another class of generalised locus configurations can be found in [F]. These configurations appear as restrictions of Coxeter root systems onto suitable subspaces (parabolic strata). They are labelled by pairs (Γ, Γ_0) of Dynkin diagrams where $\Gamma_0 \subset \Gamma$ is possibly disconnected. For a given Γ, the admissible sub-diagrams Γ_0 are characterized by a certain geometric condition (see [F], Theorems 1-3). The classical case $\Gamma = A_n, B_n, D_n$ leads to special cases of the configurations already listed above. The list of possibilities for the exceptional cases $\Gamma = F_4, E_{6-8}, H_4, H_3$ includes 43 cases and can be found in Section 6 of [F]. For example, one has the following configuration (F_4, A_1) in $\mathbb{C}^3$, see [F, (27)]:

$$\begin{cases} \pm e_i, & 1 \leq i \leq 3, \quad \text{with } k_\alpha = 2c + \frac{1}{2}\,, \\ \pm e_i \pm e_j, & 1 \leq i < j \leq 3\,, \quad \text{with } k_\alpha = c\,, \\ \pm e_1 \pm e_2 \pm e_3 & \quad \text{with } k_\alpha = 1\,. \end{cases}$$

We have checked, case-by-case, that all two-dimensional configurations in [F] satisfy the generalised locus conditions (and, in fact, can be constructed by the method of Proposition 8.1 below). Apart from the two-dimensional and Coxeter configurations, the list in [F, Section 6] contains 23 additional cases. Note that the fact that all of these are indeed generalised locus configurations does *not* follow directly from their construction in [F], and so one has to rely upon case-by-case verification. We expect that all configurations in [F] satisfy Definition 3.3, though we have not checked this for all of the cases. Note that, similar to Remark 3.2, it is sufficient to check the conditions of Definition 3.3 for all two-dimensional subconfigurations.

Remark 7.4. The configurations described in 7.1, 7.2 and 7.4 are non-twisted. This is obvious for the case 7.1; for the cases in 7.2 it can be checked directly. For those cases in 7.4 where $k_\alpha = 1$ for $\alpha \in \mathcal{A} \setminus R$, this follows from [F, Proposition 2]; for the remaining cases it can be verified directly, case by case. The $A_{n-1,2}$ configuration is twisted; also, the configurations constructed in Proposition 8.1 below are twisted in general.

Remark 7.5. Locus configurations can be obtained from the above cases by specialising parameters. The complete list of known locus configurations consists of: (1) Coxeter configurations, with all $k_\alpha \in \mathbb{Z}$, (2) $A(n, 1)$ with

$k \in \mathbb{Z}$, (3) $B(n,1)$ with $k, a, b \in \mathbb{Z}$, (4) $A_{n-1,2}$ with $k \in \mathbb{Z}$, and (5) the Berest–Lutsenko family in $\mathbb{C}^2$ (see [BL, CFV2, C2]).

8. Two-dimensional configurations

For this class of configurations, $V = \mathbb{C}^2$ and $W = I_N$, $N \geq 2$ is the dihedral group of order $2N$. A systematic way to produce generalised locus configurations in $\mathbb{C}^2$ uses Darboux transformations. Let us analyse first the case of $W = I_2$ with $R = \{\pm e_1, \pm e_2\}$. The corresponding Calogero–Moser operator is written in polar coordinates r, φ as

$$L = \frac{\partial^2}{\partial r^2} - r^{-2} L_0 \,, \qquad L_0 = -\frac{\partial^2}{\partial \varphi^2} + \frac{g(g-1)}{\sin^2 \varphi} + \frac{h(h-1)}{\cos^2 \varphi} \,,$$

where L_0 is known as the Darboux–Pöschl–Teller (DPT) operator. We used parameters g, h in L_0 instead of the multiplicities k_{e_1}, k_{e_2}, in accordance with the common tradition. It is well-known that L_0 has a family of eigenfunctions of the form

$$\psi_n(x) = (\sin x)^g (\cos x)^h P_n^{g-\frac{1}{2}, h-\frac{1}{2}}(\cos 2x) \,, \quad n = 0, 1, 2, \dots \,,$$

where $P_n^{\alpha,\beta}(z)$ are the classical Jacobi polynomials. Since L_0 is stable under the change of parameters $g \mapsto 1-g$ or $h \mapsto 1-h$, we obtain three additional families of (formal) eigenfunctions by replacing g, h accordingly in the above ψ_n. Let $\mathcal{F}$ denote the union of these four families of functions.

Proposition 8.1. *For distinct $f_1, \dots, f_m \in \mathcal{F}$, the potential defined by*

$$U = \frac{1}{r^2}\left(\frac{g(g-1)}{\sin^2 \varphi} + \frac{h(h-1)}{\cos^2 \varphi} - 2\frac{d^2}{d\varphi^2} \log \mathcal{W}\right) \,, \quad \mathcal{W} = \mathrm{Wr}(f_1, \dots, f_m) \,,$$

satisfies the conditions (3.2) *and its singularities form a generalised locus configuration in $\mathbb{C}^2$ of type $W = I_2$.*

Proof. Let

$$u_0 = \frac{g(g-1)}{\sin^2 \varphi} + \frac{h(h-1)}{\cos^2 \varphi} \,, \qquad u = u_0 - 2\frac{d^2}{d\varphi^2} \log \mathcal{W}$$

By a standard result on Darboux transformations (see e.g. [Cr]), the operators

$$L_0 = -\frac{\partial^2}{\partial \varphi^2} + u_0 \qquad L_1 = -\frac{\partial^2}{\partial \varphi^2} + u$$

are intertwined by a unique monic differential operator D of order m whose kernel is spanned by all of f_i. From the choice of f_i it is easy to see that D and L_1 will be I_2-invariant, and the intertwiner D will have meromorphic coefficients. The operator L_1 typically will have additional singular points compared to L_0. Clearly, for generic $E \in \mathbb{C}$ the intertwiner D sets up a bijection between solutions of the eigenvalue problems $L_0 \psi_0 = E \psi_0$ and $L\psi = E\psi$. Since the coefficients of D are meromorphic, both ψ_0 and ψ remain meromorphic (single-valued) away from the singular points of L_0.

Invoking Proposition 3.3 from [DG], we conclude that near every additional singular point $\varphi = \varphi_i$ of L_1 one has that

$$u(\varphi) \sim k_i(k_i+1)(\varphi-\varphi_i)^{-2} + o(1)\,, \quad \text{with some } k_i \in \mathbb{Z}_+.$$

Moreover, by the same result, one has that

$$u(\varphi) - u(s_i\varphi) \quad \text{is divisible by} \quad (\varphi-\varphi_i)^{2k_i}\,,$$

for the reflection $s_i : \varphi \mapsto 2\varphi_i - \varphi$ about $\varphi = \varphi_i$. This implies the generalised locus conditions (3.2) for $U = r^{-2}u$. □

Remark 8.2. As explained in Section 4, the existence of shift operators (and hence complete integrability) for the Calogero-Moser operators L with potentials described in Proposition 8.1 follows immediately from Lemma 4.1. The question whether Proposition 8.1 gives *all* generalised locus configurations in $\mathbb{C}^2$ is more delicate and will be discussed elsewhere.

The family constructed above can be seen as a generalisation of the Berest–Lutsenko family of locus configurations [BL, CFV2]. For comparison, in the Berest–Lutsenko family one has $W = \{e\}$ and u is constructed by applying Darboux transformations to $L_0 = \frac{\partial^2}{\partial\varphi^2}$.

In the case of $W = I_N$, the Calogero–Moser operator can be written as $L = \Delta - r^{-2}u_0$, where

$$u_0 = \begin{cases} g(g-1)N^2 \sin^{-2} N\varphi & \text{for } N \text{ odd}\,, \\ g(g-1)n^2 \sin^{-2} n\varphi + h(h-1)n^2\cos^{-2} n\varphi & \text{for } N = 2n \text{ even}\,. \end{cases}$$

The group W is generated by the reflection $\varphi \mapsto -\varphi$ and rotation $\varphi \mapsto \varphi + 2\pi/N$. Note, however, that any configuration is automatically invariant under $\varphi \mapsto \varphi + \pi$. Thus, even when N is odd, the full symmetry group can be taken as $W = I_{2N}$. This allows us to consider the case of odd N as a subcase of $W = I_{2N}$, with $h = 0$. Thus, below we restrict ourselves to the case $W = I_{2n}$.

A generalised locus configuration $\mathcal{A}$ is obtained by adding to the roots of $W = I_{2n}$ a number of vectors with integral multiplicities. The configuration $\mathcal{A}$ and multiplicities must be invariant under $\varphi \mapsto \varphi + \pi/n$. Therefore, we must have

$$L_{\mathcal{A}} = \frac{\partial^2}{\partial r^2} - r^{-2}L_1\,, \qquad L_1 = -\frac{\partial^2}{\partial\varphi^2} + \frac{g(g-1)n^2}{\sin^2 n\varphi} + \frac{h(h-1)n^2}{\cos^2 n\varphi} + \sum_i \frac{k_i(k_i+1)n^2}{\sin^2 n(\varphi-\varphi_i)}\,,$$

for some $\varphi_i \in \mathbb{C}$ and $k_i \in \mathbb{Z}_+$. The conditions (3.2) for every $\alpha = (\cos\varphi_i, \sin\varphi_i)$ translate into

$$u(\varphi) - u(s_i\varphi) \quad \text{is divisible by} \quad (\varphi-\varphi_i)^{2k_i}\,,$$

for each of the reflections $s_i : \varphi \mapsto 2\varphi_i - \varphi$. As we see from the proof the previous proposition, the generalised locus conditions (3.2) express the property that eigenfunctions of L_1 are single-valued near each of the singular points $\varphi = \varphi_i$. Clearly, this property is preserved under rescaling $\varphi \mapsto \varphi/n$. Under such rescaling, L_1 takes the form

$$L_1 = n^2 \left(-\frac{\partial^2}{\partial \varphi^2} + \frac{g(g-1)}{\sin^2 \varphi} + \frac{h(h-1)}{\cos^2 \varphi} + \sum_i \frac{k_i(k_i+1)}{\sin^2(\varphi - \varphi_i)} \right) .$$

Thus, we obtain the following result which reduces the classification problem in type $W = I_{2n}$ to that in type $W = I_2$.

Proposition 8.3. *Every generalised locus configurations $\mathcal{A}$ of type $W = I_{2n}$ is produced from some generalised Calogero–Moser operator L of type $W = I_2$,*

$$L = \frac{\partial^2}{\partial r^2} - r^{-2} \left(\frac{\partial^2}{\partial \varphi^2} - u(\varphi) \right) ,$$

by the formula

$$L_{\mathcal{A}} = \frac{\partial^2}{\partial r^2} - r^{-2} \left(\frac{\partial^2}{\partial \varphi^2} - n^2 u(n\varphi) \right) .$$

Let us illustrate this with two examples of type $W = I_2$, with the Calogero-Moser operator of the following form:

$$L = \Delta - \frac{k_1(k_1+1)}{x_1^2} - \frac{k_2(k_2+1)}{x_2^2} - \frac{k_3(k_3+1)(1+a^2)}{(x_1 - a x_2)^2} - \frac{k_3(k_3+1)(1+a^2)}{(x_1 + a x_2)^2} .$$

The parameters k_1, k_2, k_3 and a are as follows:

(1) k_1, k_2 arbitrary, $k_3 = 1$, $a = \sqrt{\frac{2k_1+1}{2k_2+1}}$;
(2) $k_1 = \frac{3a^2}{4} - \frac{1}{4}$, $k_2 = \frac{3}{4a^2} - \frac{1}{4}$, $k_3 = 2$, a arbitrary.

In both cases, the locus conditions (3.3) for $\alpha = e_1 \pm a e_2$ are easy to verify directly. The first case corresponds to $m = n = 1$ in the $B(n,m)$ family. The second case was proposed and studied in [T2], where the integrability of L (including its elliptic version) was confirmed. Note that the first case can be obtained by setting $m = 1$, $f_1 = \psi_1$ in Proposition 8.1.

Now let us apply the substitution $\varphi \mapsto 2\varphi$ in accordance with Proposition 8.3. This leads to the Calogero–Moser operators of type $W = I_4$ of the form

$$\begin{aligned} L = \Delta - \frac{k_1(k_1+1)}{x_1^2} - \frac{k_1(k_1+1)}{x_2^2} - \frac{k_2(k_2+1)}{(x_1+x_2)^2} - \frac{k_2(k_2+1)}{(x_1-x_2)^2} \\ - \frac{k_3(k_3+1)(1+b^2)}{(x_1 - b x_2)^2} - \frac{k_3(k_3+1)(1+b^2)}{(x_1 + b x_2)^2} \\ - \frac{k_3(k_3+1)(1+b^2)}{(x_2 - b x_1)^2} - \frac{k_3(k_3+1)(1+b^2)}{(x_2 + b x_1)^2} . \end{aligned}$$

Here k_1, k_2, k_3 are the same as above, while a and b are related by $a = 2b/(1-b^2)$. In the case $k_3 = 1$, the above L coincides with [F, (28)].

As another example, applying the substitution $\varphi \mapsto 3\varphi$ to the case $(k_1, k_2, k_3, a) = (\frac{1}{3}, 4, 1, \frac{\sqrt{5}}{3\sqrt{3}})$, one obtains the configuration that coincides (up to an overall rotation) with the configuration $(\mathcal{H}_4, \mathcal{A}_2)$ in [F].

Remark 8.4. Most configurations constructed in Propositions 8.1, 8.3 are twisted. For the study of non-twisted planar configurations we refer to [FJ] (see, in particular, Proposition 3.1 and Theorem 3.4 in *loc. cit.*).

9. Deformed Calogero–Moser operators with harmonic oscillator confinement

With a generalised locus configuration $\mathcal{A} \subset V$ one can associate the Calogero–Moser operator with an extra "oscillator term":

$$L_{\mathcal{A}}^{\omega} := \Delta - \omega^2 x^2 - \sum_{\alpha \in \mathcal{A}_+} \frac{k_\alpha(k_\alpha+1)(\alpha,\alpha)}{(\alpha,x)^2}\,, \tag{9.1}$$

where $x^2 = (x,x)$ is the Euclidean square norm in $V = \mathbb{R}^n$, and ω is an arbitrary parameter. In particular, when $\mathcal{A} = R$ is the root system of a Coxeter group W, we have

$$L_{W}^{\omega} := \Delta - \omega^2 x^2 - \sum_{\alpha \in R_+} \frac{k_\alpha(k_\alpha+1)(\alpha,\alpha)}{(\alpha,x)^2}\,. \tag{9.2}$$

For the classical groups $W = A_n, B_n, D_n$ the operator L_W^{ω} is known to be Liouville integrable (see [F] and references therein). For the exceptional groups this does not seem to be known (for dihedral groups the complete integrability is easy to show, see below). Still, for any Coxeter group the operator L_W^{ω} has several hallmarks of integrability, and some of them are also shared by $L_{\mathcal{A}}^{\omega}$.

Theorem 9.1. *Let $\mathcal{A} \subset \mathbb{C}^n$ be a generalised locus configuration of type W.*

(1) *Let $S^{\omega} = e^{-\omega x^2/2} S e^{\omega x^2/2}$, where S is the shift operator* (5.12). *Then*

$$L_{\mathcal{A}}^{\omega} S^{\omega} = S^{\omega}(L_W^{\omega} - 2\omega N)\,, \tag{9.3}$$

where $N = \deg \delta$ is the degree of the polynomial (5.1). *Moreover, if $(S^{\omega})^*$ is the formal adjoint of S^{ω}, then the operator $S^{\omega}(S^{\omega})^*$ commutes with $L_{\mathcal{A}}^{\omega}$.*

(2) *For any homogeneous $q \in \mathbb{C}[V]^W$ define $L_{q,0}^{\omega} = e^{-\omega x^2/2} L_{q,0}\, e^{\omega x^2/2}$, where $L_{q,0} = \mathtt{Res}(\boldsymbol{e} T_q \boldsymbol{e})$. Then*

$$L_{q,0}^{\omega} L_W^{\omega} = (L_W^{\omega} + 2\omega r) L_{q,0}^{\omega}\,,$$

where $r = \deg q$. As a result, for any $q \in \mathbb{C}[V]^W$, the operator $L_{q,0}^{\omega} L_{q,0}^{-\omega}$ commutes with L_W^{ω}.

(3) *For any homogeneous $q \in Q_{\mathcal{A}}$ define $L_q^{\omega} = e^{-\omega x^2/2} L_q\, e^{\omega x^2/2}$, where L_q is the operator constructed in Proposition 5.6. Then*

$$L_q^{\omega} L_{\mathcal{A}}^{\omega} = (L_{\mathcal{A}}^{\omega} + 2\omega r) L_q^{\omega}\,,$$

where $r = \deg q$*. As a result, for any* $q \in Q_{\mathcal{A}}$*, the operator* $L_q^{\omega} L_q^{-\omega}$ *commutes with* $L_{\mathcal{A}}^{\omega}$*.*

Proof. Define

$$L_0^{\omega} = e^{\omega x^2/2} L_W^{\omega} e^{-\omega x^2/2}\,, \qquad L^{\omega} = e^{\omega x^2/2} L_{\mathcal{A}}^{\omega} e^{-\omega x^2/2}\,.$$

By direct calculation, $L_0^{\omega} = L_W - 2\omega E$ and $L^{\omega} = L_{\mathcal{A}} - 2\omega E$, where $E = \sum_{i=1}^n x_i \partial_i$ is the Euler operator. By the homogeneity of $L_W, L_{\mathcal{A}}, S$, we have

$$[E, L_W] = -2L_W\,, \quad [E, L_{\mathcal{A}}] = -2L_{\mathcal{A}}\,, \quad [E, S] = -NS\,.$$

Thus, using that $L_{\mathcal{A}} S = S L_W$, we obtain

$$(L_{\mathcal{A}} - 2\omega E)S = S(L_W - 2\omega E) + 2\omega[E, S] = S(L_W - 2\omega E - 2\omega N)\,,$$

or $L^{\omega} S = S(L_0^{\omega} - 2\omega N)$. Conjugating this relation by $e^{\omega x^2/2}$ gives (9.3). Furthermore, taking formal adjoints in (9.3), we obtain $(S^{\omega})^* L_{\mathcal{A}}^{\omega} = (L_W^{\omega} - 2\omega N)(S^{\omega})^*$. Combining this with (9.3) gives $L_{\mathcal{A}} S^{\omega} (S^{\omega})^* = S^{\omega} (S^{\omega})^* L_{\mathcal{A}}$. This proves part (1).

For part (3), we first note that the expression for L_q given in Proposition 5.6 is obviously homogeneous of degree $-r$. Using this and $L_q L_{\mathcal{A}} = L_{\mathcal{A}} L_q$, we get

$$L_q(L_{\mathcal{A}} - 2\omega E) = (L_{\mathcal{A}} - 2\omega E)L_q + 2\omega r L_q\,,$$

or $L_q L^{\omega} = L^{\omega} L_q + 2\omega r L_q$. Conjugating this by $e^{\omega x^2/2}$ gives $L_q^{\omega} L_{\mathcal{A}}^{\omega} = (L_{\mathcal{A}}^{\omega} + 2\omega r) L_q^{\omega}$, as needed. From this we get $L_q^{-\omega} L_{\mathcal{A}}^{\omega} = (L_{\mathcal{A}}^{\omega} - 2\omega r) L_q^{-\omega}$. The fact that $L_q^{\omega} L_q^{-\omega}$ commutes with $L_{\mathcal{A}}^{\omega}$ is then obvious. This proves part (3). Part (2) is entirely similar. ☐

Remark 9.2. For the locus configurations (when $W = \{e\}$), the existence of an intertwiner S^{ω} satisfying (9.3) was established in [CO] by a considerably more involved argument.

Remark 9.3. When $W = I_N$ is the dihedral group, one additional operator commuting with L_W^{ω} is sufficient for complete integrability. The ring of invariants $\mathbb{C}[V]^W$ in this case is generated by two homogeneous elements $q_1 = x^2$ and q_2. Thus, the complete integrability of L_W^{ω} follows from part (2) of the theorem. Similarly, when $\mathcal{A} \subset \mathbb{C}^2$ is a generalised locus configuration, the operator $S^{\omega}(S^{\omega})^*$ is sufficient to conclude that $L_{\mathcal{A}}^{\omega}$ is completely integrable.

10. Affine configurations

Our main results can be easily extended to affine (i.e., noncentral) hyperplane arrangements. As before, we start with a Coxeter group W with root system R, in its reflection representation V equipped with a W-invariant scalar product $(\cdot, \cdot)$. Let $\widehat{V}$ be the vector space of affine-linear functions on V. We identify $\widehat{V}$ with $V \oplus \mathbb{C}\delta$, where vectors in V are considered as linear functionals on V via the scalar product $(\cdot, \cdot)$ and where $\delta \equiv 1$ on V. The action of W extends onto $\widehat{V}$ in an obvious way, with $w(\delta) = \delta$ for all $w \in W$.

For any $\widetilde{\alpha} = \alpha + c\delta \in \widehat{V}$ we have the orthogonal reflection with respect to the hyperplane $\widetilde{\alpha}(x) = 0$ in V,

$$s_{\widetilde{\alpha}}(x) = x - 2\widetilde{\alpha}(x)\alpha/(\alpha,\alpha)\,, \quad x \in V\,.$$

Given a finite affine hyperplane arrangement in V with prescribed multiplicities, we encode it in a finite set $\mathcal{A}_+ = \{\widetilde{\alpha}\} \subset \widehat{V}$ and a set of $k_{\widetilde{\alpha}} \in \mathbb{C}$. The hyperplanes that pass through the origin $0 \in V$ will be thus associated with vectors $\alpha \in V$. If the configuration is *central* (with all hyperplanes passing through 0), we are back to the previously considered case. As before, we extend the map $k :\ \widetilde{\alpha} \mapsto k_{\widetilde{\alpha}}$ to $\mathcal{A} := \mathcal{A}_+ \cup (-\mathcal{A}_+)$ by putting $k_{-\widetilde{\alpha}} = k_{\widetilde{\alpha}}$. With such a configuration of hyperplanes we associate a generalised Calogero–Moser operator

$$L_{\mathcal{A}} = \Delta - u_{\mathcal{A}}\,, \qquad u_{\mathcal{A}} = \sum_{\widetilde{\alpha}\in\mathcal{A}_+} \frac{k_{\widetilde{\alpha}}(k_{\widetilde{\alpha}}+1)(\alpha,\alpha)}{(\widetilde{\alpha}(x))^2}\,. \tag{10.1}$$

Definitions 3.3, 3.4 require obvious modifications in the affine case.

Definition 10.1. An affine configuration $\mathcal{A}$ is a *generalised locus configuration* if

(1) $\mathcal{A}$ contains a Coxeter configuration of type W, and both $\mathcal{A}$ and $k : \mathcal{A} \to \mathbb{C}$ are invariant under W;

(2) For any $\widetilde{\alpha} \in \mathcal{A} \setminus R$ one has $k_{\widetilde{\alpha}} \in \mathbb{Z}_+$ and

$$u_{\mathcal{A}}(x) - u_{\mathcal{A}}(s_{\widetilde{\alpha}}x) \ \text{ is divisible by } \widetilde{\alpha}^{2k_{\widetilde{\alpha}}}\,.$$

Definition 10.2. Let $\mathcal{A}$ be a generalised locus configuration of type W. A polynomial $q \in \mathbb{C}[V]^W$ is called a *generalised quasi-invariant* if, for any $\widetilde{\alpha} \in \mathcal{A} \setminus R$,

$$q(x) - q(s_{\widetilde{\alpha}}x) \ \text{ is divisible by } \widetilde{\alpha}^{2k_{\widetilde{\alpha}}}\,.$$

In the affine case the ring $Q^W_{\mathcal{A}}$ of generalised quasi-invariants is no longer graded. Let $(Q^W_{\mathcal{A}})^0$ be the associated graded ring, and $\mathtt{gr} : Q^W_{\mathcal{A}} \to (Q^W_{\mathcal{A}})^0$ be the corresponding linear map.

The following results are proved by the same arguments as Theorem 3.5 and Proposition 5.6.

Proposition 10.3. *Define $\delta \in \mathbb{C}[V]^W$ by $\delta = \prod_{\widetilde{\alpha}\in\mathcal{A}_+\setminus R} \widetilde{\alpha}^{k_{\widetilde{\alpha}}}$, and let $N = \deg\delta$. Let $S = \frac{1}{2^N N!}\mathtt{ad}^N_{L,L_0}(\delta)$, where $L = L_{\mathcal{A}}$, $L_0 = L_W$. Then $S = \prod_{\widetilde{\alpha}\in\mathcal{A}_+\setminus R}(\alpha,\partial)^{k_{\widetilde{\alpha}}} + \dots$ with lower order terms being of negative x-degree. Furthermore, $LS = SL_0$.*

Proposition 10.4. *For any homogeneous element $q \in (Q^W_{\mathcal{A}})^0$ with $\deg q = r$, define $L_q = \frac{1}{2^r r!}\mathtt{ad}^r_L(\widetilde{q})$, where $\widetilde{q} \in Q^W_{\mathcal{A}}$ is arbitrary such that $\mathtt{gr}(\widetilde{q}) = q$.*

The operators L_q, $q \in (Q^W_{\mathcal{A}})^0$ commute with $L_{\mathcal{A}}$ and with each other. Each of L_q has the form $L_q = q(\partial) + \dots$, and $L_qS = SL_{q,0}$, where $L_{q,0} = \mathtt{Res}(\boldsymbol{e}T_q\boldsymbol{e})$.

This implies that Theorem 3.5 remains true for affine configurations, with $(Q^W_{\mathcal{A}})^0$ replacing $Q^W_{\mathcal{A}}$ and with further minor modifications in the proof. Therefore, the operator $L_{\mathcal{A}}$ is completely integrable.

Let us apply this result in dimension one, with $W = \mathbb{Z}_2$, where we can see a close connection to the results of Duistermaat and Grünbaum [DG]. In this case we have an operator of the form

$$L = \frac{d^2}{dx^2} - u(x)\,, \qquad u(x) = \frac{k(k+1)}{x^2} + \sum_{p\in\mathcal{P}} \frac{k_p(k_p+1)}{(x-p)^2}\,, \tag{10.2}$$

where $\mathcal{P}$ is a finite subset of $\mathbb{C}\setminus\{0\}$, symmetric around 0, and where $k_p \in \mathbb{Z}_+$ with $k_{-p} = k_p$. The generalised locus conditions mean that (cf. [DG, (4.45)–(4.46)])

$$\frac{k(k+1)}{p^{2j+1}} + \sum_{q\in\mathcal{P}\setminus\{p\}} \frac{k_q(k_q+1)}{(p-q)^{2j+1}} = 0 \quad \text{for } 1 \le j \le k_p \text{ and all } p \in \mathcal{P}. \tag{10.3}$$

Therefore, we obtain the following result.

Proposition 10.5. *For any operator L* (10.2) *satisfying the conditions* (10.3)*, there exists a nonzero differential operator S with rational coefficients such that $LS = SL_0$, where $L_0 = \frac{d^2}{dx^2} - \frac{k(k+1)}{x^2}$. In other words, L can be obtained from L_0 by a higher order rational Darboux transformation.*

This should be compared with Proposition 4.3 from [DG]. In fact, using the results of [DG], we can prove a stronger result.

Proposition 10.6. *For any operator L* (10.2) *satisfying* (10.3) *with $\mathcal{P} \neq \emptyset$, we have* (1) *$k \in \frac{1}{2}+\mathbb{Z}$, and* (2) *$L$ can be obtained from $L_0 = \frac{d^2}{dx^2} + \frac{1}{4x^2}$ by applying a finite number of rational, first order Darboux transformations.* (*In other words, L belongs to the "even family" of bispectral operators introduced in* [DG].)

We omit the proof. The main idea is to use Proposition 10.5 to conclude that L is *bispectral.* Hence, by the classification result [DG, Theorem 0.1], it must belong either to the "KdV family" or to the "even family" (see [DG], Sections 3 and 4). Note that u in (10.2) is even, $u(x) = u(-x)$. However, from the explicit description of the KdV family (see e.g. [AM]) one can conclude that the only even KdV potentials are $u = k(k+1)x^{-2}$ with $k \in \mathbb{Z}$. Hence, L must belong to the "even family".

In view of Propositions 10.5, 10.6, we may regard the generalised locus conditions and the corresponding Calogero–Moser operators as a *multivariable generalisation* of the "even family" from [DG]. (It is interesting that in dimension > 1 the multiplicities k_α for $\alpha \in R$ do not have to be half-integers.) Unfortunately, we know very few genuinely affine generalised locus configurations in dimension > 1. Here is one two-dimensional example; it is of type A_2, and it can be viewed as a deformation of the root system of type G_2. It can be realised in $\mathbb{R}^3$ as $\widetilde{G}_2 = A_2 \cup \widetilde{A}_2$, where:

$$\begin{aligned} A_2 &= \{\pm(e_i - e_j),\ 1 \le i < j \le 3\}, \text{with} \quad k_\alpha = -1/3\,, \\ \widetilde{A}_2 &= \{\pm(3e_i - e_1 - e_2 - e_3 + c\delta),\ 1 \le i \le 3\}\,, \text{with} \quad k_\alpha = 1\,. \end{aligned}$$

The corresponding Calogero–Moser operator is

$$L_{\widetilde{G}_2} = \Delta - \frac{4}{9} \sum_{1 \le i < j \le 3} \frac{1}{(x_i - x_j)^2} - \frac{12}{(2x_1 - x_2 - x_3 + c)^2} - \frac{12}{(2x_2 - x_1 - x_3 + c)^2} - \frac{12}{(2x_3 - x_1 - x_2 + c)^2}\,. \tag{10.4}$$

Here c is arbitrary; for $c = 0$ we have a G_2 configuration.

Remark 10.7. Further examples in dimension > 1 can be obtained by taking direct sums of one-dimensional configurations. Another possibility is to apply the procedures of *isotropic reduction* and *isotropic projectivization* as described in [CFV2, Sec. 5.3].

Remark 10.8. By analogy with the results of [DG], it is natural to expect that the Calogero–Moser operators for generalised locus configurations are *bispectral*. In particular, we expect them to be bispectrally *self-dual* when the configuration is central (cf. [CFV2, Theorem 2.3]). Affine configurations, such as $\widetilde{G}_2$, should lead to examples of non-trivial bispectral duality.

Remark 10.9. In [SV2], a trigonometric version of deformed Calogero–Moser operators was considered. Our methods cannot be applied verbatim to that case and require a non-trivial modification. We hope to return to this problem elsewhere. Some results about the trigonometric locus configurations can be found in [C2, Section 4]. Let us also mention a recent paper [FVr], where a trigonometric generalisation of the operator (10.4) is proposed.

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