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Strategic investment with positive externalities

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ABSTRACT

We study strategic investment in continuous time with positive externalities of changing magnitude. Our model particularly allows for two correlated risk factors. Constructing subgame-perfect equilibria with pure and mixed strategies, we observe the novel effect that it is important for the firms to *anticipate* preemption. In fact, the presence of a second risk factor implies also an additional strategic risk. We quantify the associated extra waiting cost and show that it is ex ante uncertain whether investment will happen when there is a first- or a second-mover advantage. Our formal arguments involve several methodological contributions. In addition, we provide detailed specifications of our basic model to address various applications.

1. Introduction

In the last few decades, our understanding of firms' timing decisions under uncertainty has been much improved by explicitly considering competition. In a competitive environment, the effects of firms' decisions on each other can be at least as important as other circumstances that are beyond any firm's control. For instance, if it is crucial to be the first to use a particular investment option, the possibility to wait for optimal conditions is significantly limited. Or, if a firm faces losses in some business branch, the hope for a turning point may rest on the exit of a competitor rather than on other, external developments.

A crucial distinction for the effects of competition is, as in these two different examples, whether the firms' actions cause negative or positive externalities on each other. Investments are often associated with negative externalities, because they improve the investing firm's position, so that, relatively, its competitors are weakened. The common lesson in such cases is that the firms strategically preempt each other as soon as the investment predicts a relative gain, i.e., they compete for a *first-mover advantage*. In contrast, if there is a strong positive externality, like in the second example, the only way to win the main benefit is to wait longer than the competitor, i.e., there is a dominant *second-mover advantage*. The strategic interaction then typically resembles a war of attrition, and the firm that gives up and acts first appears as the loser.

In this paper, we are interested in interactions that differ from the essentially well-understood standard cases. We consider cases with positive externalities that are not a priori dominating. This means that taking the first action yields some

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benefit for a competitor, but the benefit for the acting firm itself may also be substantial—and possibly even higher. We think of several economically relevant settings, for instance the following three.

(1) Suppose there is a license or patent for a new product for sale, i.e., only one of several potential firms can bring this product to market. However, as soon as the product is offered, a new market for services related to that product is created. Such services are not protected, so there is a new business opportunity for other firms, too. Whether offering the product or the service is more profitable depends, amongst others, on the price of the licence or patent.

(2) Suppose two firms produce a similar good and compete in a common market. Both have the possibility to undertake a major adjustment of their facilities in order to produce another, differentiated good instead. If one of the firms uses this *replacement option*, both become monopolists in their respective markets. Which firm benefits most depends, amongst others, on the developments of the markets for the established and the differentiated good.

(3) Suppose two firms produce similar goods and both have the option to adopt a new technology, which then creates some spillovers. The relative profits again depend on various factors, e.g., the price for adopting the technology, the magnitude of spillovers, how close competitors the firms are, or if a cheaper substitute technology will become available.

An important further aspect of such settings is that there is likely more than only one risk factor. When one firm has acted (i.e., used the option), the firms' businesses are differentiated. In such cases it is natural to assume that there are some separate risk factors or that the weight of shared risk factors has changed. We, thus, consider that the different firms' risks after exercising the option are not identical—they may be correlated, but not necessarily perfectly.

And indeed, we show in this paper that modeling at least two-dimensional uncertainty is what leads to qualitatively new results compared to the aggregate of the literature (of which we provide an overview below). Our first observation is that the externalities we consider imply an interaction with phases of attrition *and* phases of preemption. Not many other settings have been studied in which this happens. The greatest distinction of our model is, then, that we observe a *feedback effect* between the two principally known strategic regimes. Specifically, we find that the anticipation of preemption creates additional costs of waiting while there is still a second-mover advantage, so that the firms should consider using the option earlier. This novel effect is driven by our two-dimensional uncertainty. At which times there is a preemption incentive depends more on the *relative* than absolute values of the risk factors that affect the investment reward and externality. There is, thus, a threat that preemption kicks in at a point where investment is less profitable. The nonlinearity of the state space then implies a region where this threat dominates the chances of further gains, so that earlier investment is optimal. In typical one-dimensional models, by contrast, such an anticipation has no observable effect. A further distinction of our model is that it is, ex ante, indeed uncertain which strategic incentive will eventually drive the option exercise decision, i.e., whether investment will happen to escape attrition or as a preemptive move.

We derive these findings by means of a reduced model (which is in fact a two-dimensional variant of classic real option models), because the mathematical problems we need to consider differ significantly from the usual ones. The reduced model allows us to focus on the novel effects. We construct subgame-perfect equilibria with pure and mixed strategies on the basis of a two-dimensional constrained optimal stopping problem. This problem is of a new type, and its solution determines the endogenous waiting cost in equilibrium. Therefore, we aim for a characterization that is as explicit as possible. For an equilibrium, the first crucial question is at which times waiting is costly at all, and we prove in terms of the solution of the stopping problem that anticipating preemption makes a difference in our model. The follow-up question is how high exactly the waiting cost is. We characterize it and then prove that it generates a strategic risk of whether investment will happen when there is a first- or a second-mover advantage. Both of these novel effects are additionally evaluated numerically, so we can quantify and visualize the firms' strategic tradeoffs.

Our results for the reduced model hold also for applications with many different features. We set up two more detailed models, one of competition for a replacement option and one of technology adoption with uncertain reward and spillovers, and show that the analysis is in each case formally equivalent to that of the reduced model. Therefore, our findings are substantially more general than at first may appear. Finally, we discuss in detail some possible extensions of our model for which we expect analogous results to hold, but which would require adjusted proofs due to the challenges of a two-dimensional state space.

Related literature

There is a large body of literature on strategic investment timing, focusing on many different aspects and applications. The strategic value of preemption was pointed out prominently in Fudenberg and Tirole (1985) within the context of technology adoption, where a negative externality caused a first-mover advantage. These ideas proved to be quite fundamental and have had a far-reaching impact, e.g., on the theory of real options, where it was shown that game-theoretic approaches imply a substantial correction to established valuation formulas. Some of the first studies of strategic investment timing under uncertainty considered FDI (Smets, 1991), real estate markets (Grenadier, 1996), or R&D competition (Weeds, 2002); early conceptual treatments are Huisman (2001) or Huisman et al. (2004); and surveys of this development can be found in Chevalier-Roignant and Trigeorgis (2011) or Azevedo and Paxson (2014). The effects in these models are essentially very similar—negative externalities imply preemptive investment—and this consequence is in fact proved within a more general framework in Steg (2018).

The interaction in settings with a systematic second-mover advantage is usually a war of attrition. Ghemawat and Nalebuff (1985) and Fudenberg and Tirole (1986), for instance, study exit timing problems of firms that are accumulating losses, but which would become profitable when holding on longer than the competitor. Strategic exit timing under uncertainty is analyzed in, e.g., Lambrecht (2001), Murto (2004), and Georgiadis et al. (2022).

Positive externalities can also arise for other reasons. Katz and Shapiro (1987) consider licensing and imitation opportunities after the leader has implemented a new technology, and Hoppe (2000) addresses the possibility to gain information regarding the profitability of an investment project by observing the leader's performance. Informational spillovers with dynamic uncertainty are studied in, e.g., Décamps and Mariotti (2004), Thijssen et al. (2006), and Kwon et al. (2016). Depending on the strength of the respective externality, some of these models may become preemption games, and the model in Agrawal et al. (2016) is actually a hybrid case, where both preemption and attrition occur in different phases of the game.

Dynamic uncertainty is usually modeled by a one-dimensional diffusion or Poisson process. The effects of different uncertainty models for preemption games are studied in, e.g., Thijssen (2010), where uncertainty is player-specific, and Hellmann and Thijssen (2018), where the players perceive probabilistic ambiguity.

We will further discuss the relation of our results to both the theoretical and the empirical literature on strategic investment in Section 9.

Organization of this paper

We set up our basic reduced model in Section 2 and define the timing game in Section 3. In Section 4, we derive the structure of our two-dimensional state space and the solution of the essential constrained optimal stopping problem. This allows us to construct subgame-perfect equilibria in Sections 5 (with pure strategies) and 6 (with mixed strategies). We quantify and illustrate our results by a numerical analysis in Section 7, for which we develop a particular finite-difference scheme. In Section 8, we present two different applications with initially more detailed models that can be reduced to our basic model. Then we discuss the relation of our findings to the existing literature in Section 9, where we in particular point out the central role of two-dimensional uncertainty. Finally, we conclude by discussing possible modifications and extensions of our model in Section 10. The Appendix collects the formal proofs of all results, and an additional online appendix contains supplementary material mentioned throughout the paper.

2. Reduced model

In this section, we set up a reduced model of strategic investment with a positive externality. We focus on our key feature: the firms face uncertainty about the evolution of both the return on investment and the magnitude of the externality. This reduction greatly simplifies the presentation of our novel methodology, makes applications in other settings easier, and helps us to point out the differences between our results and the existing literature. Examples of models with different additional features, which can all be reduced to the following one, will be presented in Section 8.

Consider a duopoly and a single investment option. Each firm $i \in \{1, 2\}$ may exercise the option at any time $t \in \mathbb{R}_+$, and as soon as one firm has exercised it, the option is gone. Exercising the option requires a total net investment $I \in \mathbb{R}_+$. The reward and the externality from this investment depend on time and some state of the world, and the firms accumulate public information about the state. Therefore, let $(\Omega, \mathscr{F}, P)$ be a fixed probability space and $(\mathscr{F}_t)$ a filtration satisfying the *usual conditions*, i.e., $(\mathscr{F}_t)$ is right-continuous and complete. If the option is exercised at time t, then the information about the state is given by the sigma-field $\mathscr{F}_t$, and the conditional expected values of reward and externality are respectively given by two $\mathscr{F}_t$-measurable random variables Y_t and X_t. Besides the externality, there may be some net cost for the other firm (e.g., an adjustment cost), given by $C \in \mathbb{R}$. We assume $C \leq I$. Both firms discount future payoffs continuously by a common factor $r \in (0, \infty)$. To summarize, if a firm uses the investment option at time t, then this firm obtains the so-called *leader* payoff

$$L_t := e^{-rt}(Y_t - I) \tag{1}$$

and the other firm the so-called *follower* payoff

$$F_t := e^{-rt}(X_t - C). \tag{2}$$

If both firms try to exercise the option simultaneously at some time t, then we assume a tie-breaking rule is applied and each firm's conditionally expected payoff is

$$M_t := \frac{1}{2}(L_t + F_t).$$

(See Appendix D for *endogenous* coordination and other generalizations.) If no firm ever uses the investment option, each firm's payoff is $M_\infty \equiv 0$.

In order to obtain analytical results as far as possible, we specifically assume that the stochastic processes (X_t) and (Y_t) are geometric Brownian motions, which can be correlated, but not perfectly.[1]

Hence, the processes (X_t) and (Y_t) have some given initial values $X_0 \equiv x_0 \in \mathbb{R}_+$ and $Y_0 \equiv y_0 \in \mathbb{R}_+$, and their dynamics follow the stochastic differential equations

$$dX_t = \mu_X X_t\, dt + \sigma_X X_t\, dB_t^{(1)} \quad \text{and} \quad dY_t = \mu_Y Y_t\, dt + \sigma_Y Y_t\, dB_t^{(2)}, \tag{3}$$

where $\mu_X, \mu_Y \in \mathbb{R}$ are given average growth rates, $\sigma_X, \sigma_Y \in \mathbb{R} \setminus \{0\}$ are given volatility parameters, and $(B_t^{(1)})$ and $(B_t^{(2)})$ are two $(\mathscr{F}_t)$-adapted Brownian motions with given correlation $\rho \in (-1, 1)$. If $x_0 > 0$, then the externality X_t stays positive for all $t \in \mathbb{R}_+$, and if $x_0 = 0$, then also $X_t \equiv 0$ for all $t \in \mathbb{R}_+$. The same holds for the reward (Y_t). To ensure finite option values, we assume $\mu_X < r$ and $\mu_Y < r$, which also implies that our payoff processes (L_t), (F_t), and (M_t) converge to the no-investment payoff $M_\infty \equiv 0$ as $t \to \infty$ (in $L^1(P)$; see Remarks 12 and 13 in Appendix E for more details).

This reduced model is rich enough to allow for explicit revenue streams and costs for both firms, which may change in different ways when the investment option is exercised, and also for some idiosyncratic components; see Section 8.

3. Timing game

Now we formalize a timing game between the two firms, in order to study their strategic conflict when to use the investment option. Because the option can be exercised by only one firm, the game ends as soon as some firm stops waiting and invests. This means that the firms cannot condition their behavior on any observed actions. Nevertheless, it is still possible that the firms make certain threats when to exercise the option or not, so we require any threats to be credible and, thus, consider only *subgame-perfect equilibria*. Specifically, we apply the framework developed in Riedel and Steg (2017), which relies on the concept of *stopping times* to allow both strategies and subgames to depend on the dynamic information about the state of the world.

Recall that a random variable $\tau : \Omega \to [0, \infty]$ is a stopping time if $\{\tau \leq t\} \in \mathscr{F}_t$ for every $t \in \mathbb{R}_+$. Any stopping time is, hence, a feasible state-dependent plan when to exercise the option. A typical example is the first time that (X_t, Y_t) hits a certain subset of the state-space $\mathbb{R}_+^2$, and for the case that this never happens, we let $\inf \emptyset := \infty$. However, stopping times can encode much more information about the state of the world, in particular concerning the "history" of the observed processes (X_t) and (Y_t). Therefore, stopping times also mark the start of subgames under the hypothesis that the option is still available. In this second role, a generic stopping time will be denoted by ϑ.[2] Let $\mathscr{T}$ denote the set of all stopping times. We first consider only *pure strategies*, but the equilibrium in pure strategies will also be the basis for an equilibrium in *mixed strategies*, which we are going to introduce in Section 6.

Definition 1. Fix any $\vartheta \in \mathscr{T}$ and consider the subgame that starts at ϑ (and in which no firm has invested, yet). A *plan* for firm $i \in \{1, 2\}$ in this subgame is another stopping time τ_i such that $\tau_i \geq \vartheta$ (a.s.). Given also a plan τ_j for the other firm $j \in \{1, 2\}$, firm i's *payoff* in this subgame is

$$V_i^\vartheta(\tau_i, \tau_j) := E\Big[\mathbf{1}_{\{\tau_i < \tau_j\}} L_{\tau_i} + \mathbf{1}_{\{\tau_j < \tau_i\}} F_{\tau_j} + \mathbf{1}_{\{\tau_i = \tau_j\}} M_{\tau_j} \Big| \mathscr{F}_\vartheta\Big]. \tag{4}$$

In equilibrium, the firms must not want to change any plans before they are realized.

Definition 2. A *pure strategy* for firm $i \in \{1, 2\}$ is a family of plans $(\tau_i^\vartheta; \vartheta \in \mathscr{T})$ for all subgames satisfying the time-consistency condition

$$\vartheta' \leq \tau_i^\vartheta \quad \Rightarrow \quad \tau_i^{\vartheta'} = \tau_i^\vartheta \quad \text{(a.s.)} \tag{5}$$

for any two $\vartheta, \vartheta' \in \mathscr{T}$ such that $\vartheta \leq \vartheta'$ (a.s.). A *subgame-perfect equilibrium in pure strategies* is a pair of pure strategies $((\tau_1^\vartheta; \vartheta \in \mathscr{T}), (\tau_2^\vartheta; \vartheta \in \mathscr{T}))$ such that

$$V_i^\vartheta(\tau_i^\vartheta, \tau_j^\vartheta) \geq V_i^\vartheta(\tau_i, \tau_j^\vartheta) \quad \text{(a.s.)}$$

for any $i, j \in \{1, 2\}$ with $i \neq j$, $\vartheta \in \mathscr{T}$, and plan τ_i for firm i in the subgame starting at ϑ.

[1] Two-dimensional models typically require customized methods (see also footnote 3 below), and geometric Brownian motion is the most common model in theory and applications of real options (and financial derivatives). In Section 10, we will discuss which of our arguments would go through more generally and which depend crucially on our model choice.

[2] The σ-field $\mathscr{F}_\vartheta$, which represents the information available at ϑ, is formally defined as $\{A \in \mathscr{F}_\infty \mid A \cap \{\vartheta \leq t\} \subseteq \mathscr{F}_t$ for all $t \in \mathbb{R}_+\}$ (where $\mathscr{F}_\infty$ is the σ-field generated by $\bigcup_{t \in \mathbb{R}_+} \mathscr{F}_t$).

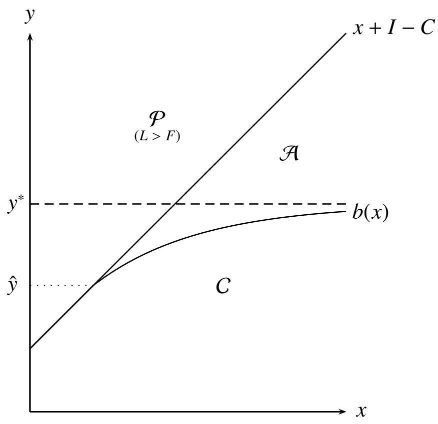


Fig. 1. Continuation, attrition, and preemption regions.

4. Regions of the state space

The subgame-perfect equilibria we are going to construct can be characterized by certain regions of the state-space of our two-dimensional process (X_t, Y_t). The first coordinate measures the strength of the investment externality and, thus, the attractiveness of becoming follower, whereas the second coordinate measures the reward from becoming leader. The first crucial distinction is which role is preferred at the moment the option is exercised. Clearly, $L_t > F_t$ if and only if

$$(X_t, Y_t) \in \mathcal{P} := \{(x, y) \in \mathbb{R}_+^2 \mid y > x + I - C\},$$

which we call the *preemption region*, because there the firms try to preempt each other in equilibrium in order to win the current first-mover advantage. On the boundary of $\mathcal{P}$, the equilibrium payoff will then be $M_t = L_t = F_t$.

The next important question is if a firm could have an incentive to exercise the option while there is a second-mover advantage $F_t > L_t$, i.e., when $(X_t, Y_t) \notin \overline{\mathcal{P}}$. Such a decision is similar to giving up in a war of attrition and letting the opponent win. Whether this is optimal depends on the cost of holding on. What distinguishes our model is that the firms know they will be "trapped" in preemption when the state hits $\mathcal{P}$, and this will create a cost already *in anticipation*. To see when a firm prefers to become leader instead of waiting for preemption, consider the hitting time $\tau_{\mathcal{P}} := \inf\{t \in \mathbb{R}_+ \mid (X_t, Y_t) \in \mathcal{P}\}$ and note that the preemption payoff is $M_{\tau_{\mathcal{P}}} = L_{\tau_{\mathcal{P}}}$ for $(x_0, y_0) \notin \mathcal{P}$. Therefore, we need to study the *constrained* optimal stopping problem of becoming leader *up to* $\tau_{\mathcal{P}}$,

$$\sup_{\tau \in \mathcal{T}:\, \tau \le \tau_{\mathcal{P}}} E\big[L_\tau\big]. \tag{6}$$

Due to the constraint, this is a stopping problem with two-dimensional state-space (depicted in Fig. 1 above). Such problems and their free boundaries are rarely studied in the literature, so characterizing the solution of this problem is our first main contribution.[3]

There are two prior estimates for the optimal stopping region for the constrained problem (6) (which will be verified formally in the proof of Proposition 3). First, a lower bound results from observing that it is optimal to wait as long as this is feasible and Y_t is less than

$$\hat{y} := \frac{r}{r - \mu_Y} I,$$

because then the reward is so low that the time effect of delaying the investment cost dominates, so that the drift of L_t is positive. Second, an upper bound is the solution of the *unconstrained* problem $\sup_{\tau \in \mathcal{T}} E[L_\tau]$, because whenever it is optimal to stop with unconstrained opportunities to wait, it must in particular be so with constraints. The unconstrained problem depends only on the one-dimensional process (Y_t). This is a standard problem, which is solved by the first hitting time $\tau_{y^*} := \inf\{t \in \mathbb{R}_+ \mid Y_t \ge y^*\}$ of the threshold

$$y^* := \frac{\beta_Y}{\beta_Y - 1} I,$$

[3] Most initially two-dimensional models are solved by a reduction to a single dimension, like in the seminal work McDonald and Siegel (1986), or in Peskir and Shiryaev (2006), Thijssen (2008), or Hackbarth and Morellec (2008). This approach is not possible in our case, because our underlying stochastic source is already two-dimensional, and the fixed investment cost prevents that we can consider only the ratio etc. of the exponential processes (X_t) and (Y_t).

where $\beta_Y > 1$ is the unique positive solution of the quadratic equation $\frac{1}{2}\sigma_Y^2\beta(\beta-1)+\mu_Y\beta-r=0$.[4] If $y^* \le I-C$, then $\tau_{y^*} \le \tau_{\mathcal{P}}$, so τ_{y^*} is in fact feasible in the constrained problem (6) and solves also this problem.

If $y^* > I-C$, however, then the optimal stopping boundary for the constrained problem (6) is the graph of a function $b(x)$ as illustrated in Fig. 1. We are going to establish the existence and general regularity properties of this function in Proposition 3 and then highlight its economically most important property in Proposition 5.[5]

Proposition 3. *If $y^* \le I-C$, then $\tau = \tau_{y^*}$ solves the constrained problem* (6)*. If $y^* > I-C$, then there exists a non-decreasing and continuous function $b : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $b(x) \le x+I-C$, such that the problem* (6) *is solved by $\tau = \tau_{C^c} := \inf\{t \in \mathbb{R}_+ \mid (X_t, Y_t) \notin C\}$ for the* continuation region

$$C = \{(x, y) \in \mathbb{R}_+^2 \mid y < b(x)\} \cup \{(0, 0)\}. \tag{7}$$

Furthermore, $b(0) = I-C$ and $b(x) \ge \min\{\hat{y}, x+I-C\}$ for all $x \in \mathbb{R}_+$.

Remark 4. The quantities y^*, $\hat{y}$, and $I-C$ in Proposition 3 and Fig. 1 are related as follows. $y^* > I-C$ if and only if $\frac{1}{\beta_Y-1}I > -C$, where $\frac{1}{\beta_Y-1}$ is positive and strictly increasing in μ_Y, σ_Y^2, and $-r$. If $y^* > I-C$, then $y^* > \hat{y} > 0$, so that $b(x) > 0$ for all $x > 0$ by $I-C \ge 0$. Also $\hat{y} > I-C$ if and only if $\frac{\mu_Y}{r-\mu_Y}I > -C$, i.e., if $\mu_Y I$ or C is sufficiently big. (See Lemma 14 in Appendix E for details.)

We now turn to the most significant property of the stopping boundary.

Proposition 5. *If $y^* > I-C$, then the function b in Proposition 3 satisfies $b(x) < y^*$ for all $x \in \mathbb{R}_+$.*

Proposition 5 means that optimal stopping must occur strictly before reaching y^* *anywhere* in the state space, even though the constraint is binding only in some *part* of the state space if $y^* > I-C$. In other words, if $\tau_{\mathcal{P}} < \tau_{y^*}$ with *any* positive probability, no matter how small, then the "naive" solution of waiting until $\min\{\tau_{\mathcal{P}}, \tau_{y^*}\}$ is *never* optimal. The consequences of these results for equilibrium behavior are explored in Sections 5 and 6.

5. SPE in pure strategies

The results from the previous section yield an equilibrium in the "subgame" starting at $\vartheta \equiv 0$ by construction: If $(x_0, y_0) \in \mathcal{P}$, then $L_0 > M_0 > F_0$, and it is optimal for both firms to engage in preemption. If $(x_0, y_0) \notin \mathcal{P}$ and firm j plans to invest only for preemption at $\tau_j = \tau_{\mathcal{P}}$, then $L_{\tau_{\mathcal{P}}} = F_{\tau_{\mathcal{P}}} = M_{\tau_{\mathcal{P}}}$ implies that τ_i is a best reply for firm i if it solves the constrained optimal stopping problem (6). But then $\tau_j = \tau_{\mathcal{P}}$ is also a best reply for firm j: By construction of τ_i, investing earlier is not worthwhile; at τ_i, there is still a second-mover advantage $F_{\tau_i} \ge L_{\tau_i}$ by $\tau_i \le \tau_{\mathcal{P}}$, and then also $F_{\tau_i} \ge M_{\tau_i}$; finally, the planned time $\tau_j = \tau_{\mathcal{P}}$ is sufficiently late to achieve the optimal payoff $E[F_{\tau_i}]$ for firm j. This reasoning can be formalized for subgames starting at arbitrary $\vartheta \in \mathscr{T}$, and since the plans are hitting times, they are time-consistent.

Theorem 6. *Suppose $y^* > I-C$ and let C be as in Proposition 3. Then, for any fixed $i, j \in \{1, 2\}$ with $i \ne j$, the pair of pure strategies consisting of the plans*

$$\tau_i^\vartheta = \tau_{C^c}(\vartheta) := \inf\{t \ge \vartheta \mid (X_t, Y_t) \notin C\} \quad \textit{and} \quad \tau_j^\vartheta = \tau_{\mathcal{P}}(\vartheta) := \inf\{t \ge \vartheta \mid (X_t, Y_t) \in \mathcal{P}\}$$

for every $\vartheta \in \mathscr{T}$ is a subgame-perfect equilibrium in pure strategies.

Eventual preemption is a binding constraint for the optimal time to exercise the option (i.e., $y^* > I-C$) in the following cases: (i) the cost C associated with the externality is positive; (ii) the necessary investment I is sufficiently big; (iii) the reward's growth rate μ_Y or variance σ_Y^2 is high enough; (iv) financing is in sufficient supply, i.e., the discount rate r low enough (cf. Remark 4).

Then, as shown in Proposition 5, already the *anticipation* of preemption implies that some firm gives in earlier and lets the other enjoy the preferred externality. There is, thus, a *feedback effect* from a preemptive continuation equilibrium on the prior phase of the game, when each firm still wishes that the other exercises the option. If there were no such competitive pressure, then the only tradeoff against the chance of a higher reward would be the risk that the state makes an excursion with low rewards; then any higher reward would be delayed, and discounting would turn it into a lower one. Preemption adds the risk of getting trapped *during* such an excursion, at a lower reward even in undiscounted terms. This risk generates an additional cost of waiting, and the boundary $b(x)$ marks where the cost of waiting becomes positive *in equilibrium*.

[4] The unconstrained problem equals the basic model of investment under uncertainty as in, e.g., Dixit and Pindyck (1994). It was first formally solved by H. P. McKean in the Appendix to Samuelson (1965), where the model was used for pricing warrants.

[5] All proofs can be found in Appendix A.

Because one firm gives in immediately when there is any cost of waiting, the payoffs are asymmetric. The higher payoff goes to the firm which commits to *not* use the option, except for preemption. This is unsatisfactory, because both firms prefer the same role, and the game may begin with a significant period of inaction in the continuation region C. Therefore, we are next going to construct a symmetric subgame-perfect equilibrium by allowing for mixed strategies. This will also yield additional insight into the strategic dynamics.

6. SPE in mixed strategies

By considering mixed strategies, it is possible to resolve the coordination problem arising from pure strategies—which firm takes the favored role—*within* the game. Constructing an equilibrium in mixed strategies also sheds more light on the strategic trade-offs, because we need to exactly quantify the equilibrium cost of waiting. (In fact, we already used this quantification for proving the feedback effect of preemption on the equilibrium investment time.) Moreover, we will obtain another novel effect: Although the firms conduct a war of attrition, typically even repeatedly if they use mixed strategies, this need not lead to investment before the preemption region is reached.

6.1. Mixed strategies and payoffs

In any subgame starting at some $\vartheta \in \mathscr{T}$, we allow the firms to randomize over their planned investment times by choosing *cumulative distribution functions* over the remaining time $t \in [\vartheta, \infty]$. These distribution functions may depend on the state of the world by the filtration $(\mathscr{F}_t)$. The payoffs are then a linear extension of the payoffs (4), and every pure plan $\tau_i^\vartheta \in \mathscr{T}$ is equivalent to the degenerate distribution function $\mathbf{1}_{\{t \geq \tau_i^\vartheta\}}$.

Definition 7. A *mixed plan* for firm $i \in \{1, 2\}$ in the subgame starting at $\vartheta \in \mathscr{T}$ is an $(\mathscr{F}_t)$-adapted stochastic process G_i^ϑ with values in $[0, 1]$ that is non-decreasing, right-continuous, and satisfying $G_i^\vartheta(t) = 0$ for all $t < \vartheta$ (a.s.). For every mixed plan, we let $G_i^\vartheta(0-) \equiv 0$ and $G_i^\vartheta(\infty) \equiv 1$. Given also a mixed plan G_j^ϑ for the other firm $j \in \{1, 2\}$, firm i's *payoff* in this subgame is

$$V_i^\vartheta(G_i^\vartheta, G_j^\vartheta) := E\bigg[\int\limits_{[0,\infty)} \big(1 - G_j^\vartheta(s)\big)L_s\, dG_i^\vartheta(s) + \int\limits_{[0,\infty)} \big(1 - G_i^\vartheta(s)\big)F_s\, dG_j^\vartheta(s) + \sum_{s\in[0,\infty]} \Delta G_i^\vartheta(s)\Delta G_j^\vartheta(s) M_s \bigg| \mathscr{F}_\vartheta\bigg]. \quad (8)$$

Time-consistency of mixed plans means that Bayes' law must be satisfied whenever possible. This is equivalent to the condition (5) for pure plans if the mixed plans are degenerate.

Definition 8. A *mixed strategy* for firm $i \in \{1, 2\}$ is a family of mixed plans $(G_i^\vartheta; \vartheta \in \mathscr{T})$ for all subgames satisfying the time-consistency condition

$$G_i^\vartheta(t) = G_i^\vartheta(\vartheta'-) + \big(1 - G_i^\vartheta(\vartheta'-)\big)G_i^{\vartheta'}(t) \quad \text{for all } t \geq \vartheta' \quad \text{(a.s.)}$$

for any two $\vartheta, \vartheta' \in \mathscr{T}$ such that $\vartheta \leq \vartheta'$. A *subgame-perfect equilibrium in mixed strategies* is a pair of mixed strategies $((G_1^\vartheta; \vartheta \in \mathscr{T}), (G_2^\vartheta; \vartheta \in \mathscr{T}))$ such that

$$V_i^\vartheta(G_i^\vartheta, G_j^\vartheta) \geq V_i^\vartheta(G_a^\vartheta, G_j^\vartheta) \quad \text{(a.s.)}$$

for any $i, j \in \{1, 2\}$ with $i \neq j$, $\vartheta \in \mathscr{T}$, and mixed plan G_a^ϑ for firm i in the subgame starting at ϑ.

The pure strategies from Theorem 6 remain a subgame-perfect equilibrium if also mixed strategies are allowed (see Proposition 2.3 in Steg (2018) for a general proof).

6.2. Waiting cost

In the equilibria in pure strategies, one firm assumes it can never enjoy the externality when this would be more profitable than going ahead with the investment itself, so it uses the option as soon as any further delay would reduce the leader payoff. This requires that the drift of L_t is negative, which it is for states $Y_t > \hat{y}$. However, such "local" decreases are not always considered losses, because the payoffs do not evolve monotonically in expectation. Waiting would be costly only outside the continuation region C, which is an *endogenous* object as the characterization in Propositions 3 and 5 has shown. On the boundary of C, future chances and risks are balanced in equilibrium.

If the firms use randomized strategies, there is an additional chance to benefit from the externality. This chance can compensate the waiting cost outside C. In order to make both firms indifferent, we are going to show now that a negative drift of L_t is considered a loss if and only if the state is outside C. From a mathematical point of view, this result is a non-trivial regularity property, because the state crosses the boundary of C very frequently (see Jacka, 1993). Our characterization in terms of the function b is instrumental for this. Additionally, we show that preemption for a first-mover advantage starts immediately when the state hits the *boundary* of $\mathcal{P}$ (which is much more straightforward).

Proposition 9.

(i) *Suppose $y^* > I - C$ and let C be as in Proposition 3. Then the cost of waiting in the constrained problem (6) is given by the rate*

$$\mathbf{1}_{\{(X_t,Y_t)\in C^c\}}e^{-rt}(r-\mu_Y)(Y_t-\hat{y})$$

for all $t \in [0, \tau_{\mathcal{P}})$ (a.s.).

(ii) $\inf\{t \geq 0 \mid (X_t, Y_t) \in \mathcal{P}\} = \inf\{t \geq 0 \mid (X_t, Y_t) \in \overline{\mathcal{P}}\}$ *(a.s.), except if $x_0 = y_0 = I - C = 0$.*

6.3. Symmetric SPE

Proposition 9 makes the features of a war of attrition visible that the firms conduct before they switch to preemption when the state hits $\mathcal{P}$. In $\overline{\mathcal{P}}^c$, there is a second-mover advantage $F_t > L_t$. Holding on is costly, however, only in the endogenous subset $\mathcal{A} = \overline{\mathcal{P}}^c \cap C^c$, which we call the *attrition region*. Knowing the exact cost of holding on, we can now make the firms indifferent by the chance that the opponent gives in at a certain rate a_t and invests first.

Theorem 10. *Suppose $y^* > I - C$. Let C be as in Proposition 3, set $\mathcal{A} = \overline{\mathcal{P}}^c \cap C^c$, and define a process (a_t) by*

$$a_t = \begin{cases} \dfrac{(r-\mu_Y)(Y_t-\hat{y})}{X_t+I-C-Y_t} & \text{if } (X_t,Y_t)\in\mathcal{A}, \\ 0 & \text{else.} \end{cases}$$

For every $\vartheta \in \mathcal{T}$, let $\tau_{\mathcal{P}}(\vartheta) := \inf\{t \geq \vartheta \mid (X_t, Y_t) \in \mathcal{P}\}$, and define a mixed plan G_i^ϑ for both $i = 1, 2$ by $G_i^\vartheta(t) = 0$ for all $t \in [0, \vartheta)$,

$$\frac{dG_i^\vartheta(t)}{1-G_i^\vartheta(t)} = a_t\,dt \tag{9}$$

for all $t \in [\vartheta, \tau_{\mathcal{P}}(\vartheta))$, and $G_i^\vartheta(t) = 1$ for all $t \geq \tau_{\mathcal{P}}(\vartheta)$. Then $(G_1^\vartheta; \vartheta \in \mathcal{T})$ and $(G_2^\vartheta; \vartheta \in \mathcal{T})$ form a subgame-perfect equilibrium in mixed strategies.

Our payoffs evolve non-monotonically and randomly, so the firms face irregular periods of attrition, and they engage in preemption when the state hits $\mathcal{P}$. In fact, it is a particular additional feature of our stochastic model that the investment is likely to happen in each of the different strategic scenarios. The reason is not simply that the state may either enter $\mathcal{A}$ or hit $\mathcal{P}$ first when it starts in C (cf. Fig. 1 again). Indeed, there are many other likely outcomes, because there is a positive probability of reaching $\mathcal{P}$ at *any* given part of its boundary from any point in C or $\mathcal{A}$. This result, stated in the following Proposition 11, is quite surprising: The attrition rate a_t becomes arbitrarily high when the state approaches the preemption boundary (where $X_t + I - C - Y_t = 0$), but it is not necessarily the case that some firm will give in and concede the externality to the other; this is only one possible outcome.

Proposition 11. *The strategies G_i^ϑ specified in Theorem 10 satisfy*

$$\tau_{\mathcal{P}}(\vartheta) < \infty \quad \Rightarrow \quad \Delta G_i^\vartheta(\tau_{\mathcal{P}}(\vartheta)) > 0 \quad \text{(a.s.)}.$$

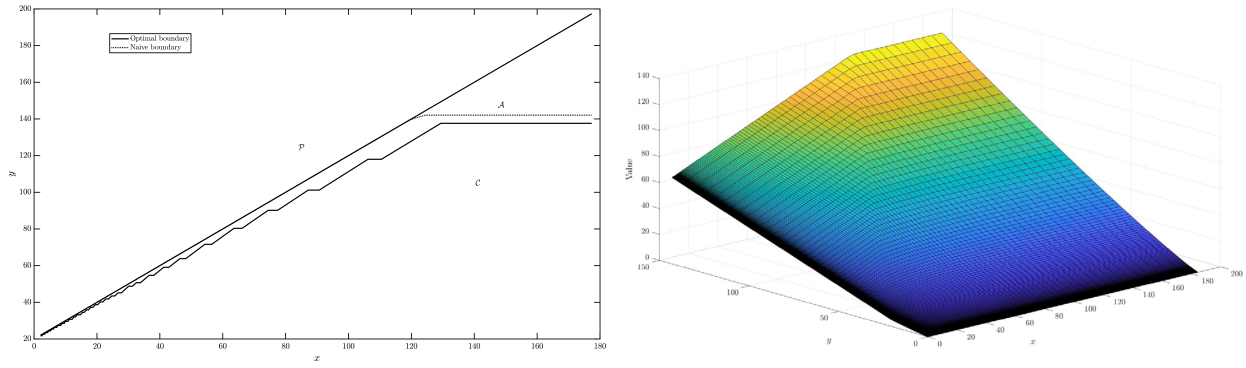
From the proof, we also obtain a by-product of independent mathematical interest, which is a certain integrability property for paths of Brownian motion; see Remark 16 in Appendix E.

7. Illustration

To illustrate the symmetric SPE in mixed strategies, we briefly study a numerical example, where we take the parameter values $I = 15$, $C = -5$, $r = 0.1$, $\mu_X = 0.05$, $\mu_Y = 0.08$, $\sigma_X = 0.2$, $\sigma_Y = 0.4$, and $\rho = -0.5$. In this case, the unconstrained leader investment threshold is $y^* \approx 142$.

Determining the exact boundary $x \mapsto b(x)$ as characterized in Proposition 3 is not trivial. It is, however, possible to construct a finite-difference scheme to approximate the boundary b and the equilibrium value function $V^*: \mathbb{R}_+^2 \to \mathbb{R}$ given by

$$V^*(x_0,y_0) = \begin{cases} M_0 & \text{if } (x_0,y_0)\in\mathcal{P}, \\ \sup_{\tau\in\mathcal{T}:\,\tau\leq\tau_{\mathcal{P}}} E\big[L_\tau\big] & \text{else,} \end{cases}$$



(a) Preemption, attrition, and continuation regions

(b) Value function

Fig. 2. Strategic regions and value function for the numerical example.

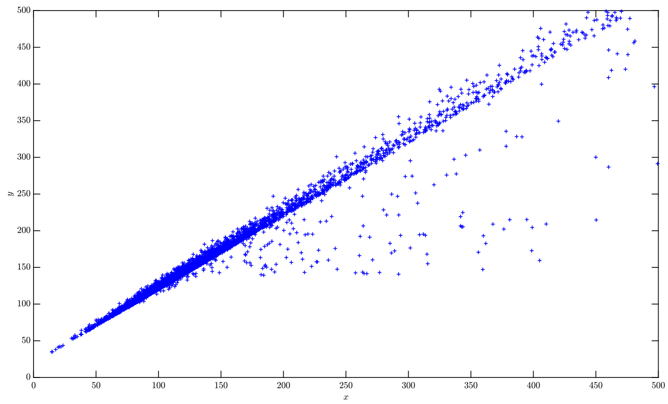


Fig. 3. Scatter plot of realizations of first-investment times.

which agrees with the value of the constrained stopping problem (6) outside $\mathcal{P}$. Details of this scheme can be found in Appendix F. Applying it to a 250×250 grid gives approximations to b and V^* as depicted in Figs. 2a and 2b, respectively.

Moreover, we run a simulation of 5,000 sample paths, all starting at $(x_0, y_0) = (150, 50)$, and record whether investment takes place in the attrition or the preemption region. We find that 88% of sample paths end in preemption, whereas 12% of sample paths end in the attrition region. A scatter diagram of the value of (X_t, Y_t) at the time of investment for each of these paths is given in Fig. 3. The average fraction of time spent in the attrition region is 7.78%, and all sample paths experience attrition before investment.

To show the qualitative importance of the feedback effect of preemption on attrition, we compare the equilibrium value of the firm with the case that the "naive" boundary

$$\hat{b}(x) = \min\left\{x + I - C, y^*\right\}$$

is chosen. This is the boundary that results from thinking that the preemption region and unconstrained attrition region can be found separately and then "glued together". Fig. 4 shows the difference between the equilibrium value function and the value function obtained by the naive boundary and gives, therefore, an indication of the amount of value that is destroyed by ignoring the feedback effect between preemption and attrition. The loss is particularly high around the "corner" of $\mathcal{A}$ that is not part of the attrition region if the "naive" boundary is used.[6] This figure shows the consequence of the feedback effect from preemption for firm values.

8. Applications

In this section, we present two economic situations that can be analyzed by our results for the reduced model. In both settings, there are positive externalities of uncertain magnitude, but the drivers of the externalities are quite different.

[6] The slightly "ragged" behavior along the estimated boundary $\hat{b}$ occurs because the true boundary b has, generically, no points that are exactly on the grid that is used in the finite-difference approximation.

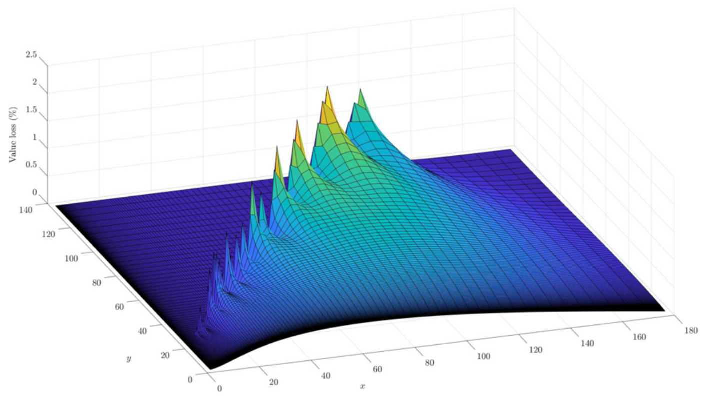


Fig. 4. Difference between equilibrium and "naive" values.

Correspondingly, also the initial models differ significantly; the first appears much simpler than the second. Nevertheless, both models are formally equivalent to the reduced one, which shows that it is possible to integrate various details in the analysis.

8.1. Competing about a replacement option

Consider a duopolistic market in which two firms produce a homogeneous product or service. The firms are in price competition à la Bertrand, so they are not making profits. There is, however, an option to replace the current good by a differentiated one. We think, e.g., of a new product, for which an exclusive license is available, or of a geographic differentiation. Using the replacement option requires a certain investment $I > 0$. By differentiation, both firms become monopolists and start making profits. How much each firm gains when one of them uses the option depends on different uncertainty factors. These factors are likely to be correlated if the degree of differentiation is low, but even then it is not certain which of the two monopoly positions is more profitable in the long run. Based on the observed relevant factors, also the estimates of the reward and the externality from the replacement option evolve. Now suppose the per-period monopoly profits for the established and the differentiated good are respectively given by geometric Brownian motions (π_t^F), (π_t^L) satisfying

$$d\pi_t^F = \mu_F \pi_t^F \, dt + \sigma_F \pi_t^F \, dB_t^{(1)} \quad \text{and} \quad d\pi_t^L = \mu_L \pi_t^L \, dt + \sigma_L \pi_t^L \, dB_t^{(2)},$$

where $(B_t^{(1)})$ and $(B_t^{(2)})$ are correlated Brownian motions as in Section 2. If the firms discount profits at a rate $r > \max\{0, \mu_F, \mu_L\}$ and if some firm uses the replacement option at time $t \in \mathbb{R}_+$, then this firm's expected net profit is

$$E\left[\int_t^\infty e^{-rs} \pi_s^L \, ds - e^{-rt} I \,\middle|\, \mathscr{F}_t\right] = e^{-rt}\left(\frac{\pi_t^L}{r - \mu_L} - I\right) \tag{10}$$

and that of the other firm is

$$E\left[\int_t^\infty e^{-rs} \pi_s^F \, ds \,\middle|\, \mathscr{F}_t\right] = e^{-rt}\left(\frac{\pi_t^F}{r - \mu_F}\right). \tag{11}$$

These payoffs are a special case of the reduced model in Section 2. When $x_0 = (r - \mu_F)^{-1}\pi_0^F$, $\mu_X = \mu_F$, $\sigma_X = \sigma_F$, $y_0 = (r - \mu_L)^{-1}\pi_0^L$, $\mu_Y = \mu_L$, $\sigma_Y = \sigma_L$, and $C = 0$, then the payoffs L_t and F_t defined in equations (1) and (2) are identical to the present payoffs (10) and (11).

8.2. Technology adoption with uncertain reward and spillovers

Now we consider an option to adopt some new technology, such that the firms stay in the same market, but exercising the option affects both firms' profit streams. This model is initially more complex, but it is also a more direct extension of standard, one-dimensional models and, thus, particularly useful for comparative studies (see Section 9.1).

Suppose both firms' revenue streams are, at first, given by some stochastic process (π_t). There might also be a certain operating cost, which is a constant $c_0 \in \mathbb{R}$. Adopting the new technology requires a fixed investment $K \in \mathbb{R}$. The new technology then increases the revenue by some markup (m_t^L) and changes the operating cost to $c_L \in \mathbb{R}$. There is some spillover from the new technology to the other firm, reflected by another revenue markup (m_t^F) and operating cost $c_F \in \mathbb{R}$.

Depending on how the perspectives for the revenue and different markups evolve, it may at any time be more favorable to adopt the technology or to wait, in order to adopt it later or to benefit from spillovers.

Suppose the initial revenue stream and the markups are geometric Brownian motions satisfying

$$\begin{aligned}&d\pi_t = \mu_\pi \pi_t\,dt + \sigma_\pi \pi_t\,dB^\pi_t, \quad dm^L_t = \mu_L m^L_t\,dt + \sigma_L m^L_t\,dB^L_t, \quad \text{and}\\ &dm^F_t = \mu_F m^F_t\,dt + \sigma_F m^F_t\,dB^F_t,\end{aligned}$$

where, similarly as before, (B^π_t), (B^L_t), and (B^F_t) are Brownian motions with pairwise correlation $\rho_{\pi,L}, \rho_{\pi,F}, \rho_{L,F} \in (-1, 1)$. Then also the post-adoption revenues $(m^L_t \pi_t)$ and $(m^F_t \pi_t)$ are geometric Brownian motions. Therefore, if some firm adopts the new technology at time $t \in \mathbb{R}_+$, this firm's expected net profit is

$$\begin{aligned}&E\Bigg[\int_0^t e^{-rs}\big(\pi_s - c_0\big)\,ds + \int_t^\infty e^{-rs}\big((m^L_s + 1)\pi_s - c_L\big)\,ds - e^{-rt}K \,\Bigg|\, \mathscr{F}_t\Bigg]\\ &= E\Bigg[\int_0^\infty e^{-rs}\big(\pi_s - c_0\big)\,ds + \int_t^\infty e^{-rs}\big(m^L_s\pi_s - (c_L - c_0)\big)\,ds \,\Bigg|\, \mathscr{F}_t\Bigg] - e^{-rt}K\\ &= E\Bigg[\int_0^\infty e^{-rs}\big(\pi_s - c_0\big)\,ds \,\Bigg|\, \mathscr{F}_t\Bigg] + e^{-rt}\left(\frac{m^L_t \pi_t}{r - \mu_Y} - \frac{c_L - c_0}{r} - K\right)\end{aligned} \tag{12}$$

for $\mu_Y = \mu_L + \mu_\pi + \sigma_L\sigma_\pi\rho_{\pi,L}$, assuming $r > \max\{\mu_\pi, \mu_Y\}$. The other firm's expected profit is

$$\begin{aligned}&E\Bigg[\int_0^t e^{-rs}\big(\pi_s - c_0\big)\,ds + \int_t^\infty e^{-rs}\big((m^F_s + 1)\pi_s - c_F\big)\,ds \,\Bigg|\, \mathscr{F}_t\Bigg]\\ &= E\Bigg[\int_0^\infty e^{-rs}\big(\pi_s - c_0\big)\,ds \,\Bigg|\, \mathscr{F}_t\Bigg] + e^{-rt}\left(\frac{m^F_t \pi_t}{r - \mu_X} - \frac{c_F - c_0}{r}\right)\end{aligned} \tag{13}$$

for $\mu_X = \mu_F + \mu_\pi + \sigma_F\sigma_\pi\rho_{\pi,F}$, assuming also $r > \mu_X$. To relate these payoffs to the reduced model in Section 2, let N_t denote the respective first term in (12) and (13). By $r > \mu_\pi$, (N_t) is a uniformly integrable martingale. Hence, *independently* of the firms' strategies, adding (N_t) to both (L_t) and (F_t) amounts to an additional payoff N_ϑ in (4) or (8) and, thus, does not alter the set of equilibria.[7] As the respective second terms in (12) and (13) are special cases of L_t and F_t defined in equations (1) and (2), all results for the reduced model apply to the present model. Given μ_X and μ_Y as already specified above, the complete remaining specification is

$$\begin{aligned}x_0 &= (r - \mu_X)^{-1}\big(m^F_0 \pi_0\big), & y_0 &= (r - \mu_Y)^{-1}\big(m^L_0 \pi_0\big),\\ \sigma_X &= \big(\sigma_F^2 + 2\sigma_F\sigma_\pi\rho_{\pi,F} + \sigma_\pi^2\big)^{1/2}, & \sigma_Y &= \big(\sigma_L^2 + 2\sigma_L\sigma_\pi\rho_{\pi,L} + \sigma_\pi^2\big)^{1/2},\\ B^{(1)}_t &= \sigma_X^{-1}\big(\sigma_F B^F_t + \sigma_\pi B^\pi_t\big), & B^{(2)}_t &= \sigma_Y^{-1}\big(\sigma_L B^L_t + \sigma_\pi B^\pi_t\big),\\ C &= \frac{c_F - c_0}{r}, & I &= K + \frac{c_L - c_0}{r}.\end{aligned}$$

Assume $\sigma_F^2 + \sigma_\pi^2 > 0$, which implies $\sigma_X > 0$, and $\sigma_L^2 + \sigma_\pi^2 > 0$, which implies $\sigma_Y > 0$. Then $(B^{(1)}_t)$ and $(B^{(2)}_t)$ are two Brownian motions with correlation $\rho = (\sigma_X\sigma_Y)^{-1}(\sigma_F\sigma_L\rho_{L,F} + \sigma_F\sigma_\pi\rho_{\pi,F} + \sigma_L\sigma_\pi\rho_{\pi,L} + \sigma_\pi^2)$. Assume also $\sigma_L^2 + \sigma_F^2 > 0$ to ensure $\rho^2 < 1$. Finally, assume $K + (c_L - c_0)/r \geq 0$, so that $I \geq 0$, and also $K + (c_L - c_F)/r \geq 0$, so that $C \leq I$.

9. Relation of our results to the literature

We are next going to relate the novel effects we have found to the existing literature in two ways. First, we will show that we assumed the minimal structure for these effects to arise. To do so, we will consider both the *one-dimensional* version of our model (in Subsection 9.1), which is much closer to many models in the literature, and also the two-dimensional *deterministic* version (in Appendix C). In these simpler models, none of the effects that we have observed occur: neither does

[7] The martingale could even be different for the two firms and capture some idiosyncratic aspects.

the anticipation of preemption affect the cost of waiting, nor is there any uncertainty whether investment will happen during attrition or preemption. Hence, we really had to consider the non-degenerate two-dimensional model, and uncertainty makes a *qualitative* difference in our case.[8]

Second, we will point out (in Subsection 9.2) how our results contribute to the discussion in the empirical literature about first- and second-mover advantages associated with strategic investments.

9.1. The role of (two-dimensional) uncertainty

In order to show that there is no feedback effect of preemption on attrition in typical one-dimensional strategic real-option models, we consider a one-dimensional version of the application from Subsection 8.2, which has the same basic structure as the models in, e.g., Smets (1991); Grenadier (1996); Weeds (2002); Thijssen et al. (2012); Riedel and Steg (2017). Nevertheless, our results can also be compared to one-dimensional models that capture different economic aspects, but where the equilibrium construction is based on similar optimal stopping problems (such as, e.g., in Agrawal et al., 2016).

Hence, consider two firms that each have an option to make an investment to increase the firm's profitability. This investment involves a sunk cost $K > 0$. The only uncertainty affecting revenues before and after the investment is now summarized by the *single* factor (π_t). Both firms' operating profits depend again on which firm already has invested, but this dependence is now reflected by *constant* multipliers $D_{k\ell} > 0$ for $k, \ell \in \{0, 1\}$, where k refers to the firm in question, ℓ to its competitor, and $k = 1$ or $\ell = 1$ means the corresponding firm has invested. Thus, in comparison to Subsection 8.2, there are constant markups $m_0^L = D_{10} - D_{00}$ and $m_0^F = D_{01} - D_{00}$ for the leader and the follower, respectively, and we set all additional operating costs to zero.

As usual, we assume $D_{10} > D_{00}$, so that there is a direct benefit from the first investment. We deviate, however, from the usual assumption that $D_{01} < D_{00}$, because this would mean a negative externality on the other firm's profits, and that the investment decisions are like *strategic substitutes*. Instead, we assume $D_{01} > D_{00}$ in order to have a positive externality and *strategic complements*. We limit the externality by assuming that $D_{01} < D_{10}$, because otherwise there would be only a second-mover advantage and no preemption at all. Finally, we assume for simplicity that there is no second investment.[9]

It is possible to analyze this model in terms of the one-dimensional geometric Brownian motion (Y_t) given by $Y_t := \frac{D_{10}-D_{00}}{r-\mu_\pi}\pi_t$, so the state space reduces to $\mathbb{R}_+$ (see Appendix B for details). Specifically, we can verify subgame-perfect equilibria both in pure and mixed strategies as in Theorems 6 and 10, but these equilibria are now fully characterized by the preemption *threshold* $y_{\mathcal{P}} := \frac{D_{10}-D_{00}}{D_{10}-D_{01}}K = \frac{m_0^L}{m_0^L-m_0^F}K$.

As before, preemption is a binding constraint for the time of the first investment if and only if the preemption region extends below the unconstrained leader's investment threshold, which is now if $y_{\mathcal{P}} < y^*$. But it is *not* optimal anymore for the leader to invest any earlier, so investment happens *only* by preemptive moves, i.e., once Y_t hits $[y_{\mathcal{P}}, \infty)$. Thus, anticipating preemption becomes pointless, and there is no doubt that investment will happen with a first-mover advantage. Both effects are also absent if $y_{\mathcal{P}} \geq y^*$. Then the equilibrium outcome is simply that some firm invests as soon as Y_t hits $[y^*, \infty)$, i.e., as if there was no competition at all, and it is ex ante clear whether there will be preemption (if the state will first hit $[y_{\mathcal{P}}, \infty)$) or a second-mover advantage for the follower (if the state will first hit $[y^*, y_{\mathcal{P}})$).

9.2. First- and second-mover advantages in the empirical literature

In the last few decades there has been a lively discussion in the management and marketing literature on whether or not there is a first-mover advantage for innovating firms. In an extensive review and synthesis of papers published in top-tier journals, Zachary et al. (2015) conclude that, while important, innovation timing is only one of several determinants. They write that "[w]e owe it to the field, and to the managers who we advise, to broaden our thinking about entry considerations or we will just perpetuate the myth that 'being first creates a competitive advantage' without [any] caveat" (Zachary et al., 2015, p. 1410).

In our model, and even in the equilibria we have characterized, it is indeed a priori uncertain whether there will be a first- or second-mover advantage at the optimal time of investment. Moreover, our analysis provides a detailed picture of the strategic trade-offs the firms have to make in their timing decisions. This can potentially be used to explain some empirically observed phenomena.

For example, Christensen et al. (1998) study strategies for survival in fast-changing industries such as the rigid disk drive industry. They find that first-mover advantages typically apply only if, at entry, firms choose architectural innovation as a technology strategy. However, this works only *before* a dominant design has been established. Once there is an established design, firms are better off adopting it. So, if first-mover advantages are not often present, the question arises whether one can count more on second-mover advantages. But that too is not generally the case. For example, Min et al. (2006, p. 30) conclude that "market pioneers are often the first to fail in really new product-markets. However, this is not true

[8] In many one-dimensional strategic real-option models, the structure of equilibria often stays the same, whether there is uncertainty or other reasons for waiting (see, e.g., Steg, 2018).

[9] This could also be justified by economic reasons (e.g., if there is patent protection in technology adoption or if the second investor cannot significantly enhance profits, so that $D_{11} \leq D_{01}$).

in incremental new markets, in which market pioneers have consistently lower survival risks than early followers." In the context of environmental innovation, Cleff and Rennings (2014) find that both first- and second-mover advantages arise in successful innovation. Summarizing the literature, they find that there are several factors leading to successful timing strategies. These factors are: *(i)* luck, *(ii)* technological leadership (Lieberman and Montgomery, 1988), *(iii)* industry, firm, and product-specific factors (Gilbert and Birnbaum–More, 1996), and *(iv)* leading time, market dynamic, and type of innovation (Min et al., 2006).

Our theoretical results can be seen at least as a partial explanation for the contradictory results in the empirical literature. Since there is an inherent *uncertainty* whether investment will entail a first- or second-mover advantage, we highlight the role that "luck" plays in successful timing decisions (Cleff and Rennings, 2014). In innovation contexts, this uncertainty implies that a firm's strategy should change over time in light of the best available evidence about an innovation's relative profitability, and such a dynamic strategy in turn impacts the likelihood of an innovation taking place while there is a first- or second-mover advantage. Several factors inherent in the type of innovation influence these likelihoods, which can provide direction for further empirical investigations.

10. Conclusion and future research

We have found that positive externalities with uncertain magnitude imply that additional strategic considerations become necessary to evaluate real options properly. First, to see when a firm needs to prepare for preemption, the value of exercising the option must be considered *relative* to the externality. This means that preemption is also possible when the expected profit is not particularly high. Consequently, there is an additional risk to take into account in the timing decision, which translates into a higher waiting cost and, hence, an earlier optimal exercise time.

These observations warrant further study. We have here extended the standard model for risk factors to a two-dimensional setting, which is still quite an explicit assumption, of course. It has allowed us to prove analytic results and to quantify the equilibrium incentives in terms of basic factors. Therefore, we have gained deeper economic and managerial insights than would have been possible at a more general level. However, it is clear that our methodological contribution can also be used to consider alternative models, e.g., where the dynamics of the two exogenous risk factors are represented by other diffusions.

A further avenue for future research is to enrich the economic environment by introducing a continuation value for the follower, so that a better understanding of the interplay between first- and second-mover advantages can be obtained. This could even be extended to a model where firms can invest repeatedly. Such an extension, however, will require a richer model of subgames and, particularly, histories (of multiple actions) in order to define time-consistent strategies appropriately, and, moreover, the solution of even more complex, interdependent stopping problems.

Finally, our game-theoretic analysis was significantly helped by assuming symmetric players, but in reality competing firms may be known to differ, e.g., with respect to their costs. Therefore, it would be a valuable task to see if and how our results change when the firms are asymmetric.

We complete this section by a more detailed discussion of the three mentioned directions to modify or extend our model, i.e., alternative stochastic dynamics, additional investment options, or asymmetric firms.

Alternative stochastic dynamics

We have chosen two geometric Brownian motions in order to be able to prove our results analytically. However, we expect that qualitatively similar results hold for other dynamics as well, because the nature of the constrained optimal stopping problem that is central to our analysis does not depend on specific assumptions regarding the underlying uncertainty. Indeed, if the setting remains symmetric with respect to the players, it is possible to construct subgame-perfect equilibria that have the same structure as ours without assuming any particular dynamics, i.e., more general—but much less explicit—versions of Theorems 6 and 10 hold.[10] In addition, many of our further arguments are at least general enough to consider alternative diffusion models. For instance, the proofs of Propositions 5 and 9 mostly exploit the fact that there is a non-decreasing stopping boundary $b(x)$. Hence, the main task would be to prove Propositions 3 and 11 for other dynamics, which would require custom methods due to the two-dimensional setting. Here we used certain path properties of geometric Brownian motion (notably the multiplicative structure to make path comparisons and the representation as functions of standard Brownian motion paths to verify a specific integrability property). We anticipate that very similar arguments will work for arithmetic Brownian motion, but other diffusions would generally need some different steps.

Additional investment option for the follower

We assumed that only one firm can invest. Hence, the follower enjoys a positive externality when the leader invests, but there is no further possibility for the follower to invest later on, e.g., when the leader's investment turns out to be very profitable and the externality not. Suppose we extend our model such that the follower has the option to invest the same sum I as the leader at an arbitrary stopping time τ that occurs after the leader's investment. This investment will yield the follower the additional reward aY_t, where $a \in (0, 1)$ is some given share. Accounting for the follower's option value does not

[10] Essentially, only certain continuity and integrability conditions are required, see Steg (2015).

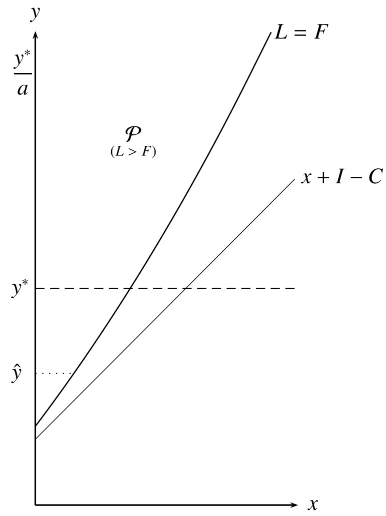


Fig. 5. Change of the preemption region due to an additional follower option.[11]

change the general shape of the preemption region, but the exact boundary of $\mathcal{P}$ is affected in three ways that can be seen in Fig. 5. Both the intercept and the slope of the boundary increase, implying that $\mathcal{P}$ shrinks. Moreover, the boundary of $\mathcal{P}$ becomes strictly convex where $y < y^*/a$, i.e., in the continuation region for the follower, but it transitions smoothly into a straight line for higher values of y. Due to the similar shape of $\mathcal{P}$, we expect that our qualitative results will persist.

Asymmetric investment costs

In parts, our results can be extended easily to asymmetric firms with individual investment costs I^i and C^i. Assume w.l.o.g. that $I^2 - C^2 \geq I^1 - C^1$. Then a first-mover advantage for firm 2 implies the same for firm 1, so that the preemption region becomes $\mathcal{P} = \{(x, y) \in \mathbb{R}^2_+ \mid y > x + I^2 - C^2\}$. Now consider the constrained problem (6) for each firm. Propositions 3 and 5 still apply, simply setting $I = I^2$ and $C = C^2$ for firm 2, but $I = I^1$ and $C = I^1 - I^2 + C^2$ for firm 1 (to fix the same $\mathcal{P}$ by $I - C = I^2 - C^2$). We can actually compare the individual solutions by ranking I^1 and I^2. Suppose in fact $I^2 \geq I^1$. Then, like in the symmetric case, whenever firm 1 is willing to wait, so is firm 2, because we have $\tau' \geq \tau \Rightarrow (L^2_{\tau'} - L^2_{\tau}) - (L^1_{\tau'} - L^1_{\tau}) = (e^{-r\tau'} - e^{-r\tau})(I^1 - I^2) \geq 0$ a.s. for any two stopping times τ and τ'. Furthermore, when firm 1 stops before the state enters the preemption region, it is still no disadvantage for firm 2 to become follower instead of leader. Therefore, we obtain a subgame-perfect equilibrium by the same arguments used to prove Theorem 6 if we ensure that firm 1 obtains the leader payoff at the stopping time that solves its constrained problem (6). For this we need a different tie-break when it is optimal to let the state hit some part of the preemption region where firm 1 has a strict first-mover advantage, i.e., if $\mathcal{P}$ extends below $\hat{y}$ (cf. Fig. 1) and the assumed inequality $I^2 - C^2 \geq I^1 - C^1$ is strict. But here we are helped by the fact that firm 2 is still indifferent on the boundary of $\mathcal{P}$, because it justifies to assume that firm 1 indeed becomes leader by strict preference.[12] Other arguments are needed, however, if $I^1 > I^2$, because then firm 2 wants to stop first, while firm 1 has a strict first-mover advantage.

Declaration of competing interest

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[11] The parameter values that were used to produce Fig. 5 are $r = 0.1$, $\mu_Y = 0.05$, $\sigma_Y = 0.3$, $I = 1$, $C = 0$, and $a = 0.5$. These values imply $\beta_Y \approx 1.436$ and $y^* \approx 3.29$.

[12] This outcome can also be obtained endogenously by extended strategies as in Riedel and Steg (2017).

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Appendix A. Proofs

Proof of Proposition 3. First, consider the degenerate case $y_0 = 0$, i.e., $Y_t \equiv 0$ for all $t \in \mathbb{R}_+$. Then the problem (6) is in fact unconstrained, because $X_t + I - C \geq 0 = Y_t$ implies $\tau_{\mathcal{P}} \equiv \infty$. The solution is, thus, given by $\tau = \tau_{y^*} \leq \tau_{\mathcal{P}}$. Specifically, L_t is now deterministic and $dL_t = e^{-rt} rI\, dt$. If $I > 0$, which is if and only if $y^* > 0$, then L_t is strictly increasing and $\tau_{y^*} \equiv \infty$ is the unique solution. If $I = 0$, then $y^* = 0$ and $\tau_{y^*} \equiv 0$ is optimal, but any other $\tau \in \mathscr{T}$ as well. Similarly, for any other value $y_0 \in \mathbb{R}_+$, if $y^* \leq I - C$, then $\tau_{y^*} \leq \tau_{\mathcal{P}}$ is still feasible and attains the value $\sup_{\tau \in \mathscr{T}} E[L_\tau] \geq \sup_{\tau \in \mathscr{T}\,:\, \tau \leq \tau_{\mathcal{P}}} E[L_\tau]$.

Now suppose $y^* > I - C$. For initial states $(X_0, Y_0) = (x_0, y_0) \notin \mathcal{P}$, the constraint in problem (6) can be formulated equivalently by "freezing" the reward $L_{\tau_{\mathcal{P}}}$ for any $t \geq \tau_{\mathcal{P}}$, i.e., by considering the auxiliary reward process $\bar{L}_t := L_{\min\{t, \tau_{\mathcal{P}}\}}$ and ignoring the constraint. Therefore, the value of the constrained problem (6) is the same as

$$V_{\bar{L}}(x_0, y_0) := \sup_{\tau \geq 0} E\big[\bar{L}_\tau\big]. \tag{A.1}$$

Defining a value function $V_{\bar{L}} \colon \mathbb{R}_+^2 \to \mathbb{R}$ by (A.1) for *arbitrary* initial values $(x_0, y_0) \in \mathbb{R}_+^2$ and noting that $\bar{L}_0 = y_0 - I$, the *continuation region* of the state space for this problem is

$$C := \{(x, y) \in \mathbb{R}_+^2 \mid V_{\bar{L}}(x, y) > y - I\} \subseteq \mathcal{P}^c.$$

By the strong Markov property and the path continuity of $(\bar{L}_t)$, it is optimal to stop as soon as (X_t, Y_t) exits C (see Krylov (1980)).

We are now going to establish the boundary of the continuation region C in the whole state space by strong path comparisons. Therefore, denote the solution to (3) for an arbitrary initial condition $(X_0, Y_0) = (x, y) \in \mathbb{R}_+^2$ by $(X^x, Y^y) = (xX^1, yY^1)$. By definition of the continuation region, for any $(x_0, y_0) \in C$, there exists a stopping time $\tau^* \in (0, \tau_{\mathcal{P}}]$ (a.s.) such that $V_{\bar{L}}(x_0, y_0) \geq E[\bar{L}_{\tau^*}] = E[e^{-r\tau^*}(Y^{y_0}_{\tau^*} - I)] > \bar{L}_0 = y_0 - I$. Now suppose $\varepsilon \in (0, y_0]$, implying that $(X^{x_0}_t, Y^{y_0-\varepsilon}_t) = (X^{x_0}_t, Y^{y_0}_t - \varepsilon Y^1_t)$ starts at $(x_0, y_0 - \varepsilon) \in \mathcal{P}^c$ and $\tau^* \leq \tau_{\mathcal{P}} \leq \inf\{t \geq 0 \mid (X^{x_0}_t, Y^{y_0-\varepsilon}_t) \in \mathcal{P}\}$. Hence, $V_{\bar{L}}(x_0, y_0 - \varepsilon) \geq E[e^{-r\tau^*}(Y^{y_0-\varepsilon}_{\tau^*} - I)] = E[e^{-r\tau^*}(Y^{y_0}_{\tau^*} - I) - e^{-r\tau^*}\varepsilon Y^1_{\tau^*}] > y_0 - I - E[e^{-r\tau^*}\varepsilon Y^1_{\tau^*}] \geq y_0 - I - \varepsilon$. The last inequality is due to $(e^{-rt}Y^1_t)$ being a supermartingale by $r > \mu_Y$. This shows that $(x_0, y_0) \in C \Rightarrow \forall \varepsilon \in (0, y_0] : (x_0, y_0 - \varepsilon) \in C$, so we can define $b(x) := \sup\{y \geq 0 \mid V_{\bar{L}}(x, y) > y - I\}$ for every $x \in \mathbb{R}_+$ for which the latter set is nonempty and then conclude $V_{\bar{L}}(x, y) > y - I$ for all $y \in [0, b(x))$. For all other $x \in \mathbb{R}_+$, we set $b(x) := 0$. Now $y < b(x) \Rightarrow (x, y) \in C$ and $y > b(x) \Rightarrow (x, y) \notin C$ hold for all $(x, y) \in \mathbb{R}_+^2$. We will address the points on the boundary in the last step of the proof.

The following bounds hold for this choice of b. First, $b(x) \leq x + I - C$, because whenever y exceeds this bound, then $(x, y) \in \mathcal{P} \subseteq C^c$, and the bound is non-negative by $I - C \geq 0$. Second, $b(x) \geq \min\{\hat{y}, x + I - C\}$, where $\hat{y} = \frac{r}{r - \mu_Y} I$. Indeed, for any $(X_0, Y_0) = (x_0, y_0) \in \mathbb{R}_+^2$ with $y_0 < x_0 + I - C$, $\tau_{\mathcal{P}} > 0$. Moreover, the drift of (L_t) is $-e^{-rt}(r - \mu_Y)(Y_t - \hat{y})\, dt$ by Itō's formula, and as $r > \mu_Y$, this drift is positive whenever $Y_t < \hat{y}$. Thus, if also $y_0 < \hat{y}$, then setting $\tau = \min\{\tau_{\mathcal{P}}, \inf\{t \geq 0 \mid Y_t \geq \hat{y}\}\}$ implies $\tau \in (0, \tau_{\mathcal{P}}]$ and $\bar{L}_0 = L_0 < E[L_\tau] = E[\bar{L}_\tau] \leq V_{\bar{L}}(x_0, y_0)$, so that $b(x_0) \geq y_0$. Third, we can establish that $b(x) \geq I - C$ by showing that, if $I - C > 0$, then $\bar{L}_0 = L_0 < E[L_\tau] \leq V_{\bar{L}}(x_0, y_0)$ for any $(x_0, y_0) \in \mathbb{R}_+^2$ with $y_0 < I - C$ and $\tau = \inf\{t \geq 0 \mid Y_t \geq I - C\} \leq \tau_{\mathcal{P}}$. For $Y_0 = y_0 < I - C$ and the given τ, standard results for geometric Brownian motion yield $E[L_\tau] = (y_0/(I - C))^{\beta_Y}(-C)$. Therefore, and since $\beta_Y > 1$, $E[L_\tau] - L_0$ is a continuous and strictly concave function of y_0 on $\mathbb{R}_+$, which vanishes for $y_0 = I - C$ (when $\tau \equiv 0$) and takes the value I for $y_0 = 0$ (when $\tau \equiv \infty$), where $I > 0$ given $y^* > I - C$, so that indeed $E[L_\tau] - L_0 > 0$ for all $y_0 \in [0, I - C)$.

To show the monotonicity of b, suppose $(X_0, Y_0) = (x_0, y_0) \in C$, let $\tau^* \leq \tau_{\mathcal{P}}$ be as before, and fix arbitrary $\varepsilon > 0$. Then $(X^{x_0+\varepsilon}, Y^{y_0}) = (X^{x_0} + \varepsilon X^1, Y^{y_0})$ starts at $(x_0 + \varepsilon, y_0) \in \mathcal{P}^c$, and $\tau^* \leq \tau_{\mathcal{P}} \leq \inf\{t \geq 0 \mid (X^{x_0+\varepsilon}_t, Y^{y_0}_t) \in \mathcal{P}\}$. Now $V_{\bar{L}}(x_0 + \varepsilon, y_0) \geq E[e^{-r\tau^*}(Y^{y_0}_{\tau^*} - I)] > y_0 - I$, whence also $(x_0 + \varepsilon, y_0) \in C$. Therefore, $b(x_0 + \varepsilon) \geq b(x_0)$.

The continuity of b in $x_0 = 0$ holds by $b(x) \in [I - C, x + I - C]$. To verify the continuity of b in $x_0 > 0$, we now show that if (x_0, y_0) belongs to C, then also the whole line $\{(x, y) \in \mathbb{R}^2_+ \,|\, x \in (0, x_0], y = xy_0/x_0\}$.[13] As the first step, we check that, given $(X_0, Y_0) = (x_0, y_0) \in \mathcal{P}^c$, $\tau_{\mathcal{P}} \leq \inf\{t \geq 0 \,|\, (X^x_t, Y^y_t) \in \mathcal{P}\}$ for any point (x, y) on the specified line. Indeed, for any $(x, y) \in \mathbb{R}^2_+$,

$$Y^y_t - X^x_t \leq I - C \quad \Leftrightarrow \quad yY^1_t - xX^1_t \leq I - C.$$

If $y = xy_0/x_0$, this becomes

$$\frac{x}{x_0}\left(y_0 Y^1_t - x_0 X^1_t\right) \leq I - C.$$

As $I - C \geq 0$, the condition for (x, y) is implied by the one for (x_0, y_0) if $x \in [0, x_0]$. Therefore, if we fix any point on the line $\{(x, y) \in \mathbb{R}^2_+ \,|\, x \in (0, x_0], y = xy_0/x_0\}$, we have $\tau^* \leq \inf\{t \geq 0 \,|\, (X^x_t, Y^y_t) \in \mathcal{P}\}$, where τ^* as before satisfies $E[e^{-r\tau^*}(Y^{y_0}_{\tau^*} - I)] > y_0 - I$ for the given $(x_0, y_0) \in C$. In particular, $(x, y) \in \mathcal{P}^c$. Now suppose that $(x, y) \notin C$. Then $y - I = V_{\bar{L}}(x, y) \geq E[e^{-r\tau^*}(Y^y_{\tau^*} - I)] = E[e^{-r\tau^*}(Y^{y_0}_{\tau^*} - I) - e^{-r\tau^*}(y_0 - y)Y^1_{\tau^*}] > y_0 - I - E[e^{-r\tau^*}(y_0 - y)Y^1_{\tau^*}] \geq y_0 - I - (y_0 - y) = y - I$, a contradiction (where we again used the fact that $(e^{-rt}Y^1)$ is a supermartingale). Thus, $(x, y) \in C$.

Finally, we argue that $(x_0, y_0) \notin C$ if $y_0 = b(x_0)$, except if $x_0 = y_0 = b(x_0) = 0$. Indeed, if $y_0 = 0$ and $y^* > I - C \geq 0$, then, as argued in the beginning of the proof, $\tau = \infty$ uniquely solves the constrained problem (6). In particular, $V_{\bar{L}}(0, 0) > \bar{L}_0 = -I$ and $(0, 0) \in C$. Next, note that $y^* > I - C \geq 0$ implies also $\hat{y} > 0$, so that $b(x) \geq \min\{\hat{y}, x + I - C\} > 0$ for all $x > 0$. Therefore, it is impossible that $x_0 > 0 = y_0 = b(x_0)$. Now suppose $y_0 = b(x_0) > 0$. Then (X_t, Y_t) exits C immediately with probability one. To show this, first consider also $x_0 > 0$. For proving the continuity of $b(x)$, we have shown that $b(x + h) \leq b(x) + hb(x)/x$ for any $x, h > 0$. Together with the monotonicity of b and $X_0 = x_0 > 0$, this implies $\{Y_t > b(X_t)\} \supseteq \{Y_t > Y_0\} \cap \{Y_t > Y_0 X_t/X_0\} = \{(\mu_Y - \sigma^2_Y/2)t + \sigma_Y B^{(2)}_t > 0\} \cap \{(\mu_Y - \sigma^2_Y/2)t + \sigma_Y B^{(2)}_t - (\mu_X - \sigma^2_X/2)t - \sigma_X B^{(1)}_t > 0\}$. Rewriting the latter two sets, the result now follows from Lemma 15. Indeed, $(B^{(1)}, B^{(2)})^\top = \Sigma W$ for an invertible matrix Σ and 2-dimensional Brownian motion W by Remark 12. Letting $\tilde{\Sigma} = \begin{pmatrix} -\sigma_X & \sigma_Y \\ 0 & \sigma_Y \end{pmatrix}$ and then $\hat{\Sigma} = \tilde{\Sigma}\Sigma$, which are both invertible, as well as $\mu = (-(\mu_X - \sigma^2_X/2) + \mu_Y - \sigma^2_Y, \mu_Y - \sigma^2_Y/2)^\top$, the intersection of the two sets is $\{\min\{Z^{(1)}_t, Z^{(2)}_t\} > 0\}$ for $(Z^{(1)}_t, Z^{(2)}_t)^\top = t\mu + \hat{\Sigma}W_t$. Finally, consider $y_0 = b(x_0) > 0 = x_0$. Then $\{Y_t > b(X_t)\} = \{Y_t > Y_0\}$, and Lemma 15 applies with $d = 1$. $\square$

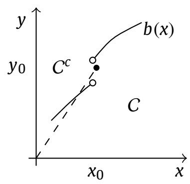
Proof of Proposition 5. Let $(\bar{L}_t)$ be defined as in the proof of Proposition 3, and recall the corresponding value function $V_{\bar{L}}$ defined by (A.1). Similarly, let $V_L \colon \mathbb{R}^2_+ \to \mathbb{R}$ be the value function of the unconstrained stopping problem, i.e., defined by $V_L(x_0, y_0) = \sup_{\tau \in \mathcal{T}} E[L_\tau]$ for arbitrary initial conditions $(X_0, Y_0) \equiv (x_0, y_0) \in \mathbb{R}^2_+$ (which is, thus, actually constant in x_0). For any (x_0, y_0) with $y_0 \geq y^*$, $\tau_{y^*} \equiv 0$ implies $y_0 - I = V_L(x_0, y_0) \geq V_{\bar{L}}(x_0, y_0) \geq y_0 - I$, so that $(x_0, y_0) \in C^c$ and $b(x_0) \leq y_0$. This shows that $b(x) \leq y^*$ for all $x \in \mathbb{R}_+$. Now suppose by way of contradiction that there is some $\hat{x} \in \mathbb{R}_+$ such that $b(\hat{x}) = y^* > I - C$, implying $y^* > 0$ and $b(x) = y^*$ for all $x > \hat{x}$ by monotonicity of b. For all $x_0 \geq \hat{x}$ and $y_0 = y^*$, we thus have $(x_0, y_0) \notin C$ and hence $V_{\bar{L}}(x_0, y_0) = \bar{L}_0 = y^* - I = V_L(x_0, y_0)$. In Proposition 9(i), we show that

$$V_{\bar{L}}(x_0, y_0) - E\Big[e^{-r\tau} V_{\bar{L}}(X_\tau, Y_\tau)\Big] = E\Bigg[\int_0^\tau \mathbf{1}_{\{(X_t, Y_t) \in C^c\}} e^{-rt}(r - \mu_Y)(Y_t - \hat{y})\, dt\Bigg]$$

for any stopping time $\tau \in [0, \tau_{\mathcal{P}}]$. Similarly, for the unconstrained problem,

$$V_L(x_0, y_0) - E\Big[e^{-r\tau} V_L(X_\tau, Y_\tau)\Big] = E\Bigg[\int_0^\tau \mathbf{1}_{\{Y_t \geq y^*\}} e^{-rt}(r - \mu_Y)(Y_t - \hat{y})\, dt\Bigg].$$

[13] A graphical illustration helps to convey the continuity argument for $b(x)$.



Let now $x_0 > \hat{x}$, $y_0 = y^*$ and $\tau = \min\{\inf\{t \geq 0 \mid X_t \leq \hat{x}\}, \tau_{\mathcal{P}}\}$, so $\tau \in (0, \tau_{\mathcal{P}}]$ and the two integrals on the respective right-hand side agree. However, $V_{\bar{L}}(X_\tau, Y_\tau) < V_L(X_\tau, Y_\tau)$ on $\{Y_\tau < y^*\}$, because then the unconstrained optimal stopping time, which is unique for $y^* > 0$, is not admissible in the constrained problem for $\max\{y_0, I - C\} < y^*$. The event $\{Y_\tau < y^*\}$ has positive probability by our non-degeneracy assumption, which contradicts that all other terms in the previous two displays are equal, respectively. □

Proof of Theorem 6. Time-consistency is easily verified, because all given plans are first hitting times of sets that do not depend on ϑ. It remains to check that τ_i^ϑ and τ_j^ϑ are mutual best replies for any $\vartheta \in \mathscr{T}$. By continuity of (L_t) and (F_t), and $L_t \geq F_t \Rightarrow M_t \geq F_t$, we have $M_{\tau_{\mathcal{P}}(\vartheta)} \geq F_{\tau_{\mathcal{P}}(\vartheta)}$. Therefore, if firm j uses the plan $\tau_j^\vartheta = \tau_{\mathcal{P}}(\vartheta)$, then $V_i^\vartheta(\tau_i, \tau_j^\vartheta) \leq V_i^\vartheta(\min\{\tau_i, \tau_{\mathcal{P}}(\vartheta)\}, \tau_j^\vartheta)$ for any plan $\tau_i \geq \vartheta$. Therefore τ_i^ϑ is optimal for firm i if it satisfies $\tau_i^\vartheta \leq \tau_{\mathcal{P}}(\vartheta)$ and attains

$$\operatorname*{ess\,sup}_{\tau \in \mathscr{T}\,:\,\tau \geq \vartheta} E\Big[\mathbf{1}_{\{\tau < \tau_{\mathcal{P}}(\vartheta)\}} L_\tau + \mathbf{1}_{\{\tau \geq \tau_{\mathcal{P}}(\vartheta)\}} M_{\tau_{\mathcal{P}}(\vartheta)} \,\Big|\, \mathscr{F}_\vartheta\Big].$$

As $\mathcal{P} \subseteq C^c$, we have indeed $\tau_i^\vartheta \leq \tau_{\mathcal{P}}(\vartheta)$. On the event $\{\tau_{\mathcal{P}}(\vartheta) = \vartheta\} \in \mathscr{F}_\vartheta$, then also $\tau_i^\vartheta = \vartheta$, which is trivially optimal. On the complement $\{\tau_{\mathcal{P}}(\vartheta) > \vartheta\} \in \mathscr{F}_\vartheta$, we have $(X_\vartheta, Y_\vartheta) \notin \mathcal{P}$ and $L_{\tau_{\mathcal{P}}(\vartheta)} = F_{\tau_{\mathcal{P}}(\vartheta)} = M_{\tau_{\mathcal{P}}(\vartheta)}$ by continuity of (L_t) and (F_t), so that τ_i^ϑ is optimal by Proposition 3 and the strong Markov property. By symmetry, τ_j^ϑ is optimal for firm j on the event $\{\tau_{\mathcal{P}}(\vartheta) = \vartheta\} \in \mathscr{F}_\vartheta$, since then $\tau_i^\vartheta = \tau_j^\vartheta = \tau_{\mathcal{P}}(\vartheta)$. On the complement $\{\tau_{\mathcal{P}}(\vartheta) > \vartheta\} \in \mathscr{F}_\vartheta$, we have $F_{\tau_i^\vartheta} \geq L_{\tau_i^\vartheta}$ by $\tau_i^\vartheta \leq \tau_{\mathcal{P}}(\vartheta)$ and continuity of (L_t) and $F(t)$, so that also $F_{\tau_i^\vartheta} \geq M_{\tau_i^\vartheta}$. Moreover, for any stopping time $\tau_j \geq \vartheta$, the optimality of τ_i^ϑ implies $L_{\tau_j} \leq E[L_{\tau_i^\vartheta} \mid \mathscr{F}_{\tau_j}]$ on the event $\{\tau_j < \tau_i^\vartheta\} \in \mathscr{F}_{\tau_j}$, so that the law of iterated expectations yields $E[\mathbf{1}_{\{\tau_j < \tau_i^\vartheta\}} L_{\tau_j} \mid \mathscr{F}_\vartheta] \leq E[\mathbf{1}_{\{\tau_j < \tau_i^\vartheta\}} L_{\tau_i^\vartheta} \mid \mathscr{F}_\vartheta]$. Together, for any stopping time $\tau_j \geq \vartheta$ then $V_j^\vartheta(\tau_j, \tau_i^\vartheta) \leq E[F_{\tau_i^\vartheta} \mid \mathscr{F}_\vartheta] = V_j^\vartheta(\tau_j^\vartheta, \tau_i^\vartheta)$. The last equality follows from $\tau_j^\vartheta \geq \tau_i^\vartheta$, where $\tau_j^\vartheta = \tau_i^\vartheta$ only on $\{\tau_i^\vartheta = \tau_{\mathcal{P}}(\vartheta)\}$, so that $M_{\tau_i^\vartheta} = F_{\tau_i^\vartheta}$ as noted before. □

Proof of Proposition 9. (i) First consider the case $y_0 = 0$, i.e., $Y_t \equiv 0$ for all $t \in \mathbb{R}_+$. Then the claimed rate is actually zero, clearly if $\hat{y} = 0$, and also if $\hat{y} > 0$, because then $(X_t, Y_t) \in C$ by $b(x) > 0$ for all $x > 0$ and $(0, 0) \in C$. But also the waiting cost in the constrained problem (6) is zero if $y_0 = 0$, because $\tau = \tau_{\mathcal{P}} \equiv \infty$ is optimal then due to $I \geq 0$. If $y_0 > x_0 = 0$, then $X_t \equiv 0$ for all $t \in \mathbb{R}_+$, so that a one-dimensional variant of the following argument for the case $x_0, y_0 > 0$ applies.

Recall, from the proof of Proposition 3, the auxiliary reward process $\bar{L} = (\bar{L}_t)$ defined by $\bar{L}_t = L_{\min\{t, \tau_{\mathcal{P}}\}}$, so that the value of the constrained problem (6) is the same as $V_{\bar{L}}(x_0, y_0) = \sup_{\tau \geq 0} E[\bar{L}_\tau]$ for arbitrary initial values $(x_0, y_0) \in \mathbb{R}_+^2$. Now we use the terminology of the general theory of optimal stopping. The *Snell envelope* $U_{\bar{L}} = (U_{\bar{L}}(t))$ is the maximal value that can still be attained if one does not stop before t, here given by $U_{\bar{L}}(t) = e^{-rt} V_{\bar{L}}(X_t, Y_t)$ for every $t \in [0, \tau_{\mathcal{P}}]$ (a.s.). It can be characterized as the smallest supermartingale dominating the payoff process $\bar{L}$, so it has a decomposition $U_{\bar{L}} = M_{\bar{L}} - D_{\bar{L}}$ with a martingale $M_{\bar{L}} = (M_{\bar{L}}(t))$ and a non-decreasing process $D_{\bar{L}} = (D_{\bar{L}}(t))$ starting from $D_{\bar{L}}(0) \equiv 0$, called the *compensator*. Hence, for every $\hat{\tau} \in \mathscr{T}$, $U_{\bar{L}}(0) - E[U_{\bar{L}}(\hat{\tau})] = E[D_{\bar{L}}(\hat{\tau})] \geq 0$ is the expected loss if one considers stopping only at stopping times $\tau \geq \hat{\tau}$. $U_{\bar{L}}$ has continuous paths, because $V_{\bar{L}}(\cdot)$ is a continuous function; see, e.g., Krylov (1980). Proposition 9 claims that $D_{\bar{L}}$ is just the negative of the drift of $\bar{L}$ in the stopping region C^c. In Jacka (1993), this is shown to be true if the local time of the non-negative semimartingale $U_{\bar{L}} - \bar{L}$ spent at zero, denoted by $L^0(U_{\bar{L}} - \bar{L}) = (L_t^0(U_{\bar{L}} - \bar{L}))$, is zero for all $t \in \mathbb{R}_+$ (a.s.).

We are going to verify that this sufficient condition holds by applying the argument used for Theorem 6 in Jacka (1993). Itō's Lemma shows that $\bar{L}$ is a continuous semimartingale with finite variation part $A_t := \int_0^t -\mathbf{1}_{\{s < \tau_{\mathcal{P}}\}} e^{-rs}(r - \mu_Y)(Y_s - \hat{y})\,ds$. Denote its decreasing part by (A_t^-), which satisfies

$$dA_t^- = \mathbf{1}_{\{Y_t > \hat{y}\}} e^{-rt}(r - \mu_Y)(Y_t - \hat{y})\,dt$$

for all $t < \tau_{\mathcal{P}}$. By Theorem 3 in Jacka (1993), $dL_t^0(U_{\bar{L}} - \bar{L}) \leq \mathbf{1}_{\{U_{\bar{L}}(t) = \bar{L}_t, t < \tau_{\mathcal{P}}\}} 2\,dA_t^-$, and as in Theorem 6 therein, the measure $dL^0(U_{\bar{L}} - \bar{L})$ is supported by $\{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid (X_t(\omega), Y_t(\omega)) \in \partial C\}$. For all $t \in \mathbb{R}_+$, it holds that

$$E\big[L_t^0(U_{\bar{L}} - \bar{L})\big] \leq E\Big[2\int_0^t \mathbf{1}_{\{(X_s, Y_s) \in \partial C, s < \tau_{\mathcal{P}}\}}\,dA_s^-\Big] \leq E\Big[2\int_0^t \mathbf{1}_{\{(X_s, Y_s) \in \partial C\}}\,dA_s^-\Big].$$

Note that dA^- has a Markovian density with respect to Lebesgue measure on $\mathbb{R}_+$ as shown above, and that our underlying diffusion (X_t, Y_t) has a log-normal transition distribution, which, thus, has a density with respect to Lebesgue measure on $\mathbb{R}_+^2$. Moreover, $\partial C = \{(x, y) \in \mathbb{R}_+^2 \mid y = b(x)\}$ has Lebesgue measure zero in $\mathbb{R}_+^2$. Therefore, like in the proof of Theorem 6 in Jacka (1993), we conclude that $L_t^0(U_{\bar{L}} - \bar{L}) = 0$ for all $t \in \mathbb{R}_+$ (a.s.).

(ii) We begin with the case $x_0 = y_0 = I - C = 0$. Then also $X_t = Y_t = I - C = 0$, so that $(X_t, Y_t) \in \overline{\mathcal{P}}$, resp. $F_t = L_t$, for all $t \in \mathbb{R}_+$. Therefore, $\tau_{\mathcal{P}} = \infty \neq \inf\{t \geq 0 \mid (X_t, Y_t) \in \overline{\mathcal{P}}\} = 0$. Next, if $y_0 = 0$ and $\max\{x_0, I - C\} > 0$, then also $Y_t = 0 < \max\{X_t, I - C\}$, so that $(X_t, Y_t) \in \overline{\mathcal{P}}^c$ for all $t \in \mathbb{R}_+$. Therefore, $\tau_{\mathcal{P}} = \inf\{t \geq 0 \mid (X_t, Y_t) \in \overline{\mathcal{P}}\} = \infty$ in this case.

Finally, we argue that if $y_0 > 0$ and $(x_0, y_0) \in \partial\mathcal{P}$, then $\tau_{\mathcal{P}} = 0$ a.s. First consider $x_0 > 0$. Then $\{(X_t, Y_t) \in \mathcal{P}\} \supseteq \{Y_t > Y_0\} \cap \{X_t < X_0\} = \{(\mu_Y - \sigma_Y^2/2)t + \sigma_Y B_t^{(2)} > 0\} \cap \{-(\mu_X - \sigma_X^2/2)t - \sigma_X B_t^{(1)} > 0\}$. Rewriting the latter two sets, the result follows from Lemma 15. Indeed, we have $(B^{(1)}, B^{(2)})^\top = \Sigma W$ for an invertible matrix Σ and 2-dimensional Brownian motion W by Remark 12. Letting $\tilde{\Sigma} = \begin{pmatrix} -\sigma_X & 0 \\ 0 & \sigma_Y \end{pmatrix}$ and then $\hat{\Sigma} = \tilde{\Sigma}\Sigma$, which are both invertible, as well as $\mu = (-(\mu_X - \sigma_X^2/2), \mu_Y - \sigma_Y^2/2)^\top$, the intersection of the two sets is $\{\min\{Z_t^{(1)}, Z_t^{(2)}\} > 0\}$ for $(Z_t^{(1)}, Z_t^{(2)})^\top = t\mu + \hat{\Sigma}W_t$. Now consider $x_0 = 0$. Then $\{(X_t, Y_t) \in \mathcal{P}\} = \{Y_t > Y_0\}$, and Lemma 15 applies with $d = 1$. □

Proof of Theorem 10. First, note that the process (a_t) is well defined, because $x + I - C - y > 0$ for all $(x, y) \in \mathcal{A} \subseteq \overline{\mathcal{P}}^c$. Second, $a_t \geq 0$, because $y \geq \hat{y}$ for all $(x, y) \in \mathcal{A}$. Indeed, by Proposition 3, $y \geq b(x)$ for all $(x, y) \in C^c$, and $b(x) < \hat{y}$ only if $b(x) = x + I - C$, so that $(x, y) \in C^c$ and $\hat{y} > y$ imply $(x, y) \in \overline{\mathcal{P}} \subseteq \mathcal{A}^c$. Third, (a_t) is adapted to $(\mathcal{F}_t)$, because a_t is defined by a measurable function of (X_t, Y_t). Therefore, the solution of the differential equation (9), which is $G_i^\vartheta(t) = 1 - \exp(-\int_\vartheta^t a_s\,ds)$ for all $t \in [\vartheta, \tau_{\mathcal{P}}(\vartheta))$, is non-decreasing, taking values in $[0, 1]$, and continuous. Thus, together with $G_i^\vartheta(t) = 1$ for all $t \geq \tau_{\mathcal{P}}(\vartheta)$, G_i^ϑ is a feasible mixed plan for firm i in the subgame starting at ϑ. Time-consistency is satisfied, because the hazard rate a_t and the set $\mathcal{P}$ do not depend on ϑ.

Now fix any $i \in \{1, 2\}$ and let j denote the respective other firm. To verify that G_i^ϑ is a best reply against G_j^ϑ, it suffices to consider only mixed plans G_i that satisfy $G_i(\tau_{\mathcal{P}}(\vartheta)) = 1$, because $G_j^\vartheta(\tau_{\mathcal{P}}(\vartheta)) = 1$ and $M_{\tau_{\mathcal{P}}(\vartheta)} \geq F_{\tau_{\mathcal{P}}(\vartheta)}$. In particular, G_i^ϑ is optimal on the event $\{\tau_{\mathcal{P}}(\vartheta) = \vartheta\} \in \mathcal{F}_\vartheta$. On the complement $\{\tau_{\mathcal{P}}(\vartheta) > \vartheta\} \in \mathcal{F}_\vartheta$, we can rewrite the payoff from G_i using $G_i(\tau_{\mathcal{P}}(\vartheta)) = G_j^\vartheta(\tau_{\mathcal{P}}(\vartheta)) = 1$, Fubini's Theorem, and the continuity of $G_j^\vartheta(t)$ in all $t < \tau_{\mathcal{P}}(\vartheta)$ as

$$V_i^\vartheta(G_i, G_j^\vartheta) = E\Bigg[\int\limits_{[0,\tau_{\mathcal{P}}(\vartheta))} \big(1 - G_j^\vartheta(s)\big) L_s\,dG_i(s) + \int\limits_{[0,\tau_{\mathcal{P}}(\vartheta)]} \Bigg(\int\limits_{[0,s)} F_u\,dG_j^\vartheta(u)\Bigg) dG_i(s)$$
$$+ \big(1 - G_i(\tau_{\mathcal{P}}(\vartheta)-)\big)\big(1 - G_j^\vartheta(\tau_{\mathcal{P}}(\vartheta)-)\big) M_{\tau_{\mathcal{P}}(\vartheta)} \Bigg| \mathcal{F}_\vartheta\Bigg].$$

Furthermore, since $M_{\tau_{\mathcal{P}}(\vartheta)} = L_{\tau_{\mathcal{P}}(\vartheta)}$ on $\{\tau_{\mathcal{P}}(\vartheta) > \vartheta\}$, $(1 - G_i(\tau_{\mathcal{P}}(\vartheta)-)) = \Delta G_i(\tau_{\mathcal{P}}(\vartheta))$, and $G_j^\vartheta(s) = G_j^\vartheta(s-)$ for all $s \in [0, \tau_{\mathcal{P}}(\vartheta))$, we obtain

$$V_i^\vartheta(G_i, G_j^\vartheta) = E\Bigg[\int\limits_{[0,\tau_{\mathcal{P}}(\vartheta)]} S_i^\vartheta(s)\,dG_i(s) \Bigg| \mathcal{F}_\vartheta\Bigg],$$

where $S_i^\vartheta(s) = (1 - G_j^\vartheta(s-))L_s + \int_{[0,s)} F_u\,dG_j^\vartheta(u)$. Therefore, $G_i = G_i^\vartheta$ is optimal if and only if it increases when it is optimal to stop the process $(S_i^\vartheta(t))$ on $[\vartheta, \tau_{\mathcal{P}}(\vartheta)]$. We now argue that this is the case anywhere in the attrition region $\mathcal{A}$ or at $\tau_{\mathcal{P}}(\vartheta)$.

Consider any stopping time $\tau \in [\vartheta, \tau_{\mathcal{P}}(\vartheta)]$ and rewrite $S_i^\vartheta(\tau) = (1 - G_j^\vartheta(\tau-))L_\tau + \int_{[0,\tau)} L_u\,dG_j^\vartheta(u) + \int_{[0,\tau)} (F_u - L_u)\,dG_j^\vartheta(u)$, so that, as $F_u - L_u = e^{-ru}(X_u + I - C - Y_u)$ and $dG_j^\vartheta(u) = (1 - G_j^\vartheta(u))a_u\,du$ for all $u \in [\vartheta, \tau_{\mathcal{P}}(\vartheta))$,

$$S_i^\vartheta(\tau) = \big(1 - G_j^\vartheta(\tau-)\big)L_t + \int\limits_{[0,\tau)} L_u\,dG_j^\vartheta(u) + \int\limits_{[0,\tau)} \big(1 - G_j^\vartheta(u)\big)\mathbf{1}_{\{(X_u,Y_u)\in\mathcal{A}\}} e^{-ru}(r - \mu_Y)(Y_u - \hat{y})\,du.$$

Denoting $\mathbf{1}_{\{t\geq\vartheta\}} \int_\vartheta^t \mathbf{1}_{\{(X_u,Y_u)\in\mathcal{A}\}} e^{-ru}(r - \mu_Y)(Y_u - \hat{y})\,du$ by D_t, applying integration by parts for the continuous and finite-variation process (D_t), and $D_\vartheta = 0$ then yield

$$S_i^\vartheta(\tau) = \big(1 - G_j^\vartheta(\tau-)\big)(L_\tau + D_\tau) + \int\limits_{[0,\tau)} (L_u + D_u)\,dG_j^\vartheta(u)$$
$$\leq \big(1 - G_j^\vartheta(\tau-)\big)\big(e^{-r\tau} V_{\bar{L}}(X_\tau, Y_\tau) + D_\tau\big) + \int\limits_{[0,\tau)} \big(e^{-ru} V_{\bar{L}}(X_u, Y_u) + D_u\big)\,dG_j^\vartheta(u),$$

where $V_{\bar{L}}(x, y)$ is the value function of the constrained problem (6) defined by equation (A.1) in the proof of Proposition 3. In order to determine the expectation of this upper bound for $S_i^\vartheta(\tau)$ by Proposition 9(i), note that D_t equals the waiting cost $\int_\vartheta^t \mathbf{1}_{\{(X_u,Y_u)\in C^c\}} e^{-ru}(r - \mu_Y)(Y_u - \hat{y})\,du$ for all $t \in [\vartheta, \tau_{\mathcal{P}}(\vartheta)]$ (a.s.). Indeed, by Proposition 9(ii), $(X_t, Y_t) \in \overline{\mathcal{P}}^c$ for all $t \in [\vartheta, \tau_{\mathcal{P}}(\vartheta))$ (a.s.), except if $x_0 = y_0 = I - C = 0$. In the latter case, however, D_t and the waiting cost are both zero, because the proof of Proposition 9(i) started by arguing that if $y_0 = 0$, i.e., $Y_t \equiv 0$ for all $t \in \mathbb{R}_+$, but $\hat{y} > 0$, then $(X_t, Y_t) \in C \subseteq \mathcal{A}^c$. Therefore, $E[e^{-r\vartheta} V_{\bar{L}}(X_\vartheta, Y_\vartheta)] = E[D_{\tau'} + e^{-r\tau'} V_{\bar{L}}(X_{\tau'}, Y_{\tau'})]$ for every stopping time $\tau' \in [\vartheta, \tau_{\mathcal{P}}(\vartheta)]$. Hence, by a change of variable as in Lemma B.2 in Riedel and Steg (2017), $E[S_i^\vartheta(\tau)] \leq E[e^{-r\vartheta} V_{\bar{L}}(X_\vartheta, Y_\vartheta)]$, and G_i^ϑ is optimal if it attains this bound. Indeed, $G_i^\vartheta(t)$ and $G_j^\vartheta(t)$ increase only when $t \in [\vartheta, \tau_{\mathcal{P}}(\vartheta)]$ and $(X_t, Y_t) \in \mathcal{A} \subseteq C^c$, so that then $L_t = e^{-rt}(Y_t - I) =$

$e^{-rt}V_{\bar{L}}(X_t, Y_t)$. Therefore, if G_i^{ϑ} increases at τ, then $E[S_i^{\vartheta}(\tau)] = E[e^{-r\vartheta}V_{\bar{L}}(X_{\vartheta}, Y_{\vartheta})]$, so that, by the change of variable, also $E[\int_{[0,\tau_{\mathcal{P}}(\vartheta)]} S_i^{\vartheta}(s)\,dG_i^{\vartheta}(s)] = E[e^{-r\vartheta}V_{\bar{L}}(X_{\vartheta}, Y_{\vartheta})]$. □

Proof of Proposition 11. The following proof applies for *any* set $\mathcal{A} \subseteq \mathbb{R}_+^2$. First, note that $\tau_{\mathcal{P}}(\vartheta) \equiv \infty$ if $Y_0 = y_0 = 0$. Therefore, consider $Y_0 = y_0 > 0$, so that $\tau_{\mathcal{P}}(\vartheta) = \inf\{t \geq \vartheta \mid Y_t \geq X_t + I - C\}$ (a.s.) by Proposition 9(ii). We need to show that

$$\int_0^{\tau_{\mathcal{P}}(\vartheta)} \frac{dG_i^{\vartheta}(t)}{1 - G_i^{\vartheta}(t)} = \int_{\vartheta}^{\tau_{\mathcal{P}}(\vartheta)} \mathbf{1}_{\{(X_t,Y_t)\in\mathcal{A}\}} \frac{(r-\mu_Y)(Y_t - \hat{y})}{X_t + I - C - Y_t}\,dt < \infty \tag{A.2}$$

on $\{\tau_{\mathcal{P}}(\vartheta) < \infty\}$ (a.s.). As (Y_t) is continuous, it is bounded on $[0, \tau_{\mathcal{P}}(\vartheta)]$ when this interval is finite. Hence, we may just use $(X_t + I - C - Y_t)^{-1}$ as the integrand in (A.2). By the strong Markov property, we set $\vartheta \equiv 0$. Hence, we will prove that

$$\int_0^{\tau_0} (X_t - Y_t + a)^{-1}\,dt < \infty \tag{A.3}$$

on $\{\tau_0 < \infty\}$ (a.s.) for any fixed level $a \geq 0$ and the stopping time $\tau_{\varepsilon} := \inf\{t \geq 0 \mid X_t - Y_t + a \leq \varepsilon\}$ with $\varepsilon = 0$. Define the process (Z_t) by $Z_t := X_t - Y_t + a$ to simplify notation, and suppose $a > 0$ (the special case $a = 0$ will be treated at the very end of the proof).

As a first step and tool, we derive the weaker result $E[\int_0^{\tau_0\wedge T} \ln(Z_t)\,dt] \in \mathbb{R}$ for any time $T > 0$ (which will imply also $\int_0^{\tau_0} |\ln(Z_t)|\,dt < \infty$ on $\{\tau_0 < \infty\}$ (a.s.) by the arguments towards the end of the proof).

Define the function $f: \mathbb{R}_+ \to \mathbb{R}$ by

$$f(x) = x\ln(x) - x \in [-1, x^2]$$

and the function $F: \mathbb{R}_+ \to \mathbb{R}$ by $F(x) = \int_0^x f(y)\,dy \in [-x, x^3]$, such that $F''(x) = \ln(x)$ for all $x > 0$. For localization purposes, fix an $\varepsilon > 0$ and a time $T > 0$. By Itō's formula,

$$\begin{aligned}
F(Z_{\tau_{\varepsilon}\wedge T}) &= F(Z_0) + \int_0^{\tau_{\varepsilon}\wedge T} f(Z_t)\,dZ_t + \frac{1}{2}\int_0^{\tau_{\varepsilon}\wedge T} \ln(Z_t)\,d[Z]_t \\
&= F(Z_0) + \int_0^{\tau_{\varepsilon}\wedge T} f(Z_t)\big(\mu_X X_t - \mu_Y Y_t\big)\,dt \\
&\quad + \int_0^{\tau_{\varepsilon}\wedge T} f(Z_t)\Big(\sigma_X X_t\,dB_t^{(1)} - \sigma_Y Y_t\,dB_t^{(2)}\Big) \\
&\quad + \frac{1}{2}\int_0^{\tau_{\varepsilon}\wedge T} \ln(Z_t)\big(\underbrace{\sigma_X^2 X_t^2 + \sigma_Y^2 Y_t^2 - 2\rho\sigma_X\sigma_Y X_t Y_t}_{=:\sigma^2(X_t,Y_t)}\big)\,dt.
\end{aligned}$$

We want to establish $\lim_{\varepsilon\searrow 0} E[F(Z_{\tau_{\varepsilon}\wedge T})]$, first in terms of the integrals, for which we need some estimates.

In order to eliminate the second, stochastic integral by taking expectations, it is sufficient to verify that $\mathbf{1}_{\{t<\tau_0\}} f(Z_t) X_t$ and $\mathbf{1}_{\{t<\tau_0\}} f(Z_t) Y_t$ are square $P\otimes dt$-integrable on $\Omega\times[0,T]$. We have $|f(Z_t)| \leq 1 + Z_t^2$. For $t \leq \tau_0$, furthermore $0 \leq Z_t \leq X_t + a$ by $Y_t \geq 0$, and hence $Z_t^2 \leq (X_t + a)^2 \leq 2X_t^2 + 2a^2$. The sought square-integrability with X follows now from the $P\otimes dt$-integrability of X_t^n on $\Omega\times[0,T]$ for any $n \in \mathbb{N}$ and analogously that for Y thanks to $0 \leq Y_t \leq X_t + a$ for $t \leq \tau_0$.

The same estimates guarantee that the expectation of the first integral converges to the finite expectation at τ_0 as $\varepsilon \searrow 0$. For the third integral, we have $\ln(Z_t) \leq Z_t \leq X_t + a$. The second term $\sigma^2(X_t, Y_t)$ is bounded from below by $(\sigma_X X_t - \sigma_Y Y_t)^2 \geq 0$ and from above by $(\sigma_X X_t + \sigma_Y Y_t)^2 \leq ((\sigma_X + \sigma_Y)X_t + \sigma_Y a)^2 \leq 2(\sigma_X + \sigma_Y)^2 X_t^2 + 2\sigma_Y^2 a^2$ for $t \leq \tau_0$ (supposing w.l.o.g. $\sigma_X, \sigma_Y > 0$). Hence, the positive part of the integrand is bounded by a $P\otimes dt$-integrable process on $\Omega\times[0,T]$, while the negative part converges monotonically, and we may take the limit of the expectation of the whole integral.

That the latter is finite follows from analyzing the limit of the LHS, $\lim_{\varepsilon\searrow 0} E[F(Z_{\tau_{\varepsilon}\wedge T})]$, directly. We have $F(Z_{\tau_{\varepsilon}\wedge T}) = \mathbf{1}_{\{\tau_{\varepsilon}\leq T\}} F(\varepsilon) + \mathbf{1}_{\{T<\tau_{\varepsilon}\}} F(Z_T)$, and it is continuous in $\varepsilon \searrow 0$. For $T < \tau_0$, again $0 \leq Z_T \leq X_T + a$ and, thus, $|F(Z_T)| \leq |Z_T| + |Z_T|^3 \leq (X_T + a) + (X_T + a)^3$. As also $|F(\varepsilon)| \leq 1$ for all $\varepsilon \leq 1$, $\big|F(Z_{\tau_{\varepsilon}\wedge T})\big|$ is bounded by an integrable random variable as $\varepsilon \searrow 0$. Consequently, $\lim_{\varepsilon\searrow 0} E[F(Z_{\tau_{\varepsilon}\wedge T})] = E[F(Z_{\tau_0\wedge T})] \in \mathbb{R}$, and also $E[\int_0^{\tau_0\wedge T} \ln(Z_t)\sigma^2(X_t, Y_t)\,dt] \in \mathbb{R}$ on the RHS. In the integral, we may ignore the term $\sigma^2(X_t, Y_t)$, which completes the first step. Indeed, with $|\rho| < 1$ we can have $\sigma^2(X_t, Y_t) = 0$

only if $X_t = Y_t = 0$, i.e., if $Z_t = a$. Therefore, $\inf\{\sigma^2(X_t, Y_t) \mid Z_t \le \varepsilon\} > 0$ for any fixed $\varepsilon \in (0, a)$, so $\sigma^2(X_t, Y_t)$ does not "kill" the downside of $\ln(Z_t)$ and

$$E\Bigg[\int_0^{\tau_0 \wedge T} \ln(Z_t)\, dt\Bigg] \in \mathbb{R}.$$

In the following, we furthermore need $E[\int_0^{\tau_0\wedge T} \ln(Z_t) X_t\, dt] \in \mathbb{R}$, which obtains as follows. With $|\rho| < 1$, $\inf\{\sigma^2(X_t, Y_t) \mid Z_t \le \varepsilon\}$ is attained only when the constraint binds, i.e., $Y_t = X_t + a - \varepsilon$. Thus, for $Z_t \le \varepsilon$, $\sigma^2(X_t, Y_t) \ge \sigma^2(X_t, X_t + a - \varepsilon)$. The latter is a quadratic function of X_t, with X_t^2 having coefficient $\sigma^2(1, 1) > 0$, given $|\rho| < 1$. The quadratic function hence exceeds X for all X sufficiently large, i.e., we can pick $K > 0$ such that $\sigma^2(X_t, Y_t) \ge X_t$ on $\{Z_t \le \varepsilon\} \cap \{X_t \ge K\}$. Thus, X_t does not "blow up" the downside of $\ln(Z_t)$ more than $\sigma^2(X_t, Y_t)$.

We are now ready to analyze

$$\begin{aligned} f(Z_{\tau_\varepsilon \wedge T}) &= f(Z_0) + \int_0^{\tau_\varepsilon \wedge T} \ln(Z_t)\, dZ_t + \frac{1}{2} \int_0^{\tau_\varepsilon \wedge T} \frac{1}{Z_t}\, d[Z]_t \\ &= f(Z_0) + \int_0^{\tau_\varepsilon \wedge T} \ln(Z_t)\big(\mu_X X_t - \mu_Y Y_t\big)\, dt \\ &\quad + \int_0^{\tau_\varepsilon \wedge T} \ln(Z_t)\Big(\sigma_X X_t\, dB_t^{(1)} - \sigma_Y Y_t\, dB_t^{(2)}\Big) \\ &\quad + \frac{1}{2} \int_0^{\tau_\varepsilon \wedge T} \frac{1}{Z_t} \sigma^2(X_t, Y_t)\, dt \end{aligned}$$

when taking the limit $\varepsilon \searrow 0$ under expectations as before. By our final observations of step one, the first integral converges (note again $Y_t \le X_t + a$ for $t \le \tau_0$). In the second, stochastic integral, we now have $|\ln(Z_t)| \le |\ln(\varepsilon)| + |Z_t|$ for $t \le \tau_\varepsilon$, an even smaller bound than above, making the expectation vanish. In the third integral, $\sigma^2(X_t, Y_t)/Z_t \ge 0$ for $t \le \tau_0$, so monotone convergence holds.

On the LHS, $\big|f(Z_{\tau_\varepsilon \wedge T})\big|$ is bounded by an integrable random variable for all $\varepsilon \le 1$ analogously to the first step, implying $\lim_{\varepsilon \searrow 0} E[f(Z_{\tau_\varepsilon \wedge T})] = E[f(Z_{\tau_0 \wedge T})] \in \mathbb{R}$ and, thus, $E[\int_0^{\tau_0 \wedge T} \sigma^2(X_t, Y_t)/Z_t\, dt] < \infty$. By the same arguments brought forward at the end of the first step, we can again ignore $\sigma^2(X_t, Y_t)$.

With $E[\int_0^{\tau_0 \wedge T} 1/Z_t\, dt] < \infty$, $P[\{\tau_0 \le T\} \cap \{\int_0^{\tau_0} 1/Z_t\, dt = \infty\}] = 0$. As T was arbitrary, we may take the union over all integer T to conclude $P[\{\tau_0 < \infty\} \cap \{\int_0^{\tau_0} 1/Z_t\, dt = \infty\}] = 0$. This completes the proof of (A.3) for the case $a > 0$ (and that of (A.2) for $I - C > 0$).

Some modification is due for the case $a = 0$ if $x_0 > y_0$ (otherwise $\tau_0 \equiv 0$ and (A.3) is trivial). Then $\sigma^2(X_t, Y_t)$ is not bounded away from 0 for Z_t small, it may be an important factor in the integrability of $\ln(Z_t)\sigma^2(X_t, Y_t)$ when (X_t, Y_t) is close to the origin. In order to infer the required well-behaved limit of $E[\int_0^{\tau_\varepsilon \wedge T} \ln(Z_t)\big(\mu_X X_t - \mu_Y Y_t\big)\, dt]$ and to finally remove $\sigma^2(X_t, Y_t)$ from $\int_0^{\tau_0} \sigma^2(X_t, Y_t)/Z_t\, dt$, it is possible to use another localization procedure: Fix a small $\delta > 0$ and use the minimum of $\sigma_\delta := \inf\{t \ge 0 \mid X_t + Y_t < \delta\}$ and $\tau_\varepsilon \wedge T$ everywhere above. Then $\sigma^2(X_t, Y_t)$ is bounded away from 0 on $[0, \sigma_\delta \wedge \tau_0]$ for $|\rho| < 1$ and again bounded below by a quadratic function with X_t^2 having coefficient $\sigma^2(1, 1) > 0$. The result now obtains as above for all paths with $X_t + Y_t \ge \delta$ on $[0, \tau_0]$. As $\delta > 0$ is indeed arbitrary, and any path with $Y_0 = y_0 > 0$ is by continuity bounded away from the origin on $[0, \tau_0]$ when the latter is finite, the claim follows. □

Appendix. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.geb.2022.11.013.

References

Agrawal, A., Kim, Y., Kwon, H., Muthulingam, S., 2016. Investment ins shared suppliers: Effect of learning, spillover, and competition. Prod. Oper. Manag. 25, 736–750.

Azevedo, A., Paxson, D., 2014. Developing real option game models. Eur. J. Oper. Res. 237, 909–920.

Chevalier-Roignant, B., Trigeorgis, L., 2011. Competitive Strategy: Options and Games. MIT Press, Cambridge, Mass.

Christensen, C., Suárez, F., Utterback, J., 1998. Strategies for survival in fast-changing industries. Manag. Sci. 44, S207–S220.

Cleff, T., Rennings, K., 2014. Are there any first and second mover advantages for eco-pioneers? Lead market strategies for environmental innovation. In: Interdisciplinary Management Research, vol. 10, pp. 164–189.

Décamps, J.P., Mariotti, T., 2004. Investment timing and learning externalities. J. Econ. Theory 118, 80–102.
Dixit, A.K., Pindyck, R.S., 1994. Investment under Uncertainty. Princeton University Press, Princeton, NJ.
Fudenberg, D., Tirole, J., 1985. Preemption and rent equalization in the adoption of new technology. Rev. Econ. Stud. 52, 383–401.
Fudenberg, D., Tirole, J., 1986. A theory of exit in duopoly. Econometrica 54, 943–960.
Georgiadis, G., Kim, Y., Kwon, H., 2022. The absence of attrition in a war of attrition under complete information. Games Econ. Behav. 131, 171–185.
Ghemawat, P., Nalebuff, B., 1985. Exit. Rand J. Econ. 16, 184–194.
Gilbert, J., Birnbaum–More, P., 1996. Innovation timing advantages; from economic theory to strategic application. J. Eng. Technol. Manag. 12, 245–266.
Grenadier, S.R., 1996. The strategic exercise of options: Development cascades and overbuilding in real estate markets. J. Finance 51, 1653–1679.
Hackbarth, D., Morellec, E., 2008. Stock returns in mergers and acquisitions. J. Finance 63, 1213–1252.
Hellmann, T., Thijssen, J.J.J., 2018. Fear of the market or fear of the competitor? Ambiguity in a real options game. Oper. Res. 66, 1744–1759.
Hoppe, H.C., 2000. Second-mover advantages in the strategic adoption of new technology under uncertainty. Int. J. Ind. Organ. 18, 315–338.
Huisman, K.J.M., 2001. Technology Investment: A Game Theoretic Real Options Approach. Kluwer Academic Publishers, Dordrecht.
Huisman, K.J.M., Kort, P.M., Pawlina, G., Thijssen, J.J.J., 2004. Strategic investment under uncertainty: Merging real options with game theory. Z. Betriebswirtsch. 67, 97–123.
Jacka, S.D., 1993. Local times, optimal stopping and semimartingales. Ann. Probab. 21, 329–339.
Katz, M.L., Shapiro, C., 1987. R& D rivalry with licensing or immitation. Am. Econ. Rev. 77, 402–420.
Krylov, N.V., 1980. Controlled Diffusion Processes. Springer, New York.
Kwon, H.D., Xu, W., Agrawal, A., Muthulingam, S., 2016. Impact of Bayesian learning and externalities on strategic investment. Manag. Sci. 62, 550–570.
Lambrecht, B.M., 2001. The impact of debt financing on entry and exit in a duopoly. Rev. Financ. Stud. 14, 765–804.
Lieberman, M., Montgomery, D., 1988. First-mover advantages. Strateg. Manag. J. 9, 51–58.
McDonald, R., Siegel, D., 1986. The value of waiting to invest. Q. J. Econ. 101, 707–728.
Min, S., Kalwani, M., Robinson, W., 2006. Market pioneer and early follower survival risks: A contingency analysis of really new versus incrementally new product markets. J. Mark. 70, 15–33.
Murto, P., 2004. Exit in duopoly under uncertainty. Rand J. Econ. 35, 111–127.
Peskir, G., Shiryaev, A., 2006. Optimal Stopping and Free-Boundary Problems. Birkhäuser Verlag, Basel.
Riedel, F., Steg, J.H., 2017. Subgame-perfect equilibria in stochastic timing games. J. Math. Econ. 72, 36–50.
Samuelson, P.A., 1965. Rational theory of warrant pricing. Ind. Manage. Rev. 6, 13–39. With Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics, by H. P. McKean, Jr.
Smets, F., 1991. Exporting versus FDI: The effect of uncertainty, irreversibilities and strategic interactions. Working Paper. Yale University, New Haven.
Steg, J.H., 2015. Symmetric Equilibria in Stochastic Timing Games. Working Paper 543. Center for Mathematical Economics, Bielefeld University.
Steg, J.H., 2018. Preemptive investment under uncertainty. Games Econ. Behav. 110, 90–119.
Thijssen, J.J.J., 2008. Optimal and strategic timing of mergers and acquisitions motivated by synergies and risk diversification. J. Econ. Dyn. Control 32, 1701–1720.
Thijssen, J.J.J., 2010. Preemption in a real option game with a first mover advantage and player-specific uncertainty. J. Econ. Theory 145, 2448–2462.
Thijssen, J.J.J., Huisman, K.J.M., Kort, P.M., 2006. The effects of information on strategic investment and welfare. Econ. Theory 28, 399–424.
Thijssen, J.J.J., Huisman, K.J.M., Kort, P.M., 2012. Symmetric equilibrium strategies in game theoretic real option models. J. Math. Econ. 48, 219–225.
Weeds, H.F., 2002. Strategic delay in a real options model of R&D competition. Rev. Econ. Stud. 69, 729–747.
Zachary, M., Gianiodis, P., Payne, G., Markman, G., 2015. Entry timing: Enduring lessons and future directions. J. Manag. 41, 1388–1415.