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Optimal rate of convergence for approximations of SPDEs with non-regular drift

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Oleg Butkovsky[*], Konstantinos Dareiotis[†], and Máté Gerencsér[‡]

Abstract

A fully discrete finite difference scheme for stochastic reaction-diffusion equations driven by a $1+1$-dimensional white noise is studied. The optimal strong rate of convergence is proved without posing any regularity assumption on the non-linear reaction term. The proof relies on stochastic sewing techniques.

Contents

1 Introduction

Consider the stochastic partial differential equation (SPDE)

$$\partial_t u = \Delta u + b(u) + \xi \qquad \text{on } (0,\infty)\times\mathbb{T}, \qquad\qquad u_0 = \psi \qquad \text{on } \mathbb{T}. \tag{1.1}$$

Here the unknown u is a random space-time stochastic process in $1+1$ dimensions, ξ is a space-time white noise, and $b : \mathbb{R} \to \mathbb{R}$ is a given function. The spatial domain is the 1-dimensional

[*]Weierstrass Institute, Mohrenstraße 39, 10117 Berlin, Germany oleg.butkovskiy@gmail.com
[†]University of Leeds, Woodhouse, LS2 9JT Leeds, United Kingdom k.dareiotis@leeds.ac.uk
[‡]TU Wien, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria mate.gerencser@tuwien.ac.at

torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, in other words, we consider the equation with periodic boundary conditions. Owing to the regularising property of the noise, equation (1.1) is well-posed even with merely bounded and measurable b, as classical results of Gyöngy and Pardoux [GP93a, GP93b] show. For a far-reaching generalisation of these results we refer to the recent work [ABLM20].

The error analysis of stochastic reaction-diffusion equations of the form (1.1) with various regularity assumptions on the drift b goes back to the early days of numerical analysis of SPDEs. In what was the first study of a fully discrete numerical scheme for SPDEs, Gyöngy [Gyö99] showed[1] that the space-time finite difference approximation of the above equation (A) strongly converges to the true solution if b is a bounded measurable function (B) converges with strong rate $1/4$ w.r.t. time and $1/2$ w.r.t. space if b is a Lipschitz continuous function. This rate was in fact shown to be sharp by Davie and Gaines [DG00]. Despite a rapidly growing literature on the numerics of SPDEs in the two decades since, the "gap" between (A) and (B) has remained and no rate of convergence has been known even if b is just shy of Lipschitz: say, $b \in \mathcal{C}^\alpha$ with $\alpha < 1$.

The aim of this paper is to resolve this question and derive the optimal rate of convergence (up to loss of arbitrarily small ε) without any regularity assumption on b. The main result can be informally summarized as follows. For the precise statement we refer to Theorem 1.1.2.

Theorem 1.0.1. *For any $\varepsilon \in (0, 1/2)$, bounded and measurable b, and any initial condition of class $\mathcal{C}^{1/2-\varepsilon}(\mathbb{T})$, the forward Euler finite difference approximation of* (1.1) *converges strongly with rate $1/4 - \varepsilon/2$ w.r.t. time and $1/2 - \varepsilon$ w.r.t. space.*

The strategy of the proof is quite different from previous works. In [Gyö99], the method for the bounded b case crucially relies on the Gyöngy-Krylov lemma [GK96, Lem. 1.1] and therefore is inherently not quantitative. As for methods in the Lipschitz (or one-sided Lipschitz) b case (see below for some references), they build on the analysis of the corresponding deterministic problem. Such approach is out of question for $b \in \mathcal{C}^\alpha$, $\alpha < 1$, since without the noise the PDE is not even well-posed, in general. Instead, our strategy uses stochastic sewing, initiated in [Lê20] and further developed in the numerical analytic direction in [BDG21, DGL21].

1.1 Formulation

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The white noise ξ on $[0,\infty) \times \mathbb{T}$ is a mapping from $\mathcal{B}_b([0,\infty) \times \mathbb{T})$, the bounded Borel sets of $[0,\infty) \times \mathbb{T}$, to $L_2(\Omega)$ such that for any collection $A_1, \ldots, A_k$ of elements of $\mathcal{B}_b([0,\infty) \times \mathbb{T})$, the vector $(\xi(A_1), \ldots, \xi(A_k))$ is Gaussian with mean 0 and covariance $\mathbb{E}(\xi(A_i)\xi(A_j)) = |A_i \cap A_j|$.

We consider the finite difference, forward Euler approximation of (1.1). To this end, we introduce the space and time grids, for each $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$

$$\Pi_n = \{0, (2n)^{-1}, \ldots, (2n-1)(2n)^{-1}\}, \qquad \Lambda_n = \{0, c(2n)^{-2}, 2c(2n)^{-2}, \ldots\},$$

where c is a constant satisfying the condition $c \in (0, 1/2)$, also commonly known as the CFL condition in the present context.

Remark 1.1.1. The restriction to look at spatial grids with even number of points (i.e. the choice of $2n$) is purely for convenience, otherwise the even and odd cases would require some notational distinction later on. The choice of focusing on the even case is motivated by the computational practice of using nested grids of mesh sizes 2^{-k}, $k = 1, 2, \ldots, N$ up to some threshold N.

[1][Gyö99] considers (1.1) with Dirichlet boundary conditions instead of periodic.

Note also that on Π_n, just like on $\mathbb{T}$, the addition is understood in a periodic way, i.e. $(2n-1)(2n)^{-1} + (2n)^{-1}$ is identified with 0. To ease notation, we also denote $h = c(2n)^{-2}$. Hence, by setting $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, one has $\Lambda_n = h\mathbb{N}_0$. Take an approximate initial condition $\psi^n : \Omega \times \mathbb{T} \to \mathbb{R}$. The approximation scheme is defined by setting $u_0^n(x) = \psi^n(x)$ for $x \in \Pi_n$ and then inductively

$$u_{t+h}^n(x) = u_t^n(x) + h\Delta_n u_t^n(x) + hb(u_t^n(x)) + h\eta_n(t,x) \tag{1.2}$$

for $t \in \Lambda_n$ and $x \in \Pi_n$, where the discrete Laplacian is defined as

$$\Delta_n f(x) = (2n)^2\big(f(x + (2n)^{-1}) - 2f(x) + f(x - (2n)^{-1})\big)\,,$$

and the discrete noise term is given by

$$\eta_n(t,x) = 2nh^{-1}\xi\Big([t, t+h] \times [x, x + (2n)^{-1}]\Big).$$

The main result of the article reads as follows.

Theorem 1.1.2. *Let $p \geq 2$, $\varepsilon \in (0, 1/4)$, and let b be bounded and measurable. Assume that the initial conditions ψ, ψ^n are $\mathcal{F}_0$-measurable $\mathcal{C}^{1/2-\varepsilon}$-valued random variables, such that for a constant $K < \infty$ they satisfy $\|\psi\|_{L_p(\Omega;\mathcal{C}^{1/2-\varepsilon}(\mathbb{T}))}, \|\psi^n\|_{L_p(\Omega;\mathcal{C}^{1/2-\varepsilon}(\Pi_n))} \leq K$. Then there exists a constant N depending only on the parameters $c, p, \varepsilon, K, \sup_{x\in\mathbb{R}}|b(x)|$ such that for all $n \in \mathbb{N}$ the following bound holds:*

$$\sup_{(t,x)\in([0,1]\cap\Lambda_n)\times\Pi_n} \|u_t(x) - u_t^n(x)\|_{L_p(\Omega)} \leq N\big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)}\big). \tag{1.3}$$

Remark 1.1.3. As will follow from the proof, with an appropriate extension of u^n from the gridpoints $\Lambda_n \times \Pi_n$ to the whole of $\mathbb{R}_+ \times \mathbb{T}$, the supremum on the left-hand side of (1.3) can be taken over $(t,x) \in [0,1] \times \mathbb{T}$.

Remark 1.1.4. The freedom of allowing different initial condition for the approximation is not a particularly important feature of the statement, but it is convenient for the proof. Indeed, it allows to easily deduce the general case from the case of short times, that is, when the supremum runs only over $t \leq S$, where S is a (small) constant depending only on the parameters of the problem.

1.2 Literature

As mentioned above, quantitative results have so far remained out of reach when b is even slightly irregular, i.e. not at least one-sided Lipschitz. Qualitative results, further to the works of Gyöngy [Gyö98, Gyö99], were obtained in the case of bounded measurable b in Pettersson and Signahl [PS05] (convergence in the nondegenerate multiplicative case) and in Anton, Cohen, and Quer-Sardanyons [ACQ20] (convergence for an exponential integrator scheme). Needless to say, in the case regular coefficients the rate of convergence of various discretisations of SPDEs is extensively studied. Even just in the context of space-time white noise driven reaction-diffusion equations the literature is rich, see among others [BG19, BGJK17, Deb10, JK08, LQ19, Pri01, Sha99, Wan20]. A wider overview can be found for example in the above mentioned work [ACQ20, Sec. 1] or in Da Prato-Zabczyk [DPZ92, Sec. 14.1.10]. The interested reader is also referred to the monographs [JK11, Kru13].

In contrast to SPDEs, which can be seen as infinite dimensional SDEs, the question of rate of convergence for finite dimensional SDEs with irregular drift coefficient is far more well-studied. As a small sample, we mention some of the most recent works [BDG21, NS20, MY20, JM21, Tag20, Yar21]. The developments of the last years are discussed in more detail in the survey [Szö21]. However, we mention that even in the finite dimensional case, the optimal strong convergence rate without any regularity assumptions has only been proved quite recently [DGL21].

1.3 Notation

For a metric space (X, d) (which for us will always be an interval, $\mathbb{T}$, or Π_n) we define the following spaces of $\mathbb{R}$-valued functions. The space of bounded and Borel-measurable functions is denoted by $\mathbb{B}(X)$ and is equipped with the norm $\|f\|_{\mathbb{B}} = \sup_x |f(x)|$. The space of continuous functions is denoted by $\mathcal{C}(X)$. For $\alpha \in (0, 1]$ we denote by $\mathcal{C}^\alpha(X)$ the space of bounded functions f that satisfy

$$[f]_{\mathcal{C}^\alpha} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\alpha} < \infty.$$

We equip $\mathcal{C}^\alpha$ with the norm $\|f\|_{\mathcal{C}^\alpha} = \|f\|_{\mathbb{B}} + [f]_{\mathcal{C}^\alpha}$. By convention, we set $\mathcal{C}^0 := \mathbb{B}$ (and not $\mathcal{C}$!).

We fix a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ such that for each $t \geq 0$, $A \in \mathcal{B}_b([0, t] \times \mathbb{T})$, and $B \in \mathcal{B}_b([t, \infty) \times \mathbb{T})$, the random variable $\xi(A)$ is $\mathcal{F}_t$-measurable, $\xi(B)$ is independent of $\mathcal{F}_t$, and $\mathcal{F}_t$ is $\mathbb{P}$-complete. For example, we may (but don't necessarily have to) take $\mathbb{F}$ to be the completion of the filtration generated by ξ. The predictable σ-algebra on $\Omega \times [0, 1]$ is denoted by $\mathscr{P}$. The conditional expectation given $\mathcal{F}_t$ is denoted by $\mathbb{E}^t$. A random process $(f(t))_{t\in[0,1]}$ taking values in some Banach space is said to be adapted if for all $t \in [0, 1]$, $f(t)$ is $\mathcal{F}_t$-measurable. The space-time stochastic integrals with respect to ξ are denoted by

$$\int_0^t \int_{\mathbb{T}} f(s, y)\, \xi(dy, ds) = \int_0^1 \int_{\mathbb{T}} \mathbf{1}_{s\in[0,t]} f(s, y)\, \xi(dy, ds).$$

Most of the time the integrand f will be deterministic, in which case the stochastic integral can simply be defined as the continuous and linear extension of the mapping $\mathbf{1}_A \mapsto \xi(A)$ from $L_2([0, 1] \times \mathbb{T})$ to $L_2(\Omega)$, which is in fact an isometry. Occasionally we consider integrands that, as a function of ω, t, are bounded and $\mathscr{P} \otimes \mathcal{B}(\mathbb{T})$-measurable, their stochastic integration can be found in e.g. [DPZ92].

We denote the convolution operator by

$$(f * g)(x) = \int f(x - y)g(y)\, dy.$$

This notation is used both when the domain of integration is $\mathbb{R}$ and $\mathbb{T}$. Since in a typical situation f will be a heat kernel either on $\mathbb{R}$ on $\mathbb{T}$, the context will make it clear which convolution we mean.

In proofs of theorems/lemmas/propositions we use the shorthand $f \lesssim g$ to mean that there exists a constant N such that $f \leq Ng$, and that N does not depend on any other parameters than the ones specified in the theorem/lemma/proposition.

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2 Estimates on heat kernels and stochastic convolutions

The evolution of the true and of the approximate solutions is very different even in the linear case $b = 0$. This is one of the main challenges compared to the finite dimensional case, where with vanishing drift the two processes are simply given by the noise process and in particular the error is 0 (see Section 3.1 for a bit more detailed comparison to the finite dimensional case). In infinite dimensions, the error of the linear problem propagates in a nontrivial way in the error analysis of the case of irregular b. The aim of this section is therefore to derive various estimates for the continuous and discrete heat kernels and the associated Ornstein-Uhlenbeck processes (i.e. the solutions of (1.1), (1.2) in the case $\psi = 0$, $b = 0$).

2.1 Definitions

We encounter three different heat kernels in the article: the continuum heat kernel on $\mathbb{R}$, the continuum heat kernel on $\mathbb{T}$, and the discrete heat kernel on $\mathbb{T}$. The first two are defined by

$$p_t^{\mathbb{R}}(x) = \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{x^2}{2t}\Big),$$
$$p_t(x) = \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{4\pi t}} \exp\Big(-\frac{(x+k)^2}{4t}\Big) = \sum_{k\in\mathbb{Z}} e^{-4\pi^2 k^2 t} e^{i2\pi k x}.$$

The difference in scaling comes from the fact that the first is chosen to be the density function in x of a centered normal random variable with variance t, while the latter is chosen to be the Green's function of the heat operator on $\mathbb{R}\times\mathbb{T}$. For sake of convenience we use separate notation for the action of the heat kernels via convolution: for $f \in \mathbb{B}(\mathbb{T})$, we denote $\mathcal{P}_t f := p_t * f$. We define $\mathcal{P}_t^{\mathbb{R}}$ analogously. The continuum heat kernels form a semigroup, that is, $\mathcal{P}_t(\mathcal{P}_s f) = \mathcal{P}_{t+s} f$, and similarly for $\mathcal{P}^{\mathbb{R}}$. The periodic heat kernel is used for the definition of the mild solution of (1.1).

Definition 2.1.1. A mild solution of (1.1) is a $\mathscr{P}\otimes\mathcal{B}(\mathbb{T})$-measurable map $u : \Omega\times[0,1]\times\mathbb{T}\to\mathbb{R}$ which is continuous in (t,x), such that almost surely for all $(t,x)\in[0,1]\times\mathbb{T}$ the following equality holds:

$$u_t(x) = \mathcal{P}_t\psi(x) + \int_0^t \mathcal{P}_{t-s} b(u_s)(x)\,ds + \int_0^t\int_{\mathbb{T}} p_{t-s}(x-y)\,\xi(dy,ds). \tag{2.1}$$

The setup of the discrete heat kernels is more complicated. The formulation below follows along the lines of [Gyö99], but for the convenience of the reader and due to various small differences we prefer to give the full details. From now on, the conjugate of a complex number $z\in\mathbb{C}$ is denoted by $\overline{z}$. Consider the functions $e_j(x) = e^{i2\pi jx}$ for $j\in\mathbb{Z}$. They are eigenfunctions of Δ with eigenvalues $\lambda_j = -4\pi^2 j^2$. It is also well-known that $(e_j)_{j\in\mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{T};\mathbb{C})$. First we have the following discrete analogue.

Proposition 2.1.2. *Let $\lambda_j^n = -16n^2\sin^2(\frac{j\pi}{2n})$ for $j\in\mathbb{Z}$. Then*

$$\Delta_n e_j(x) = \lambda_j^n e_j(x) \tag{2.2}$$

for $x\in\Pi_n$. Moreover, if $-n\le j,\ell\le n-1$, then

$$\frac{1}{2n}\sum_{x\in\Pi_n} e_j(x)\overline{e_\ell(x)} = \mathbf{1}_{j=\ell}. \tag{2.3}$$

As a consequence, $e_{-n}^n, e_{-n+1}^n, \dots, e_{n-1}^n$, as functions on Π_n form a basis of $L^2(\Pi_n;\mathbb{C})$.

Remark 2.1.3. Of course $L^2(\Pi_n;\mathbb{C})$ can be simply identified with $\mathbb{C}^{2n}$, but keeping the former viewpoint is more instructive.

Proof. We start with (2.2). For $j \in \mathbb{Z}$ and $x = k/n$, we have

$$\begin{aligned}\Delta_n e_j(x) =& 4n^2\Big(\exp\Big(\frac{i2\pi jk}{2n}+\frac{i2\pi j}{2n}\Big)-2\exp\Big(\frac{i2\pi jk}{2n}\Big)+\exp\Big(\frac{i2\pi jk}{2n}-\frac{i2\pi j}{2n}\Big)\Big)\\ =& 4n^2 e_j(x)\Big(\exp\Big(\frac{i2\pi j}{2n}\Big)-2+\exp\Big(-\frac{i2\pi j}{2n}\Big)\Big)\\ =& 8n^2 e_j(x)\Big(\cos\Big(\frac{2\pi j}{2n}\Big)-1\Big) = -16n^2 e_j(x)\sin^2\Big(\frac{\pi j}{2n}\Big) = \lambda_j^n e_j(x).\end{aligned}$$

This proves (2.2). As for (2.3), the case $j = \ell$ is trivial. For $j \neq \ell$, using that $|j-\ell| < 2n$, in the geometric series below the ratio $e^{\frac{i2\pi(j-\ell)}{2n}}$ is different from 1, therefore

$$\sum_{x\in\Pi_n} e_j(x)\overline{e_\ell(x)} = \sum_{k=0}^{2n-1} e^{\frac{i2\pi(j-\ell)k}{2n}} = \frac{1-e^{i2\pi(j-\ell)}}{1-e^{\frac{i2\pi(j-\ell)}{2n}}} = 0.$$

□

Although (2.2) holds for all $j \in \mathbb{Z}$, in the sequel we will only ever consider $-n \leq j \leq n-1$. It will be convenient to use the piecewise linear extension of the restriction of e_j to Π_n: for $-n \leq j \leq n-1$, for $x \in \Pi_n$, and $x' \in [x, x+(2n)^{-1}]$, set

$$e_j^n(x') = e_j(x) + 2n(x'-x)(e_j(x+(2n)^{-1}) - e_j(x)). \tag{2.4}$$

Let us briefly discuss how λ_j^n relates to λ_j. Defining $\gamma_0^n = 1$ and $\gamma_j^n = \frac{\lambda_j^n}{\lambda_j} = \frac{\sin^2(j\pi/2n)}{(j\pi/2n)^2}$, one has

$$4\pi^{-2} \leq \gamma_j^n \leq 1. \tag{2.5}$$

Indeed, it is elementary to see that $\frac{\sin^2(x)}{x^2}$ is even, decreasing on $[0,\pi]$, so its minimum on $[-\pi/2,\pi/2]$ equals $4\pi^{-2}$. Moreover, one has for $-n \leq j \leq n-1$,

$$|1-\gamma_j^n| \leq \frac{1}{3}\Big(\frac{j\pi}{n}\Big)^2, \tag{2.6}$$

which follows from the inequality $1-(\sin(x)/x)^2 \leq (1/3)x^2$ (whose proof we leave as an exercise to the interested reader).

Remark 2.1.4. Although we do not discuss purely spatial discretisations, it is worth remarking that the above setup would already be enough to define the heat kernel for the spatially discretised operator $\partial_t - \Delta_n$ in its spectral representation, which would take the form

$$\tilde{p}_t^n(x,y) = \sum_{j=-n}^{n-1} e^{-t\lambda_j^n} e_j^n(x)\overline{e_j^n(y)} \tag{2.7}$$

for $t \geq 0$, $x, y, \in \Pi_n$.

It remains to encode the temporal discretisation in the discrete heat kernel. Naturally, on the temporal gridpoints $t = kh$ the factor $e^{t\lambda_j^n}$ in (2.7) is simply replaced by $(1 + h\lambda_j^n)^k$. Between the gridpoints, we again interpolate linearly. More precisely, for $j = -n, \dots, n-1$, for $t \in \Lambda_n$, and $t' \in [t, t+h]$, set

$$\mu_j^n(t') = (1 + h\lambda_j^n)^{th^{-1}} + h^{-1}(t' - t)((1 + h\lambda_j^n)^{(t+h)h^{-1}} - (1 + h\lambda_j^n)^{th^{-1}}). \tag{2.8}$$

At this point we make use of the CFL condition. Notice that $-h\lambda_j^n \in [0, 4c]$, and $c < 1/2$ implies $4c < 2$. Therefore there exists a constant $\delta_0 = \delta_0(c) > 0$ such that on $[0, 4c]$ one has $|1 - x| \le e^{-\delta_0 x}$. Then there exists another constant $\delta > 0$ such that[2]

$$|1 + h\lambda_j^n|^{th^{-1}} \le e^{\delta_0 t\lambda_j^n} \le e^{-t\delta j^2}. \tag{2.9}$$

We can now define the discrete heat kernel and rewrite the approximation scheme (1.2) in a mild form. Denote by $\kappa_n(t) = \lfloor th^{-1} \rfloor h$ and $\rho_n(x) = \lfloor x2n \rfloor (2n)^{-1}$ the leftmost gridpoint from t in Λ_n and from x in Π_n, respectively. We then set

$$p_t^n(x, y) = \sum_{j=-n}^{n-1} \mu_j^n(t) e_j^n(x) \overline{e_j^n(\rho_n(y))}, \tag{2.10}$$

which is now a function of $t \ge 0$, $x, y \in \mathbb{T}$.

Remark 2.1.5. Although each e_j^n is a $\mathbb{C}$-valued function, p_t^n itself is $\mathbb{R}$-valued for all $t \ge 0$. Indeed, first consider $x \in \Pi_n$, $y \in \mathbb{T}$. Since $\lambda_j^n = \lambda_{-j}^n$ and therefore $\mu_j^n(t) = \mu_{-j}^n(t)$, one sees

$$\sum_{j=-n-1}^{n-1} \mu_j^n(t) e_j^n(x) \overline{e_j^n(\rho_n(y))} \in \mathbb{R}.$$

In addition, the restriction of e_{-n}^n to Π_n takes only ± 1 values, so $e_{-n}^n(x)\overline{e_{-n}^n(\rho_n(y))} \in \mathbb{R}$, which combined with the above shows that $p_t^n(x, y) \in \mathbb{R}$ for $x \in \Pi_n$, $y \in \mathbb{R}$. The same is true for all $x, y \in \mathbb{T}$, since $p_t^n(\cdot, y)$ is given by linear interpolation between its values on Π_n.

Let us introduce the discrete convolution

$$f *_n g(x) := \int_{\mathbb{T}} f(x, y) g(\rho_n(y))\, dy. \tag{2.11}$$

Analogously to $\mathcal{P}$, we then define the linear operators $\mathcal{P}^n$ by setting $\mathcal{P}_t^n f := p_t^n *_n f$. Most of the time we understand $\mathcal{P}_t^n$ as an operator on $\mathbb{B}(\mathbb{T})$, but it can be seen as an operator on $\mathbb{B}(\Pi_n)$ as well. In the latter case, the identity $\mathcal{P}_h^n = \mathrm{id} + h\Delta_n$ holds[3]. The inductive step (1.2) of the finite difference scheme can therefore be written as

$$u_{t+h}^n(x) = \mathcal{P}_h^n u_t^n(x) + hb(u_t^n(x)) + h\eta_n(t, x). \tag{2.12}$$

To conclude to a form similar to (2.1), it remains to show the following simple property.

Proposition 2.1.6. *For $s, t \in \Lambda_n$, we have*

$$(p_t^n *_n p_s^n(\cdot, y))(x) = p_{t+s}^n(x, y). \tag{2.13}$$

[2]For example $\delta_0 = \frac{-\log\left((4c-1)\vee(1/2)\right)}{4c}$ and $\delta = 16\delta_0$ will do.

[3]Note however that $\mathcal{P}_0^n$ as an operator on $\mathbb{B}(\mathbb{T})$ does not equal the identity.

Proof. By the definitions (2.10)-(2.11) and the orthogonality relation (2.3) we can write

$$\begin{aligned}(p_t^n *_n p_s^n(\cdot,y))(x) &= \int_{\mathbb{T}} p_t^n(x,z)p_s^n(\rho_n(z),y)\,dz \\ &= \sum_{j=-n}^{n-1}\sum_{k=-n}^{n-1}(1+h\lambda_j^n)^{th^{-1}}(1+h\lambda_k^n)^{sh^{-1}}e_j^n(x)\overline{e_k^n(\rho_n(y))}\int_{\mathbb{T}} e_k^n(\rho_n(z))\overline{e_j^n(\rho_n(z))}\,dz \\ &= \sum_{j=-n}^{n-1}(1+h\lambda_j^n)^{(s+t)h^{-1}}e_j^n(x)\overline{e_j^n(\rho_n(y))},\end{aligned}$$

as required. □

It follows that (1.2) can be equivalently written as

$$u_t^n(x) = \mathcal{P}_t^n\psi^n(x) + \int_0^t \mathcal{P}^n_{\kappa_n(t-s)}b(u^n_{\kappa_n(s)})(x)\,ds + \int_0^t\int_{\mathbb{T}} p^n_{\kappa_n(t-s)}(x,y)\,\xi(dy,ds),$$

Indeed, this clearly holds for $t=0$ and for $0<t\in\Lambda_n$ it follows inductively from (2.12) and (2.13). Recalling that p^n is defined for any space-time point, not just the ones on the grid, we then *define* an extension of u^n to the whole of $[0,1]\times\mathbb{T}$ by setting

$$u_t^n(x) = \mathcal{P}_t^n\psi^n(x) + \int_0^t \mathcal{P}^n_{\kappa_n(t-s)}b(u^n_{\kappa_n(s)})(x)\,ds + \int_0^t\int_{\mathbb{T}} p^n_{\kappa_n(t-s)}(x,y)\,\xi(dy,ds). \tag{2.14}$$

Remark 2.1.7. It is useful to note an alternative representation of $\mathcal{P}^n$ as the transition kernel of a random walk indexed by Λ_n. Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution

$$\mathbb{P}(X_1=0)=1-2c, \quad \mathbb{P}(X_1=1)=P(X_1=-1)=c.$$

One can observe that the condition $c\le 1/2$ is necessary in order for the above to be a probability distribution, while our stronger condition $c<1/2$ guarantees that the random walk is "lazy". We then define $S_n=\sum_{i=1}^n X_i$ and for $t\in\Lambda_n$, set $\widehat{S}_t^n=(2n)^{-1}S_{h^{-1}t}$. Then $\mathcal{P}^n$ is the transition semigroup of $\widehat{S}^n$: for any function $f:\Pi_n\to\mathbb{R}$ and any $x\in\Pi_n$,

$$\mathcal{P}_t^n f(x) = \mathbb{E}\tilde{f}(x+\widehat{S}_t^n), \tag{2.15}$$

where $\tilde{f}$ is the 1-periodic extension of f from Π_n to $(2n)^{-1}\mathbb{Z}$.

2.2 Discrete and continuous heat kernel bounds

We start with three classical heat kernel bounds.

Lemma 2.2.1. *Let* $(\mathcal{S},\mathbb{D})\in\{(\mathcal{P},\mathbb{T}),(\mathcal{P}^{\mathbb{R}},\mathbb{R})\}$. *The following hold.*

(i) For all $\alpha,\beta\in[0,1]$ *with* $\alpha\le\beta$, *there exists a constant* $N=N(\alpha,\beta)$, *such that for all* $f\in\mathcal{C}^\alpha(\mathbb{D})$, $0\le s\le t\le 1$ *and* $x,y\in\mathbb{D}$ *one has* [4]

$$|\mathcal{S}_t f(x)-\mathcal{S}_s f(y)| \le N\|f\|_{\mathcal{C}^\alpha}(|x-y|^\beta+|t-s|^{\beta/2})s^{(\alpha-\beta)/2}. \tag{2.16}$$

[4] with the conventions $0^0=1$ and $1/0=+\infty$

(ii) For any $\alpha \in [0,1]$, there exists a constant $N = N(\alpha)$ such that for all $f \in \mathcal{C}^\alpha(\mathbb{D})$, $t \in (0,1]$, and $x_1, x_2, x_3, x_4 \in \mathbb{D}$ one has

$$\begin{aligned}&|\mathcal{S}_t f(x_1) - \mathcal{S}_t f(x_2) - \mathcal{S}_t f(x_3) + \mathcal{S}_t f(x_4)| \\ &\quad \leq N\|f\|_{\mathcal{C}^\alpha}|x_1 - x_2||x_1 - x_3|t^{\alpha/2-1} + N\|f\|_{\mathcal{C}^\alpha}|x_1 - x_2 - x_3 + x_4|t^{(\alpha-1)/2}.\end{aligned} \tag{2.17}$$

(iii) There exists a constant N such that for all $f \in \mathbb{B}(\mathbb{D})$, $0 < s \leq t \leq 1$, and $x, y \in \mathbb{D}$, one has

$$|\mathcal{S}_t f(x) - \mathcal{S}_t f(y) - \mathcal{S}_s f(x) - \mathcal{S}_s f(y)| \leq N\|f\|_{\mathbb{B}} s^{-3/2}|t-s||x-y|. \tag{2.18}$$

Proof. The estimate in (2.16) is very standard. A proof of it and its more general variants can be found for example in [BDG21, Appendix A].

For (2.17), notice that by the fundamental theorem of calculus we have

$$\begin{aligned}&\mathcal{S}_t f(x_1) - \mathcal{S}_t f(x_2) - \mathcal{S}_t f(x_3) + \mathcal{S}_t f(x_4) \\ &\quad = \int_0^1 \Big(\nabla \mathcal{S}_t f(x_1 + \theta(x_2 - x_1)) - \nabla \mathcal{S}_t f(x_3 + \theta(x_2 - x_1))\Big)(x_1 - x_2)\, d\theta \\ &\qquad + \mathcal{S}_t f(x_3 + x_2 - x_1) - \mathcal{S}_t f(x_4).\end{aligned}$$

Consequently, we get

$$\begin{aligned}&|\mathcal{S}_t f(x_1) - \mathcal{S}_t f(x_2) - \mathcal{S}_t f(x_3) + \mathcal{S}_t f(x_4)| \\ &\quad \leq \|\nabla^2 \mathcal{S}_t f\|_{\mathbb{B}}|x_1 - x_3||x_1 - x_2| + \|\nabla \mathcal{S}_t f\|_{\mathbb{B}}|x_1 - x_2 - x_3 + x_4|\end{aligned} \tag{2.19}$$

From (2.16), with $\beta = 1$, it follows that

$$\|\nabla \mathcal{S}_t f\|_{\mathbb{B}} \leq N t^{(\alpha-1)/2}\|f\|_{\mathcal{C}^\alpha}. \tag{2.20}$$

The above also gives

$$\|\nabla^2 \mathcal{S}_t f\|_{\mathbb{B}} = \|\nabla \mathcal{S}_{t/2} \nabla(\mathcal{S}_{t/2} f)\|_{\mathbb{B}} \leq N t^{-1/2}\|\nabla(\mathcal{S}_{t/2} f)\|_{\mathbb{B}} \leq N t^{-1+\alpha/2}\|f\|_{\mathcal{C}^\alpha}. \tag{2.21}$$

Therefore, from (2.19)-(2.21) we get (2.17).

Finally, from the fundamental theorem of calculus, the identity $\partial_t \mathcal{S} = \Delta \mathcal{S}$, (2.16) with $\beta = 1$, and (2.20) with $\alpha = 0$, we get

$$\begin{aligned}&|\mathcal{S}_t f(x) - \mathcal{S}_t f(y) - \mathcal{S}_s f(x) - \mathcal{S}_s f(y)| \\ &\quad = \left|(t-s)\int_0^1 \Big(\Delta \mathcal{S}_{s+\theta(t-s)} f(x) - \Delta \mathcal{S}_{s+\theta(t-s)} f(y)\Big)\, d\theta\right| \\ &\quad = \left|(t-s)\int_0^1 \Big(\mathcal{S}_{s/2+\theta(t-s)} \Delta \mathcal{S}_{s/2} f(x) - \mathcal{S}_{s/2+\theta(t-s)} \Delta \mathcal{S}_{s/2} f(y)\Big)\, d\theta\right| \\ &\quad \leq N|x-y||t-s|s^{-1/2}\|\Delta \mathcal{S}_{s/2} f\|_{\mathbb{B}} \\ &\quad \leq N|x-y||t-s|s^{-3/2}\|f\|_{\mathbb{B}}.\end{aligned}$$

This finishes the proof. □

Before moving on, we remark a simple bound that will be frequently used.

Proposition 2.2.2. *For any* $\lambda > 0$ *and* $\gamma \geq 0$ *there exists a* $N = N(\lambda, \gamma)$ *such that for all* $t \in (0, 1]$ *the following bound holds*

$$\sum_{j\in\mathbb{Z}} |j|^{\gamma} e^{-\lambda j^2 t} \leq N t^{-(\gamma+1)/2}. \tag{2.22}$$

Proof. By the change of variables $x \mapsto t^{-1/2}x$, we have

$$\sum_{j\in\mathbb{Z}} |j|^{\gamma} e^{-\lambda j^2 t} \leq \int_{-\infty}^{\infty} (|x|+1)^{\gamma} e^{\lambda x^2 t}\, dx = t^{-(\gamma+1)/2} \int_{-\infty}^{\infty} (|x|+1)^{\gamma} e^{\lambda x^2}\, dx.$$

□

Heat kernel estimates for the discrete heat kernels $\mathcal{P}^n$ are less established. Since they are piecewise linear in time on each interval between neighbouring gridpoints, most estimates will be stated only for $t \in \Lambda_n$. For the initial time we only need the straightforward property

$$\|p_0^n(x, \cdot)\|_{L_2(\mathbb{T})}^2 = 2n. \tag{2.23}$$

For gridpoints after the initial time we recover almost the usual heat kernel bounds, at the cost of a log factor.

Lemma 2.2.3. *Let* $\alpha, \beta \in [0, 1]$ *with* $\alpha \leq \beta$. *There exists a constant* $N = N(\alpha, \beta, c)$ *such that for all* $f : \Pi_n \to \mathbb{R}$, $n \in \mathbb{N}$, $x, z \in \Pi_n$, $t \in \Lambda_n \cap (0, 1]$, *the following bound holds*

$$|\mathcal{P}_t^n f(x) - \mathcal{P}_t^n f(z)| \leq N(\log(2n))^{(\beta-\alpha)/2} |t|^{(\alpha-\beta)/2} |x - z|^{\beta} \|f\|_{C^{\alpha}(\Pi_n)}. \tag{2.24}$$

Proof. First assume $\alpha = 0$ and $\beta = 1$. Note that in this case it suffices to consider neighbouring points $x, z \in \Pi_n$ (by virtue of the triangle inequality) and in fact only the case $x = 0$ and $z = 1/2n$ (by virtue of translation invariance). For a function $g : \Pi_n \to \mathbb{R}$, denote $\|g\|_{L_1(\Pi_n)} = (2n)^{-1} \sum_{x\in\Pi_n} |g(x)|$. We then have

$$\begin{aligned} |\mathcal{P}_t^n g(0) - \mathcal{P}_t^n g(1/2n)| &= \Big| \sum_{j=-n}^{n-1} (1 + h\lambda_j^n)^{th^{-1}} (e_j^n(0) - e_j^n(1/2n)) \int_{\mathbb{T}} \overline{e_j^n(\rho_n(y))} g(\rho_n(y))\, dy \Big| \\ &\lesssim n^{-1} \|g\|_{L_1(\Pi_n)} \sum_{j=-n}^{n-1} |1 + h\lambda_j^n|^{th^{-1}} |j|. \end{aligned}$$

Using (2.9) and then Proposition 2.2.2 (with $\lambda = \delta$ and $\gamma = 1$), we get

$$|\mathcal{P}_t^n g(0) - \mathcal{P}_t^n g(1/2n)| \lesssim t^{-1} n^{-1} \|g\|_{L_1(\Pi_n)}. \tag{2.25}$$

Next, as a brief detour, take some $K > c^{-1/2}$ and recall the notations X_i, S_n, $\widehat{S}_t^n$ from Remark 2.1.7 and the representation (2.15). Denote furthermore by N_n the number of nonzero elements in $\{X_1, \ldots, X_n\}$. Then conditionally on $N_n = \ell$, $S_n + \ell$ has binomial distribution with parameters $1/2$ and ℓ. Recalling the well-known bound

$$\mathbb{P}(\text{Binom}(1/2, \ell) \geq \ell + K\sqrt{\ell}) \leq e^{-K^2},$$

we have

$$\mathbb{P}(S_n \geq K\sqrt{n}) = \sum_{\ell=1}^{n} \mathbb{P}(S_n \geq K\sqrt{n} | N_n = \ell) \mathbb{P}(N_n = \ell)$$

$$\le \sum_{\ell=1}^{n} \mathbb{P}(S_n + \ell \ge \ell k + \sqrt{\ell} | N_n = \ell)\mathbb{P}(N_n = \ell)$$
$$\le e^{-K^2}.$$

In terms of $\widehat{S}_t^n$, this means

$$\mathbb{P}(\widehat{S}_t^n \ge K\sqrt{t}) = \mathbb{P}(S_{c^{-1}n^2t} \ge K\sqrt{n^2t}) \le e^{-cK^2}. \tag{2.26}$$

For $f : \Pi_n \to \mathbb{R}$, we have

$$\begin{aligned}|\mathcal{P}_t^n f(0) - \mathcal{P}_t^n f(1/2n)| \\ \le |\mathbb{E}f(\widehat{S}_t^n)\mathbf{1}_{|\widehat{S}_t^n| \le 2K\sqrt{t}} - \mathbb{E}f(1/2n + \widehat{S}_t^n)\mathbf{1}_{|1/2n+\widehat{S}_t^n| \le 2K\sqrt{t}}| \\ + \|f\|_{L_\infty(\Pi_n)}\Big(\mathbb{P}(|\widehat{S}_t^n| \ge 2K\sqrt{t}) + \mathbb{P}(|1/2n + \widehat{S}_t^n| \ge 2K\sqrt{t})\Big).\end{aligned} \tag{2.27}$$

Notice that

$$\begin{aligned}\mathbb{P}\Big(|1/2n + \widehat{S}_t^n| \ge 2K\sqrt{t}\Big) &\le \mathbb{P}\Big(|\widehat{S}_t^n| \ge 2K\sqrt{t} - 1/2n\Big) \\ &= \mathbb{P}\Big(|\widehat{S}_t^n| \ge 2K\sqrt{t} - \sqrt{c^{-1}h}\Big) \\ &\le \mathbb{P}(|\widehat{S}_t^n| \ge K\sqrt{t}) \le \mathbb{P}(|S_t^n| \ge K\sqrt{t}) \le e^{-K^2},\end{aligned}$$

where for the second inequality we have used that $t \ge h$ and that $K > c^{-1/2}$, and for the last step we have used (2.26). This implies that

$$\Big(\mathbb{P}(|\widehat{S}_t^n| \ge 2K\sqrt{t}) + \mathbb{P}(|1/2n + \widehat{S}_t^n| \ge 2K\sqrt{t})\Big) \le 2e^{-cK^2}. \tag{2.28}$$

On the other hand, the first term on the right hand side of (2.27), we can use (2.25) with $g(\cdot) = f(\cdot)\mathbf{1}_{|\cdot| \le 2K\sqrt{t}}$. This gives

$$\begin{aligned}|\mathbb{E}f(\widehat{S}_t^n)\mathbf{1}_{\widehat{S}_t^n \le 2K\sqrt{t}} - \mathbb{E}f(1/2n + \widehat{S}_t^n)\mathbf{1}_{|1/2n+\widehat{S}_t^n| \le 2K\sqrt{t}}| \\ \lesssim t^{-1}\frac{1}{2n}\|f(\cdot)\mathbf{1}_{|\cdot| \le 2K\sqrt{t}}\|_{L_1(\Pi_n)} \lesssim t^{-1/2}\frac{1}{2n}K\|f\|_{L_\infty(\Pi_n)}\end{aligned}$$

Consequently, by (2.27), and the above, we get

$$|\mathcal{P}_t^n f(0) - \mathcal{P}_t^n f(1/2n)| \lesssim t^{-1/2}K\frac{1}{2n}\|f\|_{L_\infty(\Pi_n)} + e^{-cK^2}\|f\|_{L_\infty(\Pi_n)},$$

which by the choice $K = \sqrt{\ln(2n)/c}$, gives

$$|\mathcal{P}_t^n f(0) - \mathcal{P}_t^n f(1/2n)| \lesssim t^{-1/2}\sqrt{\ln(2n)}n^{-1}\|f\|_{L_\infty(\Pi_n)}.$$

As mentioned at the beginning of the proof, this yields (2.24) in the case $\alpha = 0$, $\beta = 1$. The case $\beta = 1$, $\alpha = 1$ follows from the trivial bound

$$|\mathcal{P}_t^n f(x) - \mathcal{P}_t^n f(z)| \le |x - z|\|f\|_{\mathcal{C}^1(\Pi_n)}.$$

The case $\beta = 1$, $\alpha \in [0, 1]$ then follows by interpolation. Finally, the case $\beta \in [0, 1]$, $\alpha \in [0, \beta]$ follows by interpolation between (2.24) with $\beta = 1$ and the trivial bound

$$|\mathcal{P}_t^n f(x) - \mathcal{P}_t^n f(z)| \le |x - z|^\alpha\|f\|_{\mathcal{C}^\alpha(\Pi_n)}.$$

This finishes the proof. □

Lemma 2.2.4. *Let $\alpha \in [0,1]$. There exists a constant $N(\alpha)$ such that for all $\psi \in \mathcal{C}^\alpha(\mathbb{T})$, $t \in [0,1]$, we have*

$$\|\mathcal{P}_t^n \psi\|_{\mathcal{C}^\alpha(\mathbb{T})} \le N \|\psi\|_{\mathcal{C}^\alpha(\mathbb{T})}.$$

Proof. One can easily see the estimate

$$\|\mathcal{P}_t^n \psi\|_{\mathbb{B}(\mathbb{T})} \le N \|\psi\|_{\mathbb{B}(\mathbb{T})}, \tag{2.29}$$

so we focus on proving that for all $t \in [0,1]$ and $x, y \in \mathbb{T}$, we have

$$|\mathcal{P}_t^n \psi(x) - \mathcal{P}_t^n \psi(y)| \le N|x-y|^\alpha \|\psi\|_{\mathcal{C}^\alpha(\mathbb{T})}.$$

Let us first prove the claim with $\alpha = 1$. In addition, let us assume for now that $t \in \Lambda_n$. There are three cases:

Case 1: $|x-y| < 1/2n$ and $\rho_n(x) = \rho_n(y)$. In this case, by (2.4) it follows that

$$\begin{aligned} |\mathcal{P}_t^n \psi(x) - \mathcal{P}_t^n \psi(y)| =& |2n(x-y)(\mathcal{P}_t^n \psi(\rho_n(x) + 1/2n) - \mathcal{P}_t^n \psi(\rho_n(x)))| \\ \le& N|x-y| \|\psi\|_{\mathcal{C}^1(\mathbb{T})}, \end{aligned}$$

where we have used the representation (2.15).

Case 2: $|x-y| < 1/2n$ and $\rho_n(x) \ne \rho_n(y)$. In this case, let us assume without loss of generality that $\rho_n(y) = \rho_n(x) + 1/2n$. By (2.4), we see that

$$\mathcal{P}_t^n \psi(x) = \mathcal{P}_t^n \psi(\rho_n(x)) + 2n(x - \rho_n(x))(\mathcal{P}_t^n \psi(\rho_n(x) + 1/2n) - \mathcal{P}_t^n \psi(\rho_n(x))),$$

which implies that

$$\begin{aligned} |\mathcal{P}_t^n \psi(x) - \mathcal{P}_t^n \psi(\rho_n(y))| &= |\mathcal{P}_t^n \psi(x) - \mathcal{P}_t^n \psi(\rho_n(x) + 1/2n)| \\ &= |(2n(x - \rho_n(x)) - 1)(\mathcal{P}_t^n \psi(\rho_n(x) + 1/2n) - \mathcal{P}_t^n \psi(\rho_n(x)))| \\ &\le |x - \rho_n(x) - 1/2n| \|\psi\|_{\mathcal{C}^1(\mathbb{T})} \\ &= |x - \rho_n(y)| \|\psi\|_{\mathcal{C}^1(\mathbb{T})}. \end{aligned}$$

We also have

$$\begin{aligned} |\mathcal{P}_t^n \psi(\rho_n(y)) - \mathcal{P}_t^n \psi(y)| &= |(2n(y - \rho_n(y)))(\mathcal{P}_t^n \psi(\rho_n(y) + 1/2n) - \mathcal{P}_t^n \psi(\rho_n(y))| \\ &\le |y - \rho_n(y)| \|\psi\|_{\mathcal{C}^1(\mathbb{T})}. \end{aligned}$$

Consequently,

$$\begin{aligned} |\mathcal{P}_t^n \psi(x) - \mathcal{P}_t^n \psi(y)| \le& |\mathcal{P}_t^n \psi(x) - \mathcal{P}_t^n \psi(\rho_n(y))| + |\mathcal{P}_t^n \psi(\rho_n(y)) - \mathcal{P}_t^n \psi(y)| \\ \le& (|x - \rho_n(y)| + |y - \rho_n(y)|) \|\psi\|_{\mathcal{C}^1(\mathbb{T})} \\ =& |x-y| \|\psi\|_{\mathcal{C}^1(\mathbb{T})}. \end{aligned}$$

Case 1: $|x-y| \ge 1/2n$. In this case, we have

$$\begin{aligned} &|\mathcal{P}_t^n \psi(x) - \mathcal{P}_t^n \psi(y)| \\ &\le |\mathcal{P}_t^n \psi(x) - \mathcal{P}_t^n \psi(\rho_n(x))| + |\mathcal{P}_t^n \psi(\rho_n(x)) - \mathcal{P}_t^n \psi(\rho_n(y))| + |\mathcal{P}_t^n \psi(y) - \mathcal{P}_t^n \psi(\rho_n(y))| \end{aligned}$$

$$\begin{aligned}&\lesssim (1/2n + |\rho_n(x) - \rho_n(y)|)\|\psi\|_{\mathcal{C}^1(\mathbb{T})}\\&\lesssim |x-y|\|\psi\|_{\mathcal{C}^1(\mathbb{T})},\end{aligned}$$

where we have used the results form the case $|x-y| < 1/2n$ for the first and the third term, the representation (2.15) for the second term, and of course the fact that $|x-y| > 1/2n$.

Hence, we have proved the claim for $t \in \Lambda_n$. If $t \geq 0$, then the claim follows form the case $t \in \Lambda_n$ virtue of the identity

$$\mathcal{P}^n_t\psi(x) = \mathcal{P}^n_{\kappa_n(t)}\psi(x) + h(t-\kappa_n(t))(\mathcal{P}^n_{\kappa_n(t)+h}\psi(x) - \mathcal{P}^n_{\kappa_n(t)}\psi(x)),$$

see (2.8).

To summarise, we have shown the desired inequality with $\alpha = 1$ for all $t \geq 0$ and $x, y \in \mathbb{T}$. From (2.29), it also follows that

$$|\mathcal{P}^n_t\psi(x) - \mathcal{P}^n_t\psi(y)| \lesssim \|\psi\|_{\mathbb{B}(\mathbb{T})},$$

for all $t \geq 0$ and $x, y \in \mathbb{T}$, which is the claim for $\alpha = 0$. Finally, the case $\alpha \in (0,1)$ follows by interpolation. ☐

Lemma 2.2.5. *Let $\beta \in [0,1]$, $\alpha \in [0,\beta]$. The following hold:*

1. *Then there exists a constant N such that for all $\psi \in \mathcal{C}^\alpha(\mathbb{T})$, $n \in \mathbb{N}$, $r \geq h$, $y \in \mathbb{T}$, $z \in \{y, \rho_n(y)\}$*
$$|\mathcal{P}^n_r\psi(y) - \mathcal{P}^n_{\kappa_n(r)}\psi(z)| \leq N\big(\log(2n)\big)^{(\beta-\alpha)/2} n^{-\beta} r^{(\alpha-\beta)/2}\|\psi\|_{\mathcal{C}^\alpha}. \tag{2.30}$$
2. *Then there exists a constant N such that for all $\psi \in \mathcal{C}^\alpha(\mathbb{T})$, $n \in \mathbb{N}$, $r \in [0,h]$, $y \in \mathbb{T}$, $z \in \{y, \rho_n(y)\}$*
$$|\mathcal{P}^n_r\psi(y) - \mathcal{P}^n_0\psi(z)| \leq N n^{-\alpha}\|\psi\|_{\mathcal{C}^\alpha}. \tag{2.31}$$

Proof. We have

$$\begin{aligned}|\mathcal{P}^n_r\psi(y) - \mathcal{P}^n_{\kappa_n(r)}\psi(\rho_n(y))| \leq{}& |\mathcal{P}^n_r\psi(y) - \mathcal{P}^n_{\kappa_n(r)}\psi(y)|\\&+ |\mathcal{P}^n_{\kappa_n(r)}\psi(y) - \mathcal{P}^n_{\kappa_n(r)}\psi(\rho_n(y))|.\end{aligned}$$

For the second term, keeping in mind the definition of e^n in (2.4), we see that

$$\begin{aligned}&|\mathcal{P}^n_{\kappa_n(r)}\psi(y) - \mathcal{P}^n_{\kappa_n(r)}\psi(\rho_n(y))|\\&\quad= \Big|2n(y-\rho_n(y))\Big(\mathcal{P}^n_{\kappa_n(r)}\psi(\rho_n(y)+(2n)^{-1}) - \mathcal{P}^n_{\kappa_n(r)}\psi(\rho_n(y))\Big)\Big|\\&\quad\lesssim \big(\log(2n)\big)^{(\beta-\alpha)/2} n^{-\beta}\big(\kappa_n(r)\big)^{(\alpha-\beta)/2}\|\psi\|_{\mathcal{C}^\alpha},\end{aligned} \tag{2.32}$$

where we have used Lemma 2.2.3 for the inequality. Similarly, by (2.8), we see that

$$\begin{aligned}|\mathcal{P}^n_r\psi(y) - \mathcal{P}^n_{\kappa_n(r)}\psi(y)| &= h^{-1}(r-\kappa_n(r))|\mathcal{P}_{\kappa_n(r)+h}\psi(y) - \mathcal{P}_{\kappa_n(r)}\psi(y)|\\&\lesssim |\mathcal{P}_{\kappa_n(r)+h}\psi(y) - \mathcal{P}_{\kappa_n(r)}\psi(y)|\\&\lesssim |\mathcal{P}_{\kappa_n(r)+h}\psi(y) - \mathcal{P}_{\kappa_n(r)+h}\psi(\rho_n(y))|\\&\quad+ |\mathcal{P}_{\kappa_n(r)+h}\psi(\rho_n(y)) - \mathcal{P}_{\kappa_n(r)}\psi(\rho_n(y))|\end{aligned}$$

$$+ |\mathcal{P}_{\kappa_n(r)}\psi(\rho_n(y)) - \mathcal{P}_{\kappa_n(r)}\psi(y)|$$

The first and the third term at the right hand side can be bounded by the right hand side of (2.30) because of (2.32). For the second term, we have that

$$|\mathcal{P}_{\kappa_n(r)+h}\psi(\rho_n(y)) - \mathcal{P}_{\kappa_n(r)}\psi(\rho_n(y))| = |h\Delta_n\mathcal{P}_{\kappa_n(r)}\psi(\rho_n(y))|,$$

which again can be estimated by the right hand side of (2.30) by virtue of Lemma 2.2.3. This shows that

$$|\mathcal{P}^n_r\psi(y) - \mathcal{P}^n_{\kappa_n(r)}\psi(y)| \lesssim \big(\log(2n)\big)^{(\beta-\alpha)/2} n^{-\beta} r^{(\alpha-\beta)/2}\|\psi\|_{\mathcal{C}^\alpha},$$

which proves (2.30) with $z = y$ and combined with (2.32) gives (2.30) with $z = \rho_n(y)$.

The proof of (2.31) is more straightforward and is left to the reader. □

The following lemma, a key error estimate between the continuous and the discrete heat kernels, is similar to [Gyö99, Lemma 3.3], where an estimate of the form (2.33) is proved for the Dirichlet setting. Our version is a bit more flexible by allowing $\beta \in [0, 2]$ in contrast to [Gyö99, Lemma 3.3] where $\beta \in [0, 1]$.

Lemma 2.2.6. *Let $\beta \in [0, 2]$. Then there exists a constant $N(\beta, c)$ such that for all $t \in [h, 1]$, $x \in \mathbb{T}$ one has the bound*

$$\|p_t(x, \cdot) - p^n_{\kappa_n(t)}(x, \cdot)\|^2_{L_2(\mathbb{T})} \le N n^{-\beta} t^{-(\beta+1)/2}. \tag{2.33}$$

Proof. Note that since $t \gtrsim n^{-2}$ (resp. $|t-s| \gtrsim n^{-2}$), it suffices to prove (2.33) (resp. (2.50)) in the $\beta = 2$ case. Let us start by defining

$$\tilde{p}^n_t(x, y) = \sum_{j=-n}^{n-1} e^{\lambda^n_j t} e^n_j(x) e_j(\rho_n(y)).$$

Writing

$$\|p_t(x, \cdot) - p^n_t(x, \cdot)\|^2_{L_2(\mathbb{T})} \le 2\|p_t(x, \cdot) - \tilde{p}^n_t(x, \cdot)\|^2_{L_2(\mathbb{T})} + 2\|\tilde{p}^n_t(x, \cdot) - p^n_{\kappa_n(t)}(x, \cdot)\|^2_{L_2(\mathbb{T})} \tag{2.34}$$

we start with an estimate for the first term at the right hand side. We write

$$\|p_t(x, \cdot) - \tilde{p}^n_t(x, \cdot)\|^2_{L_2(\mathbb{T})} \le 4\sum_{i=1}^{4} I^i_t(x),$$

with

$$I^1_t(x) = \| \sum_{\{j \le -n-1\} \cup \{j \ge n\}} e^{\lambda_j t} e_j(x)\overline{e_j}(\cdot)\|^2_{L_2(\mathbb{T})},$$
$$I^2_t(x) = \| \sum_{-n \le j \le n-1} (e^{\lambda_j t} - e^{\lambda^n_j t}) e_j(x)\overline{e_j}(\cdot)\|^2_{L_2(\mathbb{T})},$$
$$I^3_t(x) = \| \sum_{-n \le j \le n-1} e^{\lambda^n_j t} (e_j(x) - e^n_j(x))\overline{e_j}(\cdot)\|^2_{L_2(\mathbb{T})},$$

$$I^4_t(x) = \| \sum_{-n\le j\le n-1} e^{\lambda^n_j t} e^n_j(x)(\overline{e_j} - \overline{e_j \circ \rho_n})(\cdot)\|^2_{L_2(\mathbb{T})}.$$

We now show that each $I^i_t(x)$ is bounded by the right-hand side of (2.33). By the orthonormality of e_j we see that

$$I^1_t(x) = \sum_{\{j\le -n-1\}\cup\{j\ge n\}} e^{2\lambda_j t} \lesssim n^{-2} \sum_{|j|\ge 2n} j^2 e^{2\lambda_j t}. \tag{2.35}$$

Since $\lambda_j = -4\pi^2 j^2$, we can use Proposition 2.2.2 to conclude the claimed bound. Next recall that $\lambda_j \le \lambda^n_j \le 0$ and that $\lambda^n_j \le -16j^2$, we get that

$$\begin{aligned} I^2_t(x) &\le \sum_{-n\le j\le n-1} (e^{\lambda_j t} - e^{\lambda^n_j t})^2 = \sum_{-n\le j\le n-1} (e^{\lambda^n_j t}(1 - e^{-(\lambda^n_j - \lambda_j)t}))^2 \\ &\le \sum_{-n\le j\le n-1} (e^{\lambda^n_j t}(\lambda^n_j - \lambda_j)t)^2 \lesssim \sum_{-n\le j\le n-1} e^{-32j^2 t} j^4(\gamma^n_j - 1)^2 t^2 \end{aligned}$$

By (2.6) and Proposition 2.2.2 we get

$$I^2_t(x) \lesssim \sum_{-n\le j\le n-1} e^{-32j^2 t} j^6 n^{-4} t^2 \lesssim n^{-4} t^{2-9/2}. \tag{2.36}$$

Using $n^{-2} \lesssim t$ again, we get the required bound. Next, for $I^3_t(x)$ we have

$$I^3_t(x) = \sum_{-n\le j\le n-1} e^{-2\lambda^n_j t} |e_j(x) - e^n_j(x)|^2 \lesssim \sum_{-n\le j\le n-1} e^{-32j^2 t} j^2 n^{-2}. \tag{2.37}$$

As usual, Proposition 2.2.2 implies the claimed bound. Before estimating $I^4_t(x)$, we claim that for $j, \ell \in \{-n, ..., n-1\}$, with $j \ne \ell$, the functions $e_j - e_j \circ \rho_n$ and $e_\ell - e_\ell \circ \rho_n$ are orthogonal in $L_2(\mathbb{T})$. Indeed, by the orthogonality of $(e_j)_{j\in\mathbb{Z}}$ and the orthogonality of $(e^n_\ell)_{-n\le \ell\le n-1}$, we have that

$$(e_j - e_j \circ \rho_n, e_\ell - e_\ell \circ \rho_n)_{L_2(\mathbb{T})} = -(e_j, e_\ell \circ \rho_n)_{L_2(\mathbb{T})} - (e_j \circ \rho_n, e_\ell)_{L_2(\mathbb{T})}.$$

Then we see that for $j, n \in \{-n, ..., n-1\}$, $j \ne \ell$, we have

$$\begin{aligned} (e_j, e_\ell \circ \rho_n)_{L_2(\mathbb{T})} &= \sum_{k=0}^{2n} \int_{k/2n}^{(k+1)/2n} e^{i2\pi j y} e^{-i2\pi \ell k/2n}\, dy = \frac{(e^{i2\pi j/2n} - 1)}{i2\pi j} \sum_{k=0}^{2n-1} e^{i2\pi(j-\ell)k/2n} \\ &= \frac{(e^{i2\pi j/2n} - 1)}{i2\pi j} \frac{(1 - e^{i2\pi(j-\ell)})}{(1 - e^{i2\pi(j-\ell)/2n})} = 0, \end{aligned}$$

which shows the claim. Consequently,

$$I^4_t(x) = \sum_{-n\le j\le n-1} e^{-2\lambda^n_j t} \|e_j - e_j \circ \rho_n\|^2_{L_2(\mathbb{T})} \lesssim \sum_{-n\le j\le n-1} e^{-32j^2 t} j^2 n^{-2}, \tag{2.38}$$

giving the claimed bound as before. Putting (2.35)-(2.38) together, we conclude that

$$\|p_t(x, \cdot) - \tilde{p}^n_t(x, \cdot)\|^2_{L_2(\mathbb{T})} \lesssim n^{-\beta} t^{-(1+\beta)/2}. \tag{2.39}$$

Next we see that

$$\|\tilde{p}^n_t(x,\cdot) - p^n_{\kappa_n(t)}(x,\cdot)\|^2_{L_2(\mathbb{T})} \lesssim \sum_{-n\le j\le n-1} |e^{\lambda^n_j t} - e^{\lambda^n_j \kappa_n(t)}|^2 + \sum_{-n\le j\le n-1} |e^{\lambda^n_j \kappa_n(t)} - (1+h\lambda^n_j)^{\kappa_n(t)h^{-1}}|^2. \quad (2.40)$$

As before, we show that both terms are bounded by the right-hand side of (2.33). Since $t \ge h$ implies $\kappa_n(t) \ge t/2$, we have

$$\sum_{-n\le j\le n-1} |e^{\lambda^n_j t} - e^{\lambda^n_j \kappa_n(t)}|^2 \le \sum_{-n\le j\le n-1} \big(|t-\kappa_n(t)|\lambda^n_j e^{\lambda^n_j \kappa_n(t)}\big)^2 \lesssim n^{-4} \sum_{-n\le j\le n-1} j^4 e^{-16j^2 t}. \quad (2.41)$$

Proposition 2.2.2 and $n^{-2} \lesssim t$ yields the claimed bound. For the other term, we have

$$\sum_{-n\le j\le n-1} |e^{\lambda^n_j \kappa_n(t)} - (1+h\lambda^n_j)^{\kappa_n(t)h^{-1}}|^2 \lesssim \sum_{j\in J^1_n} |e^{\lambda^n_j \kappa_n(t)}|^2 + \sum_{j\in J^1_n} |(1+h\lambda^n_j)^{\kappa_n(t)h^{-1}}|^2 + \sum_{j\in J^2_n} |e^{\lambda^n_j \kappa_n(t)} - (1+h\lambda^n_j)^{\kappa_n(t)h^{-1}}|^2, \quad (2.42)$$

where

$$J^1_n = \{j \in \{-n,...,n-1\} : h\lambda^n_j \in [-(4c \vee 1/2), -1/2]\},$$
$$J^2_n = \{j \in \{-n,...,n-1\} : h\lambda^n_j \in (-1/2, 0]\}.$$

We have, using $\kappa_n(t) \ge t/2$ as before,

$$\sum_{j\in J^1_n} |e^{\lambda^n_j \kappa_n(t)}|^2 \le \sum_{j\in J^1_n} e^{-t/(2h)} \lesssim n(t/h)^{-3/2} \lesssim n^{-2}t^{-3/2},$$

and similarly, by (2.9),

$$\sum_{j\in J^1_n} |(1+h\lambda^n_j)^{\kappa_n(t)h^{-1}}|^2 \lesssim n^{-2}t^{-3/2}.$$

For the last term on the right-hand side of (2.42) we use the elementary inequalities $0 \le x - \ln(1+x) \le x^2$ for all $x \in [-1/2, 0]$, and $|1-e^y| \le |y|$ for all $y \le 0$. Therefore,

$$\begin{aligned}\sum_{j\in J^2_n} |e^{\lambda^n_j \kappa_n(t)} - (1+h\lambda^n_j)^{\kappa_n(t)h^{-1}}|^2 &= \sum_{j\in J^2_n} e^{2\lambda^n_j \kappa_n(t)} \Big|1 - e^{\kappa_n(t)h^{-1}\big(\ln(1+h\lambda^n_j) - \lambda^n_j h\big)}\Big|^2 \\ &\le \sum_{j\in J^2_n} e^{-16j^2 t} |\kappa_n(t)h^{-1}(h\lambda^n_j - \ln(1+h\lambda^n_j))|^2 \\ &\le \sum_{j\in J^2_n} e^{-16j^2 t} |\kappa_n(t)h^{-1}|^2 |h\lambda^n_j|^4 \\ &\lesssim n^{-4}|t|^2 \sum_{j\mathbb{Z}} e^{-16j^2 t} j^8.\end{aligned}$$

As before, this gives the desired bound. We conclude that

$$\sum_{-n\le j\le n-1} |e^{\lambda_j^n \kappa_n(t)} - (1+h\lambda_j^n)^{\kappa_n(t)h^{-1}}|^2 \lesssim n^{-2}t^{-3/2}. \tag{2.43}$$

Combining (2.34), (2.39), (2.40), (2.41), and (2.43) brings the proof to an end. □

For the proof of the Lemma 2.2.8 below, let us recall the following result, a nice consequence of Stein's method of normal approximation (see, for example, [CGS11, Corollary 4.2, p. 68]).

Lemma 2.2.7. *Let $\{Y_i\}_{i=1}^\infty$ be i.i.d. random variables with $\mathbb{E}Y_1 = 0$ and $\mathbb{E}Y_1^2 = 1$ and let $W \sim \mathcal{N}(0,1)$. Then, for all $k \in \mathbb{N}$ we have*

$$|\mathbb{E}f(W_k) - \mathbb{E}f(W)| \le k^{-1/2}\|f\|_{C^1}\mathbb{E}|Y_1|^3,$$

where $W_k = k^{-1/2}\sum_{i=1}^k Y_i$.

Lemma 2.2.8. *For any $\alpha \in [0,1]$ there exists a constant $N = N(\alpha)$ such that for all $\psi \in C^\alpha(\mathbb{T})$, $t \in [0,1]$, and $y \in \mathbb{T}$, we have*

$$|\mathcal{P}_t^n\psi(y) - \mathcal{P}_t\psi(y)| \le Nn^{-\alpha}\|\psi\|_{C^\alpha}. \tag{2.44}$$

Proof. We first show the desired inequality for $\alpha = 1$. We use the reformulation of the discrete heat kernel as the transition kernel of a discrete time random walk, see Remark 2.1.7.

Further, we assume for now that $t \in \Lambda_n$, $y \in \Pi_n$. Recall that in this case, by (2.15) we have that

$$\begin{aligned}|\mathcal{P}_t^n\psi(y) - \mathcal{P}_t\psi(y)| &= |\mathbb{E}\tilde\psi(y+\widehat{S}_t^n) - \mathbb{E}\tilde\psi(y+\sqrt{2}W_t)| \\ &= \Big|\mathbb{E}\tilde\psi\Big(y+\sqrt{2t}\sum_{i-1}^{h^{-1}t}\frac{X_i}{\sqrt{2c}\sqrt{h^{-1}t}}\Big) - \mathbb{E}\tilde\psi(y+\sqrt{2t}W_1)\Big|\end{aligned}$$

where W_t is a Brownian motion. By applying Lemma 2.2.7 with $Y_i = X_i/\sqrt{2c}$ and $f(\cdot) = \tilde\psi(y+\sqrt{2t}\,\cdot)$, we get

$$|\mathcal{P}_t^n\psi(y) - \mathcal{P}_t\psi(y)| \le N(h^{-1}t)^{-1/2}t^{1/2}\|\psi\|_{C^1} \le Nn^{-1}\|\psi\|_{C^1}.$$

For $t \in [0,1]$, $y \in \mathbb{T}$, we have

$$\begin{aligned}|\mathcal{P}_t^n\psi(y) - \mathcal{P}_t\psi(y)| &\le |\mathcal{P}_t^n\psi(y) - \mathcal{P}_{\kappa_n(t)}^n\psi(\rho_n(y))| + |\mathcal{P}_{\kappa_n(t)}^n\psi(\rho_n(y)) - \mathcal{P}_{\kappa_n(t)}\psi(\rho_n(y))| \\ &\quad + |\mathcal{P}_{\kappa_n(t)}\psi(\rho_n(y)) - \mathcal{P}_t\psi(y)| \\ &\lesssim n^{-1}\|\psi\|_{C^1},\end{aligned}$$

where for the first term we have used Lemma 2.2.5 ((2.30) for $t \ge h$ and (2.31) for $t \in [0,h]$) and for the last term we have used classical heat kernel estimates (for example (2.16) with $\alpha = \beta = 1$). This proves the claim for $\alpha = 1$. Also, for $\alpha = 0$ the claim trivially holds. Finally, the claim for $\alpha \in (0,1)$ follows by interpolation between the cases $\alpha = 0$ and $\alpha = 1$. □

2.3 Discrete and continuous Ornstein-Uhlenbeck processes

The last integral in (2.1) is also called the infinite dimensional Orstein-Uhlenbeck process. In the trivial case $\psi = 0$, $b = 0$, that is simply the solution process, and it also plays an important role in the general case. Let us therefore introduce a separate notation for it:

$$O_t(x) = \int_0^t \int p_{t-s}(x-y)\,\xi(dy,ds).$$

Let $(s,t) \in [0,1]_<$.Thanks to the semigroup property of $\mathcal{P}$ the Ornstein-Uhlenbeck process satisfies

$$\begin{aligned} O_t(x) &= \int_0^s \int_{\mathbb{T}} p_{t-r}(x-y)\,\xi(dy,dr) + \int_s^t \int_{\mathbb{T}} p_{t-r}(x-y)\,\xi(dy,dr) \\ &= \int_0^s \int_{\mathbb{T}} \int_{\mathbb{T}} p_{t-s}(x-z)p_{s-r}(z-y)\,dz\,\xi(dy,dr) + \int_s^t \int_{\mathbb{T}} p_{t-r}(x-y)\,\xi(dy,dr) \\ &= (\mathcal{P}_{t-s}O_s)(x) + \int_s^t \int_{\mathbb{T}} p_{t-r}(x-y)\,\xi(dy,dr). \end{aligned} \tag{2.45}$$

The second term on the right-hand side of is a Gaussian random variable that is independent of $\mathcal{F}_s$. Its variance is given by

$$\int_s^t \int_{\mathbb{T}} |p_{t-r}(x-y)|^2\,dy\,dr = \int_0^{t-s} \int_{\mathbb{T}} |p_r(y)|^2\,dy\,dr =: Q(t-s). \tag{2.46}$$

Therefore, for any continuous function g and $\mathcal{F}_s$-measurable random variable Y one has the almost sure equality

$$\mathbb{E}^s g(O_t(x) + Y) = \mathcal{P}^{\mathbb{R}}_{Q(t-s)} g((\mathcal{P}_{t-s}O_s)(x) + Y). \tag{2.47}$$

Moreover, standard heat kernel estimates give $N^{-1}s^{-1/2} \le \|\mathcal{P}_s\|^2_{L_2(\mathbb{T})} \le Ns^{-1/2}$ for all $s \in (0,1]$ with some absolute constant $N > 0$, and therefore for all $r \in (0,1]$

$$N^{-1}\sqrt{r} \le Q(r) \le N\sqrt{r}. \tag{2.48}$$

The following is well-known.

Proposition 2.3.1. *For any $p > 0$, and $\theta \in (0,1/2)$ one has $\|O\|_{L_p(\Omega;\mathcal{C}([0,1];\mathcal{C}^{1/2-\theta}(\mathbb{T}))} < \infty$.*

We similarly define the discrete Ornstein-Uhlenbeck process by setting

$$O^n_t(x) = \int_0^t \int_{\mathbb{T}} p^n_{\kappa_n(t-r)}(x,y)\,\xi(dy,dr).$$

We wish to obtain a discrete analogue of (2.47). For $s \in [0,t]$ we write

$$\begin{aligned} O^n_t(x) &= \int_0^s \int_{\mathbb{T}} p^n_{\kappa_n(t-r)}(x,y)\,\xi(dy,dr) + \int_s^t \int_{\mathbb{T}} p^n_{\kappa_n(t-r)}(x,y)\,\xi(dy,dr) \\ &=: \widehat{O}^n_{s,t}(x) + \int_s^t \int_{\mathbb{T}} p^n_{\kappa_n(t-r)}(x,y)\,\xi(dy,dr). \end{aligned}$$

Clearly, $\widehat{O}^n_{s,t}(x)$ is $\mathcal{F}_s$-measurable, while the second term in the right-hand side is a Gaussian random variable that is independent of $\mathcal{F}_s$. Its variance is given by

$$Q^n(t-s) := \int_0^{t-s} \int_{\mathbb{T}} |p^n_{\kappa_n(r)}(x,y)|^2 \, dy \, dr = \int_0^{t-s} \sum_{j=-n}^{n-1} |1 + h\lambda^n_j|^{2\kappa_n(r)h^{-1}} \, dr.$$

Therefore, for any continuous function g and $\mathcal{F}_s$-measurable random variable Y one has the almost sure equality

$$\mathbb{E}^s g(O^n_t(x) + Y) = \mathcal{P}^{\mathbb{R}}_{Q^n(t-s)} g(\widehat{O}^n_{s,t}(x) + Y). \tag{2.49}$$

In the next two statements we compare the expressions in (2.49) to the corresponding ones in (2.47).

Corollary 2.3.2. *Let $\beta \in [0,2]$ and $p > 0$. Then there exists a constant $N(p,\beta,c)$ such that for all $t, s \in [0,1]$ with $t - s \geq h$, $x \in \mathbb{T}$ one has*

$$\|\mathcal{P}_{t-s} O_s(x) - \widehat{O}^n_{s,t}(x)\|_{L_p(\Omega)} \leq N n^{-\beta/2} |t-s|^{(1-\beta)/4}. \tag{2.50}$$

Moreover, for $\beta \in [0,1)$, $p > 0$, there exists a constant $N(p,\beta,c)$ such that for all $t \in [0,1]$, $x \in \mathbb{T}$, one has

$$\|O_t(x) - O^n_t(x)\|_{L_p(\Omega)} \leq N n^{-\beta/2} t^{(1-\beta)/4}. \tag{2.51}$$

Proof. We will assume that $p = 2$, since the general case follows by the equivalence of Gaussian moments. Notice that for $t - s \geq h$, the estimate (2.50) with $\beta = 2$ follows directly by Itô's isometry and Lemma 2.2.6. Since it is true for $\beta = 2$ is is also true for any $\beta \in [0,2]$ since $1/n \leq 2c^{-1/2}|t-s|^{1/2}$.

As for (2.51), there are two cases. First, assume that $t \in [0,h]$. By Itô's isometry and (2.48), we have

$$\|O_t(x)\|_{L_2(\Omega)} = |Q(t)|^{1/2} \lesssim t^{1/4} \lesssim n^{-\beta/2} t^{(1-\beta)/4}.$$

By (2.53) (below), we have

$$\|O^n_t(x)\|_{L_2(\Omega)} = |Q^n(t)|^{1/2} = (2nt)^{1/2} \lesssim n^{-\beta/2} t^{(1-\beta)/4}.$$

Combining the two estimate above gives (2.50). In the case $t \in (h,1]$ we use Lemma 2.2.6 and the above estimates for Q and Q^n to write

$$\begin{aligned}
\|O_t(x) - O^n_t(x)\|_{L_2(\Omega)} &\leq \Big(\int_0^{t-h} \|p_{t-r}(x,\cdot) - p^n_{\kappa_n(t-r)}(x,\cdot)\|^2_{L_2(\mathbb{T})} \Big)^{1/2} \\
&\quad + \Big(\int_{t-h}^{t} \|p_{t-r}(x,\cdot) - p^n_{\kappa_n(t-r)}(x,\cdot)\|^2_{L_2(\mathbb{T})} \Big)^{1/2} \\
&\lesssim n^{-\beta/2} t^{(1-\beta)/4} + |Q(h)|^{1/2} + |Q^n(h)|^{1/2} \\
&\lesssim n^{-\beta/2} t^{(1-\beta)/4} + n^{-1/2}. \\
&\lesssim n^{-\beta/2} t^{(1-\beta)/4}.
\end{aligned}$$

This finishes the proof. □

Lemma 2.3.3. *For any $\beta \in [0,2)$ there exists a constant $N = N(c,\beta)$ such that for all $r \in [0,2]$ one has the bound*

$$|Q^n(r) - Q(r)| \leq N n^{-\beta/2} r^{1/2-\beta/4}. \tag{2.52}$$

Proof. Let us first assume $r \leq h$. Then by standard heat kernel estimates and (2.23)

$$|Q^n(r) - Q(r)| \leq \int_0^r \int_{\mathbb{T}} |p^n_{\kappa_n(s)}(x,y)|^2 + |p_s(x-y)|^2 \, dy \, ds \lesssim rn + r^{1/2}.$$

Since $r \leq h$ implies $rn \lesssim r^{1/2}$, the second term dominates. One similarly gets $r^{1/2} \lesssim r^{1/2-\beta/4}n^{-\beta/2}$ for any $\beta \geq 0$, yielding (2.52). Moving on to the $r \geq h$ case, one has

$$|Q^n(r) - Q(r)| \leq I_1 + I_2,$$

with

$$\begin{aligned} I_1 &= \int_0^h \int_{\mathbb{T}} |p^n_{\kappa_n(s)}(x,y)|^2 + |p_s(x-y)|^2 \, dy \, ds \leq n^{-1}, \\ I_2 &= \int_h^r \Big(\int_{\mathbb{T}} |p^n_{\kappa_n(s)}(x,y) - p_s(x-y)|^2 \, dx \Big)^{1/2} \Big(\int_{\mathbb{T}} |p^n_{\kappa_n(s)}(x,y) + p_s(x-y)|^2 \, dx \Big)^{1/2} ds \\ &\lesssim \int_0^r n^{-\beta/2} s^{-(\beta+1)/4} s^{1/4} \, ds, \end{aligned}$$

using (2.33) to get the last line. As long as $\beta < 2$, the last integral is finite and is of the required order. Finally, the condition $r \geq h$ implies $n^{-1} \lesssim n^{-\beta/2} r^{1/2-\beta/4}$, finishing the proof. □

For very short times, that is, $r \in [0, h]$, one has from (2.23)

$$Q^n(r) = 2nr. \tag{2.53}$$

Otherwise, we have the following control on Q^n.

Lemma 2.3.4. *For any $\beta \in [0,1]$, there exist constants $N > 0$ depending only on β and c such that for any $r \in [h, 1]$, $r' \in [r, 1]$, we have*

$$Q^n(r) \geq N^{-1}\sqrt{r}, \tag{2.54}$$

$$|Q^n(r) - Q^n(r')| \leq N|r - r'|^{\beta} |r|^{-\beta/2} |r'|^{(1-\beta)/2}. \tag{2.55}$$

Proof. First we show (2.54). Define

$$\tilde{Q}^n(r) := \int_{r/2}^r \int_{\mathbb{T}} |p^n_{\kappa_n(s)}(x,y)|^2 \, dy \, ds, \qquad \tilde{Q}(r) := \int_{r/2}^r \int_{\mathbb{T}} |p_s(x-y)|^2 \, dy \, ds.$$

Since $\tilde{Q}^n \geq Q^n$, it suffices to bound $\tilde{Q}^n$. Standard heat kernel estimates give $N_0^{-1} s^{-1/2} \leq \|\mathcal{P}_s\|^2_{L^2(\mathbb{T})} \leq N_0 s^{-1/2}$ with some absolute constant $N_0 > 0$, and therefore

$$N_1^{-1}\sqrt{r} \leq \tilde{Q}(r) \leq N_1\sqrt{r} \tag{2.56}$$

with another absolute constant $N_1 > 0$. By the triangle inequality one can write

$$|(\tilde{Q}^n(r))^{1/2} - (\tilde{Q}(r))^{1/2}| \leq \Big(\int_{r/2}^r \int_{\mathbb{T}} |p_s(x-y) - p^n_{\kappa_n(s)}(x,y)|^2 \, dy \, ds \Big)^{1/2}.$$

Note that if N' is a sufficiently large constant, then $r \geq N'n^{-2}$ implies $r/2 \geq h$. Therefore the bound (2.33) with the choice $\beta = 1$ yields

$$\int_{r/2}^{r} \int_{\mathbb{T}} |p_s(x-y) - p^n_{\kappa_n(s)}(x,y)|^2 \, dy \, ds \leq N \int_{r/2}^{r} n^{-1} s^{-1} \, ds \leq N n^{-1}.$$

Therefore, one has

$$|(\tilde{Q}^n(r))^{1/2} - (\tilde{Q}(r))^{1/2}| \leq N n^{-1/2}. \tag{2.57}$$

Combining (2.56) and (2.57) with the elementary equality $a^2 \geq 2b(a-b) + b^2$

$$\begin{aligned}\tilde{Q}^n(r) &\geq 2(\tilde{Q}(r))^{1/2}\Big((\tilde{Q}^n(r))^{1/2} - (\tilde{Q}(r))^{1/2}\Big) + Q(r) \\ &\geq -N\sqrt[4]{r} n^{-1/2} + N_1^{-1}\sqrt{r}.\end{aligned}$$

Choosing N' sufficiently large, the bound (2.54) indeed follows for $r \geq N'n^{-2}$. If $r \in [h, N'n^{-2}$, then by the monotonicity of Q^n and the fact that $Q^n(h) = 2nh$ (see (2.53)), we get

$$Q^n(r) \geq Q^n(h) = 2nh = 2cn^{-1} \geq 2c\sqrt{N'}\sqrt{r},$$

which shows (2.54) also for $r \in [h, N'n^{-2}]$.

We continue with (2.55). By (2.9) and Proposition 2.2.2 we get

$$\begin{aligned}|Q^n(r) - Q^n(r')| &= \int_r^{r'} \sum_{j=-n}^{n-1} (1 + h\lambda_j^n)^{2\kappa_n(t)h^{-1}} \, dt \lesssim 2\int_r^{r'} \sum_{j\in\mathbb{Z}} e^{-2\delta j^2 t} \, dt \\ &\lesssim \int_r^{r'} t^{-1/2} \, dt \leq |r - r'||r|^{-1/2}. \end{aligned} \tag{2.58}$$

On the other hand, by (2.52) and the standard heat kernel bounds for Q it follows that $|Q^n(r) - Q^n(r')| \lesssim (r')^{1/2}$, which combined with (2.58) finishes the proof. ☐

Lemma 2.3.5. *For any $p > 0$, $\theta \in (0, 1/2)$, there exist a constant $N(c, p, \theta)$ such that*

$$\sup_{n\in\mathbb{N}} \sup_{t\leq 1} \|O^n_t\|_{L_p(\Omega; \mathcal{C}^{1/2-\theta}(\mathbb{T}))} \leq N.$$

Proof. By the Burkholder-Davis-Gundy inequality we have

$$\|O^n_t(x) - O^n_t(z)\|^2_{L_p(\Omega)} \lesssim \int_0^t \|p^n_{\kappa_n(t-s)}(x,\cdot) - p^n_{\kappa_n(t-s)}(z,\cdot)\|^2_{L_2(\mathbb{T})} \, ds$$

We have by (2.9)

$$\begin{aligned}\|p^n_{\kappa_n(t-s)}(x,\cdot) - p^n_{\kappa_n(t-s)}(z,\cdot)\|^2_{L_2(\mathbb{T})} &= \sum_{j=-n}^{n-1} |1 + h\lambda_j^n|^{2\kappa_n(t-s)h^{-1}} |e_j^n(x) - e_j^n(z)|^2 \\ &\leq \sum_{j=-n}^{n-1} e^{-2\kappa_n(t-s)\delta j^2} |x-z|^{(1-2\theta)} n^{(1-2\theta)}.\end{aligned}$$

Consequently, by (2.9) and Proposition 2.2.2 we get

$$\begin{aligned}\|O_t^n(x) - O_t^n(z)\|_{L_p(\Omega)}^2 &\lesssim \int_0^{(t-2h)\wedge 0} \sum_{j=-n}^{n-1} e^{-2\kappa_n(t-s)\delta j^2}|x-z|^{(1-\theta)}n^{(1-\theta)}\,ds \\ &\quad + \int_{(t-2h)\vee 0}^{t} \sum_{j=-n}^{n-1} |x-z|^{(1-2\theta)}n^{(1-2\theta)}\,ds \\ &\lesssim |x-z|^{1-2\theta}\int_0^{(t-2h)\wedge 0} |t-s|^{-1+2\theta}\,ds + N(p)|x-z|^{(1-\theta)}n^{-\theta} \\ &\lesssim |x-z|^{1-2\theta}.\end{aligned}$$

Similarly, one sees that

$$\|O_t^n(0)\|_{L_p(\Omega)}^2 \lesssim 1,$$

which combined with the above estimate gives

$$\|O_t^n\|_{\mathcal{C}^{1/2-\theta}(\mathbb{T};L_p(\Omega))} \lesssim 1.$$

Since p and θ are arbitrary, by Kolmogorov's continuity criterion we get

$$\|O_t^n\|_{L_p(\Omega;\mathcal{C}^{1/2-\theta}(\mathbb{T}))} \lesssim 1,$$

and the claim follows since the bound does not depend on t or n. □

Lemma 2.3.6. *Under the assumption of Theorem 1.1.2 there exists a constant* $N(c,\varepsilon,K,p,\|b\|_{\mathbb{B}})$ *such that*

$$\sup_{n\in\mathbb{N}}\sup_{t\in[0,1]} \|u_t^n\|_{L_p(\Omega;\mathcal{C}^{1/2-\varepsilon}(\mathbb{T}))} \leq N.$$

Proof. Let us set denote $v_t^n = \mathcal{P}_t^n\psi^n + O_t^n$. The conclusion of the lemma with u^n replaced by v^n is an immediate consequence of Lemma 2.2.4 and Lemma 2.3.5.

From Girsanov's theorem (see e.g. [DPZ92, Thm 10.14] for a sufficiently general version) one has that under the measure $\tilde{\mathbb{P}}$ defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \rho = \exp\Big(-\int_0^1\int_{\mathbb{T}} b(u_t^n(x))\,d\xi(t,x) - \frac{1}{2}\int_0^1\int_{\mathbb{T}} |b(u_t^n(x))|^2\,dx\,dt\Big), \tag{2.59}$$

the mapping

$$\varphi \mapsto \xi(A) + \langle b(u^n), \mathbf{1}_A\rangle_{L_2([0,1]\times\mathbb{T})}$$

from $\mathcal{B}([0,1]\times\mathbb{T})$ to $L_2(\Omega)$ defines a white noise. In particular, the law of u^n under $\tilde{\mathbb{P}}$ and the law of v^n under $\mathbb{P}$ coincide. It is also an easy exercise that $\mathbb{E}\rho^{-1} \leq N(\|b\|_{\mathbb{B}}) < \infty$. Therefore,

$$\begin{aligned}\|u_t^n\|_{L_p(\Omega;\mathcal{C}^{1/2-\varepsilon}(\mathbb{T}))}^p &= \tilde{\mathbb{E}}(\|u_t^n\|_{\mathcal{C}^{1/2-\varepsilon}(\mathbb{T})}^p\rho^{-1}) \leq (\tilde{\mathbb{E}}\|u_t^n\|_{\mathcal{C}^{1/2-\varepsilon}(\mathbb{T})}^{2p})^{1/2}(\tilde{\mathbb{E}}\rho^{-2})^{1/2} \\ &= (\mathbb{E}\|v_t^n\|_{\mathcal{C}^{1/2-\varepsilon}(\mathbb{T})}^{2p})^{1/2}(\mathbb{E}\rho^{-1})^{1/2} \leq N.\end{aligned}$$

This finishes the proof. □

3 The sewing strategy

The goal of this section is to implement an infinite dimensional generalisation of the numerical analytic sewing approach introduced in [BDG21]. First we give an outline of the strategy, identify the various terms to be bounded, and then carry out the estimates.

3.1 Overview

Here we give a brief overview of the strategy of the proof. For reference, we will compare to the 1-dimensional additive SDE

$$dX_t = f(X_t)\,dt + dW_t$$

driven by a standard Wiener process W. Let us assume $f \in \mathcal{C}^\alpha(\mathbb{R})$ with some $\alpha \in (0,1)$. The Euler-Maruyama approximation of the SDE reads as

$$dX_t^n = f(X^n_{\hat\kappa_n(t)})\,dt + dW_t,$$

where we briefly use the notation $\hat\kappa_n(t) = \lfloor nt \rfloor n^{-1}$. Assuming identical initial conditions, one can decompose the error as

$$X_t - X_t^n = \int_0^t f(X_s) - f(X_s^n)\,ds + \int_0^t f(X_s^n) - f(X^n_{\hat\kappa_n(s)})\,ds. \tag{3.1}$$

One then aims to bound the first term by $|X - X^n|$ with *some* norm $|\cdot|$ and the second by a negative power of n, which can in fact be $n^{-(1+\alpha)/2}$. If one furthermore achieves a small constant (say, less than $1/2$) in the first bound, then the inequality buckles and the error itself is bounded by $n^{-(1+\alpha)/2}$.

Of course neither of these tasks are really obvious, since simply bounding the integrals by bringing the absolute value inside gives the bounds $t\|X - X^n\|^\alpha_{L_\infty([0,t])}$ and $n^{-\alpha}$, respectively. The former is particularly problematic, since buckling arguments (or equivalently, Gronwall-type lemmas) fail for powers strictly less than 1. In [BDG21] this issue is overcome by stochastic sewing approach, which however requires to work with a stronger norm: the choice $|\cdot| = \|\cdot\|_{\mathcal{C}^{1/2}(L_p(\Omega))}$ suffices for example. On one hand, this has the advantage of providing the final error estimates in a strong norm, the drawback is that instead of (3.1) one has to control the increments of the error as well.

In infinite dimensions there are several issues with this strategy. We have

$$u_t - u_t^n = \mathcal{P}_t\psi - \mathcal{P}_t^n\psi^n + \int_0^t p_{t-r} * (b(u_r))\,dr - \int_0^t p^n_{\kappa_n(t-r)} *_n (b(u^n_{\kappa_n(r)}))\,dr + O_t - O_t^n.$$

First, the quantity $u_t - u_s$ does not have a natural form as an integral from s to t. Second, even if one considers the "mild" increments $u_t - \mathcal{P}_{t-s}u_s$, there is no nice analogous increment for the approximate solution. Instead, we study the quantity

$$\mathcal{E}^n_{s,t} = \int_s^t p_{t-r} * (b(u_r))\,dr - \int_s^t p^n_{\kappa_n(t-r)} *_n (b(u^n_{\kappa_n(r)}))\,dr. \tag{3.2}$$

The above is not an increment (not even mild), however, it is an analogue of the increments of the right-hand side of (3.1), in the infinite dimensional case, which serves its purpose.

We will use the decomposition

$$\begin{aligned}\mathcal{E}^n_{s,t} = \mathcal{E}^{n,1}_{s,t} + \mathcal{E}^{n,2}_{s,t} + \mathcal{E}^{n,3}_{s,t} := &\int_s^t \Big(p_{t-r} * (b(u_r) - b(u^n_r))\Big)\, dr \\ &+ \int_s^t \Big(p_{t-r} * (b(u^n_r)) - p_{t-r} *_n (b(u^n_{\kappa_n(r)}))\Big)\, dr \\ &+ \int_s^t \Big((p_{t-r} - p^n_{\kappa_n(t-r)}) *_n (b(u^n_{\kappa_n(r)}))\Big)\, dr.\end{aligned} \tag{3.3}$$

Our goal will be to estimate the term $\mathcal{E}^{n,1}$ in terms of $\mathcal{E}^n$, which will lead to buckling for $\mathcal{E}^n$, and the remaining $\mathcal{E}^{n,2}, \mathcal{E}^{n,3}$ by some power of n. Both of these steps will be achieved by a infinite dimensional version of the stochastic sewing lemma. Finally, notice that the above procedure will give an estimate for $\mathcal{E}^n$ and not $u - u^n$ itself. The reason that we follow this route will become clearer later, see Remark 3.3.3.

3.2 Stochastic sewing lemma with propagators

First, we provide a variation of the stochastic sewing lemma [Lê20] using a semigroup of propagators. This version is inspired from a draft of [ABLM20] before its final version, and we thank the authors of [ABLM20] for sharing it. Let V be a Banach space with dual V^*, let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra and let $p \geq 1$. Let us set

$$\mathcal{L}^V_p(\mathcal{G}) := \mathscr{L}(V^*; L_p(\Omega; \mathcal{G})),$$

that is, the space of bounded linear operators from V^* into $L_p(\Omega; \mathcal{G})$. The space is equipped with the natural norm

$$\|f\|_{\mathcal{L}^V_p} = \sup_{\substack{v* \in V^* \\ |v*|_{V*}=1}} \|f(v^*)\|_{L_p(\Omega)}. \tag{3.4}$$

Let us denote by $L^w_p(\Omega, \mathcal{F}; V)$ the collection of all V-valued weak random variables[5] X such that $\sup_{|v^*|=1} \|v^*(X)\|_{L_p(\Omega)} < \infty$. Obviously if $X \in L^w_p(\Omega, \mathcal{F}; V)$, then it can be canonically identified with an element of $\mathcal{L}^V_p(\mathcal{F})$ which is given by $v^* \mapsto v^*(X)$. If S is a bounded linear operator on V, automatically it defines a bounded linear operator on $\mathcal{L}^V_p$ by setting $Sf(v^*) := f(S^*v^*)$. Similarly for $f \in \mathcal{L}^V_p$, we denote by $\mathbb{E}^s f$ the element of $\in \mathcal{L}^V_p$ given by $(\mathbb{E}^s f)(v^*) := \mathbb{E}^s(f(v^*))$. For $0 \leq s' < t' \leq 1$ we denote $[s', t']_< = \{(s,t) \in [s', t']^2 : s < t\}$.

Theorem 3.2.1. *Let $(s', t') \in [0,1]_<$, $p \geq 2$, and V be a Banach space. Let $(S_t)_{t\geq 0}$ be a strongly continuous semigroup on V such that the norm of S_t is bounded uniformly over $t \in [0,1]$ by some constant M. Let furthermore A be a map from $[s', t']_\leq$ to $\mathcal{L}^V_p(\mathcal{F})$ such that $A_{s,t} \in \mathcal{L}^V_p(\mathcal{F}_t)$ for all $(s,t) \in [s', t']_<$. Assume that, with the notation $\delta_S A_{s,u,t} = A_{s,t} - A_{u,t} - S_{u-s}A_{s,u}$, there exists $\varepsilon_1, \varepsilon_2 > 0$, $C_1, C_2 < \infty$ such that the bounds*

$$\|A_{s,t}\|_{\mathcal{L}^V_p} \leq C_1 |t-s|^{1/2+\varepsilon_1}, \tag{3.5}$$

$$\|\mathbb{E}^s \delta_S A_{s,u,t}\|_{\mathcal{L}^V_p} \leq C_2 |t-s|^{1+\varepsilon_2} \tag{3.6}$$

hold for all $s' \leq s \leq u \leq t \leq t'$. Then there exists a unique function $\mathcal{A} : [s', t'] \to \mathcal{L}^V_p(\mathcal{F})$ such that $\mathcal{A}_t \in \mathcal{L}^V_p(\mathcal{F}_t)$ for all $t \in [s', t']$, $\mathcal{A}_{s'} = 0$ and with some $K_1, K_2 < \infty$ the bounds

$$\|\mathcal{A}_t - S_{t-s}\mathcal{A}_s - A_{s,t}\|_{\mathcal{L}^V_p} \leq K_1 |t-s|^{1/2+\varepsilon_1} + K_2 |t-s|^{1+\varepsilon_2}, \tag{3.7}$$

[5] i.e. maps $X : \Omega \to V$ such that $v^*(X)$ is $\mathcal{F}$-measurable for all $v^* \in V^*$

$$\|\mathbb{E}^s(\mathcal{A}_t - S_{t-s}\mathcal{A}_s - A_{s,t})\|_{\mathcal{L}_p^V} \leq K_2|t-s|^{1+\varepsilon_2} \tag{3.8}$$

hold for all $(s,t) \in [s',t']_<$. *Moreover, there exists a* K *depending only on* $p, \varepsilon_1, \varepsilon_2, M$ *such that* (3.7)-(3.8) *in fact hold with* $K_1 = KC_1$*,* $K_2 = KC_2$*, as does the bound*

$$\|\mathcal{A}_t - S_{t-s}\mathcal{A}_s\|_{\mathcal{L}_p^V} \leq K_1|t-s|^{1/2+\varepsilon_1} + K_2|t-s|^{1+\varepsilon_2}. \tag{3.9}$$

Remark 3.2.2. There are a couple of infinite dimensional sewing lemmas in the recent literature. If one wishes to bound Banach-space valued processes in their natural norm (instead of (3.4)), then nontrivial issues with the Burkholder-Davis-Gundy inequality may arise. These were addressed by Lê [Lê21], where a stochastic sewing lemma in Banach spaces is formulated. Very recently Li and Sieber [LS21] also formulated a stochastic sewing lemma with semigroups, but staying in the Hilbert space setting and using the natural norms.

Proof. Throughout the proof we understand the proportionality constant in $\lesssim$ to depend on $p, \varepsilon_1, \varepsilon_2, M$. Note that from the linearity and uniform (in t) boundedness of S_t we have

$$\|v^*(S_tX)\|_{L_p(\Omega)} \lesssim \|X\|_{\mathcal{L}_p^V}, \qquad \|\mathbb{E}^s v^*(S_tX)\|_{L_p(\Omega)} \lesssim \|\mathbb{E}^s X\|_{\mathcal{L}_p^V}. \tag{3.10}$$

for any $v^* \in V^*$, $s,t \geq 0$, $X \in \mathcal{L}_p^V$. We will follow the construction from [Ger20]. Let first $(s,t) \in [s',t']_<$ be fixed. For a partition $\pi = \{s = t_0, t_1, ..., t_n = t\}$ of $[s,t]$, let us set

$$|\pi| = \max_i |t_{i+1} - t_i|, \qquad [\pi] = (t-s)/n, \qquad \Delta\pi = \min_i |t_{i+1} - t_i|.$$

Clearly $|\pi| \geq [\pi] \geq \Delta\pi$. We introduce the set of regular partitions $\Pi = \Pi_{[s,t]} = \{\pi : |\pi| \leq 2\Delta\pi\}$, and for $\pi = \{s = t_0 < t_1 < \cdots < t_n = t\} \in \Pi_{[s,t]}$, we set

$$\mathcal{A}^\pi_{s,t} = \sum_{i=1}^n S_{t-t_i} A_{t_{i-1},t_i}.$$

We claim that

(i) If $\pi \in \Pi$ is a refinement of $\pi' \in \Pi$, then

$$\|\mathcal{A}^\pi_{s,t} - \mathcal{A}^{\pi'}_{s,t}\|_{\mathcal{L}_p^V} \lesssim C_1|t-s|^{1/2}[\pi]^{\varepsilon_1} + C_2|t-s|[\pi]^{\varepsilon_2} \tag{3.11}$$

$$\|\mathbb{E}^s(\mathcal{A}^\pi_{s,t} - \mathcal{A}^{\pi'}_{s,t})\|_{\mathcal{L}_p^V} \lesssim C_2|t-s|[\pi]^{\varepsilon_2}. \tag{3.12}$$

(ii) The limit $\mathcal{A}_{s,t} = \lim_{\pi\in\Pi, |\pi|\to 0} \mathcal{A}^\pi_{s,t}$ exists in $\mathcal{L}_p^V(\mathcal{F}_t)$.

(iii) For any $(s,t) \in [s',t']_\leq$, we have $\mathcal{A}_{s,t} = \mathcal{A}_{s',t} - S_{t-s}\mathcal{A}_{s',s}$.

(iv) The process $\mathcal{A}_t := \mathcal{A}_{s',t}$ satisfies the bounds (3.7)-(3.8)-(3.9) with $K_i = KC_i$.

Notice that (ii), (iii), and (iv), follow from (i), as in the proof of [Ger20, Lem 2.2]. Therefore, only need to prove (i). Define a map ρ on Π as follows. If $\pi = \{t_0, \ldots, t_n\}$ and n is odd, first set $n' = n+1$ and $t'_i = (t_i + t_{i-1})/2$, where i is the first index such that $|\pi| = t_i - t_{i-1}$. Set furthermore $t'_j = t_j$ for $0 \leq j < i$ and $t'_j = t_{j-1}$ for $i < j \leq n$. When n is even, simply set $n' = n$ and $t'_j = t_j$ for all $0 \leq j \leq n$. Then set $\rho(\pi) = \{t'_0, t'_2, t'_4, \ldots, t'_{n'}\}$. It is clear that $\rho(\pi) \in \Pi$.

Moreover, unless π is already the trivial partition to begin with, then $n'/2 \le ((4/3)n)/2$, and so $[\rho(\pi)] \ge (3/2)[\pi]$

First we show (3.11)-(3.12) in the case of $\pi' = \rho(\pi)$. We then write

$$\mathcal{A}^{\pi}_{s,t} - \mathcal{A}^{\rho(\pi)}_{s,t} = S_{t-t_{i+1}}\delta_S A_{t_{i-1},t'_i,t_i} - \sum_{j=0}^{n'/2-1} S_{t-t'_{2j+2}}\delta_S A_{t'_{2j},t'_{2j+1},t'_{2j+2}}. \tag{3.13}$$

For the first term, we have from (3.10) and the conditions (3.5)-(3.6)

$$\|S_{t-t_{i+1}}\delta_S A_{t_{i-1},t'_i,t_i}\|_{\mathcal{L}^V_p} \lesssim C_1|t_i - t_{i-1}|^{1/2+\varepsilon_1} \lesssim C_1|t-s|^{1/2}[\pi]^{\varepsilon_1},$$
$$\|\mathbb{E}^s S_{t-t_{i+1}}\delta_S A_{t_{i-1},t'_i,t_i}\|_{\mathcal{L}^V_p} \lesssim C_2|t_i - t_{i-1}|^{1+\varepsilon_2} \lesssim C_2|t-s|[\pi]^{\varepsilon_2}.$$

It remains to prove a similar bound for the sum in (3.13). Let us denote the triples $(t'_{2j}, t'_{2j+1}, t'_{2j+2})$ by (a, b, c). At this point we employ the usual stochastic sewing strategy of [Lê20]:

$$\begin{aligned}
&\Big\|\sum v^*(S_{t-c}\delta_S A_{a,b,c})\Big\|_{L_p(\Omega)} \\
&\lesssim \sum \Big\|\mathbb{E}^a v^*(S_{t-c}\delta_S A_{a,b,c})\Big\|_{L_p(\Omega)} + \Big(\sum \Big\|(\mathrm{id} - \mathbb{E}^a) v^*(S_{t-c}\delta_S A_{a,b,c})\Big\|^2_{L_p(\Omega)}\Big)^{1/2}.
\end{aligned}$$

The first term on the right-hand side is bounded via (3.10) and the condition (3.6). For the second term we estimate each term in the identity $\delta A_{a,b,c} = A_{a,c} - A_{b,c} - S_{b-a}A_{a,b}$ separately, using the condition (3.5). This yields

$$\Big\|\sum v^*(S_{t-c}\delta_S A_{a,b,c})\Big\|_{L_p(\Omega)} \lesssim nC_2|\pi|^{1+\varepsilon_2} + (nC_1^2|\pi|^{1+2\varepsilon_1})^{1/2},$$

which yields (3.11) when $\pi' = \rho(\pi)$. Concerning (3.12), one simply has

$$\Big\|\mathbb{E}^s \sum v^*(S_{t-c}\delta_S A_{a,b,c})\Big\|_{L_p(\Omega)} \lesssim \sum \Big\|\mathbb{E}^a v^*(S_{t-c}\delta_S A_{a,b,c})\Big\|_{L_p(\Omega)},$$

and the right-hand side can be bounded just as before, yielding (3.12). Next, we show (3.11)-(3.12) in the case of $\pi' = \{s, t\}$ being the trivial partition. Define m as the smallest integer such that $\rho^m(\pi) = \pi'$. One can write

$$\begin{aligned}
\|\mathcal{A}^{\pi}_{s,t} - \mathcal{A}^{\pi'}_{s,t}\|_{\mathcal{L}^V_p} &\le \sum_{i=0}^{m-1} \|\mathcal{A}^{\rho^i(\pi)}_{s,t} - \mathcal{A}^{\rho^{i+1}(\pi)}_{s,t}\|_{\mathcal{L}^V_p} \\
&\lesssim C_1|t-s|^{1/2}\sum_{i=0}^{m-1}[\rho^{i+1}(\pi)]^{\varepsilon_1} + C_2|t-s|\sum_{i=0}^{m-1}[\rho^{i+1}(\pi)]^{\varepsilon_2}.
\end{aligned}$$

Recalling $[\rho^i(\pi)] \le (2/3)^{m-i}[\rho^m(\pi)] = (2/3)^{m-i}[\pi']$, this yields (3.11). One gets similarly (3.12) for $\pi' = \{s, t\}$. Finally, if $\pi' = \{t_0, \dots, t_n\} \in \Pi$ is arbitrary and $\pi \in \Pi$ is a refinement of it, define π_i to be the restriction of π to $[t_i, t_{i+1}]$. It is clear that $\pi_i \in \Pi_{[t_i,t_{i+1}]}$. Therefore writing

$$\mathcal{A}^{\pi}_{s,t} - \mathcal{A}^{\pi'}_{s,t} = \sum_{i=0}^{n-1} \mathcal{A}^{\pi_i}_{t_i,t_{i+1}} - A_{t_i,t_{i+1}},$$

each term in the sum is of the form that fits in the previous case, and so admits the bounds (3.11)-(3.12). Repeating the argument above, we get

$$\begin{aligned}\|\mathbb{E}^s v^*(\mathcal{A}^{\pi}_{s,t} - \mathcal{A}^{\pi'}_{s,t})\|_{L_p(\Omega)} &\lesssim \sum_{i=0}^{n-1} \|\mathbb{E}^{t_i} v^*(\mathcal{A}^{\pi_i}_{t_i,t_{i+1}} - A_{t_i,t_{i+1}})\|_{L_p(\Omega)} \\ &\lesssim \sum_{i=0}^{n-1} C_2 |t_i - t_{i+1}|^{1+\varepsilon_2} \le C_2 |t-s||\pi'|^{\varepsilon_2}.\end{aligned}$$

This yields (3.12) and the argument for (3.11) is again just as above.

Concerning uniqueness, by linearity it suffices to treat the case $A_{s,t} \equiv 0$. Note that the estimates (3.7)-(3.8) allow one to bound the quantities $\delta_S \mathcal{A}_{s,t,t}$. Fix $t \in [s', t']$, let $n \in \mathbb{N}$, and set $t_i = s' + i\kappa$ with $\kappa = (t - s')/n$. We then have

$$\mathcal{A}_t = \mathcal{A}_t - S_{t-s'}\mathcal{A}_{s'} = \sum_{i=1}^{n} S_{t-t_i}(\mathcal{A}_{t_i} - S_\kappa \mathcal{A}_{t_{i-1}}),$$

and so by the usual stochastic sewing argument and the boundedness of S we get

$$\begin{aligned}\|\mathcal{A}_t\|_{\mathcal{L}^V_p} &\lesssim \sum_{i=1}^{n} \|\mathbb{E}^{t_{i-1}}(\mathcal{A}_{t_i} - S_\kappa \mathcal{A}_{t_{i-1}})\|_{\mathcal{L}^V_p} + \Big(\sum_{i=1}^{n} \|(\mathcal{A}_{t_i} - S_\kappa \mathcal{A}_{t_{i-1}})\|^2_{\mathcal{L}^V_p}\Big)^{1/2} \\ &\lesssim n\kappa^{1+\varepsilon_2} + (n\kappa^{1+2\varepsilon_1})^{1/2}.\end{aligned}$$

Letting $n \to \infty$ yields $\|\mathcal{A}_t\|_{\mathcal{L}^V_p} = 0$. □

3.3 Estimate for $\mathcal{E}^{n,1}$

The purpose of this section is to provide the estimate for the term $\mathcal{E}^{n,1}_{s,t}$ in the decomposition (3.3). Theorem 3.2.1 will be applied with $V = \mathcal{C}(\mathbb{T})$, and $S_t = \mathcal{P}_t$. In this setting, we know that V^* coincides with $\mathcal{M}(\mathbb{T})$, the space of all finite, signed Borel measures on $\mathbb{T}$ equipped with the total variation norm.

Remark 3.3.1. Since linear combinations of Dirac measures are dense in $\mathcal{M}(\mathbb{T})$ in the weak* topology, it follows that for a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ and a map $f : \Omega \to \mathcal{C}(\mathbb{T})$, f is a weakly $\mathcal{G}$-measurable if and only if $f(x)$ is $\mathcal{G}$-measurable for all $x \in \mathbb{T}$. Moreover, this density combined with the convexity of the norm and Fatou's lemma gives

$$\|f\|_{\mathcal{L}^V_p} = \sup_{x \in \mathbb{T}} \|f(x)\|_{L_p(\Omega)}.$$

Let us also introduce the following (semi)norms. Let $(s', t') \in [0,1]_<$ and $p \in [2, \infty]$. For a map $\varphi : [s', t'] \to L^w_p(\Omega, \mathcal{F}; \mathcal{C}(\mathbb{T}))$ we set

$$\|\varphi\|_{\mathscr{C}^0_p[s',t']} = \sup_{x \in \mathbb{T}} \sup_{s \in [s',t']} \|\varphi_s(x)\|_{L_p(\Omega)}.$$

Furthermore, for $\alpha \in (0,1]$ and a map $\varphi : [s', t']_< \to L^w_p(\Omega, \mathcal{F}; \mathcal{C}(\mathbb{T}))$ we set

$$[\varphi]_{\mathscr{C}^\alpha_p[s',t']_<} = \sup_{x \in \mathbb{T}} \sup_{(s,t) \in [s',t']_<} \frac{\|\varphi_{s,t}(x)\|_{L_p(\Omega)}}{|t-s|^\alpha}.$$

Although our goal is to bound $\mathcal{E}^{n,1}_{s,t}$, it is useful to introduce the generalised quantity

$$\mathcal{E}^{n,1}_{s,t}[f] = \int_s^t \mathcal{P}_{t-r}(f(u_r) - f(u^n_r))\, dr.$$

Note that for fixed $s, t, f \in \mathbb{B}(\mathbb{T}), n, \mathcal{E}^{n,1}_{s,t}[f]$ is an element of $L^w_p(\Omega, \mathcal{F}; \mathcal{C}(\mathbb{T}))$ (for any $p \in [0, \infty]$) and $\mathcal{E}^{n,1}_{s,t}[b] = \mathcal{E}^{n,1}_{s,t}$.

Lemma 3.3.2. *Let $\tau \in (1/4, 3/4)$ and $(s', t') \in [0, 1]_<$. Then, under the assumption of Theorem 1.1.2, for all $f \in \mathbb{B}$, $n \in \mathbb{N}$, and $(s, t) \in [s', t']_<$ the following bound holds*

$$\begin{aligned}\sup_{x\in\mathbb{T}} \|\mathcal{E}^{n,1}_{s,t}[f](x)\|_{L_p(\Omega)} \leq & N\|f\|_{\mathbb{B}}|t-s|^{3/4}(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']}) \\ & + N\|f\|_{\mathbb{B}}|t-s|^{3/4+\tau}[\mathcal{E}^n]_{\mathscr{C}^\tau_p[s',t']_<}, \end{aligned} \tag{3.14}$$

where the constant N depends only on $\|b\|_{\mathbb{B}}, c, \varepsilon, p, K$, and τ.

Remark 3.3.3. Notice that the right-hand side contains the term $[\mathcal{E}^n]_{\mathscr{C}^\tau_p[s',t']_<}$, where $\tau > 1/4$. This is the reason that we aim to buckle for $\mathcal{E}^n$ and not $u - u^n$ itself, as the latter has no more than $1/4$ regularity in time, because of the term $O - O^n$.

Proof. By linearity in f, we may and will assume $\|f\|_{\mathbb{B}} = 1$. We first assume that f is in addition Lipschitz, derive the bound (3.14) that does not depend on its Lipschitz norm, and then conclude with a standard approximation argument.

Define for $(s, t) \in [s', t']_<$ we define an element $A_{s,t} \in L^p_w(\Omega, \mathcal{F}_t; \mathcal{C}(T)) \subseteq \mathcal{L}^{\mathcal{C}(\mathbb{T})}_p(\mathcal{F}_t)$ by

$$A_{s,t}(x) := \mathbb{E}^s \int_s^t \Big(\mathcal{P}_{t-r}(f(O_r + \phi_{s,r})) - \mathcal{P}_{t-r}(f(O^n_r + \phi^n_{s,r}))\Big)(x)\, dr,$$

where

$$\begin{aligned}\phi_{s,r}(y) &= \mathcal{P}_r\psi(y) + \mathbb{E}^s \int_0^r \mathcal{P}_{r-\theta}(b(u_\theta))(y)\, d\theta, \\ \phi^n_{s,r}(y) &= \mathcal{P}^n_r\psi^n(y) + \mathbb{E}^s \int_0^r \mathcal{P}^n_{\kappa_n(r-\theta)}(b(u^n_{\kappa_n(\theta)}))(y)\, d\theta.\end{aligned}$$

We aim to verify the conditions of Theorem 3.2.1. We start by (3.5), that is, by obtaining an estimate for $\sup_{x\in\mathbb{T}} \|A_{s,t}(x)\|_{L_p(\Omega)}$. First of all, notice that we can interchange the action of $\mathbb{E}^u$ and $\mathcal{P}_{t-r}$, and therefore by (2.47) and (2.49) one can write

$$A_{s,t}(x) = \int_s^t \mathcal{P}_{t-r}B_r(x)\, dr,$$

with

$$B_r(y) := \mathcal{P}^{\mathbb{R}}_{Q(r-s)}f(\mathcal{P}_{r-s}O_s(y) + \phi_{s,r}(y)) - \mathcal{P}^{\mathbb{R}}_{Q^n(r-s)}f(\widehat{O}^n_{s,r}(y) + \phi^n_{s,r}(y)).$$

First consider the case $t \geq s + h$. We then have

$$A_{s,t}(x) = I_1(x) + I_2(x) := \Big(\int_s^{s+h} + \int_{s+h}^t\Big)\mathcal{P}_{t-r}B_r(x)\, dr.$$

For I_1 we have the trivial estimate

$$\|I_1(x)\|_{L_p(\Omega)} \lesssim h \lesssim n^{-1/2}(t-s)^{3/4}. \tag{3.15}$$

As for I_2, by applying (2.16) we get

$$\begin{aligned}|B_r(y)| \lesssim & (|\mathcal{P}_{r-s}O_s(y) + \phi_{s,r}(y) - \widehat{O}^n_{s,r}(y) + \phi^n_{s,r}(y))| + |Q(r-s) - Q^n(r-s)|^{1/2}) \\ & \times (Q(r-s) \wedge Q^n(r-s))^{-1/2}\end{aligned} \tag{3.16}$$

Next, using (2.50) with $\beta = 1 - 2\varepsilon$ gives,

$$\|\mathcal{P}_{r-s}O_s(y) - \widehat{O}^n_{s,r}(y)\|_{L_p(\Omega)} \lesssim n^{-1/2+\varepsilon}|r-s|^{\varepsilon/2} \lesssim n^{-1/2+\varepsilon}. \tag{3.17}$$

By using (2.44) with $\alpha = 1/2 - \varepsilon$, and the assumption of the theorem, we get

$$\begin{aligned}&\|\phi_{s,r}(y) - \phi^n_{s,r}(y)\|_{L_p(\Omega)} \\ \leq & \|\mathcal{P}^n_r\psi^n(y) - \mathcal{P}_r\psi(y)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']} \\ \leq & \|\mathcal{P}^n_r\psi^n(y) - \mathcal{P}_r\psi^n(y)\|_{L_p(\Omega)} + \|\mathcal{P}_r\psi^n(y) - \mathcal{P}_r\psi(y)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']} \\ \lesssim & n^{-1/2+\varepsilon}\|\psi^n\|_{L_p(\Omega;C^{1/2-\varepsilon}(\mathbb{T}))} + (\mathcal{P}_r\|\psi(\cdot) - \psi^n(\cdot)\|_{L_p(\Omega)})(y) + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']} \\ \lesssim & n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}}\|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']}\end{aligned} \tag{3.18}$$

Moreover, by using (2.52) with $\beta = 2 - 2\varepsilon$ we get

$$|Q(r-s) - Q^n(r-s)|^{1/2} \lesssim n^{-1/2+\varepsilon}|r-s|^{\varepsilon/4} \lesssim n^{-1/2+\varepsilon}. \tag{3.19}$$

We now combine (3.16) with (3.17)-(3.19), and by keeping in mind that $Q(r-s) \gtrsim |r-s|^{1/2}$ for all $r, s \in [s', t']_<$, and that by (2.54) we also have $Q^n(r-s) \gtrsim |r-s|^{1/2}$ for $r \geq s + h$, we conclude that

$$\sup_{y\in\mathbb{T}}\|B_r(y)\|_{L_p(\Omega)} \lesssim |r-s|^{-1/4}(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}}\|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']}).$$

This in turn implies that

$$\|I_2(x)\|_{L_p(\Omega)} \lesssim |t-s|^{3/4}(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}}\|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']}). \tag{3.20}$$

Hence, in the regime $t \geq s + h$ we have from (3.15) and (3.20) that

$$\sup_{x\in\mathbb{T}}\|A_{s,t}(x)\|_{L_p(\Omega)} \lesssim |t-s|^{3/4}(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}}\|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']}). \tag{3.21}$$

If $t \in [s, s+h)$, we can simply use a trivial bound:

$$\sup_{x\in\mathbb{T}}\|A_{s,t}(x)\|_{L_p(\Omega)} \leq 2|t-s| \lesssim n^{-1/2}|t-s|^{3/4}.$$

We conclude that (3.5) is satisfied with

$$C_2 = N(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}}\|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']})$$

and $\varepsilon_1 = 1/4$ (recall also Remark 3.3.1).

Next, let us bound the term $\|\mathbb{E}^s \delta_{\mathcal{P}} A_{s,u,t}(x)\|_{L_p(\Omega)}$. A simple calculation shows that

$$\begin{aligned}\mathbb{E}^s \delta_{\mathcal{P}} A_{s,u,t}(x) =& \mathbb{E}^s \mathbb{E}^u \int_u^t \Big(\mathcal{P}_{t-r}(f(O_r + \phi_{s,r})) - \mathcal{P}_{t-r}(f(O_r^n + \phi_{s,r}^n))\Big)(x)\, dr \\ &- \mathbb{E}^s \mathbb{E}^u \int_u^t \Big(\mathcal{P}_{t-r}(f(O_r + \phi_{u,r})) - \mathcal{P}_{t-r}(f(O_r^n + \phi_{u,r}^n))\Big)(x)\, dr.\end{aligned}$$

Similarly to before, we write

$$\mathbb{E}^s \delta_{\mathcal{P}} A_{s,u,t}(x) = \mathbb{E}^s \int_u^t \mathcal{P}_{t-r} D_r(x)\, dr, \tag{3.22}$$

with

$$\begin{aligned}D_r(y) :=& \mathcal{P}^{\mathbb{R}}_{Q(r-u)} f(\mathcal{P}_{r-u} O_u(y) + \phi_{s,r}(y)) - \mathcal{P}^{\mathbb{R}}_{Q^n(r-u)} f(\widehat{O}^n_{u,r}(y) + \phi^n_{s,r}(y)) \\ &- \mathcal{P}^{\mathbb{R}}_{Q(r-u)} f(\mathcal{P}_{r-u} O_u(y) + \phi_{u,r}(y)) + \mathcal{P}^{\mathbb{R}}_{Q^n(r-u)} f(\widehat{O}^n_{u,r}(y) + \phi^n_{u,r}(y)).\end{aligned}$$

Let us start by a rough estimate when $|r-u| \le h$. We pair up the first and third, and the second and fourth terms in $D_r(y)$ and apply (2.16) (with $\beta = 1$). We get

$$\begin{aligned}|D_r(y)| &\lesssim |Q(r-u)|^{-1/2} |\phi_{s,r}(y) - \phi_{u,r}(y)| + |Q^n(r-u)|^{-1/2} |\phi^n_{s,r}(y) - \phi^n_{u,r}(y)| \\ &\lesssim |r-u|^{-1/4} |\phi_{s,r}(y) - \phi_{u,r}(y)| + n^{-1/2} |r-u|^{-1/2} |\phi^n_{s,r}(y) - \phi^n_{u,r}(y)|,\end{aligned}$$

where we have used (2.53). Recall that for all $X, Y \in L_\infty(\Omega)$ with Y being $\mathcal{F}_s$-measurable, we have $\|\mathbb{E}^s X - X\|_{L_\infty(\Omega)} \le 2\|Y - X\|_{L_\infty(\Omega)}$. By using this, we see that

$$\begin{aligned}\|\phi_{s,r}(y) - \phi_{u,r}(y)\|_{L_\infty(\Omega)} &= \Big\| \mathbb{E}^s \mathbb{E}^u \int_0^r \mathcal{P}_{r-\theta}(b(u_\theta))(y)\, d\theta - \mathbb{E}^u \int_0^r \mathcal{P}_{r-\theta}(b(u_\theta))(y)\, d\theta \Big\|_{L_\infty(\Omega)} \\ &\le 2 \Big\| \mathbb{E}^u \int_0^s \mathcal{P}_{r-\theta}(b(u_\theta))(y)\, d\theta - \mathbb{E}^u \int_0^r \mathcal{P}_{r-\theta}(b(u_\theta))(y)\, d\theta \Big\|_{L_\infty(\Omega)} \\ &\lesssim |r-s|. \end{aligned} \tag{3.23}$$

Similarly, we get

$$\|\phi^n_{s,r}(y) - \phi^n_{u,r}(y)\|_{L_\infty(\Omega)} \lesssim |r-s|. \tag{3.24}$$

Therefore

$$\sup_{y \in \mathbb{T}} \|D_r(y)\|_{L_p(\Omega)} \lesssim \Big((r-u)^{-1/4}(r-s) + n^{-1/2}(r-u)^{-1/2}(r-s)\Big). \tag{3.25}$$

Let us now first deal with the case $t \in [u, u+h)$. Putting the above bound into (3.22) we get

$$\begin{aligned}\sup_{x \in \mathbb{T}} \|\mathbb{E}^s \delta_{\mathcal{P}} A_{s,u,t}(x)\|_{L_p(\Omega)} &\le \int_u^t \sup_{y \in \mathbb{T}} \|D_r(y)\|_{L_p(\Omega)}\, dr \\ &\lesssim \Big((t-u)^{3/4}(t-s) + n^{-1/2}(t-s)^{3/2}\Big) \\ &\lesssim n^{-1/2}(t-s)^{3/2}. \end{aligned} \tag{3.26}$$

Moving on to the case $t \geq u+h$, we write

$$\|\mathbb{E}^s \delta_{\mathcal{P}} A_{s,u,t}(x)\|_{L_p(\Omega)} \leq \|I_1\|_{L_p(\Omega)} + \|I_2\|_{L_p(\Omega)} := \Big(\int_u^{u+h} + \int_{u+h}^t \Big) \sup_{y \in \mathbb{T}} \|D_r(y)\|_{L_p(\Omega)} \, dr.$$

For I_1 we may use (3.25) again to get

$$\|I_1\|_{L_p(\Omega)} \lesssim n^{-3/2}|t-s| \lesssim n^{-1/2}|t-s|^{3/2}.$$

As for I_2, we decompose the integrand as $D_r = D_r^1 + D_r^2$, where

$$\begin{aligned} D_r^1(y) : &= \mathcal{P}^{\mathbb{R}}_{Q(r-u)} f(\widehat{O}^n_{u,r}(y) + \phi^n_{s,r}(y)) - \mathcal{P}^{\mathbb{R}}_{Q(r-u)} f(\widehat{O}^n_{u,r}(y) + \phi^n_{u,r}(y)) \\ &\quad - \mathcal{P}^{\mathbb{R}}_{Q^n(r-u)} f(\widehat{O}^n_{u,r}(y) + \phi^n_{s,r}(y)) + \mathcal{P}^{\mathbb{R}}_{Q^n(r-u)} f(\widehat{O}^n_{u,r}(y) + \phi^n_{u,r}(y)); \\ D_r^2(y) : &= \mathcal{P}^{\mathbb{R}}_{Q(r-u)} f(\mathcal{P}_{r-u} O_u(y) + \phi_{s,r}(y)) - \mathcal{P}^{\mathbb{R}}_{Q(r-u)} f(\widehat{O}^n_{u,r}(y) + \phi^n_{s,r}(y)) \\ &\quad - \mathcal{P}^{\mathbb{R}}_{Q(r-u)} f(\mathcal{P}_{r-u} O_u(y) + \phi_{u,r}(y)) + \mathcal{P}^{\mathbb{R}}_{Q(r-u)} f(\widehat{O}^n_{u,r}(y) + \phi^n_{u,r}(y)) \end{aligned}$$

For D_r^1 we use (2.18) to obtain

$$\begin{aligned} |D_r^1(y)| &\lesssim (Q(r-u) \wedge Q^n(r-u))^{-3/2} |Q(r-u) - Q^n(r-u)| |\phi^n_{s,r}(y) - \phi^n_{u,r}(y)| \\ &\lesssim |r-u|^{-1/2} n^{-1/2} |\phi^n_{s,r}(y) - \phi^n_{u,r}(y)|, \end{aligned}$$

where for the second inequality we have used (2.54) and (2.52) (the latter with $\beta = 1$). Consequently, by (3.24), we get

$$\sup_{y \in \mathbb{T}} \|D_r^1(y)\|_{L_p(\Omega)} \lesssim |r-u|^{-1/2} n^{-1/2} |t-s|. \tag{3.27}$$

For D_r^2 we use (2.17), to get

$$\begin{aligned} |D_r^2(y)| &\lesssim |Q(r-u)|^{-1} |\mathcal{P}_{r-u} O_u(y) + \phi_{s,r}(y) - \widehat{O}^n_{u,r}(y) - \phi^n_{s,r}(y)| |\phi_{s,r}(y) - \phi_{u,r}(y)| \\ &\quad + |Q(r-u)|^{-1/2} |\phi_{s,r}(y) - \phi^n_{s,r}(y) - \phi_{u,r}(y) + \phi^n_{u,r}(y)|. \end{aligned} \tag{3.28}$$

From (3.17), (3.18) and (3.23) we get that

$$\begin{aligned} &\Big\| |\mathcal{P}_{r-u} O_u(y) + \phi_{s,r}(y) - \widehat{O}^n_{u,r}(y) - \phi^n_{s,r}(y)| |\phi_{s,r}(y) - \phi_{u,r}(y)| \Big\|_{L_p(\Omega)} \\ &\qquad \lesssim (n^{-1/2+\varepsilon} + \sup_{x \in \mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']}\|) |r-s|. \end{aligned} \tag{3.29}$$

Moreover, similarly to the argument for (3.23), we get

$$\begin{aligned} &\|\phi_{s,r}(y) - \phi^n_{s,r}(y) - \phi_{u,r}(y) + \phi^n_{u,r}(y)\|_{L_p(\Omega)} \\ &\leq 2 \Big\| \mathbb{E}^u \int_s^r \mathcal{P}_{r-\theta}(b(u_\theta))(y) \, d\theta - \mathbb{E}^u \int_s^r \mathcal{P}^n_{\kappa_n(r-\theta)}(b(u^n_{\kappa_n(\theta)}))(y) \, d\theta \Big\|_{L_p(\Omega)} \\ &\leq 2 \|\mathcal{E}^n_{s,r}(y)\|_{L_p(\Omega)} \end{aligned}$$

Therefore, we get that

$$\sup_{y \in \mathbb{T}} \|D_r^2(y)\|_{L_p(\Omega)} \tag{3.30}$$

$$\lesssim (n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']} + [\mathcal{E}^n]_{\mathscr{C}^\tau_p[s',t']_<})|r-s|^{-1/4+\tau},$$

where we used that $\tau < 3/4$. Integrating the bounds (3.27) and (3.30) with respect to r, we conclude that

$$\sup_{x\in\mathbb{T}} \|\mathbb{E}^s \delta_{\mathcal{P}} A_{s,u,t}(x)\|_{L_p(\Omega)}$$
$$\lesssim \Big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']} + [\mathcal{E}^n]_{\mathscr{C}^\tau_p[s',t']_<}\Big)(t-s)^{3/4+\tau}.$$

This shows that (3.6) is satisfied with

$$C_2 = N\Big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + \|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[s',t']} + [\mathcal{E}^n]_{\mathscr{C}^\tau_p[s',t']_<}\Big)$$

and $\varepsilon_2 = 3/4 + \tau - 1 > 0$, where we used that $\tau > 1/4$. Therefore Theorem 3.2.1 applies.

We claim that the map $\mathcal{A} : [s',t'] \to \mathcal{L}^{\mathcal{C}(\mathbb{T})}_p(\mathcal{F})$ constructed in Theorem 3.2.1 coincides with $\mathcal{E}^{n,1}_{s',\cdot}[f] : [s',t'] \to L^w_p(\Omega,\mathcal{F};\mathcal{C}(\mathbb{T})) \subseteq \mathcal{L}^{\mathcal{C}(\mathbb{T})}_p(\mathcal{F})$. First of all, it is obvious that $\mathcal{E}^{n,1}_{s',t}[f] \in L^w_p(\Omega,\mathcal{F}_t;\mathcal{C}(\mathbb{T})) \subseteq \mathcal{L}^{\mathcal{C}(\mathbb{T})}_p(\mathcal{F}_t)$ for each $t\in[s',t']$ and that $\mathcal{E}^{n,1}_{s',s'}[f] = 0$. Hence, we only have to check that $\mathcal{E}^{n,1}_{s',\cdot}[f]$ satisfies (3.7) (3.8) with some constants K_1 and K_2. Notice that (3.7) trivially holds with $K_1 = 2\|f\|_{\mathbb{B}}$. Concerning (3.8), we have by a simple application of the conditional Jensen and triangle inequalities

$$\begin{aligned}
&\|\mathbb{E}^s(\mathcal{E}^{n,1}_{s',t}[f] - \mathcal{P}_{t-s}\mathcal{E}^{n,1}_{s',s}[f] - A_{s,t})(x)\|_{L_p(\Omega)}\\
&\quad\le \int_s^t \|\mathcal{P}_{t-r}(f(u_r) - f(O_r + \phi_{s,r})(x)\|_{L_p(\Omega)}\,dr\\
&\qquad+ \int_s^t \|\mathcal{P}_{t-r} * (f(u^n_r) - f(O^n_r + \phi^n_{s,r})(x)\|_{L_p(\Omega)}\,dr.
\end{aligned}$$

Since

$$\begin{aligned}
u_r - (O_r + \phi_{s,r}) &= \int_s^r \mathcal{P}_{r-\theta}(b(u_\theta) - \mathbb{E}^s b(u_\theta))\,d\theta,\\
u^n_r - (O^n_r + \phi^n_{s,r}) &= \int_s^r \mathcal{P}_{\kappa_n(r-\theta)}(b(u^n_{\kappa_n(\theta)}) - \mathbb{E}^s b(u^n_{\kappa_n(\theta)}))\,d\theta,
\end{aligned}$$

it is then clear that (3.8) is satisfied with $K_2 = 4\|f\|_{\mathcal{C}^1}\|b\|_{\mathbb{B}}$, and therefore the claim is proved.

It only remains to remove the additional Lipschitz assumption on f. To this end take a smooth approximation of f, for instance $f_m := \mathcal{P}^{\mathbb{R}}_{1/m} f$. Then $f_m \to f$ almost everywhere and $\|f_m\|_{\mathbb{B}} \le \|f\|_{\mathbb{B}}$. From Girsanov's theorem (see e.g. [DPZ92, Thm 10.14] for a sufficiently general version) we have that for all the law of $u_r(x)$ and that of $\mathcal{P}_r\psi(x) + O_r(x)$ are mutually absolutely continuous, and therefore for $r>0$, $x\in\mathbb{T}$, the law of $u_r(x)$ is absolutely continuous with respect to the Lebesque measure. The same holds for $u^n_r(x)$. Therefore, for fixed s,t,x,n, $\mathcal{E}^{n,1}_{s,t}[f_m](x) \to \mathcal{E}^{n,1}_{s,t}[f](x)$ almost surely, and so an application of Fatou's lemma finishes the proof. □

3.4 Estimate for $\mathcal{E}^{n,2}$

The purpose of this section is to provide the estimate for the term $\mathcal{E}^{n,2}_{s,t}$ in the decomposition (3.3). We set

$$v^n_t(x) = \mathcal{P}^n_t\psi^n(x) + O^n_t(x).$$

Lemma 3.4.1. *Under the assumption of Theorem 1.1.2, for any $p > 0$ there exists a constant $N = N(p, \varepsilon, c, K)$ such that for all $g \in \mathbb{B}$, and all $0 \le s \le t \le 1$, $n \in \mathbb{N}$, one has the bound*

$$\sup_{x\in\mathbb{T}} \Big\| \int_s^t p_{t-r} * (g(v_r^n))(x) - p_{t-r} *_n (g(v^n_{\kappa_n(r)}))(x)\, dr \Big\|_{L_p(\Omega)} \le N\|g\|_{\mathbb{B}} n^{-1+3\varepsilon}|t-s|^{1/2+\varepsilon/2}. \tag{3.31}$$

Proof. It clearly suffices to prove the claim for $p \ge 2$ and $\|g\|_{\mathbb{B}} = 1$. Define for $(s,t) \in [s',t']_<$ we define an element $A_{s,t} \in L^p_w(\Omega, \mathcal{F}_t; \mathcal{C}(T)) \subseteq \mathcal{L}_p^{\mathcal{C}(\mathbb{T})}(\mathcal{F}_t)$ by

$$A_{s,t}(x) = \mathbb{E}^s \int_s^t \int_{\mathbb{T}} p_{t-r}(x-y)(g(v_r^n(y)) - g(v^n_{\kappa_n(r)}(\rho_n(y)))\, dy\, dr.$$

We aim to verify the conditions of Theorem 3.2.1. It is easy to check that

$$\mathbb{E}^s \delta_{\mathcal{P}} A_{s,u,t}(x) = \mathbb{E}^s(A_{s,t}(x) - \mathcal{P}_{t-u}A_{s,u}(x) - A_{u,t}(x)) = 0.$$

This shows that (3.6) is satisfied with $C_2 = 0$.

Moving on to (3.5), we separate two cases. When $t \ge s + 2h$, we write

$$A_{s,t}(x) = I_1 + I_2 := \Big(\int_s^{s+2h} + \int_{s+2h}^t\Big)\int_{\mathbb{T}} p_{t-r}(x,y)\mathbb{E}^s(g(v_r^n(y)) - g(v^n_{\kappa_n(r)}(\rho_n(y)))\, dy\, dr.$$

For $r \in [s+2h, t]$ we have $\kappa_n(r) \ge s$. Therefore, we have by (2.49)

$$\begin{aligned} J_{r,s}(y) &:= \mathbb{E}^s(g(v_r^n(y)) - g(v^n_{\kappa_n(r)}(\rho_n(y))) \\ &= \mathcal{P}^{\mathbb{R}}_{Q^n(r-s)} g(\mathcal{P}^n_r\psi^n(y) + \widehat{O}^n_{s,r}(y)) - \mathcal{P}^{\mathbb{R}}_{Q^n(\kappa_n(r)-s)} g(\mathcal{P}^n_{\kappa_n(r)}\psi^n(\rho_n(y)) + \widehat{O}^n_{s,\kappa_n(r)}(\rho_n(y))). \end{aligned}$$

Applying (2.16) for the outer heat kernels with $\alpha = 0$ and $\beta = 1$ we have

$$\begin{aligned} |J_{r,s}(y)| \lesssim \Big(&|\mathcal{P}^n_r\psi^n(y) - \mathcal{P}^n_{\kappa_n(r)}\psi^n(\rho_n(y))| + |\widehat{O}^n_{s,r}(y) - \widehat{O}^n_{s,\kappa_n(r)}(\rho_n(y))| \\ &+ |Q^n(r-s) - Q^n(\kappa_n(r)-s)|^{1/2}\Big)|Q^n(\kappa_n(r)-s)|^{-1/2}. \end{aligned} \tag{3.32}$$

For the first term we apply Lemma 2.2.5 (with $\beta = 1$, $\alpha = 1/2 - \varepsilon$) to get that

$$|\mathcal{P}^n_r\psi^n(y) - \mathcal{P}^n_{\kappa_n(r)}\psi^n(\rho_n(y))| \lesssim n^{-1+\varepsilon} r^{-1/4-\varepsilon/2}. \tag{3.33}$$

For the second term on the right hand side of (3.32)we first write

$$\begin{aligned} \widehat{O}^n_{s,r}(y) - \widehat{O}^n_{s,\kappa_n(r)}(\rho_n(y)) = (\widehat{O}^n_{s,r}(y) - \mathcal{P}_{r-s}O_s(y)) + (\mathcal{P}_{\kappa_n(r)-s}O_s(\rho_n(y)) - \widehat{O}^n_{s,\kappa_n(r)}(\rho_n(y))) \\ + (\mathcal{P}_{r-s}O_s(y) - \mathcal{P}_{\kappa_n(r)-s}O_s(\rho_n(y))). \end{aligned}$$

The $L_p(\Omega)$ norms of the first two terms are readily bounded by (2.50) (with $\beta = 2$). As for the third, by (2.16) (with $\beta = 1$ and $\alpha = 1/2 - \varepsilon$) and Proposition 2.3.1 we obtain

$$\begin{aligned} &\|\mathcal{P}_{r-s}O_s(y) - \mathcal{P}_{\kappa_n(r)-s}O_s(\rho_n(y))\|_{L_p(\Omega)} \\ &\quad\lesssim \|O_s\|_{L_p(\Omega;C^{1/2-\varepsilon})}(|y-\rho_n(y)| + |r-\kappa_n(r)|^{1/2})|\kappa_n(r)-s|^{-1/4-\varepsilon/2} \\ &\quad\lesssim n^{-1}|r-s|^{-1/4-\varepsilon/2}, \end{aligned}$$

where in the last line we used that $|r-s| \lesssim |\kappa_n(r)-s|$ for $r \in [s+2h,t]$. Consequently, we have

$$\|\widehat{O}^n_{s,r}(y) - \widehat{O}^n_{s,\kappa_n(r)}(\rho_n(y))\|_{L_p(\Omega)} \lesssim n^{-1}|r-s|^{-1/4-\varepsilon/2}. \tag{3.34}$$

Next, by (2.55) (with $\beta = 1$) we have

$$|Q_n(r-s) - Q_n(\kappa_n(r)-s)|^{1/2} \lesssim n^{-1}|r-s|^{-1/4}. \tag{3.35}$$

Finally, by (2.54) we have

$$|Q_n(\kappa_n(r)-s)|^{-1/2} \lesssim |r-s|^{-1/4}. \tag{3.36}$$

Substituting the bounds (3.33)-(3.34)-(3.35)-(3.36) into (3.32), we get

$$\|J_{r,s}(y)\|_{L_p(\Omega)} \lesssim n^{-1+\varepsilon}|r-s|^{-1/2-\varepsilon/2} \lesssim n^{-1+3\varepsilon}|r-s|^{-1/2+\varepsilon/2}.$$

It remains to integrate with respect to y and r to get

$$\|I_2\|_{L_p(\Omega)} \lesssim n^{-1+3\varepsilon}|t-s|^{1/2+\varepsilon/2}.$$

The term I_1 is trivial: by using the boundedness of g we get

$$\|I_1\|_{L_p(\Omega)} \lesssim h \lesssim n^{-1+3\varepsilon}|t-s|^{1/2+3\varepsilon/2} \lesssim n^{-1+3\varepsilon}|t-s|^{1/2+\varepsilon/2}.$$

Consequently, for $t \geq s+2h$ we have shown that

$$\sup_{x\in\mathbb{T}} \|A_{s,t}(x)\|_{L_p(\Omega)} \lesssim n^{-1+3\varepsilon}|t-s|^{1/2+\varepsilon/2}. \tag{3.37}$$

In addition, note that (3.37) also holds in the $t \in [s, s+2h]$ case, since it follows from the trivial bound $\|A_{s,t}(x)\|_{L_p(\Omega)} \lesssim |t-s|$. We can conclude that that (3.5) is satisfied with $C_1 = Nn^{-1+3\varepsilon}$ and $\varepsilon_1 = \varepsilon/2$. Therefore, Theorem 3.2.1 applies.

Let us now define

$$\mathcal{A}_t := \int_0^t \int_{\mathbb{T}} p_{t-r}(x-y)(g(O^n_r(y)) - g(O^n_{\kappa_n(r)}(\rho_n(y)))\,dy\,dr.$$

By the adaptedness of O^n and the fact that $\|g\|_{\mathbb{B}} = 1$ it is obvious that $\mathcal{A}_t \in L^w_p(\Omega, \mathcal{F}_t; \mathcal{C}(\mathbb{T})) \subseteq \mathcal{L}^{\mathcal{C}(\mathbb{T})}_p(\mathcal{F}_t)$ for all $t \in [0,1]$. Moreover, since $\|g\|_{\mathbb{B}} = 1$, it is obvious that it satisfies (3.7) with $K_1 = 4$. In addition, by definition, it satisfies (3.8) with $K_2 = 0$. It is therefore the unique process with this properties and the desired estimate now follows from (3.9). ☐

Corollary 3.4.2. *Under the assumption of Theorem 1.1.2, for any $p > 0$ there exists a constant $N = N(p, \varepsilon, c, \|b\|_{\mathbb{B}}, K)$ and all $0 \leq s \leq t \leq 1$, $n \in \mathbb{N}$, one has the bound*

$$\sup_{x\in\mathbb{T}} \|\mathcal{E}^{n,2}_{s,t}(x)\|_{L_p(\Omega)} \leq Nn^{-1+3\varepsilon}|t-s|^{1/2+\varepsilon/2}. \tag{3.38}$$

Proof. Fix $p \in (0,\infty)$, $x \in \mathbb{T}$, $0 \leq s \leq t \leq 1$, $n \in \mathbb{N}$. For any random field $(Y_t(x))_{(t,x)\in[0,1]\times\mathbb{T}}$ denote the random variable

$$h(Y) = \int_s^t p_{t-r} * b(Y_r)(x) - p_{t-r} *_n b(Y_{\kappa_n(r)})(x)\,dr.$$

From Girsanov's theorem, as in Lemma 2.3.6, we have

$$\begin{aligned}\mathbb{E}|h(u^n)|^p = \tilde{\mathbb{E}}(|h(u^n)|^p\rho^{-1}) &\leq (\tilde{\mathbb{E}}|h(u^n)|^{2p})^{1/2}(\tilde{\mathbb{E}}\rho^{-2})^{1/2} \\ &= (\mathbb{E}|h(v^n)|^{2p})^{1/2}(\mathbb{E}\rho^{-1})^{1/2} \\ &\lesssim (n^{-1+3\varepsilon}|t-s|^{1/2+\varepsilon/2})^p,\end{aligned}$$

where $\tilde{\mathbb{E}}$ denotes the expectation under the measure $\tilde{\mathbb{P}}$ defined in (2.59) and $\rho := d\mathbb{P}/d\tilde{\mathbb{P}}$. To get the last line we used Lemma 3.4.1 with $2p$ in place of p and b in place of g. ☐

3.5 Proof of Theorem 1.1.2

As indicated in Section 3.1, we first aim to derive a buckling inequality for $\mathcal{E}^n$. In the decomposition (3.3) the only term not treated so far is $\mathcal{E}^{n,3}$, for which however it is easy to see the almost sure bound

$$\sup_{x\in\mathbb{T}} \mathcal{E}^{n,3}_{s,t}(x) \lesssim n^{-1/2}|t-s|^{1/2}. \tag{3.39}$$

Indeed, when $|t-s| \leq h$, then simply using the boundedness of b yields a bound of order $|t-s|$, which even implies a bound of order $n^{-1}|t-s|^{1/2}$. In the regime $|t-s| \geq h$ we split the integral into two as usual, and the trivial estimates

$$\Big|\int_{t-h}^{t} \Big(p_{t-r} *_n b(u^n_{\kappa_n(r)}) - p^n_{\kappa_n(t-r)} *_n b(u^n_{\kappa_n(r)})\Big)\,dr\Big| \lesssim n^{-2},$$
$$\Big|\int_{s}^{t-h} \Big(p_{t-r} *_n b(u^n_{\kappa_n(r)}) - p^n_{\kappa_n(t-r)} *_n b(u^n_{\kappa_n(r)})\Big)\,dr\Big| \lesssim \int_{s}^{t-h} \|p_{t-r} - p^n_{\kappa_n(t-r)}\|_{L_1(\mathbb{T})}\,dr$$

indeed imply (3.39), using (2.33) (with $\beta = 1$) to bound the last integral.

Fix $\tau = 3/8$. Denote briefly by $\tilde{K}$ the constant N obtained from Lemma 2.3.6. When we apply below Lemma 3.3.2 and Corollary 3.4.2, we do so with $\tilde{K}$ in place of K. For a parameter $N_0 > 0$ to be specified later take $S \in \Lambda_n \cap [N_0/2, N_0]$, which is certainly possible for large enough n. Putting together (3.14), (3.38), (3.39), and the trivial inequality $\|\mathcal{E}^n_{0,\cdot}\|_{\mathscr{C}^0_p[0,S]} \leq [\mathcal{E}^n]_{\mathscr{C}^{3/8}_p[0,S]_<}$, we have for all $(s,t) \in [0,S]_<$

$$\sup_{x\in\mathbb{T}} \|\mathcal{E}^n_{s,t}\|_{L_p(\Omega)} \leq N_1\big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + [\mathcal{E}^n]_{\mathscr{C}^{3/8}_p[0,S]_<}\big)|t-s|^{1/2}, \tag{3.40}$$

where the constant N_1 does not depend on n or N_0 (in our usual notation, $N_1 \lesssim 1$). Dividing by $|t-s|^{3/8}$ and taking supremum over $(s,t) \in [0,S]_<$, we get

$$[\mathcal{E}^n]_{\mathscr{C}^{3/8}_p[0,S]_<} \leq N_1\big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)} + [\mathcal{E}^n]_{\mathscr{C}^{3/8}_p[0,S]_<}\big)S^{1/8}.$$

We now fix $N_0 = (2N_1)^{-8}$. Since $S \leq N_0$, the inequality buckles and we get

$$[\mathcal{E}^n]_{\mathscr{C}^{3/8}_p[0,S]_<} \leq N_1\big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)}\big). \tag{3.41}$$

Returning to the main error, we have

$$u_t(x) - u^n_t(x) = (\mathcal{P}_t\psi(x) - \mathcal{P}_t\psi^n(x)) + (\mathcal{P}_t\psi^n(x) - \mathcal{P}^n_t\psi^n(x)) + \mathcal{E}^n_{0,t}(x) + (O_t(x) - O^n_t(x)).$$

These terms are bounded by trivially, (2.44), (3.41), and (2.50), respectively. This gives the bound

$$\sup_{t,x\in[0,S]\times\mathbb{T}} \|u_t(x) - u^n_t(x)\|_{L_p(\Omega)} \leq N_2\big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)}\big)$$

with another constant $N_2 \lesssim 1$. Repeating the same argument on $[S, 2S]$, with viewing u_S and u^n_S as initial conditions, we get

$$\begin{aligned}\sup_{t,x\in[S,2S]\times\mathbb{T}} \|u_t(x) - u^n_t(x)\|_{L_p(\Omega)} &\leq N_2\big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|u_S(x) - u^n_S(x)\|_{L_p(\Omega)}\big) \\ &\leq (N_2^2 + N_2)\big(n^{-1/2+\varepsilon} + \sup_{x\in\mathbb{T}} \|\psi(x) - \psi^n(x)\|_{L_p(\Omega)}\big).\end{aligned}$$

By iterating the argument at most $2/N_0$ times and recalling that N_0 does not depend on n, the proof is finished. $\square$

References

[ABLM20] S. ATHREYA, O. BUTKOVSKY, K. LÊ, and L. MYTNIK. Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation. *ArXiv e-prints* (2020). http://arxiv.org/abs/2011.13498v1.

[ACQ20] R. ANTON, D. COHEN, and L. QUER-SARDANYONS. A fully discrete approximation of the one-dimensional stochastic heat equation. *IMA Journal of Numerical Analysis* **40**, no. 1, (2020), 247–284. http://dx.doi.org/10.1093/imanum/dry060.

[BDG21] O. BUTKOVSKY, K. DAREIOTIS, and M. GERENCSÉR. Approximation of SDEs: a stochastic sewing approach. *Probability Theory and Related Fields* (2021). http://dx.doi.org/10.1007/s00440-021-01080-2.

[BG19] C.-E. BRÉHIER and L. GOUDENÈGE. Analysis of some splitting schemes for the stochastic Allen-Cahn equation. *Discrete & Continuous Dynamical Systems - B* **24**, no. 8, (2019), 4169–4190. http://dx.doi.org/10.3934/dcdsb.2019077.

[BGJK17] S. BECKER, B. GESS, A. JENTZEN, and P. E. KLOEDEN. Strong convergence rates for explicit space-time discrete numerical approximations of stochastic Allen-Cahn equations. *arXiv e-prints* arXiv:1711.02423. http://arxiv.org/abs/1711.02423.

[CGS11] L. H. Y. CHEN, L. GOLDSTEIN, and Q.-M. SHAO. *Normal approximation by Stein's method*. Probability and its Applications (New York). Springer, Heidelberg, 2011. http://dx.doi.org/10.1007/978-3-642-15007-4.

[Deb10] A. DEBUSSCHE. Weak approximation of stochastic partial differential equations: the nonlinear case. *Mathematics of Computation* **80**, no. 273, (2010), 89–117. http://dx.doi.org/10.1090/s0025-5718-2010-02395-6.

[DG00] A. M. DAVIE and J. G. GAINES. Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations. *Mathematics of Computation* **70**, no. 233, (2000), 121–135. http://dx.doi.org/10.1090/s0025-5718-00-01224-2.

[DGL21] K. DAREIOTIS, M. GERENCSÉR, and K. LÊ. Quantifying a convergence theorem of gyöngy and krylov. *ArXiv e-prints* (2021). http://arxiv.org/abs/2101.12185.

[DPZ92] G. DA PRATO and J. ZABCZYK. *Stochastic equations in infinite dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992. http://dx.doi.org/10.1017/CBO9780511666223.

[Ger20] M. GERENCSÉR. Regularisation by regular noise. *arXiv e-prints* (2020). http://arxiv.org/abs/2009.08418.

[GK96] I. GYÖNGY and N. KRYLOV. Existence of strong solutions for Itô's stochastic equations via approximations. *Probability Theory and Related Fields* **105**, no. 2, (1996), 143–158. http://dx.doi.org/10.1007/BF01203833.

[GP93a] I. GYÖNGY and E. PARDOUX. On quasi-linear stochastic partial differential equations. *Probability Theory and Related Fields* **94**, no. 4, (1993), 413–425. http://dx.doi.org/10.1007/BF01192556.

[GP93b] I. GYÖNGY and E. PARDOUX. On the regularization effect of space-time white noise on quasi-linear parabolic partial differential equations. *Probability Theory and Related Fields* **97**, no. 1-2, (1993), 211–229. http://dx.doi.org/10.1007/bf01199321.

[Gyö98] I. GYÖNGY. Lattice Approximations for Stochastic Quasi-Linear Parabolic Partial Differential Equations Driven by Space-Time White Noise I. *Potential Analysis* **9**, no. 1, (1998), 1–25. http://dx.doi.org/10.1023/a:1008615012377.

[Gyö99] I. GYÖNGY. Lattice Approximations for Stochastic Quasi-Linear Parabolic Partial Differential Equations driven by Space-Time White Noise II. *Potential Analysis* **11**, no. 1, (1999), 1–37. http://dx.doi.org/10.1023/a:1008699504438.

[JK08] A. JENTZEN and P. E. KLOEDEN. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space–time noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **465**, no. 2102, (2008), 649–667. http://dx.doi.org/10.1098/rspa.2008.0325.

[JK11] A. JENTZEN and P. KLOEDEN. *Taylor Approximations for Stochastic Partial Differential Equations*. SIAM, 2011.

[JM21] B. JOURDAIN and S. MENOZZI. Convergence Rate of the Euler-Maruyama Scheme Applied to Diffusion Processes with $L^Q - L^\rho$ Drift Coefficient and Additive Noise. *arXiv e-prints* (2021). http://arxiv.org/abs/2105.04860.

[Kru13] R. KRUSE. *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations*. Springer International Publishing, 2013.

[Lê20] K. LÊ. A stochastic sewing lemma and applications. *Electron. J. Probab.* **25**, (2020), Paper No. 38, 55. http://dx.doi.org/10.1214/20-ejp442.

[Lê21] K. LÊ. Stochastic sewing in Banach space. *arXiv e-prints* (2021). http://arxiv.org/abs/2105.09364.

[LQ19] Z. LIU and Z. QIAO. Strong approximation of monotone stochastic partial differential equations driven by white noise. *IMA Journal of Numerical Analysis* **40**, no. 2, (2019), 1074–1093. http://dx.doi.org/10.1093/imanum/dry088.

[LS21] X.-M. LI and J. SIEBER. Mild Stochastic Sewing Lemma, SPDE in Random Environment, and Fractional Averaging. *arXiv e-prints* (2021). http://arxiv.org/abs/2108.05573.

[MY20] T. MÜLLER-GRONBACH and L. YAROSLAVTSEVA. Sharp lower error bounds for strong approximation of sdes with discontinuous drift coefficient by coupling of noise. *ArXiv e-prints* (2020). http://arxiv.org/abs/2010.00915v1.

[NS20] A. NEUENKIRCH and M. SZÖLGYENYI. The euler–maruyama scheme for SDEs with irregular drift: convergence rates via reduction to a quadrature problem. *IMA Journal of Numerical Analysis* (2020). http://dx.doi.org/10.1093/imanum/draa007.

[Pri01] J. PRINTEMS. On the discretization in time of parabolic stochastic partial differential equations. *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique* **35**, no. 6, (2001), 1055–1078.

[PS05] R. PETTERSSON and M. SIGNAHL. Numerical approximation for a white noise driven SPDE with locally bounded drift. *Potential Analysis* **22**, no. 4, (2005), 375–393. http://dx.doi.org/10.1007/s11118-004-1329-4.

[Sha99] T. SHARDLOW. Numerical methods for stochastic parabolic PDEs. *Numerical Functional Analysis and Optimization* **20**, no. 1-2, (1999), 121–145. http://dx.doi.org/10.1080/01630569908816884.

[Szö21] M. SZÖLGYENYI. Stochastic differential equations with irregular coefficients: mind the gap! *Internationale Mathematische Nachrichten* **246**, (2021), 43–56.

[Tag20] D. TAGUCHI. A generalized Avikainen's estimate and its applications. *arXiv e-prints* arXiv:2001.05608. http://arxiv.org/abs/2001.05608.

[Wan20] X. WANG. An efficient explicit full-discrete scheme for strong approximation of stochastic allen–cahn equation. *Stochastic Processes and their Applications* **130**, no. 10, (2020), 6271–6299. http://dx.doi.org/10.1016/j.spa.2020.05.011.

[Yar21] L. YAROSLAVTSEVA. An adaptive strong order 1 method for SDEs with discontinuous drift coefficient. *arXiv e-prints* arXiv:2102.13155. http://arxiv.org/abs/2102.13155.