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Quantisations of the Volterra hierarchy

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Abstract

In this paper, we explore a recently emerged approach to the problem of quantisation based on the notion of quantisation ideals. We explicitly prove that the nonabelian Volterra together with the whole hierarchy of its symmetries admits a deformation quantisation. We show that all odd-degree symmetries of the Volterra hierarchy admit also a non-deformation quantisation. We discuss the quantisation problem for periodic Volterra hierarchy including their quantum Hamiltonians, central elements of the quantised algebras, and demonstrate super-integrability of the quantum systems obtained. We show that the Volterra system with period 3 admits a bi-quantum structure, which can be regarded as a quantum deformation of its classical bi-Hamiltonian structure.

Keywords The quantum Volterra equation · Quantum integrability · Super integable systems · Non-deformation quantisation · Quantised algebra

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1 Introduction

The problem of quantisation has a century long history. In 1925, inspired by Heisenberg's commutation relations between coordinates and momenta [1], namely,

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$$\hat{q}_n \hat{p}_m - \hat{p}_m \hat{q}_n = i\hbar\delta_{n,m}, \quad \hat{q}_n \hat{q}_m - \hat{q}_m \hat{q}_n = 0,$$
$$\hat{p}_n \hat{p}_m - \hat{p}_m \hat{p}_n = 0, \quad n, m = 1, \ldots, N, \tag{1}$$

Dirac proposed the concept of *quantum algebra* and noticed that in the limit $\hbar \to 0$ the commutators of observables are proportional to their Poisson brackets in classical mechanics $[\hat{q}_n, \hat{p}_m] \to i\hbar\{q_n, p_m\}$. He raised the issue of consistency of the commutation relations (1) with each other and with the equations of motion for a finite Plank constant $\hbar \neq 0$ [2]. In fact, Dirac proposed the problem of non-commutative deformations of multiplication on Poisson manifolds that is presently an active research area. Important results in this direction have been obtained by Kontsevich [3]. Witten, in his recent lectures [4], pointed out that due to "the operator ordering problem, there is no natural, general procedure to quantise a classical system", and described some partial remedies to this problem. The general problem of quantisation is still open.

Recently, a fresh approach to the quantisation problem was proposed in [5]. It is proposed to start from a dynamical system defined on a free associative algebra $\mathfrak{A}$ with a finite or infinite number of multiplicative generators. The dynamical system defines a derivation $\partial_t : \mathfrak{A} \mapsto \mathfrak{A}$. By quantisation, it is understood a reduction in the dynamical system on $\mathfrak{A}$ to the system defined on a quotient algebra $\mathfrak{A}_\mathfrak{I} = \mathfrak{A} / \mathfrak{I}$ over a two-sided ideal $\mathfrak{I} \subset \mathfrak{A}$ satisfying the following properties:

(i) the ideal $\mathfrak{I}$ is ∂_t–stable, that is, $\partial_t(\mathfrak{I}) \subset \mathfrak{I}$;
(ii) the quotient algebra $\mathfrak{A}_\mathfrak{I}$ admits an additive basis of normally ordered monomials.

In [5], an ideal satisfying the above two conditions is called a *quantisation ideal* , and $\mathfrak{A}_\mathfrak{I}$ is called a *quantised algebra*.

The condition (i) is crucial. The reduction in a dynamical system corresponding to the derivation ∂_t to the quotient algebra $\mathfrak{A}_\mathfrak{I}$ is well defined if and only if the ideal is ∂_t–stable.

The second condition (ii) enables one to define commutation relations between any two elements of the quotient algebra and uniquely represent elements of $\mathfrak{A}_\mathfrak{I}$ in the basis of normally ordered monomials (similar to a normal ordering in quantum physics). Finitely generated algebras, admitting a Poincaré–Birkhoff–Witt basis, and their quotients, satisfy the condition (ii). They have a wide range of applications, and share some properties with the commutative polynomial rings (see [6, 7] and references in).

Any finitely generated associative algebra can be presented as (is isomorphic to) a quotient of a free associative algebra over a suitable two-sided ideal. For example, Dirac's quantum algebra is a quotient of the free algebra $\mathbb{C}\langle q_1, p_1, \ldots, q_N, p_N\rangle$ over the two-sided ideal generated by the commutation relations (1).

We emphasise that quantisation proposed in [5] guarantees the consistency of the "commutation relations" with each other and with the equations of motion (resolving the issue raised by Dirac) and the associativity of the non-commutative multiplication in the quantised algebra (which potentially could be an issue in the deformation quantisation). This new approach also results in examples of non-deformation quantisations.

In order to apply this method of quantisation to a classical dynamical system with commutative variables, one needs to lift it to a system on a nonabelian free associative algebra. Such lifting is not unique (on the quantum level it has been noted already by

Dirac [2] and highlighted by Witten in his lectures [4]). The guiding principle here is to preserve the most important properties of the classical system in the lifted one. For example, integrable systems admit hierarchies of symmetries and we would like to have this property for the corresponding systems defined on a free associative algebras and for the quantised systems as well. Fortunately many integrable systems admit such liftings [8–12], and can be quantised by the method proposed in [5]. Recently, the hierarchies of stationary Korteweg de–Vries equation and Novikov's equations have been quantised using the method of quantisation ideals [13].

In this paper, we study the quantisation problem for the integrable nonabelian Volterra system

$$\partial_{t_1}(u_n) = \varrho K^{(1)}, \quad K^{(1)} = u_{n+1}u_n - u_n u_{n-1}, \quad n \in \mathbb{Z} \tag{2}$$

and its hierarchy of symmetries. Here $\varrho \in \mathbb{C}$ is a constant which can be set to be equal to 1 by the re-scaling $u_n \to \varrho u_n$. In the classical (commutative) case system (2) was introduced by Zakharov et al. for the description of the fine structure of the spectra of Langmuir oscillations in a plasma [14]. Its integrability and Lax representation were discovered by Manakov [15] and independently by Kac and van Moerbeke [16]. The nonabelian version of the system (2), with variables $u_n(t_1)$ taking values in a free associative algebra, was studied by Bogoyavlensky [17].

The Volterra system (2) is the first member of the infinite hierarchy of commuting symmetries

$$\partial_{t_\ell}(u_n) = K^{(\ell)}(u_{n+\ell}, \dots, u_{n-\ell}), \quad \ell = 1, 2, \dots, \quad n \in \mathbb{Z},$$

where $K^{(\ell)}(u_{n+\ell}, \dots, u_{n-\ell})$ are homogeneous polynomials of degree $\ell + 1$ which can be found explicitly [12]. The second member of the hierarchy

$$\begin{aligned}\partial_{t_2}(u_n) = K^{(2)} &= u_{n+2}u_{n+1}u_n + u_{n+1}^2 u_n + u_{n+1}u_n^2 \\ &\quad - u_n^2 u_{n-1} - u_n u_{n-1}^2 - u_n u_{n-1} u_{n-2}\end{aligned} \tag{3}$$

is given by the cubic polynomial. It can be straightforwardly verified that $\partial_{t_2}(\partial_{t_1}(u_n)) = \partial_{t_1}(\partial_{t_2}(u_n))$, and thus, (3) is a cubic symmetry of (2).

In the new approach, the quantisation problem for equation (2) reduces to the problem of finding two-sided ideals in the free associative algebra $\mathfrak{A} = \mathbb{C}\langle u_n\,;\ n \in \mathbb{Z}\rangle$ generated by an infinite number of non-commuting variables such that the above conditions (i) and (ii) are satisfied. It is obvious that the ideal $\mathfrak{I}$ generated by the infinite set of polynomials

$$\mathfrak{I} = \langle u_n u_m - \omega_{n,m} u_m u_n\,;\ n, m \in \mathbb{Z}, \omega_{n,m} \in \mathbb{C}^*\rangle \tag{4}$$

satisfies the condition (ii) for any choice of the parameters $\omega_{n,m} = \omega_{m,n}^{-1}$. In [5], it was stated that the ideal $\mathfrak{I}$ satisfies the condition (i) if and only if

$$\omega_{n,n+1} = \omega_{n+1,n}^{-1} = \omega, \quad \omega_{n,m} = 1 \ \text{ if } \ |n - m| \geq 2.$$

Thus, the quantisation ideal suitable for the Volterra system (2) is

$$\mathfrak{I}_a=\langle\{u_n u_{n+1}-\omega u_{n+1}u_n\ ;\ n\in\mathbb{Z}\}\cup\{u_n u_m-u_m u_n\ ;\ |n-m|>1,\ n,m\in\mathbb{Z}\ \}\rangle, \quad (5)$$

leading to the commutation relations

$$u_n u_{n+1}=\omega u_{n+1}u_n,\quad u_n u_m=u_m u_n \text{ if } |n-m|\geq 2,\quad n,m\in\mathbb{Z} \quad (6)$$

in the quotient algebra $\mathfrak{A}\diagup\mathfrak{I}_a$. It was verified by direct computations that the ideal $\mathfrak{I}_a$ is invariant with respect to derivations defined by a few first symmetries of the Volterra hierarchy and conjectured that it is also true for the whole hierarchy. In this paper, we give an explicit proof for the above conjecture (Theorem 9). The ideal $\mathfrak{I}_a$ corresponds to a deformation quantisation. In the limit $\omega\to 1$, it leads to the classical commutative case.

It was claimed in [5] that the cubic symmetry of the Volterra system, Eq. (3), admits two distinct quantisations ideals of the form (4). The first one coincides with $\mathfrak{I}_a$ defined by (5), while the second one is

$$\mathfrak{I}_b=\langle\{u_n u_{n+1}-(-1)^n\omega u_{n+1}u_n\ ;\ n\in\mathbb{Z}\}\cup\{u_n u_m+u_m u_n\ ;\ |n-m|>1,\ n,m\in\mathbb{Z}\}\rangle\,. \quad (7)$$

Note that the quantisation corresponding to the ideal $\mathfrak{I}_b$ is not a deformation of a commutative or Grassmann algebra. It is a new and *non-deformation* quantisation of Eq. (3) with the commutation relations

$$u_n u_{n+1}=(-1)^n\omega u_{n+1}u_n,\quad u_n u_m+u_m u_n=0 \text{ if } |n-m|\geq 2,\quad n,m\in\mathbb{Z} \quad (8)$$

in the quotient algebra $\mathfrak{A}\diagup\mathfrak{I}_b$. The ideal $\mathfrak{I}_b$ given by (7) is not invariant with respect to the Volterra system (2), and thus, it is not suitable for its quantisation. In [5], it was claimed that the ideal $\mathfrak{I}_b$ is invariant with respect to a first few odd degree symmetries of the Volterra equation. In this paper, we prove that the ideal $\mathfrak{I}_b$ (7) is a quantisation ideal for all odd degree members of the Volterra hierarchy (Theorem 14).

In the quantum theory, we replace real valued commutative variables u_n by Hermitian elements. Their commutation relations are defined by the quantisation ideal, which should be stable with respect to the Hermitian conjugation (Definition 3). In the case of the ideals $\mathfrak{I}_a$ and $\mathfrak{I}_b$, it implies that $\omega=e^{2\mathrm{i}\hbar}$, where $\hbar$ is an arbitrary real parameter, an analogue of the Plank constant, and $\mathrm{i}^2=-1$. Moreover, in the quantised equations of the Volterra hierarchy, we should introduce the factors $e^{\mathrm{i}\ell\hbar}$ which make the right-hand side of the equations self-adjoint, that is,

$$\partial_{t_\ell}(u_n)=e^{\mathrm{i}\ell\hbar}K^{(\ell)}(u_{n+\ell},\ldots,u_{n-\ell}),\quad \ell=1,2,\ldots,\quad n\in\mathbb{Z}. \quad (9)$$

In the algebra $\mathfrak{A}_{\mathfrak{I}_a}$ with commutation relations (6), the quantised Volterra equation and its symmetry can be represented in the Heisenberg form

$$\partial_{t_1}(u_n) = e^{\mathrm{i}\hbar}K^{(1)} = \frac{\mathrm{i}}{2\sin(\hbar)}[H_1, u_n], \tag{10}$$

$$\partial_{t_2}(u_n) = e^{2\mathrm{i}\hbar}K^{(2)} = \frac{\mathrm{i}}{2\sin(2\hbar)}[H_2, u_n], \tag{11}$$

where

$$H_1 = \sum_{k\in\mathbb{Z}} u_k, \quad H_2 = \sum_{k\in\mathbb{Z}}(u_k^2 + u_{k+1}u_k + u_k u_{k+1}).$$

In the algebra $\mathfrak{A}_{\mathfrak{I}_b}$ with commutation relations (8), the first member of the quantised Volterra sub-hierarchy of odd degree symmetries has the same Heisenberg form (11). Moreover, in the case of the algebra $\mathfrak{A}_{\mathfrak{I}_b}$ we have $H_2 = H_1^2$, which is not true for the algebra $\mathfrak{A}_{\mathfrak{I}_a}$.

The quantisation of the Volterra system was studied by Volkov and Babelon in the frame of the quantum inverse scattering method [18, 19]. In the paper by Inoue and Hikami [20], the commutation relations (6) as well as a first few Hamiltonians of the classical and quantum Volterra hierarchy were found using ultra-local Lax representation and R–matrix technique. Our alternative approach does not rely on the existence of a Lax or Hamiltonian structures, and it enables us to reproduce the results presented in [20] and to find a non-deformation quantisation (8) for odd degree members of the Volterra hierarchy which is new and rather surprising.

The Volterra equation and its hierarchy admit periodic reductions with arbitrary positive integer period $M \in \mathbb{N}$. The periodic reduction is the identification $u_{n+M} = u_n$ for all $n \in \mathbb{Z}$. It reduces the infinite system of equations (2) to a system of M equations on a finitely generated free algebra $\mathfrak{A}_M = \mathbb{C}\langle u_1, \ldots, u_M\rangle$. The problem of quantisation of the periodic Volterra hierarchies is discussed in Section 4. In particular, we show that the Volterra system with period 3 admits bi-quantum structure, which is a quantum analogue of its bi-Hamiltonian structure in the classical case. In the case $M = 4$, we obtain three possible quantisations, and show that the obtained quantised systems are super-integrable, whose first integrals and central elements are explicitly presented.

2 Integrable nonabelian Volterra hierarchy

In this section we introduce some basic notations required for this paper and present the Volterra hierarchy on a free associative algebra in an explicit form.

Let $\mathfrak{A} = \mathbb{C}\langle u_n\,;\, n \in \mathbb{Z}\rangle$ be a free associative algebra generated by an infinite number of non-commuting variables. There is a natural automorphism $\mathcal{S} : \mathfrak{A} \mapsto \mathfrak{A}$, which we call the *shift operator*, defined as

$$\mathcal{S} : a(u_k, \ldots, u_r) \mapsto a(u_{k+1}, \ldots, u_{r+1}), \quad \mathcal{S} : \alpha \mapsto \alpha, \quad a(u_k, \ldots, u_r) \in \mathfrak{A},\ \alpha \in \mathbb{C}.$$

Thus, $\mathfrak{A}$ is a difference algebra. Let $\mathcal{T}$ denote the antiautomorphism of $\mathfrak{A}$ defined by

$$\mathcal{T}(u_k) = u_{-k}, \quad \mathcal{T}(a \cdot b) = \mathcal{T}(b) \cdot \mathcal{T}(a), \quad \mathcal{T}(\alpha) = \alpha, \quad a, b \in \mathfrak{A}, \quad \alpha \in \mathbb{C}.$$

The involution $\mathcal{T}$ is a composition of the reflection in the alphabet index $u_k \mapsto u_{-k}$ and the transposition of the monomials. For example:

$$\mathcal{T}(uu_1 + u_4u_1u_{-3}u_{-2}) = u_{-1}u + u_2u_3u_{-1}u_{-4}.$$

A derivation $\mathcal{D}$ of the algebra $\mathfrak{A}$ is a $\mathbb{C}$–linear map satisfying Leibniz's rule

$$\mathcal{D}(\alpha a + \beta b) = \alpha\mathcal{D}(a) + \beta\mathcal{D}(b), \quad \mathcal{D}(a \cdot b) = \mathcal{D}(a) \cdot b + a \cdot \mathcal{D}(b), \quad a, b \in \mathfrak{A}, \quad \alpha, \beta \in \mathbb{C}.$$

Thus, a derivation $\mathcal{D}$ can be uniquely defined by its action on the generators and $\mathcal{D}(\alpha) = 0,\ \alpha \in \mathbb{C}$.

A derivation $\mathcal{D}$ is called evolutionary if it commutes with the automorphism $\mathcal{S}$. An evolutionary derivation is completely characterised by its action on the generator u (we often write u instead of u_0), that is,

$$\mathcal{D}(u) = a \quad \text{and} \quad \mathcal{D}(u_k) = \mathcal{S}^k(a), \quad a \in \mathfrak{A}.$$

Thus, it is natural to adopt the notation $\mathcal{D}_a$, such that $\mathcal{D}_a(u) = a$, for an evolutionary derivation with the characteristic a. A commutator of evolutionary derivations $\mathcal{D}_a, \mathcal{D}_b$ is also the evolutionary derivation $[\mathcal{D}_a, \mathcal{D}_b] = \mathcal{D}_c$ with the characteristic $c = \mathcal{D}_a(b) - \mathcal{D}_b(a)$, which is called the Lie bracket of the elements a and b. Evolutionary derivations form a Lie subalgebra of the Lie algebra of derivations of $\mathfrak{A}$.

Assuming that the generators u_k depend on $t \in \mathbb{C}$, we can identify an evolutionary $\mathcal{D}_a$ with an infinite system of differential difference equations

$$\partial_t(u_n) = \mathcal{D}_a(u_n) = \mathcal{S}^n(a), \quad n \in \mathbb{Z}.$$

Therefore, we can say that $\partial_t(u) = a$ defines a derivation of $\mathfrak{A}$.

The Volterra system (2) defines the derivation $\partial_{t_1} : \mathfrak{A} \mapsto \mathfrak{A}$, which commutes with the automorphism and anti-commute with the involution $\mathcal{T}$, i.e.

$$\mathcal{S} \cdot \partial_{t_1} = \partial_{t_1} \cdot \mathcal{S}, \quad \mathcal{T} \cdot \partial_{t_1} = -\partial_{t_1} \cdot \mathcal{T}.$$

The differential difference system (3) defines another evolutionary derivation ∂_{t_2} commuting with $\mathcal{S}$ and anti-commuting with $\mathcal{T}$. Evolutionary derivations commuting with ∂_{t_1} are symmetries of the Volterra system. It can be straightforwardly verified that $[\partial_{t_1}, \partial_{t_2}] = 0$, and thus, Eq. (3) is a symmetry of the Volterra system.

It is well known that the Volterra system has an infinite hierarchy of commuting symmetries. They can be found using Lax representations both in commutative [15] and non-commutative [17] cases, or the recursion operators [12, 21]. Remarkably, the explicit expressions for generalised symmetries of the Volterra system (2) can be presented in terms of a family of nonabelian homogeneous difference polynomials

[12], which is inspired by the polynomials in the commutative case discovered in [22, 23].

Let us assume that the generators u_k of the free associative algebra $\mathfrak{A}$ depend on an infinite set of "times" $t_1, t_2, \ldots$. It follows from Casati and Wang [12] that the hierarchy of commuting symmetries of the Volterra system (2) can be written in the following explicit form

$$\partial_{t_\ell}(u) = \mathcal{S}(X^{(\ell)})u - u\mathcal{S}^{-1}(X^{(\ell)}), \quad \ell \in \mathbb{N}, \tag{12}$$

where the (noncommutative) polynomials $X^{(\ell)}$ are given by explicit formulae

$$X^{(\ell)} = \sum_{0\le\lambda_1\le\cdots\le\lambda_\ell\le\ell-1} \left(\prod_{j=1}^{\rightarrow \ell} u_{\lambda_j+1-j} \right). \tag{13}$$

Here $\prod_{j=1}^{\rightarrow\ell}$ denotes the order of the values j, from 1 to ℓ in the product of the non-commutative generators u_{λ_j+1-j}. For example, we have $X^{(1)} = u$ and

$$X^{(2)} = u_1 u + u^2 + u u_{-1}; \tag{14}$$

$$\begin{aligned} X^{(3)} &= u_2 u_1 u + u_1^2 u + u u_1 u + u_1 u^2 + u^3 + u u_{-1} u \\ &\quad + u_1 u u_{-1} + u^2 u_{-1} + u u_{-1}^2 + u u_{-1} u_{-2}. \end{aligned} \tag{15}$$

Note that $\mathcal{T}(X^{(\ell)}) = X^{(\ell)}$, and thus, we have $\mathcal{T} \cdot \partial_{t_\ell} = -\partial_{t_\ell} \cdot \mathcal{T}$ for all ℓ. Clearly, we get the Volterra equation (2) when $\ell = 1$ and the system (3) when $\ell = 2$.

3 Quantisation ideals of the Volterra equation and its symmetry

In this section, we prove the statements on quantisation ideals for the Volterra equation (2) itself and its symmetry (3) stated in [5].

Let $\mathfrak{I} \subset \mathfrak{A}$ be a two-sided ideal generated by the infinite set of polynomials $\mathfrak{f}_{i,j}$:

$$\mathfrak{I} = \langle \mathfrak{f}_{i,j}\,;\; i < j,\; i, j \in \mathbb{Z} \rangle, \quad \mathfrak{f}_{i,j} = u_i u_j - \omega_{i,j} u_j u_i, \tag{16}$$

where $\omega_{i,j} \in \mathbb{C}^*$ are arbitrary non-zero complex parameters. Given an ideal $\mathfrak{I}$, we denote the projection on the quotient algebra by $\pi_{\mathfrak{I}} : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$. The quotient algebra $\mathfrak{A}/\mathfrak{I}$ has an additive basis of *standard normally ordered monomials*

$$u_{i_1} u_{i_2} \cdots u_{i_n}\,; \quad i_1 \ge i_2 \ge \cdots \ge i_n,\; i_k \in \mathbb{Z},\; n \in \mathbb{N}.$$

Indeed, in $\mathfrak{A}/\mathfrak{I}$ any polynomial can be represented in this basis by recursive replacements $u_n u_m \to \omega_{n,m} u_m u_n$ if $m > n$ in the monomials. Thus, the condition (ii) for the ideal $\mathfrak{I}$ is satisfied. The condition (i) imposes constraints on the structure constants $\omega_{n,m}$ of the ideal.

Proposition 1 *The ideal* $\mathfrak{I}$ (16) *is invariant with respect to the Volterra dynamics* (2) *if and only if*

$$\omega_{n,n+1} = \omega_{0,1}, \quad \omega_{n,m} = 1 \text{ if } m - n \geq 2, \quad n, m \in \mathbb{Z}.$$

Denoting $\omega_{0,1} = \omega$, we arrive to the commutation relations (6) and the ideal $\mathfrak{I}_a$ given by (5).

Proof Let us differentiate $\mathfrak{f}_{i,j}$ $(i < j)$ by the derivation ∂_{t_1} associated with the Volterra equation (2). We have

$$\begin{aligned}\partial_{t_1}\left(\mathfrak{f}_{i,j}\right) = {} & u_{i+1}u_iu_j - u_iu_{i-1}u_j + u_iu_{j+1}u_j - u_iu_ju_{j-1} \\ & - \omega_{i,j}(u_{j+1}u_ju_i - u_ju_{j-1}u_i + u_ju_{i+1}u_i - u_ju_iu_{i-1}).\end{aligned}$$

We project this equation on the quotient algebra and require

$$\begin{aligned}0 = \pi_{\mathfrak{I}}\left(\partial_{t_1}(\mathfrak{f}_{i,j})\right) = {} & \omega_{i,j}(\omega_{i+1,j} - 1)u_ju_{i+1}u_i + \omega_{i,j}(1 - \omega_{i-1,j})u_ju_iu_{i-1} \\ & + \omega_{i,j}(\omega_{i,j+1} - 1)u_{j+1}u_ju_i + \omega_{i,j}(1 - \omega_{i,j-1})u_ju_{j-1}u_i, \end{aligned} \tag{17}$$

where we use the convention $\omega_{i,i} = 1$. When $j > i+2$, the four monomials $u_{j+1}u_ju_i$, $u_ju_iu_{i-1}$, $u_ju_{i+1}u_i$ and $u_ju_{j-1}u_i$ are linearly independent. Thus, $\pi_{\mathfrak{I}}\left(\partial_{t_1}(\mathfrak{f}_{i,j})\right) = 0$ if and only if all their coefficients vanish since $\omega_{i,j} \neq 0$. This leads to

$$\omega_{i+1,j} = \omega_{i-1,j} = \omega_{i,j+1} = \omega_{i,j-1} = 1.$$

Hence, we must have $\omega_{i,j} = 1$ whenever $i + 1 < j$. Using this result, it follows from (17) that

$$0 = \pi_{\mathfrak{I}}\left(\partial_{t_1}(\mathfrak{f}_{i,i+2})\right) = \omega_{i,i+2}(\omega_{i+1,i+2} - \omega_{i,i+1})u_{i+2}u_{i+1}u_i.$$

This implies that all the $\omega_{i,i+1}$ are equal to each other. Let $\omega = \omega_{i,i+1}$. It remains to check that (17) is valid for $j = i + 1$. Indeed,

$$\pi_{\mathfrak{I}}\left(\partial_{t_1}(\mathfrak{f}_{i,i+1})\right) = \omega(1 - \omega_{i-1,i+1})u_{i+1}u_iu_{i-1} + \omega(\omega_{i,i+2} - 1)u_{i+2}u_{i+1}u_i = 0,$$

and we proved the statement. □

Proposition 2 *The ideal* $\mathfrak{I}$ (16) *is invariant with respect to the dynamical system* (3), *i.e.* $\partial_{t_2}(u) = \mathcal{S}(X^{(2)})u - u\mathcal{S}^{-1}(X^{(2)})$ *only in two cases:*

(a) $\omega_{n,n+1} = \omega, \quad \omega_{n,m} = 1 \text{ if } m - n \geq 2, \quad n, m \in \mathbb{Z};$
(b) $\omega_{n,n+1} = (-1)^n\omega, \quad \omega_{n,m} = -1 \text{ if } m - n \geq 2, \quad n, m \in \mathbb{Z},$

where $\omega \in \mathbb{C}^*$ *is an arbitrary non-zero complex parameter.*

Thus, Eq. (3) admits the same quantisation $\mathfrak{A}/\mathfrak{I}_a$ (5) as the Volterra system. Additionally, it admits the quantisation with the ideal $\mathfrak{I}_b$ (7), which is not invariant with respect to the Volterra system (2). The latter quantisation is not a deformation of a commutative system.

Proof We differentiate $\mathfrak{f}_{i,j}$ $(i < j)$ by the derivation ∂_{t_2} defined by equation (3) and project on the quotient algebra. When $i + 2 \leq j$, we have

$$
\begin{aligned}
&\omega_{i,j}^{-1}\pi_{\mathfrak{J}}\left(\partial_{t_2}(\mathfrak{f}_{i,j})\right)\\
&\quad= (\omega_{i+1,j}\omega_{i+2,j}-1)u_ju_{i+2}u_{i+1}u_i + (\omega_{i+1,j}^2-1)u_ju_{i+1}^2u_i\\
&\qquad+(\omega_{i,j}\omega_{i+1,j}-1)u_ju_{i+1}u_i^2-(\omega_{i,j}\omega_{i-1,j}-1)u_ju_i^2u_{i-1}-(\omega_{i-1,j}^2-1)u_ju_iu_{i-1}^2\\
&\qquad-(\omega_{i-1,j}\omega_{i-2,j}-1)u_ju_iu_{i-1}u_{i-2}+(\omega_{i,j+1}\omega_{i,j+2}-1)u_{j+2}u_{j+1}u_ju_i\\
&\qquad+(\omega_{i,j+1}^2-1)u_{j+1}^2u_ju_i+(\omega_{i,j}\omega_{i,j+1}-1)u_{j+1}u_j^2u_i-(\omega_{i,j}\omega_{i,j-1}-1)u_j^2u_{j-1}u_i\\
&\qquad-(\omega_{i,j-1}^2-1)u_ju_{j-1}^2u_i-(\omega_{i,j-1}\omega_{i,j-2}-1)u_ju_{j-1}u_{j-2}u_i,
\end{aligned}
\tag{18}
$$

where we use the convention $\omega_{i,i}=1$. If $i+3<j$ all monomials in (18) are distinct and one deduces from $\pi_{\mathfrak{J}}\left(\partial_{t_2}(\mathfrak{f}_{i,j})\right)=0$ that

$$
\begin{aligned}
\omega_{i+1,j}\omega_{i+2,j} &= \omega_{i+1,j}^2=\omega_{i,j}\omega_{i+1,j}=\omega_{i,j}\omega_{i-1,j}=\omega_{i-1,j}^2=\omega_{i-1,j}\omega_{i-2,j}\\
&=\omega_{i,j+1}\omega_{i,j+2}=\omega_{i,j+1}^2=\omega_{i,j}\omega_{i,j+1}\\
&=\omega_{i,j}\omega_{i,j-1}=\omega_{i,j-1}^2=\omega_{i,j-1}\omega_{i,j-2}=1
\end{aligned}
$$

It follows that $\omega_{i,j}=\epsilon$ for all $i+1<j$ where $\epsilon=\pm1$. Next let us look at $\partial_{t_2}(\mathfrak{f}_{i,i+3})$. When $j=i+3$, (18) becomes

$$
\epsilon\pi_{\mathfrak{J}}\left(\partial_{t_2}(\mathfrak{f}_{i,i+3})\right)=\epsilon(\omega_{i+2,i+3}-\omega_{i,i+1})u_{i+3}u_{i+2}u_{i+1}u_i,
$$

which leads to $\omega_{i,i+1}=\omega_{i+2,i+3}$ for all $i\in\mathbb{Z}$. So the ideal is invariant under the automorphism $\mathcal{S}^2$. We now look at $\partial_{t_2}(\mathfrak{f}_{i,i+2})$. Substituting $j=i+2$ into (18), we get

$$
\begin{aligned}
\epsilon\pi_{\mathfrak{J}}\left(\partial_{t_2}(\mathfrak{f}_{i,i+2})\right)&=(\omega_{i+1,i+2}-\epsilon\omega_{i,i+1})u_{i+2}^2u_{i+1}u_i\\
&\quad+(\omega_{i+1,i+2}^2-\omega_{i,i+1}^2)u_{i+2}u_{i+1}^2u_i+(\epsilon\omega_{i+1,i+2}-\omega_{i,i+1})u_{i+2}u_{i+1}u_i^2,
\end{aligned}
$$

which vanishes if and only if $\omega_{i,i+1}=\epsilon\omega_{i+1,i+2}$. Combining all the constraints obtained on $\omega_{i,j}$, we obtain the two cases listed in the statement. Finally, we check

$$
\begin{aligned}
&\omega_{i,i+1}^{-1}\pi_{\mathfrak{J}}\left(\partial_{t_2}(\mathfrak{f}_{i,i+1})\right)=(\omega_{i,i+1}\epsilon-\omega_{i+1,i+2})u_{i+2}u_{i+1}^2u_i-(\omega_{i,i+1}\epsilon-\omega_{i-1,i})\\
&u_{i+1}u_i^2u_{i-1}=0.
\end{aligned}
$$

Thus, we complete the proof. □

In Sect. 5, we will show that every member of the Volterra hierarchy (12) admits the quantisation $\mathfrak{A}/\mathfrak{I}_a$ (Theorem 9) and that every even member of the Volterra hierarchy

$$
\partial_{t_{2\ell}}(u)=\mathcal{S}(X^{(2\ell)})u-u\mathcal{S}^{-1}(X^{(2\ell)}),\quad \ell\in\mathbb{N}
$$

also admits the quantisation $\mathfrak{A}/\mathfrak{I}_b$ (Theorem 14).

In the classical commutative case, the variables u_n are usually assumed to be real valued. Thus, in the quantum case they should be presented by self-adjoint operators with respect to the Hermitian conjugation $\dagger$.

Definition 3 The Hermitian conjugation $\dagger$ in algebra $\mathfrak{A}$ is defined by the following rules

$$u_n^\dagger = u_n, \quad \alpha^\dagger = \bar{\alpha}, \quad (a+b)^\dagger = a^\dagger + b^\dagger, \quad (ab)^\dagger = b^\dagger a^\dagger, \quad u_n, a, b \in \mathfrak{A}, \quad \alpha \in \mathbb{C},$$

where $\bar{\alpha}$ is the complex conjugate of $\alpha \in \mathbb{C}$.

The algebra $\mathfrak{A}$ is $\mathbb{Z}_2$-graded as a linear space. It can be represented as a direct sum of self-adjoint and anti-self-adjoint subspaces

$$\mathfrak{A} = \mathfrak{A}^+ \bigoplus \mathfrak{A}^-, \quad \mathfrak{A}^+ = \{a \in \mathfrak{A}\,;\, a^\dagger = a\}, \quad \mathfrak{A}^- = \{a \in \mathfrak{A}\,;\, a^\dagger = -a\}\,.$$

The Hermitian conjugation $\dagger$ can be extended to the quantised algebra $\mathfrak{A}/\mathfrak{I}$ if the ideal $\mathfrak{I}$ is $\dagger$-stable: $\mathfrak{I}^\dagger = \mathfrak{I}$.

Proposition 4 *The quantisation ideals $\mathfrak{I}_a$ (5) and $\mathfrak{I}_b$ (7) are $\dagger$–stable if and only if $\omega^\dagger = \omega^{-1}$.*

Proof Indeed, in the case of the ideal $\mathfrak{I}_a$ we have

$$\begin{aligned}&(u_n u_{n+1} - \omega u_{n+1} u_n)^\dagger = u_{n+1} u_n - \omega^\dagger u_n u_{n+1} = -\omega^\dagger (u_n u_{n+1} - (\omega^\dagger)^{-1} u_{n+1} u_n) \in \mathfrak{I}_a\\ &\Leftrightarrow \omega^\dagger = \omega^{-1}.\end{aligned}$$

In the case for $\mathfrak{I}_b$, the proof is similar. □

It suggests to represent $\omega = q^2$, $q = e^{i\hbar}$, where $\hbar \in \mathbb{R}$ is a real constant (an analog of the Plank constant). Thus $(u_{n+1}u_n)^\dagger = u_n u_{n+1} = q^2 u_{n+1} u_n$. The quantum Volterra hierarchy, which is consistent with the condition $u_n^\dagger = u_n$, can be presented in the form

$$u_{t_1} = q(u_1 u - u u_{-1}), \quad u_{t_\ell} = q^\ell \left(\mathcal{S}(X^{(2\ell)})u - u\mathcal{S}^{-1}(X^{(2\ell)})\right), \quad \ell \in \mathbb{N}. \tag{19}$$

Finally, we present the Volterra system and its first symmetry in the Heisenberg form in the quotient algebras. In the algebra $\mathfrak{A}/\mathfrak{I}_a$ with commutation relations (6), the Volterra Eq. (2) and its symmetry (3) can be represented in the Heisenberg form

$$\begin{aligned}\partial_{t_1}(u_n) &= \frac{1}{q^{-1} - q}[H_1, u_n], \quad H_1 = \sum_{k\in\mathbb{Z}} u_k;\\ \partial_{t_2}(u_n) &= \frac{1}{q^{-2} - q^2}[H_2, u_n], \quad H_2 = \sum_{k\in\mathbb{Z}} (u_k^2 + u_{k+1}u_k + u_k u_{k+1}),\end{aligned} \tag{20}$$

where H_1 and H_2 are self-adjoint algebraically independent and commuting Hamiltonians $[H_1, H_2] = 0$ in $\mathfrak{A}/\mathfrak{I}_a$.

The quantisation $\mathfrak{A}/\mathfrak{I}_b$ with commutation relations (8) also enables us to present Eq. (3) in the Heisenberg form

$$\partial_{t_2}(u_n) = \frac{1}{q^{-2} - q^2}[H_2, u_n]. \tag{21}$$

Note that in the quantised algebra $\mathfrak{A}/\mathfrak{I}_b$ we have $H_2 = H_1^2$ and $H_2^\dagger = H_2$.

4 Periodic Volterra hierarchy

In the Volterra system (2), we can assume that the function $u_n(t_1)$ is periodical in n with an integer period $M \in \mathbb{N}$, that is, $u_n = u_{n+M}$, $n \in \mathbb{Z}$. In this case, the infinite dimensional system (2) reduces to the M-dimensional dynamical system on $\mathfrak{A}_M = \mathbb{C}\langle u_1, \dots u_M\rangle = \mathfrak{A}/\mathcal{I}_M$, where the ideal $\mathcal{I}_M = \langle u_n - u_{n+M}\,;\ n \in \mathbb{Z}\rangle$. The ideal $\mathcal{I}_M$ is obviously stable with respect to evolutionary derivations. We can take u_n, $n = 1, \dots M$ as canonical representatives of the cosets $u_k + \mathcal{I}_M$, $k \in \mathbb{Z}$. The algebra $\mathfrak{A}_M$ is a difference algebra with the induced automorphism $\mathcal{S}(u_k) = u_{(k+1) \bmod M}$ of order M.

The hierarchy of symmetries (12) of the Volterra system (2) reduces to the hierarchy of symmetries of the M-periodic system provided we count the subscript k in u_k modulo M. The cases $M = 1, 2$ lead to trivial equations.

In the case $M = 3$, the periodic Volterra system takes the form

$$\partial_{t_1}(u_1) = u_2u_1 - u_1u_3, \quad \partial_{t_1}(u_2) = u_3u_2 - u_2u_1, \quad \partial_{t_1}(u_3) = u_1u_3 - u_3u_2. \tag{22}$$

It has an infinitely hierarchy of commuting symmetries:

$$\begin{aligned}
\partial_{t_2}(u_1) &= u_1^2u_3 + u_1u_3u_2 + u_1u_3^2 - u_2u_1^2 - u_2^2u_1 - u_3u_2u_1,\\
\partial_{t_3}(u_1) &= u_1^3u_3 + u_1^2u_3u_2 + u_1^2u_3^2 + u_1u_2u_1u_3 + u_1u_3u_1u_3 + u_1u_3u_2^2\\
&\quad + u_1u_3u_2u_3 + u_1u_3^2u_2 + u_1u_3^3 - u_2u_1^3 - u_2u_1u_2u_1 - u_2u_1u_3u_1\\
&\quad - u_2^2u_1^2 - u_2^3u_1 - u_2u_3u_2u_1 - u_3u_2u_1^2 - u_3u_2^2u_1 - u_3^2u_2u_1\,,\\
&\quad \dots
\end{aligned}$$

For any M, the nonabelian Volterra hierarchy has a common first integral $H = \sum_{k=1}^{M} u_k$.

In the case of the finitely generated free algebra $\mathfrak{A}_M$, we consider more general inhomogeneous ideals $\mathfrak{I}_M \subset \mathfrak{A}_M$ (than (4)) generated by the polynomials $\mathfrak{f}_{i,j}$:

$$\begin{aligned}
\mathfrak{I}_M &= \langle \mathfrak{f}_{i,j}, 1 \le i < j \le M, i, j \in \mathbb{N}\rangle,\\
\mathfrak{f}_{i,j} &= u_iu_j - \omega_{i,j}u_ju_i - \sigma_{i,j}^r u_r - \eta_{i,j},
\end{aligned} \tag{23}$$

where $\omega_{i,j} \neq 0$, $\omega_{i,j}, \sigma_{i,j}^r, \eta_{i,j} \in \mathbb{C}$ and we use Einstein summation convention, namely $\sigma_{i,j}^r u_r$ denotes $\sum_{r=1}^{M} \sigma_{i,j}^r u_r$. In this section, we explore the quantisation problem for periodic reductions in the Volterra system and its cubic symmetry.

4.1 Quantisation of the periodic Volterra system

Similar to what we did in Sect. 3, we are able to prove the following statement for the periodic Volterra equation:

Theorem 5 *A nonabelian periodical Volterra chain with period M admits a* $\mathfrak{I}_M$–*quantisation if and only if the following commutation relations hold:*

$$M=3: u_n u_{n+1} = \alpha u_{n+1} u_n + \beta(u_1+u_2+u_3)+\eta, \quad n\in\mathbb{Z}_3; \tag{24}$$

$$\begin{aligned} M=4: u_1u_2 &= \alpha u_2 u_1 + \beta u_2 + \gamma u_1 - \beta\gamma, \\ u_1u_3 &= u_3u_1 - \beta u_2 + \beta u_4, \\ u_4u_1 &= \alpha u_1 u_4 + \beta u_4 + \gamma u_1 - \beta\gamma, \\ u_2u_3 &= \alpha u_3 u_2 + \beta u_2 + \gamma u_3 - \beta\gamma, \\ u_2u_4 &= u_4u_2 - \gamma u_3 + \gamma u_1, \\ u_3u_4 &= \alpha u_4 u_3 + \beta u_4 + \gamma u_3 - \beta\gamma; \end{aligned} \tag{25}$$

$$\begin{aligned} M\geq 5: u_n u_{n+1} &= \alpha u_{n+1} u_n, \\ u_n u_m &= u_m u_n, \quad |n-m|>1, \ n,m\in\mathbb{Z}_M. \end{aligned} \tag{26}$$

The constants $\alpha,\beta,\gamma,\eta\in\mathbb{C},\ \alpha\neq 0$ *are arbitrary.*

Proof When $M=3$, the ideal $\mathfrak{I}_3$ is generated by three polynomials $\mathfrak{f}_{1,2}$, $\mathfrak{f}_{1,3}$ and $\mathfrak{f}_{2,3}$. We differentiate them by the derivation ∂_{t_1} associated with the Volterra Eq. (22) and project it on the quotient algebra. We have

$$\begin{aligned} \pi_{\mathfrak{I}_3}\left(\partial_{t_1}(\mathfrak{f}_{1,2})\right) = {} & \omega_{1,2}(\omega_{1,3}\omega_{2,3}-1)u_3u_2u_1 + \left(\sigma_{1,2}^2+\omega_{1,2}\sigma_{1,3}^2\right)u_2^2 + (\omega_{1,2}\omega_{1,3}\sigma_{2,3}^1-\sigma_{1,2}^1)u_1^2 \\ & + \left(\omega_{1,2}\omega_{2,3}\sigma_{1,3}^3+\omega_{2,3}\sigma_{1,2}^3+\sigma_{1,2}^3-\sigma_{1,2}^2\right)u_3u_2 \\ & + \left(\omega_{1,2}\omega_{1,3}\sigma_{2,3}^3+\omega_{1,3}\sigma_{1,2}^1-\omega_{1,3}\sigma_{1,2}^3-\sigma_{1,2}^3\right)u_3u_1 \\ & + \omega_{1,2}\left(\omega_{1,3}\sigma_{2,3}^2+\sigma_{1,3}^1\right)u_2u_1 + \left(\omega_{1,2}\sigma_{1,3}^3\sigma_{2,3}^3+\sigma_{1,2}^1\sigma_{1,3}^3-\sigma_{1,2}^3\sigma_{1,3}^3+\sigma_{1,2}^3\sigma_{2,3}^3\right)u_3 \\ & + \left(\omega_{1,2}\sigma_{1,3}^3\sigma_{2,3}^2+\omega_{1,2}\eta_{1,3}+\sigma_{1,2}^1\sigma_{1,3}^2-\sigma_{1,2}^3\sigma_{1,3}^2+\sigma_{2,3}^2\sigma_{1,2}^3+\eta_{1,2}\right)u_2 \\ & + \left(\omega_{1,2}\omega_{1,3}\eta_{2,3}+\omega_{1,2}\sigma_{1,3}^3\sigma_{2,3}^1+\sigma_{1,2}^1\sigma_{1,3}^1-\sigma_{1,2}^3\sigma_{1,3}^1+\sigma_{2,3}^1\sigma_{1,2}^3-\eta_{1,2}\right)u_1 \\ & + \left(\omega_{1,2}\sigma_{1,3}^3\eta_{2,3}+\sigma_{1,2}^1\eta_{1,3}-\sigma_{1,2}^3\eta_{1,3}+\sigma_{1,2}^3\eta_{2,3}\right). \end{aligned}$$

In the same way, we compute $\pi_{\mathfrak{I}_3}\left(\partial_{t_1}(\mathfrak{f}_{2,3})\right)$ and $\pi_{\mathfrak{I}_3}\left(\partial_{t_1}(\mathfrak{f}_{1,3})\right)$. If $\mathfrak{I}_3$ is preserved under the derivation ∂_{t_1}, all coefficients in these expressions should vanish, which leads to an algebraic system for $\omega_{i,j}, \sigma_{i,j}^r, \eta_{i,j}$, $1\leq i<j\leq 3$ and $r\in\{1,2,3\}$. The only nontrivial solution of this system is

$$\omega_{1,2}=\omega_{2,3}=\frac{1}{\omega_{1,3}}; \quad \sigma_{1,2}^r=\sigma_{2,3}^r=-\omega_{1,2}\sigma_{1,3}^r, \quad r=1,2,3; \quad \eta_{1,2}=\eta_{2,3}=-\omega_{1,2}\eta_{1,3},$$

which is the ideal presented in the statement by setting $\omega_{1,2}=\alpha$, $\sigma_{1,2}^1=\beta$ and $\eta_{1,2}=\eta$.

The proof of the statement for the case when $M = 4$ is similar, and we do not present it here. Let us now prove the last part of the statement concerning the case $M \geq 5$. The condition $M \geq 5$ implies that $u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}$ are algebraically independent in $\mathfrak{A}_M/\mathfrak{I}_M$ for all $n \in \mathbb{Z}$. In the quotient algebra $\mathfrak{A}_M/\mathfrak{I}_M$, $\pi_{\mathfrak{I}_M}\left(\partial_{t_1}(\mathfrak{f}_{i,j})\right) = 0$ for all $i < j$ is equivalent to all terms with the same degree vanishing. We denote its cubic terms as $Q^{(3)}_{i,j}$. Note that the cubic terms of $\partial_{t_1}(\mathfrak{f}_{i,j})$ are

$$\begin{aligned} &u_{i+1}u_iu_j - u_iu_{i-1}u_j + u_iu_{j+1}u_j - u_iu_ju_{j-1} \\ &\quad - \omega_{i,j}\left(u_{j+1}u_ju_i - u_ju_{j-1}u_i + u_ju_{i+1}u_i - u_ju_iu_{i-1}\right). \end{aligned} \tag{27}$$

It is clear that $Q^{(3)}_{n,n+1} = 0$ if and only if $\omega_{n,n+2} = 1$ for all n. We have

$$\begin{aligned} Q^{(3)}_{n,n+2} = &(\omega_{n+1,n+2} - \omega_{n,n+1})u_{n+2}u_{n+1}u_n + (\omega_{n,n+3} - 1)u_{n+3}u_{n+2}u_n \\ &+ (1 - \omega_{n-1,n+2})u_{n+2}u_nu_{n-1}, \end{aligned}$$

which vanishes when $\omega_{n,n+3} = \omega_{n-1,n+2} = 1$ and $\omega_{n,n+1} = \omega_{n+1,n+2}$. We set $\omega_{n,n+1} = \alpha$.

Let k be the distance between i and j modulo M. If $k > 2$, the sets $\{i+1, i, j\}$, $\{i, i-1, j\}$, $\{i, j+1, j\}$ and $\{i, j, j-1\}$ are all distinct (elements are taken modulo M). It follows from (27) that, for $k > 2$,

$$\begin{aligned} Q^{(3)}_{i,j} = &\omega_{i,j}((\omega_{i+1,j} - 1)u_ju_{i+1}u_i - (\omega_{i-1,j} - 1)u_ju_iu_{i-1}) \\ &+ \omega_{i,j}((\omega_{i,j+1} - 1)u_{j+1}u_ju_i - (\omega_{i,j-1} - 1)u_ju_{j-1}u_i) \end{aligned}$$

implying that $\omega_{i+1,j} = \omega_{i,j+1} = 1$ for all i and j. This leads to $\omega_{i,j} = 1$ for all i and j. So far we have proved that $\omega_{n,n+1} = \alpha$ for all n and $\omega_{i,j} = 1$ otherwise.

We are now ready to look at the rest terms in $\pi_{\mathfrak{I}_M}\left(\partial_{t_1}(\mathfrak{f}_{i,j})\right)$. The condition $\pi_{\mathfrak{I}_M}\left(\partial_{t_1}(\mathfrak{f}_{n,n+1})\right) = 0$ is equivalent to the following equation (we imply sums over r):

$$\begin{aligned} \pi_{\mathfrak{I}_M}\left(\sigma^r_{n,n+1}(u_{r+1}u_r - u_ru_{r-1})\right) = &\pi_{\mathfrak{I}_M}\left(\sigma^r_{n,n+1}(u_{n+1} + u_{n+2})u_r - \sigma^r_{n-1,n+1}u_nu_r\right) \\ &+ \pi_{\mathfrak{I}_M}\left(\sigma^r_{n,n+2}u_ru_{n+1} - \sigma^r_{n,n+1}u_r(u_n + u_{n-1})\right) \\ &+ \eta_{n,n+1}(u_{n+2} + u_{n+1} - u_n - u_{n-1}) \\ &+ \eta_{n,n+2}u_{n+1} - \eta_{n-1,n+1}u_n. \end{aligned} \tag{28}$$

In this expression, if we look at quadratic terms not containing u_l, $n-1 \leq l \leq n+2$ as a factor, we get $\sigma^r_{n,n+1} = 0$ if $r \notin \{n-1, n, n+1, n+2\}$. We substitute them into (28) and get $\sigma^{n-1}_{n,n+1} = \sigma^{n+2}_{n,n+1} = 0$ after comparing to the quadratic terms in its both sides. We denote the sum over r of $\sigma^r_{n-1,n+1}u_r$ by Σ_n. The quadratic terms in (28) becomes

$$\begin{aligned} 0 = &\sigma^{n+1}_{n,n+1}u^2_{n+1} - \sigma^{n+1}_{n,n+1}u_{n+1}u_{n-1} + \Sigma_{n+1}u_{n+1} \\ &+ \sigma^n_{n,n+1}u_{n+2}u_n - u_n\Sigma_n - \sigma^n_{n,n+1}u^2_n, \end{aligned}$$

which implies that Σ_n is proportional to u_n and further leads to $\sigma_{n,n+1}^{n+1} = \sigma_{n,n+1}^{n} = \Sigma_n = 0$. Finally from the vanishing of linear terms in (28), we have $\eta_{n,n+1} = \eta_{n,n+2} = 0$. Thus, we have that for all n, $\mathfrak{f}_{n,n+1} = u_n u_{n+1} - \alpha u_{n+1} u_n$ and $\mathfrak{f}_{n,n+2} = u_n u_{n+2} - u_{n+2} u_n$.

We will prove that $\mathfrak{f}_{n,n+m} = u_n u_{n+m} - u_{n+m} u_n$ for $m > 2$ by induction. Assume that we have for all $2 \leq l \leq k$ that $\mathfrak{f}_{n,n+l} = u_n u_{n+l} - u_{n+l} u_n$. We now compute $\partial_{t_1}(\mathfrak{f}_{n,n+k})$. Using the induction assumption, we have

$$\begin{aligned} 0 = \pi_{\mathfrak{I}_M}\left(\partial_{t_1}(\mathfrak{f}_{n,n+k})\right) &= \pi_{\mathfrak{I}_M}\left(u_n u_{n+k+1} u_{n+k} - u_{n+k+1} u_{n+k} u_n \right.\\ &\quad \left. + u_{n+k} u_n u_{n-1} - u_n u_{n-1} u_{n+k}\right) \\ &= \sigma_{n,n+k+1}^{r} u_r u_{n+k} - \sigma_{n-1,n+k}^{r} u_n u_r - \eta_{n-1,n+k} u_n + \eta_{n,n+k+1} u_{n+k}. \end{aligned}$$

Thus, the coefficient $\sigma_{n,n+k+1}^{r}$ should be zero whenever r is not n but also whenever r is not $n+k+1$ hence the σ's are identically zeros, from which it follows that $\eta_{n,n+k+1} = 0$. Hence, we conclude the induction and complete the proof. □

Note that the proof for the case $M \geq 5$ can be directly generalised to the non-periodic case which means that the ideal $\mathfrak{I}$ is the only stable ideal for the nonabelian Volterra flow within the class of ideals where $\mathfrak{f}_{i,j}$ has the form (23). This justifies our choice of the ideal $\mathfrak{I}$ (4) in the case of infinite Volterra chain (2).

4.2 Bi-quantum structure of the periodic Volterra system with period 3

In the classical commutative case, the $M = 3$ periodic Volterra system (22) is bi-Hamiltonian [24]. There are two compatible Poisson brackets defined by

$$\{u_{n+1}, u_n\}_0 = 1, \quad \{u_n, u_{n+1}\}_1 = u_{n+1} u_n, \quad n \in \mathbb{Z}_3$$

such that a linear combination of the Poisson brackets, called a Poisson pencil,

$$\{\cdot,\cdot\}_\kappa = (1-\kappa)\{\cdot,\cdot\}_0 + \kappa\{\cdot,\cdot\}_1$$

is also a Poisson bracket for any choice of κ, i.e. the bracket $\{\cdot,\cdot\}_\kappa$ is skew-symmetric and satisfies the Jacobi identity. The system admits two first integrals

$$H_1 = u_1 + u_2 + u_3, \quad H_2 = u_3 u_2 u_1, \tag{29}$$

such that Eq. (22) with commutative variables can be written in a bi-Hamiltonian form

$$\partial_{t_1}(u_k) = \{u_k, H_2\}_0 = \{u_k, H_1\}_1, \quad k \in \mathbb{Z}_3. \tag{30}$$

These first integrals Poisson commute with each other, and moreover, H_1 is in the kernel of the first Poisson bracket (is a Casimir element), while H_2 is in the kernel of the second one

$$\{u_k, H_1\}_0 = \{u_k, H_2\}_1 = 0, \quad k \in \mathbb{Z}_3.$$

and $H_\kappa = (1-\kappa)H_1 - \kappa H_2$ is a Casimir element of the bracket $\{\cdot,\cdot\}_\kappa$.

According to Proposition 4 and Theorem 5, the periodic Volterra system (22) on the free algebra $\mathfrak{A}_3$ admits a ∂_{t_1} and $\dagger$ stable difference ideal $\mathfrak{I}_{\theta,\hbar} = \langle f_n^{(\theta,\hbar)}; n \in \mathbb{Z}_3\rangle$, generated by the polynomials

$$f_n^{(\theta,\hbar)} = q^{-1}u_n u_{n+1} - q u_{n+1} u_n - \mathrm{i}\theta, \quad n \in \mathbb{Z}_3, \quad q = e^{\mathrm{i}\hbar},$$

depending on the two real parameters $0 \le \hbar < \pi, \theta \in \mathbb{R}$. Thus, we have a pencil of quantised algebras $\mathfrak{A}^{(\theta,\hbar)} = \mathfrak{A}_3 / \mathfrak{I}_{\theta,\hbar}$. Algebra $\mathfrak{A}^{(\theta,\hbar)}$ has a central element

$$\mathcal{H}(\theta,\hbar) = \sin(\hbar)H_2 + \theta(2+\cos(2\hbar))H_1,$$

where the self-adjoint elements

$$H_1 = u_1 + u_2 + u_3, \tag{31}$$

$$\begin{aligned} H_2 &= \sum_{\sigma\in\mathcal{S}_3} u_{\sigma(1)}u_{\sigma(2)}u_{\sigma(3)} \\ &= 3(q^2+1)u_3u_2u_1 + \mathrm{i}\theta\left((2q+q^{-1})(u_1+u_3) - (q+2q^{-1})u_2\right) \end{aligned} \tag{32}$$

are first integrals for the quantum Volterra system

$$(u_n)_{t_1} = q(u_{n+1}u_n - u_n u_{n-1}), \quad n \in \mathbb{Z}_3. \tag{33}$$

Moreover, system (33) in algebra $\mathfrak{A}^{(\theta,\hbar)}$ can be represented in the Heisenberg form

$$(u_n)_{t_1} = \frac{\mathrm{i}}{2\sin\hbar}[H_1, u_n] = -\frac{\mathrm{i}}{2\theta(2+\cos(2\hbar))}[H_2, u_n].$$

With two quotient algebras $\mathfrak{A}^{(\theta,0)}$ and $\mathfrak{A}^{(0,\hbar)}$, we associate the following *bi-quantum* structure (a quantum deformation of the bi-Hamiltonian structure (30)) as follows:

choice of parameters	$\theta \neq 0,\ \hbar = 0,\ q = 1$	$\theta = 0,\ 0 < \hbar < \pi,\ q = e^{\mathrm{i}\hbar}$
stable ideal in $\mathfrak{A}_3$	$\mathfrak{I}_{\theta,0}$	$\mathfrak{I}_{0,\hbar}$
quantised algebra	$\mathfrak{A}^{(\theta,0)} = \mathfrak{A}_3 / \mathfrak{I}_{\theta,0}$	$\mathfrak{A}^{(0,\hbar)} = \mathfrak{A}_3 / \mathfrak{I}_{0,\hbar}$
self-adjoint central element	$H_1 = u_1 + u_2 + u_3$	$H_2 = 3(1+q^2)u_3u_2u_1$
the Heisenberg form of (33)	$(u_n)_{t_1} = -\frac{\mathrm{i}}{6\theta}[H_2, u_n]$	$(u_n)_{t_1} = \frac{\mathrm{i}}{2\sin\hbar}[H_1, u_n]$

More work is required to study the quantum periodic Volterra systems with $M \ge 4$ (25), (26) as we did for $M = 3$ above, which is not included in this paper.

4.3 Quantisation of periodic reductions in the cubic symmetry

In this section, we study the quantisation problem for periodical reductions in the cubic symmetry (3). In the infinite case, this system admits two distinct quantisations (Proposition 2).

We claim that:

1. In the case $M = 3$, the quantisation ideal (23) is generated by relations (24).
2. For odd $M \geq 5$, the quantisation ideal (23) is generated by relations (26).
3. For even $M \geq 6$, there are two distinct quantisations corresponding to the ideal $\mathfrak{I}_a$ generated by the relations (26) and $\mathfrak{I}_b$ generated by relations

$$\begin{aligned} &u_n u_{n+1} = (-1)^n \omega u_{n+1} u_n, \\ &u_n u_m + u_m u_n = 0 \text{ if } |n - m| \geq 2, \quad n, m \in \mathbb{Z}_M . \end{aligned} \tag{34}$$

The case $M = 4$ is exceptional, it admits three distinct quantisation ideals. One quantisation ideal is generated by commutation relations (25), and the other two are generated by homogeneous quadratic commutation relations. The periodical reduction in the system (3) with the period $M = 4$ can be written in the form (Here we also add the constant q^2 following (19)):

$$\begin{aligned} &\partial_{t_2} u_n = q^2 \left(u_{n+2} u_{n+1} u_n + u_{n+1}^2 u_n + u_{n+1} u_n^2 - u_n^2 u_{n+3} - u_n u_{n+3} u_{n+2} - u_n u_{n+3}^2 \right), \\ &q = e^{i\hbar} \end{aligned} \tag{35}$$

where the lower index $n \in \mathbb{Z}_4$. In the free algebra $\mathfrak{A}_4 = \mathbb{C}\langle u_1, \dots u_4 \rangle$, we consider the ideal $\mathfrak{I}$

$$\mathfrak{I} = \langle \mathfrak{f}_{i,j} \, ; \, 1 \leq i < j \leq 4 \rangle, \quad \mathfrak{f}_{i,j} = u_i u_j - \omega_{i,j} u_j u_i, \tag{36}$$

generated by six homogeneous quadratic polynomials $\mathfrak{f}_{i,j}$, which depend on six nonzero constants $\omega_{i,j}$. The ideal $\mathfrak{I}$ is ∂_{t_2}–stable if and only if $\partial_{t_2}(\mathfrak{f}_{i,j}) \in \mathfrak{I}, \ 1 \leq i < j \leq 4$. This is equivalent to the following system of equations on the parameters $\omega_{i,j}$

$$\begin{aligned} &\omega_{2,4}^2 = 1, \quad \omega_{1,4}^2 \omega_{3,4}^2 = 1, \quad \omega_{2,3} = \omega_{2,4} \omega_{3,4}, \\ &\omega_{1,2} = \omega_{1,4} \omega_{2,4} \omega_{3,4}^2, \quad \omega_{1,3} = \omega_{1,4} \omega_{3,4}. \end{aligned} \tag{37}$$

Solving the above system of equations, we obtain the following statement:

Theorem 6 *A nonabelian system* (35) *admits a* $\mathfrak{I}$*–quantisation of the form* (36) *if and only if the six constants* $\omega_{i,j}$ *take values as in one of four cases:*

$$\begin{array}{lcccccc} & \omega_{1,2} & \omega_{1,3} & \omega_{2,3} & \omega_{1,4} & \omega_{2,4} & \omega_{3,4} \\ \text{(a)}: & \omega, & 1, & \omega, & \omega^{-1}, & 1, & \omega; \\ \text{(b)}: & \omega, & -1, & -\omega, & -\omega^{-1}, & -1, & \omega; \\ \text{(c)}: & -\omega, & -1, & \omega, & -\omega^{-1}, & 1, & \omega; \\ \text{(d)}: & -\omega, & 1, & -\omega, & \omega^{-1}, & -1, & \omega, \end{array}$$

and $\omega = q^2 = e^{2\mathrm{i}\hbar}$*, where* $\hbar \in \mathbb{R}$*. Moreover, in each of the above four cases the system* (35) *is a super-integrable quantum system.*

The first and second solutions correspond to the cases (a) and (b) in Proposition 2. Solutions (c) and (d) are new; they are related by the automorphism $\mathcal{S}$ of $\mathfrak{A}_4$ and thus equivalent. The commutation relations in the case (a) can be extended by non-homogeneous terms (25), while commutation relations (b)–(d) do not admit non-homogeneous extensions.

Proof First note that the four cases listed in the statement correspond to the four solutions of the system (37). It is obvious that in each case the ideal is †–stable if and only if $\omega^\dagger = \omega^{-1}$. Thus, we can set $\omega = e^{2\mathrm{i}\hbar},\ \hbar \in \mathbb{R}$.

We now prove the super-integrability of the obtained system in each case. Let

$$H = u_1 + u_2 + u_3 + u_4,$$

which is a first integral for the quantum system (35) in all four cases. Moreover, in all four cases the quantum system (35) for self-adjoint variables u_n can be written in the same Heisenberg form (21):

$$\partial_{t_2}(u_n) = \frac{\mathrm{i}}{2\sin(2\hbar)}[H^2, u_n], \quad n \in \mathbb{Z}_4. \tag{38}$$

In the case (a), corresponding to the quantisation of the Volterra system, the quantisation ideal $\mathfrak{I}_a$ is generated by the commutation relations between the variables u_k as follows:

$$\begin{aligned} &u_1u_2 = \omega u_2u_1, \quad u_1u_3 = u_3u_1, \quad u_4u_1 = \omega u_1u_4, \\ &u_2u_3 = \omega u_3u_2, \quad u_2u_4 = u_4u_2, \quad u_3u_4 = \omega u_4u_3. \end{aligned} \tag{39}$$

The algebra $\mathfrak{A}_4 / \mathfrak{I}_a$ has two central elements

$$\mathcal{H}_1 = u_3u_1, \quad \mathcal{H}_2 = u_4u_2.$$

Since the central elements of the algebra commute with the Hamiltonian, they are first integrals of the system (38). The system of four Eq. (35) admits three commuting first integrals, and therefore, it is super–integrable.

In the case (b), the quantisation ideal $\mathfrak{I}_b$ is generated by the commutation relations between the variables u_k as follows

$$\begin{aligned} &u_1u_2 = \omega u_2u_1, \quad u_1u_3 = -u_3u_1, \quad u_4u_1 = -\omega u_1u_4, \\ &u_2u_3 = -\omega u_3u_2, \quad u_2u_4 = -u_4u_2, \quad u_3u_4 = \omega u_4u_3. \end{aligned} \tag{40}$$

The dynamical system (35) on $\mathfrak{A}_4 / \mathfrak{I}_b$ admits two first integrals

$$H_1 = u_3u_1, \quad H_2 = u_4u_2.$$

Elements H_1, H_2 anti-commute with H, but H^2, H_1 and H_2 commute with each other. Thus, the system (35) is super–integrable on $\mathfrak{A}_4 / \mathfrak{I}_b$. Taking H_1 and H_2 as Hamiltonians, we can find two commuting symmetries of the quantum Eq. (35) on $\mathfrak{A}_4 / \mathfrak{I}_b$, i.e.

$$\partial_\xi(u_n) = [H_1, u_n] = 2u_3u_1u_n, \quad \partial_\eta(u_n) = [H_2, u_n] = 2u_4u_2u_n.$$

The algebra $\mathfrak{A}_4 / \mathfrak{I}_b$ has three central elements

$$\mathcal{H} = u_4u_3u_2u_1, \quad \mathcal{H}_1 = u_3^2u_1^2, \quad \mathcal{H}_2 = u_4^2u_2^2.$$

In the case (c), which is new, the quantisation ideal $\mathfrak{I}_c$ is generated by the commutation relations between the variables u_k as follows:

$$\begin{aligned} &u_1u_2 = -\omega u_2u_1, \quad u_1u_3 = -u_3u_1, \quad u_4u_1 = -\omega u_1u_4, \\ &u_2u_3 = \omega u_3u_2, \quad u_2u_4 = u_4u_2, \quad u_3u_4 = \omega u_4u_3. \end{aligned} \tag{41}$$

The dynamical system (35) on $\mathfrak{A}_4 / \mathfrak{I}_c$ admits the first integral $H_1 = u_3u_1$ commuting with H^2. The algebra $\mathfrak{A}_4 / \mathfrak{I}_c$ has two central elements

$$\mathcal{H}_1 = u_3^2u_1^2, \quad \mathcal{H}_2 = u_4u_2.$$

The first integrals H^2, H_1 and $\mathcal{H}_2$ are obviously independent, and therefore, system (35) on $\mathfrak{A}_4 / \mathfrak{I}_c$ is super–integrable.

The last case (d) can be obtained from the case (c) by the cyclic permutation of the variables $\{u_1, u_2, u_3, u_4\} \mapsto \{u_2, u_3, u_4, u_1\}$. □

In the case $M = 5$, the only ∂_{t_2}–stable ideal is defined by (26). The system admits three commuting first integrals

$$H_1 = \sum_{k\in\mathbb{Z}_5} u_k, \quad H_2 = \sum_{k\in\mathbb{Z}_5} (u_k^2 + u_ku_{k+1} + u_{k+1}u_k), \quad \mathcal{H} = u_5u_4u_3u_2u_1,$$

where $\mathcal{H}$ is a central element of the algebra. The Heisenberg equations corresponding to H_1 and H_2 result in the periodic Volterra system and its cubic symmetry, respectively.

5 Quantisation of the nonabelian Volterra hierarchy

In this section, we extend Propositions 1 and 2 in Sect. 2 to the whole nonabelian Volterra hierarchy. We show that the quantum ideal $\mathfrak{I}_a$ (5) is invariant with respect to every member of the hierarchy (12) (Theorem 9) and that the quantum ideal $\mathfrak{I}_b$ (7) is invariant with respect to every even member of the nonabelian Volterra hierarchy

$$\partial_{t_{2\ell}}(u) = \mathcal{S}(X^{(2\ell)})u - u\mathcal{S}^{-1}(X^{(2\ell)}), \quad \ell \in \mathbb{N},$$

that is, odd degree symmetries of the nonabelian Volterra equation (Theorem 14).

We are going to use the explicit expressions given by (12) to prove these statements. First we introduce some notations and definitions inspired by the monomials appearing in $X^{(l)}$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{Z}^k$ be a k-component vector. For each $\alpha \in \mathbb{Z}^k$, we define the k-degree monomial $u_\alpha = u_{\alpha_1}u_{\alpha_2}\cdots u_{\alpha_k}$. We denote the degree of α by $|\alpha| = k$. Conventionally, we write $(\alpha_1+1, \alpha_2+1, \cdots, \alpha_k+1)$ as $\alpha+1$. Thus, we have $\mathcal{S}^i u_\alpha = u_{\alpha+i}$ for $i \in \mathbb{Z}$. The number of variable u_i in monomial u_α is denoted by $\nu(\alpha, i)$. Similarly, we denote by $\nu(\alpha, \geq i)$ the number of $k \geq i$ such that u_k appears in u_α, counted with multiplicities. We say that two monomials u_α and u_β are *similar* written as $\alpha \sim \beta$ if $\nu(\alpha, i) = \nu(\beta, i)$ for all $i \in \mathbb{Z}$.

We introduce two sets of distinguished monomials, for $k \geq 1$

$$\mathcal{A}^k = \left\{\alpha \in \mathbb{Z}^k \middle| 1-k \leq \alpha_k \leq 0,\ k-1 \geq \alpha_1 \geq 0,\ \alpha_{i+1}+1 \geq \alpha_i,\ i = 1, ..., k-1\right\};$$
$$\mathcal{Z}^k_{\geq} = \left\{\alpha \in \mathbb{Z}^k \middle| \alpha_{i+1}+1 \geq \alpha_i \geq \alpha_{i+1},\ i = 1, ..., k-1\right\}.$$

We say that a k-degree monomial u_α is admissible if $\alpha \in \mathcal{A}^k$ and is nonincreasing if $\alpha \in \mathcal{Z}^k_{\geq}$.

Using these notations, we can simply write the expression $X^{(k)}$ given by (13) as

$$X^{(k)} = \sum_{\alpha \in \mathcal{A}^k} u_\alpha. \tag{42}$$

Given an ideal $\mathfrak{I}$, either $\mathfrak{I}_a$ or $\mathfrak{I}_b$, the canonical projection $\pi_{\mathfrak{I}} : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ acts on $X^{(k)}$ as follows:

$$\pi_{\mathfrak{I}}(X^{(k)}) = \sum_{\alpha \in \mathcal{A}^k \cap \mathcal{Z}^k_{\geq}} P^{\mathfrak{I}}_\alpha(\omega) u_\alpha,$$

where $P^{\mathfrak{I}}_\alpha(\omega)$ is the unique polynomial in $\mathbb{Z}[\omega]$ such that for $\alpha \in \mathcal{A}^k \cap \mathcal{Z}^k_{\geq}$,

$$P^{\mathfrak{I}}_\alpha(\omega) u_\alpha = \pi_{\mathfrak{I}}\left(\sum_{\beta \in \mathcal{A}^k, \beta \sim \alpha} u_\beta\right). \tag{43}$$

We often write it as $P_\alpha(\omega)$ if there is no ambiguity.

We say that two polynomials $f, g \in \mathfrak{A}$ are $\mathfrak{I}$–equivalent denoted by $f \overset{\mathfrak{I}}{\simeq} g$ if $f - g \in \mathfrak{I}$. Polynomials f and g are $\mathfrak{I}$ equivalent if and only if $\pi_{\mathfrak{I}}(f) = \pi_{\mathfrak{I}}(g)$.

5.1 Quantisation of the Volterra hierarchy

In this section, we will prove that the ideal $\mathfrak{I}_a$ defined by (5) is preserved by the symmetry flows (12), for all $\ell \in \mathbb{N}$.

To do so, we need to study the polynomials $P_\alpha^{\mathfrak{I}_a}(\omega)$. Here we focus on the quantum ideal $\mathfrak{I}_a$. For the sake of simplicity, we write the polynomials as $P_\alpha(\omega)$, which are in $\mathbb{Z}_+[\omega]$. For example, we have

$$\begin{aligned}
&\pi_{\mathfrak{I}_a}(X^{(1)}) = X^{(1)} = u; \quad \pi_{\mathfrak{I}_a}(X^{(2)}) = X^{(2)} = u_1 u + u^2 + u u_{-1};\\
&\pi_{\mathfrak{I}_a}(X^{(3)}) = u_2 u_1 u + u_1^2 u + (1+\omega) u_1 u^2 + u^3 + (1+\omega) u^2 u_{-1}\\
&\qquad\qquad + u_1 u u_{-1} + u u_{-1}^2 + u u_{-1} u_{-2}.
\end{aligned}$$

This defines the polynomials $P_\alpha(\omega)$, e.g. $P_{(0,0,-1)}(\omega) = 1 + \omega$. In general, we prove the following identity:

Proposition 7 *Let $\alpha \in \mathcal{Z}^k_{\geq}$. Then, we have*

$$P_\alpha(\omega) + \omega^{\nu(\alpha,0)} P_{\alpha-1}(\omega) = P_{\alpha-1}(\omega) + \omega^{\nu(\alpha,1)} P_\alpha(\omega). \tag{44}$$

Proof First note that this formula holds whenever $\alpha \notin \mathcal{A}^k$ or $\alpha - 1 \notin \mathcal{A}^k$ since for $\alpha \in \mathcal{Z}^k_{\geq}$, $\alpha \in \mathcal{A}^k$ if and only if $\nu(\alpha, 0) \neq 0$. If $\alpha \notin \mathcal{A}^k$, then $P_\alpha(\omega) = 0$ and $\nu(\alpha, 0) = 0$. Similarly, if $\alpha - 1 \notin \mathcal{A}^k$, then $P_{\alpha-1}(\omega) = 0$ and $\nu(\alpha, 1) = 0$. Thus, the formula holds in both cases.

We now assume that $\alpha \in \mathcal{A}^k$ and $\alpha - 1 \in \mathcal{A}^k$. Consider the set E_α defined as

$$E_\alpha = \left\{\beta \in \mathbb{Z}^k \middle| \beta \sim \alpha,\ \beta_1 \geq 0,\ \beta_k \leq 1,\ \beta_i \leq \beta_{i+1} + 1, i = 1, \ldots, k-1\right\}.$$

We split E_α in two different ways by defining four subsets of E_α:

$$\begin{aligned}
A_\alpha &= \{\beta \in E_\alpha \mid \beta_k \leq 0\}, \quad B_\alpha = \{\beta \in E_\alpha \mid \beta_1 \geq 1\},\\
C_\alpha &= \{\beta \in E_\alpha \mid \beta_k = 1\}, \quad D_\alpha = \{\beta \in E_\alpha \mid \beta_1 = 0\}.
\end{aligned}$$

It is clear that $E_\alpha = A_\alpha \cup C_\alpha = B_\alpha \cup D_\alpha$, $A_\alpha \cap C_\alpha = \emptyset$ and $B_\alpha \cap D_\alpha = \emptyset$. We now have

$$\begin{aligned}
\pi_{\mathfrak{I}_a}\left(\sum_{\beta \in E_\alpha} u_\beta\right) &= \pi_{\mathfrak{I}_a}\left(\sum_{\beta \in A_\alpha} u_\beta\right) + \pi_{\mathfrak{I}_a}\left(\sum_{\beta \in C_\alpha} u_\beta\right)\\
&= \pi_{\mathfrak{I}_a}\left(\sum_{\beta \in B_\alpha} u_\beta\right) + \pi_{\mathfrak{I}_a}\left(\sum_{\beta \in D_\alpha} u_\beta\right).
\end{aligned} \tag{45}$$

We are going to evaluate each term in it. Note that $A_\alpha = \mathcal{A}^k$ is the set of all elements equivalent to α. Thus, by definition (43), we have

$$\pi_{\mathfrak{I}_a}\left(\sum_{\beta\in A_\alpha} u_\beta\right) = P_\alpha(\omega)u_\alpha. \tag{46}$$

For any $\beta \in B_\alpha$, we have $\beta - 1 \in \mathcal{A}^k$ and $\beta - 1 \sim \alpha - 1$ and thus

$$\begin{aligned}\pi_{\mathfrak{I}_a}\left(\sum_{\beta\in B_\alpha} u_\beta\right) &= \mathcal{S}\pi_{\mathfrak{I}_a}\left(\sum_{\beta-1\in\mathcal{A}^k,\beta-1\sim\alpha-1} u_{\beta-1}\right)\\ &= \mathcal{S}\left(P_{\alpha-1}(\omega)u_{\alpha-1}\right) = P_{\alpha-1}(\omega)u_\alpha.\end{aligned} \tag{47}$$

Let $\beta \in D_\alpha$. There is $\beta_i > 0$ for some $0 < i < k$ since $\beta \sim \alpha$ and $\alpha - 1 \in \mathcal{A}^k$. Assume that there are $0 < m \le k$ positive components at positions $i_1 \le i_2 \le \cdots \le i_m$ in β. Starting from i_1, we find the first zero entry on the left of i_1, that is, $l_1 = \max_{1\le j\le i_1-1}\left\{\beta_j = 0\right\}$ and move the components from l_1 to $i_1 - 1$ to the right of i_1 and obtain β^1 with

$$\begin{aligned}&\beta^1_j = \beta_j,\ 1 \le j \le l_1 - 1;\quad \beta^1_{l_1} = \beta_{i_1};\\ &\beta^1_j = \beta_{j-1},\ l_1 + 1 \le j \le i_1;\quad \beta^1_j = \beta_j,\ i_1 + 1 \le j \le k.\end{aligned}$$

For β^1, we find the first zero entry on the left of i_2, that is, $l_2 = \max_{l_1+1\le j\le i_2-1}\left\{\beta^1_j = 0\right\}$ and move the components from l_2 to $i_2 - 1$ to the right of i_2 and obtain β^2. We repeat this procedure for all positive components in β. Thus, we obtain a k-component vector $\gamma = \beta^l \in A_\alpha$. This leads to

$$\begin{aligned}\pi_{\mathfrak{I}_a}\left(\sum_{\beta\in D_\alpha} u_\beta\right) &= \pi_{\mathfrak{I}_a}\left(\sum_{\gamma\in A_\alpha} \omega^{\nu(\beta,1)}u_\gamma\right)\\ &= \omega^{\nu(\alpha,1)}\pi_{\mathfrak{I}_a}\left(\sum_{\gamma\in A_\alpha} u_\gamma\right) = \omega^{\nu(\alpha,1)}P_\alpha(\omega)u_\alpha.\end{aligned} \tag{48}$$

Similarly, let $\beta \in C_\alpha$. There is $\beta_i \le 0$ for some $0 < i < k$ since $\beta \sim \alpha$ and $\alpha \in \mathcal{A}^k$. For all nonpositive components, we move the first component being 1 on its right to its left, taking with all the components of β on its left that are larger than 1. Thus, we obtain a k-component vector $\gamma \in B_\alpha$. This leads to

$$\pi_{\mathfrak{I}_a}\left(\sum_{\beta\in C_\alpha} u_\beta\right) = \pi_{\mathfrak{I}_a}\left(\sum_{\gamma\in B_\alpha} \omega^{\nu(\beta,0)} u_\gamma\right)$$
$$= \omega^{\nu(\alpha,0)}\pi_{\mathfrak{I}_a}\left(\sum_{\gamma\in B_\alpha} u_\gamma\right) = \omega^{\nu(\alpha,0)} P_{\alpha-1}(\omega) u_\alpha. \tag{49}$$

We substitute (46)–(49) into (45), and thus, we obtain the required identity (44). □

In the same way as the proof of Proposition 7, we are able to show that

$$P_{\alpha+m}(\omega) + \omega^{\nu(\alpha,-m)} P_{\alpha+m-1}(\omega) = P_{\alpha+m-1}(\omega) + \omega^{\nu(\alpha,1-m)} P_{\alpha+m}(\omega) \quad \text{for all} m \in \mathbb{Z}. \tag{50}$$

This leads to the following statement:

Corollary 8 *Let $\alpha \in \mathcal{Z}^k_{\geq}$. There exists a nonzero rational function $R_\alpha(\omega) \in \mathbb{Q}(\omega)$ such that*

$$P_{\alpha+m}(\omega) = R_\alpha(\omega)(1 - \omega^{\nu(\alpha,-m)}) \quad \textit{for all } m \in \mathbb{Z}. \tag{51}$$

Proof For $\alpha \in \mathcal{Z}^k_{\geq}$, there exists $l \in \mathbb{Z}$ such that $\nu(\alpha + l, 0) = \nu(\alpha, -l) \neq 0$. By iterating (50), we get

$$P_{\alpha+m}(\omega)(1 - \omega^{\nu(\alpha,-l)}) = P_{\alpha+l}(\omega)(1 - \omega^{\nu(\alpha,-m)}) \quad \text{for all} m \in \mathbb{Z}.$$

Hence, choosing

$$R_\alpha(\omega) = P_{\alpha+l}(\omega)(1 - \omega^{\nu(\alpha,-l)})^{-1},$$

we obtain the required result. □

Theorem 9 *The quantisation ideal $\mathfrak{I}_a$ is stable with respect to every member of the Volterra hierarchy $\partial_{t_\ell}(u) = \mathcal{S}(X^{(\ell)})u - u\mathcal{S}^{-1}(X^{(\ell)}),\ \ell \in \mathbb{N}$.*

Proof We fix k and let $u_\tau = Q^{(k)}$ be the $(k + 1)$-degree symmetry of the Volterra equation given by (12). Since $\mathcal{S}(\mathfrak{I}) = \mathfrak{I}$, we only need to show that

$$\pi_{\mathfrak{I}_a}\left(\partial_\tau(uu_m - \omega^{\delta_{1,m}} u_m u)\right) = 0, \quad m \in \mathbb{N}.$$

This means that

$$\pi_{\mathfrak{I}_a}\left(Q^{(k)}u_m + uQ^{(k)}_m - \omega^{\delta_{1,m}} Q^{(k)}_m u - \omega^{\delta_{1,m}} u_m Q^{(k)}\right) = 0.$$

We rewrite it in terms of X. Here we simply drop its upper index of $X^{(k)}$.

$$\pi_{\mathfrak{I}_a}\left(uX_{m+1}u_m - \omega^{\delta_{1,m}}X_{m+1}u_m u - uu_m X_{m-1} + \omega^{\delta_{1,m}}u_m X_{m-1}u\right.$$
$$\left. + X_1 uu_m - \omega^{\delta_{1,m}}u_m X_1 u - uX_{-1}u_m + \omega^{\delta_{1,m}}u_m uX_{-1}\right) = 0. \quad (52)$$

It is clear that, for any $\alpha \in \mathcal{Z}^k_{\geq}$, we have

$$uu_\alpha u_m \overset{\mathfrak{I}_a}{\simeq} \omega^{\nu(\alpha,1)-\nu(\alpha,-1)}u_\alpha uu_m,$$
$$u_m u_\alpha u \overset{\mathfrak{I}_a}{\simeq} \omega^{\delta_{1,m}}\omega^{\nu(\alpha,m+1)-\nu(\alpha,m-1)}u_\alpha uu_m,$$
$$uu_m u_\alpha \overset{\mathfrak{I}_a}{\simeq} \omega^{\nu(\alpha,m+1)+\nu(\alpha,1)-\nu(\alpha,-1)-\nu(\alpha,m-1)}u_\alpha uu_m.$$

Note that for all $l \in \mathbb{Z}$, we have

$$\pi_{\mathfrak{I}_a}(X_l) = \pi_{\mathfrak{I}_a}(\mathcal{S}^l X) = \mathcal{S}^l \pi_{\mathfrak{I}_a}(X) = \mathcal{S}^l\left(\sum_{\alpha\in\mathcal{Z}^k_{\geq}} P_\alpha(\omega)u_\alpha\right) = \sum_{\alpha\in\mathcal{Z}^k_{\geq}} P_\alpha(\omega)u_{\alpha+l}$$
$$= \sum_{\alpha\in\mathcal{Z}^k_{\geq}} P_{\alpha-l}(\omega)u_\alpha.$$

Here the sum is over all $\alpha \in \mathcal{Z}^k_{\geq}$ including the ones not in $\mathcal{A}^k$. Hence, the left-handed side of (52) becomes

$$\sum_{\alpha\in\mathcal{Z}^k_{\geq}}\left(P_{\alpha-m-1}(\omega) - P_{\alpha-m+1}(\omega)\omega^{\nu(\alpha,m+1)-\nu(\alpha,m-1)}\right)\left(\omega^{\nu(\alpha,1)-\nu(\alpha,-1)} - 1\right)\pi_{\mathfrak{I}_a}(u_\alpha uu_m)$$
$$+ \sum_{\alpha\in\mathcal{Z}^k_{\geq}}\left(P_{\alpha-1}(\omega) - P_{\alpha+1}(\omega)\omega^{\nu(\alpha,1)-\nu(\alpha,-1)}\right)\left(1 - \omega^{\nu(\alpha,m+1)-\nu(\alpha,m-1)}\right)\pi_{\mathfrak{I}_a}(u_\alpha uu_m)$$

For any $\alpha \in \mathcal{Z}^k_{\geq}$, we need to check that the coefficient of $\pi_{\mathfrak{I}_a}(u_\alpha uu_m)$ vanishes. Using Corollary 8, it amounts to compute

$$\left(1 - \omega^{\nu(\alpha,m+1)} - (1 - \omega^{\nu(\alpha,m-1)})\omega^{\nu(\alpha,m+1)-\nu(\alpha,m-1)}\right)\left(\omega^{\nu(\alpha,1)-\nu(\alpha,-1)} - 1\right)$$
$$+ \left(1 - \omega^{\nu(\alpha,1)} - (1 - \omega^{\nu(\alpha,-1)})\omega^{\nu(\alpha,1)-\nu(\alpha,-1)}\right)\left(1 - \omega^{\nu(\alpha,m+1)-\nu(\alpha,m-1)}\right),$$

which equals zero after the simplification, and thus, we complete the proof. □

5.2 Non-deformation quantisation for all odd-degree Volterra symmetries

In this section, we will prove that all odd-degree symmetries of the nonabelian Volterra hierarchy admit the quantisation $\mathfrak{I}_b$, that is, the ideal $\mathfrak{I}_b$ defined by (7) is preserved by the symmetry flows (12) when ℓ is even. We extend the automorphism $\mathcal{S}$ and the antiautomorphism $\mathcal{T}$ to the algebra $\mathfrak{A}[\omega]$ by letting $\mathcal{S}(\omega) = \mathcal{T}(\omega) = -\omega$ so that these operators are well-defined on the quotient $\mathfrak{A}/\mathfrak{I}_b$.

The ideas guiding the proof essentially are the same as in the previous section with the notable difference of the equivalence of Proposition 7, which is much harder in this case.

As in the previous section, for an ideal $\mathfrak{I}_b$, we define uniquely $P_\alpha(\omega) \in \mathbb{Z}[\omega]$ by the canonical projection $\pi_{\mathfrak{I}_b} : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}_b$ acting on $X^{(k)}$. For example, we have

$$\begin{aligned}
\pi_{\mathfrak{I}_b}(X^{(1)}) &= X^{(1)} = u; \quad \pi_{\mathfrak{I}_b}(X^{(2)}) = X^{(2)} = u_1 u + u^2 + u u_{-1};\\
\pi_{\mathfrak{I}_b}(X^{(3)}) &= u_2 u_1 u + u_1^2 u + (1+\omega) u_1 u^2 + u^3 + (1-\omega) u^2 u_{-1}\\
&\quad + u_1 u u_{-1} + u u_{-1}^2 + u u_{-1} u_{-2}.
\end{aligned}$$

This leads to the polynomials $P_\alpha(\omega)$, e.g. $P_{(0,0,-1)}(\omega) = 1 - \omega$.

To prove that the ideal $\mathfrak{I}_b$ defined by (7) is preserved by the symmetry flows $Q^{(2k)}$, we first prove the equivalents of Proposition 7 only in this case for $\alpha \in \mathcal{Z}_{\geq}^{2k}$. We now assume that $\alpha \in \mathcal{A}^{2k}$ and $\alpha - 1 \in \mathcal{A}^{2k}$. In the same way as we prove Proposition 7, we define the set E_α as

$$E_\alpha = \left\{ \beta \in \mathbb{Z}^{2k} \middle| \beta \sim \alpha,\ \beta_1 \geq 0,\ \beta_{2k} \leq 1,\ \beta_i \leq \beta_{i+1} + 1, i = 1, ..., 2k-1 \right\}$$

and split E in two different ways by defining four subsets of E_α:

$$\begin{aligned}
A_\alpha &= \{\beta \in E_\alpha \mid \beta_{2k} \leq 0\}, \quad B_\alpha = \{\beta \in E_\alpha \mid \beta_1 \geq 1\},\\
C_\alpha &= \{\beta \in E_\alpha \mid \beta_{2k} = 1\}, \quad D_\alpha = \{\beta \in E_\alpha \mid \beta_1 = 0\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\pi_{\mathfrak{I}_b}\left(\sum_{\beta \in E_\alpha} u_\beta\right) &= \pi_{\mathfrak{I}_b}\left(\sum_{\beta \in A_\alpha} u_\beta\right) + \pi_{\mathfrak{I}_b}\left(\sum_{\beta \in C_\alpha} u_\beta\right)\\
&= \pi_{\mathfrak{I}_b}\left(\sum_{\beta \in B_\alpha} u_\beta\right) + \pi_{\mathfrak{I}_b}\left(\sum_{\beta \in D_\alpha} u_\beta\right). \qquad (53)
\end{aligned}$$

We need to evaluate each term under the ideal $\mathfrak{I}_b$. Since $A_\alpha = \mathcal{A}^{2k}$ is the set of all elements equivalent to α, it follows from (43) that

$$\pi_{\mathfrak{I}_b}\left(\sum_{\beta \in A_\alpha} u_\beta\right) = P_\alpha(\omega) u_\alpha. \qquad (54)$$

For any $\beta \in B$, note that $\beta - 1 \in \mathcal{A}^{2k}$ and $\beta - 1 \sim \alpha - 1$ and thus

$$\pi_{\mathfrak{I}_b}\left(\sum_{\beta \in B} u_\beta\right) = \pi_{\mathfrak{I}_b}\mathcal{S}\left(\sum_{\beta-1\in\mathcal{A}^{2k},\beta-1\sim\alpha-1} u_{\beta-1}\right)$$
$$= P_{\alpha-1}(-\omega)\mathcal{S}u_{\alpha-1} = P_{\alpha-1}(-\omega)u_\alpha. \tag{55}$$

We are now left to evaluate the terms for D_α and for C_α, and we do so in Propositions 10 and 11, respectively.

Proposition 10 *Let $u_\alpha = u_\mu u^n u_\gamma$, where $\alpha = (\mu, 0, \cdots, 0, \gamma) \in \mathbb{Z}^{2k}_{\geq}$. Then we have*

$$\pi_{\mathfrak{I}_b}\left(\sum_{\beta \in D_\alpha} u_\beta\right) = (-1)^{\nu(\alpha,\geq 2)}\omega^{\nu(\alpha,1)}P_\alpha(\omega)u_\alpha. \tag{56}$$

Proof We divide μ and γ into n parts and denote each part by a_i for μ and b_i for γ, where $i = 1, 2, \cdots, n$, such that $\vec{a} = (a_1, ..., a_n) \sim \mu$ and $\vec{b} = (b_1, ..., b_n) \sim \gamma$. Note that it is possible that the length of some a_j (and/or b_j) is zero, in which case we take the convention $u_{a_j} = 1, |a_j| = 0$. Clearly we have

$$p = (0, b_1, a_1, 0, b_2, a_2 \ldots, 0, b_n, a_n) \in D_\alpha; \quad q = (a_1, 0, b_1, a_2, 0, b_2 \ldots, a_n, 0, b_n) \in A_\alpha.$$

Thus, in the quotient algebra, we obtain

$$\pi_{\mathfrak{I}_b}\left(\prod_{i=1}^n u u_{b_i} u_{a_i}\right) = \prod_{i=1}^n (-1)^{\nu(a_i,\geq 2)+|a_i||b_i|}\omega^{\nu(a_i,1)}u_{a_i}uu_{b_i}$$
$$= \omega^{\nu(\mu,1)}(-1)^{\nu(\mu,\geq 2)}(-1)^{\sum_{i=1}^n |a_i||b_i|}\prod_{i=1}^n u_{a_i}uu_{b_i}.$$

We denote $\sum_{i=1}^n |a_i||b_i|$ by $\vec{a}\cdot\vec{b}$ and note that $\nu(\mu, 1) = \nu(\alpha, 1)$ and $\nu(\mu, \geq 2) = \nu(\alpha, \geq 2)$. Hence,

$$\pi_{\mathfrak{I}_b}\left(\sum_{p\in D_\alpha} u_p\right) = \pi_{\mathfrak{I}_b}\left(\sum_{(\vec{a},\vec{b})}\prod_{i=1}^n uu_{b_i}u_{a_i}\right)$$
$$= \omega^{\nu(\alpha,1)}(-1)^{\nu(\alpha,\geq 2)}\pi_{\mathfrak{I}_b}\left(\sum_{\vec{a}\cdot\vec{b}=0 \bmod 2}\prod_{i=1}^n u_{a_i}uu_{b_i} - \sum_{\vec{a}\cdot\vec{b}=1 \bmod 2}\prod_{i=1}^n u_{a_i}uu_{b_i}\right)$$
$$= \omega^{\nu(\alpha,1)}(-1)^{\nu(\alpha,\geq 2)}\pi_{\mathfrak{I}_b}\left(\sum_{q\in A_\alpha}\prod_{i=1}^n u_{a_i}uu_{b_i} - 2\sum_{\vec{a}\cdot\vec{b}=1 \bmod 2}\prod_{i=1}^n u_{a_i}uu_{b_i}\right).$$

Note that the first term gives us the required identity (56) using (54). Thus, we are left to prove that

$$\pi_{\mathfrak{I}_b}\left(\sum_{\vec{a}\cdot\vec{b}=1 \bmod 2}\prod_{i=1}^{n} u_{a_i} u u_{b_i}\right) = 0.$$

From now on, we identify a pair of vectors $(\vec{a}, \vec{b})$ with $\prod_{i=1}^{n} u_{a_i} u u_{b_i}$. Let

$$\Sigma = \{(\vec{a}, \vec{b}),\ \vec{a}\cdot\vec{b} = 1 \mod 2\}.$$

We split this set in two equal parts Y and Z after the following remarks. Let c be the number of indices i such that $|a_i|$ and $|b_i|$ are both odd and d the number of indices such that $|a_i|$ and $|b_i|$ are both even. When none of this is true, the parity of $|a_i| + |b_i|$ is odd.

Since the length of α is even , the parity of $|\mu| + |\gamma|$ is the same as n. Hence,

$$n = \sum_{i=1}^{n} |a_i| + |b_i| \mod 2 = n - c - d \mod 2,$$

which implies that $c + d$ is even. Moreover, we know that $\vec{a}\cdot\vec{b}$ is odd, that is,

$$1 = \sum_{i=1}^{n} |a_i||b_i| \mod 2 = c \mod 2.$$

Thus, we have that both c and d are odd.

Let $\mathcal{I} = \{i_1, ..., i_{c+d}\}$ be the set of indices i such that $|a_i| + |b_i|$ is even (We know that this set has cardinal $c + d$). Let l be minimal so that $|a_{i_l}|$ and $|a_{i_{c+d+1-l}}|$ have different parity. Such l exists and is unique. Indeed, if it did not exist, we would have $|a_{i_l}| \equiv |a_{i_{l+c+d-1}}|$ for all l implying that c and d are even.

We denote i_l by $k(\vec{a}, \vec{b})$ and $i_{c+d+l-1}$ by $m(\vec{a}, \vec{b})$. However, in the sequel we will abuse notation and simply write k and m, knowing that we have fixed the element $(\vec{a}, \vec{b})$ in the set Σ. Based on these definitions, we put the pair $(\vec{a}, \vec{b})$ in the set Y if $|a_k|$ is odd and we put it in Z if $|a_k|$ is even.

Let $q \in Y$ and $u_q = \prod_{i=1}^{n} u_{a_i} u u_{b_i}$. We are going to construct a bijective map $\phi : Y \mapsto Z$ such that $\phi(u_q) \overset{\mathfrak{I}_b}{\simeq} -u_q$ in the quotient algebra for all $q \in Y$. Define

$$\phi(u_q) = (\xi_{m-1} ... \xi_k)(u_q),$$

where the maps ξ_i are defined in Lemma 17 in Appendix. Thus, ϕ only transforms the product from the block k to the block m, i.e. $\prod_{i=k}^{m} u_{a_i} u u_{b_i}$.

By definition of the maps ξ_i, if we represent $\phi(u_q)$ as $(\vec{c}, \vec{d})$ we see that c_k and d_k will have even length and that c_m and d_m will have odd length. It means that $\phi(u_q)$ is an element of Z, but also that we still have $k(\phi(u_q)) = k$ and $m(\phi(u_q)) = m$. That is because we have left the first $k-1$ blocks and the last $n-m$ blocks intact. Since the values of k and m are unchanged by ϕ and that all the ξ_i's are bijections, it follows that ϕ is a bijection as well. So it only remains to check that $\phi(u_q) \overset{\mathfrak{I}_b}{\simeq} -u_q$. By Lemma 17, we have

$$\phi(u_q) \overset{\mathfrak{I}_b}{\simeq} (-1)^{\eta} u_q$$

with

$$\eta = |b_k| + |a_{k+1}| + |b_{k+1}| + 1 + |a_{k+2}| + \dots + |b_{m-1}| + 1 + |a_m|$$

We know that $|b_k| = 1 \mod 2$ and $|a_m| = 0 \mod 2$. Hence,

$$\eta = 1 + \sum_{i=k+1}^{m-1} (|a_i| + |b_i| + 1) \mod 2 = 1 \mod 2$$

since there is a even number of indices i for which $|a_i| \equiv |b_i|$ between k and m. □

Below we give an example to illustrate this proposition.

Example 1 Let $\alpha = (1, 1, 0, 0, 0, \text{-}1)$. We write as $\alpha = 11000\text{-}1$ for short. There are 18 elements in the set A_α. Indeed, to get an admissible monomial equivalent to α one needs to pick an element in

$$\{11000, 10100, 10010, 01100, 01010, 00110\}$$

and an element in

$$\{000-1, 00-10, 0-100\}.$$

Under the ideal $\mathfrak{I}_b$, we have

$$P_\alpha(\omega) = 1 + 2\omega^2 + 2\omega^4 + \omega^6.$$

Similarly there are 18 elements in D_α since they are determined by the choice of an element in $\{01100, 01010, 01001, 00110, 00101, 00011\}$ and an element in $\{000-1, 00-10, 0-100\}$. So we have

$$\pi_{\mathfrak{I}_b}\left(\sum_{\beta \in D_\alpha} u_\beta\right) = \omega^2 + 2\omega^4 + 2\omega^6 + \omega^8 = \omega^2 P_\alpha(\omega),$$

which is consistent with (56) since $\nu(\alpha, \geq 2) = 0$ and $\nu(\alpha, 1) = 2$.

Following the line of Proposition 10's proof, with this example we first give a full description of the set Σ, then split it as $\Sigma = Y \cup Z$. An admissible monomial is given by a partition of $|a_1|+|a_2|+|a_3|=2$ and a partition $|b_1|+|b_2|+|b_3|=1$. For this monomial to be in Σ, we need $|a_1||b_1|+|a_2||b_2|+|a_3||b_3|$ to be odd. It must be that $(|b_1|,|b_2|,|b_3|)$ is one of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Hence, there are 6 elements in Σ:

$$\Sigma=\{10-1100, 1010-10, 010-110, 01010-1, 10-1010, 10010-1\},$$

where 3 elements belong to Y, namely

$$Y=\{10{-}1100, 1010{-}10, 10{-}1010\}.$$

For each element in Y, we first identify the blocks k and m, to remove a 1 and a -1 from the block k and to add them to the block m. We now write Z in the same order, that is, $Z=\phi(Y)$:

$$Z=\{010\text{-}110, 01010\text{-}1, 10010\text{-}1\}.$$

One can check that $\pi_{\mathfrak{I}_b}(\sum_{\beta\in\Sigma}u_\beta)=0$ and $\pi_{\mathfrak{I}_b}(\sum_{\beta\in Y}u_\beta)=-\pi_{\mathfrak{I}_b}(\sum_{\beta\in Z}u_\beta)$.

Proposition 11 *Let $u_\alpha=u_\mu u^n u_\gamma$, where $\alpha=(\mu,0,\cdots,0,\gamma)\in\mathbb{Z}^{2k}_{\geq}$. Then, we have*

$$\pi_{\mathfrak{I}_b}\left(\sum_{\beta\in C_\alpha}u_\beta\right)=(-1)^{\nu(\alpha,\geq 0)}\omega^{\nu(\alpha,0)}P_{\alpha-1}(-\omega)u_\alpha. \tag{57}$$

Proof Note that $\beta\in C_\alpha$ if and only if $\mathcal{T}\mathcal{S}^{-1}(\beta)\in D_{\mathcal{T}(\alpha-1)}$, where $\mathcal{T}$ is the antiautomorphism. Hence, we have

$$\mathcal{T}\mathcal{S}^{-1}(C_\alpha)=D_{\mathcal{T}(\alpha-1)}.$$

Moreover, by definition of the map $\mathcal{T}$, it is clear that $\mathcal{T}(A_{\alpha-1})=A_{\mathcal{T}(\alpha-1)}$. Using these facts and Proposition 10, we obtain

$$\begin{aligned}
\sum_{\beta\in C_\alpha}u_\beta &= \mathcal{S}\mathcal{T}\left(\sum_{\beta\in C_\alpha}\mathcal{T}\mathcal{S}^{-1}(u_\beta)\right)=\mathcal{S}\mathcal{T}\left(\sum_{\beta\in D_{\mathcal{T}(\alpha-1)}}u_\beta\right)\\
&\overset{\mathfrak{I}_b}{\simeq}\mathcal{S}\mathcal{T}\left((-1)^{\nu(\mathcal{T}(\alpha-1),\geq 2)}\omega^{\nu(\mathcal{T}(\alpha-1),1)}\sum_{\beta\in A_{\mathcal{T}(\alpha-1)}}u_\beta\right)\\
&\overset{\mathfrak{I}_b}{\simeq}(-1)^{\nu(\alpha,\leq -1)}\omega^{\nu(\alpha,0)}\mathcal{S}\mathcal{T}\sum_{\beta\in A_{\mathcal{T}(\alpha-1)}}u_\beta\overset{\mathfrak{I}_b}{\simeq}(-1)^{\nu(\alpha,\geq 0)}\omega^{\nu(\alpha,0)}\mathcal{S}\sum_{\beta\in A_{\alpha-1}}u_\beta\\
&\overset{\mathfrak{I}_b}{\simeq}(-1)^{\nu(\alpha,\geq 0)}\omega^{\nu(\alpha,0)}\mathcal{S}\left(P_{\alpha-1}(\omega)u_{\alpha-1}\right)\overset{\mathfrak{I}_b}{\simeq}(-1)^{\nu(\alpha,\geq 0)}\omega^{\nu(\alpha,0)}P_{\alpha-1}(-\omega)u_\alpha,
\end{aligned}$$

which leads to (57) since $\alpha\in\mathbb{Z}^{2k}_{\geq}$. □

Having evaluated all terms in (53), we are now in the position to prove the similar result as Proposition 7 for the ideal $\mathfrak{I}_b$ defined by (7).

Proposition 12 *Let* $\alpha \in \mathcal{Z}^{2k}_{\geq}$*. Then, we have*

$$P_\alpha(\omega) + (-1)^{\nu(\alpha,\geq 0)}\omega^{\nu(\alpha,0)}P_{\alpha-1}(-\omega) = P_{\alpha-1}(-\omega) + (-1)^{\nu(\alpha,\geq 2)}\omega^{\nu(\alpha,1)}P_\alpha(\omega). \tag{58}$$

Proof First note that this formula holds whenever $\alpha \notin \mathcal{A}^{2k}$ or $\alpha - 1 \notin \mathcal{A}^{2k}$ in the same reason as in the proof for Proposition 7. When $\alpha \in \mathcal{A}^{2k}$ and $\alpha - 1 \in \mathcal{A}^{2k}$, we substitute (54)–(57) into (53) and this leads to the required identity (58). □

Similar to Corollary 8 for the case of ideal $\mathfrak{I}_a$, we have the following statement for the case of ideal $\mathfrak{I}_b$:

Corollary 13 *Let* $\alpha \in \mathcal{Z}^{2k}_{\geq}$*. There exists a nonzero rational function* $R_\alpha(\omega) \in \mathbb{Q}(\omega)$ *such that*

$$P_{\alpha+m}((-1)^m\omega) = R_\alpha(\omega)(1 - (-1)^{\nu(\alpha,\geq -m)+m\nu(\alpha,-m)}\omega^{\nu(\alpha,-m)}) \quad \textit{for all} m \in \mathbb{Z}. \tag{59}$$

Proof Without the loss of generality, we assume that $\alpha - l \in \mathcal{A}^{2k}$, for $0 \leq l \leq q$. Let

$$R_\alpha(\omega) = \frac{P_\alpha(\omega)}{1 - (-1)^{\nu(\alpha,\geq 0)}\omega^{\nu(\alpha,0)}}.$$

The identity (58) implies that

$$R_{\alpha-1}(-\omega) = R_\alpha(\omega).$$

Thus, for $0 \leq l \leq q$ we have

$$\begin{aligned} P_{\alpha-l}((-1)^l\omega) &= R_{\alpha-l}((-1)^l\omega)\left(1 - (-1)^{\nu(\alpha,\geq l)+l\nu(\alpha,l)}\omega^{\nu(\alpha,l)}\right) \\ &= R_\alpha(\omega)\left(1 - (-1)^{\nu(\alpha,\geq l)+l\nu(\alpha,l)}\omega^{\nu(\alpha,l)}\right). \end{aligned}$$

When $\alpha + m \notin \mathcal{A}^{2k}$, we have $P_{\alpha+m}(\omega) = 0$ following the definition of (43). □

Theorem 14 *The quantisation ideal* $\mathfrak{I}_b$ *is stable with respect to every even member of the Volterra hierarchy* $\partial_{t_{2\ell}}(u) = \mathcal{S}(X^{(2\ell)})u - u\mathcal{S}^{-1}(X^{(2\ell)}),\ \ell \in \mathbb{N}$.

Proof Let $u_\tau = G = X_1^{(2\ell)}u - uX_{-1}^{(2\ell)}$, where $X^{(2\ell)}$ is the sum of all admissible monomials of size 2ℓ, $\ell \geq 1$. Let $k \geq 2$. We want to show that $\partial_\tau(uu_k + u_ku)$ is in the ideal $\mathfrak{I}_b$. By definition of u_τ, this means that

$$\pi_{\mathfrak{I}_b}(Gu_k + uG_k + G_ku + u_kG) = 0, \tag{60}$$

or, in terms of X (we drop its upper index):

$$\begin{aligned} & uX_{k+1}u_k + X_{k+1}u_k u - uu_k X_{k-1} - u_k X_{k-1} u \\ & \quad + X_1 uu_k + u_k X_1 u - uX_{-1} u_k - u_k u X_{-1} \overset{\mathfrak{I}_b}{\simeq} 0. \end{aligned} \tag{61}$$

Let us fix an element $\beta \in \mathcal{Z}^{2\ell}$. We are going to show that the terms equivalent to $u_\beta u u_k$ modulo multiplication by an element of $\mathbb{Z}[\omega]$ in (61) cancel out. It is clear that

$$\begin{aligned} uu_\beta u_k &\overset{\mathfrak{I}_b}{\simeq} (-1)^{\nu(\beta,0)+\nu(\beta,1)} \omega^{\nu(\beta,1)-\nu(\beta,-1)} u_\beta u u_k, \\ u_k u_\beta u &\overset{\mathfrak{I}_b}{\simeq} (-1)^{\nu(\beta,k)+\nu(\beta,k+(-1)^k)} \omega^{\nu(\beta,k+1)-\nu(\beta,k-1)} u_\beta u_k u, \\ uu_k u_\beta &\overset{\mathfrak{I}_b}{\simeq} (-1)^{\nu(\beta,k)+\nu(\beta,k+(-1)^k)+\nu(\beta,0)+\nu(\beta,1)} \\ &\quad \omega^{\nu(\beta,k+1)+\nu(\beta,1)-\nu(\beta,0)-\nu(\beta,k-1)} u_\beta u u_k. \end{aligned}$$

We know that for all $m \in \mathbb{Z}$,

$$\pi_{\mathfrak{I}_b}(X_m) = \sum_{\alpha \in \mathcal{Z}^{2n}_{\geq}} P_\alpha((-1)^m \omega) u_{\alpha+m}.$$

Hence, the $\mathbb{Z}[\omega]$ coefficient of $u_\beta u u_k$ in $uX_{k+1}u_k + X_{k+1}u_k u$ is

$$P_{\beta-k-1}((-1)^{k+1}\omega)((-1)^{\nu(\beta,0)+\nu(\beta,1)} \omega^{\nu(\beta,1)-\nu(\beta,-1)} - 1).$$

We compute the terms coming from X_{-k-1}, X_1 and X_{-1} in a similar way. Thus, to prove that the coefficient of $u_\beta u u_k$ in (61) is zero amounts to check that

$$\begin{aligned} 0 &= P_{\beta-k-1}((-1)^{k+1}\omega)((-1)^{\nu(\beta,0)+\nu(\beta,1)} \omega^{\nu(\beta,1)-\nu(\beta,-1)} - 1) \\ &\quad + P_{\beta-k+1}((-1)^{k-1}\omega)(-1)^{\nu(\beta,k)+\nu(\beta,k+(-1)^k)} \omega^{\nu(\beta,k+1)-\nu(\beta,k-1)} \\ &\quad (1 - (-1)^{\nu(\beta,0)+\nu(\beta,1)} \omega^{\nu(\beta,1)-\nu(\beta,0)}) \\ &\quad + P_{\beta-1}(-\omega)(1 - (-1)^{\nu(\beta,k)+\nu(\beta,k+(-1)^k)} \omega^{\nu(\beta,k+1)-\nu(\beta,k-1)}) \\ &\quad + P_{\beta+1}(-\omega)(-1)^{\nu(\beta,0)+\nu(\beta,1)} \omega^{\nu(\beta,1)-\nu(\beta,-1)} \\ &\quad ((-1)^{\nu(\beta,k)+\nu(\beta,k+(-1)^k)} \omega^{\nu(\beta,k+1)-\nu(\beta,k-1)} - 1). \end{aligned}$$

Using Corollary 13, we need to verify

$$
\begin{aligned}
&(1-(-1)^{\nu(\beta,\geq k+1)+(k+1)\nu(\beta,k+1)}\omega^{\nu(\beta,k+1)})((-1)^{\nu(\beta,0)+\nu(\beta,1)}\omega^{\nu(\beta,1)-\nu(\beta,-1)}-1)\\
&\quad+(1-(-1)^{\nu(\beta,\geq k-1)+(k+1)\nu(\beta,k-1)}\omega^{\nu(\beta,k-1)})(-1)^{\nu(\beta,k)+\nu(\beta,k+(-1)^k)}\omega^{\nu(\beta,k+1)-\nu(\beta,k-1)}\\
&\quad\times(1-(-1)^{\nu(\beta,0)+\nu(\beta,1)}\omega^{\nu(\beta,1)-\nu(\beta,0)})\\
&\quad+(1-(-1)^{\nu(\beta,\geq 1)+\nu(\beta,1)}\omega^{\nu(\beta,1)})(1-(-1)^{\nu(\beta,k)+\nu(\beta,k+(-1)^k)}\omega^{\nu(\beta,k+1)-\nu(\beta,k-1)})\\
&\quad+(1-(-1)^{\nu(\beta,\geq -1)+\nu(\beta,-1)}\omega^{\nu(\beta,-1)})(-1)^{\nu(\beta,0)+\nu(\beta,1)}\omega^{\nu(\beta,1)-\nu(\beta,-1)}\\
&\quad\times((-1)^{\nu(\beta,k)+\nu(\beta,k+(-1)^k)}\omega^{\nu(\beta,k+1)-\nu(\beta,k-1)}-1)\\
&=0
\end{aligned}
$$

, and thus, the identity (60) holds. The proof that $\pi_{\mathfrak{I}_b}\left(\partial_\tau(u_k u_{k+1}-(-1)^k\omega u_{k+1}u_k)\right)=0$ for all $k\in\mathbb{Z}$ is similar and we will not repeat it. □

6 Summary and discussion

In this paper, we develop the method of quantisation of dynamical systems defined on free associative algebras based on the concept of quantisation ideals [5]. It enables us to determine possible commutation relations between the dynamical variables which are consistent with the dynamical system and define associative multiplication in the quotient algebra. The method does not use any information on the Poisson structure of the dynamical system and enables us to find non-deformation quantisations of the system. To determine commutation relations consistent with a system is a very first step to its quantum theory. Next steps will require the development of the representation theory for the quantised algebras obtained and study the spectral theory of the operators involved.

In this paper, we explicitly proved that the nonabelian Volterra system (2) and its infinite hierarchy of symmetries admit the deformation quantisation with commutation relations (6). We also proved that the sub-hierarchy, consisting of all odd degree symmetries, admits a non-deformation quantisation with commutation relations (8). The existence of non-deformation quantisations is quite surprising. Further study is required to explore the properties of these new remarkable quantum algebra and quantum integrable equations.

Recently, when the paper has already been submitted to the journal, we found explicit expressions for the infinite sequence of quantum Hamiltonians H_n corresponding to the $\mathfrak{I}_a$ quantisation of the Volterra hierarchy

$$H_\ell=\sum_{k\in\mathbb{Z}}\sum_{\alpha\in\mathcal{A}_0^\ell}\frac{\omega^\ell-1}{\omega^{\nu(\alpha,0)}-1}P_\alpha(\omega)u_{\alpha+k},$$

where $\mathcal{A}_0^\ell=\{\alpha\in\mathcal{A}^\ell\cap\mathcal{Z}_\geq^\ell;\ \alpha_\ell=0\}$. Assuming that $\omega=e^{2\mathrm{i}\hbar}$, $\hbar\in\mathbb{R}$, the Hamiltonians H_ℓ are self-adjoint $H_\ell^\dagger=H_\ell$. They commute with each other, and the dynamical equations of the quantum hierarchy can be written in the Heisenberg form [compare with (20)]:

$$\partial_{t_\ell}(u_n) = \frac{\mathrm{i}}{2\sin(\ell\hbar)}[H_\ell, u_n], \quad n \in \mathbb{Z},\ \ell \in \mathbb{N}\,.$$

We have also found explicit expressions for self-adjoint commuting quantum Hamiltonians corresponding to non-deformation quantisation (8) and present the quantum hierarchy with even times in the Heisenberg form. A detail proof of these results will be published elsewhere soon.

The Volterra hierarchy admits periodic reductions with any positive integer period M. We have shown that the Volterra system with periods $M = 3, 4$ admit quantisations with non-homogeneous commutation relations (Theorem 5). When $M = 3$, we proved the resulting quantum system is not only super integrable but also admits bi-quantum structure, similar to its bi-Hamiltonian structure in the classical case. The cubic symmetry of the Volterra system with period $M = 4$ admits three distinct quantisations. In each case, the quantum system is a super-integrable systems (Theorem 6). Systems with periods $M \geq 5$ require more work, they have not been studied in this paper in any detail.

The methods developed in [5] and this paper can be applied to the nonabelian Narita–Itoh–Bogoyavlensky lattice [17]

$$u_t = \sum_{k=1}^{p} (u_k u - u u_{-k})\,, \quad p \in \mathbb{N}. \tag{62}$$

The Volterra equation is corresponding to the case when $p = 1$. Our study shows that system (62), and all equations of its hierarchy admit the quantisation with commutation relations

$$u_n u_{n+k} = \omega u_{n+k} u_n\,,\ \ 1 \leq k \leq p, \quad u_n u_m = u_m u_n\,, \quad |n-m| > p \quad n, m \in \mathbb{Z},$$

where ω is a nonzero constant. The proof of this statement will be published elsewhere. These commutation relations were also obtained by Inoue and Hikami [20] using ultra-local Lax representation and R–matrix technique.

Besides quadratic ideals, our computations for the nonabelian Volterra equation and its lower degree symmetries suggest that there is a ∂_{t_ℓ}–stable ideal generated by quadratic and cubic homogeneous polynomials. For example, as far as we have checked, the first few symmetries in the nonabelian Volterra hierarchy leave the following cubic ideal invariant:

$$\tilde{\mathfrak{I}} = \langle u_n u_{n+1} u_{n-1} - u_{n+1} u_{n-1} u_n\,,\ u_n u_m - u_m u_n\,;\ |n-m| > 1,\ n, m \in \mathbb{Z}\rangle.$$

Further research is needed to study the properties of the Volterra chain which is well defined on the quotient algebra $\mathfrak{A}\big/\tilde{\mathfrak{I}}$. Very little is known about this new invariant ideal and the quotient algebra which does not satisfy the condition (ii).

The concept of quantisation ideals has not been linked yet with Lax representations, recursion operators, master symmetries and other objects associated with the theory of integrable systems. We think that further development of this theory will enable us to embrace a wide range of integrable systems as well as to clarify and simplify rather technical proofs of the statements presented in this paper.

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Declarations

Conflict of interest We declare that there is no conflict of interests.

Appendix: Lemmas used for the proof of Proposition 10

In appendix, we are going to prove the lemmas used in constructing the bijection map between sets A_α and D_α (Proposition 10) in Sect. 5.2.

Let l be any integer. We denote by Λ_l the set of admissible monomials of the form $u_a u_l u_b$ satisfying

(i) both a and b have components greater than l if they are not empty.
(ii) there exists a suffix d of a of odd length $a = cd$ where c is either empty or ends with $l+1$.
(iii) if b is non-empty, then it ends with $l+1$.

If the length of d in (ii) is minimal, we say that d is the minimal odd suffix of a.

We denote by Γ_l the set of admissible monomials of the form $u_a u_l u_b$ where

(i) both a and b have components greater than l.
(ii) there exists a prefix c of b of odd length $b = cd$ where c ends with $l+1$.
(iii) b ends with $l+1$.

If the length of c in (ii) is minimal, we say that c is the minimal odd prefix of b.

Lemma 15 *For all $l \in \mathbb{Z}$, we construct a bijection $\psi : \Lambda_l \to \Gamma_l$ such that for all $x \in \Lambda_l$, $\pi_{\mathfrak{J}_b}(\psi(x)) = (-1)^l \omega\, x$. Moreover, if $x = u_a u_l u_b$ and $\psi(x) = u_c u_l u_d$, then $|c| = |a| - |m|$ and $|d| = |b| + |m|$, where m is the minimal odd suffix of a.*

Proof We construct ψ by induction on $|a| + |b|$. The only element of length 2 in Λ_l is $u_{l+1}u_l$, while the only element of length 2 in Γ_l is $u_l u_{l+1}$. We let $\psi(u_{l+1}u_l) = u_l u_{l+1}$.

The minimal odd suffix of u_{l+1} is itself, and we have $\pi_{\mathfrak{J}_b}(u_l u_{l+1}) = (-1)^l \omega u_{l+1} u_l$; hence, the statement of the lemma holds for elements of length 2.

Suppose that we have constructed ψ for all lengths strictly less than n satisfying the statement. We now construct ψ for elements of length n and prove it satisfies the statement. Let $u_a u_l u_b$ be an element of Λ_l of length n. Let d be the minimal odd suffix of a. Explicitly, this u_d has the form $u_e u_{l+1} u_{d_1} u_{l+1} \cdots u_{d_p} u_{l+1}$, where the $|d_i|$'s are odd and $|e|$ is even (hence possibly e is empty). Note that in this decomposition of u_d, the elements d_i and e do not contain any $j < l+2$ and all end with $l+2$ (except if e is empty). Hence for all $i = 1, ..., p$, $u_{l+1} u_{d_i}$ is an element of Γ_{l+1} whose length is strictly less than n. By the induction hypothesis, there exist f_i of odd length and g_i of even length such that

$$\psi^{-1}(u_{l+1} d_i) = u_{f_i} u_{l+1} u_{g_i}.$$

Note that f_i does not have a proper odd suffix due to the last assertion in the lemma. Recall that all elements in f_i and g_i are greater than $l+1$. The element $u_e \psi^{-1}(u_{l+1} u_{d_1}) \cdots \psi^{-1}(u_{l+1} u_{d_p}) u_{l+1}$ is well defined. It has exactly the same (odd) length as d without any proper odd prefix and

$$\pi_{\mathfrak{J}_b}(u_e \psi^{-1}(u_{l+1} u_{d_1}) \cdots \psi^{-1}(u_{l+1} u_{d_p}) u_{l+1}) = ((-1)^{l+1}\omega)^{-p} u_e u_{l+1} u_{d_1} u_{l+1} \cdots u_{d_p} u_{l+1}.$$

We let

$$\psi(u_a u_l u_b) = u_c u_l u_e \psi^{-1}(u_{l+1} u_{d_1}) \cdots \psi^{-1}(u_{l+1} u_{d_p}) u_{l+1} u_b.$$

Note that the last statement in the lemma is satisfied. Let

$$\chi = u_e \psi^{-1}(u_{l+1} u_{d_1}) \cdots \psi^{-1}(u_{l+1} u_{d_p}) u_{l+1}.$$

It has odd length and the number of u_{l+1} in χ is $p+1$. Thus, we have in the quotient algebra

$$\pi_{\mathfrak{J}_b}(u_l \chi) = (-1)^{1+(l+1)(p+1)} \omega^{p+1} \chi u_l;$$

hence,

$$\begin{aligned}\pi_{\mathfrak{J}_b}&(u_l u_e \psi^{-1}(u_{l+1} u_{d_1}) \cdots \psi^{-1}(u_{l+1} u_{d_p}) u_{l+1}) \\ &= \pi_{\mathfrak{J}_b}(u_l \chi) = (-1)^l \omega u_e u_{l+1} u_{d_1} u_{l+1} \cdots u_{d_p} u_{l+1} u_l\end{aligned}$$

and a fortiori,

$$\pi_{\mathfrak{J}_b}(\psi(u_a u_l u_b)) = (-1)^l \omega u_a u_l u_b.$$

We know that there are as many elements of length n in Γ_l as in Λ_l; hence, it remains to check the injectivity of ψ for length n. Suppose that we have $\psi(u_a u_l u_b) = \psi(u_{\tilde{a}} u_l u_{\tilde{b}})$.

In other words, we have

$$
\begin{aligned}
&u_c u_l u_e \psi^{-1}(u_{l+1}u_{d_1})\cdots\psi^{-1}(u_{l+1}u_{d_p})u_{l+1}u_b \\
&= u_{\tilde{c}} u_l u_{\tilde{e}} \psi^{-1}(u_{l+1}u_{\tilde{d}_1})\cdots\psi^{-1}(u_{l+1}u_{\tilde{d}_q})u_{l+1}u_{\tilde{b}}
\end{aligned}
$$

This equality implies that $c = \tilde{c}$ so we can simplify it slightly:

$$
\begin{aligned}
&u_e \psi^{-1}(u_{l+1}u_{d_1})\cdots\psi^{-1}(u_{l+1}u_{d_p})u_{l+1}u_b \\
&= u_{\tilde{e}} \psi^{-1}(u_{l+1}u_{\tilde{d}_1})\cdots\psi^{-1}(u_{l+1}u_{\tilde{d}_q})u_{l+1}u_{\tilde{b}}
\end{aligned}
$$

Recall that $u_e\psi^{-1}(u_{l+1}u_{d_1})\cdots\psi^{-1}(u_{l+1}u_{d_p})u_{l+1}$ is the minimal odd prefix of the left hand side and that $u_{\tilde{e}}\psi^{-1}(u_{l+1}u_{\tilde{d}_1})\cdots\psi^{-1}(u_{l+1}u_{\tilde{d}_q})u_{l+1}$ is the minimal odd prefix of the right hand side. By unicity of the minimal odd prefix, they are equal. In particular, we have $b = \tilde{b}$ and $p = q$. Recall the definition of f_i and g_i such that $\psi^{-1}(u_{l+1}u_{d_i}) = u_{f_i}u_{l+1}u_{g_i}$. Similarly, we write

$$
\psi^{-1}(u_{l+1}u_{\tilde{d}_i}) = u_{\tilde{f}_i}u_{l+1}u_{\tilde{g}_i}.
$$

We have

$$
u_{g_0}u_{f_1}u_{l+1}u_{g_1}u - f_2u_{l+1}\cdots u_{f_p}u_{l+1}u_{g_p} = u_{\tilde{g}_0}u_{\tilde{f}_1}u_{l+1}u_{\tilde{g}_1}u_{\tilde{f}_2}u_{l+1}\cdots u_{\tilde{f}_p}u_{l+1}u_{\tilde{g}_p},
$$

where we have let $g_0 = e$ and $\tilde{g}_0 = \tilde{e}$. Therefore, we have for all $i = 0, ..., p-1$

$$
g_i f_{i+1} = \tilde{g}_i \tilde{f}_{i+1}.
$$

Recall that both f_{i+1} and $\tilde{f}_{i+1}$ are their own minimal odd suffix. Hence, f_{i+1} is the minimal odd suffix of $g_i f_{i+1}$ and $\tilde{f}_{i+1}$ is the minimal odd suffix of $\tilde{g}_i\tilde{f}_{i+1}$. By unicity of the minimal odd suffix we have $f_{i+1} = \tilde{f}_{i+1}$, from where it follows that $g_i = \tilde{g}_i$. Hence,

$$
u_{l+1}u_{d_i} = \psi(u_{f_i}u_{l+1}u_{g_i}) = \psi(u_{\tilde{f}_i}u_{l+1}u_{\tilde{g}_i}) = u_{l+1}u_{\tilde{d}_i},
$$

and thus, we complete the proof. □

Let l be any integer. We denote by Θ_l the set of admissible monomials of the form $u_a u_l u_b$ where

(i) both a and b have components strictly smaller than l.
(ii) there exists a suffix d of a of odd length $a = cd$ where d starts with $l-1$.
(iii) a starts with $l-1$.

If the length of d in (ii) is minimal, we say that d is the minimal odd suffix of a.

We denote by Φ_l the set of admissible monomials of the form $u_a u_l u_b$ where

(i) both a and b have components strictly smaller than l.

(ii) there exists a prefix c of b of odd length $b = cd$ where d is either empty or starts with $l - 1$.
(iii) a is either empty or starts with $l - 1$.

If the length of c in (ii) is minimal, we say that c is the minimal odd prefix of b.

Lemma 16 *For all $l \in \mathbb{Z}$, we construct a bijection $\rho : \Theta_l \to \Phi_l$ such that $\pi_{\mathfrak{I}_b}(\rho(x)) = (-1)^{l+1}\omega^{-1}x$ for all $x \in \Theta_l$. Moreover, if $x = u_a u_l u_b$ and $\psi(x) = u_c u_l u_d$, then $|c| = |a| - |m|$ and $|d| = |b| + |m|$, where m is the minimal odd suffix of a.*

Proof Take $\rho = \mathcal{T}\psi^{-1}\mathcal{T}$, where $\mathcal{T}$ maps Θ_l to Γ_l and maps Λ_l to Φ_l. Let $u_a u_l u_b \in \Theta_l$. We have

$$\psi^{-1}(\mathcal{T}(b)u_{-l}\mathcal{T}(a)) \equiv (-1)^l \omega^{-1}\mathcal{T}(b)u_{-l}\mathcal{T}(a)$$

and since $\mathcal{T}(\omega) = -\omega$,

$$\mathcal{T}(\psi^{-1}(\mathcal{T}(b)u_{-l}\mathcal{T}(a))) \equiv (-1)^{l+1}\omega^{-1}au_l b.$$

Let m be the minimal odd prefix of $\mathcal{T}(a)$. We know that $\psi^{-1}(\mathcal{T}(b)u_{-l}\mathcal{T}(a)) = cu_{-l}d$ with $|c| = |\mathcal{T}(b)| + |m|$ and $|d| = |\mathcal{T}(a)| - |m|$. We have $\rho(au_l b) = \mathcal{T}(d)u_l\mathcal{T}(c)$. We conclude by noting that $\mathcal{T}(m)$ is the minimal odd suffix of a. □

Recall that we identify an element of Σ, that is a pair $(\vec{a}, \vec{b})$ such that $\vec{a} \cdot \vec{b} = 1 \mod 2$ with the product $\prod_{i=1}^n u_{a_i} u u_{b_i}$. We denote a subset of X consisting of a part of Σ such that $u_{a_j} u \in \Lambda_0$ and $u_{b_j} u \in \Theta_0$ for some $1 \leq j \leq n$ by Σ_j . We are going to construct bijections $\xi_j : \Sigma_j \to \Sigma_{j+1}$.

Lemma 17 *There exists a bijection $\xi_j : \Sigma_j \to \Sigma_{j+1}$, $1 \leq j \leq n - 1$, so that*

$$\xi_j(u_p) \overset{\mathfrak{I}_b}{\simeq} (-1)^{|a_{j+1}|+|b_j|}u_p, \quad p \in \Sigma_j.$$

Proof Let $(\vec{a}, \vec{b})$ be an element of Σ_j. Consider the product of block j with block $j + 1$, i.e.

$$u_{a_j} u u_{b_j} u_{a_{j+1}} u u_{b_{j+1}}.$$

We have $a_j 0 a_{j+1} \in \Lambda_0$ and $b_j 0 b_{j+1} \in \Theta_0$. Hence, there exist $\tilde{a}_j, \tilde{\tilde{a}}_j, \tilde{b}_j, \tilde{\tilde{b}}_j$ such that,

$$\psi(u_{a_j} u u_{a_{j+1}}) = u_{\tilde{a}_j} u u_{\tilde{\tilde{a}}_j}, \quad \rho(u_{b_j} u u_{b_{j+1}}) = u_{\tilde{b}_j} u u_{\tilde{\tilde{b}}_j}.$$

From the definitions of ρ and ψ, it follows that $\tilde{\tilde{a}}_j 0 \in \Lambda_0$, $\tilde{\tilde{b}}_j 0 \in \Theta_0$ and

$$(|\tilde{a}_j|, |\tilde{\tilde{a}}_j|, |\tilde{b}_j|, |\tilde{\tilde{b}}_j|) = (|a_j| + 1, |a_{j+1}| + 1, |b_j| + 1, |b_{j+1}| + 1) \mod 2.$$

We now define $\xi_j : ((\vec{a}, \vec{b}) \mapsto (\vec{c}, \vec{d})$ as follows:

$$c_i = a_i \text{ and } d_i = b_i \text{ if } i \neq j \quad \text{and} \quad i \neq j+1$$
$$c_j = \tilde{a}_j,\ d_j = \tilde{b}_j,\ c_{j+1} = \tilde{\tilde{a}}_j, \quad \text{and} \quad d_{j+1} = \tilde{\tilde{b}}_j.$$

It is clear that $(\vec{c}, \vec{d})$ is in the subset Σ_{j+1}. The map ξ_j is a bijection since both ψ and ρ are bijections. Moreover, we have

$$\pi_{\mathfrak{I}_b}(u_{\tilde{a}_j} u u_{\tilde{\tilde{a}}_j}) = \omega u_{a_j} u u_{a_{j+1}}, \quad \pi_{\mathfrak{I}_b}(u_{\tilde{b}_j} u u_{\tilde{\tilde{b}}_j}) = -\omega^{-1} u_{b_j} u u_{b_{j+1}}.$$

We know $\pi_{\mathfrak{I}_b}(u_{a_j} u u_{b_j} u_{a_{j+1}} u b_{j+1}) = (-1)^{|b_j||a_{j+1}|} u_{a_j} u u_{a_{j+1}} u_{b_j} u u_{b_{j+1}}$. Therefore, we obtain

$$\pi_{\mathfrak{I}_b}(\xi_j(u_p))_{p \in X_j} = (-1)^{1+|b_|| a_{j+1}|} u_{a_1} u u_{b_1} \cdots u_{\tilde{a}_j} u u_{\tilde{\tilde{a}}_j} u_{\tilde{b}_j} u u_{\tilde{\tilde{b}}_j} \cdots u_{a_n} u u_{b_n} = (-1)^{|b_j|+|a_{j+1}|} u_q,$$

where $q = (\vec{c}, \vec{d}) \in \Sigma_{j+1}$, and thus, we complete the proof. □

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