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EQUAL RANK LOCAL THETA CORRESPONDENCE AS A STRONG MORITA EQUIVALENCE

BRAM MESLAND AND MEHMET HALUK ŞENGÜN

ABSTRACT. Let (G, H) be one of the equal rank reductive dual pairs (Mp_{2n}, O_{2n+1}) or (U_n, U_n) over a nonarchimedean local field of characteristic zero. It is well-known that the theta correspondence establishes a bijection between certain subsets, say \widehat{G}_θ and \widehat{H}_θ , of the tempered duals of G and H . We prove that this bijection arises from an equivalence between the categories of representations of two C^* -algebras whose spectra are \widehat{G}_θ and \widehat{H}_θ . This equivalence is implemented by the induction functor associated to a Morita equivalence bimodule (in the sense of Rieffel) which we construct using the oscillator representation. As an immediate corollary, we deduce that the bijection is functorial and continuous with respect to weak inclusion. We derive further consequences regarding the transfer of characters and preservation of formal degrees.

CONTENTS

1. Introduction	1
2. The local theta correspondence	5
3. Some $*$ -algebras associated to groups	7
4. Strong Morita equivalence	9
5. The oscillator bimodule	15
6. The θ -subalgebras and induced representations	19
7. Application: transfer of characters	23
8. Application: preservation of formal degrees	24
References	27

1. INTRODUCTION

In this paper, we shed new light onto the fundamental mechanism of the equal rank tempered local theta correspondence by approaching it via the framework of the representation theory of C^* -algebras.

Local theta correspondence, founded by Roger Howe in the mid 1970's, is a major theme in the theory of automorphic forms and representation theory. In a nutshell, local theta correspondence establishes a bijection between certain sets of smooth irreducible representations of reductive groups G and H which form a “dual pair”, that is, G and H sit inside a large enough symplectic group in such a way that they form each others' centralisers. Roughly speaking, this bijection is obtained by considering how the so-called oscillator representation of the ambient symplectic group decomposes as a $G \times H$ -representation. When the two groups have “the same size”, the local theta correspondence enjoys several attractive properties, in particular, it preserves temperedness. It is this tempered correspondence in the equal rank case that we will consider.

The second ingredient of our paper is the notion of strong Morita equivalence for C^* -algebras introduced by Mark Rieffel, again, in the mid 1970's, as part of his C^* -algebraic generalisation of Mackey's theory of induced representations of locally compact groups. Given two C^* -algebras A and B , roughly put, an “equivalence A - B -bimodule” X is an A - B -bimodule which is equipped with

an A -valued inner product and a B -valued inner product such that these inner products satisfy certain compatibility and continuity conditions. If such a bimodule exists, then A and B are said to be “*strongly Morita equivalent*”. This is an equivalence relation.

Given a representation π of B realized on a Hilbert space V , one can “induce” it to a representation $\text{Ind}_A^B(X, \pi)$ of A captured on the Hilbert space $X \otimes_B V$ obtained by “interior tensor product” (this process is sometimes called “Rieffel induction”). This association is functorial and has an inverse implemented by the dual module of X , thus leading to an equivalence of categories of representations of A and B . It identifies the lattices of two-sided closed ideals of A and B and furthermore, preserves weak containment and direct integrals.

1.1. Description of the main result. We bring together the two themes above in the case where (G, H) is an equal rank dual pair of the form (Mp_{2n}, O_{2n+1}) or (U_n, U_n) over a nonarchimedean local field of characteristic zero. In this case, the theta correspondence $\pi \mapsto \theta(\pi)$ establishes a bijection between certain subsets of the tempered duals of G and H . Let us name these subsets \widehat{G}_θ and \widehat{H}_θ . We consider the reduced C^* -algebras associated to the groups G and H . These are algebras of operators, going back to Irving Segal, which are obtained from the convolution action of the L^1 -algebra of a locally compact Hausdorff group on its L^2 -space. As such, these C^* -algebras are directly related to tempered representations. We exhibit ideals $C_\theta^*(G)$ and $C_\theta^*(H)$ of the reduced C^* -algebras of G and H whose spectra are homeomorphic to \widehat{G}_θ and \widehat{H}_θ respectively. We show that the (smooth) oscillator representation of $G \times H$ provides a natural bimodule for the reduced C^* -algebras of G and H and that this bimodule can be promoted to an equivalence $C_\theta^*(G)$ - $C_\theta^*(H)$ -bimodule in the sense above. Remarkably, the crucial compatibility property between the $C_\theta^*(G)$ -valued and $C_\theta^*(H)$ -valued inner products turns out to be precisely the so-called “*local Rallis inner product formula*” of Gan and Ichino [8].

We call the equivalence $C_\theta^*(G)$ - $C_\theta^*(H)$ -bimodule above the “*oscillator bimodule*” and denote it Θ . The key point is that given an irreducible representation of $C_\theta^*(H)$, which is the same as an element of \widehat{H}_θ , the induced representation $\text{Ind}_{C_\theta^*(H)}^{C_\theta^*(G)}(\Theta, \pi)$ of $C_\theta^*(G)$ is the (integrated form) of G -representation $\theta(\pi^*)$. In fact, $\text{Ind}_{C_\theta^*(H)}^{C_\theta^*(G)}(\Theta, \pi)$ is precisely the (integrated form) of the G -representation obtained from π via the “averaging of matrix coefficients” construction of Jian-Shu Li introduced in [23] in the so-called “stable range” case (roughly speaking, when G is at least twice the size of H). In the equal rank cases, it is known that Li’s construction, hence ours, agrees with $\theta(\pi^*)$.

As an immediate corollary, we see that the tempered theta correspondence, in the equal rank set-up, is simply the restriction of an equivalence of categories of representations of two C^* -algebras to the irreducible objects. As such it is functorial. Moreover, as it is implemented by an equivalence bimodule, it enjoys various properties, such as the preservation of weak containment. In particular, the tempered theta correspondence is a homeomorphism between \widehat{G}_θ and \widehat{H}_θ .

The oscillator bimodule interpolates the oscillator representation of one group with the regular representation of the other group. This immediately implies that \widehat{G}_θ lies in the support of the oscillator representation viewed as a G -representation. In fact, we show with an elementary analysis that the latter is precisely the closure of the former. This is also follows from a Plancherel decomposition result of Sakellaridis [36].

The oscillator bimodule Θ can be viewed as generalization of the *Heisenberg module* of Rieffel. Let W be a symplectic vector space. Given a closed subgroup Γ of W , Rieffel shows in [35] that the (twisted) C^* -algebras associated to Γ and its dual/annihilator group Γ^\perp are strongly Morita equivalent. The equivalence is implemented by the Heisenberg module based on the (projective)

Heisenberg representation of W . The critical compatibility condition for the two operator valued inner products on the bimodule reduces to the Poisson transformation in this case.

In the local theta correspondence set-up, we operate inside the symplectic group $\mathrm{Sp}(W)$ with the roles of Γ, Γ^\perp played by the equal rank dual pair (G, H) . Accordingly we consider not the Heisenberg representation but the more complicated oscillator representation. In this sense, the local Rallis inner product formula of Gan and Ichino that we used in our proof of the critical compatibility property can be viewed as a non-commutative generalisation of the Poisson transform.

1.2. Applications. After promoting the equal rank tempered theta correspondence to a categorical equivalence, we move on to illustrate the fact that an equivalence bimodule allows the transfer of a lot of information between the two sides. We do this with two applications that are attractive in the simplicity of their statements and the elementary nature of their short proofs.

1.2.1. Explicit transfer of characters. Let us continue with the set-up of the above section. If π is a tempered irreducible representation of H , the character of π is the tempered distribution on H , that is, the continuous linear functional

$$\mathrm{ch}(\pi) : \mathcal{S}(H) \rightarrow \mathbb{C}$$

on Harish-Chandra's Schwartz algebra $\mathcal{S}(H)$ of H given by the trace

$$\mathrm{ch}(\pi)(\varphi) := \mathrm{tr} \pi(\varphi).$$

The oscillator bimodule forms a connection between parts of the Schwartz algebras of G and H , and as such gives a meaningful way of expressing the character one representation in terms of that of its theta lift. An elementary half-a-page long argument based on a concrete representation of the oscillator bimodule Θ as a space of operators gives us the following.

Corollary 1.2.2. *Let π be a tempered irreducible representation of H that enters the theta correspondence. Let \mathbb{S} denote the Frechét space which affords the smooth oscillator representation of $G \times H$. Given $x, y \in \mathbb{S}$, let $\langle x, y \rangle_H \in \mathcal{S}(H)$ and ${}_G \langle x, y \rangle \in \mathcal{S}(G)$ be the matrix coefficient functions defined below in (5.2.2) and (5.3.2). We have*

$$\mathrm{ch}(\theta(\pi))({}_G \langle x, y \rangle) = \mathrm{ch}(\pi)(\langle y, x \rangle_H).$$

The inner products ${}_G \langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_H$ span ideals in the Schwartz algebras of G and H respectively and the above result explicitly relates the two characters when they are restricted to these ideals. To extend the above transfer formula beyond these ideals, various convergence issues in the theory of operators on Hilbert C^* -modules need to be addressed. This is subject of further research.

Investigations on the question of how characters of representations relate, if at all, under the theta correspondence go back to the late 1980s. Notably, Przebinda studied the stable range case¹ over the reals (e.g. [30, 31]). For a more recent result in this direction, see [26] which treats real cases in which one group is compact and [25] which adapts earlier works of Przebinda to the non-archimedean stable range setting. Our result in the non-archimedean equal rank case has been recently announced by Wee Teck Gan [11, 12]. While his proof seems different than ours, it can be said that it philosophically agrees with ours in that matrix coefficients of the oscillator representation play a central role.

1.2.3. Preservation of formal degrees. It is well-known (see [7]) that the local theta correspondence takes discrete series representations to discrete series representations in the setting of equal rank pairs. Recall that the *formal degree* of a discrete series representation π of, say, H is the positive real number $\mathrm{deg}(\pi)$ such that

$$\int_H \langle v, \pi(h)(v') \rangle \overline{\langle w, \pi(h)(w') \rangle} ds = \frac{1}{\mathrm{deg}(\pi)} \langle v, w \rangle \langle v', w' \rangle$$

¹He also proposed an explicit formula which is conjectured to hold beyond the stable range case.

for all $v, v', w, w' \in V_\pi$. It depends on the chosen Haar measure on H .

Using a cohomological argument mixed with some known facts regarding the transfer of trace maps under equivalence bimodules, we obtain the following.

Corollary 1.2.4. *Let π be a discrete series representation of H which enters the theta correspondence. Then*

$$\deg(\pi) = \deg(\theta(\pi)).$$

The Haar measures used in the above result are the ones that we use for the proof the compatibility property (Prop. 5.4.1) of the oscillator bimodule. The key point of the proof is that discrete series give generators of K_0 of the reduced group C^* -algebra and one can access their formal degrees using the canonical trace. The canonical trace is given by the orbital integral associated to the trivial conjugacy class. We did not pursue this as we did not need it, but one could also explicitly transfer traces arising for orbital integrals of other conjugacy classes.

The above result on preservation of formal degrees result is not new: Gan and Ichino proved² it in [8]. However, we think that our proof is of interest as it is a simple illustration of the usefulness of the K -theoretic tools that are now made available by the oscillator bimodule approach for the study of theta correspondence.

1.3. Remarks.

- (1) Our approach also applies to the stable range case which was mentioned earlier. This case will be treated in a companion paper. It should be possible to treat the almost equal rank cases as well.
- (2) We should point out that an equivalence of categories in the naive sense does not hold at the level of full smooth (as opposed to tempered) equal rank theta correspondence; this was explained to us by Dipendra Prasad who recently has been pursuing the idea of interpreting the full smooth equal rank theta correspondence as a ‘derived equivalence’, see [29].
- (3) As pointed out to us by Wee Teck Gan, Our C^* -algebraic approach to theta correspondence could be interpreted to lie within the general C^* -algebraic framework for symplectic quantization theory [22] developed by Klaas Landsman in the 1990’s, see [13].

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Remark 1.4.1. All Hilbert spaces in this paper will be right Hilbert spaces.

²Their methods are very different but it is interesting to note that their local Rallis inner product formula is a key input for both their and our proofs.

2. THE LOCAL THETA CORRESPONDENCE

Let F be a non-archimedean local field of characteristic 0. Let E be F or a quadratic extension of F . Put $\varepsilon = \pm 1$ and set

$$\varepsilon_0 = \begin{cases} \varepsilon, & \text{if } E = F, \\ 0, & \text{if } E \neq F. \end{cases}$$

Following the conventions of [9], we set

$$W = W_n = \text{a } -\varepsilon\text{-Hermitian space over } E \text{ of dimension } n,$$

$$V = V_m = \text{a } \varepsilon\text{-Hermitian space over } E \text{ of dimension } m.$$

We define the associated groups as follows:

$$G = G(W) = \begin{cases} \text{the metaplectic group}^3 Mp(W) & \text{if } W \text{ is symplectic and } \dim(V) \text{ is odd,} \\ \text{the isometry group of } W, & \text{otherwise.} \end{cases}$$

We define $H = H(V)$ similarly by switching the roles of W and V . If $E = F$ and $\varepsilon = 1$, then $G = Sp(W)$ or $Mp(W)$ depending on the parity of the dimension of V . If $E = F$ and $\varepsilon = -1$, then $G = O(W)$. If $E \neq F$, then $G = U(W)$.

We fix a non-trivial additive character φ of F . Associated to φ , and to a pair of auxiliary characters χ_V, χ_W of E^\times that we will not need to make precise (see [8, 3.3]), there is a distinguished unitary representation ω of the group $G \times H$ called⁴ the **oscillator representation**. We will mainly consider the underlying smooth representation and still denote it by ω .

2.1. The Theta lift. Given a smooth representation π of G (always assumed to be of finite length), the maximal π -isotypic quotient of ω has the form

$$\pi \otimes \Theta(\pi)$$

for some smooth representation $\Theta(\pi) = \Theta_{W,V}(\pi)$ of H , known as the “**big theta lift**” of π . Alternatively, we can describe $\Theta(\pi)$ as the representation $\omega \otimes \mathbf{1}$ of H on the space of G -coinvariants

$$(\omega \otimes \pi^*)_G. \tag{2.1.1}$$

The maximal semisimple quotient of $\Theta(\pi)$ is denoted by $\theta(\pi) = \theta_{W,V}(\pi)$ and is called the “**small theta lift**” of π .

In the case $G = Mp(W)$, we call π genuine if it does not factor through $Sp(W)$. If π is not genuine, it is easy to see that its big theta lift is zero. We now state the fundamental result for local theta correspondence theory.

Theorem 2.1.1. (*Howe Duality*) $\Theta(\pi)$ is either zero or has a unique irreducible quotient, so that $\theta(\pi)$ is irreducible. Moreover, if $\theta(\pi) \simeq \theta(\pi')$ then $\pi \simeq \pi'$.

This was originally conjectured by Howe [18] and proven by him [19] in the archimedean setting. In our nonarchimedean set-up, it was proven by Waldspurger [40] when the residue characteristic p was not equal to 2. Much later, Gan and Takeda [9] proved this for all p .

2.2. Equal rank correspondence. We will now specialize the discussion to **equal rank pairs** (G, H) , that is, pairs for which we have $m = n + \varepsilon_0$. Precisely, these are

$$(Mp_{2k}, O_{2k+1}), (O_{2k+1}, Mp_{2k}), (U_k, U_k).$$

Theorem 2.2.1. Assume that $m = n + \varepsilon_0$. Let π be a tempered irreducible representation of G . If $\Theta(\pi)$ is not zero, then it is irreducible (thus $\Theta(\pi) = \theta(\pi)$) and tempered.

Proof. This can be found in [10]: see their Thm 1.2 and Lemma 4.1. □

⁴Also known as the “Weil representation” or the “metaplectic representation”.

In passing, we mention that the non-vanishing of the above theta lifts has an elegant characterisation in terms of the “*standard ϵ -factors*”.

2.3. Li’s form. Let π be a *tempered* irreducible representation of G . Following Li [23], one introduces a sesquilinear form $(\cdot, \cdot)_\pi$ on $\omega \otimes \pi$ as follows

$$(\phi \otimes v, \phi' \otimes v')_\pi := \int_G \langle \phi, \omega(g)(\phi') \rangle \langle v, \pi(g)(v') \rangle dg \quad (2.3.1)$$

The defining integral is well-known to be absolutely convergent in our equal rank case; see [23, Cor. 3.2]. One can also conclude its convergence from that of the doubling zeta integral at $s = 0$, see [8, Lemma 9.5 (ii)].

Using the fact that G is unimodular, it is easy to see that this form is Hermitian and H -invariant with respect to the natural action $\omega \otimes \mathbf{1}$ of H . Let N denote the radical of $(\cdot, \cdot)_\pi$, namely

$$N := \{\Phi \in \omega \otimes \pi \mid (\Phi, \Psi)_\pi = 0 \ \forall \Psi \in \omega \otimes \pi\}.$$

Then N is stabilised by H and thus the quotient

$$(\omega \otimes \pi) / N \quad (2.3.2)$$

affords a H -representation that we will denote by $L(\pi)$.

A straightforward calculation shows that the subspace spanned by elements of the form

$$\Phi - (\omega \otimes \pi)(g)(\Phi)$$

with $\Phi \in \omega \otimes \pi$ and $g \in G$ lies inside the radical N . Therefore we have a projection

$$\Theta(\pi^*) \simeq (\omega \otimes \pi)_G \twoheadrightarrow L(\pi).$$

2.4. The next result is well-known to specialists. In fact, most of it can be found in the literature, alas not completely and not in the way we want. So we give a quick proof.

Proposition 2.4.1. *Assume that $m = n + \varepsilon_0$. Let π be a tempered irreducible representation of G . We have*

$$L(\pi) \simeq \theta(\pi^*).$$

Moreover, if the form $(\cdot, \cdot)_\pi$ is non-zero, then

$$(\cdot, \cdot)_\pi \geq 0.$$

Proof. Recall from Theorem 2.2.1 that $\Theta(\pi^*) = \theta(\pi^*)$. Assume that $(\cdot, \cdot)_\pi$, and hence $L(\pi)$, is non-zero. Since π , and hence π^* , is tempered, by Thm. 2.2.1, we know that $\Theta(\pi^*)$ is either zero or is irreducible. Therefore $L(\pi)$ being a nonzero quotient of $\Theta(\pi^*)$ immediately implies that $L(\pi) \simeq \Theta(\pi^*)$.

Now assume that $(\cdot, \cdot)_\pi$ is zero. We will show that $\Theta(\pi^*)$ is zero. This is proven⁵ in [15, Prop. B.4.1] for the unitary pairs. Their proof adapts easily to the metaplectic/orthogonal pairs as well; indeed, this is essentially done in [8, Prop. 16.1.3 (iii)] which treats both cases simultaneously. In [8, Prop. 16.1.3 (iii)], the authors consider only discrete series π but the proof still works if π is tempered as we indicate now. First thing to point out is that, in the notation of [8], the submodule $R(V, \chi_W) \oplus R(V', \chi_W)$ equals all of the degenerate principal series $I_{\mathbf{P}}^G(0, \chi_V)$ (see [8, Prop. 7.2.(i)], compare with [15, Prop. B.3.2]). As mentioned above, the doubling zeta integral \mathcal{Z} is convergent at $s = 0$. It is also well-known (see [8, Thm. 9.1.(iii)]) that \mathcal{Z} is non-trivial on $I_{\mathbf{P}}^G(0, \chi_V)$. Our assumption that $(\cdot, \cdot)_\pi$ is zero on $\omega_{V,W} \otimes \pi$ implies that \mathcal{Z} is zero on the submodule $R(V, \chi_W)$, therefore it is non-zero on the complement $R(V', \chi_W)$. This implies that $(\cdot, \cdot)_\pi$ is non-zero on $\omega_{V',W} \otimes \pi$. From the previous paragraph, it follows then that $\Theta_{V',W}(\pi^*)$ is nonzero. Now the theta dichotomy principle (see [8, Cor. 9.2]) tells us that $\Theta_{V,W}(\pi^*)$ is zero as claimed.

⁵They work with $\text{Hom}_{U(V)}(\omega \otimes \pi, \mathbb{C})$. Note that $\text{Hom}_{U(V)}(\omega \otimes \pi, \mathbb{C}) \simeq \text{Hom}((\omega \otimes \pi)_{U(V)}, \mathbb{C}) \simeq \text{Hom}(\Theta(\pi^*), \mathbb{C})$.

For the second claim, assume again that $(\cdot, \cdot)_\pi$ is non-zero. Non-negativity of $(\cdot, \cdot)_\pi$ follows immediately from Thm A.5 of [15]. One sets the groups G and H in the statement of Thm. A.5 to be equal to our G . Similarly, one sets the representations π_H and π_G in the statement of Thm. A.5 to be equal to our ω and π respectively. The hypothesis (i) of Thm. A.5 is automatically satisfied since in the non-archimedean set-up all smooth vectors are K -finite, and hypothesis (ii) is also satisfied thanks to the fast decay of the matrix coefficients of the oscillator representation that we alluded to above, see Lemma 5.1.1 below. Now non-negativity of $(\cdot, \cdot)_\pi$ is precisely the conclusion of Thm. A.5. One could also directly refer to [36, Prop. 3.3.1] for non-negativity. \square

3. SOME *-ALGEBRAS ASSOCIATED TO GROUPS

In this section we discuss various topological algebras associated to a locally compact group. They will be the reduced C^* -algebra $C_r^*(G)$, whose spectrum coincides with the tempered dual of G . The Schwartz algebra $\mathcal{S}(G)$ is a dense subalgebra of functions on G , and is itself a Fréchet $*$ -algebra. The latter algebra is more susceptible to the explicit constructions and calculations that we present in Section 4.

3.1. C^* -algebras of groups. Given a locally compact Hausdorff topological group G , we let $L^1(G)$ denote the Banach $*$ -algebra of integrable functions. It is well-known that there is a bijection between unitary representations of G and non-degenerate $*$ -representations of $L^1(G)$: given a (strongly continuous) unitary representation $\pi : G \rightarrow U(V_\pi)$ of G on a Hilbert space V_π , we obtain a $*$ -representation of $L^1(G)$ (still denoted π) by integrating

$$\pi(f) := \int_G f(s)\pi(s)ds, \tag{3.1.1}$$

where $f \in L^1(G)$.

Let (π, V_π) be a unitary representation of G . We denote the C^* -algebra generated⁶ by the image of $L^1(G)$ under the $*$ -representation $\pi : L^1(G) \rightarrow \mathcal{L}(V_\pi)$ by

$$C_\pi^*(G).$$

It is called the C^* -algebra of G associated to π . A most important example is when we take π to be the regular representation of G on $L^2(G)$; in that case $C_\pi^*(G)$ is the so-called **reduced C^* -algebra of G** and it has the established notation

$$C_r^*(G).$$

Recall that the spectrum of $L^1(G)$ is in bijection with the unitary dual \widehat{G} of G . As is well-known, \widehat{G} comes equipped with a topology that is typically described via weak containment, or uniform approximation of matrix coefficients on compacta. It follows that the above bijection gives a homeomorphism between the spectrum of $C_\pi^*(G)$ and the *support of π* , denoted $\text{supp}(\pi)$, that is, the subset of \widehat{G} that are made of those representations which are weakly contained in π . A special case of this is the well-known the fact that the spectrum of $C_r^*(G)$ is homeomorphic to the tempered dual of G :

$$\widehat{C_r^*(G)} \xleftarrow{\text{homeom.}} \widehat{G}_{\text{temp}}. \tag{3.1.2}$$

3.2. Schwartz algebra. Given a connected reductive linear algebraic group \mathbf{G} over F (which, we recall, is non-archimedean), let us put $G = \mathbf{G}(F)$. We fix a minimal parabolic subgroup P and a “good”⁷ maximal compact subgroup K so that $G = PK$. Consider the smooth normalized induced representation $I_P^G(\mathbf{1})$ of the trivial representation of P to G . Let e_K denote the unique vector in

⁶This is the closure of the image of $L^1(G)$ with respect to the operator norm.

⁷Roughly put, it needs to be the stabilizer of a well-chosen vertex in the building associated to G . See [8, p544]

$I_P^G(\mathbf{1})$ such that $e_K(k) = 1$ for all $k \in K$. We define *Harish-Chandra's function* as the diagonal matrix coefficient of v_K :

$$\Xi(g) := \langle I_P^G(\mathbf{1})(g)e_K, e_K \rangle, \quad g \in G.$$

It is well-known that Ξ is a positive, K -biinvariant function that satisfies $\Xi(g) = \Xi(g^{-1})$ (see [41, Section II.1]).

We say that a continuous function $f : G \rightarrow \mathbb{C}$ is **rapidly decreasing** if for all $n > 0$ we have

$$v_n(f) := \sup_{g \in G} |f(g)| \Xi(g)^{-1} (1 + \log \|g\|)^n < \infty, \quad (3.2.1)$$

where $\|\cdot\|$ is the standard norm on G arising from a good choice of embedding $\iota : G \hookrightarrow GL_m(F)$ where m is the F -rank of G (see, e.g. [8, p544]). The space $\mathcal{S}(G)$ of all rapidly decreasing, uniformly locally constant⁸ functions on G is an algebra under convolution and it is called the (Harish-Chandra) **Schwartz algebra** of G .

Given a compact open subgroup K , let $\mathcal{S}(G//K)$ denote the subspace of functions in $\mathcal{S}(G)$ which are constant on the double cosets of K . Then the space $\mathcal{S}(G//K)$ is a nuclear, unital Fréchet $*$ -algebra under convolution, with the topology given by the seminorms v_n above. We have that

$$\mathcal{S}(G) = \bigcup_K \mathcal{S}(G//K)$$

where K ranges over compact open subgroups of G (the right hand side is a vector space direct limit). We equip $\mathcal{S}(G)$ with the direct limit topology. Let $C_c^\infty(G//K)$ denote the subspace of functions in $\mathcal{S}(G//K)$ that are compactly supported. Then $C_c^\infty(G//K)$ is a unital convolution algebra that acts on $L^2(G)$, again via convolution. If we denote by $C_r^*(G//K)$ the C^* -algebra generated by $C_c^\infty(G//K)$ inside $\mathcal{L}(L^2(G))$, then

$$C_r^*(G) = \varinjlim C_r^*(G//K),$$

where the right hand side a direct limit of the C^* -subalgebras $C_r^*(G//K)$ partially ordered by inclusion.

We single out some properties of $\mathcal{S}(G)$ that will be of importance to us. A unitary representation π of G is called *tempered* if for any smooth vectors $v, v' \in \pi$, there exists a constant d such that we have

$$|\langle \pi(g)(v), v' \rangle| \leq d \Xi(g)$$

for all $g \in G$ (see [2, Eq. 2.2.3]).

Theorem 3.2.1. *The Schwartz algebra $\mathcal{S}(G)$ enjoys the following properties.*

- (1) *If π is a tempered representation and π^∞ its associated smooth representation, then the G -action on π^∞ integrates to an action of $\mathcal{S}(G)$.*
- (2) *If K is a compact open subgroup of G , then $\mathcal{S}(G//K)$ is a dense $*$ -subalgebra of $C_r^*(G//K)$, in particular the inclusion $\mathcal{S}(G) \rightarrow C_r^*(G)$ is continuous and has dense range;*
- (3) *If an element of $\mathcal{S}(G//K)$ is invertible in $C_r^*(G//K)$, then it is already invertible in $\mathcal{S}(G//K)$;*

Proof. The first claim is classical. It follows from the factorization $\mathcal{S}(G) = C_c^\infty(G) \star \mathcal{S}(G)$ (see [2, (2.1.1) and (2.2.7)]). The other two claims are due to Vignéras [39]. An alternative proof is given in [3, Lemma 2] for $G = GL_m(F)$. \square

⁸A function is *uniformly locally constant* if it is B bi-invariant for a compact open subgroup B .

3.3. Metaplectic and orthogonal groups. The definition of the Schwartz algebra can be adapted to the non-linear group⁹ $Mp(W)$ and the disconnected $O(V)$ in a straight-forward way. For $Mp(W)$, we pull-back the Harish-Chandra function Ξ and the standard norm $\|\cdot\|$ from $Sp(W)$ to $Mp(W)$. For $O(V)$, noting that $O(V) \simeq SO(V) \times \{\pm 1\}$, we extend the Ξ and $\|\cdot\|$ from $SO(V)$ to $O(V)$ by declaring that $\Xi(-g) = \Xi(g)$ and $\|-g\| = \|g\|$. Then in both cases, Ξ continues to enjoy the usual properties and $1 + \log \|\cdot\|$ still defines a length function¹⁰. The definition of the Schwartz algebra now applies. A careful treatment for the case of the metaplectic group can be found in [24, Section 2.3].

Thm. 3.2.1 stays valid when G is $Mp(W)$ or $O(V)$. The first part of Thm. 3.2.1 is clear (see [24, Section 2.3] for the metaplectic case). For the third part, the key property needed for Vignéras' results to apply are

- (i) given a compact open K of G , the double coset space $K \backslash G / K$ has “polynomial growth” ([39, p237,]) with respect to the “scale” $\sigma := L - 1 = \log \|\cdot\|$, (this is required for results in [39, Section 6])
- (ii) $\Xi \sigma^{-r} \in L^2(G)$ for large enough $r > 0$. (required in order to be able to apply [39, Thm. 20], see [39, Lem. 27 and Prop. 28])

It is easy to see that these conditions are both satisfied for the cases of $G = Mp(W)$ and $G = O(V)$. For example, let K be a compact open in $Mp(W)$. Then if B' is the image of K (again compact open) in $Sp(W)$ under the covering map, then the natural map $K \backslash Mp(W) / K \rightarrow K' \backslash Sp(W) / K'$ has fibers of size at most 2, and it follows that polynomial growth of the latter implies the same for the former. This addresses item (i). For item (ii), we simply observe that the integral over $Mp(W)$ is twice that over $Sp(W)$. Similar reasoning applies to the case of $O(V)$.

4. STRONG MORITA EQUIVALENCE

In this section, we introduce the notion of strong Morita equivalence for C^* -algebras. A Morita equivalence between C^* -algebras A and B induces a bijection between their Hilbert space representations. Good references for the C^* -theory include [32, 21].

Using the C^* -algebras $C_r^*(H)$ and $C_r^*(G)$, we will exploit this in the context of local theta correspondence. However, as the matrix coefficients of the oscillator representation live in the Schwartz algebras $\mathcal{S}(G)$ and $\mathcal{S}(H)$ in our equal rank set-up, our constructions naturally start at the level of these algebras.

4.1. Local subalgebras of C^* -algebras. Let \mathcal{A} be a complex $*$ -algebra. The **spectrum** of $a \in \mathcal{A}$ is the set

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C} : a - \lambda \text{ is not invertible in } \mathcal{A}\}.$$

If \mathcal{A} is nonunital, its *unitisation* is the space

$$\mathcal{A}^+ := \mathcal{A} \oplus \mathbb{C},$$

equipped with coordinatewise addition and multiplication

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda \mu).$$

For nonunital algebras, we define $\sigma_{\mathcal{A}}(a) := \sigma_{\mathcal{A}^+}((a, 0))$. An element $a \in \mathcal{A}$ is *positive* if $a = a^*$ and $\sigma_{\mathcal{A}}(a) \subset \mathbb{R}_{\geq 0}$.

If $\mathcal{A}_0 \subset \mathcal{A}$ is a $*$ -subalgebra we say that \mathcal{A}_0 is **spectral invariant** in \mathcal{A} if for all $a \in \mathcal{A}_0$ we have

$$\sigma_{\mathcal{A}_0}(a) = \sigma_{\mathcal{A}}(a).$$

⁹While $Mp(W)$ is not linear like $Sp(W)$, it still is an “ ℓ -group” like $Sp(W)$: i.e. it is a Hausdorff topological group with a basis of neighborhoods of the identity consisting of compact open subgroups.

¹⁰A *length function* on a group G is a continuous function $L : G \rightarrow [0, \infty]$ such that $L(e) = 1$, $L(g^{-1}) = L(g)$ and $L(gh) \leq L(g) + L(h)$ for all $g, h \in G$.

A Fréchet $*$ -algebra admits **holomorphic functional calculus** ([27, Lemma 1.3]): if $a \in \mathcal{A}$, $U \subset \mathbb{C}$ an open set containing $\sigma_{\mathcal{A}}(a)$ and $f : U \rightarrow \mathbb{C}$ a holomorphic function, then

$$f(a) := \int_C f(\lambda)(a - \lambda)^{-1} d\lambda \in \mathcal{A},$$

where C is simple closed curve in U enclosing $\sigma(a)$. Now suppose that A is a C^* -algebra and $\mathcal{A} \subset A$ a $*$ -subalgebra. We say that \mathcal{A} is **stable under holomorphic functional calculus** in A if for all $a \in \mathcal{A}$ and f a holomorphic function on neighbourhood of $\sigma_{\mathcal{A}}(a)$, we have $f(a) \in \mathcal{A}$.

Definition 4.1.1. Let A be a C^* -algebra and $\mathcal{A} \subset A$ a dense $*$ -subalgebra. We say that \mathcal{A} is **local** in A if \mathcal{A} is stable under holomorphic functional calculus.

Lemma 4.1.2 (cf. Lemma 1.2 in [37]). *Suppose that \mathcal{A} is a Fréchet $*$ -algebra, A a C^* -algebra and $i : \mathcal{A} \rightarrow A$ a continuous injective $*$ -homomorphism with dense range. Then $i(\mathcal{A}) \subset A$ is local if and only if $i(\mathcal{A})$ is spectral invariant in A .*

In the above situation we identify \mathcal{A} with its image $i(\mathcal{A})$ and simply say that $\mathcal{A} \subset A$ is local.

Proposition 4.1.3. *Let F be a non-archimedean local field of characteristic 0. Let G be either the F -points of a connected reductive group, or be $Mp(W)$ or $O(W)$ where W is over F . Then $\mathcal{S}(G) \subset C_r^*(G)$ is local.*

Proof. Since $\mathcal{S}(G) \subset C_r^*(G)$ is dense, it remains to show spectral invariance. Since $\mathcal{S}(G) \subset C_r^*(G)$ and a both algebras are nonunital, we have

$$\sigma_{C_r^*(G)}(a) \subset \sigma_{\mathcal{S}(G)}(a).$$

We have argued in Section 3.2 that for any compact open subgroup K of G , an element of $\mathcal{S}(G//K)$ is invertible in $C_r^*(G//K)$, then it is already invertible in $\mathcal{S}(G//K)$ (see Theorem 3.2.1(iii)). It follows from the definitions then that $\mathcal{S}(G//K)$ is local in $C_r^*(G//K)$ and hence spectral invariant by Lemma 4.1.2.

Now let $a \in \mathcal{S}(G) = \bigcup_K \mathcal{S}(G//K)$, so $a \in \mathcal{S}(G//K)$ for some K . Since $C_r^*(G)$ and $\mathcal{S}(G)$ are nonunital, whereas $\mathcal{S}(G//K)$ and $C_r^*(G//K)$ are unital, and inclusions of C^* -algebras such as $C_r^*(G//K) \rightarrow C_r^*(G)$ are spectral invariant away from 0, we find

$$\sigma_{\mathcal{S}(G)}(a) \subset \sigma_{\mathcal{S}(G//K)}(a) \setminus \{0\} = \sigma_{C_r^*(G//K)}(a) \setminus \{0\} = \sigma_{C_r^*(G)}(a).$$

We deduce that $\sigma_{C_r^*(G)}(a) = \sigma_{\mathcal{S}(G)}(a)$ as desired. Now if f is holomorphic on $\sigma(a)$ and $a \in \mathcal{S}(G//K)$ then $f(a) \in \mathcal{S}(G//K) \subset \mathcal{S}(G)$, proving that $\mathcal{S}(G) \subset C_r^*(G)$ is local. \square

Remark 4.1.4. We will use the above result that the Schwartz algebras are local inside the reduced group C^* -algebras in the proof of Prop. 4.2.3 and for our second application in Section 8 where we use the fact that localness implies that they have the same K -theory as the reduced group C^* -algebras. Prop. 4.2.3 is critical in proving that the oscillator bimodule is an equivalence bimodule. However we would like to point out that one could prove the continuity statement (4.2.2) for the oscillator bimodule without exploiting the locality of the Schwartz algebras but instead using the fact that the theta correspondence preserves temperedness.

4.2. Inner product modules. Let B be a C^* -algebra and $\mathcal{B} \subset B$ a $*$ -subalgebra. A (complex) vector space \mathcal{X} is called a right **inner product \mathcal{B} -module** if \mathcal{X} is a right \mathcal{B} -module and it is equipped with a \mathcal{B} -valued positive-definite Hermitian form that is compatible with the right \mathcal{X} -module structure. More precisely, there is a sesquilinear map

$$\langle \cdot, \cdot \rangle_{\mathcal{B}} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{B},$$

satisfying the following properties:

- (1) $\langle x, y \rangle_{\mathcal{B}}^* = \langle x, y \rangle_{\mathcal{B}}$ for all $x, y \in \mathcal{X}$,

- (2) $\langle x, yb \rangle_{\mathcal{B}} = \langle x, y \rangle_{\mathcal{B}} b$ for all $x, y \in \mathcal{X}$ and $b \in \mathcal{B}$.
(3) $\langle x, x \rangle_{\mathcal{B}}$ is a positive as an element of B for every $x \in \mathcal{X}$.

The span of the set $\{\langle x, y \rangle_{\mathcal{B}} \mid x, y \in \mathcal{X}\}$ is an ideal of \mathcal{B} . We call \mathcal{X} **full** if this ideal is dense in the ambient C^* -algebra B . The inner product module \mathcal{X} is **nondegenerate** if

$$\langle x, x \rangle_{\mathcal{B}} = 0 \Leftrightarrow x = 0.$$

We define left inner product modules in a similar way using left linear inner products.

If \mathcal{X} is an inner product \mathcal{B} -module, then

$$\|x\|^2 := \left\| \langle x, x \rangle_{\mathcal{B}} \right\|_B, \quad (4.2.1)$$

defines a norm on \mathcal{X} .

In the above, making the particular choice $\mathcal{B} = B$, we arrive at the following definition.

Definition 4.2.1. Let B be a C^* -algebra. An inner product B -module X is a **Hilbert C^* -module** if X is complete with respect to the norm (4.2.1).

If $\mathcal{B} \subset B$ is a dense $*$ -subalgebra and \mathcal{X} a nondegenerate inner product \mathcal{B} -module, then the completion X of \mathcal{X} in the norm (4.2.1) is a Hilbert C^* -module over B ([32, Lemma 2.16]). In the sequel we will construct inner product modules over the Schwartz algebra $\mathcal{S}(G)$ of a topological group G . Since $\mathcal{S}(G) \subset C_r^*(G)$ is dense, such modules admit a completion as Hilbert C^* -modules over $C_r^*(G)$.

Definition 4.2.2. Let $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ be dense $*$ -subalgebras and \mathcal{X} a right inner product \mathcal{B} -module. We say that \mathcal{X} is an **$(\mathcal{A}, \mathcal{B})$ -correspondence** if \mathcal{X} is a left \mathcal{A} -module such that

$$\langle ax, y \rangle_{\mathcal{B}} = \langle x, a^*y \rangle_{\mathcal{B}}, \quad \forall x, y \in \mathcal{X}, b \in \mathcal{B}.$$

In case $\mathcal{A} = A$, $\mathcal{B} = B$ and X is an (A, B) -correspondence that is a Hilbert C^* -module over B , we say that X is a **C^* -correspondence** for (A, B) .

Given a Hilbert C^* -module over a C^* -algebra B , its algebra of adjointable operators is the space

$$\text{End}^*(X) := \{T : X \rightarrow X : \exists T^* : X \rightarrow X \forall x, y \in X \langle Tx, y \rangle = \langle x, T^*y \rangle\}.$$

Elements of $\text{End}^*(X)$ are automatically bounded and right B -linear. In fact $\text{End}^*(X)$ forms a C^* -algebra in the operator norm it derives from the norm on X . Thus, for a C^* -correspondence X , we in fact have a $*$ -homomorphism $A \rightarrow \text{End}^*(X)$ between C^* -algebras.

Let \mathcal{X} be an $(\mathcal{A}, \mathcal{B})$ -correspondence for dense subalgebras $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$. We have discussed above that \mathcal{X} can be completed to a right Hilbert B -module X . We will now see that if \mathcal{A} is a *local* subalgebra of A , then the X can be promoted to a C^* -correspondence for (A, B) .

Proposition 4.2.3. *Let $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ be dense subalgebras \mathcal{X} an $(\mathcal{A}, \mathcal{B})$ -correspondence. If \mathcal{A} is local, then for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$ the inequality*

$$\langle a \cdot x, a \cdot x \rangle_{\mathcal{B}} \leq \|a\|_A^2 \langle x, x \rangle_{\mathcal{B}}, \quad (4.2.2)$$

holds true in the C^ -algebra B .*

Proof. Let $\varepsilon > 0$ and for $a \in \mathcal{A}$ and $x \in \mathcal{X}$ consider

$$(\|a\|_A^2 + \varepsilon) \langle x, x \rangle_{\mathcal{B}} - \langle a \cdot x, a \cdot x \rangle_{\mathcal{B}} = \langle (\|a\|_A^2 + \varepsilon - a^*a)x, x \rangle_{\mathcal{B}} \quad (4.2.3)$$

and observe that by spectral invariance the element $\|a\|_A^2 + \varepsilon - a^*a \in \mathcal{A}^+$ is positive invertible, and thus has spectrum contained in $[\varepsilon, M]$ for some $M > 0$. Since the square root is a holomorphic function on an open neighborhood of $[\varepsilon, M]$, we have that $(\|a\|_A^2 + \varepsilon - a^*a)^{1/2} \in \mathcal{A}^+$, and thus that $(\|a\|_A^2 + \varepsilon - a^*a)^{1/2} \cdot x \in \mathcal{X}$. We find that

$$\langle (\|a\|_A^2 + \varepsilon - a^*a) \cdot x, x \rangle_{\mathcal{B}} = \langle (\|a\|_A^2 + \varepsilon - a^*a)^{1/2} \cdot x, (\|a\|_A^2 + \varepsilon - a^*a)^{1/2} \cdot x \rangle_{\mathcal{B}} \geq 0,$$

and therefore by (4.2.3)

$$\langle a \cdot x, a \cdot x \rangle_B \leq \|a\|_A^2 \langle x, x \rangle + \varepsilon \langle x, x \rangle_B. \quad (4.2.4)$$

Since (4.2.4) holds for all $\varepsilon > 0$ we conclude that

$$\langle a \cdot x, a \cdot x \rangle_B \leq \|a\|_A^2 \langle x, x \rangle_B,$$

as desired. \square

Remark 4.2.4. Inequality (4.2.2) is one of the defining properties of a “pre-imprimitivity bimodule”, see [32, Definition 3.9].

Corollary 4.2.5. *Let $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ be dense $*$ -subalgebras, \mathcal{X} an $(\mathcal{A}, \mathcal{B})$ -correspondence and*

$$\mathcal{N} := \left\{ x \in \mathcal{X} : \langle x, x \rangle_B = 0 \right\},$$

the radical of \mathcal{X} . Assume that \mathcal{A} is local. Then \mathcal{X}/\mathcal{N} can be completed into a C^ -correspondence for (A, B) .*

Proof. By Proposition 4.2.3, \mathcal{A} maps \mathcal{N} into itself, so \mathcal{X}/\mathcal{N} becomes a nondegenerate $(\mathcal{A}, \mathcal{B})$ -correspondence. Let X denote its completion as a right Hilbert C^* -module over B . Then applying Proposition 4.2.3 once more, we deduce that the action of \mathcal{A} on \mathcal{X}/\mathcal{N} is bounded with respect to the C^* -module norm on \mathcal{X}/\mathcal{N} , and thus extends to an action of \mathcal{A} on X by adjointable operators. Since $\mathcal{A} \subset A$ is dense in the C^* -norm on A , another application of Proposition 4.2.3 shows that the left \mathcal{A} -module structure extends to a left A -module structure satisfying $\langle ax, y \rangle = \langle x, a^*y \rangle$ for all $a \in A$ and $x, y \in X$. Hence X is a C^* -correspondence for (A, B) . \square

4.3. Induction of representations of C^* -algebras. For C^* -correspondences over a pair of C^* -algebras (A, B) there is a far reaching theory of induced representations. Due to the Gelfand-Naimark-Segal theorem, a C^* -algebra usually admits numerous Hilbert space representations. A C^* -correspondence for (A, B) allows one to construct a representation of A given a representation of B . In the case of group C^* -algebras, this gives a functorial correspondence between the associated group representations.

4.3.1. The interior tensor product. Let A and B be C^* -algebras and X a C^* -correspondence for (A, B) . Then we can “induce” representations of B to A via X via the following tensor product construction.

Proposition 4.3.2 ([21, Proposition 4.5]). *Let X be a C^* -correspondence for (A, B) and $\pi : B \rightarrow \mathcal{L}(V_\pi)$ a representation of B on a Hilbert space V_π . Consider $X \otimes^{\text{alg}} V_\pi$ the algebraic tensor product of vector spaces. The right sesquilinear form*

$$(x \otimes v, x' \otimes v') := \langle v, \pi(\langle x, x' \rangle_B) v' \rangle_{V_\pi}, \quad (4.3.1)$$

is positive and its radical

$$N_\pi := \left\{ \xi \in X \otimes^{\text{alg}} V_\pi \mid (\xi, \xi) = 0 \right\}$$

is equal to the balancing subspace spanned by elements of the form

$$xb \otimes v - x \otimes \pi(b)(v) \quad (x \in X, v \in V_\pi, b \in B).$$

*The completion of $(X \otimes^{\text{alg}} V_\pi)/N_\pi$ with respect to the inner product (4.3.1) is a Hilbert space that we denote by $X \otimes_B V_\pi$ and is commonly called is the **internal tensor product of X and V_π over B** .*

Proof. The proof of positivity is based on the fact that for $x_1, \dots, x_n \in X$ the matrix $\varepsilon := \pi(\langle x_i, x_j \rangle_B)_{ij} \in M_n(\mathcal{L}(V))$ is a positive operator on V^n , so that for $\xi = \sum_{i=1}^n x_i \otimes v_i$ we have

$$(\xi, \xi) = (v, \varepsilon \cdot v) \geq 0, \quad v := \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in V^n.$$

The fact that the radical coincides with the balancing subspace uses positivity of the matrix ε and the fact the $M_n(B)$ is again a C^* -algebra, so we can extract square roots. \square

It can then be shown that that action

$$a(x \otimes v) := a \cdot x \otimes v$$

of A on the space $X \otimes V$ gives rise to representation of A on the Hilbert space $X \otimes_B V$ which we will denote

$$\text{Ind}_B^A(X, \pi),$$

and refer to as the A -representation **induced from π via X** .

4.3.3. Functoriality of induction. The induction procedure that we described above is functorial (see [32, Prop. 2.69]).

Proposition 4.3.4. *Let A, B be two C^* -algebras and let X be a C^* -correspondence for (A, B) . Assume that the action of A on X is non-degenerate (i.e. $A \cdot X = X$). Then the map*

$$\pi \mapsto \text{Ind}_B^A(X, \pi)$$

is a functor from the category of non-degenerate representations of B with bounded intertwining operators to the corresponding category of A , which at the level of morphisms takes the form

$$\mathcal{L}(V_\pi) \ni T \mapsto \mathbf{1} \otimes T \in \mathcal{L}(X \otimes_B V_\pi).$$

It follows that induction respects unitary equivalence and direct sums.

4.3.5. Continuity of induction. Let C be a C^* -algebra. Given a $*$ -representation π and a set of representations S of C , we say that π is **weakly contained** in S (denoted $\pi \prec S$) if

$$\ker(\pi) \supset \bigcap_{\sigma \in S} \ker(\sigma). \tag{4.3.2}$$

Let $\text{Rep}(C)$ denote the collection¹¹ of equivalence classes of all $*$ -representations of C . This space comes equipped with a second-countable topology (due to Fell) that is characterized as follows: a net $\{T_i\}$ of elements of $\text{Rep}(C)$ converge to T if and only if T is weakly contained in every subnet of $\{T_i\}$.

Let \widehat{C} denote the subset of irreducible elements of $\text{Rep}(C)$. We call \widehat{C} the *spectrum* of C . Relativized to \widehat{C} , closure in Fell topology agrees weak closure: a subset $S \subset \widehat{C}$ is closed if and only if $S = \{\pi \in \widehat{C} \mid \pi \prec S\}$. In this case, the topology agrees with the pull-back of the hull-kernel topology on the space of primitive ideals of C . See [5] for a discussion of these topologies.

We have the following continuity result (see [32, Prop. 2.72])

Proposition 4.3.6. *Let A, B be two C^* -algebras and let X be a C^* -correspondence for (A, B) . Then the map*

$$\pi \mapsto \text{Ind}_B^A(X, \pi)$$

is continuous with respect to the topologies on $\text{Rep}(B)$ and on $\text{Rep}(A)$.

¹¹In order to ensure that the collection is a set, we actually fix a cardinal \aleph and consider $*$ -representations on Hilbert spaces of cardinality $\leq \aleph$. For us, considering separable Hilbert spaces will suffice.

4.4. Equivalence bimodules.

Definition 4.4.1. Let $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ be local subalgebras and \mathcal{X} an algebraic $(\mathcal{A}, \mathcal{B})$ -bimodule in the usual sense that the two actions commute. We call \mathcal{X} an **inner product bimodule** if

- (1) \mathcal{X} is a left inner product \mathcal{A} -module and a right inner product \mathcal{B} -module,
- (2) for all $x, y \in \mathcal{X}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we have

$$\langle a \cdot x, y \rangle_{\mathcal{B}} = \langle x, a^* \cdot y \rangle_{\mathcal{B}} \quad \mathcal{A} \langle x \cdot b, y \rangle = \mathcal{A} \langle x, y \cdot b^* \rangle \quad (4.4.1)$$

- (3) for all $x, y, z \in \mathcal{X}$, we have

$$\mathcal{A} \langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_{\mathcal{B}}. \quad (4.4.2)$$

4.4.2. *Strong Morita equivalence of C^* -algebras.* Given an inner product bimodule \mathcal{X} for local subalgebras $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$, we obtain a norm on \mathcal{X} for each of the inner products. The following is well-known (see e.g. [32, Prop. 3.11]). We give the short proof here for the convenience of the reader.

Proposition 4.4.3. *Let $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ be local subalgebras and \mathcal{X} an $(\mathcal{A}, \mathcal{B})$ inner product bimodule. Then the norms*

$$\|x\|_A := \|\mathcal{A} \langle x, x \rangle\|_A^{1/2}, \quad \|x\|_B := \|\langle x, x \rangle_{\mathcal{B}}\|_B^{1/2}, \quad (4.4.3)$$

on \mathcal{X} are equal.

Proof. Using Definition 4.4.1 and the Cauchy-Schwartz inequality for Hilbert C^* -modules, we have

$$\begin{aligned} \|\langle x, x \rangle_{\mathcal{B}}\|_B^2 &= \|\langle x, x \rangle_{\mathcal{B}} \langle x, x \rangle_{\mathcal{B}}\|_B = \|\langle x, x \cdot \langle x, x \rangle_{\mathcal{B}} \rangle_{\mathcal{B}}\|_B = \|\langle x, \mathcal{A} \langle x, x \rangle \cdot x \rangle_{\mathcal{B}}\|_B \\ &\leq \|\langle x, x \rangle_{\mathcal{B}}\|_B^{1/2} \|\langle \mathcal{A} \langle x, x \rangle \cdot x, \mathcal{A} \langle x, x \rangle \cdot x \rangle_{\mathcal{B}}\|_B^{1/2}, \end{aligned}$$

for all $x \in \mathcal{X}$. Applying Proposition 4.2.3 to the second factor in the last term, we conclude that

$$\|\langle x, x \rangle_{\mathcal{B}}\|_B^2 \leq \|\langle x, x \rangle_{\mathcal{B}}\|_B^{1/2} \|\mathcal{A} \langle x, x \rangle\|_A \|\langle x, x \rangle_{\mathcal{B}}\|_B^{1/2}.$$

Cancelling a factor of $\|\langle x, x \rangle_{\mathcal{B}}\|_B$ gives us $\|\langle x, x \rangle_{\mathcal{B}}\|_B \leq \|\mathcal{A} \langle x, x \rangle\|_A$. Swapping the roles of A and B , we obtain the opposite inequality and obtain the desired equality of norms. \square

Definition 4.4.4. Let A and B be C^* -algebras. A **Hilbert C^* -bimodule** for (A, B) is an inner product bimodule that is complete in the norms

$$\|x\|_A^2 = \|\mathcal{A} \langle x, x \rangle\|_A = \|x\|_B^2 = \|\langle x, x \rangle_{\mathcal{B}}\|_B.$$

A Hilbert C^* -bimodule for (A, B) is a **(Morita) equivalence bimodule** if both inner products are full. Two C^* -algebras A and B are called **strongly Morita equivalent** if there exists an (A, B) Morita equivalence bimodule X .

While for unital C^* -algebras A and B , this notion coincides with classical Morita equivalence as rings, in general, it is stronger than the classical notion (hence the name).

4.5. We have the notion of “dual” of an $(\mathcal{A}, \mathcal{B})$ -inner product bimodule. This is a $(\mathcal{B}, \mathcal{A})$ -inner product bimodule.

Definition 4.5.1. Let \mathcal{X} be an $(\mathcal{A}, \mathcal{B})$ -inner product bimodule and \mathcal{X}^* its conjugate vector space. By definition, we have an anti-linear bijection $\beta : \mathcal{X} \rightarrow \mathcal{X}^*$ such that $\beta(\lambda \cdot x) = \bar{\lambda} \cdot \beta(x)$ for every $x \in \mathcal{X}$ and $\lambda \in \mathbb{C}$ a complex scalar. The **dual module of \mathcal{X}** is \mathcal{X}^* equipped with the following $(\mathcal{B}, \mathcal{A})$ -inner product bimodule structure

$$\begin{aligned} b \cdot \beta(x) &:= \beta(x \cdot b^*), & \mathcal{B} \langle \beta(x), \beta(y) \rangle &:= \langle x, y \rangle_{\mathcal{B}}, \\ \beta(x) \cdot a &:= \beta(a^* \cdot x), & \langle \beta(x), \beta(y) \rangle_{\mathcal{A}} &:= \mathcal{A} \langle x, y \rangle. \end{aligned}$$

If X is an (A, B) -equivalence bimodule, then dual module X^* is a (B, A) -equivalence bimodule. In fact there are isomorphisms of interior tensor products

$$X \otimes_B X^* \simeq A, \quad X^* \otimes_A X \simeq B,$$

as (A, A) and (B, B) C^* -bimodules, respectively. It follows that the induction functors associated to X and to X^* are inverses to each other. Thus the categories of representations of A and B are equivalent.

4.6. Given a two-sided closed ideal J of B , the space X_J given by the closure of the linear span of all $x \cdot b$ with $x \in X$ and $b \in J$ forms an (I, J) -equivalence bimodule where I is the two-sided closed ideal of A given by the closure of the linear span of all ${}_A \langle x \cdot b, y \rangle$ with $x, y \in X$ and $b \in J$. We will view this association as an *induction of ideals* implemented by X and accordingly denote I by

$$\text{Ind}_B^A(X, J).$$

It is well-known (see [32, Props. 3.24 and 3.25]) that the map $J \mapsto \text{Ind}_B^A(X, J)$ sets up a bijection between the two-sided closed ideals of B and A which respects inclusion of ideals (thus, it identifies the ‘‘lattices of ideals’’ of B and A). Moreover, the induction of ideals is compatible with the induction of representations, that is, if π is a representation of B , then

$$\text{Ind}_B^A(X, \ker(\pi)) = \ker(\text{Ind}_B^A(X, \pi)).$$

4.7. Lastly, we observe that any Hilbert C^* -bimodule induces a Morita equivalence between certain associated C^* -algebras. Suppose that X is a Hilbert C^* -bimodule for (A, B) . The sets

$$\begin{aligned} I_X &= {}_A \langle X, X \rangle := \overline{\text{span} \{ {}_A \langle x, y \rangle : x, y \in X \}} \subset A, \\ J_X &= \langle X, X \rangle_B := \overline{\text{span} \{ \langle x, y \rangle_B : x, y \in X \}} \subset B, \end{aligned}$$

form closed two-sided ideals (in particular, C^* -subalgebras) of A and B respectively. Since X is an (A, B) Hilbert C^* -bimodule and the inner products in fact take their values in the ideals $I_X \subset A$ and $J_X \subset B$, we can view X as Hilbert C^* -bimodule over (I_X, J_X) . By construction, the inner products are now full so X is a Morita equivalence bimodule for (I_X, J_X) . We now summarise our findings for future reference.

Proposition 4.7.1. *Let A and B be C^* -algebras and $\mathcal{A} \subset A$ and $\mathcal{B} \subset B$ local subalgebras. Suppose that \mathcal{X} is a nondegenerate $(\mathcal{A}, \mathcal{B})$ inner product bimodule and denote by X the completion of \mathcal{X} in the norm (4.4.3). Then X is an (A, B) Hilbert C^* -bimodule and hence a Morita equivalence bimodule for the pair of ideals (I_X, J_X) .*

5. THE OSCILLATOR BIMODULE

We consider the smooth oscillator representation ω of $G \times H$ realized on the space of smooth vectors that will denote by \mathbb{S} . In this section we will equip \mathbb{S} with the structure of an $(\mathcal{S}(G), \mathcal{S}(H))$ inner product bimodule in the sense of Definition 4.4.1. We will then use Proposition 4.7.1 to complete \mathbb{S} into a C^* -bimodule for $(C_r^*(G), C_r^*(H))$.

5.1. **Matrix coefficients.** Matrix coefficients of the oscillator representation are critical to our construction. So we start by recording a well-known fast decay property that our equal rank case enjoys.

Proposition 5.1.1. *For all $x, y \in \mathbb{S}$, the matrix coefficient functions*

$$g \mapsto \langle x, \omega(g)(y) \rangle, \quad h \mapsto \langle x, \omega(h)(y) \rangle,$$

belong to the Schwartz algebras $\mathcal{S}(G), \mathcal{S}(H)$ of G and H respectively.

Proof. This is well-known to the experts. It follows from the matrix coefficient estimates of Li (see Cor. 3.4 and proof of Thm 3.2 in [23]). See also Prop. 3.1.1 of [36] (where G_2 denotes the “smaller” group of the dual pair and hence applies to both groups in our equal rank case). The same observation is made in [20, Lemma 7.4] for the case of real unitary groups. \square

Observe that in particular, the oscillator representation, when restricted to G or H , is tempered (see Section 3.2).

5.2. Right inner product module structure. We equip \mathbb{S} with a *right* $\mathcal{S}(H)$ -module structure as follows: for $x \in \mathbb{S}$

$$x \cdot b := \int_H b(h) \omega(h^{-1})(x) dh, \quad b \in \mathcal{S}(H). \quad (5.2.1)$$

Note that $x \cdot b$ is well-defined and belongs to \mathbb{S} . Next, we equip \mathbb{S} with an $\mathcal{S}(H)$ -valued right linear form

$$\langle x, y \rangle_H(h) := \langle x, \omega(h)(y) \rangle, \quad x, y \in \mathbb{S}, h \in H \quad (5.2.2)$$

It is easy to check that this form is Hermitian (i.e. $\langle x, y \rangle_H^* = \langle y, x \rangle_H$) and compatible with the right $\mathcal{S}(H)$ -module structure given above (i.e. $\langle x, y \cdot b \rangle_H = \langle x, y \rangle_H b$).

Proposition 5.2.1. *Equipped with the right module structure (5.2.1) and the form (5.2.2), the space \mathbb{S} becomes a nondegenerate right inner product module over $\mathcal{S}(H)$.*

Proof. We just need to prove that the form $\langle \cdot, \cdot \rangle_H$ is positive definite, that is, for any $x \in \mathbb{S}$, we have $\langle x, x \rangle_H \geq 0$ as an element of the C^* -algebra $C_r^*(H)$ and that $\langle x, x \rangle_H = 0$ only when $x = 0$.

The latter is clear. To show the former, it is enough to show that given an injective representation Π of $C_r^*(H)$, the operator $\Pi(\langle \varphi, \varphi \rangle_H)$ is positive for every $\varphi \in \mathbb{S}$. If we prove that $\pi(\langle \varphi, \varphi \rangle_H)$ is positive for every π in the spectrum of $C_r^*(H)$, then we will be done by considering representation

$$\Pi = \bigoplus_{\pi \in \widehat{C_r^*(H)}} \pi$$

which is injective. Therefore it suffices to prove that

$$\pi(\langle x, x \rangle_H) \geq 0$$

as an operator on V_π for every π in the spectrum of $C_r^*(H)$.

$$\langle v, \pi(\langle x, x \rangle_H)v \rangle = \int_H \langle x, \omega(h)(x) \rangle \langle v, \pi_h(v) \rangle dh.$$

Let $x, x' \in \mathbb{S}$ and consider the operator $\pi(\langle x, x' \rangle_H)$ on V_π . This operator is determined by the bilinear form

$$\langle v, \pi(\langle x, x' \rangle_H)(v') \rangle \geq 0$$

for $v, v' \in V_\pi$. We unfold the left hand side

$$\left\langle v, \pi(\langle x, x' \rangle_H)(v') \right\rangle = \left\langle v, \int_H \langle x, x' \rangle_H(h) \pi_h(v') dh \right\rangle \quad (5.2.3)$$

$$= \int_H \langle x, x' \rangle_H(h) \langle v, \pi_h(v') \rangle dh \quad (5.2.4)$$

$$= \int_H \langle x, \omega(h)(x') \rangle \langle v, \pi_h(v') \rangle dh \quad (5.2.5)$$

$$= (x \otimes v, x' \otimes v')_\pi \quad (5.2.6)$$

where $(\cdot, \cdot)_\pi$ is the Hermitian form on $\omega \otimes \pi$ (see 2.3.1). The latter is non-negative thanks to Prop. 2.4.1. Therefore, we conclude that for $x \in \mathbb{S}$ and $v \in V_\pi$ we have

$$\langle v, \pi(\langle x, x \rangle_H)v \rangle = (x \otimes v, x \otimes v)_\pi \geq 0,$$

which implies that $\langle x, x \rangle_H \geq 0$ in $C_r^*(H)$. Now suppose $\langle x, x \rangle_H = 0$, so that

$$\langle \omega(h)x, x \rangle = 0, \quad \forall h \in H.$$

Then in particular, for $h = e$ we find that $\langle x, x \rangle = 0$, so that $x = 0$ in the oscillator representation. Since \mathbb{S} injects into the oscillator representation, we conclude that $x = 0$ in \mathbb{S} . \square

5.3. Left inner product module structure. We will show that Θ can also be obtained by equipping \mathbb{S} with a left $\mathcal{S}(G)$ inner product module structure. As before, we first equip \mathbb{S} with a left $\mathcal{S}(G)$ -module structure: for $x \in \mathbb{S}$

$$a \cdot x := \omega(a)(x) = \int_G a(g)\omega(g)(x) dg, \quad a \in \mathcal{S}(G). \quad (5.3.1)$$

Notice that $a \cdot x$ is well-defined and belongs to \mathbb{S} since ω , as a G -representation, is tempered (as evidenced by Prop. 5.1.1). Next, we equip \mathbb{S} with an $\mathcal{S}(G)$ -valued left linear form

$${}_G\langle x, y \rangle(g) := \langle \omega(g)(y), x \rangle, \quad x, y \in \mathbb{S}, g \in G \quad (5.3.2)$$

It is straight-forward to check that this $\mathcal{S}(G)$ -valued form is Hermitian and compatible with the left $\mathcal{S}(G)$ -module structure given above.

Proposition 5.3.1. *Equipped with the left module structure (5.3.1) and the form (5.3.2), the space \mathbb{S} becomes a nondegenerate left inner product module over $\mathcal{S}(G)$.*

Proof. We just need to prove that the form ${}_G\langle \cdot, \cdot \rangle$ is positive definite. It suffices to prove that $\pi({}_G\langle x, x \rangle)$ is positive as an operator on V_π for every π in the spectrum of $C_r^*(G)$.

Given $v \in \pi$, we have seen in the said proof that $\langle v, \pi({}_G\langle x, x \rangle)(v) \rangle$ equals $\overline{\langle x \otimes v, x \otimes v \rangle_{\bar{\pi}}}$ on $\omega \otimes \bar{\pi}$. Positivity again follows from Prop. 2.4.1. \square

5.4. Key compatibility property. In fact \mathbb{S} is an inner product bimodule for $(\mathcal{S}(G), \mathcal{S}(H))$ in the sense of Definition 4.4.1. We will prove the following compatibility between the two inner products: for $x, y, z \in \mathbb{S}$

$${}_G\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_H \quad (5.4.1)$$

A convenient reformulation is as follows, for $x, y, z, u \in \mathbb{S}$,

$$\left\langle {}_G\langle x, y \rangle \cdot z, u \right\rangle = \left\langle x \cdot \langle y, z \rangle_H, u \right\rangle$$

Unfolding two sides, we obtain

$$\left\langle {}_G\langle x, y \rangle \cdot z, u \right\rangle = \left\langle \int_G {}_G\langle x, y \rangle(g) \omega(g)(z) dg, u \right\rangle = \int_G \langle x, \omega(g)(y) \rangle \overline{\langle u, \omega(g)(z) \rangle} dg$$

and

$$\left\langle x \cdot \langle y, z \rangle_H, u \right\rangle = \left\langle \int_H \langle y, z \rangle_H(h) \omega(h^{-1})(x) dh, u \right\rangle = \int_H \overline{\langle y, \omega(h)(z) \rangle} \langle x, \omega(h)(u) \rangle dh$$

Thus we arrive at the formulation

$$\int_G \langle x, \omega(g)(y) \rangle \overline{\langle u, \omega(g)(z) \rangle} dg = \int_H \langle x, \omega(h)(u) \rangle \overline{\langle y, \omega(h)(z) \rangle} dh. \quad (5.4.2)$$

Proposition 5.4.1. *Equation (5.4.2), and hence (5.4.1), hold for any $x, y, z \in \mathbb{S}$.*

Proof. Let $\bar{\omega}$ denote the complex conjugate representation of ω . Consider the maps

$$P_G, P_H : \omega \otimes \bar{\omega} \otimes \omega \otimes \bar{\omega} \rightarrow \mathbb{C},$$

given by

$$P_G(x \otimes \bar{z} \otimes y \otimes \bar{u}) := \int_G \langle x, \omega(g)(y) \rangle \overline{\langle u, \omega(g)(z) \rangle} dg,$$

$$P_H(x \otimes \bar{z} \otimes y \otimes \bar{u}) := \int_H \langle x, \omega(h)(u) \rangle \overline{\langle y, \omega(h)(z) \rangle} dh.$$

Straight-forward calculations show that

$$P_G, P_H \in \text{Hom}_{H \times (G \times G)}((\omega \otimes \bar{\omega}) \otimes (\omega \otimes \bar{\omega}), \mathbb{C})$$

where we consider $\omega \otimes \bar{\omega}$ with the diagonal action of H and with the natural action of $G \times G$.

Now let $\mathbb{W} = W + (-W)$ where $-W$ denotes the space W with the form $-\langle \cdot, \cdot \rangle_W$. We have an oscillator representation ω of $H \times G(\mathbb{W})$ which satisfies¹² (see [8, Section 4])

$$\omega \otimes (\bar{\omega} \otimes \chi_V) \simeq \omega$$

as $G \times G$ -representations. Here we embed $G(W) \times G(-W)$ in $G(\mathbb{W})$ and identify $G(-W) = G(W)$.

In Section 17 of [8], Gan and Ichino introduce two forms

$$\mathcal{I}, \mathcal{E} \in \text{Hom}_{H \times (G \times G)}(\omega \otimes \bar{\omega} \otimes \bar{\chi}_V \otimes \chi_V, \mathbb{C}).$$

Note that roles of G and H in their Section 17 have to be swapped, as we did in the previous display and below, in order to make it compatible with ours.

Remarkably, our forms P_G and P_H are essentially equal to the forms \mathcal{I} and \mathcal{E} of Gan and Ichino. To see this, observe that for $x, y, z, u \in \mathbb{S}$, we have

$$\begin{aligned} \mathcal{I}(x \otimes \bar{z}, y \otimes \bar{u}) &= \int_G \langle \omega(g)(x), y \rangle \overline{\langle \omega(g)(z), u \rangle} dg \\ &= \int_G \langle x, \omega(g^{-1})(y) \rangle \overline{\langle z, \omega(g^{-1})(u) \rangle} dg \\ &= \int_G \langle x, \omega(g)(y) \rangle \overline{\langle z, \omega(g)(u) \rangle} dg \\ &= P_G(x \otimes \bar{u} \otimes y \otimes \bar{z}) \end{aligned}$$

(for the first equality, see the middle of page 593 of [8], plug in $g = g' = e$ and swap G with H). Moreover, we have (plugging in $g = g' = e$, and swapping G with H , in line 7 of p 594 of [8]):

$$\mathcal{E}(x \otimes \bar{z}, y \otimes \bar{u}) = \int_H \mathcal{F}_{x \otimes \bar{z}}(i(h, 1)) \overline{\mathcal{F}_{y \otimes \bar{u}}(i(h, 1))} dh$$

where by definition (see top of page 587 of [8]) we have

$$\mathcal{F}_{\phi \otimes \bar{\psi}}(i(h, 1)) = \langle \omega(h)(\phi), \psi \rangle, \quad h \in H, \phi, \psi \in \mathbb{S}.$$

So plugging this in, we have

$$\begin{aligned} \mathcal{E}(x \otimes \bar{z}, y \otimes \bar{u}) &= \int_H \langle \omega(h)(x), z \rangle \overline{\langle \omega(h)(y), u \rangle} dh \\ &= \int_H \langle x, \omega(h^{-1})(z) \rangle \overline{\langle y, \omega(h^{-1})(u) \rangle} dh \\ &= \int_H \langle x, \omega(h)(z) \rangle \overline{\langle y, \omega(h)(u) \rangle} dh \\ &= P_H(x \otimes \bar{u} \otimes y \otimes \bar{z}). \end{aligned}$$

Gan and Ichino prove ([8, Thm. 17.2]) that the space $\text{Hom}_{G \times H \times H}(\omega \otimes \bar{\omega} \otimes \bar{\chi}_V \otimes \chi_V, \mathbb{C})$ is one dimensional, so that \mathcal{I} and \mathcal{E} are proportional. The proportionality constant depends on the choice of Haar measures on G and H . For a specific choice of Haar measures ([8, Section 20.1], Gan and Ichino calculate that $C = 1$ in the unitary/unitary case, and in the metaplectic/orthogonal case,

¹²Recall that χ_V is one of the two auxillary characters of E^\times that we fixed at the very beginning to make sure that the oscillator representation factors through $G \times H$.

$C = 2$ or $C = 1/2$ depending on ε (see [8, Thm. 20.1]). Scaling the Haar measure they use by a factor of $1/2$ or 2 , we make $C = 1$ in the metaplectic/orthogonal case. This gives us

$$P_G(x \otimes \bar{u} \otimes y \otimes \bar{z}) = \mathcal{I}(x \otimes \bar{z}, y \otimes \bar{u}) = \mathcal{E}(x \otimes \bar{z}, y \otimes \bar{u}) = P_G(x \otimes \bar{u} \otimes y \otimes \bar{z})$$

as desired. \square

Putting together Propositions 5.2.1, 5.3.1 and 5.4.1 together with Proposition 4.7.1 gives us the following.

Theorem 5.4.2. *The space \mathbb{S} is a nondegenerate inner product bimodule for $(\mathcal{S}(G), \mathcal{S}(H))$. Its C^* -module completion is a Hilbert C^* -bimodule for $(C_r^*(G), C_r^*(H))$.*

We will denote this Hilbert C^* -module by Θ and will call it the **oscillator bimodule** as an homage to Rieffel's Heisenberg module.

6. THE θ -SUBALGEBRAS AND INDUCED REPRESENTATIONS

6.1. Truncation. We will now apply the final statement of Proposition 4.7.1 to make Θ into an equivalence bimodule for the ideals $C_\theta^*(G) \subset C_r^*(G)$ and $C_\theta^*(H) \subset C_r^*(H)$ generated by the span of the left- and right inner products, respectively. We will then analyse their spectra in terms of the local theta correspondence.

6.1.1. Truncate $C_r^*(H)$. The linear span of the range of $\langle \cdot, \cdot \rangle_H$ is a two-sided ideal of $\mathcal{S}(H)$. Let us denote this linear span by $\langle \mathbb{S}, \mathbb{S} \rangle_H$. Consider its C^* -closure

$$\boxed{C_\theta^*(H) := \overline{\langle \mathbb{S}, \mathbb{S} \rangle_H}^{C_r^*(H)}}. \quad (6.1.1)$$

Thus $C_\theta^*(H)$ is closed two-sided ideal of $C_r^*(H)$.

Proposition 6.1.2. *The spectrum of $C_\theta^*(H)$ can be identified with tempered irreducible representations π of H for which $\theta(\pi) \neq 0$.*

Proof. Basic theory tells us that the spectrum of $C_\theta^*(H)$ is simply the subset of the spectrum of $C_r^*(H)$ made precisely of those elements which do not vanish on $C_\theta^*(H)$.

Let π be a tempered irreducible representation of H (in other words, an elements of the spectrum of $C_r^*(H)$). Observe that π vanishes on $C_\theta^*(H)$ if and only if it vanishes on the range of $\langle \cdot, \cdot \rangle_H$, thanks to density of the latter in the former. Let $x, x' \in \mathbb{S}$. Then $\pi(\langle x, x' \rangle_H)$ is the zero operator on V_π if and only if

$$\langle v, \pi(\langle x, x' \rangle_H)(v') \rangle = 0$$

for all $v, v' \in V_\pi$. We unfold the left hand side

$$\langle v, \pi(\langle x, x' \rangle_H)(v') \rangle = \left\langle v, \int_H \langle x, x' \rangle_H(h) \pi_h(v') dh \right\rangle \quad (6.1.2)$$

$$= \int_H \langle x, x' \rangle_H(h) \langle v, \pi_h(v') \rangle dh \quad (6.1.3)$$

$$= \int_H \langle x, \omega(h)(x') \rangle \langle v, \pi_h(v') \rangle dh \quad (6.1.4)$$

$$= (x \otimes v, x' \otimes v')_\pi \quad (6.1.5)$$

where $(\cdot, \cdot)_\pi$ is the Hermitian form on $\omega \otimes \pi$ (see 2.3.1). Therefore, we conclude that π vanishes on $C_\theta^*(H)$ if and only the form $(\cdot, \cdot)_\pi$, hence $L(\pi)$ is zero. However, by Prop. 2.4.1, $L(\pi)$ is isomorphic to $\theta(\pi^*) = \Theta(\pi^*)$. The claim now follows from the fact that $\Theta(\pi^*)$ is non-zero if and only if $\Theta(\pi)$ is non-zero, an immediate corollary of Lemma 6.1 of [8]. \square

6.1.3. *Truncate $C_r^*(G)$.* The linear span of the range of ${}_G\langle \mathbb{S}, \mathbb{S} \rangle$ is a two-sided ideal of $\mathcal{S}(G)$. Consider its C^* -closure

$$C_\theta^*(G) := \overline{{}_G\langle \mathbb{S}, \mathbb{S} \rangle}^{C_r^*(G)}. \quad (6.1.6)$$

Then $C_\theta^*(G)$ is a closed two-sided ideal of $C_r^*(G)$, and hence is a C^* -subalgebra.

Proposition 6.1.4. *The spectrum of $C_\theta^*(G)$ can be identified with tempered irreducible (necessarily genuine) representations π of G for which $\theta(\pi) \neq 0$.*

Proof. The proof is the same as that of Prop. 6.1.2. Given tempered irreducible representation π of G , $x, x' \in \mathbb{S}$ and $v, v' \in \pi$, we observe that

$$\langle v, \pi({}_G\langle x, x' \rangle)(v') \rangle = \int_G \langle \omega_g(x), x' \rangle \langle v, \pi_g(v') \rangle dg = \int_G \overline{\langle x, \omega_g(x') \rangle} \langle v, \pi_g(v') \rangle dg. \quad (6.1.7)$$

This is the conjugate of the Hermitian form $(\cdot, \cdot)_{\bar{\pi}}$ on $\omega \otimes \bar{\pi}$ (see 2.3.1) where $\bar{\pi}$ is the conjugate representation on $\overline{V_\pi}$. Therefore, we conclude that π vanishes on $C_\theta^*(G)$ if and only if $(\cdot, \cdot)_{\bar{\pi}}$, hence $(\cdot, \cdot)_\pi$, is zero. The claim now follows from Prop. 2.4.1 as explained in the proof of Prop. 6.1.2. \square

6.2. **The induced G -representation.** Consider the action of G on \mathbb{S} via the oscillator representation. As the action of G and H commute, the G -action preserves the $C_\theta^*(H)$ -valued inner product $\langle \cdot, \cdot \rangle_H$ on \mathbb{S} , that is

$$\langle \omega(g)x, \omega(g)y \rangle_H = \langle x, y \rangle_H, \quad x, y \in \mathbb{S}.$$

It follows that $\|\omega(g)\|_{\text{End}^*(\Theta)} = 1$, so $\omega(g)$ can be extended to a unitary operator on all of Θ .

Now, given an irreducible representation of $C_\theta^*(H)$, in other words, a tempered irreducible representation (π, V_π) of H with $\theta(\pi) \neq 0$, consider $\Theta \otimes_{C_\theta^*(H)} V_\pi = \Theta \otimes_{C_r^*(H)} V_\pi$. Following the previous paragraph, G acts on $\Theta \otimes_{C_\theta^*(H)} V_\pi$ via the formula

$$g \cdot (x \otimes v) := \omega(g)(x) \otimes v$$

where $g \in G$, $x \in \Theta$ and $v \in V_\pi$. Recall from Section 4.3 that the space $\Theta \otimes_{C_\theta^*(H)} V_\pi$ comes equipped with a positive Hermitian form

$$(x \otimes v, x' \otimes v') := \left\langle v, \pi(\langle x, x' \rangle_B)(v') \right\rangle_{V_\pi}. \quad (6.2.1)$$

As the action of G commutes with that of $C_\theta^*(H)$ on Θ , the above form and also its radical are preserved under the action of G and we obtain a unitary representation of G on the Hilbert space $\Theta \otimes_{C_\theta^*(H)} V_\pi$ which we will denote

$$\text{Ind}_H^G(\Theta, \pi).$$

Proposition 6.2.1. *The unitary representation $\text{Ind}_H^G(\Theta, \pi)$ of G is precisely the unitarization of $\theta(\pi^*)$.*

Proof. By Prop. 2.4.1, we can replace $\theta(\pi^*)$ by $L(\pi)$. Consider the map

$$W : \mathbb{S} \otimes V_\pi^\infty \longrightarrow \Theta \otimes_{C_\theta^*(H)} V_\pi$$

sending

$$x \otimes v \mapsto x \otimes_{C_\theta^*(H)} v$$

where we view \mathbb{S} as a dense subspace of Θ .

As we have already observed in the proof of Prop. 6.1.2, we have

$$(x \otimes v, x' \otimes v')_\pi = \int_H \langle \omega(h)(x), x' \rangle \langle \pi(h)(v), v' \rangle_{V_\pi} dh = \left\langle v, \pi(\langle x, x' \rangle_B)(v') \right\rangle_{V_\pi}.$$

Thus the map W preserves the forms on the two sides. Therefore, the kernel of W is precisely the radical N of $(\cdot, \cdot)_\pi$, so that W descends to a linear embedding

$$(\mathbb{S} \otimes V_\pi^\infty) / N \hookrightarrow \Theta \otimes_{C_\theta^*(H)} V_\pi,$$

giving us the desired injective G -intertwiner. \square

We summarize our results.

Theorem 6.2.2. *The oscillator bimodule Θ is an equivalence $(C_\theta^*(G), C_\theta^*(H))$ -bimodule. Moreover, the associated induction map*

$$\pi \mapsto \text{Ind}_{C_\theta^*(H)}^{C_\theta^*(G)}(\Theta, \pi)$$

captures the tempered local theta correspondence in the sense that if π is a tempered irreducible representation of H then $\text{Ind}_{C_\theta^(H)}^{C_\theta^*(G)}(\Theta, \pi)$ is (the integrated form of) the unitarization of $\theta(\pi^*)$.*

Proof. The Morita equivalence statement follows from Propositions 4.7.1 and 5.4.2. For the induction part, let π be a tempered irreducible representation of H . Recall that π belongs to the spectrum of $C_\theta^*(H)$ (i.e. π restricted to $C_\theta^*(H)$ is not zero) if and only if $\theta(\pi) \neq 0$. If $\theta(\pi) = 0$ then $\pi(C_\theta^*(H)) = 0$ and hence $\text{Ind}_{C_\theta^*(H)}^{C_\theta^*(G)}(\Theta, \pi) = 0$. So we can assume that $\theta(\pi) \neq 0$. We have seen in Prop. 6.2.1 that the G -representation $\text{Ind}_H^G(\Theta, \pi)$ is the unitarization of $\theta(\pi^*)$. It is clear that the $C_\theta^*(G)$ -representation $\text{Ind}_{C_\theta^*(H)}^{C_\theta^*(G)}(\Theta, \pi)$ is nothing but the integrated form of $\text{Ind}_H^G(\Theta, \pi)$. Therefore, the induction of representations of $C_\theta^*(H)$ to $C_\theta^*(G)$ implemented via Θ captures the local theta correspondence as claimed. \square

6.3. Functoriality. The induction of representations implemented by Θ establishes an equivalence between the categories of representations of the ideal $C_\theta^*(G)$ of $C_r^*(G)$ and of the ideal $C_\theta^*(H)$ of $C_r^*(H)$. Recall that the spectra of $C_\theta^*(G)$ and $C_\theta^*(H)$ capture those tempered irreducible representations of G and H which enter the theta correspondence and that the induction map, once restricted to the irreducible representations, capture the theta correspondence. Therefore the theta correspondence is functorial.

6.4. Continuity. The spectra of $C_\theta^*(G)$ and $C_\theta^*(H)$ are homeomorphic. Thus the tempered theta correspondence is a homeomorphism with respect to weak containment of representations.

6.5. Support of the oscillator representation. In this section, we make some elementary observations regarding the role played by the oscillator representation in our picture of the theta correspondence. Recall that the oscillator representation in our equal rank case is tempered both as a G -representation and an H -representation. We first show that the induction functor associated to the oscillator bimodule, when viewed as a $(C_r^*(G), C_r^*(H))$ -correspondence, sends the regular representation of H to the oscillator representation (viewed as a representation of G). Using this, we show that $C_\theta^*(G)$ sits as an essential ideal in $C_\omega^*(G)$. Next we prove that the closure of the set of tempered representations that enter the theta correspondence equal the support of the oscillator representation.

Proposition 6.5.1. *Denote by (ω, V_ω) the oscillator representation and by $(\rho, L^2(H))$ the left regular representation of H . The map*

$$\begin{aligned} \mathcal{U}_\Theta &: \mathbb{S} \otimes_{\mathcal{S}(H)}^{\text{alg}} \mathcal{S}(H) \rightarrow \mathbb{S} \\ x \otimes f &\mapsto x \cdot f, \end{aligned}$$

extends to a G -equivariant unitary isomorphism

$$U_\Theta : \Theta \otimes_{C_r^*(H)} L^2(H) \rightarrow V_\omega.$$

Proof. The map \mathcal{U}_Θ has dense range because the oscillator representation is H -essential. It thus suffices to show that it is an isometry. This is established by the following calculation.

$$\langle \mathcal{U}_\Theta(x \otimes f), \mathcal{U}_\Theta(y \otimes g) \rangle = \langle x \cdot f, y \cdot g \rangle = \int_H \int_H \langle \omega(s)f(s^{-1})x, \omega(t)g(t^{-1})y \rangle ds dt$$

$$\begin{aligned}
 &= \int_H \int_H \langle f(s^{-1})x, \omega(t)g(t^{-1}s^{-1})y \rangle ds dt = \int_H \int_H \langle f(s)x, \omega(t)g(t^{-1}s)y \rangle ds dt \\
 &= \int_H \int_H \overline{f(s)} \langle x, \omega(t)y \rangle g(t^{-1}s) ds dt = \int_H \int_H \overline{f(s)} \langle x, \omega(t)y \rangle (\rho(t)g)(s) ds dt \\
 &= \int_H \overline{f(s)} \rho(\langle x, y \rangle_{\Theta}) g(s) ds dt = \langle f, \rho(\langle x, y \rangle_{\Theta}) g \rangle_{L^2(H)}.
 \end{aligned}$$

The G -equivariance now follows since $\mathcal{U}_{\Theta}(gx \otimes f) = (gx) \cdot f = g(x \cdot f) = g(\mathcal{U}_{\Theta}(x \otimes f))$. \square

In accordance with the notation we introduced in Section 3.1, let $C_{\omega}^*(G)$ denote the image of $C^*(G)$ under the oscillator representation ω .

Corollary 6.5.2. *Denote by $L_{\Theta} : C_r^*(G) \rightarrow \text{End}^*(\Theta)$ the $*$ -homomorphism determined by the left module action. We have $\ker L_{\Theta} = \ker \omega$ and $C_{\theta}^*(G)$ embeds into $C_{\omega}^*(G)$ as a closed two sided essential ideal. In particular the support of ω contains $\widehat{C_{\theta}^*(G)}$.*

Proof. The regular representation is faithful, so we have an injection

$$\text{End}^*(\Theta) \rightarrow B(\Theta \otimes_{C_r^*(H)} L^2(H)) \simeq B(V_{\omega}).$$

Moreover this injection is compatible with the $C_r^*(G)$ representation on both sides. Therefore the image of L_{Θ} is equal to $C_{\omega}^*(G)$. Since Θ carries a left $C_{\theta}^*(G)$ -valued inner product, we obtain an injection $C_{\theta}^*(G) \rightarrow C_{\omega}^*(G)$. In particular $\ker \omega \cap C_{\theta}^*(G) = 0$ and $\ker \omega \subset C_{\theta}^*(G)^{\perp}$, where the latter denotes the annihilator of $C_{\theta}^*(G)$.

For $b \in C_r^*(G)$ and $x \in \mathbb{S}$ we have $b \cdot x \in \Theta \cap V_{\omega}$ since b can be approximated in norm by a sequence $b_n \in \mathcal{S}(H)$, so that $b_n \cdot x \rightarrow b \cdot x$ in both V_{ω} and Θ . For $b \in C_{\theta}^*(G)^{\perp}$ and $x \in \mathbb{S}$ we have $0 = b^*_G \langle x, x \rangle b =_G \langle b \cdot x, b \cdot x \rangle$. Since the left $C_r^*(G)$ -valued inner product is nondegenerate we find that $b \cdot x = 0 \in V_{\omega} \cap \Theta$ and since \mathbb{S} is dense in V_{ω} we have $b \in \ker \omega$. The statement follows. \square

Recall that the closure of $\widehat{C_{\theta}^*(G)}$ in the tempered dual of G is the set of those irreducible representations of G which are weakly contained in $\widehat{C_{\theta}^*(G)}$.

Lemma 6.5.3. *The oscillator representation ω is contained in the closure of $\widehat{C_{\theta}^*(G)}$.*

Proof. We need to show that (see (4.3.2))

$$\bigcap_{\pi \in \widehat{C_{\theta}^*(G)}} \ker \pi \subset \ker \omega.$$

Let $b \in \bigcap_{\pi \in \widehat{C_{\theta}^*(G)}} \ker \pi$, $x \in \mathbb{S}$ and consider $b \cdot x \in \Theta \cap V_{\omega}$. Then for all $\pi \in \widehat{C_{\theta}^*(G)}$ we have

$$\pi(G \langle b \cdot x, b \cdot x \rangle) = \pi(b^*) \pi(G \langle x, x \rangle) \pi(b) = 0,$$

so it follows that $_G \langle b \cdot x, b \cdot x \rangle = 0 \in C_{\theta}^*(G)$. Therefore $b \cdot x = 0$ for all $x \in \mathbb{S}$, that is $b \in \ker \omega$. \square

Corollary 6.5.4. *The closure of $\widehat{C_{\theta}^*(G)}$ in the tempered dual of G equals the support of the oscillator representation ω .*

Proof. Note that the support of ω is simply the closure of the singleton $\{\omega\}$. Thus Lemma 6.5.3 gives us that the support of ω is contained in the closure of $\widehat{C_{\theta}^*(G)}$. The converse containment is given by Corollary 6.5.2. \square

The above corollary is known; it follows alternatively from Thm. 3.0.2 of Sakellaridis' paper [36] (see his Remark 3.0.3).

7. APPLICATION: TRANSFER OF CHARACTERS

Let π be an irreducible representation of $C_\theta^*(H)$. Let us abbreviate, temporarily, $\text{Ind}_{C_\theta^*(H)}^{C_\theta^*(G)}(\Theta, \pi)$ as $\text{Ind}(\pi)$ for convenience so that $V_{\text{Ind}(\pi)} = \Theta \otimes_{C_\theta^*(H)} V_\pi$. Consider the map

$$T : \Theta \rightarrow \mathcal{L}(V_\pi, V_{\text{Ind}(\pi)}), \quad T(x)(v) := x \otimes v$$

for $x \in \Theta$ and $v \in V_\pi$. The map T is linear and satisfies¹³ (see e.g. [42, Lemma 2.6])

$$T(a \cdot x \cdot b) = \text{Ind}(\pi)(a) T(x) \pi(b).$$

Moreover, we have $T(x)^*(y \otimes w) = \pi(\langle x, y \rangle_H)(w)$ and

$$T(x)^*T(y) = \pi(\langle x, y \rangle_H), \quad T(y)T(x)^* = \text{Ind}(\pi)({}_G\langle y, x \rangle). \quad (7.0.1)$$

To see the latter, observe that

$$T(y)T(x)^*(z \otimes w) = y \otimes \pi(\langle x, z \rangle_H)(w) = y \cdot \langle x, z \rangle_H \otimes w = {}_G\langle y, x \rangle \cdot z \otimes w = \text{Ind}(\pi)({}_G\langle y, x \rangle)(z \otimes w).$$

Recall that the elements $\langle x, y \rangle_H$ and ${}_G\langle y, x \rangle$ lie in the Schwartz algebras of H and G respectively. As both π and $\text{Ind}(\pi)$ are tempered, the operators $\pi(\langle x, y \rangle_H)$ and $\text{Ind}(\pi)({}_G\langle y, x \rangle)$ are of trace class.

Lemma 7.0.1. *Let $x, y \in \mathbb{S}$. We have*

$$\text{tr } \pi(\langle x, y \rangle_H) = \text{tr } \text{Ind}(\pi)({}_G\langle x, y \rangle).$$

Proof. The case where $x = y$ can be found in [1, Cor. 5]: it follows directly from (7.0.1) together with the fact that the traces of the operators SS^* and S^*S are the same for any $S \in \mathcal{L}(V_\pi, V_{\text{Ind}(\pi)})$. For the case $x \neq y$, one uses the polarization identity

$$4\langle x, y \rangle_H = \sum_{k=0}^3 i^k \langle x + i^k y, x + i^k y \rangle_H$$

to reduce to the case where $x = y$. □

Now recall that $\theta(\pi) = \text{Ind}(\pi^*)$ so that for every $x, y \in \mathbb{S}$, we have

$$\text{tr } \pi^*(\langle x, y \rangle_H) = \text{tr } \theta(\pi)({}_G\langle x, y \rangle).$$

Observing that $\pi^*(\langle x, y \rangle_H) = \pi(\overline{\langle x, y \rangle_H}) = \pi(\langle y, x \rangle_H)$, we obtain the following corollary where we use the terminology of the Section 1.2.1.

Corollary 7.0.2. *Let π be a tempered irreducible representation of H that enters the theta correspondence. Given $x, y \in \mathbb{S}$, let $\langle x, y \rangle_H \in \mathcal{S}(H)$ and ${}_G\langle x, y \rangle \in \mathcal{S}(G)$ be the matrix coefficient functions defined earlier in (5.2.2) and (5.3.2). We have*

$$\text{ch}(\theta(\pi))({}_G\langle x, y \rangle) = \text{ch}(\pi)(\langle y, x \rangle_H).$$

The above result has been recently announced by Wee Teck Gan [11, 12]. While his proof seems different than ours, it can be said that it philosophically agrees with ours in that matrix coefficients of the oscillator representation play a central role.

¹³We are essentially considering the C^* -counterpart of $\text{Hom}_{G \times H}(\omega, \pi \otimes \theta(\pi^*))$. The intertwiner space $\text{Hom}_{G \times H}(\omega, \pi \otimes \theta(\pi))$ is well-known to be one dimensional.

8. APPLICATION: PRESERVATION OF FORMAL DEGREES

In equal rank local theta correspondence, discrete series representations are sent to discrete series representations, see [7]. In this section, we will reprove a well-known result of Gan and Ichino about the preservation of formal degrees of discrete series. The main point of interest will be our method which will feature K -theory and transfer of trace maps.

Let (G, H) be an equal rank dual pair as in Section 2. We will *assume* in this section that in the metaplectic-orthogonal case, H denotes the orthogonal group $O(V)$.

Recall that an irreducible unitary representation of H is called *discrete series* if its matrix coefficients lie in $L^2(H)$ ¹⁴. The *formal degree* of a discrete series representation π of, say, H if the positive real number $\deg(\pi)$ such that

$$\int_H \langle v, \pi(h)(v') \rangle \overline{\langle w, \pi(h)(w') \rangle} dh = \frac{1}{\deg(\pi)} \langle v, w \rangle \langle v', w' \rangle$$

for all $v, v', w, w' \in V_\pi$. Note that the formal degree depends on the chosen Haar measure dh . Gan and Ichino proved in [8] there exist a choice of Haar measures on G and on H such that for every discrete series representation π of H which enters the theta correspondence, we have

$$\deg(\pi) = \deg(\theta(\pi)). \tag{8.0.1}$$

8.1. K -theory. Given a unital complex algebra C , one defines the abelian group $K_0(C)$ as the group of (formal differences of) Murray-von Neumann equivalence classes of idempotents in

$$M_\infty(C) := \varinjlim_n M_n(C).$$

When C is C^* -algebra, we can alternatively describe $K_0(C)$ using (homotopy classes of) projections instead of idempotents. For a non-unital complex algebra C , one defines $K_0(C)$ as the kernel of map

$$K_0(\tilde{C}) \rightarrow K_0(\mathbb{C}) \simeq \mathbb{Z}$$

induced by the natural map $C^+ \rightarrow \mathbb{C}$ where C^+ is the minimal unitization of C .

Note that K_0 of a C^* -algebra is a naturally ordered group.

8.2. Discrete series and K -theory. Let π be a discrete series representation of H such that $\theta(\pi) \neq 0$, so that π belongs to the spectrum of $C_\theta^*(H)$. In order to show that π defines an element in $K_*(C_\theta^*(H))$, we need the following lemma.

Lemma 8.2.1. *The singleton $\{\pi\}$ is a clopen (closed and open) subset of the spectrum of $C_\theta^*(H)$.*

Proof. In case (G, H) is a unitary dual pair, H is a connected reductive p -adic group with compact center, and the Plancherel formula of Harish-Chandra is well-known to imply that $\{\pi\}$ is clopen in the whole tempered dual \widehat{H}_{temp} of H for every discrete series representation. Since the topology of $\widehat{C_\theta^*(H)}$ is simply the subspace topology inherited from \widehat{H}_{temp} , the point $\{\pi\}$ is clopen in $\widehat{C_\theta^*(H)}$ as well.

Let us now consider the case $(G, H) = (Mp_{2n}, O_{2n+1})$, so that $H = O(V)$ with V odd dimensional. We have $O(V) \simeq SO(V) \times \{\pm 1\}$. The restriction map gives us a 2-to-1 surjection from the tempered dual of $O(V)$ to that of $SO(V)$. Restriction of π to $SO(V)$, say σ , is again a discrete series representation and the Plancherel formula argument above tells us that $\{\sigma\}$ is clopen in the tempered dual of $SO(V)$. As the restriction map is continuous ([5, Lemma 1.11]), the preimage of $\{\sigma\}$ is clopen in the tempered dual of $O(V)$. It is well-known ([7, Prop. 6.3]) that in this preimage, which has size two, only π enters the theta correspondence. Therefore this preimage, which is clopen, intersects the spectrum of $C_\theta^*(H)$ only in the singleton $\{\pi\}$ giving us the claim. \square

¹⁴Recall that H is unimodular.

The fact that $\{\pi\}$ is a clopen subset of the spectrum of $C_\theta^*(H)$ implies that the closed two-sided ideal $\ker(\pi)$ (here π is restricted to $C_\theta^*(H)$) is complemented (see [38]):

$$C_\theta^*(H) \simeq \ker(\pi) \oplus J_\pi,$$

where J_π is the closed two-sided ideal

$$J_\pi := \bigcap_{\substack{\sigma \in \widehat{C_\theta^*(H)} \\ \sigma \neq \pi}} \ker(\sigma),$$

and the sum is a direct sum of C^* -algebras.

Lemma 8.2.2. *Any discrete series representation π of H defines a class $[\pi] \in K_0(C_\theta^*(H))$ of infinite order.*

Proof. Recall that $\pi : C_r^*(H) \rightarrow \mathbb{K}(V_\pi)$ is surjective¹⁵. Since its restriction to $C_\theta^*(H)$ is non-zero and $\mathbb{K}(V_\pi)$ is simple, we conclude that $\pi : C_\theta^*(H) \rightarrow \mathbb{K}(V_\pi)$ is still surjective. It follows that

$$J_\pi \simeq C_\theta^*(H)/\ker(\pi) \simeq \mathbb{K}(V_\pi).$$

As J_π is a direct summand, the injection $J_\pi \hookrightarrow C_\theta^*(H)$ leads to an injection

$$\iota : K_0(J_\pi) \rightarrow K_0(C_\theta^*(H)).$$

Since $K_0(\mathbb{K}(V_\pi)) \simeq \mathbb{Z}$, we conclude that there is a copy of \mathbb{Z} in $K_0(C_\theta^*(H))$ that is contributed by π . We fix the positive generator $[\pi]$ of $K_0(J_\pi) \simeq \mathbb{Z}$ and call it “**the class associated to π** ” viewing it inside $K_0(C_\theta^*(H))$. \square

Now, we can use the oscillator bimodule to induce ideals as well (see Section 4.6), leading to an isomorphism of the lattices of ideal of A and B . Consider the discrete series representation $\theta(\pi^*)$ of G . As $\theta(\pi^*)$ belongs to the spectrum of $C_\theta^*(G)$, and the spectra of $C_\theta^*(G)$ and $C_\theta^*(H)$ are homeomorphic, we deduce that $\{\theta(\pi^*)\}$ is isolated in $\widehat{C_\theta^*(G)}$. It follows from the previous paragraph that we have a direct sum of C^* -algebras

$$C_\theta^*(G) \simeq \ker(\theta(\pi^*)) \oplus J_{\theta(\pi^*)}$$

with

$$J_{\theta(\pi^*)} := \bigcap_{\substack{\sigma \in \widehat{C_\theta^*(G)} \\ \sigma \neq \theta(\pi^*)}} \ker(\sigma).$$

As the induction of ideals is compatible with the induction of representations, for any $\sigma \in \widehat{C_\theta^*(H)}$, we have that

$$\text{Ind}_{C_\theta^*(H)}^{C_\theta^*(G)}(\Theta, \ker(\sigma)) = \ker(\text{Ind}_{C_\theta^*(H)}^{C_\theta^*(G)}(\Theta, \sigma)).$$

It follows that

$$\text{Ind}(\Theta, \ker(\pi)) = \ker(\theta(\pi^*)), \quad \text{Ind}(\Theta, J_\pi) = J_{\theta(\pi^*)}. \quad (8.2.1)$$

A (C, D) -equivalence bimodule X gives rise to an isomorphism of K -groups (see, for example, Prop. 2.4 and the paragraph following that in [34] for the unital case):

$$\Psi_X : K_0(C) \xrightarrow{\cong} K_0(D)$$

¹⁵This is true for any irreducible unitary representation of H since H is a ‘CCR group’.

as ordered groups. Assuming the set-up of the above paragraph, $\Theta_\pi := \Theta J_\pi$ is a $(J_{\theta(\pi^*)}, J_\pi)$ -equivalence bimodule. We are led to below commutative diagram

$$\begin{array}{ccc} K_0(J_{\theta(\pi^*)}) & \xrightarrow{\iota} & K_0(C_\theta^*(G)) \\ \simeq \downarrow \Psi_{\Theta_\pi} & & \simeq \downarrow \Psi_\Theta \\ K_0(J_\pi) & \xrightarrow{\iota} & K_0(C_\theta^*(H)) \end{array} \quad (8.2.2)$$

Lemma 8.2.3. *Let π be a discrete series representation of H such that $\theta(\pi) \neq 0$. Then the class of $\theta(\pi^*)$ in $K_0(C_\theta^*(G))$ is taken to the class of π in $K_0(C_\theta^*(H))$ under the map Ψ_Θ .*

Proof. As Ψ_{Θ_π} is an isomorphism of ordered groups, it takes $[\theta(\pi^*)]$ to $[\pi]$. Our claim now follows from the commutativity of diagram (8.2.2). \square

8.3. Traces. Let C be a C^* -algebra and let C_+ denote its cone of positive elements. By a **trace** on C , we mean a linear map $\chi : C_+ \rightarrow [0, \infty]$ such that $\chi(0) = 0$ and $\chi(cc^*) = \chi(c^*c)$ for all $a \in C$. If χ takes finite values, then it is called *bounded*. In this case, χ can be extended to a linear functional on C (since C_+ spans C) satisfying the usual trace property: $\chi(cd) = \chi(dc)$ for all $c, d \in C$.

We will be mainly interested unbounded traces. We say that χ is densely defined if its domain $\{c \in C_+ \mid \chi(c) < \infty\}$ is dense in C_+ . The “**canonical trace**” τ_H on $C_r^*(H)$ is a densely defined, unbounded trace that is determined uniquely by the property $\tau_H(f) = f(e)$ for any $f \in \mathcal{S}(H)$.

Upon restriction, one obtains a linear map $\tau_H : \mathcal{S}(H) \rightarrow \mathbb{C}$ that satisfies the usual trace property. The map τ_H on the algebra $\mathcal{S}(H)$ induces¹⁶ a linear functional, denoted τ_H^* , on $K_0(\mathcal{S}(H))$. It follows from Thm. 3.2.1 (ii) that $\mathcal{S}(H)$ is spectral invariant in $C_r^*(H)$, which in turn implies that the inclusion $i : \mathcal{S}(H) \rightarrow C_r^*(H)$ induces an isomorphism $i_* : K_0(\mathcal{S}(H)) \xrightarrow{\sim} K_0(C_r^*(H))$ in K -theory. Thus we obtain a linear map

$$\tau_H^* : K_0(C_r^*(H)) \rightarrow \mathbb{C}.$$

They key fact is that if π is a discrete series representation of H then

$$\tau_H^*([\pi]) = \deg(\pi). \quad (8.3.1)$$

It is worth mentioning that τ_H^* vanishes on all other classes which do not correspond to discrete series representations.

8.4. Transfer of traces. Given a densely defined trace χ on $C_\theta^*(H)$, one can construct, using the oscillator bimodule, a densely defined $\hat{\chi}$ on $C_\theta^*(G)$ which satisfies

$$\hat{\chi}_G(\langle x, x \rangle) = \chi(\langle x, x \rangle_H)$$

for all $x \in \Theta$ (see [34, Section 2] for bounded traces over unital algebras, and [28, Section 1, Prop. 1.3.11] for the densely defined case, see also [4, Section 2.1] for the case of ‘lower semi-continuous’ traces which our canonical traces are examples of). For the canonical trace, we have

$$\hat{\tau}_H(\langle x, x \rangle) = \tau_H(\langle x, x \rangle_H) = \langle x, x \rangle = \tau_G(\langle x, x \rangle) \quad (8.4.1)$$

for all $x \in \mathbb{S}$. Therefore $\hat{\tau}_H$ equals the canonical trace τ_G on $C_\theta^*(G)$.

We have the following cohomological aspect of the transfer of traces. Restrict τ_G, τ_H to $C_\theta^*(G)$ and $C_\theta^*(H)$ respectively, and restrict τ_G^*, τ_H^* to $K_0(C_\theta^*(G))$ and $K_0(C_\theta^*(H))$ respectively.

Lemma 8.4.1. *The pull-back map $\tau_H^* \circ \Psi_\Theta : K_0(C_\theta^*(G)) \rightarrow \mathbb{C}$ agrees with the map $\hat{\tau}_H^* : K_0(C_\theta^*(G)) \rightarrow \mathbb{C}$ associated to the transfer $\hat{\tau}_H$ of τ_H from $C_\theta^*(H)$ to $C_\theta^*(G)$.*

¹⁶One can extend the trace to $M_n(\mathcal{S}(H))$ in the obvious way and this will give a well-defined map on the classes of projections.

Proof. For bounded traces on unital algebras, this is already noted by Rieffel in Prop. 2.5 of [34]. More generally, this is recorded by Pierrot in Cor. 1.3.12 of [28]: to see this, just set his A to be our $C_\theta^*(H)$ and his E to be our Θ so that his $\mathbf{K}(E)$ becomes isomorphic our $C_\theta^*(G)$. \square

Corollary 8.4.2. *Let π be a discrete series representation of H such that $\theta(\pi)$ is non-zero. Assume that $\{\pi\}$ is isolated in the tempered dual of H . Then the formal degree of $\theta(\pi)$ equals that of π .*

Proof. Consider the pull-back $\tau_H^* \circ \Psi_\Theta : K_0(C_\theta^*(G)) \rightarrow \mathbb{C}$. We have

$$(\tau_H^* \circ \Psi_\Theta)([\theta(\pi^*)]) = \tau_H^*(\Psi_\Theta([\theta(\pi^*)])) = \tau_H^*([\pi]) \tag{8.4.2}$$

where we used Lemma 8.2.3 in the middle equality. By Lemma 8.4.1, the map $\tau_H^* \circ \Psi_\Theta$ equals the map $K_0(C_\theta^*(G)) \rightarrow \mathbb{C}$ that is induced by the transfer $\widehat{\tau}_H$ of τ_H to $C_\theta^*(G)$. By (8.4.1), we have $\widehat{\tau}_H = \tau_G$; thus

$$\tau_G^*([\theta(\pi^*)]) = \widehat{\tau}_H^*([\theta(\pi^*)]) = (\tau_H^* \circ \Psi_\Theta)([\theta(\pi^*)]). \tag{8.4.3}$$

Recall from (8.3.1) that $\tau_H^*([\pi])$ equals $\deg(\pi)$ and that $\tau_G^*([\theta(\pi^*)])$ equals $\deg(\theta(\pi^*))$. Thus, combining (8.4.2) and (8.4.3), we deduce that the degree of $\theta(\pi^*)$ equals that of π . The claim follows since π^* and π have the same degree. \square

We point out the Haar measures used above are the ones that we employed during the proof of Prop. 5.4.1.

Remark 8.4.3. Here we only dealt with the canonical trace as we were interested in the formal degrees. One could however consider the transfer of traces given orbital integrals (see, e.g. [17]) associated to conjugacy classes other than the trivial element (which gives the canonical trace).

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