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Consistent and inconsistent inequality indices for ordinal variables ☆

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ABSTRACT

Should inequality comparisons with ordinal variables be sensitive to alternative sorting of the categories? We introduce the consistency property whereby an inequality or bipolarisation index regards frequency distribution $\mathbf{r}$ less unequal than $\mathbf{s}$ if and only if it ranks the reverse-ordered distribution $\mathbf{r}'$ less unequal than $\mathbf{s}'$ for every pair of comparable frequency distributions. We characterise the class of consistent indices with a functional equation that serves as a useful test of the property. Applying the test to the most popular indices in the literature, we identify the respective consistent and inconsistent sets.

1. Introduction

Some socioeconomic indicators allow for alternative representations which essentially convey the same information. For instance, infant mortality and survival rates, adult literacy and illiteracy rates, or overcrowding ratios in the form of people per room and rooms per capita. Since at least the end of the XXth Century, theoreticians and practitioners have been gauging the consistency of inequality comparisons to such alternative choices of representation and proposing consistent measurement methods, accordingly.[1] Part of the discussion has also revolved around the desirability of the consistency property itself.

Meanwhile, building on the seminal work of Mendelson (1987) and Allison and Foster (2004) a growing body of literature has proposed methods to measure inequality and bipolarisation when the indicators are ordered categorical variables (henceforth ordinal variables),[2] such as measures of self-reported general health, life satisfaction, educational levels, or sanitation ladders. Remarkably though, the literature has not yet noted that ordinal variables can also admit alternative representations for inequality assessments, which in turns begs the question as to the consistency of inequality and bipolarisation comparisons based on ordered multinomial distributions (henceforth 'distributions').

Indeed, consider the following distribution of self-assessed general health: $\mathbf{p} = (0.1, 0.3, 0.2, 0.1, 0.3)$, which means that 10% of the population are in the least desirable category (e.g. 'very bad' or alternatively 'poor'), 30% are in the most desirable category (e.g. 'very good' or alternatively 'excellent'), and so forth regarding the middle categories (with similar corresponding labels). For some inequality or bipolarisation index I, let $I(\mathbf{p}) > I(\mathbf{q})$, where $\mathbf{q} = (0.05, 0.4, 0.2, 0, 0.35)$. Now consider reversing the order of the categories to obtain reverse-ordered distributions $\mathbf{p}' = (0.3, 0.1, 0.2, 0.3, 0.1)$ and $\mathbf{q}' = (0.35, 0, 0.2, 0.4, 0.05)$. Should we demand from an index like I that $I(\mathbf{p}') > I(\mathbf{q}')$ too? In other words, should it matter *for inequality* whether the (common) ordering direction of self-reported health categories is 'very bad', 'bad', 'medium', 'good', 'very good' (e.g. $\mathbf{p}$ and $\mathbf{q}$) as

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[1] In the case of indicators admitting attainment and shortfall representations, some examples include Micklewright and Stewart (1999), Clarke et al. (2002), Kenny (2004), Erreygers (2009), Lambert and Zheng (2011), Lasso de la Vega and Aristondo (2012), Aristondo and Lasso de la Vega (2013), Silber (2015), Chakravarty et al. (2015), Kjellsson et al. (2015), Bosmans (2016), Permanyer (2016), Permanyer et al. (2022). In the case of ratio indicators admitting alternative roles for numerator and denominator the most recent contributions are Bosmans (2016), Yalonetzky (2020).

[2] Key contributions include Apouey (2007), Abul Naga and Yalcin (2008), Kobus and Milos (2012), Lazar and Silber (2013), Lv et al. (2015), Kobus (2015), Chakravarty and Maharaj (2015), Cowell and Flachaire (2017), Sarkar and Santra (2020), Jenkins (2021). Additionally, measures of multiple polarisation have also been proposed (e.g. Permanyer and D'Ambrosio, 2015; Mussini, 2018), but they will not be covered in this paper.

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opposed to 'very good', 'good', 'medium', 'bad', 'very bad' (e.g. $\mathbf{p}'$ and $\mathbf{q}'$)?[3]

There may be good reasons to answer these questions either positively or negatively. If, for a particular inequality evaluation function I and every pair of *comparable* distributions $\mathbf{p}$ and $\mathbf{q}$,[4] we expect $I(\mathbf{p}) \geq I(\mathbf{q})$ if and only if $I(\mathbf{p}') \geq I(\mathbf{q}')$ then we are *demanding that the inequality comparison be consistent with respect to alternative admissible representations of the distributions.*[5]

The original consistency property in the literature on inequality measurement with cardinal bounded variables was deemed controversial.[6] Arguably, there is no reason to expect consensus regarding the desirability of the proposed consistency property for inequality comparisons with ordinal variables either. Perhaps such desirability should depend on the nature of the variable? Yet, notwithstanding one's position on the matter, some inequality and bipolarisation indices for ordinal variables *are* consistent, whereas others *are* inconsistent; as this paper demonstrates.

We provide a simple characterisation of consistent inequality and bipolarisation indices for ordinal variables, which serves as a useful test of the property. As it turns out, complete inequality and bipolarisation indices are consistent if and only if they satisfy a property that we call *reversal invariance*. Applying the test to the most popular indices of inequality and bipolarisation for ordinal variables we find that only the proposals by Apouey (2007), Lazar and Silber (2013) and Lv et al. (2015) are consistent across their whole parameter domains, respectively. By contrast, the measures of status inequality introduced by Cowell and Flachaire (2017) are inconsistent throughout their parameter domain. As far as the proposals by Abul Naga and Yalcin (2008), Kobus and Milos (2012) and Chakravarty and Maharaj (2015) are concerned, they are only consistent when their parameter domains are suitably restricted.

The rest of the paper proceeds as follows. Section 2 introduces the main measurement framework. Section 3 states the consistency axiom and then provides a characterisation of all consistent inequality and bipolarisation indices which serves as a straightforward consistency test. It also classifies existing inequality and bipolarisation indices in terms of their satisfaction of consistency. Section 4 provides some concluding remarks.

2. Notation and preliminaries

Let $\mathbf{p} \equiv (p_1, \ldots, p_C)$ be a vector of relative frequencies (henceforth frequencies) where $C > 1$ is the natural number of categories (each labelled by a natural number between 1 and C). Hence $0 \leq p_i \leq 1$ and $\sum_{i=1}^{C} p_i = 1$. We call this vector a distribution. The set of all possible distributions with C categories is denoted by $\mathcal{O}^C$.

Then we define the cumulative distribution function (henceforth CDF) as $\mathbf{P} \equiv (P_1, \ldots, P_C)$, where $P_k \equiv \sum_{i=1}^{k} p_i$ for any $k = 1, \ldots, C$. Earlier, we motivated the relevance of a consistency property by alluding to a reverse-ordered distribution wherein the categories are sorted in reverse order. Let $\mathbf{X}$ be a C-dimensional reversal matrix.[7] If $\mathbf{p}' \equiv (p'_1, \ldots, p'_C) = \mathbf{pX}$ is the reverse-ordered distribution of $\mathbf{p}$ then it must be the case that $p'_{C-i+1} = p_i$ for $i = 1, \ldots, C$. Therefore, $P'_{C-i} = 1 - P_i$ for $i = 1, \ldots, C-1$. These relations are useful for the derivation of results in Table 1 below.

In Section 3 we consider some inequality and bipolarisation comparisons of distributions sharing the same median or medians, i.e. based on the notion of median-preserving spreads (Allison and Foster, 2004; Kobus, 2015), or distributions whose different medians can still be equalised following suitable transformations (Sarkar and Santra, 2020). Hence, following Kobus (2015) we should define the median category (or categories) as any set of natural numbers $\{m_1, \ldots, m_l\} \subseteq \{1, \ldots, C\}$ with $l \in \{1, \ldots, C\}$ such that two types of distributions are possible in terms of number of medians (Kobus, 2015, Remark 1): (1) distributions with one median, where there exists a single $m \in \{1, \ldots, C\}$ such that $P_{m-1} < 0.5$, $P_m > 0.5$ for $1 < m$, or $m = 1$ if $P_1 > 0.5$; (2) distributions with two or more medians if $P_{m_1-1} < 0.5$, $P_{m_l} \geq 0.5$, $P_{m_{l+1}} > 0.5$, and $P_{m_i} = 0.5$ for all $i \in \{1, \ldots, l-1\}$ and $m_1 > 1$, or $P_{m_i} = 0.5$ for all $i \in \{1, \ldots, l-1\}$ if $m_1 = 1$.[8]

3. Consistency of inequality and bipolarisation indices for ordinal variables

Let $I : \mathcal{O}^C \to \mathbb{R}_+$ be an inequality (or bipolarisation) index for ordinal variables. Then we can consider the following consistency requirement:

Axiom 3.1. *Consistency: An inequality (or bipolarisation index) index I is consistent if and only if, for every pair of distributions $\mathbf{p}, \mathbf{q} \in \mathcal{O}^C$, $I(\mathbf{p}) \geq I(\mathbf{q})$ if and only if $I(\mathbf{p}') \geq I(\mathbf{q}')$.*

Then Proposition 3.1 characterises the set of consistent inequality and bipolarisation indices:

Proposition 3.1. *For any natural number $C > 1$, and every $\mathbf{p} \in \mathcal{O}^C$, I is consistent if and only if $I(\mathbf{p}) = I(\mathbf{p}')$.*

Proof. "If" part: Let $I(\mathbf{p}) = I(\mathbf{p}')$ for every $\mathbf{p} \in \mathcal{O}^C$. Then for every pair $\mathbf{p}, \mathbf{q} \in \mathcal{O}^C$, if $I(\mathbf{p}) \geq I(\mathbf{q})$ then it must be the case that $I(\mathbf{p}') \geq I(\mathbf{q}')$ because $I(\mathbf{p}') = I(\mathbf{p}) \geq I(\mathbf{q}) = I(\mathbf{q}')$. By the same reasoning if $I(\mathbf{p}') \geq I(\mathbf{q}')$ then it must be the case that $I(\mathbf{p}) \geq I(\mathbf{q})$. That is, I is consistent.

"Only if" part: Let there be a distribution $\mathbf{g} \in \mathcal{O}^C$ for which $I(\mathbf{g}) \neq I(\mathbf{g}')$. Then to prove the inconsistency of I we need to find a pair $\mathbf{p}, \mathbf{q} \in \mathcal{O}^C$ such that $I(\mathbf{p}) \geq I(\mathbf{q})$ but $I(\mathbf{p}') \leq I(\mathbf{q}')$. Consider, for instance, $\mathbf{p} = \mathbf{g}$ and $\mathbf{q} = \mathbf{p}'$. Then, if $I(\mathbf{p}) \geq I(\mathbf{q})$ we get $I(\mathbf{p}') = I(\mathbf{q}) \leq I(\mathbf{p}) = I(\mathbf{q}')$ because $\mathbf{q} = \mathbf{p}'$ if and only if $\mathbf{q}' = \mathbf{p}$. Therefore I is inconsistent. ■

Proposition 3.1 is useful to test the satisfaction of consistency. Evaluating a given index I at several arbitrary pairs of comparable distributions, $\mathbf{p}$ and $\mathbf{q}$, to test whether $I(\mathbf{p}) \geq I(\mathbf{q})$ if and only if $I(\mathbf{p}') \geq I(\mathbf{q}')$ holds every time (or not) is a pointless exercise because if consistency holds for a given number of such pairs there is no guarantee that it holds for every other pair. Instead, Proposition 3.1 suggests the following procedure which we apply to the most prominent classes of inequality and bipolarisation indices for ordinal variables in the literature: first, choose a *specific* $\mathbf{p} \in \mathcal{O}^2$ such that $\mathbf{p} \neq \mathbf{p}'$ and check whether $I(\mathbf{p}) = I(\mathbf{p}')$. This efficiently provides the simplest necessary consistency test for any particular index I; namely if $I(\mathbf{p}) \neq I(\mathbf{p}')$ for one $\mathbf{p} \in \mathcal{O}^2$ such that $\mathbf{p} \neq \mathbf{p}'$, then we conclude that I is inconsistent. Otherwise,

[3] Or what if the order of political preferences in a survey is 'strong liberal', 'liberal', 'moderate', 'conservative', 'strong conservative', as opposed to the reverse alternative? Should that matter for inequality?

[4] Most inequality and bipolarisation evaluation criteria demand minimum standards of comparability, e.g. equal number of categories or, in some cases, identical median categories.

[5] That is, the distributions themselves and their counterpart constructed by sorting the categories in reverse order.

[6] Though most of that literature supports the property, some notable examples of dissent can be found in Kenny (2004), Bosmans (2016).

[7] A reversal matrix is a permutation matrix where all elements are equal to 0, except for the elements of the counterdiagonal which are equal to 1 (Horn and Johnson, 2013, p. 33).

[8] For ease of presentation, and following the literature, we assume distributions with one median category only whenever considering median-dependent indices in Section 3. Otherwise, the assumption, is not made.

Table 1
Consistency of inequality and bipolarisation indices for ordinal variables.

Class	Consistent?
$I^{AP}(\mathbf{p},\alpha)=1-\frac{2^{\alpha}}{C-1}\sum_{i=1}^{C-1}\lvert P_i-0.5\rvert^{\alpha},\ \alpha>0$ Apouey (2007)	Yes
$I^{LWX}(\mathbf{p})=\sum_{i=1}^{C}\sum_{j\neq i}^{C}g(\lvert i-j\rvert)p_ip_j$; g is an increasing function Lv et al. (2015)	Yes
$I^{AY}(\mathbf{p};\beta,\gamma)=\frac{\sum_{i<m}P_i^{\gamma}-\sum_{i\geq m}P_i^{\beta}+(C+1-m)}{(m-1)(\frac{1}{2})^{\gamma}-[1+(C-m)(\frac{1}{2})^{\beta}]+(C+1-m)},\ \beta,\gamma>0$ Abul Naga and Yalcin (2008)	No Unless $\beta=\gamma=1$
$I^{KM}(\mathbf{p};a,b)=\frac{a\sum_{i<m}P_i-b\sum_{i\geq m}P_i+b(C+1-m)}{(a(m-1)+b(C-m))/2},\ a\geq 0;b\geq 0$ Kobus and Milos (2012)	No Unless $a=b$
$I_1^{LS}(\mathbf{p})=\frac{1}{C-1}\sum_{i=1}^{C-1}\log_2 P_i^{-P_i}(1-P_i)^{-(1-P_i)}$ Lazar and Silber (2013)	Yes
$I_2^{LS}(\mathbf{p})=\frac{1}{C-1}\sum_{i=1}^{C-1}[4P_i(1-P_i)]^{\alpha},\ \alpha>0$ Lazar and Silber (2013)	Yes
$I_2^{LS}(\mathbf{p})=\frac{1}{C-1}\sum_{i=1}^{C-1}(1-\lvert 2P_i-1\rvert)$ Lazar and Silber (2013)	Yes
$I^{CM}(\mathbf{p})=\sum_{i=1}^{m-1}w_iP_i+\sum_{i=m}^{C-1}w_i(1-P_i)$; $w_i>0\ \forall i=1,\dots,C-1$ Chakravarty and Maharaj (2015)	No Unless $w_i=w_{C-i}\ \forall i=1,\dots,C-1$
$I^{CF}(\mathbf{p};\alpha)=\frac{1}{\alpha(1-\alpha)}[\sum_{i=1}^{C}p_iP_i^{\alpha}-1],\ \alpha\in(-\infty,1)/\{0\}$ $I^{CF}(\mathbf{p};0)=-\sum_{i=1}^{C}p_i\log P_i$ Cowell and Flachaire (2017)	No

if we obtain $I(\mathbf{p})=I(\mathbf{p}')$ for one $\mathbf{p}\in\mathcal{O}^2$ such that $\mathbf{p}\neq\mathbf{p}'$, then we proceed to the general necessary and sufficient consistency test for I, which requires checking that the functional equation $I(\mathbf{p})=I(\mathbf{p}')$ holds for I and any $\mathbf{p}\in\mathcal{O}^C$ with $C>1$ such that $\mathbf{p}\neq\mathbf{p}'$.

Table 1 shows the consistency test results based on the aforementioned procedure. Among the most important proposals, the classes of indices proposed by Apouey (2007), Lazar and Silber (2013) and Lv et al. (2015) satisfy consistency across their whole parameter domains, respectively. For instance, the necessary and sufficient test for the class proposed by Apouey (2007) yields: $I^{AP}(\mathbf{p},\alpha)=1-\frac{2^{\alpha}}{C-1}\sum_{i=1}^{C-1}\lvert P_i-0.5\rvert^{\alpha}=1-\frac{2^{\alpha}}{C-1}\sum_{i=1}^{C-1}\lvert 1-P'_{C-i}-0.5\rvert^{\alpha}=I^{AP}(\mathbf{p}',\alpha)$ for all $\alpha>0$. These are all indices satisfying the median-preserving-spreads principle (henceforth 'MPS').[9] By contrast, other classes satisfying the same principle, such as those by Abul Naga and Yalcin (2008), Kobus and Milos (2012) and Chakravarty and Maharaj (2015), only fulfil the consistency requirement if their respective parameter domains are duly restricted. Among indices sensitive to both bipolarisation transformations,[10] some classes are consistent (e.g. I^{AP}) whereas others violate the property, such as the class I^{CM} when the parameter restriction in Table 1 is not met. Meanwhile, I^{CF}, the class of peer-inclusive downward-looking status inequality indices (Cowell and Flachaire, 2017), which is not based on the MPS principle, is inconsistent throughout its whole parameter domain.[11] Indeed, let $\mathbf{p}\in\mathcal{O}^2$ such that $\mathbf{p}\neq\mathbf{p}'$, then the necessary consistency test yields: $I^{CF}(\mathbf{p};\alpha)=\frac{1}{\alpha(1-\alpha)}[p_1^{1+\alpha}+p_2-1]\neq\frac{1}{\alpha(1-\alpha)}[p_2^{1+\alpha}+p_1-1]=I^{CF}(\mathbf{p}';\alpha)$ for $\alpha\in(-\infty,1)/\{0\}$ and $I^{CF}(\mathbf{p};0)=-p_1\log p_1\neq -p_2\log p_2=I^{CF}(\mathbf{p}';0)$.

Why are some inequality indices consistent whereas others satisfying similar desirable properties are not? Most of the indices in Table 1 rely on weighted sums of cumulative frequencies (or functions thereof). These weights (and functional parameters like the powers β and γ attached to the cumulative frequencies in I^{AY}) regulate the sensitivity to frequency transfers (e.g., clustering, MPS, etc.) in different parts of the distribution. Such indices are consistent only when their weights are *symmetric* (and the powers β and γ equal to 1 in the case of I^{AY}, etc.). Otherwise, reversing the order of categories without correspondingly reversing the vector of weights leads to a violation of the consistency test in Proposition 3.1; namely, $I(\mathbf{p})=I(\mathbf{p}')$. These weights are explicit in some indices (e.g., I^{CM}) and implicit in others (e.g., I^{AP}, I^{CF}, those by Lazar and Silber, 2013). Thus, some indices are consistent because their implicit weights are always equal (e.g. I^{AP}), which is a special case of symmetry.

Moreover, with those parameter restrictions, consistent measures do not satisfy ordinal-variable adaptations of the transfer sensitivity axiom. Indeed, Lambert and Zheng (2011, theorem 6) proved that consistency and transfer sensitivity are incompatible for bounded cardinal variables. Something similar occurs with ordinal variables, wherein the role of Pigou–Dalton or regressive 'income' transfers is replaced with different types of egalitarian or regressive frequency transfers such as MPS, clustering and Hammond transfers.[12] Adapting Lambert and Zheng (2011), let $\mathbf{p}=(0.1,0.2,0.3,0.2,0.2)$ and $\mathbf{q}=(0,0.4,0.3,0,0.3)$ where $\mathbf{q}$ is obtained from $\mathbf{p}$ through a (progressive) Hammond transfer involving 0.1 moving from category 1 to 2 coupled with 0.1 from

9 Following Kobus (2015), this principle states that, for every pair $\mathbf{p},\mathbf{q}\in\mathcal{O}^C$, $I(\mathbf{q})\geq I(\mathbf{p})$ whenever $\mathbf{q}$ is obtained from $\mathbf{p}$ through a finite sequence of MPS. To define the latter, let m be the median of $\mathbf{p}\in\mathcal{O}^C$, where $1\leq m\leq C$. Then $\mathbf{q}$ is obtained from $\mathbf{p}$ through an MPS if and only if $q_i-p_i=p_j-q_j=\delta>0$ for some pair $\{i,j\}$ such that $i<j\leq m$ or $m\leq j<i$.

10 Namely MPS and clustering transfers on one side of the median (henceforth 'CT'). The increased-clustering principle (Apouey, 2007; Chakravarty and Maharaj, 2015) states that, for every pair $\mathbf{p},\mathbf{q}\in\mathcal{O}^C$, $I(\mathbf{q})\geq I(\mathbf{p})$ whenever $\mathbf{q}$ is obtained from $\mathbf{p}$ through a finite sequence of CT. To define CT, let $1\leq m\leq C$ be the median of $\mathbf{p}$. Then $\mathbf{q}$ is obtained from $\mathbf{p}$ through a CT if and only if $q_{i+1}-p_{i+1}=p_i-q_i=q_{j-1}-p_{j-1}=p_j-q_j=\delta>0$ for some pair $\{i,j\}$ such that $i+1\leq j-1<m$ or $m<i+1\leq j-1$.

11 Cowell and Flachaire (2017) proposed measuring inequality with ordinal variables in terms of dispersion of personal status. They suggested operationalising personal status in four possible ways, including peer-inclusive downward-looking status in the form of the proportion of people in the same category or worse.

12 $\mathbf{q}\in\mathcal{O}^C$ is obtained from $\mathbf{p}\in\mathcal{O}^C$ through a Hammond transfer if and only if $p_i-q_i=q_j-p_j=q_k-q_k=p_l-q_l=\frac{1}{n}$ for some quadruplet $\{i,j,k,l\}$ such that $1\leq i<j\leq k<l\leq C$. Then, the associated Hammond principle (Gravel et al., 2021) states that, for every pair $\mathbf{p},\mathbf{q}\in\mathcal{O}^C$, $I(\mathbf{q})\leq I(\mathbf{p})$ whenever $\mathbf{q}$ is obtained from $\mathbf{p}$ through a finite sequence of Hammond transfers.

category 3 to 2, and a reversed (regressive) Hammond transfer involving 0.1 from category 4 to 3 coupled with 0.1 from category 4 to 5. This is an example of the Hammond-transfer equivalent of a favourable composite transfer (FACT) which combines a Pigou–Dalton transfer at the bottom of the distribution with an equally sized regressive transfer at the top, for cardinal variables (see Lambert and Zheng, 2011, p. 217.). If I is transfer-sensitive in the sense that it decreases in response to a so-called Hammond FACT, thus prioritising the progressive Hammond transfer at the bottom of the distribution over the regressive one at the top, then we would get $I(\mathbf{p}) > I(\mathbf{q})$. Now, consider $\mathbf{p}' = (0.2, 0.2, 0.3, 0.2, 0.1)$ and $\mathbf{q}' = (0.3, 0, 0.3, 0.4, 0)$. Clearly, $\mathbf{p}'$ is obtained from $\mathbf{q}'$ through a Hammond FACT. Hence, if I is transfer-sensitive in the aforementioned sense, we would get $I(\mathbf{p}') < I(\mathbf{q}')$. But then I would not be consistent. Hence consistent inequality indices for ordinal variables are not transfer-sensitive, as they do not always decrease in the presence of the ordinal-variable versions of FACTs.

4. Conclusions

Applied to inequality (or bipolarisation) assessments with ordinal variables, the consistency property stresses that it should not matter whether the categories are sorted in ascending or descending order, *if* we judge that both representations convey the same information regarding distributional dispersion. We proved the theoretical relevance of this property by showing that several inequality and bipolarisation indices violate it, whereas many others satisfy it. Any future new index of inequality or bipolarisation for ordinal variables can be subjected to the proposed consistency test ensuing from the characterisation of the class of consistent measures.

Remarkably, Proposition 3.1 states that complete inequality and bipolarisation orderings are consistent if and only if the underlying inequality indices satisfy a property which we call *reversal invariance*, whereby $I(\mathbf{p}) = I(\mathbf{p}')$. Note that reversal invariance is *not* identical to functional symmetry.[13] In fact most inequality indices for ordinal variables are not symmetric and cannot be so, because that property clashes with axioms reflecting sensitivity to dispersion-inducing transformations. For example, a symmetric index would violate the MPS principle because, ultimately, symmetry demands dismissing the order between the categories altogether. Consider $\mathbf{p} = (0.2, 0.1, 0.3, 0.4)$ and $\mathbf{q} = (0.1, 0.2, 0.4, 0.3)$. A symmetric index would rank both distributions equally, whereas an inequality or bipolarisation index satisfying the MPS principle deems $\mathbf{p}$ more unequal than $\mathbf{q}$.

We note that, in the status inequality measurement framework, the consistency property essentially demands inequality rankings to be insensitive between peer-inclusive downward-looking status (operationalised by the CDF) and its peer-inclusive upward-looking counterpart (operationalised by the survival function, which is identical to the reverse-ordered CDF). While we can convincingly explain why downward-looking and upward-looking status are conceptually different, this paper showed that the two approaches to measuring status with ordinal variables yield different distributional rankings too.

Meanwhile, although inequality indices for ordinal variables satisfying the Hammond principle have not yet been purposefully introduced in the literature, it is easy to show that some of the consistent indices in Table 1 do comply with the Hammond principle, e.g. those by Lazar and Silber (2013). Thus, practitioners interested in performing consistent inequality comparisons across distributions with *different* medians would do well to consider measures satisfying the Hammond principle. Otherwise, they may still be able to use measures respecting some of the other aforementioned principles (e.g. MPS) but with adjustments such as those proposed by Sarkar and Santra (2020).

Inequality and bipolarisation comparisons respecting the MPS principle apply to distributions with a common median. However, Sarkar and Santra (2020) showed that we can compare pairs of distributions with different medians in compliance with the MPS principle if, first, we transform the original distributions to obtain new pairs with a common median. This is generally accomplished with combinations of *additions of unpopulated categories at extremities*, which involve adding new categories with zero frequencies either next to the top or bottom categories, and *slides*, whereby the frequencies are slid sequentially toward adjacent categories (either to the right or the left of the original categories) as long as the destination categories are initially unpopulated. Then, we compare the transformed distributions with inequality indices that respect the MPS principle but need to be insensitive to both additions and slides. As we know from Table 1 above, several among these suitably insensitive indices, such as I^{AP} and those proposed by Lazar and Silber (2013) (see Sarkar and Santra, 2020, Table 1), are also consistent. Moreover, it is not difficult to prove that we can obtain $\mathbf{q}$ from $\mathbf{p}$ through a combination of additions and slides if and only if $\mathbf{q}'$ can be obtained from $\mathbf{p}'$ through a similar combination of additions and slides, but with the additions on the other extremes and the slides in the opposite direction.[14] Hence consistent inequality comparisons respecting the MPS principle are feasible even when the medians differ.

Finally, it is worth recalling that the desirability of consistency may not carry consensus, just like other consistency properties in distributional analysis. In the context of ordinal variables, a bone of contention may lie in whether imposing consistency in an inequality comparison entails a loss of information. Advocates of consistency may claim that the property respects the order of categories, but the latter admits two representations, namely ascending or descending, and the sorting direction should not matter when performing inequality comparisons. By contrast, arguably few people would dispute that applying inequality measures for nominal variables (such as the Simpson index) to ordinal variables incurs an unwarranted loss of information, as these ranking criteria are insensitive to any sorting of categories.

Data availability

No data was used for the research described in the article.

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[13] In our context, I is symmetric if and only if $I(\mathbf{p}) = I(\mathbf{q})$ for any $\mathbf{q} = \mathbf{pY}$ such that $\mathbf{Y}$ is a C-dimensional permutation matrix.

[14] For instance, let $\mathbf{p} = (0.2, 0.4, 0.3, 0.1)$ and $\mathbf{q} = (0.1, 0.2, 0.3, 0.4)$ with respective medians $m_p = 2$ and $m_q = 3$. Then, adding one unpopulated category to the left extreme of $\mathbf{p}$ and another one to the right extreme of $\mathbf{q}$ we obtain, respectively, $\mathbf{pp} = (0, 0.2, 0.4, 0.3, 0.1)$ and $\mathbf{qq} = (0.1, 0.2, 0.3, 0.4, 0)$, both with median $m = 3$ (and preserving common number of categories); hence comparable with indices respecting the MPS principle. Likewise, we can compare $\mathbf{p}'$ to $\mathbf{q}'$ by generating $\mathbf{pp}'$ and $\mathbf{qq}'$ with additions on the right and left extremes, respectively.

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