Mathematische Zeitschrift
https://doi.org/10.1007/s00209-022-03106-4

Mathematische Zeitschrift

Sheaf homology of hyperplane arrangements, Boolean covers and exterior powers

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Received: 16 March 2022 / Accepted: 28 July 2022

Abstract
We compute the sheaf homology of the intersection lattice of a hyperplane arrangement with coefficients in the graded exterior sheaf $\Lambda^\bullet F$ of the natural sheaf F. This builds on the results of our previous paper Everitt and Turner (Adv Math 402:Paper No. 108354, 2022. https://doi.org/10.1016/j.aim.2022.108354) where this homology was computed for $\Lambda^1 F = F$, itself a generalisation of an old result of Lusztig. The computational machinery we develop in this paper is quite different though: sheaf homology is lifted to what we call Boolean covers, where we instead compute homology cellularly. A number of tools are given for the cellular homology of these Boolean covers, including a deletion–restriction long exact sequence.

Introduction

The combinatorics of a hyperplane arrangement is encapsulated by its intersection lattice. The homology of this lattice, with constant coefficients, was first determined in [2, 7], with Quillen [11] showing that it has the homotopy type of a wedge of spheres. Interest in homology may be revived though by taking coefficients in a more interesting local system, that is to say, in a *sheaf* on the lattice. The resulting sheaf homology $H_*(L\backslash\mathbf{0};\, F)$, where L is the intersection lattice of a hyperplane arrangement and F is some interesting (naturally occuring) sheaf, then becomes worthy of investigation.

Intersection lattices of hyperplanes arrangements come equipped with a canonical sheaf as the elements of the lattice are vector spaces. We call this the natural sheaf, and in [4] we showed that the reduced sheaf homology is trivial in all degrees, except the top one, whose dimension is related to the β-invariant of the arrangement, i.e. the derivative of the character-

Dedicated to Mary, Emmanuelle and Adrien

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Published online: 23 August 2022

istic polynomial of L evaluated at 1—see [15]. This generalises, to an arbitrary arrangement, an old result of Lusztig [9] where he considers the arrangement of all hyperplanes in a vector space over a finite field. There are various other sheaves that can be put on an intersection lattice—see [10]—but they turn out to be what Yuzvinsky [14] calls local sheaves, and so the homology vanishes for general reasons. The natural sheaf is not local.

In this paper our principal object of interest is the sheaf homology of L with coefficients in the graded sheaf $\Lambda^\bullet F$, where F is the natural sheaf and $\Lambda^j F$ is the jth exterior power of F. We concentrate first on the case where the arrangement is *essential*, meaning that the intersection of all the hyperplanes is trivial. Our result here is:

Theorem 9 *Let L be the intersection lattice of an essential hyperplane arrangement in a space V. Let F be the natural sheaf on L and $\Lambda^j F$ be the jth exterior power of F. If $rk(L) \geq 2$ then $H_i(L\backslash\mathbf{0}; \Lambda^j F)$ is trivial unless:*

– *either* $0 < i < rk(L) - 1$ *and* $i + j = rk(L) - 1$, *in which case*

$$\dim H_i(L\backslash\mathbf{0}; \Lambda^j F) = \frac{(-1)^{i+1}}{j!}\chi_L^{(j)}(1)$$

– *or,* $i = 0$ *and or* $j = rk(L) - 1$, *in which case*

$$\dim H_0(L\backslash\mathbf{0}; \Lambda^j F) = \binom{rk(L)}{j} - \frac{1}{j!}\chi_L^{(j)}(1)$$

– *or,* $i = 0$ *and* $j < rk(L) - 1$, *in which case*

$$\dim H_0(L\backslash\mathbf{0}; \Lambda^j F) = \binom{rk(L)}{j}$$

where $\chi_L^{(j)}(t)$ *is the* j*th derivative of the characteristic polynomial of* L.

The case $j = 1$ reproduces the main result of [4], and the appearance there of the β-invariant of the arrangement is expanded to the appearance of higher derivatives of the characteristic polynomial that are related to the dimensions of the higher exterior powers. The graded Euler characteristic of this (bi-graded) homology is (see Corollary 5)

$$\chi_q H_*(L\backslash\mathbf{0}; \Lambda^\bullet F) = -\chi_L(1+q) + (1+q)^{\dim V}$$

The homology $H_*(L\backslash\mathbf{0}; \Lambda^\bullet F)$ can thus be interpreted as a categorification of the characteristic polynomial of the hyperplane arrangement, although we do not pursue this point of view. We extend the results above to non-essential arrangements in Theorem 11.

Our main computational tool is given by what we call *Boolean covers*. These are Boolean lattices that keep track of all the expressions of elements as joins of atoms. As lattices they are particularly amenable to having their homology computed *cellularly*—a philosophy that we adopted in [6]. We then make the connection betwen this cellular homology of Boolean covers and the sheaf homology of the lattices being covered.

This is a two step process. Writing $\widetilde{L}$ for the Boolean cover of L, a number of spectral sequence arguments establish:

Theorem 3 *Let L be a graded atomic lattice with sheaf F and let $f : \widetilde{L} \to L$ be its Boolean cover. Then*

$$H_*(L\backslash\mathbf{0}; F) \cong H_*(\widetilde{L}\backslash\mathbf{0}; F).$$

This result also appears in [9, §1.2]. The second step is:

Theorem 4 *If B is a Boolean lattice and F is a sheaf on B then*

$$H_*(B\backslash\mathbf{0};\, F) \cong H_*^{cell}(B\backslash\mathbf{0};\, F).$$

If L is an intersection lattice, then for a hyperplane a the deletion L_a and restriction L^a are lattices of "smaller" arrangements—see Sect. 1.1. The characteristic polynomial of L satisfies a deletion–restriction relation in terms of L_a and L^a, and our main technical tool is a lift of this to the setting of the cellular homology of Boolean covers.

Theorem 7 *Let L be a geometric lattice equipped with a sheaf F and let $f : \widetilde{L} \to L$ be its Boolean cover. Then for any atom $a \in L$ there is a long exact sequence*

$$\cdots \to H_i^{cell}(\widetilde{L^a};\, F) \to H_i^{cell}(\widetilde{L_a};\, F) \to H_i^{cell}(\widetilde{L};\, F) \to H_{i-1}^{cell}(\widetilde{L^a};\, F) \to H_{i-1}^{cell}(\widetilde{L_a};\, F) \to \cdots$$

This allows us to prove the analogue of Theorem 9 for the cellular homology of Boolean covers:

Theorem 8 *Let L be the intersection lattice of an essential hyperplane arrangement in a space V, let F be the natural sheaf on L and $\Lambda^j F$ be the jth exterior power of F. If $rk(L) \geq 2$ and $\widetilde{L} \to L$ is the Boolean cover of L, then $H_i^{cell}(\widetilde{L};\, \Lambda^j F)$ is trivial unless $0 \leq i < rk(L)$ and $i + j = rk(L) = \dim V$, in which case:*

$$\dim H_i^{cell}(\widetilde{L};\, \Lambda^j F) = \frac{(-1)^i}{j!}\chi_L^{(j)}(1)$$

where $\chi_L^{(j)}(t)$ is the jth derivative of the characteristic polynomial of L.

Indeed this is proved first, and Theorem 9 is a corollary. It is extended to non-essential sheaves in Theorem 10.

The theorems above, indeed all the results of this paper, hold for lattices in a range of generalities. The broadest class—for example in Theorem 3—are the graded atomic lattices. The proof of the long exact sequence in Theorem 7 requires the restriction L^a to also be graded atomic; to ensure this we restrict to the smaller class of geometric lattices. Specific computations of homology, such as Theorems 8 and 9, are done for the natural sheaf on the further restricted class of arrangement lattices. Finally, for our cellular calculations we restrict yet further to the Boolean lattices, although this is purely for conciseness and convenience—an analogous result to Theorem 4 holds for the class of cellular posets; see [6, Theorem 2].

Working with the Boolean cover takes us quite close to the perspective of Dansco and Licata [3]. Motivated by Khovanov homology-style constructions, they make a number of decorated hypercubes (some using exterior powers) which give rise to homologies which categorify the characteristic polynomial, among other things, of a hyperplane arrangement. Our cellular homology of the Boolean cover is very much of this type, but in fact the resulting decorated hypercube is not one they consider. They initiate some computations of the homology for their examples and it would be interesting to see further (or full) computations. The techniques we develop for Boolean covers may be of some use in this regard.

The structure of the paper is as follows. In Sect. 1 we discuss the basics of lattices, arrangements and sheaves. We recall the necessary background on hyperplane arrangements and their intersection lattices, sheaves on lattices, and characteristic polynomials. We also introduce Boolean covers. In Sect. 2 we move to homology, first discussing sheaf homology and its basic properties and calculating the Euler characteristic in the example of interest. We then discuss a Leray–Serre type spectral sequence needed to make the connection between a lattice and its Boolean cover. In Sect. 3 we introduce the cellular homology of a Boolean lattice with coefficients in a sheaf. We show that cellular homology computes sheaf homology and give a number of technical results about cellular homology, of which the most important is the deletion–restriction long exact sequence. Section 4 studies the main example of the homology of an arrangement lattice with coefficients in the exterior powers of the natural sheaf. After a brief discussion of graded Euler characteristics, we state and prove our main results first for essential arrangements and then in the non-essential case.

1 Lattices, arrangements and sheaves

This section summarises the basics of posets, lattices and sheaves. Section 1.1 presents basic poset notions and terminology along with the examples that preoccupy this paper: the intersection lattices of hyperplane arrangements. Section 1.2 gives basic sheaf notions and constructions and the principal examples: the natural sheaf of a hyperplane arrangement and its exterior powers. Section 1.3 recalls the characteristic polynomial and finally Sect. 1.4 introduces a key tool in the computation of sheaf homology: the Boolean cover of a graded atomic lattice.

1.1 Posets, lattices and arrangements

Let $P = (P, \leq)$ be a finite graded poset with rank function $rk : P \to \mathbb{Z}$ (see [13, Chapter 3] for this and other basic poset terminology in this section). A *minimum* is an element $\mathbf{0} \in P$ with $\mathbf{0} \leq x$ for all $x \in P$ and a *maximum* is an element $\mathbf{1} \in P$ with $x \leq \mathbf{1}$ for all $x \in P$. We assume $rk(\mathbf{0}) = 0$. The *atoms* of P are the elements of rank 1. A poset map $f : Q \to P$ is a set map such that $fx \leq fy \in P$ if $x \leq y \in Q$.

A subset $K \subset P$ is *upper convex* if $x \in K$ and $x \leq y$ implies $y \in K$. If $x \leq y$, the *interval* $[x, y]$ consists of those $z \in P$ such that $x \leq z \leq y$; if $x \in P$ the interval $P_{\geq x}$ consists of those $z \in P$ such that $z \geq x$; one defines $P_{\leq x}$, $P_{>x}$ and $P_{<x}$ similarly.

A lattice is a poset equipped with a join $\vee$ and a meet $\wedge$. A finite lattice has a minimum $\mathbf{0}$, equal to the meet of all its elements, and a maximum $\mathbf{1}$, equal to the join. A graded lattice is *atomic* if every element can be expressed (not necessarily uniquely) as a join of atoms, with the convention that the empty join is the minimum $\mathbf{0}$. The rank $rk(L)$ of a graded lattice L is $rk(L) := rk(\mathbf{1})$.

If A is a finite set then the *Boolean lattice* $B = B(A)$ consists of the subsets of A ordered by inclusion. The result is a graded atomic lattice with $rk(x) = |x|$, join $x \vee y = x \cup y$, meet $x \wedge y = x \cap y$, minimum $\mathbf{0} = \varnothing$, maximum $\mathbf{1} = A$ and atoms A. Any element has a unique expression as a join of atoms.

This paper is about arrangement lattices. If V is a finite dimensional vector space over a field k, then an *arrangement* in V is a finite set $A = \{a_i\}$ of linear hyperplanes, i.e. codimension one subspaces. The corresponding *arrangement lattice* $L = L(A)$ has elements all possible intersections of hyperplanes in A—with the empty intersection taken to be V—and ordered by *reverse* inclusion. Then L is a graded atomic lattice with atoms the hyperplanes A, rank function $rk(x) = \operatorname{codim} x$, minimum $\mathbf{0} = V$, maximum $\mathbf{1} = \bigcap_{a \in A} a$,

$$x \vee y = x \cap y, \text{ and } x \wedge y = \bigcap z$$

where the intersection on the right is indexed by the set $\{z \in L : x \cup y \subseteq z\}$. Moreover, L is *geometric*, in that the rank function satisfies $rk(x \vee y) + rk(x \wedge y) \leq rk(x) + rk(y)$. An arrangement is *essential* when $\bigcap_{a \in A} a$ is the trivial subspace, or equivalently, $rk(L) = \dim V$. The arrangement lattices on at most three hyperplanes are shown in Fig. 1. The first three are Boolean—realised by arrangements of coordinate hyperplanes with respect to a basis in 1, 2 or 3-dimensions—and the last is a braid arrangement (see for instance [12]) combinatorially isomorphic to the partition lattice $\Pi(3)$ of a set of size 3.

If $a \in A$ is a hyperplane of an arrangement in V, then the *deletion* arrangement in V has hyperplanes $A \backslash \{a\}$. Its intersection lattice L_a consists of the elements of L that can be expressed as a join of the atoms $A \backslash \{a\}$. The *restriction* arrangement in a has hyperplanes the subspaces $a \cap b$ for $b \in A \backslash \{a\}$. Its intersection lattice L^a is the interval $L_{\geq a} = \{x \in L : x \geq a\}$.

In any graded atomic lattice, a set $S \subset A$ of atoms is *independent* if $\bigvee T < \bigvee S$ for all proper subsets T of S, and *dependent* otherwise. An atom a in a dependent set of atoms S with the property that $\bigvee S \backslash \{a\} = \bigvee S$ is called a *dependent atom*. A schematic of L, L_a and L^a, when a is dependent, is shown in Fig. 2. It is well known (see for instance [1, 5]) that the only

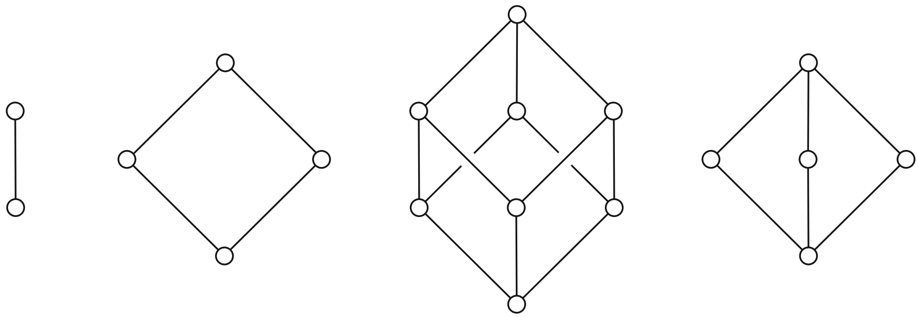


Fig. 1 The arrangement lattices $L(A)$ where $|A| \leq 3$

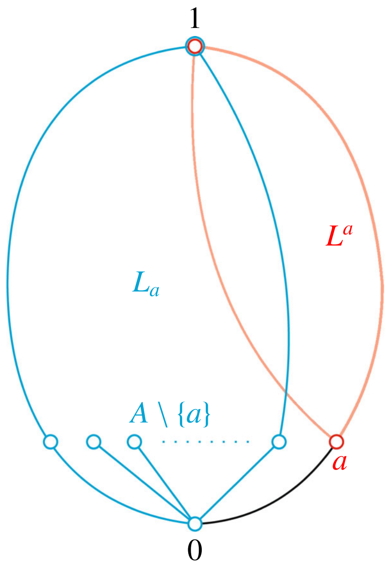


Fig. 2 The decomposition of L into the deletion L_a and restriction L^a for a dependent atom a

graded atomic lattices without dependent atoms are the Booleans. Moreover, in a geometric lattice L we have $rk(\bigvee S) \leq |S|$, and S is independent if and only if $rk(\bigvee S) = |S|$.

1.2 Sheaves on lattices

A *sheaf* on a poset P is a *contra*variant functor $F : P \to {}_R\mathbf{Mod}$ to the category of R-modules, where R is a commutative ring with 1, and P is interpreted as a category in the usual way (having a unique morphism $x \to y$ whenever $x \leq y$). A *morphism* of sheaves is a natural transformation of functors $\kappa : F \to G$ and an isomorphism is a natural isomorphism. We write F_x^y for the *structure map* of the sheaf given by $F(x \leq y) : F(y) \to F(x)$.

For example, if $M \in {}_R\mathbf{Mod}$ is fixed, then the *constant* sheaf ΔM has $\Delta M(x) = M$ for every $x \in P$ and $(\Delta M)_x^y = id : M \to M$ for every $x \leq y$ in P.

Many sheaf constructions can be done locally, or "pointwise". For example, the direct sum $F \oplus G$ of sheaves F and G has $(F \oplus G)(x) = F(x) \oplus G(x)$ and structure maps $F_x^y \oplus G_x^y$ when $x \leq y$. The tensor product $F \otimes G$ can be formed in an analogous way. An ($\mathbb{N}$-) *graded* sheaf $F^\bullet$ is a direct sum $\bigoplus_{i \geq 0} F_i$ of sheaves F_i.

If $Z : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ is a functor then we write ZF for the sheaf arising from the composite $Z \circ F : P \to {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$. For example, if F is a sheaf and $j \geq 0$, we have the exterior powers $\Lambda^j F$ of F, and hence the graded sheaf:

$$\Lambda^\bullet F = \bigoplus_{j \geq 0} \Lambda^j F.$$

It is easy to check that $\Lambda^j \Delta M = \Delta \Lambda^j M$, and that the standard module result:

$$\Lambda^j(F \oplus G) \cong \bigoplus_{s+t=j} \Lambda^t F \otimes \Lambda^s G$$

carries straight through to sheaves of modules.

1.3 The characteristic polynomial

Recall that if k is a field and L is a lattice then the *Möbius function* $\mu = \mu_L$ of L is the k-valued function on the intervals $[x, y]$ defined recursively by $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$, for all $x < y$ in L and $\mu(x, x) = 1$. If L is an arrangement lattice then the *characteristic polynomial* $\chi_L(t)$ is defined by $\chi_L(t) = \sum_{x \in L} \mu_L(\mathbf{0}, x) t^{\dim(x)}$. The kth derivative of χ_L is denoted $\chi_L^{(k)}$; the value $(-1)^{rk(L)-1}\chi^{(1)}(1)$ of the derivative at 1 is called the β-invariant of the arrangement [15, 7.3].

We generalise to when there is a sheaf F on L. The *characteristic polynomial of the pair* (L, F), denoted $\chi_{(L,F)}(t)$, is defined by

$$\chi_{(L,F)}(t) = \sum_{x \in L} \mu_L(\mathbf{0}, x) t^{\dim F(x)}.$$

If F is the natural sheaf on L then $\chi_{(L,F)}(t) = \chi_L(t)$.

1.4 Boolean covers

Let L be a graded atomic lattice with atoms A and let $B = B(A)$ be the Boolean lattice on A. There is a canonical lattice map $f : B \to L$ given by

$$f : \bigvee_B a_i \mapsto \bigvee_L a_i$$

and we refer to the pair (B, f) as the *Boolean cover* of L. We usually write $\widetilde{L}$, instead of B, for the Boolean cover of L. If F is a sheaf on L, then there is an induced sheaf $\widetilde{F}$ on the Boolean cover defined at $x \in \widetilde{L}$ by $\widetilde{F}(x) = F(fx)$ and with structure maps $\widetilde{F}_x^y = F_{fx}^{fy} : F(fy) \to F(fx)$. To simplify the notation we will drop the tilde, writing[1] F for $\widetilde{F}$.

For a Boolean lattice B we have $\mu_B(\mathbf{0}, x) = (-1)^{rk(x)}$; see [13, Example 3.8.3]. Thus, the characteristic polynomial for (B, F) is given by

$$\chi_{(B,F)}(t) = \sum_{x \in B} (-1)^{rk(x)} t^{\dim F(x)}$$

Proposition 1 *If $\widetilde{L}$ is the Boolean cover of L then $\chi_{(\widetilde{L},F)}(t) = \chi_{(L,F)}(t)$.*

Proof Unpacking [10, Lemma 2.35] gives $\mu_L(\mathbf{0}, x) = \sum_{y \in f^{-1}(x)} (-1)^{rk(y)}$. Hence

$$\begin{aligned}\chi_{(\widetilde{L},F)}(t) &= \sum_{y \in \widetilde{L}} (-1)^{rk(y)} t^{\dim F(y)} \\ &= \sum_{x \in L} \sum_{y \in f^{-1}(x)} (-1)^{rk(y)} t^{\dim F(y)} = \sum_{x \in L} \mu_L(\mathbf{0}, x) t^{\dim F(x)} = \chi_{(L,F)}(t)\end{aligned}$$

□

2 Homology

In Sect. 2.1 we recall the basics of the homology of posets with coefficients in a sheaf and in Sect. 2.2 we discuss the (graded) Euler characteristic of the resulting homology. Section 2.3 gives some spectral sequences that will prove useful in the next section where we compare (sheaf) homology with the cellular homology defined in Sect. 3.

2.1 Sheaf homology

For a fixed poset P let $\varinjlim^P$ be the colimit functor from sheaves on P to ${}_R\mathbf{Mod}$, and let

$$\varinjlim\nolimits_*^P := L_* \varinjlim^P$$

be the left derived functors, or *higher colimits*. The *homology $H_*(P; F)$ of P with coefficients in the sheaf F* are these higher colimits evaluated at the sheaf F. The homology can be computed using a chain complex $S_*(P; F)$ whose group of n-chains is

$$S_n(P; F) = \bigoplus_{\sigma} F(x_0)$$

[1] In [6] we wrote $f^* F$ for this induced sheaf.

where the direct sum is over the totally ordered chains $\sigma = x_n \leq \cdots \leq x_0$ in P. For such a chain σ and $s \in F(x_0)$ we write s_σ for the element of S_n that has value s in the component indexed by σ and value 0 in all other components. The differential $d : S_n(P; F) \to S_{n-1}(P; F)$ is then given by

$$ds_\sigma = F_{x_1}^{x_0}(s)_{d_0\sigma} + \sum_{i=1}^{n}(-1)^i s_{d_i\sigma} \tag{1}$$

where as usual $d_i\sigma = x_n \leq \cdots \leq \widehat{x_i} \leq \cdots \leq x_0$ for $0 \leq i \leq n$.

We have (see [8, Appendix II])

$$H_*(P; F) = \varinjlim{}_*^P F \cong HS_*(P; F).$$

The following are some well-known properties of homology.

Lemma 1 *1. If ΔM is a constant sheaf then $H_*(P; \Delta M) \cong H_*(|P|, M)$, the ordinary simplicial homology of the order complex $|P|$, which is the geometrical realisation of the simplicial complex whose vertices are elements of P and n-simplicies are chains $\sigma = x_n \leq \cdots \leq x_0$.*

2. If P has a minimum or maximum, and ΔM is a constant sheaf, then $H_0(P; \Delta M) = M$ and $H_i(P; \Delta M)$ vanishes for $i > 0$.

3. If P has a minimum $\mathbf{0}$, and F is any sheaf, then $H_0(P; F) = F(\mathbf{0})$ and $H_i(P; F)$ vanishes for $i > 0$.

Let $T_*(P; F)$ be the chain complex whose n-chains are $T_n(P; F) = \bigoplus_\sigma F(x_0)$, the sum is over the non-degenerate chains $\sigma = x_n < \cdots < x_0$, and with differential given by the formula (1). Then $T^*(P; F)$ is a sub-complex of S_* homotopy equivalent to it (see for example [4, 2.1]). We will interchange between the S_* and T_* complexes as convenience dictates.

If $F^\bullet$ is a graded sheaf then $H_*(P; F^\bullet) = \bigoplus_j H_*(P; F^j)$ has the structure of a bi-graded vector space.

2.2 Euler characteristics

As usual, the *Euler characteristic* of homology is defined to be

$$\chi H_*(P; F) = \sum_n (-1)^n \dim H_n(P; F).$$

If $V_\bullet$ is an graded vector space then its *graded dimension* is $\dim_q V_\bullet = \sum_k \dim V_k\, q^k$, and if $F^\bullet$ is a graded sheaf then the *graded Euler characteristic* of the homology $H_*(P; F^\bullet)$ is given by

$$\begin{aligned}\chi_q H_*(P; F^\bullet) &= \sum_n (-1)^n \dim_q H_n(P; F^\bullet) = \sum_{n,k} (-1)^n \dim H_n(P; F^k) q^k \\ &= \sum_k \chi H_*(P; F^k) q^k.\end{aligned}$$

Proposition 2 *The Euler characteristic* $\chi H_*(L\backslash\mathbf{0}; F) = -\sum_{x\in L\backslash\mathbf{0}} \mu_L(\mathbf{0},x)\dim F(x)$

Proof Let $x \in L$ and define $ch_n(x)$ to be the set of (strict) n-chains in $L\backslash\mathbf{0}$ of the form $x_n < \cdots < x_1 < x_0$, where $x_0 = x$. If σ is such a chain we write $\ell(\sigma) = n$. Then by [13, Proposition 3.8.5] we have

$$\mu_L(\mathbf{0},x) = \sum_{\sigma\in ch_*(x)} (-1)^{\ell(\sigma)-1}.$$

Recall that the Euler characteristic is the same as the alternating sum of the dimensions of the chain groups in a complex computing the homology, so

$$\chi H_*(L\backslash\mathbf{0}; F) = \chi T_*(L\backslash\mathbf{0}; F) = \sum_{n\geq 0}(-1)^n \dim T_n(L\backslash\mathbf{0}; F).$$

The dimension of $T_n(L\backslash\mathbf{0}; F)$ can calculated as

$$\dim T_n(L\backslash\mathbf{0}; F) = \sum_{x_n<\cdots<x_1<x} \dim F(x) = \sum_{x\in L\backslash\mathbf{0}} |ch_n(x)| \dim F(x)$$

giving

$$\chi H_*(L\backslash\mathbf{0}; F) = \sum_{n\geq 0}\sum_{x\in L\backslash\mathbf{0}} (-1)^n |ch_n(x)| \dim F(x) = \sum_{x\in L\backslash\mathbf{0}}\sum_{n\geq 0}(-1)^n |ch_n(x)| \dim F(x).$$

The value of the Möbius function $\mu_L(\mathbf{0},x)$ may be expressed as

$$\mu_L(\mathbf{0},x) = -\sum_n (-1)^n |ch_n(x)|$$

(see, for example, [13, Proposition 3.8.5]), from which we get

$$\chi H_*(L\backslash\mathbf{0}; F) = \sum_{x\in L\backslash\mathbf{0}}\sum_{n\geq 0}(-1)^n |ch_n(x)| \dim F(x) = -\sum_{x\in L\backslash\mathbf{0}} \mu_L(\mathbf{0},x)\dim F(x).$$

□

Corollary 1 *Writing* $\chi'_{(L,F)}(t)$ *for the derivative of the characteristic polynomial* $\chi_{(L,F)}(t)$*, we have*

$$\chi H_*(L\backslash\mathbf{0}; F) = \dim F(\mathbf{0}) - \chi'_{(L,F)}(1).$$

Proof From the definition of the characteristic polynomial we have

$$\chi'_{(L,F)}(t) = \sum_{x\in L} \mu_L(\mathbf{0},x)\dim F(x) t^{\dim F(x)-1}$$

so that

$$\chi'_{(L,F)}(1) = \sum_{x\in L} \mu_L(\mathbf{0},x)\dim F(x).$$

This then gives

$$\chi'_{(L,F)}(1) = \mu_L(\mathbf{0},\mathbf{0})\dim F(\mathbf{0}) + \sum_{x\in L\backslash\mathbf{0}} \mu_L(\mathbf{0},x)\dim F(x) = \dim F(\mathbf{0}) - \chi H_*(L\backslash\mathbf{0}; F).$$

□

2.3 Some spectral sequences

There is a Leray–Serre style spectral sequence associated to a poset map. The following is an adaptation of [8, Appendix II, Theorem 3.6]—see also [4, §2.3].

Let $f: P \to Q$ be a poset map and let F be a sheaf on P. For each $q \geq 0$ define a sheaf H_q^{fib} on Q by

$$H_q^{\text{fib}}(x) = H_q(f^{-1}Q_{\geq x}; F)$$

for $x \in Q$. If $x \leq y$ in Q then the structure map $H_q^{\text{fib}}(y) \to H_q^{\text{fib}}(x)$ is induced by the inclusion $Q_{\geq y} \hookrightarrow Q_{\geq x}$.

Theorem 1 (Leray–Serre) *There is a spectral sequence*

$$E^2_{p,q} = H_p(Q; H_q^{\mathit{fib}}) \Rightarrow H_{p+q}(P; F)$$

We are interested in a special case of this spectral sequence which we now describe. Let P be a poset equipped with sheaf F and let

$$P = \bigcup_{\alpha \in K} P_\alpha$$

be a covering of P by upper convex subposets. We define a poset N, the *nerve* of the covering, that mimics the simplicial complex nerve of a covering of a space. If X is a non-empty subset of the indexing set K, let

$$P_X = \bigcap_{\alpha \in X} P_\alpha \tag{2}$$

Then N is the sub-poset of the Boolean lattice $B(K)$ consisting of those X for which $P_X \neq \varnothing$.

For each $q \geq 0$, define a sheaf $\mathcal{H}_q$ on N by

$$\mathcal{H}_q(X) = H_q(P_X; F)$$

and with structure map $\mathcal{H}_q(X \subset Y) : H_q(P_Y; F) \to H_q(P_X; F)$ induced by the inclusion $P_Y \hookrightarrow P_X$.

Theorem 2 *Given the set-up of the previous paragraph, there is a spectral sequence*

$$E^2_{p,q} = H_p(N; \mathcal{H}_q) \Rightarrow H_{p+q}(P; F).$$

Proof Define a map $f : P \to N$ by $f(x) = \{\alpha \in K : x \in P_\alpha\}$. Let $x \leq y$ in P and suppose that $\alpha \in f(x)$, hence $x \in P_\alpha$. As P_α is upper convex we have $y \in P_\alpha$ too, hence $\alpha \in f(y)$. Thus $f(x) \subseteq f(y)$ in N, and f is a poset map.

We now claim that for $X \in N$, the fiber $f^{-1}N_{\geq X}$ is the subposet P_X in (2). It then follows that the fiber sheaves H_*^{fib} of f are the $\mathcal{H}_*$ above, and hence the result after applying Theorem 1. To see the claim, we have $x \in P_X$ if and only if $x \in P_\alpha$ for all $\alpha \in X$; this in turn happens if and only if $X \subseteq f(x)$, or equivalently, $x \in f^{-1}N_{\geq X}$. □

Lusztig [9, §1.2] gives a simplicial complex version of this result which he describes as “well known”, although the reader might struggle to find a reference.

2.4 Passing to the Boolean cover

The spectral sequence of a covering from the previous section (Theorem 2) allows us to pass from a lattice L to its Boolean cover $\widetilde{L}$ when computing homology. The following result can be found in [9, §1.2].

Theorem 3 *Let L be a graded atomic lattice with sheaf F and let $f : \widetilde{L} \to L$ be its Boolean cover. Then*

$$H_*(L\backslash\mathbf{0}; F) \cong H_*(\widetilde{L}\backslash\mathbf{0}; F).$$

Proof We cover $L\backslash\mathbf{0}$ and apply the spectral sequence of Theorem 2. If A is the set of atoms of L, then $L\backslash\mathbf{0} = \bigcup_{a\in A} L_{\geq a}$, is a covering by upper convex sets. If $X \subseteq A$ then

$$L_X = \bigcap_{a\in X} L_{\geq a} = L_b \text{ where } b = \bigvee_{a\in X} a$$

as $L_{\geq x} \cap L_{\geq y} = L_{\geq x\vee y}$. Thus $L_X \neq \varnothing$ for all non-empty $X \subseteq A$, and the nerve poset N is just the Boolean lattice minus its minimum, i.e. $N = B(A)\backslash\mathbf{0} = \widetilde{L}\backslash\mathbf{0}$. The sheaf $\mathcal{H}_q$ is given by

$$\mathcal{H}_q(X) = H_q(L_X; F) = H_q(L_{\geq X}; F) = \begin{cases} F(X), & q = 0, \\ 0, & q > 0 \end{cases}$$

for $\varnothing \neq X \subseteq A$. Thus $\mathcal{H}_q$ is the trivial sheaf when $q > 0$ and $\mathcal{H}_0 = F$. The E^2-page of the sequence of Theorem 2 is thus zero except for the $q = 0$ line, where $E^2_{p,0} = H_p(\widetilde{L}\backslash\mathbf{0}; F)$. This gives the desired isomorphism $H_*(L\backslash\mathbf{0}; F) \cong H_*(\widetilde{L}\backslash\mathbf{0}; F)$. □

3 Cellular homology

The ordinary singular homology of a space can be computed cellularly. In [6] we define a cellular cohomology that computes, for a large class of posets, the cohomology of a poset with coefficients in a sheaf. In this section we recall the basics we need (for homology rather than cohomology), restricting ourselves to the setting of Boolean lattices, and then reprove a theorem of Lusztig relating the homology of a lattice equipped with a sheaf to the cellular homology of the Boolean cover. Sections 3.3 and 3.4 contain technical results that give a useful splitting theorem in Sect. 3.5. This leads to the first main theorem of the paper, the deletion–restriction long exact sequence for cellular homology—Theorem 7 of Sect. 3.6.

3.1 Basics

Let $B = B(A)$ be the Boolean lattice on the finite set A and let F be a sheaf on B. Pick an ordering on A and write $A = \{a_1, a_2, \dots, a_n\}$. An element $x \in B$ is a subset of A, say $x = \{a_{i_1}, \dots, a_{i_k}\}$, which we write as $x = a_{i_1} \dots a_{i_k}$ assuming that $i_m < i_n$ for $m < n$. If

$$y = a_{i_1} \dots \widehat{a_{i_j}} \dots a_{i_k}$$

for some j then define $\varepsilon^x_y := (-1)^{j-1}$.

The *cellular chain complex* $C_*(B; F)$ has k-chains

$$C_k(B; F) := \bigoplus_{rk(x)=k} F(x)$$

where the direct sum is over the subsets x of size k. The differential $d : C_k \to C_{k-1}$ is given by $d = \sum_{y<x} d_y^x$, where the sum is over the pairs $y < x$ with y of size $k-1$ and x of size k, and with $d_y^x = \varepsilon_y^x F_y^x$.

If z is a subset of size $k-2$ with $z < x$ and y_1, y_2 are the two subsets of size $k-1$ with $z < y_1, y_2 < x$, then

$$\varepsilon_z^{y_1}\varepsilon_{y_1}^x + \varepsilon_z^{y_2}\varepsilon_{y_2}^x = 0. \tag{3}$$

It follows that $d^2 = 0$ and $C_*(B; F)$ is a chain complex. Call $H_*^{\text{cell}}(B; F)$ the *cellular homology* of B with coefficients in F. Up to isomorphism this construction is independent of the order chosen on A and of the sign assignment used (any collection of ε_y^x satisfying (3) will do).

3.2 Sheaf = Cellular

For a Boolean lattice we now have two kinds of homology—sheaf and cellular—and we now show these are isomorphic. The proof of the following is adapted from [6, Theorem 2].

Theorem 4 *If B is a Boolean lattice and F is a sheaf on B then*

$$H_*(B\backslash\mathbf{0}; F) \cong H_*^{cell}(B\backslash\mathbf{0}; F).$$

The proof filters the complex $S_*(L; F)$ so that the standard spectral sequence has E^1-page with single non-zero row the cellular chain complex $C_*(L; F)$.

Proof Write $L = B\backslash\mathbf{0}$ and filter the complex $S_*(L; F)$ by defining $F_pS_* = S_*(L_p; F)$, where $L_p = \{x \in L : rk_L(x) \leq p\}$, the elements whose rank *in* L is at most p. The E^0-page of the standard spectral sequence of a filtration is then

$$E^0_{pq} = \frac{S_{p+q}(L_p; F)}{S_{p+q}(L_{p-1}; F)}$$

a quotient complex that we denote by $S_*(L_p, L_{p-1}; F)$.

The E^1-page is $E^1_{pq} = H_{p+q}(L_p, L_{p-1}; F)$. Analysing this homology a little further, the arguments of [6, §2] can be adapted to show

$$H_*(L_p, L_{p-1}; F) \cong \bigoplus_{rk_L(x)=p} H_*(L_{\leq x}, L_{<x}; \Delta F(x)) \cong \bigoplus_{rk_L(x)=p} \widetilde{H}_{*-1}(L_{<x}; \Delta F(x)).$$

Here, as we have the constant sheaf $\Delta F(x)$, the homology $\widetilde{H}_*(L_{<x}; \Delta F(x))$ is just the ordinary reduced singular homology of the order complex of $L_{<x}$.

The poset $L_{<x}$ is isomorphic to a Boolean lattice of rank $rk_L(x)$ minus its minimum and maximum. This, in turn, may be identified with the poset of sub-simplices of the boundary of a standard $rk_L(x)$-simplex. Thus, the order complex $|L_{<x}|$ is a $(rk(x)-1)$-sphere and

$$\widetilde{H}_{i-1}(L_{<x}; \Delta F(x)) \cong \begin{cases} F(x), & i = rk(x) \\ 0, & \text{else.} \end{cases}$$

It follows that $H_i(L_p, L_{p-1}; F) = 0$ when $i \neq p$, so the E^1-page is trivial except along the $q = 0$ line, where

$$E^1_{p,0} = H_p(L_p, L_{p-1}; F) = \bigoplus_{rk_L(x)=p} F(x) = C_p(L; F)$$

is the module of cellular p-chains. The differential $H_{p-1}(L_{p-1}, L_{p-2}; F) \leftarrow H_p(L_p, L_{p-1}; F)$ coincides with the cellular differential $C_{p-1} \leftarrow C_p$, and thus $H_*(L; F) \cong H_*^{\text{cell}}(L; F)$. ◻

As a corollary to Theorems 3 and 4 we obtain a result of Lusztig [9, Chapter 1], who proves that the homology of a lattice with coefficients in a sheaf is isomorphic to the *cellular* homology of the Boolean cover equipped with the induced sheaf:

Corollary 2 *(Lusztig) Let L be a graded atomic lattice with sheaf F and let $\widetilde{L} \to L$ be its Boolean cover. Then,*

$$H_*(L\backslash\mathbf{0}; F) \cong H_*^{cell}(\widetilde{L}\backslash\mathbf{0}; F),$$

3.3 Short exact sequences for cellular homology

There are two short exact sequences of cellular chain complexes that will prove useful.

The sequence induced by a sub-Boolean. Let $B = B(A)$ be the Boolean lattice on the set A and let $x \in B$. As x is a subset of A we can consider the Boolean $B(x)$—consisting of the subsets of x ordered by inclusion—and this is naturally a sub-poset of B with minimum $\mathbf{0}$ and maximum x. If $x = A\backslash\{a\}$ then $B(x)$ is just the deletion B_a; if $x = \varnothing$ then $B(x) = \mathbf{0}$.

If F is a sheaf on B, then (up to choice of signage in constructing the differential) the cellular complex $C_*(B(x); F)$ is a subcomplex of $C_*(B; F)$. Moreover, the quotient complex can be easily described: it is a "cellular like" complex of $B\backslash B(x)$. Specifically, let

$$C_k = \bigoplus_y F(y)$$

the direct sum over the subsets y of size $k+1$ such that $y \not\leq x$. Define $d : C_k \to C_{k-1}$ as before: $d = \sum_{w<y} d_w^y$, where y has one more element than w, but where now both $w, y \not\leq x$. If $z \not\leq x$ has size $k-1$ and $z \leq y$, then the y_1, y_2 of size k with $z < y_1, y_2 < y$ are also such that $y_1, y_2 \not\leq x$. It follows from (3) that $d^2 = 0$. Write $C_*(B\backslash B(x); F)$ for the resulting complex.

There is then a short exact sequence of cellular complexes

$$0 \to C_*(B(x); F) \to C_*(B; F) \to C_{*-1}(B\backslash B(x); F) \to 0 \tag{4}$$

If $x = A\backslash\{a\}$ then $B\backslash B(x)$ is the restriction B^a, which is again a Boolean lattice.

The sequence induced by a short exact sequence of sheaves. Let F and G be sheaves on the Boolean $B = B(A)$ and $\kappa = \{\kappa_y\} : G \to F$ a map of sheaves. Then there is an induced map $\kappa_* : C_*(B; G) \to C_*(B; F)$ defined by

$$\kappa_* : s_y \mapsto \kappa_y(s_y)$$

where $s_y \in C_k(B; G)$ has value $s \in F(y)$ in the coordinate indexed by the k-subset y, and value 0 elsewhere. Then κ_* is a chain map and moreover, the cellular chain complex $C_*(B; –)$ is an exact functor from the category of sheaves on B to the category of chain complexes. Thus, a short exact sequence of sheaves

$$0 \to G \to F \to H \to 0$$

induces a short exact sequence of cellular complexes

$$0 \to C_*(B; G) \to C_*(B; F) \to C_*(B; H) \to 0. \tag{5}$$

Corollary 3 *Let F and G be sheaves on the Boolean lattice B. Then,*

$$H_*^{cell}(B; F \oplus G) \cong H_*^{cell}(B; F) \oplus H_*^{cell}(B; G)$$

3.4 Fiddling with 0

We saw in Lemma 1 that a minimum needs to be removed for sheaf homology to be meaningful. Corollary 2 above transfers this requirement to the Boolean cover. Nevertheless, it will turn out to be more convenient to leave the miniumum *in* when performing calculations with Boolean covers. This section marries the two points of view.

Proposition 3 *Let B be a Boolean lattice and let F be a sheaf on B. Then $H_i^{cell}(B\backslash\mathbf{0}; F) \cong H_{i+1}^{cell}(B; F)$ for $i > 0$, and in low degrees there is an exact sequence*

$$0 \to H_1^{cell}(B; F) \to H_0^{cell}(B\backslash\mathbf{0}; F) \to F(\mathbf{0}) \to H_0^{cell}(B; F) \to 0$$

Proof If we take $x = \mathbf{0}$ in the sequence induced by a sub-Boolean in Sect. 3.3 we get a short exact sequence

$$0 \to C_*(\mathbf{0}; F) \to C_*(B; F) \to C_{*-1}(B\backslash\mathbf{0}; F) \to 0.$$

The result follows immediately from the associated long exact sequence. □

Putting this together with Corollary 2, we get the sheaf homology $H_*(L\backslash\mathbf{0}; F)$ in terms of the cellular homology $H_*^{\text{cell}}(\widetilde{L}; F)$:

Proposition 4 *If L is a graded, atomic lattice then $H_i(L\backslash\mathbf{0}; F) \cong H_{i+1}^{cell}(\widetilde{L}; F)$ for $i > 0$, and*

$$\dim H_0(L\backslash\mathbf{0}; F) = \dim H_1^{cell}(\widetilde{L}; F) - \dim H_0^{cell}(\widetilde{L}; F) + \dim F(\mathbf{0}).$$

Proof For $i > 0$ apply Corollary 2 and Proposition 3. For degree zero, consider the low degree short exact sequence of Proposition 3:

$$0 \to H_1^{\text{cell}}(\widetilde{L}; F) \to H_0^{\text{cell}}(\widetilde{L}\backslash\mathbf{0}; F) \to F(\mathbf{0}) \to H_0^{\text{cell}}(\widetilde{L}; F) \to 0$$

Now use Corollary 2 to replace $H_0^{\text{cell}}(\widetilde{L}\backslash\mathbf{0}; F)$ by $H_0(L\backslash\mathbf{0}; F)$ and recall that for an exact sequence the alternating sum of dimensions is zero. □

Corollary 4 *Let $\chi'_{(L,F)}(t)$ be the derivative of the characteristic polynomial of the pair (L, F). Then the Euler characteristic of the cellular homology of the Boolean cover is*

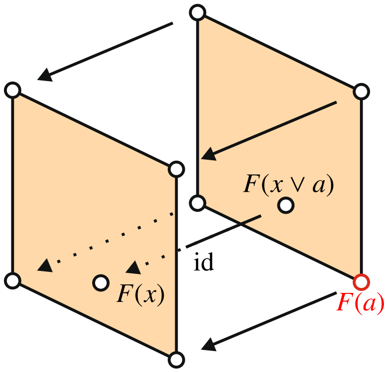
$$\chi H_*^{cell}(\widetilde{L}; F) = \chi'_{(L,F)}(1) = \sum_{x \in L} \mu_L(\mathbf{0}, x) \dim F(x).$$

Proof From the above and Corollary 1 we have

$$\chi H_*^{\text{cell}}(\widetilde{L}; F) = \dim F(\mathbf{0}) - \chi H_*(L\backslash\mathbf{0}; F) = \chi'_{(L,F)}(1)$$

□

Fig. 3 A double



3.5 Splitting Booleans

An atom a splits a Boolean B into the deletion B_a and the resriction B^a, both of which are themselves Booleans of rank $rk(B) - 1$. Proposition 5 and Theorem 5 below describe two situations where such a splitting can give useful information about the homology of B itself.

Doubling: Let F be a sheaf on B for which there is an atom $a \in A$ such that for all $x \in B_a$ the structure map

$$F_x^{x \vee a} : F(x \vee a) \to F(x) \tag{6}$$

is the identity. The restrictions of F to B_a and B^a are consequently exactly the same sheaf and so we call $(B; F)$ a *double* (see Fig. 3).

Proposition 5 *Let (B, F) be a double. Then $C_*(B; F)$ is acyclic, i.e. $H_i^{cell}(B; F) = 0$ for all i.*

Proof Taking $x = A \backslash \{a\}$ in (4), gives the short exact sequence

$$0 \to C_*(B_a; F) \to C_*(B; F) \to C_{*-1}(B^a; F) \to 0$$

from which there results a long exact sequence

$$\cdots \to H_i^{\text{cell}}(B^a; F) \xrightarrow{\delta} H_i^{\text{cell}}(B_a; F)$$
$$\to H_i^{\text{cell}}(B; F) \to H_{i-1}^{\text{cell}}(B^a; F) \xrightarrow{\delta} H_{i-1}^{\text{cell}}(B_a; F) \to \cdots$$

Recall that the signs in the definition of the differential of cellular homology required a choice of ordering on A. By reordering if necessary we may place a in first position. It follows that the signs $\varepsilon_x^{x \vee a}$ are equal to 1 for all $x \in B_a$ and consequently the connecting homomorphism δ is the map in homology induced by the identity map $\text{id} : C_*(B^a; F) \to C_*(B_a; F)$. Thus, δ is an isomorphism and the result follows. □

Decomposing: A small generalisation of the doubling idea gives a very useful recursive procedure for computing cellular homology. We will use it for example in Sect. 4.2 in the computation of $H_*(L \backslash \mathbf{0}; \Lambda^\bullet F)$. Let F be a sheaf for which there is an atom $a \in A$ such that for all $x \in B_a$ the structure map

$$F_x^{x \vee a} : F(x \vee a) \to F(x) \tag{7}$$

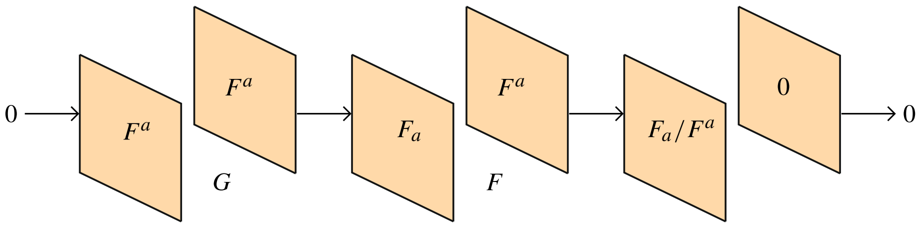
is injective. We will call such a sheaf *decomposable*.

Let F^a denote the restriction of F to B^a and F_a the restriction to B_a. Since $B_a = B^a$ we may also equip B_a with the sheaf F^a: for $x \in B_a$ set $F^a(x) = F(x \vee a)$ and for $x \leq y$ define $F^a(x \leq y) = F_{x\vee a}^{y\vee a}$. The maps $F^a(x) = F(x \vee a) \to F(x) = F_a(x)$ define a morphism $F^a \to F_a$ of sheaves on B_a, which is injective by condition (7), and we will denote the quotient sheaf by F_a/F^a.

Theorem 5 *Let B be Boolean and F a decomposable sheaf on B. Then*

$$H_*^{cell}(B; F) \cong H_*^{cell}(B_a; F_a/F^a)$$

Proof There is a short exact sequence of sheaves on B:



The structure maps between the elements of B^a and B_a in the leftmost sheaf G are all the identity; the middle sheaf is just F; the rightmost sheaf is trivial on B^a and the quotient sheaf F_a/F^a on B_a. By (5) we have a short exact sequence of cellular chain complexes with resulting long exact sequence:

$$\cdots \to H_i^{\text{cell}}(B; G) \to H_i^{\text{cell}}(B; F) \to H_i^{\text{cell}}(B_a; F_a/F^a) \to \cdots$$

after identifying the cellular homology of the rightmost sheaf with $H_i^{\text{cell}}(B_a; F_a/F^a)$. But $(B; G)$ is a double (6), and so the result follows from Proposition 5. □

3.6 The deletion–restriction long exact sequence

Let L be a geometric lattice and let $f : \widetilde{L} \to L$ be its Boolean cover. We will write $B = \widetilde{L}$. If a is an atom of L then the Boolean cover $\widetilde{L_a}$ of the deletion L_a can be identified with the sub-Boolean B_a of $\widetilde{L}$. Under this identification $\widetilde{F_a}$ on $\widetilde{L_a}$ is the restriction of F (on B) to B_a. Consequently, we just write F for the sheaf on $\widetilde{L_a}$ induced by a sheaf F on L, and

$$H_i^{\text{cell}}(B_a; F) = H_i^{\text{cell}}(\widetilde{L_a}; F). \tag{8}$$

The Boolean cover $\widetilde{L^a}$ of the restriction L^a is not, however, the sub-Boolean B^a of B: the rank of the cover is in general less than that of the sub-Boolean. Nevertheless, they have the same cellular homology. If $\widetilde{L^a} \to L^a$ is the Boolean cover of the restriction we also just write F for the sheaf induced on $\widetilde{L^a}$ by the restriction of F to L^a.

Theorem 6 *Let $L, \widetilde{L}$ and $a \in L$ be as above. Then, for all i,*

$$H_i^{cell}(B^a; F) \cong H_i^{cell}(\widetilde{L^a}; F).$$

Proof Write $\mathbb{B} = B^a = (\widetilde{L})^a$. Let A be the set of atoms of L and let $A_a = A\backslash\{a\}$ be the atoms of L_a. The set of atoms of the restriction L^a is $A^a = \{b \vee_L a : b \in A_a\}$. The atoms of the sub-Boolean $\mathbb{B}$ are the elements $b \vee_B a$ where $b \in A_a$. Note that these are all distinct.

If the elements $b \vee_L a$ for $b \in A_a$ are all are distinct (in L itself), then the Boolean cover $\widetilde{L^a}$ is precisely the sub-Boolean $\mathbb{B}$, and the result follows.

Otherwise, there exist distinct atoms $s = a \vee_B b$ and $s' = a \vee_B b'$ of $\mathbb{B}$ that are mapped by f to the same atom $a \vee_L b = a \vee_L b'$ of L^a. As usual let $\mathbb{B}_s$ and $\mathbb{B}^s$ denote the deletion and restriction of $\mathbb{B}$ with respect to the atom s. We claim that, for all i,

$$H_i^{\text{cell}}(\mathbb{B}; F) \cong H_i^{\text{cell}}(\mathbb{B}_s; F) \tag{9}$$

To prove this we will show that $\mathbb{B}^s$ is a double. Let $\alpha = s \vee_B s' = a \vee_B b \vee_B b'$. This is an atom of $\mathbb{B}^s$ and we may consider the deletion $(\mathbb{B}^s)_\alpha$ and the restriction $(\mathbb{B}^s)^\alpha$. Note that α and s are mapped by f to the same element of L^a:

$$\begin{aligned} f(\alpha) &= f(a \vee_B b \vee_B b') = a \vee_L b \vee_L b' = (a \vee_L b') \vee_L b \\ &= (a \vee_L b) \vee_L b = a \vee_L b = f(a \vee_B b) = f(s). \end{aligned}$$

Let $y \in (\mathbb{B}^s)_\alpha$. There is a corresponding element $y' = y \vee_B \alpha$ in $(\mathbb{B}^s)^\alpha$. We may write $y = x \vee_B s$ for some $x \in \mathbb{B}_s$ and since $s \vee_B \alpha = s \vee_B s \vee_B s' = s \vee_B s' = \alpha$ we have

$$y' = x \vee_B s \vee_B \alpha = x \vee_B \alpha.$$

Applying f, while recalling that $f\alpha = fs$, gives

$$f(y') = f(x \vee_B \alpha) = fx \vee_L f\alpha = fx \vee_L fs = f(x \vee_B s) = f(y).$$

It follows from the definition of the induced sheaf on the Boolean cover that the map $F_y^{y'}$ is the identity. This shows that $\mathbb{B}^s$ is a double with respect to the atom α.

To finish the proof of (9), we use Proposition 5 and the long exact sequence resulting from

$$0 \to C_*(\mathbb{B}_s; F) \to C_*(\mathbb{B}; F) \to C_*(\mathbb{B}^s; F) \to 0.$$

We may now repeat this process by taking a sequence of deletions of $\mathbb{B}$ until we arrive at $\widetilde{L^a}$. Courtesy of (9), the homology remains unchanged at each step, giving the required result. □

The previous result allows us to relate the cellular homology of the Boolean cover of a lattice with the homology of the Boolean covers of the restriction and deletion.

Theorem 7 (Deletion–restriction long exact sequence) *Let L be a geometric lattice equipped with a sheaf F and let $f : \widetilde{L} \to L$ be its Boolean cover. Then for any atom $a \in L$ there is a long exact sequence*

$$\begin{aligned} \cdots &\to H_i^{cell}(\widetilde{L^a}; F) \to H_i^{cell}(\widetilde{L_a}; F) \to H_i^{cell}(\widetilde{L}; F) \to H_{i-1}^{cell}(\widetilde{L^a}; F) \\ &\to H_{i-1}^{cell}(\widetilde{L_a}; F) \to \cdots \end{aligned}$$

Proof If A are the atoms of L then we can use the short exact sequence (4), induced by a sub-Boolean with $x = A \backslash \{a\}$, to get a short exact sequence

$$0 \to C_*(B_a; F) \to C_*(\widetilde{L}; F) \to C_{*-1}(B^a; F) \to 0$$

where, as above, $B = \widetilde{L}$. The result follows by applying Theorem 6 and (8) to the resulting long exact sequence. □

4 Sheaves on hyperplane arrangements

We return to the arrangement lattices of Sect. 1.1, the natural sheaf and its exterior powers from Sect. 1.2. In Sect. 4.1 we discuss graded Euler characteristics and their computation in terms of characteristic polynomials. Section 4.2 gives a complete calculation of the homology of an essential hyperplane arrangement with coefficients in the exterior natural sheaf. Section 4.3 extends this to the non-essential case.

Throughout, V is a finite dimensional vector space over a field k (initially arbitrary, then restricted to a subfield of $\mathbb{C}$ in Sect. 4.3); A is an arrangement in V and $L = L(A)$ is the intersection lattice; F is the natural sheaf associated to A.

4.1 Graded Euler characteristics

For F the natural sheaf on the arrangement lattice L we have the exterior sheaf $\Lambda^\bullet F$. The graded Euler characteristic of the cellular homology of $\widetilde{L}$ with coefficients in $\Lambda^\bullet F$ turns out to be very close to the characteristic polynomial of the arrangement lattice L.

Proposition 6 *The graded Euler chracteristic* $\chi_q H_*^{cell}(\widetilde{L}; \Lambda^\bullet F) = \chi_L(1+q)$.

Proof This is a straight-forward calculation (recalling that $\dim F(x) = \dim x$):

$$\begin{aligned}\chi_q H_*^{\text{cell}}(\widetilde{L}; \Lambda^\bullet F) &= \sum_k q^k \chi H_*^{\text{cell}}(\widetilde{L}; \Lambda^k F) = \sum_k q^k \sum_{x\in L} \mu_L(\mathbf{0}, x) \dim \Lambda^k F(x)\\ &= \sum_k q^k \sum_{x\in L} \mu_L(\mathbf{0}, x) \binom{\dim x}{k} = \sum_{x\in L} \mu_L(\mathbf{0}, x) \sum_k q^k \binom{\dim x}{k}\\ &= \sum_{x\in L} \mu_L(\mathbf{0}, x)(1+q)^{\dim x} = \chi_L(1+q)\end{aligned}$$

where we have used Corollary 4 at the second equality. □

From this, another application of Corollary 4 gives the graded Euler characteristic for the sheaf homology:

Corollary 5 $\chi_q H_*(L\backslash\mathbf{0}; \Lambda^\bullet F) = -\chi_L(1+q) + (1+q)^{\dim V}$.

4.2 The exterior sheaf on an essential arrangement

We focus first on *essential* arrangements—those for which the intersection of all hyperplanes is trivial.

Theorem 8 *Let L be the intersection lattice of an essential hyperplane arrangement in a space V, let F be the natural sheaf on L and $\Lambda^j F$ be the jth exterior power of F. If $rk(L) \geq 2$ and $\widetilde{L} \to L$ is the Boolean cover of L, then $H_i^{cell}(\widetilde{L}; \Lambda^j F)$ is trivial unless $0 \leq i < rk(L)$ and $i + j = rk(L) = \dim V$, in which case:*

$$\dim H_i^{cell}(\widetilde{L}; \Lambda^j F) = \frac{(-1)^i}{j!} \chi_L^{(j)}(1)$$

where $\chi_L^{(j)}(t)$ is the jth derivative of the characteristic polynomial of L.

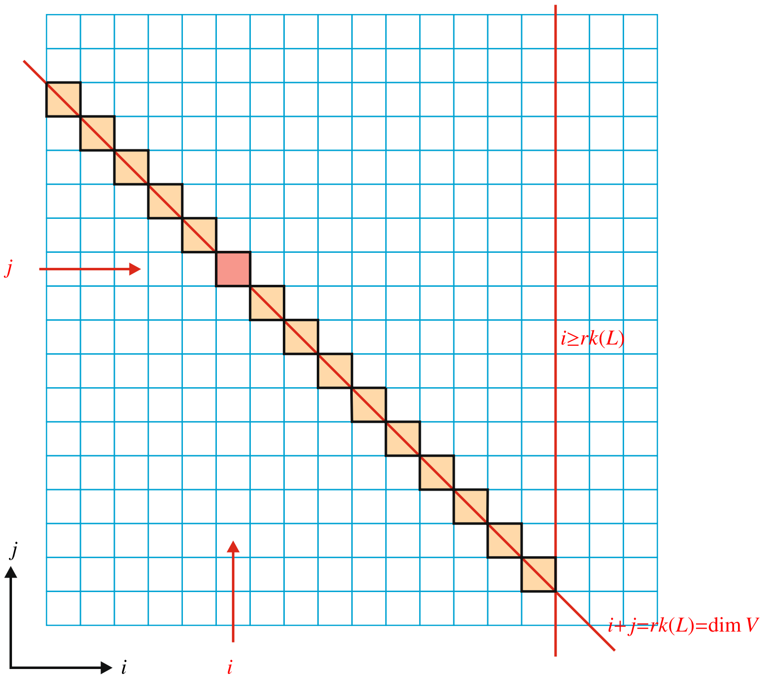


Fig. 4 Support of $H_i^{\text{cell}}(\widetilde{L};\Lambda^j F)$ for L essential in the (i,j)-plane, with the dimension of the red square given in Theorem 8

The support of $H_i^{\text{cell}}(\widetilde{L};\Lambda^j F)$ is shown in Fig. 4. The remainder of the section is devoted to the proof of Theorem 8, which is broken down into several subparts:

The proof of Theorem 8 when L itself is Boolean. We prove the result where L itself is Boolean (and hence $\widetilde{L}=L$) separately from the general case. The characteristic polynomial is given by

$$\chi_L(t)=(t-1)^{rk(L)}$$

and we require:

Proposition 7 *Let L be a Boolean intersection lattice of an essential hyperplane arrangement in a space V, let F be the natural sheaf on L and $\Lambda^j F$ be the jth exterior power of F. If $rk(L)\geq 1$ then*

$$\dim H_i^{cell}(L;\Lambda^j F)=\begin{cases}1 & \textit{if } i=0 \textit{ and } j=rk(L)\\ 0 & \textit{else.}\end{cases}$$

Proof The proof is an induction on the rank of L. The case where $rk(L)=1$ (and so we have a single hyperplane the trivial space in a 1-dimensional V) can be verified by brute force. Otherwise, for $rk(L)\geq 2$, there is a basis $Z=\{v_1,\ldots,v_n\}$ for V such that L is the lattice of subsets of Z ordered via reverse inclusion, with the subset corresponding to $x\in L$ giving a basis for $F(x)$. If $a\in A$ is a hyperplane with basis $\{v_1,\ldots,\hat{v}_i,\ldots,v_n\}$, then for $x\in L_a$ the space $F(x\vee a)$ has basis some subset $\{u_1,\ldots,u_m\}$ of $Z\backslash\{v_i\}$ and $F(x)$ has basis $\{v_i,u_1,\ldots,u_m\}$.

A basis vector $u_{i_1} \wedge \cdots \wedge u_{i_j}$ of $\Lambda^j F(x)$ may or may not contain the element v_i, leading to a decomposition

$$\Lambda^j F(x) \cong \Lambda^{j-1} F(x \vee a) \oplus \Lambda^j F(x \vee a). \tag{10}$$

Writing $G = \Lambda^j F$ for $j \geq 1$ (and similarly G_a and G^a) the structure maps

$$G_x^{x \vee a} \colon G(x \vee a) = \Lambda^j F(x \vee a) \to \Lambda^j F(x) = G(x)$$

are the obvious inclusions, so G is decomposable in the sense of Sect. 3.5. Moreover, the isomorphism (10) leads to an isomorphism of sheaves

$$G_a / G^a \cong \Lambda^{j-1} F^a \tag{11}$$

(some care is needed in checking what happens when j is close to $\dim x$, as some of the spaces become 0). Theorem 5 thus gives

$$H_*^{\text{cell}}(L; \Lambda^j F) \cong H_*^{\text{cell}}(L_a; \Lambda^{j-1} F^a) \tag{12}$$

with F^a the natural sheaf of the essential arrangement A^a on $L_a \cong L^a$ (both are Boolean of one smaller rank than L). Hence, by induction

$$\dim H_i^{\text{cell}}(L; \Lambda^j F) = \dim H_i^{\text{cell}}(L_a; \Lambda^{j-1} F^a) = \begin{cases} 1 & \text{if } i = 0 \text{ and } j - 1 = rk(L) - 1 \\ 0 & \text{else.} \end{cases}$$

□

Suppose now that L is not Boolean. We argue by induction on the number $|A|$ of hyperplanes. Throughout, if $j > rk(L) = \dim V$ then the sheaf $\Lambda^j F = 0$; we thus need only consider j in the range $0 \leq j \leq rk(L) = \dim V$. When $|A| = 1$ or 2, the intersection lattice $L(A)$ is Boolean of rank $|A|$, and so these cases have been handled already.

The base case $|A| = 3$. We saw in Sect. 1.1 that the only non-Boolean L on three hyperplanes is realised by a braid arrangement, and with L isomorphic to the partition lattice $\Pi(3)$. When essential, the arrangement lives in a 2-dimensional V with basis $\{v_1, v_2\}$ and consists of the lines spanned by v_1, v_2 and $-v_1 - v_2$.

The characteristic polynomial (see [13, §3.10.4]) is $\chi_L(t) = (t-1)(t-2)$ and Theorem 8 becomes: $H_i^{\text{cell}}(\widetilde{L}; \Lambda^0 F)$ are all trivial; $\dim H_1^{\text{cell}}(\widetilde{L}; \Lambda^1 F) = 1$ and the remaining groups $H_i^{\text{cell}}(\widetilde{L}; \Lambda^1 F)$ are trivial; $\dim H_0^{\text{cell}}(\widetilde{L}; \Lambda^2 F) = 1$ and the remaining groups $H_i^{\text{cell}}(\widetilde{L}; \Lambda^2 F)$ are trivial. To prove this, each case is treated separately. For $j = 0$ the sheaf is constant $\Lambda^0 F = \Delta k$ and so the induced sheaf on the Boolean cover is also constant and by applying Theorem 5 we have

$$H_i^{\text{cell}}(\widetilde{L}; \Delta k) = H_i^{\text{cell}}(\widetilde{L}_a; \text{zero sheaf}) = 0, \text{ for all } i.$$

For $j = 1$, we have the natural sheaf and the induced sheaf on the Boolean cover $\widetilde{L}$ has constant value $F(\mathbf{1})$ on all the elements of ranks 2 and 3. Two applications of Theorem 5 give the required result. For $j = 2$, the sheaf $\Lambda^2 F$ is trivial except at $\mathbf{0} \in L$ where it is 1-dimensional with basis $v_1 \wedge v_2$, once again in happy agreement with Theorem 8.

The vanishing degrees in the general case $|A| > 3$. We may assume that L is non-Boolean and so, by the results of Sect. 1.1, it has a dependent atom $a \in A$.

The deletion L_a is then an essential arrangement lattice having $|A| - 1$ hyperplanes and $rk(L_a) = rk(L)$, by the dependence of a. The sheaf F_a, which is just F restricted to L_a, is the natural sheaf of this arrangement, and the restriction of $\Lambda^j F$ to L_a is just $\Lambda^j F_a$. The restriction L^a is an essential arrangement lattice having at most $|A| - 1$ hyperplanes and

$rk(L^a) = rk(L) - 1$. The sheaf F^a is the natural sheaf of this arrangement and $\Lambda^j F^a$ is the restriction of $\Lambda^j F$ to L^a.

Both L_a and L^a are either Boolean, or essential arrangement lattices on fewer than $|A|$ hyperplanes, hence come under the auspices of the inductive hypothesis. The deletion–restriction long exact sequence, Theorem 7, gives

$$\cdots \to H_i^{\text{cell}}(\widetilde{L_a}; \Lambda^j F_a) \to H_i^{\text{cell}}(\widetilde{L}; \Lambda^j F) \to H_{i-1}^{\text{cell}}(\widetilde{L}^a; \Lambda^j F^a) \to \cdots$$

If $i \neq rk(L) - j$ or $i = rk(L)$, then both the left and right terms vanish, hence by induction we get $H_i^{\text{cell}}(\widetilde{L}; \Lambda^j F) = 0$ as required.

The non-vanishing degree in the general case $|A| > 3$.

For fixed j we have shown that there is only one non-trivial group among the $H_i^{\text{cell}}(\widetilde{L}; \Lambda^j F)$, namely when $i = rk(L) - j$. This reduces the task to an Euler characteristic computation. We have

$$\chi H_*^{\text{cell}}(\widetilde{L}; \Lambda^j F) = \sum_n (-1)^n \dim H_n^{\text{cell}}(\widetilde{L}; \Lambda^j F) = (-1)^{rk(L)-j} \dim H_{rk(L)-j}^{\text{cell}}(\widetilde{L}; \Lambda^j F).$$

From this we get

$$\chi_q H_*^{\text{cell}}(\widetilde{L}; \Lambda^\bullet F) = \sum_j q^j \chi H_*^{\text{cell}}(\widetilde{L}; \Lambda^j F) = \sum_j q^j (-1)^{rk(L)-j} \dim H_{rk(L)-j}^{\text{cell}}(\widetilde{L}; \Lambda^j F).$$

Thus, for $i = rk(L) - j$,

$$\begin{aligned} \dim H_i^{\text{cell}}(\widetilde{L}; \Lambda^j F) &= (-1)^i \times (\text{coefficient of } q^j \text{in} \chi_q H_*^{\text{cell}}(\widetilde{L}; \Lambda^\bullet F)) \\ &= (-1)^i \times (\text{coefficient of } q^j in \chi_L(1+q)) \end{aligned}$$

where the last equality is due to Proposition 6. The Taylor expansion of (the polynomial) $\chi_L(1+q)$ immediately reveals the coefficient of q^j in $\chi_L(1+q)$ to be $\frac{1}{j!}\, \chi_L^{(j)}(1)$ giving

$$\dim H_i^{\text{cell}}(\widetilde{L}; \Lambda^j F) = \frac{(-1)^i}{j!} \chi_L^{(j)}(1).$$

This completes the proof of Theorem 8 computing the cellular homology. Our real interest is in the sheaf homology, which we can now compute by applying Proposition 4 to Theorem 8.

Theorem 9 *Let L be the intersection lattice of an essential hyperplane arrangement in a space V. Let F be the natural sheaf on L and $\Lambda^j F$ be the jth exterior power of F. If $rk(L) \geq 2$ then $H_i(L \backslash \mathbf{0}; \Lambda^j F)$ is trivial unless:*

– either $0 < i < rk(L) - 1$ and $i + j = rk(L) - 1$, in which case

$$\dim H_i(L \backslash \mathbf{0}; \Lambda^j F) = \frac{(-1)^{i+1}}{j!} \chi_L^{(j)}(1)$$

– or, $i = 0$ and or $j = rk(L) - 1$, in which case

$$\dim H_0(L \backslash \mathbf{0}; \Lambda^j F) = \binom{rk(L)}{j} - \frac{1}{j!} \chi_L^{(j)}(1)$$

– or, $i = 0$ and $j < rk(L) - 1$, in which case

$$\dim H_0(L \backslash \mathbf{0}; \Lambda^j F) = \binom{rk(L)}{j}$$

where $\chi_L^{(j)}(t)$ is the jth derivative of the characteristic polynomial of L.

4.3 The exterior sheaf for a non-essential arrangement

In this section we assume our field to be a sub-field of $\mathbb{C}$ and so we may assume that the vector space V comes equipped with an inner product $\langle -, - \rangle$. Let L be the intersection lattice of a hyperplane arrangement in a space V with $U = \bigcap_{a \in A} a$. Let F be the natural sheaf on L.

For a subspace $B \subset V$ such that $U \subset B \subset V$ we define the orthogonal complement of U in B to be

$$U^{\perp B} = \{b \in B \mid \langle b, u \rangle = 0, \text{ for all } u \in U\}.$$

Note that $U^{\perp B}$ is a subspace of B and $B = U \oplus U^{\perp B}$. (This last condition may fail in finite characteristic.) Moreover, if $U \subset B \subset B' \subset V$, there is an inclusion $\iota\colon U^{\perp B} \subset U^{\perp B'}$ and with respect to the decompositions $B = U \oplus U^{\perp B}$ and $B' = U \oplus U^{\perp B'}$ the inclusion $B \subset B'$ decomposes as $1 \oplus \iota$.

We define a sheaf $F^{\perp}$ on L as follows. For $x \in L$ we set $F^{\perp}(x) = U^{\perp F(x)}$. If $x \leq y$ then the structure map $F^{\perp}(x \leq y)$ is the inclusion $F^{\perp}(y) = U^{\perp F(y)} \subset U^{\perp F(x)} = F^{\perp}(x)$ induced by the inclusion $F(y) \subset F(x)$.

We have:

Lemma 2 *There is a direct sum decomposition of sheaves $F = \Delta U \oplus F^{\perp}$.*

By definition $F^{\perp}$ is a sub-sheaf of F on L, but it is also the natural sheaf of an essential hyperplane arrangement in $U^{\perp V}$. There is one hyperplane $U^{\perp H}$ for each hyperplane H of the original arrangement and courtesy of the relation $U^{\perp (B \cap C)} = U^{\perp B} \cap U^{\perp C}$ we see that the lattice of this new arrangement is again L. The natural sheaf on the new arrangement is precisely $F^{\perp}$, seen immediately from the definition of $F^{\perp}$. We refer to $F^{\perp}$ as the *essentialisation* of F.

We will continue to write $\chi_L(t)$ for the characteristic polynomial of L equipped with the natural sheaf F, that is to say, $\chi_L(t) = \sum \mu_L(\mathbf{0}, x) t^{\dim x}$. Writing $\chi_{(L, F^{\perp})}(t)$ for the characteristic polynomial of the essentialisation we easily see

$$\chi_L(t) = t^{\dim U} \chi_{(L, F^{\perp})}(t).$$

Let $\Lambda^j F$ be the jth exterior power of F (the natural sheaf). Theorem 10 below gives the cellular homology $H_i^{\text{cell}}(\widetilde{L}; \Lambda^j F)$ and the support in the (i, j)-plane is illustrated in Fig. 5.

Theorem 10 *Let L be the intersection lattice of a hyperplane arrangement in a space V with $U = \bigcap_{a \in A} a$. Let F be the natural sheaf on L and $\Lambda^j F$ be the jth exterior power of F. If $rk(L) \geq 2$ and $\widetilde{L} \to L$ is the Boolean cover of L, then $H_i^{cell}(\widetilde{L}; \Lambda^j F)$ is trivial unless $0 \leq i < rk(L)$ and $rk(L) \leq i + j \leq \dim V$, in which case:*

$$\dim H_i^{cell}(\widetilde{L}; \Lambda^j F) = \frac{(-1)^i}{(rk(L) - i)!} \binom{\dim U}{i + j - rk(L)} \chi_{(L, F^{\perp})}^{(rk(L) - i)}(1)$$

with $\chi_{(L, F^{\perp})}^{(k)}(t)$ the kth derivative of the characteristic polynomial of the essentialisation of L.

Proof The decomposition $F = \Delta U \oplus F^{\perp}$ allows us to write

$$\Lambda^j F = \Lambda^j (\Delta U \oplus F^{\perp}) = \bigoplus_{s+t=j} \Lambda^s \Delta U \otimes \Lambda^t F^{\perp} = \bigoplus_{s+t=j} \Delta \Lambda^s U \otimes \Lambda^t F^{\perp}.$$

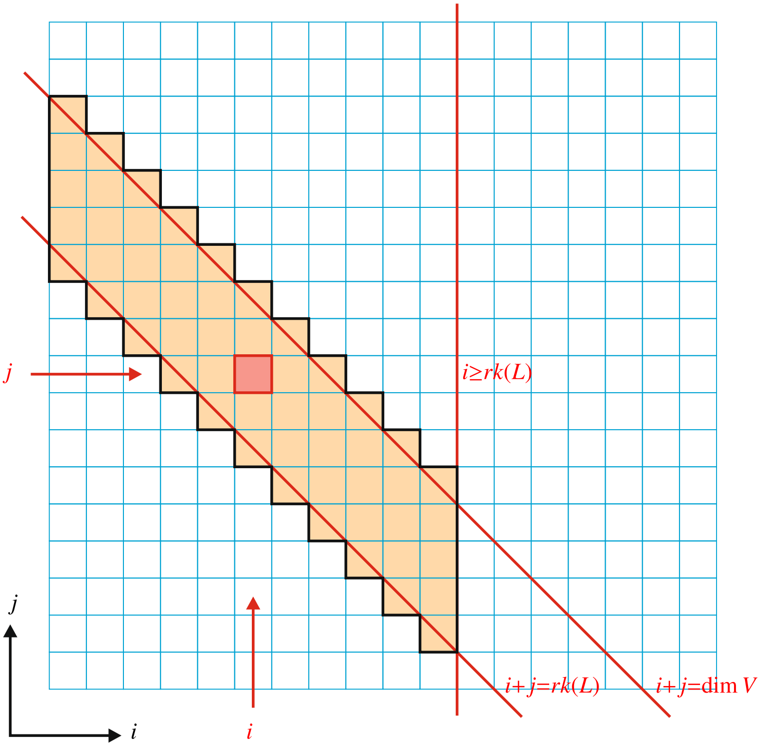


Fig. 5 Support of $H_i^{\text{cell}}(\widetilde{L}; \Lambda^j F)$ in the (i, j)-plane, with the dimension of the red square given by Theorem 10

Applying, Corollary 3 gives

$$H_i^{\text{cell}}(\widetilde{L}; \Lambda^j F) = \bigoplus_{s+t=j} H_i^{\text{cell}}(\widetilde{L}; \Delta\Lambda^s U \otimes \Lambda^t F^{\perp}).$$

Recalling that we are working in characteristic zero, the universal coefficient theorem tells us that

$$H_*^{\text{cell}}(\widetilde{L}; \Delta\Lambda^s U \otimes \Lambda^t F^{\perp}) \cong \Lambda^s U \otimes H_*^{\text{cell}}(\widetilde{L}; \Lambda^t F^{\perp})$$

and we have

$$\dim H_i^{\text{cell}}(\widetilde{L}; \Delta\Lambda^s U \otimes \Lambda^t F^{\perp}) = \binom{\dim U}{s} \dim H_i^{\text{cell}}(\widetilde{L}; \Lambda^t F^{\perp}). \tag{13}$$

The dimension on the right can be computed from Theorem 8 because $F^{\perp}$ is essential: we have that $\dim H_i^{\text{cell}}(\widetilde{L}; \Lambda^t F^{\perp}) = 0$ unless $t = rk(L) - i$, in which case

$$\dim H_i^{\text{cell}}(\widetilde{L}; \Lambda^{rk(L)-i} F^{\perp}) = \frac{(-1)^i}{(rk(L)-i)!} \chi_{(L,F^{\perp})}^{(rk(L)-i)}(1).$$

Since $s + t = j$ we have $s = i + j - rk(L)$ which means given i, j the values of s and t must be taken to be $t = rk(L) - i$ and $s = i + j - rk(L)$ and so we get

$$\dim H_i^{\mathrm{cell}}(\widetilde{L}; \Lambda^j F) = \frac{(-1)^i}{(rk(L) - i)!} \binom{\dim U}{i + j - rk(L)} \chi_{(L,F^\perp)}^{(rk(L)-i)}(1).$$

What remains is to find the values of i and j for which this computation is valid. The conditions are (i) $0 \le i < rk(L)$ (in order to be able to apply Theorem 8), (ii) $s \le \dim U$ (otherwise $\Lambda^s U$ is trivial) and (iii) $t \le j$ (since $s + t = j$ in the sum above). Condition (i) is seen in the statement of the theorem; since $\dim U = \dim V - rk(L)$, condition (ii) implies $i + j \le \dim V$; condition (iii) implies $rk(L) \le i + j$. So conditions (ii) and (iii) together give the other condition in the statement of the theorem, namely $rk(L) \le i + j \le \dim V$. □

As in the essential case, we can convert this into a result about sheaf homology. As before we write $\chi_{(L,F^\perp)}^{(k)}(t)$ for the kth derivative of the characteristic polynomial of the essentialisation of L.

Theorem 11 *Let L be the intersection lattice of a hyperplane arrangement in a space V with $U = \bigcap_{a\in A} a$. Let F be the natural sheaf on L and $\Lambda^j F$ be the jth exterior power of F. If $rk(L) \ge 2$ then $H_i(L\backslash \mathbf{0}; \Lambda^j F)$ is trivial unless:*

– *either $0 < i < rk(L) - 1$ and $rk(L) \le i + j + 1 \le \dim V$, in which case*

$$\dim H_i(L\backslash \mathbf{0}; \Lambda^j F) = \frac{(-1)^{i+1}}{(rk(L) - i - 1)!} \binom{\dim U}{i + 1 + j - rk(L)} \chi_{(L,F^\perp)}^{(rk(L)-i-1)}(1)$$

– *or, $i = 0$ and $rk(L) \le j < \dim V$, in which case $\dim H_0(L\backslash \mathbf{0}; \Lambda^j F)$ equals*

$$\binom{\dim V}{j} - \frac{1}{rk(L)!} \binom{\dim U}{j - rk(L)} \chi_{(L,F^\perp)}^{(rk(L))}(1) - \frac{1}{(rk(L)-1)!} \binom{\dim U}{j + 1 - rk(L)} \chi_{(L,F^\perp)}^{(rk(L)-1)}(1)$$

– *or, $i = 0$ and $j = rk(L) - 1$, in which case*

$$\dim H_0(L\backslash \mathbf{0}; \Lambda^{rk(L)-1} F) = \binom{\dim V}{rk(L) - 1} - \frac{1}{(rk(L) - 1)!} \chi_L^{(rk(L)-1)}(1)$$

– *or, $i = 0$ and $j < rk(L) - 1$, in which case*

$$\dim H_0(L\backslash \mathbf{0}; \Lambda^j F) = \binom{\dim V}{j}.$$

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