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ABSTRACT

We introduce a new family of central elements of the Yangian of the queer Lie superalgebra $\mathfrak{q}_1$.

I. INTRODUCTION

In this article, we work over the complex field $\mathbb{C}$. The family of *strange* Lie superalgebras consists of the queer Lie superalgebras $\mathfrak{q}_N$ and periplectic Lie superalgebras $\mathfrak{p}_N$, where N is any positive integer. Both $\mathfrak{q}_N$ and $\mathfrak{p}_N$ are fixed point subalgebras of the general linear Lie superalgebra $\mathfrak{gl}_{N|N}$ relative to certain involutive automorphisms. For $\mathfrak{q}_N$, this automorphism is denoted by π (see Sec. III).

Take the *twisted* polynomial current Lie superalgebra

$$\mathfrak{g} = \{X(u) \in \mathfrak{gl}_{N|N}[u] : \pi(X(u)) = X(-u)\}. \tag{1}$$

Then, the Yangian $\mathrm{Y}(\mathfrak{q}_N)$ is a deformation of the universal enveloping algebra of $\mathfrak{g}$ in the class of Hopf superalgebras.

The Yangian $\mathrm{Y}(\mathfrak{q}_N)$ has been discovered by the present author by extending to $\mathfrak{q}_N$ the centralizer construction[1] of the Yangian of the general linear Lie algebra $\mathfrak{gl}_N$. The resulting definition of $\mathrm{Y}(\mathfrak{q}_N)$ was published in Ref. 2 where the Yangian of the Lie superalgebra $\mathfrak{p}_N$ was also defined. The Yangian $\mathrm{Y}(\mathfrak{q}_N)$ from Ref. 2 was further studied in Ref. 3. Details of the original centralizer construction of $\mathrm{Y}(\mathfrak{q}_N)$ involving the invariant theory of Lie superalgebras were later published in Ref. 4. There is no alternative definition of the Yangian of $\mathfrak{p}_N$, however, other than that given in Ref. 2.

Due to the centralizer construction of $\mathrm{Y}(\mathfrak{q}_N)$, it appears in the theory of W-algebras. For any positive integer M, take the finite W-algebra of $\mathfrak{q}_{MN}$ defined by a non-regular nilpotent odd element with the Jordan blocks of size M each. This W-algebra[5] is a quotient of $\mathrm{Y}(\mathfrak{q}_N)$.

The definition of the Yangian $\mathrm{Y}(\mathfrak{q}_N)$ is stated in Sec. V. It is based on a new solution of the quantum Yang–Baxter equation (9) given in Ref. 2. This solution is a rational function (8) of two variables u, v with values in the supercommutant in $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 2}$ of the image of $\mathfrak{q}_N$. Unlike all other rational solutions of Eq. (9) known before,[2] it is *not* a function of only the difference $u - v$ of the variables. In Sec. IV, we prove that (8) satisfies (9). The proof was not published previously.

In Secs. VII and VIII, we study in more detail the Yangian $\mathrm{Y}(\mathfrak{q}_1)$. Take the finite W-algebra of $\mathfrak{q}_N$ defined by any regular nilpotent odd element. This W-algebra[6] is a quotient of $\mathrm{Y}(\mathfrak{q}_1)$. Here, we introduce a new family of central elements of $\mathrm{Y}(\mathfrak{q}_1)$. Its generating function $C(u)$ is a match to the generating function of central elements of the Yangian of $\mathfrak{gl}_{1|1}$, called the quantum Berezinian.[7] We relate $C(u)$ to another generating function of central elements of $\mathrm{Y}(\mathfrak{q}_1)$. The latter is just the specialization to $N = 1$ of the generating function $Z(u)$ of central elements of $\mathrm{Y}(\mathfrak{q}_N)$ given in Ref. 2 and reviewed in our Sec. IV.

II. GENERAL CONVENTIONS

We will use the following general conventions. Let A and B be any two associative $\mathbb{Z}_2$-graded algebras. Their tensor product $A \otimes B$ is also an associative $\mathbb{Z}_2$-graded algebra such that for any homogeneous elements X, X' of A and Y, Y' of B, we have

$$(X \otimes Y)(X' \otimes Y') = XX' \otimes YY' \, (-1)^{\deg X' \deg Y},$$
$$\deg(X \otimes Y) = \deg X + \deg Y.$$

Furthermore, for any two $\mathbb{Z}_2$-graded modules U and V over A and B, respectively, the vector space $U \otimes V$ is also a $\mathbb{Z}_2$-graded module over $A \otimes B$ such that for any homogeneous elements $x \in U$ and $y \in V$,

$$(X \otimes Y)(x \otimes y) = Xx \otimes Yy \, (-1)^{\deg x \, \deg Y}, \tag{2}$$
$$\deg(x \otimes y) = \deg x + \deg y. \tag{3}$$

As usual, a homomorphism $\alpha: \mathrm{A} \to \mathrm{B}$ is a linear map such that $\alpha(XX') = \alpha(X)\alpha(X')$ for all $X, X' \in \mathrm{A}$. However, an antihomomorphism $\beta: \mathrm{A} \to \mathrm{B}$ is a linear map such that for all homogeneous elements $X, X' \in \mathrm{A}$,

$$\beta(XX') = \beta(X')\,\beta(X)\,(-1)^{\deg X \deg X'}.$$

Let n be any positive integer. If the algebra A is unital, let ι_p be its embedding into the tensor product $\mathrm{A}^{\otimes n}$ as the pth tensor factor,

$$\iota_p(X) = 1^{\otimes(p-1)} \otimes X \otimes 1^{\otimes(n-p)} \qquad \text{for} \quad p = 1, \dots, n.$$

We will also employ various embeddings of $\mathrm{A}^{\otimes m}$ to $\mathrm{A}^{\otimes n}$ for $m = 1, \dots, n$. For any choice of m pairwise distinct indices $p_1, \dots, p_m \in \{1, \dots, n\}$ and of an element W of $\mathrm{A}^{\otimes m}$ of the form $W = X^{(1)} \otimes \cdots \otimes X^{(m)}$, we will denote

$$W_{p_1 \dots p_m} = \iota_{p_1}(X^{(1)}) \dots \iota_{p_m}(X^{(m)}) \in \mathrm{A}^{\otimes n}. \tag{4}$$

We will extend the notation $W_{p_1 \dots p_m}$ to all elements W of $\mathrm{A}^{\otimes m}$ by linearity.

III. THE QUEER LIE SUPERALGEBRA

Let the indices i, j run through $\pm 1, \dots, \pm N$. We will write $\bar{\imath} = 0$ if $i > 0$ and $\bar{\imath} = 1$ if $i < 0$. Hence, $\bar{\imath}$ will take values in $\mathbb{Z}_2$. Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{N|N}$. Let $e_i \in \mathbb{C}^{N|N}$ be an element of the standard basis. The $\mathbb{Z}_2$-grading on $\mathbb{C}^{N|N}$ is defined by $\deg e_i = \bar{\imath}$.

Let $E_{ij} \in \operatorname{End} \mathbb{C}^{N|N}$ be the standard matrix unit. It is defined by setting $E_{ij} e_k = \delta_{jk} e_i$. Then, the associative algebra $\operatorname{End} \mathbb{C}^{N|N}$ is $\mathbb{Z}_2$-graded so that $\deg E_{ij} = \bar{\imath} + \bar{\jmath}$. Hence, $\mathbb{C}^{N|N}$ is a $\mathbb{Z}_2$-graded module over $\operatorname{End} \mathbb{C}^{N|N}$. For any n, we can identify the tensor product $(\operatorname{End} \mathbb{C}^{N|N})^{\otimes n}$ with the algebra $\operatorname{End}((\mathbb{C}^{N|N})^{\otimes n})$ acting on the vector space $(\mathbb{C}^{N|N})^{\otimes n}$ by using conventions (2) and (3). An involutive automorphism π of $\operatorname{End} \mathbb{C}^{N|N}$ is defined by

$$\pi : E_{ij} \mapsto E_{-i,-j}.$$

Consider the general linear Lie superalgebra $\mathfrak{gl}_{N|N}$. To avoid confusion, denote by e_{ij} the element of $\mathfrak{gl}_{N|N}$ corresponding to $E_{ij} \in \operatorname{End} \mathbb{C}^{N|N}$. Then, $\deg e_{ij} = \bar{\imath} + \bar{\jmath}$,

$$[e_{ij}, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{li}\, e_{kj}\, (-1)^{(\bar{\imath} + \bar{\jmath})(\bar{k} + \bar{l})}.$$

Therefore, π is also an involutive automorphism of $\mathfrak{gl}_{N|N}$.

Now, the *queer* Lie superalgebra $\mathfrak{q}_N$ is the fixed point subalgebra of $\mathfrak{gl}_{N|N}$ relative to the automorphism π. This subalgebra is spanned by the elements

$$f_{ij} = e_{ij} + \pi(e_{ij}) = e_{ij} + e_{-i,-j}.$$

In the Lie superalgebra $\mathfrak{q}_N$, we have $f_{-i,-j} = f_{ij}$ and

$$[f_{ij}, f_{kl}] = \delta_{jk}\, f_{il} + \delta_{j,-k}\, f_{-i,l} - (\delta_{li}\, f_{kj} + \delta_{-l,i}\, f_{k,-j})(-1)^{(\bar{\imath} + \bar{\jmath})(\bar{k} + \bar{l})}.$$

Note that the elements f_{ij} with $i > 0$ form a basis of $\mathfrak{q}_N$.

We will also work with the universal enveloping algebra $\mathrm{U}(\mathfrak{q}_N)$. This is a $\mathbb{Z}_2$-graded associative algebra generated by the elements f_{ij} with $\deg f_{ij} = \bar{\imath} + \bar{\jmath}$ and the same relations as above, where the square brackets now stand for the supercommutator, however.

Note that $\mathrm{U}(\mathfrak{q}_N)$ is a $\mathbb{Z}_2$-graded Hopf algebra where the counit homomorphism $\mathrm{U}(\mathfrak{q}_N) \to \mathbb{C}$, comultiplication homomorphism $\mathrm{U}(\mathfrak{q}_N) \to \mathrm{U}(\mathfrak{q}_N) \otimes \mathrm{U}(\mathfrak{q}_N)$ and antipodal antihomomorphism $\mathrm{U}(\mathfrak{q}_N) \to \mathrm{U}(\mathfrak{q}_N)$ are defined by

$$f_{ij} \mapsto \delta_{ij}, \quad f_{ij} \mapsto f_{ij} \otimes 1 + 1 \otimes f_{ij} \quad \text{and} \quad f_{ij} \mapsto -f_{ij}.$$

Let us consider in more detail the Lie superalgebra $\mathfrak{q}_1$. We will choose the basis of $\mathfrak{q}_1$ consisting of two elements,

$$a = f_{11} = f_{-1,-1} = e_{11} + e_{-1,-1},$$
$$b = f_{1,-1} = f_{-1,1} = e_{1,-1} + e_{-1,1}.$$

By the above general relations, we get Lie brackets in $\mathfrak{q}_1$,

$$[a,a] = [a,b] = 0 \quad \text{and} \quad [b,b] = 2\,a. \tag{5}$$

Hence, $\mathrm{U}(\mathfrak{q}_1)$ is the $\mathbb{Z}_2$-graded associative algebra generated by the elements a and b, where $\deg a = 0$ and $\deg b = 1$. The defining relations in $\mathrm{U}(\mathfrak{q}_1)$ are the same (5) but with the square brackets now meaning the supercommutator. Hence, element a of $\mathrm{U}(\mathfrak{q}_1)$ is central and $b^2 = a$. Element a generates the center of the superalgebra $\mathrm{U}(\mathfrak{q}_1)$; see Ref. 4 for the corresponding general result about $\mathrm{U}(\mathfrak{q}_N)$.

IV. THE R-MATRIX

Take the element of the algebra $(\operatorname{End}\mathbb{C}^{N|N})^{\otimes 2}$,

$$P = \sum_{i,j} E_{ij} \otimes E_{ji}\,(-1)^{\bar{\jmath}}.$$

It acts on the vector space $(\mathbb{C}^{N|N})^{\otimes 2}$ so that

$$e_i \otimes e_j \mapsto e_j \otimes e_i\,(-1)^{\bar{\imath}\bar{\jmath}}.$$

We identify the algebra $(\operatorname{End}\mathbb{C}^{N|N})^{\otimes 2}$ with the algebra $\operatorname{End}((\mathbb{C}^{N|N})^{\otimes 2})$ by using (2). Note that $P^2 = 1$ and

$$(\pi \otimes \pi)(P) = -P.$$

Furthermore, take the element of the algebra $(\operatorname{End}\mathbb{C}^{N|N})^{\otimes 2}$,

$$Q = \sum_{i,j} E_{-i,-j} \otimes E_{ji}\,(-1)^{\bar{\jmath}}.$$

Then, we have the equalities

$$Q = (\pi \otimes \mathrm{id})(P) = (-\,\mathrm{id} \otimes \pi)(P). \tag{6}$$

Note that by the above definitions of P and Q, we have

$$P\,Q + Q\,P = 0 \quad \text{and} \quad Q^2 = 1. \tag{7}$$

Now consider a function of complex variables u, v with values in the algebra $(\operatorname{End}\mathbb{C}^{N|N})^{\otimes 2}$,

$$R(u,v) = 1 - \frac{P}{u-v} + \frac{Q}{u+v}. \tag{8}$$

By (6), we have

$$(\pi \otimes 1)(R(u,v)) = R(-u,v),$$
$$(1 \otimes \pi)(R(u,v)) = R(u,-v).$$

Furthermore, we have

$$R(u,v)\,R(-u,-v) = 1 - \frac{1}{(u-v)^2} - \frac{1}{(u+v)^2}.$$

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Indeed, due to the relation $P^2 = 1$ and to (7),

$$\left(1 - \frac{P}{u-v} + \frac{Q}{u+v}\right)\left(1 + \frac{P}{u-v} - \frac{Q}{u+v}\right) = 1 - \frac{P^2}{(u-v)^2} + \frac{PQ + QP}{(u-v)(u+v)} - \frac{Q^2}{(u+v)^2} = 1 - \frac{1}{(u-v)^2} - \frac{1}{(u+v)^2}.$$

Let us now verify that the function $R(u,v)$ obeys the *Yang–Baxter equation* in $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 3}(u,v,w)$,

$$R_{12}(u,v)R_{13}(u,w)R_{23}(v,w) = R_{23}(v,w)R_{13}(u,w)R_{12}(u,v). \tag{9}$$

Using definition (8), the equality in (9) will follow from the following relations in the algebra $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 3}$:

$$P_{12}\,P_{13} = P_{23}\,P_{12} = P_{13}\,P_{23}, \tag{10}$$

$$P_{13}\,P_{12} = P_{12}\,P_{23} = P_{23}\,P_{13}, \tag{11}$$

$$Q_{12}\,Q_{13} = P_{23}\,Q_{12} = Q_{13}\,P_{23}, \tag{12}$$

$$Q_{13}\,Q_{12} = Q_{12}\,P_{23} = P_{23}\,Q_{13}, \tag{13}$$

$$Q_{12}\,P_{13} = Q_{23}\,Q_{12} = -\,P_{13}\,Q_{23}, \tag{14}$$

$$P_{13}\,Q_{12} = Q_{12}\,Q_{23} = -\,Q_{23}\,P_{13}, \tag{15}$$

$$P_{12}\,Q_{13} = Q_{23}\,P_{12} = -\,Q_{13}\,Q_{23}, \tag{16}$$

$$Q_{13}\,P_{12} = P_{12}\,Q_{23} = -\,Q_{23}\,Q_{13}, \tag{17}$$

$$P_{12}\,P_{13}\,P_{23} = P_{23}\,P_{13}\,P_{12}, \tag{18}$$

$$P_{12}\,Q_{13}\,Q_{23} = Q_{23}\,Q_{13}\,P_{12}, \tag{19}$$

$$Q_{12}\,Q_{13}\,P_{23} = P_{23}\,Q_{13}\,Q_{12}, \tag{20}$$

$$Q_{12}\,P_{13}\,Q_{23} = Q_{23}\,P_{13}\,Q_{12}, \tag{21}$$

$$P_{12}\,P_{13}\,Q_{23} = P_{23}\,Q_{13}\,P_{12} = Q_{12}\,P_{13}\,P_{23} = Q_{23}\,Q_{13}\,Q_{12}, \tag{22}$$

$$Q_{23}\,P_{13}\,P_{12} = P_{12}\,Q_{13}\,P_{23} = P_{23}\,P_{13}\,Q_{12} = Q_{12}\,Q_{13}\,Q_{23}. \tag{23}$$

Relations (10) and (11) are used with the identity

$$\frac{1}{u-v}\,\frac{1}{u-w} - \frac{1}{u-v}\,\frac{1}{v-w} + \frac{1}{u-w}\,\frac{1}{v-w} = 0,$$

which is easy to verify. Relations (12) and (13) are used with the identity

$$\frac{1}{u+v}\,\frac{1}{u+w} + \frac{1}{u+v}\,\frac{1}{v-w} - \frac{1}{u+w}\,\frac{1}{v-w} = 0$$

obtained from the previous one by changing the sign of u. Relations (14) and (15) are used with the identity

$$\frac{1}{u+v}\,\frac{1}{u-w} + \frac{1}{u+v}\,\frac{1}{v+w} - \frac{1}{u-w}\,\frac{1}{v+w} = 0$$

obtained from the previous one by changing the sign of w. Relations (16) and (17) are used with the identity

$$\frac{1}{u-v}\,\frac{1}{u+w}-\frac{1}{u-v}\,\frac{1}{v+w}+\frac{1}{u+w}\,\frac{1}{v+w}=0$$

obtained from the previous one by changing the sign of u once again. Finally, relations (22) and (23) are used along with another identity, which is easy to verify,

$$\frac{1}{u-v}\,\frac{1}{u-w}\,\frac{1}{v+w}-\frac{1}{u-v}\,\frac{1}{u+w}\,\frac{1}{v-w}+\frac{1}{u+v}\,\frac{1}{u-w}\,\frac{1}{v-w}-\frac{1}{u+v}\,\frac{1}{u+w}\,\frac{1}{v+w}=0.$$

Let us verify relations (10)–(17). Relations (10) and (11) follow from the description of the action of P on the vector space $(\mathbb{C}^{N|N})^{\otimes 2}$ given in the beginning of this section. In turn, relations (12)–(17) follow from (10) and (11) by using the observation below. Let

$$J=\sum_i E_{i,-i}\,(-1)^{\,\bar{\imath}}\ \in \mathrm{End}\,\mathbb{C}^{N|N}.$$

Then, $Q=P\,(J\otimes J)$ by the definitions of P and Q. Note the equality $J^2=-1$. Because $\deg J=1$, in the algebra $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 3}$ for any $p\neq q$, we also have the equality

$$J_p\,J_q=-\,J_q\,J_p\,,$$

where we use the notation (4) for $n=3$ and $m=1$. We omit the details of verifying (12)–(17) in this fashion.

Next, let us verify relations (18)–(23). It follows from the above-mentioned description of the action of P on $(\mathbb{C}^{N|N})^{\otimes 2}$ that either side of (18) is equal to P_{13}. In turn, relations (19)–(23) follow from (18). Here, we again use our observation involving the element J. In particular, either side of (21) is equal to $-P_{13}$. Thus, the function (8) obeys Eq. (9).

V. THE YANGIAN

The *Yangian* of the Lie superalgebra $\mathfrak{q}_N$ is a complex associative unital algebra $\mathrm{Y}(\mathfrak{q}_N)$ with a set of generators

$$T_{ij}^{(r)}\quad\text{where}\quad r=1,2,\,\dots\quad\text{and}\quad i,j=\pm 1,\dots,\pm N.$$

The algebra $\mathrm{Y}(\mathfrak{q}_N)$ is $\mathbb{Z}_2$-graded so that $\deg T_{ij}^{(r)}=\bar{\imath}+\bar{\jmath}$ for all indices r. To write down defining relations for these generators of $\mathrm{Y}(\mathfrak{q}_N)$, we will use the formal power series in u^{-1} with coefficients from $\mathrm{Y}(\mathfrak{q}_N)$,

$$T_{ij}(u)=\delta_{ij}\cdot 1+T_{ij}^{(1)}u^{-1}+T_{ij}^{(2)}u^{-2}+\cdots.$$

Let us combine all these series into a single element

$$T(u)=\sum_{i,j}E_{ij}\otimes T_{ij}(u)$$

of the algebra $(\mathrm{End}\,\mathbb{C}^{N|N})\otimes \mathrm{Y}(\mathfrak{q}_N)[[u^{-1}]]$. Then, we will impose the relation

$$(\pi\otimes\mathrm{id})(T(u))=T(-u).$$

In terms of the series $T_{ij}(u)$, it means that for all i and j,

$$T_{-i,-j}(u)=T_{ij}(-u). \tag{24}$$

In terms of the generators $T_{ij}^{(r)}$, it simply means that

$$T_{-i,-j}^{(r)}=(-1)^r\,T_{ij}^{(r)}. \tag{25}$$

For any n and any $p=1,\dots,n$, we will denote

$$T_p(u)=(\iota_p\otimes\mathrm{id})(T(u))$$

in the algebra $(\operatorname{End}\mathbb{C}^{N|N})^{\otimes n}\otimes \mathrm{Y}(\mathfrak{q}_N)[[u^{-1}]]$. By using this notation for $n=2$, the remaining defining relations of the algebra $\mathrm{Y}(\mathfrak{q}_N)$ can be written as the single equation

$$(R(u,v)\otimes 1)\,T_1(u)\,T_2(v)=T_2(v)\,T_1(u)\,(R(u,v)\otimes 1). \tag{26}$$

By using definition (8), expanding Eq. (26) in the basis of $(\operatorname{End}\mathbb{C}^{N|N})^{\otimes 2}$ consisting of the vectors

$$E_{ij}\otimes E_{kl}\,(-1)^{\bar{\jmath}\bar{k}+\bar{\jmath}\bar{l}+\bar{k}\bar{l}}$$

with $i,j,k,l=\pm1,\ldots,\pm N$ yields the relations

$$[T_{ij}(u),T_{kl}(v)]\,(-1)^{\bar{\imath}\bar{k}+\bar{\imath}\bar{l}+\bar{k}\bar{l}}=\frac{T_{kj}(u)\,T_{il}(v)-T_{kj}(v)\,T_{il}(u)}{u-v}-\frac{T_{-k,j}(u)\,T_{-i,l}(v)-T_{k,-j}(v)\,T_{i,-l}(u)}{u+v}\,(-1)^{\bar{k}+\bar{l}} \tag{27}$$

in $\mathrm{Y}(\mathfrak{q}_N)[[u^{-1},v^{-1}]]$. The square brackets above stand for the supercommutator. The first fraction in (27) belongs to $\mathrm{Y}(\mathfrak{q}_N)[[u^{-1},v^{-1}]]$ because its numerator vanishes at $u-v=0$. The second fraction in (27) belongs to $\mathrm{Y}(\mathfrak{q}_N)[[u^{-1},v^{-1}]]$ because its numerator vanishes at $u+v=0$ by relations (24).

By comparing this definition of $\mathrm{Y}(\mathfrak{q}_N)$ with the above relations in the algebra $\mathrm{U}(\mathfrak{q}_N)$, it is direct to verify that a homomorphism $\mathrm{U}(\mathfrak{q}_N)\to\mathrm{Y}(\mathfrak{q}_N)$ can be defined by

$$f_{ij}\mapsto -T^{(1)}_{ji}\,(-1)^{\bar{\imath}}. \tag{28}$$

It is also straightforward to verify that a homomorphism $\mathrm{Y}(\mathfrak{q}_N)\to\mathrm{U}(\mathfrak{q}_N)$ can be defined by

$$T_{ij}(u)\mapsto \delta_{ij}-f_{ji}\,u^{-1}\,(-1)^{\bar{\jmath}}. \tag{29}$$

The homomorphism (29) is clearly surjective. Note that the composition of (28) with (29) is just the identity map $\mathrm{U}(\mathfrak{q}_N)\to\mathrm{U}(\mathfrak{q}_N)$. This implies that the homomorphism (28) is injective.

Furthermore, it follows from our definition of $\mathrm{Y}(\mathfrak{q}_N)$ that an antiautomorphism of $\mathrm{Y}(\mathfrak{q}_N)$ can be defined by mapping

$$T_{ij}(u)\mapsto \widetilde{T}_{ij}(u), \tag{30}$$

where the series $\widetilde{T}_{ij}(u)\in\mathrm{Y}(\mathfrak{q}_N)[[u^{-1}]]$ is defined by

$$T(u)^{-1}=\sum_{i,j}E_{ij}\otimes\widetilde{T}_{ij}(u). \tag{31}$$

Indeed, by dividing (26) on the left and right by $T_2(u)$ and then by $T_1(u)$, we get the relation

$$(R(u,v)\otimes 1)\,T_2(v)^{-1}\,T_1(u)^{-1}=T_1(u)^{-1}\,T_2(v)^{-1}\,(R(u,v)\otimes 1).$$

Comparing this with (26) verifies the antiautomorphism property of the map (30); see also the relation (24).

It also follows from (24) and (31) that for all i and j,

$$\widetilde{T}_{-i,-j}(u)=\widetilde{T}_{ij}(-u). \tag{32}$$

There is a natural Hopf algebra structure on $\mathrm{Y}(\mathfrak{q}_N)$. A coassociative comultiplication homomorphism,

$$\Delta:\mathrm{Y}(\mathfrak{q}_N)\to\mathrm{Y}(\mathfrak{q}_N)\otimes\mathrm{Y}(\mathfrak{q}_N),$$

is defined by

$$\Delta:T_{ij}(u)\;\mapsto\;\sum_k T_{ik}(u)\otimes T_{kj}(u)\,(-1)^{(\bar{\imath}+\bar{k})(\bar{\jmath}+\bar{k})}$$

where the tensor product is over the subalgebra $\mathbb{C}[[u^{-1}]]$ of $\mathrm{Y}(\mathfrak{q}_N)[[u^{-1}]]$. Furthermore, the counit homomorphism $\mathrm{Y}(\mathfrak{q}_N)\to\mathbb{C}$ is defined by mapping $T_{ij}(u)\mapsto\delta_{ij}$. The antipodal map $\mathrm{Y}(\mathfrak{q}_N)\to\mathrm{Y}(\mathfrak{q}_N)$ is the antiautomorphism (30). Justification of all these definitions is standard.[3]

The antipodal map of any Hopf algebra is a coalgebra antihomomorphism as well. Hence, for any indices i and j,

$$\Delta : \widetilde{T}_{ij}(u) \mapsto \sum_k \widetilde{T}_{kj}(u) \otimes \widetilde{T}_{ik}(u). \tag{33}$$

Note that (28) is a homomorphism of $\mathbb{Z}_2$-graded Hopf algebras $\mathrm{U}(\mathfrak{q}_N) \to \mathrm{Y}(\mathfrak{q}_N)$. However, (29) is a homomorphism of $\mathbb{Z}_2$-graded associative algebras $\mathrm{Y}(\mathfrak{q}_N) \to \mathrm{U}(\mathfrak{q}_N)$ only, *not* a homomorphism of Hopf algebras.

We can naturally define two ascending $\mathbb{Z}$-filtrations on the algebra $\mathrm{Y}(\mathfrak{q}_N)$. The first $\mathbb{Z}$-filtration is defined by setting to r the degree of $T_{ij}^{(r)}$ for every i and j. Consider the corresponding $\mathbb{Z}$-graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N)$. It is also $\mathbb{Z}_2$-graded. It follows from (27) that the algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N)$ is supercommutative. By Ref. 3 (Corollary 2.4), the elements corresponding to $T_{ij}^{(r)}$ with $i > 0$ are free generators of this supercommutative algebra. Their freeness will also follow from the argument at the end of this section.

The second $\mathbb{Z}$-filtration on the algebra $\mathrm{Y}(\mathfrak{q}_N)$ is defined by setting the degree of $T_{ij}^{(r)}$ to $r-1$. Let $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{q}_N)$ be the corresponding $\mathbb{Z}$-graded algebra. It is $\mathbb{Z}_2$-graded too. Let $t_{ij}^{(r)}$ be the element of $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{q}_N)$ defined by $T_{ij}^{(r)}$. Hence,

$$t_{-i,-j}^{(r)} = (-1)^r\, t_{ij}^{(r)}$$

by (25). For $r, s \geqslant 1$, by taking coefficients at $u^{-r}v^{-s}$ in (27), we get the supercommutation relations in $\mathrm{gr}'\,\mathrm{Y}(\mathfrak{q}_N)$,

$$[t_{ij}^{(r)}, t_{kl}^{(s)}]\,(-1)^{\bar{\imath}\bar{k} + \bar{\imath}\bar{l} + \bar{k}\bar{l}} = \delta_{kj}\, t_{il}^{(r+s-1)} - t_{kj}^{(r+s-1)}\,\delta_{il} + (\delta_{-k,j}\, t_{-i,l}^{(r+s-1)} - t_{k,-j}^{(r+s-1)}\,\delta_{i,-l})\,(-1)^{\bar{k}+\bar{l}+r}.$$

These imply that for the Lie superalgebra (1), a surjective homomorphism $\mathrm{U}(\mathfrak{g}) \to \mathrm{gr}'\,\mathrm{Y}(\mathfrak{q}_N)$ is defined by mapping

$$e_{ij}\, u^{r-1} + e_{-i,-j}\,(-u)^{r-1} \mapsto -\, t_{ji}^{(r)}\,(-1)^{\bar{\imath}}. \tag{34}$$

By Ref. 3 (Theorem 2.3), this homomorphism is injective too. The injectivity will also follow from the argument below.

Denote by γ_n the homomorphism $\mathrm{Y}(\mathfrak{q}_N) \to \mathrm{U}(\mathfrak{q}_N)^{\otimes n}$ defined by using the comultiplication $\mathrm{Y}(\mathfrak{q}_N) \to \mathrm{Y}(\mathfrak{q}_N)^{\otimes n}$ first and then applying the homomorphism (29) to each tensor factor of $\mathrm{Y}(\mathfrak{q}_N)^{\otimes n}$. Let us prove that the kernels of all the homomorphisms γ_n with $n = 1, 2, \ldots$ have zero intersection. We will follow Ref. 8 where the Yangian of the general linear Lie superalgebra $\mathfrak{gl}_{M|N}$ was considered.

The algebra $\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$ is generated by the elements

$$\iota_p(f_{ij}) \quad \text{where} \quad p = 1, \ldots, n \quad \text{and} \quad i, j = \pm 1, \ldots, \pm N.$$

Here, we use the notation of Sec. II with $\mathrm{A} = \mathrm{U}(\mathfrak{q}_N)$. Define an ascending $\mathbb{Z}$-filtration on the algebra $\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$ in the standard way, that is, by setting to 1 the degrees of all the above generators. Consider the corresponding $\mathbb{Z}$-graded algebra $\mathrm{gr}\,\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$. It is also $\mathbb{Z}_2$-graded and then supercommutative. Let $x_{ij}^{(p)}$ be the elements of this algebra, corresponding to the above displayed generators of $\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$. Note that for any indices i and j, we have

$$x_{-i,-j}^{(p)} = x_{ij}^{(p)}. \tag{35}$$

Because of the Poincaré–Birkhoff–Witt theorem for the Lie superalgebra $\mathfrak{q}_N$, the elements $x_{ij}^{(p)}$ with $i > 0$ are free generators of the supercommutative algebra $\mathrm{gr}\,\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$.

By the definition of comultiplication on $\mathrm{Y}(\mathfrak{q}_N)$, our γ_n maps the series $T_{ij}(u)$ to $(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}$ times the sum over $k_1, \ldots, k_{n-1} = \pm 1, \ldots, \pm N$ of the tensor products,

$$\begin{aligned}
&(\delta_{ik_1} - f_{k_1 i}\, u^{-1}(-1)^{\bar{\imath}\bar{k}_1}) \otimes \\
&(\delta_{k_1 k_2} - f_{k_2 k_1}\, u^{-1}(-1)^{\bar{k}_1 \bar{k}_2}) \otimes \\
&\qquad \vdots \\
&(\delta_{k_{n-2} k_{n-1}} - f_{k_{n-1} k_{n-2}}\, u^{-1}(-1)^{\bar{k}_{n-2}\bar{k}_{n-1}}) \otimes \\
&(\delta_{k_{n-1} j} - f_{j k_{n-1}}\, u^{-1}(-1)^{\bar{k}_{n-1}\bar{\jmath}}),
\end{aligned}$$

where we also used the definition (29). Hence, $T_{ij}^{(r)}$ with $r \leqslant n$ obtained by γ_n to $(-1)^{\bar{\imath}\,\bar{\jmath}\,+\bar{\jmath}\,+r}$ times the sum over $k_1,\dots,k_{r-1} = \pm 1,\dots,\pm N$ and $1 \leqslant p_1 < \dots < p_r \leqslant n$ of the products in $\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$,

$$\iota_{p_1}(f_{k_1 i})\,\iota_{p_2}(f_{k_2 k_1})\,\dots\,\iota_{p_{r-1}}(f_{k_{r-1}k_{r-2}})\,\iota_{p_r}(f_{jk_{r-1}})(-1)^{\bar{\imath}\,\bar{k}_1+\bar{k}_1\bar{k}_2+\dots+\bar{k}_{r-2}\,\bar{k}_{r-1}+\bar{k}_{r-1}\,\bar{\jmath}}.$$

If $r > n$, then $T_{ij}^{(r)}$ is annihilated by γ_n. Note that γ_n is also a homomorphism of $\mathbb{Z}$-filtered algebras relative to the first filtration on $\mathrm{Y}(\mathfrak{q}_N)$.

The element of the algebra $\mathrm{gr}\,\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$ corresponding to the last displayed product in $\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$ is by definition

$$x_{k_1 i}^{(p_1)}\,x_{k_2 k_1}^{(p_2)}\,\dots\,x_{k_{r-1}k_{r-2}}^{(p_{r-1})}\,x_{jk_{r-1}}^{(p_r)}(-1)^{\bar{\imath}\,\bar{k}_1+\bar{k}_1\bar{k}_2+\dots+\bar{k}_{r-2}\,\bar{k}_{r-1}+\bar{k}_{r-1}\,\bar{\jmath}}. \tag{36}$$

Let $y_{ij}^{(r)}$ be the sum over all $k_1,\dots,k_{r-1} = \pm 1,\dots,\pm N$ and all $1 \leqslant p_1 < \dots < p_r \leqslant n$ of the products (36) in the algebra $\mathrm{gr}\,\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$ multiplied by $(-1)^{\bar{\imath}\,\bar{\jmath}\,+\bar{\jmath}}$. We have

$$y_{-i,-j}^{(r)} = (-1)^r\,y_{ij}^{(r)}.$$

We can also take the element of the $\mathbb{Z}$-graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N)$ corresponding to $(-1)^r\,T_{ij}^{(r)}$. Its image relative to the homomorphism $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N) \to \mathrm{gr}\,\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$ defined by γ_n coincides with $y_{ij}^{(r)}$. However, we do not need this fact.

We will prove that the supercommutative monomials in the elements $y_{ij}^{(r)}$ with $i > 0$ and $r \leqslant n$ are all linearly independent. Hence, the kernels of the homomorphisms γ_n with $n = 1,2,\dots$ will have zero intersection. Moreover, the freeness property of the supercommutative algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_N)$ stated above will then follow too. Furthermore, the injectivity of homomorphism (34) follows from linear independence of those monomials for every n.

Let $i = 1,\dots,N$ and $j = \pm 1,\dots,\pm N$. Fix any total ordering of the triples (i,j,r) where $r = 1,\dots,n$. Using this ordering, form a matrix of the left superderivatives,

$$\partial\,y_{ij}^{(r)}\,/\,\partial\,x_{kl}^{(p)}, \tag{37}$$

where the triples (k,l,p) range over the same ordered set as the triples (i,j,r). Fix complex numbers $x_1,\dots,x_n$ so that $x_r \pm x_p \neq 0$ for all $r < p$. Due to freeness of the supercommutative algebra $\mathrm{gr}\,\mathrm{U}(\mathfrak{q}_N)^{\otimes n}$, we can specialize

$$x_{kl}^{(p)} = x_p\,\delta_{kl} \tag{38}$$

in the matrix of superderivatives (37). It suffices to show that the determinant of the specialized matrix is not zero.

For $r \geqslant 1$, take the *elementary* symmetric polynomial

$$\sigma_r(x_1,\dots,x_n) = \sum_{p_1<\dots<p_r} x_{p_1}\dots x_{p_r}.$$

We will assume that $\sigma_0(x_1,\dots,x_n) = 1$. If $j > 0$, then the specialization of superderivative (37) and (38) equals

$$\sigma_{r-1}(x_1,\dots,x_{p-1},x_{p+1},\dots,x_n)\,\delta_{il}\,\delta_{kj}.$$

If $j < 0$, then the specialization of (37) and (38) equals

$$-\,\sigma_{r-1}(x_1,\dots,x_{p-1},-\,x_{p+1},\dots,\,-x_n)\,\delta_{i,-l}\,\delta_{k,-j}.$$

Here, we used (35) and the condition that $i,k > 0$ in (37).

A detailed calculation from the proof of Theorem 1 of Ref. 8 shows that the determinant of the matrix formed by

$$\sigma_{r-1}(x_1,\dots,x_{p-1},x_{p+1},\dots,x_n),$$

with $r,p = 1,\dots,n$, is equal to the product

$$\prod_{r<p}(x_r - x_p).$$

Similarly, the determinant of the matrix formed by

$$\sigma_{r-1}(x_1,\dots,x_{p-1},-\,x_{p+1},\dots,\,-x_n), \tag{39}$$

with $r, p = 1, \ldots, n$, is equal to the product

$$\prod_{r<p} (x_r + x_p).$$

One reduces the latter calculation by taking differences of adjacent columns of the matrix of (39). Both products are not zero due to our choice of the numbers $x_1, \ldots, x_n$. Hence, the determinant of the matrix of (37) is not zero.

VI. THE CENTER

There is a natural family of central elements of $\mathrm{Y}(\mathfrak{q}_N)$. To define it, let us consider the antiautomorphism τ of the $\mathbb{Z}_2$-graded associative algebra $\mathrm{End}\,\mathbb{C}^{N|N}$ defined by

$$\tau : E_{ij} \mapsto E_{ji}\,(-1)^{\,\bar\imath\,\bar\jmath\,+\,\bar\imath}.$$

Take the element of the algebra $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 2}$,

$$K = \sum_{i,j} E_{ij} \otimes E_{ij}\,(-1)^{\,\bar\imath\,\bar\jmath}.$$

Then, we have the equalities

$$K = (\tau \otimes \mathrm{id})(P) = (\mathrm{id} \otimes \tau)(P). \tag{40}$$

Note that the image of the action of K on the vector space $(\mathbb{C}^{N|N})^{\otimes 2}$ is one dimensional. This image is spanned by

$$\sum_i e_i \otimes e_i.$$

Here, we again identify the algebra $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 2}$ with $\mathrm{End}\,((\mathbb{C}^{N|N})^{\otimes 2})$ by using (2). Furthermore, take the element

$$L = \sum_{i,j} E_{-i,-j} \otimes E_{ij}\,(-1)^{\,\bar\imath\,\bar\jmath}.$$

Similar to (40), we have the equalities in $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 2}$,

$$L = (\tau \otimes \mathrm{id})(Q) = (\mathrm{id} \otimes \tau)(Q). \tag{41}$$

The image of the action of L on $(\mathbb{C}^{N|N})^{\otimes 2}$ is again one dimensional and spanned by the vector

$$\sum_i e_i \otimes e_{-i}\,(-1)^{\,\bar\imath}.$$

Now, consider a function of complex variables u, v with values in the algebra $(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 2}$,

$$S(u,v) = 1 - \frac{K}{u-v} + \frac{L}{u+v}.$$

Then, due to (8) and to (40) and (41), we have the equalities

$$S(u,v) = (\tau \otimes \mathrm{id})(R(u,v)) = (\mathrm{id} \otimes \tau)(R(u,v)).$$

Note that by the above definitions of K and L, we have

$$K\,L = L\,K = 0 \quad \text{and} \quad K^2 = L^2 = 0.$$

These equalities imply that

$$S(u,v)\,S(-u,-v) = 1. \tag{42}$$

We will use the notation

$$T'(u) = (\tau \otimes \mathrm{id})(T(u)^{-1})).$$

Let us now divide (26) on the left and right by $T_2(u)$ and apply to the resulting relation in the algebra,

$$(\mathrm{End}\,\mathbb{C}^{N|N})^{\otimes 2} \otimes \mathrm{Y}(\mathfrak{q}_N)\,[[u^{-1}, v^{-1}]],$$

the antiautomorphism τ relative to the second tensor factor $\mathrm{End}\,\mathbb{C}^{N|N}$. We get the relation

$$(S(-u,-v) \otimes 1)\, T_1(u)\, T_2'(v) = T_2'(v)\, T_1(u)\, (S(-u,-v) \otimes 1) \tag{43}$$

where we also used equality (42). Next, let us multiply relation (43) by $v - u$ and then set $v = u$. We get

$$(K \otimes 1)\, T_1(u)\, T_2'(u) = T_2'(u)\, T_1(u)\, (K \otimes 1). \tag{44}$$

Because the image of the action of K on $(\mathbb{C}^{N|N})^{\otimes 2}$ is one dimensional, either side of relation (44) equals $K \otimes Z(u)$ for some series $Z(u) \in \mathrm{Y}(\mathfrak{q}_N)\,[[u^{-1}]]$. By using the definition of K and expanding (44), we get

$$\sum_k T_{ki}(u)\, \widetilde{T}_{jk}(u) = Z(u)\, \delta_{ij}, \tag{45}$$
$$\sum_k \widetilde{T}_{jk}(u)\, T_{ki}(u) = Z(u)\, \delta_{ij}.$$

Relations (24) now imply that $Z(u) = Z(-u)$. Hence,

$$Z(u) = 1 + Z_2 u^{-2} + Z_4 u^{-4} + \cdots$$

for some elements $Z_2, Z_4, \ldots \in \mathrm{Y}(\mathfrak{q}_N)$ of $\mathbb{Z}_2$-degree 0. By Proposition 3.1 of Ref. 3, all these elements are central in $\mathrm{Y}(\mathfrak{q}_N)$. The centrality of any $X \in \mathrm{Y}(\mathfrak{q}_N)$ means that the supercommutator $[X, Y] = 0$ for every $Y \in \mathrm{Y}(\mathfrak{q}_N)$. By Proposition 3.5 of Ref. 3, for $r \geqslant 1$, the element of the algebra $\mathrm{gr}'\mathrm{Y}(\mathfrak{q}_N)$ corresponding to $Z_{2r} \in \mathrm{Y}(\mathfrak{q}_N)$ is $(2r-1)$ times

$$\sum_i t_{ii}^{(2r-1)}\, (-1)^{\bar{\imath}}. \tag{46}$$

Through the isomorphism $\mathrm{U}(\mathfrak{g}) \to \mathrm{gr}'\mathrm{Y}(\mathfrak{q}_N)$ defined by (34), the element (46) corresponds to

$$-\sum_i (e_{ii} + e_{-i,-i})\, u^{2r-2} \in \mathfrak{g}. \tag{47}$$

By Proposition 3.6 of Ref. 3, the elements (47) with $r = 1, 2, \ldots$ freely generate the center of $\mathrm{U}(\mathfrak{g})$. Hence, the elements $Z_2, Z_4, \ldots$ freely generate the center of $\mathrm{Y}(\mathfrak{q}_N)$.

Our $Z(u)$ is also comultiplicative for $\mathrm{Y}(\mathfrak{q}_N)$, that is,

$$\Delta : Z(u) \mapsto Z(u) \otimes Z(u). \tag{48}$$

Indeed, by setting $j = i$ in (45) and then employing (33), the comultiplication maps $Z(u)$ to

$$\begin{aligned}\sum_{h,k,l} (T_{kl}(u) \otimes T_{li}(u))\, (\widetilde{T}_{hk}(u) \otimes \widetilde{T}_{ih}(u))(-1)^{(\bar{\imath}+\bar{l})(\bar{k}+\bar{l})} &= \sum_{h,k,l} (T_{kl}(u)\, \widetilde{T}_{hk}(u)) \otimes (T_{li}(u))\, \widetilde{T}_{ih}(u)\,)(-1)^{(\bar{\imath}+\bar{l})(\bar{h}+\bar{l})} \\ &= \sum_{h,l} (Z(u)\, \delta_{hl}) \otimes (T_{li}(u))\, \widetilde{T}_{ih}(u)\,)\, (-1)^{(\bar{\imath}+\bar{l})(\bar{h}+\bar{l})},\end{aligned}$$

which is equal to the right-hand side of (48) as we stated. Here, we used relations (45) again.

Note that $Z(u) \mapsto 1$ by the counit map $\mathrm{Y}(\mathfrak{q}_N) \to \mathbb{C}$. Due to the axioms of a Hopf algebra, it now follows from (48) that $Z(u) \mapsto Z(u)^{-1}$ under the antipodal map. The square of the antipodal map is always a homomorphism of associative algebras. By Proposition 3.2 of Ref. 3, under this homomorphism of $\mathrm{Y}(\mathfrak{q}_N)$, for any indices i and j,

$$T_{ij}(u) \mapsto Z(u)^{-1}\, T_{ij}(u).$$

VII. THE CASE OF $N = 1$

Let us now consider in more detail the Yangian $\mathrm{Y}(\mathfrak{q}_1)$. For short, we will denote

$$A(u) = T_{11}(u) \ \text{and} \ B(u) = T_{1,-1}(u).$$

The coefficients of the series $A(u)$ and $B(u)$ in $\mathrm{Y}(\mathfrak{q}_1)$ are of $\mathbb{Z}_2$-degrees 0 and 1, respectively. It follows from (24) that the algebra $\mathrm{Y}(\mathfrak{q}_1)$ is generated by these coefficients.

By (27), the supercommutation relations below define the Yangian $\mathrm{Y}(\mathfrak{q}_1)$ as an associative $\mathbb{Z}_2$-graded algebra,

$$[A(u), A(v)] = \frac{A(u)\,A(v) - A(v)\,A(u)}{u-v} - \frac{B(-u)\,B(-v) - B(v)\,B(u)}{u+v}, \tag{49}$$

$$[A(u), B(v)] = \frac{A(u)\,B(v) - A(v)\,B(u)}{u-v} + \frac{B(-u)\,A(-v) - B(v)\,A(u)}{u+v}, \tag{50}$$

$$[B(u), A(v)] = \frac{B(u)\,A(v) - B(v)\,A(u)}{u-v} - \frac{A(-u)\,B(-v) - A(v)\,B(u)}{u+v}, \tag{51}$$

$$[B(u), B(v)] = \frac{B(u)\,B(v) - B(v)\,B(u)}{u-v} + \frac{A(-u)\,A(-v) - A(v)\,A(u)}{u+v}. \tag{52}$$

By our definitions, the comultiplication on $\mathrm{Y}(\mathfrak{q}_1)$ maps

$$A(u) \mapsto A(u) \otimes A(u) - B(u) \otimes B(-u), \tag{53}$$

$$B(u) \mapsto A(u) \otimes B(u) + B(u) \otimes A(-u). \tag{54}$$

Furthermore, let us denote

$$\widetilde{A}(u) = \widetilde{T}_{11}(u) \ \text{and} \ \widetilde{B}(u) = \widetilde{T}_{1,-1}(u).$$

Hence, $\widetilde{A}(u)$ and $\widetilde{B}(u)$ are, respectively, the images of $A(u)$ and $B(u)$ by the antiautomorphism (30) of $\mathrm{Y}(\mathfrak{q}_1)$. Due to (32) and (33), the comultiplication on $\mathrm{Y}(\mathfrak{q}_1)$ maps

$$\widetilde{A}(u) \mapsto \widetilde{A}(u) \otimes \widetilde{A}(u) + \widetilde{B}(-u) \otimes \widetilde{B}(u), \tag{55}$$

$$\widetilde{B}(u) \mapsto \widetilde{B}(u) \otimes \widetilde{A}(u) + \widetilde{A}(-u) \otimes \widetilde{B}(u). \tag{56}$$

Consider the homomorphism $\mathrm{Y}(\mathfrak{q}_1) \to \mathrm{U}(\mathfrak{q}_1)$ defined by (29). It maps

$$A(u) \mapsto \frac{u-a}{u} \quad \text{and} \quad B(u) \mapsto \frac{b}{u}. \tag{57}$$

By definition (31), we have the following two equations:

$$A(u)\,\widetilde{A}(u) - B(u)\,\widetilde{B}(-u) = 1, \tag{58}$$

$$B(-u)\,\widetilde{A}(u) + A(-u)\,\widetilde{B}(-u) = 0. \tag{59}$$

These two equations determine $\widetilde{A}(u)$ and $\widetilde{B}(-u)$ uniquely by $A(u), B(u)$ and $A(-u), B(-u)$. Using these equations along with (57), the homomorphism (29) for $N = 1$ maps

$$\widetilde{A}(u) \mapsto \frac{(u+a)\,u}{u^2 - a^2 - a}, \tag{60}$$

$$\widetilde{B}(-u) \mapsto \frac{b\,u}{u^2 - a^2 - a}. \tag{61}$$

We also employ the centrality of a and the relation $b^2 = a$ in $\mathrm{U}(\mathfrak{q}_1)$ but omit the details of this direct calculation.

For $N = 1$, by setting $i = j = 1$ in (45), we obtain

$$Z(u) = A(u)\,\widetilde{A}(u) + B(-u)\,\widetilde{B}(u).$$

By (57), (60), and (61), the homomorphism (29) maps $Z(u)$ to

$$\frac{u-a}{u}\,\frac{(u+a)\,u}{u^2-a^2-a} - \frac{b}{u}\,\frac{-\,b\,u}{u^2-a^2-a} = \frac{u^2-a^2}{u^2-a^2-a} + \frac{b^2}{u^2-a^2-a} = \frac{u^2-a^2+a}{u^2-a^2-a}.$$

VIII. THE QUANTUM BEREZINIAN

In this section, we will introduce a family of generators of the center of $\mathrm{Y}(\mathfrak{q}_1)$, different from the family provided for $N = 1$ by the coefficients of the series $Z(u)$. Denote

$$C(u) = A(u)\,\widetilde{A}(-u) \quad \text{and} \quad D(u) = B(u)\,\widetilde{B}(u). \tag{62}$$

The coefficients of these two series are of $\mathbb{Z}_2$-degree zero. The series $C(u)$ will be called the *quantum Berezinian* for the Yangian $\mathrm{Y}(\mathfrak{q}_1)$. To justify this terminology, consider the $\mathbb{Z}$-graded algebra $\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_1)$. Take the image of $C(u)$ in the supercommutative algebra $(\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_1))[[u^{-1}]]$. Relations (58) and (59) imply that the matrix

$$\begin{bmatrix} T_{11}(u) & -\,T_{1,-1}(u) \\ T_{-1,1}(u) & T_{-1,-1}(u) \end{bmatrix} = \begin{bmatrix} A(u) & -B(u) \\ B(-u) & A(-u) \end{bmatrix} \tag{63}$$

with entries from $\mathrm{Y}(\mathfrak{q}_1)[[u^{-1}]]$ has the inverse matrix

$$\begin{bmatrix} \widetilde{T}_{11}(u) & -\,\widetilde{T}_{1,-1}(u) \\ \widetilde{T}_{-1,1}(u) & \widetilde{T}_{-1,-1}(u) \end{bmatrix} = \begin{bmatrix} \widetilde{A}(u) & -\widetilde{B}(u) \\ \widetilde{B}(-u) & \widetilde{A}(-u) \end{bmatrix}.$$

Hence, the image of $C(u)$ is the Berezinian of the matrix with the entries from $(\mathrm{gr}\,\mathrm{Y}(\mathfrak{q}_1))[[u^{-1}]]$ corresponding to the entries of (63). It is also called the *superdeterminant*.

We will show that coefficients of each of series $C(u)$ and $D(u)$ generate the center of $\mathrm{Y}(\mathfrak{q}_1)$. We will also link the two series to each other and to $Z(u)$ at $N = 1$. We will use the fact that the comultiplication on $\mathrm{Y}(\mathfrak{q}_1)$ maps

$$C(u) \mapsto C(u) \otimes C(u) + D(u) \otimes D(-u), \tag{64}$$

$$D(u) \mapsto C(u) \otimes D(u) + D(u) \otimes C(-u). \tag{65}$$

Indeed, by (53)–(56), the comultiplication on $\mathrm{Y}(\mathfrak{q}_1)$ maps $C(u)$ to the product in $\mathrm{Y}(\mathfrak{q}_1)^{\otimes 2}[[u^{-1}]]$,

$$\begin{aligned} &(A(u) \otimes A(u) - B(u) \otimes B(-u))\,(\widetilde{A}(-u) \otimes \widetilde{A}(-u) + \widetilde{B}(u) \otimes \widetilde{B}(-u)) \\ &\quad = (A(u)\,\widetilde{A}(-u)) \otimes (A(u)\,\widetilde{A}(-u)) + (B(u)\,\widetilde{B}(u)) \otimes (B(-u)\,\widetilde{B}(-u)) + (A(u)\,\widetilde{B}(u)) \otimes (A(u)\,\widetilde{B}(-u)) \\ &\qquad - (B(u)\,\widetilde{A}(-u)) \otimes (B(-u)\,\widetilde{A}(-u)). \end{aligned}$$

The last two displayed tensor products cancel each other due to relation (59) and the relation

$$A(u)\,\widetilde{B}(-u) + B(-u)\,\widetilde{A}(-u) = 0 \tag{66}$$

obtained by setting $i = 1$ and $j = 2$ in (45) when $N = 1$. The sum of the preceding two tensor products is the right-hand side of (64) by definition. Similarly, the comultiplication maps $D(u)$ to

$$\begin{aligned} &(A(u) \otimes B(u) + B(u) \otimes A(-u))\,(\widetilde{B}(u) \otimes \widetilde{A}(u) + \widetilde{A}(-u) \otimes \widetilde{B}(u)) \\ &\quad = (B(u)\,\widetilde{A}(-u)) \otimes (A(-u)\,\widetilde{B}(u)) - (A(u)\,\widetilde{B}(u)) \otimes (B(u)\,\widetilde{A}(u)) + (A(u)\,\widetilde{A}(-u)) \otimes (B(u)\,\widetilde{B}(u)) \\ &\qquad + (B(u)\,\widetilde{B}(u)) \otimes (A(-u)\,\widetilde{A}(u)). \end{aligned}$$

The first two tensor products at the right-hand side of the last display cancel each other by the relations (59) and (66). The sum of the next two tensor products is the right-hand side of (65). Hence, we have (64) and (65).

Now, consider the matrix with entries from $\mathrm{Y}(\mathfrak{q}_1)[[u^{-1}]]$,

$$\begin{bmatrix} C(u) & D(u) \\ D(-u) & C(-u) \end{bmatrix}. \tag{67}$$

The two assignments (64) and (65) imply that applying the comultiplication on $\mathrm{Y}(\mathfrak{q}_1)$ to this matrix amounts to multiplying (67) by itself as a matrix while taking tensor products of entries instead of usual multiplication. This means that the matrix (67) is *comultiplicative* for $\mathrm{Y}(\mathfrak{q}_1)$.

Due to (62), the antiautomorphism (30) of $\mathrm{Y}(\mathfrak{q}_1)$ maps

$$C(u) \mapsto (Z(-u)^{-1}A(-u))\,\widetilde{A}(u) = Z(u)^{-1}C(-u),$$

$$D(u) \mapsto -(Z(u)^{-1}B(u))\,\widetilde{B}(u) = -Z(u)^{-1}D(u).$$

Here, we used the relation $Z(-u) = Z(u)$. We also used the description of the square of the antiautomorphism (30) of $\mathrm{Y}(\mathfrak{q}_1)$; see the very end of Sec. IV. Hence, the antiautomorphism (30) of $\mathrm{Y}(\mathfrak{q}_1)$ maps the matrix (67) to

$$Z(u)^{-1}\begin{bmatrix} C(-u) & -D(u) \\ -D(-u) & C(u) \end{bmatrix}.$$

On the other hand, by the axioms of the anipodal, map the comultiplicativity of the matrix (67) for $\mathrm{Y}(\mathfrak{q}_1)$ implies that (30) inverts this matrix. By equating to the identity matrix the product of (67) with the last displayed matrix, we obtain the relations $C(u)D(u) = D(u)C(u)$ and

$$C(u)\,C(-u) - D(u)\,D(-u) = Z(u). \tag{68}$$

Thus, $Z(u)$ is equal to the determinant of the matrix (67). This yields explicit expressions for the coefficients of the series $Z(u)$ in terms of those of the series $C(u)$ and $D(u)$. Pairwise commutativity of all entries of the matrix (67) will follow from centrality of their coefficients in $\mathrm{Y}(\mathfrak{q}_1)$.

Due to (57), (60), and (61), the homomorphism (29) maps

$$C(u) \mapsto \frac{u-a}{u}\,\frac{(u-a)\,u}{u^2-a^2-a} = \frac{(u-a)^2}{u^2-a^2-a},$$

$$D(u) \mapsto \frac{b}{u}\,\frac{-\,b\,u}{u^2-a^2-a} = \frac{-\,a}{u^2-a^2-a}.$$

Hence, the matrix (67) gets mapped by (29) to the matrix

$$\frac{1}{u^2-a^2-a}\begin{bmatrix} (u-a)^2 & -\,a \\ -\,a & (u+a)^2 \end{bmatrix}.$$

According to general conventions of Sec. II for each $p = 1,\dots,n$, denote $a_p = \iota_p(a)$ in the algebra $\mathrm{U}(\mathfrak{q}_1)^{\otimes n}$. Then, $\deg a_p = 0$ relative to the $\mathbb{Z}_2$-grading on $\mathrm{U}(\mathfrak{q}_1)^{\otimes n}$. Hence, the elements $a_1,\dots,a_n$ commute with each other.

Consider the homomorphism $\gamma_n : \mathrm{Y}(\mathfrak{q}_1) \to \mathrm{U}(\mathfrak{q}_1)^{\otimes n}$ defined as in Sec. V but for $N = 1$. The arguments above prove that γ_n maps the matrix (67) to the product over $p = 1,\dots,n$ of the matrices

$$\frac{1}{u^2-a_p^2-a_p}\begin{bmatrix} (u-a_p)^2 & -\,a_p \\ -\,a_p & (u+a_p)^2 \end{bmatrix}. \tag{69}$$

The matrices (69) commute, so the ordering of the factors in the product does not matter. Moreover, the entries of the product of all these n matrices are rational functions of u with values in the ring symmetric of polynomials in $a_1,\dots,a_n$ with complex coefficients.

Note that conjugating each matrix (69) by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

amounts to changing the sign of u in it. Thus, the product of the n matrices (69) can be written as the matrix

$$\begin{bmatrix} \varphi_n(u) & \psi_n(u) \\ \psi_n(-u) & \varphi_n(-u) \end{bmatrix}$$

for certain $\varphi_n(u)$ and $\psi_n(u)$. Here, $\psi_n(-u) = \psi_n(u)$ as all n matrices (69) are symmetric and pairwise commute. Moreover, it is easy to verify by induction on n that

$$\varphi_n(u) - \varphi_n(-u) = 4\,u\,\psi_n(u).$$

Indeed, for $n = 1$, this relation is obvious. If $n > 1$ and this relation holds for $n - 1$ instead of n, then we have

$$\begin{aligned}(u^2 - a_p^2 - a_p)\,(\varphi_n(u) - \varphi_n(-u)) &= (u - a_p)^2\,\varphi_{n-1}(u) - a_p\,\psi_n(u)\; - (u + a_p)^2\,\varphi_{n-1}(-u) + a_p\,\psi_n(-u)\\ &= (u^2 + a_p^2)\,(\varphi_{n-1}(u) - \varphi_{n-1}(-u))\; - 2\,u\,a_p\,(\varphi_{n-1}(u) + \varphi_{n-1}(-u))\\ &= (u + a_p)^2\,(\varphi_{n-1}(u) - \varphi_{n-1}(-u))\; - 2\,u\,a_p\,(\varphi_{n-1}(u) - \varphi_{n-1}(-u)) - 2\,u\,a_p\,(\varphi_{n-1}(u) + \varphi_{n-1}(-u))\\ &= 4\,u\,(u + a_p)^2\,\psi_{n-1}(u) - 4\,u\,a_p\,\varphi_{n-1}(u)\\ &= 4\,u\,(u^2 - a_p^2 - a_p)\,\psi_n(u).\end{aligned}$$

The kernels of homomorphisms γ_n with $n = 1, 2, \dots$ have zero intersection in $\mathrm{Y}(\mathfrak{q}_1)$, see the end of Sec. V. Hence we get the relations $D(-u) = D(u)$ and

$$C(u) - C(-u) = 4\,u\,D(u) \tag{70}$$

in $\mathrm{Y}(\mathfrak{q}_1)[[u^{-1}]]$. Moreover, because for any n the images of the coefficients of the series $C(u)$ and $D(u)$ relative to the homomerphism γ_n belong to the centre of $\mathrm{U}(\mathfrak{q}_1)^{\otimes n}$, the coefficients themselves belong to the centre of $\mathrm{Y}(\mathfrak{q}_1)$.

Just by the definitions (62) we have the expansions

$$C(u) = 1 + C_1 u^{-1} + C_2 u^{-2} + \cdots$$

for certain central elements $C_1, C_2, \dots \in \mathrm{Y}(\mathfrak{q}_1)$ and

$$D(u) = D_2 u^{-2} + D_4 u^{-4} + \cdots$$

for another central elements $D_2, D_4, \dots \in \mathrm{Y}(\mathfrak{q}_1)$. Here, we also used the relation $D(-u) = D(u)$. By (70), we get

$$C_1 = 2D_2,\;\; C_3 = 2D_4,\;\; \dots .$$

Now consider the $\mathbb{Z}$-graded algebra $\mathrm{gr}'\mathrm{Y}(\mathfrak{q}_1)$ defined as in Sec. V but for $N = 1$. Take the twisted current Lie superalgebra (1) with $N = 1$. For $r \geqslant 1$, consider the element of $\mathrm{gr}'\mathrm{Y}(\mathfrak{q}_1)$ corresponding to $C_{2r-1} \in \mathrm{Y}(\mathfrak{q}_1)$. Through the isomorphism $\mathrm{U}(\mathfrak{g}) \to \mathrm{gr}'\mathrm{Y}(\mathfrak{q}_1)$ defined by (34), this element of $\mathrm{gr}'\mathrm{Y}(\mathfrak{q}_1)$ corresponds to

$$-\,2\,(e_{11} + e_{-1,-1})\,u^{2r-2} = -\,2\,a\,u^{2r-2} \in \mathfrak{g}.$$

Due to Proposition 3.6 of Ref. 3, the latter elements of $\mathfrak{g}$ with $r = 1, 2, \dots$ freely generate the center of $\mathrm{U}(\mathfrak{g})$. Therefore, $C_1, C_3, \dots$ freely generate the center of $\mathrm{Y}(\mathfrak{q}_1)$. They get degrees $0, 2, \dots$ by the $\mathbb{Z}$-filtration defining $\mathrm{gr}'\mathrm{Y}(\mathfrak{q}_1)$.

By Theorem 3.4 of Ref. 3, the coefficients $Z_2, Z_4, \dots$ of $Z(u)$ for $N = 1$ also freely generate the center of $\mathrm{Y}(\mathfrak{q}_1)$. They have degrees $0, 2, \dots$ by the same $\mathbb{Z}$-filtration on $\mathrm{Y}(\mathfrak{q}_1)$. The left-hand side of (68) involves both $C_1, C_3, \dots$ and $C_2, C_4, \dots$. To express $Z_2, Z_4, \dots$ in $C_1, C_3, \dots$ only, we will use the homomorphisms $\gamma_n : \mathrm{Y}(\mathfrak{q}_1) \to \mathrm{U}(\mathfrak{q}_1)^{\otimes n}$.

By our argument using the matrix (67), the image of the series $C(u)$ by γ_n equals $\varphi_n(u)$. Consider $\varphi_n(u)$ as a formal power series in u^{-1} with coefficients being some polynomials in $a_1, \dots, a_n$. By taking only the top degree components of these coefficients, we obtain from $\varphi_n(u)$,

$$\prod_p \frac{(u - a_p)^2}{u^2 - a_p^2} = \prod_p \frac{u - a_p}{u + a_p} = \exp\left(-\sum_{r\geqslant 1} \frac{2a_1^{2r-1} + \cdots + 2a_n^{2r-1}}{(2r-1)\,u^{2r-1}}\right).$$

The latter equality is obtained by taking the logarithm of the product and then exponentiating. The coefficients of the above series at $u^{-1}, u^{-3}, \dots, u^{1-2n}$ are algebraically independent. Consequently, the coefficients of $\varphi_n(u)$ at $u^{-1}, u^{-3}, \dots, u^{1-2n}$ are also algebraically independent. This provides another proof of algebraic independence of the central elements $C_1, C_3, \dots$ of the Yangian $\mathrm{Y}(\mathfrak{q}_1)$.

Now denote by $\omega_n(u)$ the image of the series $Z(u)$ by the homomorphism γ_n. Due to (48), our $\omega_n(u)$ equals

$$\prod_p \frac{u^2 - a_p^2 + a_p}{u^2 - a_p^2 - a_p} = \prod_p \left(1 + \frac{2a_p}{u^2 - a_p^2 - a_p}\right) = \prod_p \left(1 + \sum_{r\geqslant 1} \frac{2a_p(a_p^2 + a_p)^{r-1}}{u^{2r}}\right)$$

where again $p = 1, \ldots, n$. Also see the end of Sec. VII. Consider $\omega_n(u)$ as a formal power series in u^{-1} with the coefficients being polynomials in $a_1, \ldots, a_n$. For $r \geqslant 1$ the top degree component of the coefficient at u^{-2r} is

$$2a_1^{2r-1} + \cdots + 2a_n^{2r-1}.$$

Therefore, the coefficients of $\omega_n(u)$ at $u^{-2}, u^{-4}, \ldots, u^{-2n}$ are algebraically independent polynomials in $a_1, \ldots, a_n$. Without relying on Ref. 3, the latter fact implies the freeness of the generators $Z_2, Z_4, \ldots$ of the center of $\mathrm{Y}(\mathfrak{q}_1)$.

We can uniquely express the coefficients of $\omega_n(u)$ and $\varphi_n(u)$ at u^{-2n} in the coefficients of the same series $\varphi_n(u)$ at $u^{-1}, u^{-3}, \ldots, u^{1-2n}$. Hence, we express Z_{2n} and C_{2n} in $C_1, C_3, \ldots, C_{2n-1}$. We used the fact that C_{2n} is of degree $2n-2$ relative to the second $\mathbb{Z}$-filtration on $\mathrm{Y}(\mathfrak{q}_1)$.

We can also uniquely express the coefficients of $\varphi_n(u)$ at u^{1-2n} and u^{-2n} in the coefficients of the series $\omega_n(u)$ at $u^{-2}, u^{-4}, \ldots, u^{-2n}$. Thus, we express C_{2n-1} and C_{2n} in $Z_2, Z_4, \ldots, Z_{2n}$.

It would be interesting to deduce relation (70) and the centrality of the coefficients of $C(u)$ directly from the defining relations of the algebra $\mathrm{Y}(\mathfrak{q}_1)$, without invoking its representation theory. It would be also interesting to relate the coefficients of $C(u)$ to the central elements of $\mathrm{Y}(\mathfrak{q}_1)$, which were recently introduced in Ref. 9.

Toward the end of our Introduction, we mentioned the quantum Berezinian for the Yangian of $\mathfrak{gl}_{1|1}$. This is the specialization to $M = N = 1$ of the quantum Berezinian[7] for the Yangian of any general linear Lie superalgebra $\mathfrak{gl}_{M|N}$. It would be fascinating to extend the definition of the series $C(u)$ to the Yangian $\mathrm{Y}(\mathfrak{q}_N)$ for any $N > 1$.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Maxim Nazarov: Investigation (lead).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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