MULTIPLICATION BY A FINITE BLASCHKE PRODUCT ON WEIGHTED BERGMAN SPACES: COMMUTANT AND REDUCING SUBSPACES

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ABSTRACT. We provide a characterization of the commutant of analytic Toeplitz operators T_B induced by finite Blaschke products B acting on weighted Bergman spaces which, as a particular instance, yields the case $B(z) = z^n$ on the Bergman space solved recently by by Abkar, Cao and Zhu [2]. Moreover, it extends previous results by Cowen and Wahl in this context and applies to other Banach spaces of analytic functions such as Hardy spaces H^p for $1 < p < \infty$. Finally, we apply this approach to study the reducing subspaces of T_B in weighted Bergman spaces.

1. INTRODUCTION

A general question in Operator Theory in Hilbert spaces consists in describing the commutant of a bounded linear operator, since it provides a better understanding of its action on the whole space. Unless the operator is normal, very little is known about this general question and even for subnormal operators the question is far from being solved completely.

Clearly, such a question has been addressed for particular classes of operators. Indeed, in the pioneering work [22], Shields and Wallen considered the commutants of a special class of operators which can be viewed as the coordinate operators on some reproducing kernel Hilbert spaces of holomorphic functions. Later on, Deddens and Wong [9] studied the communtant of analytic Toeplitz operators T_f on the classical Hardy space H^2 whenever the inducing symbol f was univalent. Both works were the starting point of a fruitful line of research where relevant results were proved by Cowen [4, 5, 6] or Thomson [25, 26, 27].

When the underlying space is the Bergman space, Thomson's theorem states, in particular, that if f is holomorphic on a neighborhood of the closed unit disc $\overline{\mathbb{D}}$, then there is always a finite Blaschke product B and an H^∞-function g on $\mathbb{D}$ such that $f = g \circ B$ and the commutants of T_f and T_B coincide. Thus, studying the commutant of T_B whenever B is a finite Blaschke provides a clear insight on the commutator of a larger class of functions which turns out to be a difficult question in general.

Likewise, knowing the commutant of T_B provides information about reducing subspaces. Recall that a non-trivial subspace M in H is reducing for a bounded linear operator T if both M and its orthogonal complement $M^\perp$ are invariant under T. Equivalently, the orthogonal projection P_M onto

M commutes with T. At this regard, it is worthy to point out that while the commutant of the analytic Toeplitz operator induced by the monomial z^n for $n \geq 2$ acting on the Hardy space H^2 has infinitely many minimal reducing subspaces, such an operator acting on the Bergman space has exactly n distinct minimal reducing subspaces.

Indeed, for Blaschke products of order two, it was shown independently in [24] and [29] that T_B has exactly two distinct minimal reducing subspaces in the Bergman space. For finite Blaschke products, there has been an active line of research to determine the conmutant, where the latest progress was due to Douglas, Putinar and Wang [11]. We refer to the recent monograph [19] for relevant results and more on the subject as well as the references therein.

The aim of the present work is twofold. On one hand, we characterize the commutant of analytic Toeplitz operators T_B induced by finite Blaschke products B on an scale of weighted Bergman spaces, when B is a finite Blaschke product. The case $B(z) = z^n$ was recently solved by Abkar, Cao and Zhu [2] (following earlier work by Čučković [8]), and we shall later see their result as a special case of our general theorem. Moreover, the characterization extends the results of Cowen and Wahl [7] who proved that on a large scale of Hilbert spaces $\mathcal{H}_k$ of analytic functions on $\mathbb{D}$ satisfying *natural* properties, among them the Bergman space, the commutants of T_B on $\mathcal{H}_k$ and on H^2 are the same in the sense that they have the same formulas in their actions on these spaces, and therefore the operators on these spaces are all extensions or restrictions of each other.

In addition, as it is well-known, a large class of weighted shift operators W can be realized as the analytic Toeplitz operator of multiplication by z on some weighted Hardy spaces $H^2(\beta)$ of analytic functions on a disk centered at the origin. It is a classical result due to Shields [21] that in such a case every operator in the commutant of T_z on $H^2(\beta)$ is the strong limit of a sequence of polynomials in T_z and thus a member of the multiplier algebra. Accordingly, Abkar, Cao and Zhu's result state that for such weighted shift W operators, the commutant of W^n can be identified with n copies of the multiplier algebra in a natural sense. They ask to what extent this result can be generalized pointing out that the main difficulty is [2, Proposition 3], which is no longer true for general weighted Hardy spaces. From this standpoint, our approach provides such a generalization for analytic Toeplitz operators induced by finite Blaschke products on weighted Bergman spaces. Indeed, the method works also for other Banach spaces of analytic functions, not necessarily Hilbert spaces, such as the Hardy spaces H^p, $1 < p < \infty$.

On the other hand, our second aim deals with reducing subspaces for analytic Toeplitz operators induced by finite Blaschke products on weighted Bergman spaces. As we have already pointed out, this is related to study the self-adjoint projections belonging to the commutant $\{T_B\}'$. Nevertheless, by considering appropriate isometries intertwining T_B and the shift operator, it is possible to exhibit reducing subspaces for T_B.

The rest of this manuscript is organized as follows. In what follows, we introduce the weighted Bergman spaces where our work takes place, recalling some important issues about them which will be of interest throughout the manuscript. In Section 2 we provide the characterization of the commutant while in Section 3 we deal with the study of reducing subspaces.

1.1. **The setting.** Given a real number α, the weighted Bergman space $\mathcal{A}_\alpha$ consists of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ on the unit disc $\mathbb{D}$ such that its norm

$$\|f\|_\alpha := \left(\sum_{k=0}^{\infty} |a_k|^2 (k+1)^\alpha \right)^{1/2}$$

is finite. Observe that for particular choices of α, we recover well-known Hilbert spaces of analytic functions. Namely, for $\alpha = -1$ we have the classical Bergman space A^2, $\alpha = 0$ corresponds to the Hardy space H^2, and $\alpha = 1$ to the Dirichlet space $\mathcal{D}$. For $\alpha > 1$ every function $f \in \mathcal{A}_\alpha$ admits a continuous extension to the closure of $\mathbb{D}$.

As introduced, weighted Bergman spaces are particular instances of the so-call weighted Hardy spaces $H^2(\beta)$ where β is a sequence of positive numbers. Such spaces were introduced by Shields [21] in order to study weighted shift operators on Hilbert spaces, though other classes of operators have been extensively studied on them.

The *multiplier algebra* $\mathcal{M}_\alpha$ of $\mathcal{A}_\alpha$ consists of analytic functions φ on $\mathbb{D}$ such that $\varphi f \in \mathcal{A}_\alpha$ for every $f \in \mathcal{A}_\alpha$. It is well known that for $\alpha = -1, 0$ the multiplier algebra $\mathcal{M}_\alpha$ can be identified with H^∞, the space of bounded analytic functions on $\mathbb{D}$ endowed with the sup-norm. For other instances of α this is no longer the case, and it is an open question to identify them for any α.

An important feature which will be needed in what follows is that $\mathcal{A}_\alpha$ spaces admit an expansion in terms of finite dimensional spaces. More precisely, if B is a finite Blaschke product of degree n, namely

$$B(z) = e^{i\theta} \prod_{i=1}^{n} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z}, \qquad (z \in \mathbb{D})$$

where $\theta \in \mathbb{R}$ and each $\alpha_i \in \mathbb{D}$ is counted according to its (prescribed) multiplicity, we denote by K_B the finite-dimensional model space $H^2 \ominus BH^2$. In [3, 14] the authors give the following theorem:

Theorem 1.1. *Let $\alpha \in \mathbb{R}$ and B a finite nonconstant Blaschke product. Then $f \in \mathcal{A}_\alpha$ if and only if $f = \sum_{k=0}^{\infty} h_k B^k$ (convergence in $\mathcal{A}_\alpha$ norm) with $h_k \in K_B$ and*

$$(1.1) \qquad \sum_{k=0}^{\infty} (k+1)^\alpha \|h_k\|_0^2 < \infty.$$

Before proceeding further two remarks are in order. First, in (1.1) we use the H^2 norm $\|.\|_0$, but since all norms on finite-dimensional vector spaces are equivalent we could equally use $\|.\|_\alpha$.

Secondly, though the result was only given for $\alpha \in [-1, 1]$ in [3, 14], the argument is based on two facts that hold for all $\alpha \in \mathbb{R}$:

(1) For B a finite Blaschke product the composition operator $f \mapsto f \circ B$ is bounded on $\mathcal{A}_\alpha$.

(2) For B a finite Blaschke product and $u \in K_B$ the analytic Toeplitz operator T_u of multiplication by u is bounded on $\mathcal{A}_\alpha$.

These can be deduced from the fact that for any non-negative integer N with $2N + \alpha > -1$, $\mathcal{A}_\alpha$ is equal to the space of analytic functions f on the unit disc $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f^{(N)}(z)|^2 (1 - |z|^2)^{2N + \alpha} dA(z) < \infty,$$

where $dA(z)$ is two-dimensional Lebesgue measure (see [28, 30] for more on these spaces).

To prove (1) we use Faà di Bruno's formula for derivatives of compositions, and for (2) the Leibniz formula for derivatives of products.

2. A characterization of $\{T_B\}'$ on $\mathcal{A}_\alpha$

In this section, we provide the characterization of $\{T_B\}'$ on $\mathcal{A}_\alpha$ identifying it with n copies of the multiplier algebra in a natural sense, answering the question posed by Abkar, Cao and Zhu in [2].

Let us consider the space $(\mathcal{A}_\alpha)^n = \mathcal{A}_\alpha \oplus \cdots \oplus \mathcal{A}_\alpha$ with the ℓ^2 direct sum norm. Define the shift S on $(\mathcal{A}_\alpha)^n$ by $S(f_1, \ldots, f_n)(z) = (zf_1(z), \ldots, zf_n(z))$. First, we observe the following

Lemma 2.1. *An operator $A = (A_{jk})_{j,k=1}^n$ on $(\mathcal{A}_\alpha)^n$ commutes with S if and only if each A_{jk} lies in the multiplier algebra of $\mathcal{A}_\alpha$.*

Proof. Taking each f_k to be 0 apart from $k = m$, say, we have

$$SAf = (SA_{1m}f_m, \ldots, SA_{nm}f_m)^t$$

and

$$ASf = (A_{1m}Sf_m, \ldots, A_{nm}Sf_m)^t.$$

so that each A_{jm} commutes with S.

Conversely, it is easy to see by multiplying out matrices, that the condition is sufficient for A to commute with S. $\qquad\square$

Let $u_1, \ldots, u_n$ be an (algebraic) basis for the model space K_B. For example, if B is a finite Blaschke product with distinct zeros $z_1, \ldots, z_n$, then the functions $u_n(z) = 1/(1 - \bar{z}_n z)$ can be used. For the case $B(z) = z^n$, the functions $u_k(z) = z^{k-1}$, $k = 1, \ldots, n$, form a basis of K_B.

Now, we have the following commutative diagram:

$$
\begin{array}{ccc}
(\mathcal{A}_\alpha)^n & \xrightarrow{\;S\;} & (\mathcal{A}_\alpha)^n \\
J\downarrow & & \downarrow J \\
\mathcal{A}_\alpha & \xrightarrow[T_B]{} & \mathcal{A}_\alpha
\end{array}
$$

where the mapping J is defined by

$$J\left(\sum_{m=0}^\infty a_{1,m}z^m, \ldots, \sum_{m=0}^\infty a_{n,m}z^m\right) = \sum_{j=1}^n \sum_{m=0}^\infty a_{j,m}u_j B^m.$$

Clearly $JS = T_B J$. Moreover J is an isomorphism, since the (squared) norms

$$\sum_{m=0}^\infty \left\| \sum_{j=1}^n a_{j,m}u_j \right\|^2 (m+1)^\alpha \qquad \text{and} \qquad \sum_{m=0}^\infty \sum_{j=1}^n |a_{j,m}|^2 (m+1)^\alpha$$

are equivalent. Accordingly, we have the following

Lemma 2.2. *Let α be a real number. A bounded linear operator W on $\mathcal{A}_\alpha$ commutes with T_B if and only if it has the expression $W = JVJ^{-1}$ where*

$$V(f_1, \ldots, f_n) = \left(\sum_{k=1}^{n} \phi_{jk} f_k \right)_{k=1}^{n}$$

with $(\phi_{jk})_{j,k=1}^{n} \subset \mathcal{M}_\alpha$.

From Theorem 1.1, we have a unique decomposition $f = \sum_{j=1}^{n} u_j f_j(B)$ for $f \in \mathcal{A}_\alpha$, where $f_1, \ldots, f_n$ also lie in $\mathcal{A}_\alpha$. Using this, we can obtain an explicit formula from Lemma 2.2.

Theorem 2.3. *Let $\alpha \in \mathbb{R}$. A bounded linear operator W on $\mathcal{A}_\alpha$ commutes with T_B if and only if*

(2.1)
$$W\left(\sum_{j=1}^{n} u_j f_j(B) \right) = \sum_{k=1}^{n} \phi_k f_k(B)$$

for some $\phi_1, \ldots, \phi_n$ in $\mathcal{M}_\alpha$.

Proof. We see first that every expression as in (2.1) defines an operator that commutes with T_B.

For the converse, we have $W = JVJ^{-1}$ for some V commuting with S, as above, and

$$\begin{aligned}
JVJ^{-1}\left(\sum_{j=1}^{n} u_j f_j(B) \right) &= JV(f_1, \ldots, f_n) \\
&= J\left(\sum_{k=1}^{n} \phi_{jk} f_k \right) \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} u_j \phi_{jk}(B) f_k(B).
\end{aligned}$$

Writing $\phi_k = \sum_{j=1}^{n} u_j \phi_{jk}(B)$, we see easily that $\phi_k \in \mathcal{M}_\alpha$, since each u_j is a rational function and also a multiplier of $\mathcal{A}_\alpha$. $\square$

From this we have the special case $B(z) = z^n$ given in [2].

Corollary 2.4. *Let $\alpha \in \mathbb{R}$ and let n be a positive integer. A bounded linear operator W on $\mathcal{A}_\alpha$ commutes with T_{z^n} if and only if, on decomposing a function $f \in \mathcal{A}_\alpha$ uniquely as*

$$f(z) = \sum_{j=0}^{n-1} z^j f_j(z^n),$$

with $f_0, f_1, \ldots, f_{n-1}$ in $\mathcal{A}_\alpha$, we have

(2.2)
$$W\left(\sum_{j=0}^{n-1} z^j f_j(z^n) \right) = \sum_{k=0}^{n-1} \phi_k f_k(z^n)$$

for some $\phi_0, \ldots, \phi_{n-1}$ in $\mathcal{M}_\alpha$.

Proof. This follows from Theorem 2.3 since $\{1, z, z^2 \ldots, z^{n-1}\}$ is a basis for K_{z^n}. $\square$

2.1. **A remark on a theorem of Cowen and Wahl.** As we noted in the Introduction, Cowen and Wahl [7] by means of a generalization of a theorem of Cowen [4] proved that on a large scale of Hilbert spaces $\mathcal{H}_k$ of analytic functions on $\mathbb{D}$, among them the Bergman space, the commutants of T_B on $\mathcal{H}_k$ and on H^2 are the same in the sense that they have the same formulas in their actions on these spaces.

In particular, for $\alpha \in \mathbb{D}$ and k_α denoting the reproducing kernel for H^2, Cowen showed the following result.

Theorem (Cowen, 1978). Let T be a linear bounded operator acting on the Hardy space H^2 and $f \in H^\infty$. The following are equivalent:

  (1) T commutes with T_f.
  (2) For all $\alpha \in \mathbb{D}$, $T^* k_\alpha \perp (f - f(\alpha))H^2$.
  (3) There exists $J \subset \mathbb{D}$ such that $\sum_{\alpha \in J}(1 - |\alpha|) = \infty$ and for all $\alpha \in J$, $T^* k_\alpha \perp (f - f(\alpha))H^2$.

In the case where $f = B$, a finite Blaschke product, we may show easily that Cowen's conditions are equivalent to our condition (2.1).

In particular, by taking $W = T$ we have that for every $g \in H^2$,

$$\langle W^* k_\alpha, (B - B(\alpha)g) \rangle \quad = \quad \overline{W((B - B(\alpha))g)(\alpha)}$$

Now, assuming (2.1), we obtain that $W(Bg) = BW(g)$ so that

$$W(Bg)(\alpha) = B(\alpha)W(g)(\alpha) = W(B(\alpha)g)(\alpha)$$

for all $\alpha \in \mathbb{D}$, from which we see that $\langle W^* k_\alpha, (B - B(\alpha)g) \rangle = 0$.

Conversely, suppose that

$$W(Bg - B(\alpha)g)(\alpha) = 0$$

for all $\alpha \in J \subset \mathbb{D}$ where J is a non-Blaschke subset of $\mathbb{D}$, and consider $W(u_j f_j(B))$ where we use the notation of Theorem 2.3, so that $u_j \in K_B$ is fixed and $f_j \in H^2$ is arbitrary.

We begin by identifying $W(u_j B^n)$ for $n = 0, 1, 2, \ldots$. Observe that

$$W(u_j B^{n+1})(\alpha) = B(\alpha)W(u_j B^n)(\alpha)$$

for all $\alpha \in J$.

But if two functions $h_1, h_2 \in H^2$ satisfy $h_1(\alpha) = B(\alpha)h_2(\alpha)$ for all $\alpha \in J$, then we clearly have $h_1 = Bh_2$; thus in our case, $W(u_j B^n) = B^n W(u_j)$ for all $n = 0, 1, 2, \ldots$ and we conclude by continuity and linearity that $W(u_j f(B)) = \phi_j f(B)$ for some $\phi_j \in H^\infty$. Accordingly, (2.1) follows.

Analogously, the Cowen–Wahl Theorem can be derived as an application of Theorem 2.3.

2.2. **On $\{T_B\}'$ on H^p spaces.** The previous approach may be considered as well on other spaces of analytic functions, not necessarily Hilbert spaces. This is the case of the classical Hardy spaces H^p for $1 < p < \infty$. Recall that an analytic function f on $\mathbb{D}$ belongs to H^p if the norm

$$\|f\|_p = \left(\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

is finite. A classical result due to Fatou (see [12], for instance) states that the radial limit $f(e^{it}) := \lim_{r \to 1^-} f(re^{it})$ exists a.e. on the boundary $\mathbb{T}$ of the unit disc. In this regard, it is well known that

H^p can be regarded as a closed subspace of $L^p(\mathbb{T})$. In particular, for $p = 2$ the Lebesgue space $L^2(\mathbb{T})$ may be decomposed as the orthogonal sum

$$L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}, \tag{2.3}$$

where $\overline{H_0^2} = \{f \in L^2(\mathbb{T}) : \overline{f} \in H^2 \text{ and } f(0) = 0\}$. Note that in the identity (2.3) we are identifying H^2 through the non-tangential boundary values of the H^2 functions.

Let B be a finite Blaschke product of degree n and denote by K_B the associated *model space* in H^p, namely,

$$H^p \cap B\overline{H_0^p},$$

which turns out to be an n-dimensional space which does not depend on p.

For $p = 2$, the Wold Decomposition Theorem yields that

$$H^2 = K_B \oplus BK_B \oplus B^2K_B \oplus \dots,$$

as an orthogonal decomposition. Nevertheless, in the case of general p, we take an orthonormal basis $u_1, \dots, u_n$ of K_B (in the H^2 sense) and we claim that

$$f \in H^p \iff f = \sum_{k=1}^{n} u_k f_k(B) \qquad \text{for some} \quad f_1, \dots, f_n \in H^p.$$

Moreover, there is an equivalence of norms between $\|f\|_p$ and $\sum_{k=1}^{n} \|f_k\|_p$.

Since the composition operator C_B is bounded we have, for finite sums,

$$\left\| \sum_{j=0}^{m} \sum_{k=1}^{n} a_{jk} u_k B^j \right\|_p \leq \sum_{k=1}^{n} \|u_k\|_\infty \left\| \sum_{j=0}^{m} a_{jk} B^j \right\|_p \leq \max \|u_k\|_\infty \sum_{k=1}^{n} \left\| \sum_{j=0}^{m} a_{jk} z^j \right\|_p.$$

This means that there is a continuous mapping $H^p \times \dots \times H^p \to H^p$ given by

$$(f_1, \dots, f_n) \mapsto \sum_{k=1}^{n} u_k f_k(B), \tag{2.4}$$

with dense range (since the union of the subspaces $B^j K_B$, $j = 0, 1, 2, \dots$ is dense).

Using the isomorphic duality $(H^p)^* \simeq H^q$ induced by the H^2 inner product we see that in fact the mapping (2.4) is also bounded below, and so it has closed range and is a surjection. Accordingly, the identification of the commutant of T_B on H^p provided by the formula in Theorem 2.3 also holds.

Remark 2.5. Observe that in the previous discussion it is not necessary that $u_1, \dots, u_n$ to be orthonormal, simply an algebraic basis, since in such a case, the duality argument would involve taking the biorthogonal basis $v_1, \dots, v_n$ and the duality formula

$$\left\langle \sum_{k=1}^{n} u_k f_k(B), \sum_{k=1}^{n} v_k g_k(B) \right\rangle_{H^p \times H^q} = \sum_{k=1}^{n} \langle f_k, g_k \rangle_{H^p \times H^q}.$$

3. MINIMAL REDUCING SUBSPACES OF T_B ON THE BERGMAN SPACE

As it was pointed out in the Introduction, recently there has been an increasing interest on reducing subspaces of multiplication operators whenever they act on spaces of analytic functions, and in particular on the Bergman space A^2. Clearly the works by Douglas and coauthors [10], [11] or those by Guo and Huang [16], [17], [18] are good examples in this regard.

In this setting, if M is a reducing subspace of the multiplication operator T_B, the orthogonal projection P_M onto M commutes with both T_B and its adjoint T_B^*. Accordingly, if $\mathcal{W}(B)$ denotes the von Neumann algebra generated by the multiplication T_B on A^2, that is, the unital C^*-algebra on A^2 which is closed in the weak operator topology, it is plain that P_M belongs to the commutant $\mathcal{W}(B)'$. Note that, in particular, $\mathcal{W}(B)' = \{T_B, T_B^\star\}'$.

As noted in [19], on the Hardy space every non-trivial closed invariant subspace N in $H^2 \ominus BH^2$ satisfies that $B^m N \perp B^n N$ for $n \neq m$, and hence the direct sum of all $B^k N$, $k \geq 0$ gives a reducing subspace of T_B. On the Bergman space, in [20], the authors showed that there is always a canonical reducing subspace M_0 such that T_B restricted on M_0 is unitarily equivalent to the Bergman shift. Their main tool relies on the Hardy space of the bidisk. We shall provide some similar results in the context of weighted Bergman spaces.

On the other hand, recently Douglas, Putinar and Wang [11] proved that the von Neumann algebra $\mathcal{W}(B)'$ is abelian for any finite Blaschke product B whenever T_B acts on the Bergman space. Indeed, a clear consequence in this context is the following

Proposition 3.1. *For every finite Blaschke product, T_B has a non-trivial closed hyperinvariant subspace in the Bergman space.*

Note that if $A \in \{T_B\}'$, having into account that T_B has a non-trivial reducing subspace M, the orthogonal projection onto M, P_M, belongs to $\{T_B\}'$ and hence, $AP_M = P_M A$. So the subspace M is invariant for every operator commuting with T_B.

In order to find minimal reducing subspaces of T_B it is helpful to understand the minimal rank-1 projections that commute with T_B.

Since $1 \in \mathcal{A}_\alpha$ we clearly have that every multiplier $\phi \in M_\alpha$ corresponds to a function in $\mathcal{A}_\alpha$. We therefore carry on from Theorem 2.3 and write $\phi_k = \sum_{j=1}^n u_j \phi_{jk}(B)$, where the ϕ_{jk} lie in $\mathcal{A}_\alpha$.

Then (2.1) can be written as

$$W\left(\sum_{j=1}^n u_j f_j(B)\right) = \sum_{k=1}^n \sum_{j=1}^n u_j \phi_{jk}(B) f_k(B),$$

or, in an obvious notation

$$\begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{pmatrix} \mapsto \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{pmatrix} = \Phi \begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{pmatrix}, \quad \text{say.}$$

Thus the space of operators commuting with T_B is isomorphic to the space of $n \times n$ matrices of multipliers of $\mathcal{A}_\alpha$. (Of course, to get a natural isometry one would need to work with an equivalent norm on $\mathcal{A}_\alpha$.)

However, algebraically at least, a minimal projection P such that $P^2 = P$, corresponds to a rank-one matrix Φ such that $\Phi^2 = \Phi$. The following example exhibits some rank-one projections in the commutant of T_{z^2}.

Example 3.2. Let $B(z) = z^2$, so that every $f \in \mathcal{A}_\alpha$ decomposes as $f(z) = f_1(z^2) + z f_2(z^2)$. This is mapped to the function

$$\phi_{11}(z) f_1(z^2) + \phi_{12}(z) f_2(z^2) + z \phi_{21}(z) f_1(z^2) + z \phi_{22}(z) f_2(z^2),$$

where the ϕ_{ij} are in M_α.

Some examples of rank-1 idempotents are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ p(z) & 0 \end{pmatrix}$, where p is a polynomial, or indeed any matrix with rank 1 and trace equal to 1, such as $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

In the Hardy space the decomposition in Theorem1.1 is orthogonal, a fact which we have exploited already. This is not the case in the Bergman space, but next we propose an alternative.

3.1. An orthogonal decomposition in the Bergman space $\mathcal{A}_{-1}$.
Note that if B is a finite Blaschke product of degree N, then the multiplication operator T_B is not an isometry on $\mathcal{A}_{-1}$ and in the decomposition of Theorem 1.1 we do not have K_B orthogonal to $B\mathcal{A}_{-1}$ in general. However, T_B is an injective operator whose range is closed with finite codimension (it is Fredholm), and moreover it is bounded below. Thus we can construct an orthogonal decomposition

$$(3.1) \qquad \mathcal{A}_{-1} = K_0 \oplus B K_1 \oplus B^2 K_2 \oplus \ldots$$

as follows. We take K_0 to be the orthogonal complement of the closed subspace $B\mathcal{A}_{-1}$, a finite-dimensional subspace which, in the case of distinct zeros, is spanned by Bergman reproducing kernels. We then take $K_0 + B K_1$ to be the orthogonal complement of $B^2 \mathcal{A}_{-1}$ (the span of reproducing kernels and derivative kernels) in such a way that $B K_1$ is orthogonal to K_0. We continue in this way, and for each k the space K_k has dimension equal to N, the degree of B.

If B is a monomial, $B(z) = z^N$, such a decomposition provides a straightforward description of reducing subspaces as follows. For each $0 \le j \le N - 1$, let

$$\mathcal{M}_j = \overline{\text{span} \{z^{j+kN} : \ k \ge 0\}}^{\mathcal{A}_{-1}}.$$

Clearly, each $\mathcal{M}_j$ is a reducing subspace for T_{z^N} and Stessin and Zhu proved that they are the only minimal reducing subspaces for T_{z^N} in the Bergman space. Moreover, T_{z^N} has exactly 2^N distinct reducing subspaces counting the trivial ones (see [23, Theorem B]).

If $B(z) = \left(\dfrac{z - a}{1 - \bar{a}z} \right)^N$ for $a \in \mathbb{D}, a \ne 0$, taking

$$K_0 = \text{span} \{z^j / (1 - \bar{a}z)^{j+2} : \ 0 \le j \le N - 1\}$$

namely, to within constants, the successive derivative kernels, we have K_0 is orthogonal to $B\mathcal{A}_{-1}$. Next, $B K_0$ is orthogonal to both K_0 and $B^2 \mathcal{A}_{-1}$, since the span of K_0 and $B K_0$ is the same as

$$\text{span} \{z^j / (1 - \bar{a}z)^{j+2} : 0 \le j \le 2N - 1\}.$$

Arguing similarly, we deduce that $B^i K_0$ is also orthogonal to $B^{i-1} K_0$ and $B^i \mathcal{A}_{-1}$ for every $i \geq 1$. Accordingly, (3.1) turns out to be

$$(3.2) \qquad\qquad \mathcal{A}_{-1} = K_0 \oplus B K_0 \oplus B^2 K_0 \oplus \dots$$

and the following statement follows as a consequence:

Proposition 3.3. *Let* $B(z) = \left(\dfrac{z - a}{1 - \overline{a} z} \right)^N$ *for* $a \in \mathbb{D}, a \neq 0$. *For each* $0 \leq j \leq N - 1$, *the subspaces*

$$\mathcal{M}_j = \overline{\mathrm{span} \left\{ \left(\frac{z}{(1 - \overline{a}z)^2} \right)^{j+kN} : k \geq 0 \right\}}^{\mathcal{A}_{-1}}.$$

are reducing for T_B *in* $\mathcal{A}_{-1}$.

The main theorem in [20] (see, more recently, [15] for a multi-dimensional variant) asserts that if B is a finite Blaschke product of degree $N > 1$, then the analytic Toeplitz operator T_B on the Bergman space is reducible and there is a reducing subspace on which the restriction of T_B is unitarily equivalent to the shift $S = T_z$ on $\mathcal{A}_{-1}$. In the scale of weighted Bergman spaces we have the following.

Theorem 3.4. *For* $\alpha \in \mathbb{R}$ *and* $B(z) = z^n$ *for an integer* $n \geq 1$ *the space* $\mathcal{A}_\alpha$ *has a closed* T_B-*reducing subspace on which* T_B *is unitarily equivalent to the shift* $S = T_z$ *on* $\mathcal{A}_\alpha$. *For a general finite Blaschke product* B *there is an equivalent Hilbert space norm* $\| \cdot \|_B$ *on* $\mathcal{A}_\alpha$ *under which* T_B *has a reducing subspace on which it is unitarily equivalent to the shift on* $\mathcal{A}_\alpha$.

Proof. For $B(z) = z^n$ we have an obvious reducing subspace V_B spanned by $z^{n-1}, z^{2n-1}, \dots$ (its orthogonal complement is clearly invariant under T_B), and since $\|z^{mn-1}\|_\alpha^2 = (mn)^\alpha$ we see that we have the following commutative diagram, where J is the unitary mapping with $J(z^k) = z^{(k+1)n-1}/n^{\alpha/2}$ for $k = 0, 1, 2, \dots$.

$$
\begin{array}{ccc}
 & S & \\
\mathcal{A}_\alpha & \longrightarrow & \mathcal{A}_\alpha \\
J \downarrow & & \downarrow J \\
V_B & \longrightarrow & V_B \\
 & T_B &
\end{array}
$$

For a general B we return to the decomposition in Theorem 1.1. Writing $f = \sum_{k=0}^\infty h_k B^k$ (convergence in $\mathcal{A}_\alpha$ norm) with $h_k \in K_B$ we define

$$\|f\|_B^2 = \sum_{k=0}^\infty (k+1)^\alpha \|h_k\|_0^2 < \infty.$$

We may choose $h \in K_B$ to have $\mathcal{A}_\alpha$ norm 1, and then let the subspace V_B be spanned by $h, hB, hB^2, \dots$, noting that $\|hB^k\|_B^2 = (k+1)^\alpha$, and hence we have the same commutative diagram as before with $J(z^k) = hB^k$. $\qquad\square$

Observe that it we endow V_B with the usual $\mathcal{A}_\alpha$ norm instead, then the operator J is merely an isomorphism, and in general not unitary.

Remark 3.5. In [14] the authors showed that in a scale of weighted Dirichlet spaces including the Bergman space $\mathcal{A}_{-1}$, for every finite Blaschke product B, it is always possible to renorm the space (with an equivalent norm) such that B enjoys the *wandering subspace property* (see [14, Theorem 3.3]). Accordingly, both Theorem 3.3 in [14] and Theorem 3.4 show, in particular, that the geometry of the space plays a significant role in order to study such questions since the answer depends on the norm expression.

3.2. **A final remark: An orthogonal decomposition in the Bergman space related to the T_B action.**

As a final observation, we note that most of the previous results relied on *ad-hoc* orthogonal decompositions of the underlying space which allowed us to simplify strategies. At this regard, we finish by exhibiting another orthogonal decomposition of the Bergman space $\mathcal{A}_{-1}$ which is closer to the action of T_B as a shift.

Let B be a Blaschke product of degree N. Observe that $T_B^k \mathcal{A}_{-1}$ is a closed subspace of $\mathcal{A}_{-1}$ of codimension Nk for $k = 1, 2, \ldots$.

Let us write $X_k = T_B^k \ominus T_B^{k+1} \mathcal{A}_{-1}$ for $k = 0, 1, 2, \ldots$, where T_B^0 denotes the identity operator. Note that each X_k has dimension N. We therefore have

$$(3.3) \qquad \mathcal{A}_{-1} = X_0 \oplus X_1 \oplus X_2 \oplus \ldots,$$

which is an orthogonal decomposition. Clearly, it allows one to exhibit suitable orthonormal bases of $\mathcal{A}_{-1}$ by combining orthonormal bases of the X_k.

Observe that, in particular,

$$T_B^k \mathcal{A}_{-1} = X_k \oplus X_{k+1} \oplus \ldots$$

for $k = 0, 1, 2, \ldots$ Accordingly, T_B acts as a shift respect to the orthogonal decomposition (3.3).

On the other hand, note that if W commutes with T_B then for a vector $x = B^k y$ in $T_B^k \mathcal{A}_{-1}$ we have

$$W x = W B^k y = B^k W y \in T_B^k \mathcal{A}_{-1}.$$

In particular, this implies that W has a lower triangular block matrix with respect to the decomposition (3.3). Moreover, if W is also self-adjoint then the matrix is block-diagonal with every block self-adjoint.

Consequently, every projection in $\{T_B\}'$ admits a matrix representation respect to (3.3) which is block-diagonal with every block self-adjoint. Such matrix representation may be of help in understanding the reducing subspaces for T_B.

References

[1] M. Abrahamse and J. Ball, *Analytic Toeplitz operators with automorphic symbol II*, Proc. Amer. Math. Soc. 59 (1976), 323–328.

[2] A. Abkar, G. Cao and K. Zhu, *The commutant of some shift operators*, Complex Analysis and Operator Theory, 14 (2020), no. 6, art. 58.

[3] I. Chalendar, E.A. Gallardo-Gutiérrez, and J.R. Partington, *Weighted composition operators on the Dirichlet space: Boundedness and spectral properties*, Math. Ann. 363 (2015), 1265–1279.

[4] C. C. Cowen, *The commutant of an analytic Toeplitz operator*, Trans. Am. Math. Soc. 239 (1978), 1–31.

[5] C. C. Cowen, *The commutant of an analytic Toeplitz operator, II*, Indiana Univ. Math. J. 29 (1980), 1–12.

[6] C. C. Cowen, *An analytic Toeplitz operator that commutes with a compact operator and a related class of Toeplitz operators*, J. Funct. Anal. 36 (1980), 169–184.

[7] C. C. Cowen and R. G. Wahl, *Commutants of finite Blaschke product multiplication operators*, Invariant subspaces of the shift operator, 99–114, Contemp. Math., 638, Amer. Math. Soc., Providence, RI, 2015.

[8] Ž. Čučković, *Commutants of Toeplitz operators on the Bergman space*, Pacific J. Math. 162 (1994), no. 2, 277–285.

[9] J. Deddens and T. Wong, *The commutant of analytic Toeplitz operators* Trans. Am. Math. Soc. 184 (1973), 261–273.

[10] R. G. Douglas, S. Sun and D. Zheng, *Multiplication operators on the Bergman space via analytic continuation*, Adv. Math. 226 (2012), 541–583.

[11] R. G. Douglas, M. Putinar and K. Wang, *Reducing subspaces for analytic multipliers of the Bergman space*, J. Funct. Anal. 263 (2012), 1744–1765.

[12] P. Duren, Theory of H^p spaces, Academic Press, New York (1970).

[13] P. Duren, D. Khavinson, H.S. Shapiro and C. Sundberg, Contractive zero-divisors in Bergman spaces. *Pacific J. Math.* 157 (1993), no. 1, 37–56.

[14] E.A. Gallardo-Gutiérrez, J.R. Partington and D. Seco, *On the Wandering Property in Dirichlet spaces*, Integral Equations Operator Theory 92 (2020), no. 2, Paper No. 16.

[15] G. Ghosh, *Multiplication operators on the Bergman space by proper holomorphic mappings*, J. Math. Anal. Appl. 510 (2022), Paper No. 126026.

[16] K. Guo and H. Huang, *On multiplication operators of the Bergman space: similarity, unitary equivalence and reducing subspaces*, J. Operator Theory 65 (2011), 355–378.

[17] K. Guo and H. Huang, *Multiplication operators defined by covering maps on the Bergman space: the connection between operator theory and von Neumann algebras*, J. Funct. Anal. 260 (2011), 1219–1255.

[18] K. Guo and H. Huang, *Geometric constructions of thin Blaschke products and reducing subspace problem*, Proc. Lond. Math. Soc. 109 (2014), 1050–1091.

[19] K. Guo and H. Huang, Multiplication operators on the Bergman space. Lecture Notes in Mathematics, 2145. Springer, Heidelberg, 2015.

[20] J. Hu, S. Sun, X. Xu and D. Yu, *Reducing subspace of analytic Toeplitz operators on the Bergman space*, Integral Equations Operator Theory 49 (2004), 387–395.

[21] A.L. Shields, Weighted shift operators and analytic function theory. *Topics in operator theory*, 49–128. Math. Surveys, No. 13, Amer. Math. Soc., Providence, R.I., 1974.

[22] A. Shields and L. Wallen, *The commutants of certain Hilbert space operators*, Indiana Univ. Math. J. 20 (1970/1971), 777–788.

[23] M. Stessin and K. Zhu, *Reducing subspace of weighted shift operators*, Proc. Amer. Math. Soc., 130 (2002) 2631–2639.

[24] S.L. Sun and Y. Wang, *Reducing subspaces of certain analytic Toeplitz operators on the Bergman space*, Northeast. Math. J. 14 (1998), 147–158.

[25] J. Thomson, *The commutant of a class of analytic Toeplitz operators*, American J. Math. 99 (1977), 522–529.

[26] J. Thomson, *The commutant of a class of analytic Toeplitz operators II*, Indiana Univ. Math. J. 25 (1976), 793–800.

[27] J. Thomson, *The commutant of certain analytic Toeplitz operators*, Proc. Amer. Math. Soc. 54 (1976), 165–169.

[28] R. Zhao and K. Zhu, *Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$*, Mém. Soc. Math. Fr. (N.S.) No. 115 (2008), vi+103 pp.

[29] K. Zhu, *Reducing subspaces for a class of multiplication operators*, J. Lond. Math. Soc. 62 (2000), 553–568.

[30] K. Zhu, *Operator theory in function spaces.* Second edition. Mathematical Surveys and Monographs, 138. American Mathematical Society, Providence, RI, 2007.

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