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On the existence and uniqueness of equilibria in meshed DC microgrids with CPLs

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Abstract— **In this paper, we analyse the existence and uniqueness of equilibria of constant power loads (CPLs) in meshed DC microgrid architectures. Given the CPLs' nonlinear characteristic and negative impedance behaviour, they are commonly known to introduce a destabilising effect into the system, effect intuitively coined as *negative impedance instability*. In the present approach, we start by deriving the characteristic polynomial from the power balance equation aiming to observe the nature of the CPLs voltage solutions, and assess their feasibility. Then, the algebraic expression is transformed into a problem of existence and uniqueness of a fixed point, and further tested by means of contraction mapping theory. A sufficient condition for the sources' voltage references is obtained to guarantee the existence and uniqueness of equilibria. This provides a useful guidance in selecting the voltage references in the control design process. A numerical investigation on a meshed DC microgrid is carried out to verify the acquired sufficient condition and its underlying developed theory.**

I. INTRODUCTION

Given the direct current nature of renewable energy generation and increasing number of DC end consumers, DC microgrids have started to stand out as a solution for rapid integration of green energy resources into the existing grid, as they can forgo some AC/DC and DC/AC conversion steps [1]. The consumers are often connected to the DC bus through power electronics interfaced circuits, behaving as constant power loads. when the coupling power converter response is fast. This occurrence is not uncommon, as similar behaviour has also been encountered in tightly regulated electric motors and downstream power converters operating in distribution feeders in conventional grid networks [2].

The V-I characteristic of the CPLs is depicted in Fig. 1, where one can notice that the rate of change (slope) between voltage and current is negative ($\Delta V/\Delta i < 0$), while the instantaneous value of the impedance always remains positive ($V/i > 0$). However, increasing the constant power of the CPLs causes a potential drop in the CPL voltage, while the current will increase to satisfy the power demand. The problem arises when the load voltage becomes less than the voltage value of the stable point. If the system does not have proper control in place to prevent it, the voltage will continue to drop until zero, while the current will go to infinity [3]. In simple terms, the challenges that CPLs impose stem from their power conditioning at the load side, and the nonlinearity introduced into the power balance dynamics could result in voltage collapse [4]–[6] when the power of the CPLs increases above a certain level.

Therefore, the existence of a steady-state behaviour, that is in the form of CPLs' voltage equilibria, is critical for the normal and safe operation of DC microgrids. This requirement, however, is not a straightforward challenge to take on analytically and it has posed considerable difficulties.

A. Literature review

CPLs have been extensively studied in the literature (see [3], [7], [8] and the references within), with a more or less comprehensive characterisation of their equilibria. The challenges related with the existence of voltage equilibria in microgrid systems that include CPLs have been reported previously in DC microgrids and distribution level grid applications [9]–[11]. The existence of the solutions is generally ensured upon compliance with a certain inequality condition for the system voltage, and it has been reported in the literature either when having a single or multiple CPLs [12], [13].

Constant impedance, current and power (ZIP) loads have been considered in [14], where an approximate characterisation of the high-voltage of the CPLs is given when the demanded power is "small". In [15] the high-voltage solution of CPLs is chosen based on the argument that the high-voltage is the feasible choice. The selection of the high-voltage solution in [16]–[18] is guaranteed

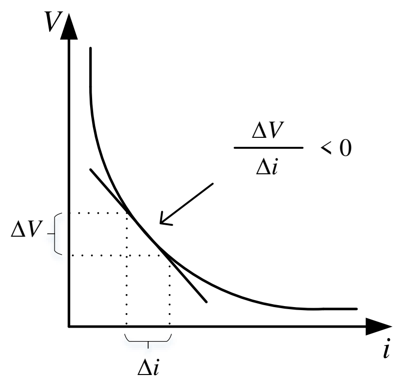

Fig. 1. Constant power load I-V characteristic

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by the employment of a current-limiting control approach. Nevertheless, selecting the CPLs feasible voltage solution, without any pre-determined assumptions or control strategy in place, is currently lacking the supporting theoretical analysis.

A general approach of analysing the existence of solutions of the power-balance equation is to transform the nonlinear equation solvability problem into the existence problem of a fixed point for a derived function [4], [19]–[21]. Constructing regions described by norm-like constraints for the existence of a solution within a given distance of the nominal solution, by means of Brouwer's fixed point theorem, has been proposed in [19]. Equivalent results and sufficient conditions have also been reported in [4], [20]. A less conservative sufficient condition based on Tarski's fixed-point theorem is derived in [21] by constructing an increasing fractional mapping. An extensive study on the existence and stability of equilibria in DC microgrids with CPLs has been carried out in [22] by using the properties of M-matrices.

Regardless of control strategy and network configuration, the existence and uniqueness equilibrium points of CPLs are vital prerequisites for the operation of microgrid systems, which is why the choice should be based on formal proofs.

B. Main contributions

In this paper, the scope revolves around analysing the existence and uniqueness of high-voltage equilibria of CPLs. This work is complementary to existing studies in the literature, but also unique in the sense that it proposes an approach to guarantee both existence and uniqueness of the high-voltage solution of a second order polynomial in a feasible set, pending a straightforward condition.

The main contributions of this paper are highlighted below:

1) By carefully deriving the CPLs power balance equation, the existence and uniqueness of the high-voltage solution (equilibrium) is rigorously demonstrated by means of complete normed linear (Banach) spaces and contraction mapping theory.
2) A sufficient condition for selecting the reference voltage V_{ref} in DC microgrids with CPLs is obtained to guide the control design and likewise ensure a safe and reliable microgrid operation.
3) A numerical case-study is being carried out to test the developed condition and gain additional insights.

The remainder of this section introduces useful notations and theoretical preliminaries, while the rest of the paper is organised in the following manner. Section 2 describes the meshed microgrid model network, while the constant power loads model is detailed in Section 3, where the characteristic polynomial and the voltage solutions are also brought to attention. In Section 4, the proof of existence and uniqueness of the high-voltage equilibria is presented in the form of a theorem. Finally, a numerical case-study has been investigated in Section 5 to test the sufficient condition and assess the feasible solution set, followed by conclusions being drawn in Section 6.

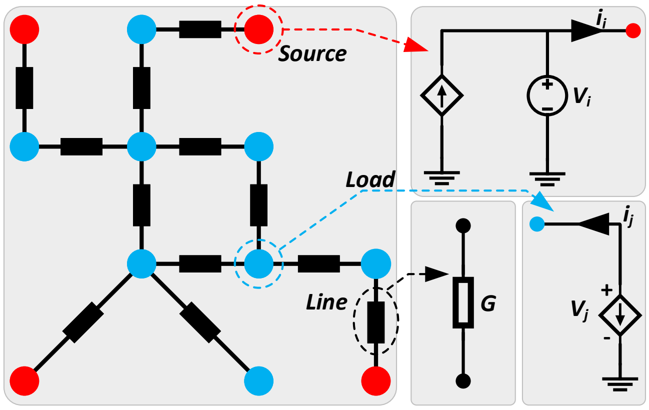

Fig. 2. Meshed DC microgrid network with source and load nodes ($i \in \{1, \ldots, n_S\}, j \in \{n_S + 1, \ldots, n\}$)

C. Notation and preliminaries

1) Vectors and matrices
Let $\mathbf{1}_n \in \mathbf{R}^n$, $\mathbf{0}_n \in \mathbf{R}^n$ and $\mathbf{1}_{n\times n} \in \mathbf{R}^{n\times n}$, $\mathbf{0}_{n\times n} \in \mathbf{R}^{n\times n}$ be the n-dimensional vectors and square matrices of all 1's and 0's, respectively. Given an n-tuple sequence $(x_1, \ldots, x_n)$, let $\mathbf{x} \in \mathbf{R}^n$ be the associated vector and $[\mathbf{x}] \in \mathbf{R}^{n\times n}$ the diagonal matrix whose diagonal terms are the elements of vector $\mathbf{x}$. If A and B are matrices then *blkdiag*$\{A \ \ B\}$ represents the block-diagonal matrix having A and B as diagonal block entries. In particular, if they have the same number of columns, then col $(A \ \ B)$ denotes the matrix $[A \ \ B]^T$.
2) Contraction mapping
Definition 1. *(Infinite norm) The infinite norm of a vector* $\mathbf{x}$*, denoted* $||\mathbf{x}||_\infty$*, is defined as the scalar equal to the maximum of the absolute value of all vector entries, i.e.*
$$||\mathbf{x}||_\infty = \max|x_i|.$$
Definition 2. *(Banach space) A normed linear space* $\mathcal{X}$ *is a Banach space if every Cauchy sequence in* $\mathcal{X}$ *converges to a vector in* $\mathcal{X}$*.*
Theorem 1. *(Contraction mapping theorem) With* $\mathcal{X}$ *a Banach space, let* S *be a closed subset of* $\mathcal{X}$ *and* T *a mapping that maps* S *into* S*. If*
$$\|T(x) - T(y)\|_\infty \leq \gamma \|x - y\|_\infty, \ \forall\, x, y \in S,$$
with $0 \leq \gamma < 1$*, then there exists a unique vector* $x^* \in S$ *that satisfies* $x^* = T(x^*)$*.*
Proof. Presented in [23, Appx.B].

II. DC Microgrid Model

The DC micro-grid modelled in Fig. 2, induces an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V} \in \mathcal{I}$ being the set of vertices, represented by bus nodes, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges, representing interconnecting lines in the microgrid, here assumed resistive. The set of nodes $\mathcal{V}$ is divided in two subsets; the sources subset n_S, and the loads subset n_L, such that $n = n_S + n_L$. Then, the vectors of currents and voltages at each node of $\mathcal{G}$ can be denoted as $i = \text{col}\,(i_S \; i_L) \in \mathbf{R}^n$ and $V = \text{col}\,(V_S \; V_L) \in \mathbf{R}^n$, with $i_S = \text{col}\,(i_1, \ldots, i_{n_S})$, $i_L = \text{col}\,\big(i_{n_{S+1}}, \ldots, i_n\big)$, and $V_S = \text{col}\,(V_1, \ldots, V_{n_S})$, $V_L = \text{col}\,\big(V_{n_{S+1}}, \ldots, V_n\big)$. The current-voltage relation will be given by $i = YV$, as follows

$$\begin{bmatrix} i_S \\ i_L \end{bmatrix} = \begin{bmatrix} Y_{SS} & Y_{SL} \\ Y_{LS} & Y_{LL} \end{bmatrix} \begin{bmatrix} V_S \\ V_L \end{bmatrix} \tag{1}$$

Note that matrices Y_{SS} and Y_{LL} are symmetric, with Y_{LL} having, as we have shown in a previous work [24], the following expression

$$Y_{LL} = Y_{LN} + [V_L]^{-2}\,[P] \tag{2}$$

where matrix $[V_L]^{-2}\,[P]$ represents the equivalent conductance of the CPLs. One can infer that matrices Y_{SS} and Y_{LN} are positive-definite[1], as they are submatrices of the conductance matrix without the CPLs, i.e. $\mathcal{L}_Y = Y - blkdiag\{\mathbf{0}_{n_S \times n_S} \; [V_L]^{-2}\,[P]\}$. The row-sums of matrix $\mathcal{L}_Y$ are null, that is

$$Y_{SS}\mathbf{1}_{n_S} + Y_{SL}\mathbf{1}_{n_L} = \mathbf{0}_{n_S} \tag{3}$$

$$Y_{LN}\mathbf{1}_{n_L} + Y_{LS}\mathbf{1}_{n_S} = \mathbf{0}_{n_L} \tag{4}$$

since the network induces a connected and undirected graph $\mathcal{G}$, whose corresponding Laplacian matrix is symmetric and balanced ($\mathcal{L}_Y \mathbf{1}_n = \mathbf{0}_n$).

III. Constant power loads

For the constant power loads, the power balance equation in vector-form is expressed as

$$[V_L]\, i_L = -P \tag{5}$$

where $P = \text{col}\,(P_{n_L+1}, \ldots, P_n)$ is constant and represents the requested power at each load node. Note that, since the CPL current direction is opposite to the reference direction (as seen in Fig. 2), the negative sign appears on the right side of equation (5).

By combining equations (1) and (5), one obtains

$$Y_{LS}V_S + Y_{LL}V_L = -\,[V_L]^{-1}\,P, \tag{6}$$

or, equivalently,

$$[V_L]\, Y_{LS}V_S + [V_L]\, Y_{LL}V_L + P = \mathbf{0}_{n_L}. \tag{7}$$

It is clear that V_L has two solutions given by the roots of the above polynomial. The low-voltage and high-voltage solutions have also been reported in [15], with the feasible solution being the high-voltage.

Remark 1. *In [15] the high voltage solution has been chosen based on the feasibility argument. In [16], [17] the high-voltage solution was shown to be feasible by using a current-limiting approach that can also ensure that the output current $|i_S| \leq i^{max}$ when having boost converters. However, so far, the argument to support this claim seems insufficiently supported by rigorous theoretical proofs.*

[1]This statement is more or less obvious, nonetheless, it can be trivially proven using Cauchy's interlacing theorem.

Moreover, following Remark 1, the existence and uniqueness of the high-voltage solution equilibria, is a necessary prerequisite for a reliable DC microgrid operation.

By substituting matrix Y_{LL} from (2) in equation (6), the latter becomes

$$[V_L]\, Y_{LS}V_S + [V_L]\, Y_{LN}V_L + 2P = \mathbf{0}_{n_L}, \tag{8}$$

and by left-multiplying (8) with $Y_{LN}^{-1}[V_L]^{-1}$, it yields

$$Y_{LN}^{-1}Y_{LS}V_S + V_L + 2Y_{LN}^{-1}\,[V_L]^{-1}\,P = \mathbf{0}_{n_L}. \tag{9}$$

Consider the following theorem.

Theorem 2. *For a connected DC microgrid configuration, the following statement holds*

$$Y_{LN}^{-1}Y_{LS} = -Q, \tag{10}$$

where $Q \in \mathbf{R}^{n_L \times n_S}$ is a matrix with row-sums equal to 1.

Proof. One could equivalently prove the rewritten equation (10), which is

$$Y_{LN}Q + Y_{LS} = \mathbf{0}_{n_L \times n_S}. \tag{11}$$

By right-multiplication with vector $\mathbf{1}_{n_S}$, one gets

$$Y_{LN}Q\mathbf{1}_{n_S} + Y_{LS}\mathbf{1}_{n_S} = \mathbf{0}_{n_L \times n_S}. \tag{12}$$

Notice that $Q\mathbf{1}_{n_S} = \mathbf{1}_{n_L}$, and based on the row-sum identity (4), equation (12) holds. Thus, statement (10) is proven. The proof is complete. □

Hence, by virtue of Theorem 2, equation (9) becomes

$$V_L = QV_S - 2Y_{LN}^{-1}\,[V_L]^{-1}\,P. \tag{13}$$

Remark 2. *In a dynamical system V_L and/or i_L are state variables or state-dependent variables. Hence, as the solutions of the polynomial would also represent the equilibria of the system, the terms solutions and equilibria are used interchangeably throughout the entirety of this paper.*

IV. Existence and uniqueness of equilibria

Prior to presenting the main result, let us make the following assumption.

Assumption 1. *At steady state, the source voltages V_S are assumed to be equal to their respective reference values $V_{S,ref} = [V_{ref,1} \ldots V_{ref,n_S}]^T$. To make the subsequent analysis more straightforward, one can assume, without loss of generality and without influencing the analysis and the end results, same value for all entries of the reference voltage vector, i.e. $V_{ref}\mathbf{1}_{n_S}$.*

Remark 3. *It is clear that this is a sensible assumption to make. By keeping different voltage references one would eventually get $QV_{S,ref}$ which would be equal to a vector with the average values of the voltage references as entries (i.e. $QV_{S,ref} = \overline{V}_{S,ref}$), since matrix Q has unit row-sums. But, for simplicity one assumes same voltage reference. A trivial control design can be put in place and guarantee steady-state voltage regulation such that $V_S = V_{ref}\mathbf{1}_{n_S}$ (see* *for instance [25]); hence satisfying Assumption 1. However, since control design does not fall within the scope of this paper, the presence of Assumption 1 is required.*

Thus, when $V_S = V_{ref}\mathbf{1}_{n_S}$, the following vector relation is obtained $QV_S = QV_{ref}\mathbf{1}_{n_S} = V_{ref}\mathbf{1}_{n_L}$, yielding the modified equation (13) as follows

$$V_L = V_{ref}\mathbf{1}_{n_L} - 2Y_{LN}^{-1}\left[V_L\right]^{-1}P. \tag{14}$$

Next, consider the function $T : S \to S$ defined as

$$T\left(V_L\right) = V_{ref}\mathbf{1}_{n_L} - 2Y_{LN}^{-1}\left[V_L\right]^{-1}P. \tag{15}$$

According to Theorem 1, if there exists a non-empty compact set S such that $T\left(V_L\right)$ is a contraction mapping, then there exists a unique $V_L \in S$ such that $T\left(V_L\right) = V_L$. Following this latter argument, one can obtain a sufficient condition for the existence and uniqueness of equilibria in equation (9).

Theorem 3. *Consider* $\frac{V_{ref}}{\sqrt{2}} < V^* < V_{ref}$ *and define* $S = \{V_L \big| |V_L - V^*| \leq \rho V^*\}$. *Equation* (9) *admits a unique solution in* S *if the following condition is satisfied*

$$V_{ref} > \frac{2}{1-\rho}\sqrt{||Y_{LN}^{-1}P||_\infty}, \tag{16}$$

for any selection $\frac{1}{2} \leq \rho < 1$.

Proof. By virtue of Theorem 1, when $T\left(V_L\right)$ is both a self-mapping and a contraction mapping in S, function (15) admits a unique solution in S.

One starts by proving $T\left(V_L\right)$ is self-mapping, that is $T\left(V_L\right) \in S$ for any $V_L \in S$, i.e.

$$\|T\left(V_L\right)\|_\infty \leq \left(1+\rho\right)V^*. \tag{17}$$

Given $V^* \in \left(\frac{V_{ref}}{\sqrt{2}}, V_{ref}\right)$, $|V_L| \leq \left(1+\rho\right)V^*$, and if condition (16) is satisfied, it yields

$$\begin{aligned}\|T\left(V_L\right) - V_{ref}\mathbf{1}_{n_L}\|_\infty &\leq \frac{2\|Y_{LN}^{-1}P\|_\infty}{\min\{V_L\}} \\ &\leq \frac{\left(1-\rho\right)^2 V_{ref}^2}{2\left(1-\rho\right)V^*} \leq \frac{\left(1-\rho\right)^2 V^{*2}}{\left(1-\rho\right)V^*} \leq \left(1-\rho\right)V^*\end{aligned} \tag{18}$$

According to inequality (17), the above needs to satisfy

$$\begin{aligned}\left(1-\rho\right)V^* &\leq \left(1+\rho\right)V^* - V_{ref} \\ &\leq \left(1+\rho\right)V^* - V^* \leq \rho V^*,\end{aligned} \tag{19}$$

which is always ensured given $\rho \in \left[\frac{1}{2}, 1\right)$, thus, $T\left(V_L\right)$ is self-mapping. The first part of the proof is accomplished.

To show that function $T\left(V_L\right)$ is also a contraction mapping over S, let $V_X, V_Y \in S$ and consider

$$\begin{aligned}&\|T\left(V_X\right) - T\left(V_Y\right)\|_\infty \\ &\leq -\frac{2\min\{Y_{LN}^{-1}P\}}{\|V_X\|_\infty} + \frac{2\|Y_{LN}^{-1}P\|_\infty}{\min\{V_Y\}} \\ &\leq 2\|Y_{LN}^{-1}P\|_\infty\|[V_X]^{-1}\mathbf{1}_{n_L}\|_\infty\|[V_Y]^{-1}\mathbf{1}_{n_L}\|_\infty\|V_X - V_Y\|_\infty\end{aligned} \tag{20}$$

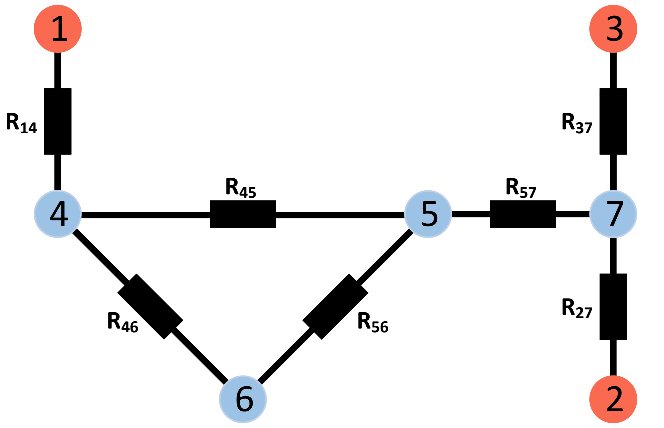

Fig. 3. Considered case-study DC microgrid network with seven source and load nodes

Since $V_X, V_Y \in S$, the following inequalities are obtained

$$\begin{cases}\|\left[V_X\right]^{-1}\mathbf{1}_{n_L}\|_\infty \leq \frac{1}{(1-\rho)V^*} \\ \|\left[V_Y\right]^{-1}\mathbf{1}_{n_L}\|_\infty \leq \frac{1}{(1-\rho)V^*}\end{cases} \tag{21}$$

By combining (20)-(21), it yields

$$||T\left(V_X\right) - T\left(V_Y\right)|| \leq \underbrace{\frac{2||Y_{LN}^{-1}P||_\infty}{\left(1-\rho\right)^2 V^{*2}}}_{\gamma} ||V_X - V_Y||. \tag{22}$$

Note that, since $V^* > \frac{V_{ref}}{\sqrt{2}}$ and based on condition (16), the following inequality takes place

$$\gamma = \frac{2||Y_{LN}^{-1}P||_\infty}{\left(1-\rho\right)^2 V^{*2}} < \frac{4||Y_{LN}^{-1}P||_\infty}{\left(1-\rho\right)^2 V_{ref}^2} < 1. \tag{23}$$

With the term $\gamma < 1$, and by virtue of Theorem 1, function $T\left(V_L\right)$ is also a contraction mapping in S. Thus, there exists a unique solution $V_L \in S$ such that $T\left(V_L\right) = V_L$. The proof of the theorem is completed. □

Remark 4. *Assuming the condition in Theorem 3 is satisfied, one can appropriately select the reference voltage for any* $\frac{1}{2} \leq \rho < 1$. *For instance, by choosing* $\rho = 0.5$ *and the nominal voltage* $V^* = V_{ref}$, *one would get a condition similar to the ones obtained or put as assumptions in [14]–[17]. However, considering the nominal voltage equal to the reference voltage is not a realistic assumption.*

V. Numerical Testing of the Equilibria Existence and Uniqueness

The meshed DC microgrid studied in this section is depicted in Fig. 3, with the system parameters specified in Table I. The network consists of 7 vertices/ nodes, having 3 source nodes (amber) and 4 load nodes (blue), interconnected via a resistive network. The corresponding Laplacian matrix $\mathcal{L}_Y$ looks as expressed below

$$\mathcal{L}_Y = \begin{bmatrix} 1.125 & 0 & 0 & -1.125 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 4.6607 & -1.2857 & -2.25 & 0 \\ -1.125 & 0 & 0 & -1.2857 & 4.8857 & -1.8 & -1.8 \\ 0 & 0 & 0 & -2.25 & -1.8 & 4.05 & 0 \\ 0 & -3 & -1 & 0 & -1.8 & 0 & 5.8 \end{bmatrix}.$$

With the nominal voltage $V^* = 45\,V$, the end goal is to identify graphically the voltage equilibria of the 4 constant power loads, and to observe the implications of choosing the reference voltage V_{ref} as specified by the condition given in Theorem 3, since it must be ensured that $V^* \in \left(\frac{V_{ref}}{\sqrt{2}}, V_{ref}\right)$. If it cannot be guaranteed, then the nominal voltage V^* must be increased to avoid voltage collapse as the power of the CPLs is too large.

A plot of the voltage polynomials appear in Fig. 4, where the power of the loads is $P = [110\ \ 150\ \ 95\ \ 125]\ W$. By computing condition (16), one obtains a lower value for V_{ref}, which requires the voltage reference to satisfy $V_{ref} > 55.0182\,V$. Hence, as shown in Fig. 4, to comply with this condition and also make sure that $\frac{V_{ref}}{\sqrt{2}} < V^*$, the voltage reference is selected as $V_{ref} = 60\,V$. The set of the high-voltage solutions is also computed as $S = [22.5\ \ 67.5]$ since one has $\rho = 0.5$. It can be seen, in Fig. 4, that all high-voltage solutions are contained in the set S, as expected and proved in the theoretical sections.

Some key observations and insights that could be helpful in future testings are worth mentioning here. The developed theory guarantees the existence and uniqueness of high-voltage equilibria given the condition is satisfied, but one needs to keep in mind that

- when choosing the voltage reference V_{ref} to satisfy the condition in Section 4, one also needs to remember that the nominal voltage has to be within the following bounds $\frac{V_{ref}}{\sqrt{2}} < V^* < V_{ref}$. If such a V_{ref} does not exist to satisfy the latter condition as well, then either V^* needs to be increased, or the CPLs' power decreased.
- if the parameter ρ is chosen close to maximum, i.e. $\rho \approx 1$, one might be tempted to think that the solution set will then include the low-voltage equilibria as well, since it will get very close to including the zero value, i.e. $S \in (0,\ 2V^*)$. However, parameter $\rho \to 1$ will affect condition (16) for the voltage reference, essentially rendering $V_{ref} \to \infty$.

TABLE I

NUMERICAL CASE-STUDY PARAMETERS

System Parameters	Numerical Values
R_{14}	$0.8889\,\Omega$
R_{27}	$0.3333\,\Omega$
R_{37}	$1\,\Omega$
R_{45}	$0.7778\,\Omega$
R_{46}	$0.4444\,\Omega$
R_{56}	$0.5556\,\Omega$
R_{57}	$0.5556\,\Omega$
P	$[110\ \ 150\ \ 95\ \ 125]\ W$
V^*	$45\,V$
V_{ref}	$60\,V$
ρ	0.5

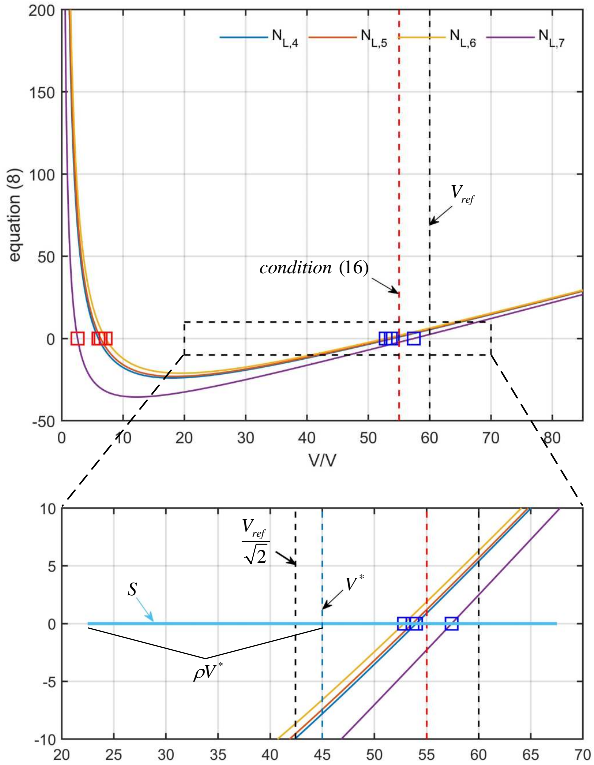

Fig. 4. Characteristic polynomial plots corresponding to the 4 load nodes, with a clear marking of the low ( ) and high-voltage ( ) equilibria

Still, note that in the preceding sections we obtained a sufficient condition for the reference voltage. Hence, V_{ref} not complying with condition (16) does not automatically imply that the equilibria does not exist, nor that is not unique in a particular solutions set. However, guaranteeing analytically that a unique solution exists is a well sought-after result.

VI. Conclusions

The voltage equilibria of CPLs in DC microgrids is paramount for the overall system operation, as it is an essential requirement for safe and stable functioning of the microgrid network. A sufficient condition has been developed by means of contraction theory to suitably select the reference voltage. Complying with the given condition guarantees the existence and uniqueness of the high-voltage solution of CPLs. The current findings and sufficient condition have also been tested to showcase the validity of the theoretical analysis.

A possible extension of the current work would be to study the stability of the high-voltage equilibria, which is an active research pursuit, and several steps in this direction have already been taken, and more can be found in the literature.

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