

This is a repository copy of *Dirichlet is not just bad and singular*.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/186080/>

Version: Published Version

Article:

Beresnevich, Victor orcid.org/0000-0002-1811-9697, Guan, Lifan, Marnat, Antoine et al. (2 more authors) (2022) *Dirichlet is not just bad and singular*. *Advances in Mathematics*. 108316. ISSN 0001-8708

<https://doi.org/10.1016/j.aim.2022.108316>

Reuse

This article is distributed under the terms of the Creative Commons Attribution (CC BY) licence. This licence allows you to distribute, remix, tweak, and build upon the work, even commercially, as long as you credit the authors for the original work. More information and the full terms of the licence here:

<https://creativecommons.org/licenses/>

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.

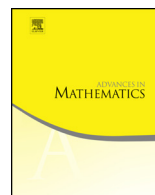


ELSEVIER

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim



Dirichlet is not just bad and singular

Victor Beresnevich^{a,1}, Lifan Guan^{b,2}, Antoine Marnat^{c,3},
Felipe Ramírez^{d,4}, Sanju Velani^{a,*,5}^a Department of Mathematics, University of York, Heslington, York, YO10 5DD, England, United Kingdom^b Institute for Theoretical Sciences and School of Science, Westlake University, Hangzhou, Zhejiang, China^c Moscow Center of Fundamental and Applied Mathematics, Leninskie Gory, Moscow, 119991, Russian Federation^d Department of Mathematics and Computer Science, Wesleyan University, 265 Church Street Middletown, CT 06459, United States of America

ARTICLE INFO

Article history:

Received 13 March 2021

Received in revised form 15

February 2022

Accepted 18 February 2022

Available online 17 March 2022

Communicated by George Andrews

Dedicated to Pru and Chim - both
60no and going strong!

MSC:

11J13

11H06

ABSTRACT

It is well known that in dimension one the set of Dirichlet improvable real numbers consists precisely of badly approximable and singular numbers. We show that in higher dimensions this is not the case by proving that there exist continuum many Dirichlet improvable vectors that are neither badly approximable nor singular. This is a consequence of a stronger statement that involves very well approximable points. In the last section we formulate the notion of intermediate Dirichlet improvable sets concerning approximations by rational planes of every intermediate dimension and show that they coincide. This naturally extends a classical theorem of Davenport & Schmidt (1969) which states that the simultaneous form of Dirichlet's theorem is improvable if and only if the dual form is improvable. Consequently, our main "continuum" result is

* Corresponding author.

E-mail addresses: victor.beresnevich@york.ac.uk (V. Beresnevich), guanlifan@westlake.edu.cn (L. Guan), antoine.marnat@univie.ac.at (A. Marnat), framirez@wesleyan.edu (F. Ramírez), sanju.velani@york.ac.uk (S. Velani).

¹ Research partly supported by EPSRC Programme grant EP/J018260/1.² Research supported by EPSRC Programme grant EP/J018260/1 and ERC Consolidator grant 648329.³ Research supported by EPSRC Programme grant EP/J018260/1 and FWF Project I 3466-N35.⁴ Research partly supported by EPSRC Programme grant EP/J018260/1.⁵ Research partly supported by EPSRC Programme grant EP/J018260/1.<https://doi.org/10.1016/j.aim.2022.108316>0001-8708/© 2022 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

Keywords:

Dirichlet improvable numbers
Intermediate Diophantine sets

equally valid for the corresponding intermediate Diophantine sets of badly approximable, singular and Dirichlet improvable points.

© 2022 The Authors. Published by Elsevier Inc. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

1.1. Background and motivation

The main goal of this paper is to investigate the relation between three basic sets arising from Dirichlet's fundamental theorem in the classical theory of Diophantine approximation. It is therefore natural to start with the statement of the theorem and in turn describe the associated sets. For $x \in \mathbb{R}$, let $\langle x \rangle := \min\{|x - m| : m \in \mathbb{Z}\}$ denote the distance from x to the nearest integer and for $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ let

$$\langle \mathbf{x} \rangle := \max_{1 \leq i \leq n} \langle x_i \rangle.$$

Theorem 1.1 (*Dirichlet*). *For any $\mathbf{x} \in \mathbb{R}^n$ and $N \in \mathbb{N}$, there exists $q \in \mathbb{Z}$ such that*

$$\langle q\mathbf{x} \rangle < N^{-\frac{1}{n}} \quad \text{and} \quad 1 \leq q \leq N. \quad (1.1)$$

An important consequence of Dirichlet's theorem is the following statement.

Corollary 1.1 (*Dirichlet*). *For any $\mathbf{x} \in \mathbb{R}^n$ there exist infinitely many $q \in \mathbb{N}$ such that*

$$\langle q\mathbf{x} \rangle < q^{-\frac{1}{n}}. \quad (1.2)$$

The above foundational theorem from the theory of simultaneous Diophantine approximation prompts the following natural question:

Question I. Can Dirichlet's theorem be improved?

The following notions help make the question more precise. Following Davenport & Schmidt [11], for a particular $\mathbf{x} \in \mathbb{R}^n$ we say that improvement in Dirichlet's theorem is possible if there exists a constant $\varepsilon \in (0, 1)$ such that, for all $N > N_0(\mathbf{x}, \varepsilon)$ sufficiently large there exists $q \in \mathbb{Z}$ such that

$$\langle q\mathbf{x} \rangle < \varepsilon N^{-\frac{1}{n}} \quad \text{and} \quad 1 \leq q \leq N. \quad (1.3)$$

For obvious reasons such an \mathbf{x} is referred to as *Dirichlet improvable* and we let \mathbf{DI}_n denote the set of Dirichlet improvable points in \mathbb{R}^n . Furthermore, we say that $\mathbf{x} \in \mathbb{R}^n$ is *singular* if it is Dirichlet improvable with $\varepsilon > 0$ arbitrarily small; that is, for any $\varepsilon > 0$

for all sufficiently large N there exists $q \in \mathbb{Z}$ satisfying (1.3). We let \mathbf{Sing}_n denote the set of singular points in \mathbb{R}^n . By definition, we clearly have that

$$\mathbf{Sing}_n \subseteq \mathbf{DI}_n$$

and it is easily verified that \mathbf{Sing}_n contains every rational hyperplane in \mathbb{R}^n . Thus

$$n - 1 \leq \dim \mathbf{Sing}_n \leq n.$$

Here and throughout, $\dim X$ will denote the Hausdorff dimension of a subset X of \mathbb{R}^n . In the case $n = 1$, a nifty argument due to Khintchine [17] dating back to the twenties shows that a real number is singular if and only if it is rational; that is

$$\mathbf{Sing}_1 = \mathbb{Q}. \tag{1.4}$$

Recently, Cheung & Chevallier [7], building on the spectacular $n = 2$ work of Cheung [6], have shown that

$$\dim \mathbf{Sing}_n = \frac{n^2}{n + 1} \quad \text{for all } n \geq 2.$$

Note that since $\frac{n^2}{n+1} > n - 1$, this immediately implies that in higher dimensions \mathbf{Sing}_n does not simply correspond to rationally dependent $\mathbf{x} \in \mathbb{R}^n$ as in the one-dimensional case – the theory is much richer.

In [11], Davenport & Schmidt established various results concerning the set \mathbf{DI}_n of Dirichlet improvable points and the set \mathbf{Bad}_n of simultaneously badly approximable points. In particular they showed (see [11, Theorem 2]) that

$$\mathbf{Bad}_n \subseteq \mathbf{DI}_n. \tag{1.5}$$

Recall, $\mathbf{x} \in \mathbb{R}^n$ is said to be *badly approximable* if there exists a constant $\varepsilon = \varepsilon(\mathbf{x}) \in (0, 1)$ so that

$$\langle q\mathbf{x} \rangle > \varepsilon q^{-\frac{1}{n}} \quad \forall \quad q \in \mathbb{N}. \tag{1.6}$$

In other words, \mathbf{Bad}_n corresponds to those $\mathbf{x} \in \mathbb{R}^n$ for which the right hand side of the inequality appearing in Dirichlet’s corollary, namely (1.2), cannot be improved by $\varepsilon > 0$ arbitrarily small. By definition, we clearly have that $\mathbf{Bad}_n \cap \mathbf{Sing}_n = \emptyset$. It is worth mentioning the well known fact that \mathbf{Bad}_n is a set of n -dimensional Lebesgue measure zero but of full dimension; i.e.

$$\dim \mathbf{Bad}_n = n.$$

In view of (1.5) it thus follows that

$$\dim \mathbf{DI}_n = n.$$

In a follow-up paper [12], Davenport & Schmidt showed that \mathbf{DI}_n is a set of n -dimensional Lebesgue measure zero and thus in terms of measure and dimension it has the same properties as the set \mathbf{Bad}_n . In the case $n = 1$, much more is true: any irrational $x \in \mathbb{R}$ is Dirichlet improvable if and only if it is badly approximable. This for example follows directly from [11, Theorem 1] and together with (1.4) implies that

$$\mathbf{DI}_1 = \mathbf{Bad}_1 \cup \mathbf{Sing}_1. \quad (1.7)$$

Thus, in view of (1.7) we have a complete characterisation in dimension one. In higher dimensions, we know that

$$\mathbf{DI}_n \supseteq \mathbf{Bad}_n \cup \mathbf{Sing}_n$$

but surprisingly it seems unknown whether or not equality is possible. In other words, the answer to the following basic problem seems unknown. As far as we are aware, it first appeared in print in Fabian Süess' beautifully written PhD thesis [28, Section 4.1].

Problem 1.1. *Is the set $\mathbf{DI}_n \setminus (\mathbf{Bad}_n \cup \mathbf{Sing}_n)$ empty when $n \geq 2$?*

The key goal of this paper is to show that it is not. Maybe it is “folklore” that in higher dimensions there exist Dirichlet improvable points that are neither badly approximable nor singular. However, we would like to stress that we are unaware of any such a statement.

Theorem 1.2. *For $n \geq 2$, the set*

$$\mathbf{FS}_n := \mathbf{DI}_n \setminus (\mathbf{Bad}_n \cup \mathbf{Sing}_n)$$

has continuum many points.

We suspect that our theorem is far from the truth. Indeed, it may well be the case that for $n \geq 2$

$$\dim \mathbf{FS}_n = n.$$

As we shall see in the next section, we actually prove a more general and effective version of Theorem 1.2. Unfortunately, the effective Theorem 1.5 sheds no light on the dimension of \mathbf{FS}_n . However, it does imply various natural strengthenings of Theorem 1.2 such as the following in which we also remove the set \mathbf{VW}_n of very well approximable numbers. Recall, $\mathbf{x} \in \mathbb{R}^n$ is said to be *very well approximable* if there exists a constant $\varepsilon = \varepsilon(\mathbf{x}) > 0$ such that

$$\langle q\mathbf{x} \rangle < q^{-\left(\frac{1}{n} + \varepsilon\right)} \quad \text{for infinitely many } q \in \mathbb{N}.$$

Theorem 1.2*. For $n \geq 2$, the set

$$\mathbf{FS}_n := \mathbf{DI}_n \setminus (\mathbf{Bad}_n \cup \mathbf{Sing}_n \cup \mathbf{VW}_n)$$

has continuum many points.

Remark 1.1. Following the appearance of the first pre-print version of this paper on the arXiv, Nikolay Moshchevitin kindly pointed out the work of Akhunzhanov and Shatskov [2]. For $n = 2$, they compute the Dirichlet spectrum for simultaneous approximation by rational points with respect to the Euclidean norm. In short, their method uses the theory of best approximations and can be adapted to construct Dirichlet improvable points in \mathbb{R}^2 that are not simultaneously singular or badly approximable. At the point of revising this paper for publication, the details of the adaptation have since appeared in the paper [1] involving Akhunzhanov and Moshchevitin – see the “in particular” statement following [1, Theorem 1].

1.2. The setup, further background and main results

Recall that from the classical point of view there are two forms of Diophantine approximation in \mathbb{R}^n ; one corresponding to (simultaneous) approximation by rational points as considered in the previous section and the other corresponding to (dual) approximation by rational hyperplanes. Concerning the latter, the dual version of Dirichlet’s theorem states that for any $\mathbf{x} \in \mathbb{R}^n$ and $N \in \mathbb{N}$ there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\langle \mathbf{q} \cdot \mathbf{x} \rangle \leq N^{-n} \quad \text{and} \quad \|\mathbf{q}\| \leq N. \tag{1.8}$$

Here and elsewhere $\mathbf{q} \cdot \mathbf{x} := q_1x_1 + \dots + q_nx_n$ is the standard inner product and $\|\mathbf{q}\| := \max\{|q_1|, \dots, |q_n|\}$ is the supremum norm of \mathbf{q} . Also, recall that $\langle x \rangle = \min\{|x - m| : m \in \mathbb{Z}\}$. In [12, Theorem 2], Davenport & Schmidt proved that the dual form of Dirichlet’s theorem is improvable if and only if the simultaneous form of Dirichlet’s theorem (Theorem 1.1) is improvable. So it follows that $\mathbf{x} \in \mathbf{DI}_n$ if and only if there exists $\varepsilon \in (0, 1)$ such that, for all $N > N_0(\mathbf{x}, \varepsilon)$ sufficiently large there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that

$$\langle \mathbf{q} \cdot \mathbf{x} \rangle \leq \varepsilon N^{-n} \quad \text{and} \quad \|\mathbf{q}\| \leq N. \tag{1.9}$$

Indeed, the same transference between the simultaneous and dual forms of approximation is true when considering the set of singular points \mathbf{Sing}_n . However, this dual versus simultaneous equivalence for singular points (and indeed badly approximable points) holds in a much wider context. This we now describe since it will be the setting of our main result.

Let d be integer satisfying $0 \leq d \leq n - 1$. The setup we now consider is one in which we approximate points $\mathbf{x} \in \mathbb{R}^n$ by d -dimensional rational affine subspaces $L \subset \mathbb{R}^n$. With this in mind, we let

$$d(\mathbf{x}, L) := \min_{\mathbf{y} \in L} \|\mathbf{x} - \mathbf{y}\| = \min_{\mathbf{y} \in L} \max_{1 \leq i \leq n} |x_i - y_i| \tag{1.10}$$

denote the minimal distance between \mathbf{x} and L . We also let $H(L)$ denote the height of L . In short, $H(L)$ is the co-volume of the sub-lattice $\mathbb{Z}^{n+1} \cap L_0$ where L_0 is the unique $(d + 1)$ -dimensional subspace of \mathbb{R}^{n+1} containing the d -dimensional embedding $\{(y_1, \dots, y_n, 1) : \mathbf{y} \in L\}$ of L into \mathbb{R}^{n+1} . This notion of height is relatively standard and is usually referred to as the projective or Weil height of L – see [8,20,24] for more details. Note that when $d = 0$, L reduces to a rational point $\frac{\mathbf{p}}{q} := (\frac{p_1}{q}, \dots, \frac{p_n}{q})$ for some $(\mathbf{p}, q) \in \mathbb{Z}^n \times \mathbb{Z} \setminus \{0\}$. In turn, we have that

$$H(L) \asymp \max\{\|\mathbf{p}\|, |q|\} \quad \text{and} \quad d(\mathbf{x}, L) = \max_{1 \leq i \leq n} \frac{|qx_i - p_i|}{|q|}.$$

Also note that when $d = n - 1$, L reduces to a rational affine hyperplane $\{\mathbf{y} \in \mathbb{R}^n : \mathbf{q} \cdot \mathbf{y} = p\}$ for some $(\mathbf{q}, p) \in \mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}$. In turn, we have that

$$H(L) \asymp \max\{|p|, \|\mathbf{q}\|\} \quad \text{and} \quad d(\mathbf{x}, L) \asymp \frac{|q_1x_1 + \dots + q_nx_n - p|}{\|\mathbf{q}\|}.$$

To simplify notation the symbols \ll and \gg will be used to indicate an inequality with an unspecified positive multiplicative constant. If $a \ll b$ and $a \gg b$ we write $a \asymp b$, and say that the quantities a and b are *comparable*. In the above, the implied ‘comparability’ constants are dependent on n . Thus, up to some multiplicative constants, the extreme cases $d = 0$ and $d = n - 1$ correspond to the standard simultaneous and dual forms of Diophantine approximation. We now consider the natural analogues of the sets \mathbf{Bad}_n and \mathbf{Sing}_n introduced within the framework of simultaneous Diophantine approximation in §1.1. With this in mind, we start by stating a Dirichlet type theorem for approximation by d -dimensional rational subspaces. Throughout, given $n \in \mathbb{N}$ and $d \in \{0, 1, \dots, n - 1\}$, we let

$$\omega_d := \frac{d + 1}{n - d}.$$

Theorem 1.3. *Let $n \in \mathbb{N}$ and d be integer satisfying $0 \leq d \leq n - 1$. Then for any $\mathbf{x} \in \mathbb{R}^n$ there exists a constant $c = c(n, d, \mathbf{x}) > 0$, such that for any $N \in \mathbb{N}$ there exists a d -dimensional rational affine subspace $L \subset \mathbb{R}^n$, such that*

$$d(\mathbf{x}, L) \leq cH(L)^{-1}N^{-\omega_d} \quad \text{and} \quad H(L) \leq N. \tag{1.11}$$

The above statement is a consequence of standard tools from the geometry of numbers such as Minkowski’s second convex body theorem and Mahler’s theory for compound

bodies. For completeness, the details are given in §4 (see Proposition 4.1 in §4.1.2 and Proposition 4.3 in §4.2.1). In turn, the theorem gives rise to the following statement.

Corollary 1.2. *Let $n \in \mathbb{N}$ and d be integer satisfying $0 \leq d \leq n - 1$. Then for any $\mathbf{x} \in \mathbb{R}^n$ with at least $(d + 1)$ -rationally independent coordinates, there exists a constant $c = c(n, d, \mathbf{x}) > 0$ and infinitely many d -dimensional rational affine subspaces $L \subset \mathbb{R}^n$ such that*

$$d(\mathbf{x}, L) \leq c H(L)^{-1-\omega_d}. \tag{1.12}$$

Remark 1.2. Proposition 4.3 in §4.2.1 together with the remark immediately preceding it, enables us to explicitly compute the constant $c = c(n, d, \mathbf{x})$ appearing in the above results. Observe that if we restrict \mathbf{x} to a bounded subset of \mathbb{R}^n , then the constant c can be made to be independent of \mathbf{x} . In particular, if $\mathbf{x} \in [0, 1]^n$ then in the simultaneous (resp. dual) case we can replace $H(L)$ by $|q|$ (resp. $\|\mathbf{q}\|$) in the theorem and corollary, and the inequalities corresponding to (1.11) and (1.12) remain valid if we translate \mathbf{x} by an integer vector. Thus, up to a constant dependent only on the dimension n , Theorem 1.3 and its corollary coincide with the classical simultaneous and dual forms of Dirichlet theorem and its corollary.

Taking our lead from the classical simultaneous and dual settings, we say that a point $\mathbf{x} \in \mathbb{R}^n$ is *d-singular* if for any given $\varepsilon \in (0, 1)$ and $N > N_0(\mathbf{x}, \varepsilon, d)$ sufficiently large, there exists a d -dimensional rational affine subspace $L \subset \mathbb{R}^n$ such that

$$d(\mathbf{x}, L) \leq \varepsilon H(L)^{-1} N^{-\omega_d} \quad \text{and} \quad H(L) \leq N. \tag{1.13}$$

On the other hand, we say that a point $\mathbf{x} \in \mathbb{R}^n$ is *d-badly approximable* if there exists a constant $\varepsilon = \varepsilon(\mathbf{x}) \in (0, 1)$ so that

$$d(\mathbf{x}, L) > \varepsilon H(L)^{-1-\omega_d} \tag{1.14}$$

for all d -dimensional rational affine subspaces $L \subset \mathbb{R}^n$. Finally, we let \mathbf{Sing}_n^d (resp. \mathbf{Bad}_n^d) denote the set of d -singular (resp. d -badly approximable) points in \mathbb{R}^n .

The following shows that the well known classical equivalence between the simultaneous and dual singular points (and indeed badly approximable points) holds in the general context of approximation by d -dimensional rational affine subspaces. We provide a proof in §2.2.

Proposition 1.1. *Let $n \in \mathbb{N}$ and d be integer satisfying $0 \leq d \leq n - 1$. Then,*

$$\mathbf{Sing}_n := \mathbf{Sing}_n^0 = \mathbf{Sing}_n^d \quad \text{and} \quad \mathbf{Bad}_n := \mathbf{Bad}_n^0 = \mathbf{Bad}_n^d.$$

Remark 1.3. Note that for the purpose of defining d -singular and d -badly approximable points it makes no difference whether the minimal distance $d(\mathbf{x}, L)$ is defined via the maximum norm (as in (1.10)) or some other norm (such as the Euclidean norm). The point is that these notions are not sensitive to the actual value of the constant $c = c(n, d, \mathbf{x})$ appearing in Theorem 1.3 and its corollary. Thus, most importantly, the set \mathbf{Sing}_n^d (resp. \mathbf{Bad}_n^d) coincides with the classical simultaneous singular (resp. badly approximable) set when $d = 0$ and the dual singular (resp. badly approximable) set when $d = n - 1$. However, when it comes to defining the ‘right’ notion of d -Dirichlet improvable points it is paramount that c is optimal and that (an appropriate version of) Theorem 1.3 coincides with the classical simultaneous and dual forms of Dirichlet theorem. Clearly, in its current form it fails to do so. In the last section of this paper we shall propose two versions (algebraic and geometric) of Theorem 1.3 leading to corresponding notions of Dirichlet improvable points that rectify this issue. Although it is not particularly relevant within the context of our main result, we hope the last section is of independent interest. In short, we show that within either setting the corresponding d -Dirichlet improvable sets \mathbf{DI}_n^d are all equivalent and coincide with the classical set \mathbf{DI}_n ; that is, the set of Dirichlet improvable points in \mathbb{R}^n defined via either the classical simultaneous ($d = 0$) or dual ($d = n - 1$) form of Dirichlet’s theorem (both gives rise to the same set thanks to the aforementioned statement of Davenport & Schmidt). It thus follows that the d -Dirichlet improvable sets defined via the algebraic and geometric settings also coincide.

In order to state our main result, it is convenient to introduce the notion of exponents of Diophantine approximation.

Definition 1.1. Let d be an integer with $0 \leq d \leq n - 1$ and let $\mathbf{x} \in \mathbb{R}^n$. We define the d^{th} ordinary exponent $\omega_d(\mathbf{x})$ (resp. the d^{th} uniform exponent $\hat{\omega}_d(\mathbf{x})$) as the supremum of the real numbers ω for which there exist d -dimensional rational affine subspaces $L \subset \mathbb{R}^n$ such that

$$d(\mathbf{x}, L) \leq H(L)^{-1} N^{-\omega} \quad \text{and} \quad H(L) \leq N,$$

for arbitrarily large real numbers N (resp. for every sufficiently large real number N).

Remark 1.4. By definition, whenever $\omega_d(\mathbf{x})$ is finite, there exists infinitely many d -dimensional rational affine subspaces $L \subset \mathbb{R}^n$ such that

$$d(\mathbf{x}, L) \leq H(L)^{-1-\omega}$$

if $\omega < \omega_d(\mathbf{x})$, and if $\omega > \omega_d(\mathbf{x})$ there are at most finitely many such subspaces $L \subset \mathbb{R}^n$.

Remark 1.5. In [10] a point $\mathbf{x} \in \mathbb{R}^n$ satisfying $\hat{\omega}_0(\mathbf{x}) > 1/n$ (equivalently $\hat{\omega}_{n-1}(\mathbf{x}) > n$) is called very singular and the set of such points is denoted by \mathbf{VSing}_n . In the context of approximation by d -dimensional rational affine subspaces, it is natural to define the notion of d -very singular points as points in the set

$$\mathbf{VSing}_n^d := \{\mathbf{x} \in \mathbb{R}^n : \hat{\omega}_d(\mathbf{x}) > \omega_d\}.$$

As is the case of badly approximable and singular sets, it turns out that the sets \mathbf{VSing}_n^d ($0 \leq d \leq n - 1$) are the same – see Remark 1.6 below. By definition, a d -very singular point is d -singular and since both notions are independent of d we can simply write $\mathbf{VSing}_n \subseteq \mathbf{Sing}_n$.

Within the classical simultaneous and dual forms of Diophantine approximation, the above exponents were introduced by Khintchine [17,18] and Jarník [16] in the nineteen twenties and thirties. For $n \geq 3$, the intermediate exponents (i.e., those corresponding to $1 \leq d \leq n - 2$) were formally introduced by Laurent [20] in 2009 but had implicitly been studied by Schmidt [24] some fifty years earlier. Clearly, for any $\mathbf{x} \in \mathbb{R}^n$ we have that $\omega_d(\mathbf{x}) \geq \hat{\omega}_d(\mathbf{x})$ and a direct consequence of Theorem 1.3 is that

$$\omega_d(\mathbf{x}) \geq \hat{\omega}_d(\mathbf{x}) \geq \omega_d := \frac{d + 1}{n - d}.$$

Observe that $\omega_0 = \frac{1}{n}$ and $\omega_{n-1} = n$. Thus, when $d = 0$ (resp. $d = n - 1$) the quantity ω_d coincides with the exponent appearing in the classical simultaneous (resp. dual) form of Dirichlet’s theorem. Another reasonably straightforward consequence, this time of the Borel-Cantelli lemma from probability theory, is that

$$\omega_d(\mathbf{x}) = \omega_d \quad \text{for almost all } \mathbf{x} \in \mathbb{R}^n. \tag{1.15}$$

The following elegant transference principle enables us to transfer information between the ordinary Diophantine exponents $\omega_d(\mathbf{x})$ associated with approximating points in $\mathbf{x} \in \mathbb{R}^n$ by d -dimensional rational subspaces of \mathbb{R}^n . It makes sense to include the statement at this point since one of the conditions turns up in the statement of our main theorem.

Theorem 1.4 (Laurent & Roy). *Let $n \geq 2$. For any point $\mathbf{x} \in \mathbb{R}^n$ with $1, x_1, \dots, x_n$ linearly independent over \mathbb{Q} , we have that $\omega_0(\mathbf{x}) \geq \omega_0$ and*

$$\frac{d\omega_d(\mathbf{x})}{\omega_d(\mathbf{x}) + d + 1} \leq \omega_{d-1}(\mathbf{x}) \leq \frac{(n - d)\omega_d(\mathbf{x}) - 1}{n - d + 1} \quad (1 \leq d \leq n - 1). \tag{1.16}$$

If $\omega_d(\mathbf{x}) = \infty$, the left hand side in (1.16) is replaced by d . Furthermore, given any $\tau_0, \dots, \tau_{n-1} \in [0, \infty]$ with $\tau_0 \geq \omega_0$ and

$$\frac{d\tau_d}{\tau_d + d + 1} \leq \tau_{d-1} \leq \frac{(n - d)\tau_d - 1}{n - d + 1} \quad (1 \leq d \leq n - 1), \tag{1.17}$$

there exists a point $\mathbf{x} \in \mathbb{R}^n$ with $1, x_1, \dots, x_n$ linearly independent over \mathbb{Q} such that $\omega_d(\mathbf{x}) = \tau_d$ and $\hat{\omega}_d(\mathbf{x}) = \omega_d$ for $0 \leq d \leq n - 1$.

The transference inequalities (1.16) are due to Laurent [20]. Equivalently, they can be re-written in the language of Schmidt [24] as the *Going-up transfer*

$$\omega_{d+1}(\mathbf{x}) \geq \frac{(n-d)\omega_d(\mathbf{x}) + 1}{n-d-1} \quad (0 \leq d \leq n-2) \tag{1.18}$$

and the *Going-down transfer*

$$\omega_{d-1}(\mathbf{x}) \geq \frac{d\omega_d(\mathbf{x})}{\omega_d(\mathbf{x}) + d + 1} \quad (1 \leq d \leq n-1). \tag{1.19}$$

As pointed out [20], on iterating (1.18) and (1.19) we obtain Khintchine’s classical transference principle [17]:

$$\frac{\omega_{n-1}(\mathbf{x})}{(n-1)\omega_{n-1}(\mathbf{x}) + n} \leq \omega_0(\mathbf{x}) \leq \frac{\omega_{n-1}(\mathbf{x}) - n + 1}{n}.$$

Thus the transference inequalities (1.16) of Laurent naturally split those of Khintchine relating the simultaneous and dual exponents $\omega_0(\mathbf{x})$ and $\omega_{n-1}(\mathbf{x})$. The furthermore part of Theorem 1.4, shows that transference inequalities of Laurent are optimal and was proved by Roy [23]. It extends the classical work of Jarník [16] showing that Khintchine’s transference principle is optimal.

Remark 1.6. The Laurent transference inequalities (1.16) are equally valid for the uniform exponents. Indeed, Laurent’s proof for the ordinary exponents can be naturally adapted to the uniform setting – see for example [13]. With (1.16) for uniform exponents at hand, it is easily seen that for any $\mathbf{x} \in \mathbb{R}^n$ and $1 \leq d \leq n$, the statement that $\hat{\omega}_d(\mathbf{x}) = \omega_d$ is equivalent to $\hat{\omega}_{d-1}(\mathbf{x}) = \omega_{d-1}$. Hence, it follows that the very singular sets \mathbf{VSing}_n^d ($0 \leq d \leq n-1$) discussed within Remark 1.5 are equivalent.

As usual let d be an integer with $0 \leq d \leq n-1$. Then given a real number $\tau \geq 0$, consider the Diophantine sets

$$\mathbf{W}_n^d(\tau) := \{\mathbf{x} \in \mathbb{R}^n : \omega_d(\mathbf{x}) \geq \tau\}$$

and

$$\mathbf{E}_n^d(\tau) := \{\mathbf{x} \in \mathbb{R}^n : \omega_d(\mathbf{x}) = \tau\}.$$

In dimension one, the latter corresponds to the exact order sets first studied by Güting within the context of Mahler’s classification of transcendental numbers – see [3] and references within for further details. Note that by definition, for any $0 \leq d \leq n-1$ we have that

$$\mathbf{W}_n^d(\tau) = \mathbb{R}^n \quad \text{if} \quad \tau \leq \omega_d$$

and

$$\mathbf{Bad}_n \cap \mathbf{W}_n^d(\tau) = \emptyset \quad \text{if } \tau > \omega_d.$$

Note that in view of (1.15), the set $\mathbf{E}_n^d(\omega_d)$ is of full n -dimensional Lebesgue measure and since $\mathbf{E}_n^d(\tau) \subseteq \mathbf{W}_n^d(\tau)$, it follows that $\mathbf{Bad}_n \cap \mathbf{E}_n^d(\tau) = \emptyset$ if $\tau > \omega_d$.

Using the parametric geometry of numbers a la Schmidt & Summerer [25,26] and Roy [22], we prove a stronger version of Theorem 1.2 that involves the exact order sets $\mathbf{E}_n^d(\tau)$ and the following quantitative form of the set of Dirichlet improvable points \mathbf{DI}_n . Given $\varepsilon \in (0, 1)$, let $\mathbf{DI}_n(\varepsilon)$ denote the set of $\mathbf{x} \in \mathbb{R}^n$ such that, for all $N > N_0(\mathbf{x}, \varepsilon)$ sufficiently large there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ such that (1.9) holds.

Remark 1.7. By definition, it follows that

$$\mathbf{DI}_n = \bigcup_{\varepsilon \in (0,1)} \mathbf{DI}_n(\varepsilon).$$

Recall, that in view of Davenport & Schmidt [12, Theorem 2] we can define \mathbf{DI}_n via either the ‘simultaneous’ inequality (1.3) or the ‘dual’ inequality (1.9) – both give rise to the same set. However, if we fix $\varepsilon > 0$ and a point $x \in \mathbf{DI}_n(\varepsilon)$, then it is not necessarily true that for all N sufficiently large there exists $q \in \mathbb{Z}$ such that (1.3) holds (for the same ε). In view of this, we emphasise the fact that when referring to the quantitative Dirichlet improvable set $\mathbf{DI}_n(\varepsilon)$ it will always be via the ‘dual’ inequality (1.9).

The following theorem constitutes our main result.

Theorem 1.5. *Let $n \geq 2$ and $\varepsilon \in (0, 1)$. Then, given any n -tuple of real numbers $\tau_0, \dots, \tau_{n-1} \in [0, \infty]$ with $\tau_0 \geq \omega_0$ and τ_d ($1 \leq d \leq n - 1$) satisfying (1.17), the set*

$$\left(\bigcap_{d=0}^{n-1} \mathbf{E}_n^d(\tau_d) \cap \left(\mathbf{DI}_n(\varepsilon) \setminus \mathbf{DI}_n(\varepsilon \kappa_n) \right) \right) \setminus (\mathbf{Bad}_n \cup \mathbf{Sing}_n)$$

where

$$\kappa_n := e^{-20(n+1)^3(n+10)}$$

has continuum many points. In particular, for any $0 \leq d \leq n - 1$ and $\tau \geq \omega_d$, the set $(\mathbf{DI}_n \cap \mathbf{E}_n^d(\tau)) \setminus (\mathbf{Bad}_n \cup \mathbf{Sing}_n)$ has continuum many points.

Note that on taking $\tau = \omega_d$ in the ‘in particular’ part of the Theorem 1.5, we immediately obtain the statement of Theorem 1.2* which in turn clearly implies Theorem 1.2.

It is worth mentioning that the constant κ_n appearing in the statement of the theorem is far from optimal. Indeed, our proof can be tightened to yield a larger but more complicated expression. Nevertheless, we feel there is not much gained in doing so and for

the sake of aesthetics we are content with κ_n as defined. Also, we should point out that if we had defined the quantitative Dirichlet improvable set $\mathbf{DI}_n(\varepsilon)$ via the ‘simultaneous’ inequality (1.3) then the statement of the theorem remains completely unchanged apart from the choice of the κ_n . The modified constant can be directly deduced from [12].

2. Preliminaries

In this section we start by recalling aspects of the theory of parametric geometry of numbers that will be used in establishing Theorem 1.5. We then use this to essentially reformulate the Diophantine sets appearing in the statement of Theorem 1.5 in terms of successive minima. Moreover, we will see that the proof of Proposition 1.1 is a pretty straightforward application of this reformulation.

2.1. The parametric geometry of numbers

Fix $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^n$. For each real number $t \geq 0$, consider the convex body

$$\mathcal{C}_{\mathbf{x}}(e^t) := \left\{ \mathbf{y} \in \mathbb{R}^{n+1} : |y_i| \leq 1 \ (1 \leq i \leq n), \left| \sum_{i=1}^n y_i x_i + y_{n+1} \right| \leq e^{-t} \right\}. \tag{2.1}$$

Then, for each $i = 1, \dots, n + 1$ and $t > 0$, let

$$\lambda_{\mathbf{x},i}(t) := \lambda_i(\mathbb{Z}^{n+1}, \mathcal{C}_{\mathbf{x}}(e^t)) \tag{2.2}$$

denotes the i -th successive minima of the convex body $\mathcal{C}_{\mathbf{x}}(e^t)$ with respect to the lattice \mathbb{Z}^{n+1} . In other words, $\lambda_i(\mathbb{Z}^{n+1}, \mathcal{C}_{\mathbf{x}}(e^t))$ is the smallest real number λ such that the rescaled convex body $\lambda \mathcal{C}_{\mathbf{x}}(e^t)$ contains at least i linearly independent points of \mathbb{Z}^{n+1} . In turn, we let

$$L_{\mathbf{x},i}(t) := \log \lambda_{\mathbf{x},i}(t) = \log \lambda_i(\mathbb{Z}^{n+1}, \mathcal{C}_{\mathbf{x}}(e^t)) \quad (t \geq 0, 1 \leq i \leq n + 1) \tag{2.3}$$

and consider the map

$$\mathbf{L}_{\mathbf{x}} : [0, \infty) \rightarrow \mathbb{R}^{n+1} : t \rightarrow \mathbf{L}_{\mathbf{x}}(t) := (L_{\mathbf{x},1}(t), \dots, L_{\mathbf{x},n+1}(t)). \tag{2.4}$$

The following notion was introduced by Roy in [23, Definition 4.5]. It generalises the $(n + 1)$ -systems of Schmidt & Summerer [26]. In short, these ‘systems’ incorporate desirable behaviour of the maps $\mathbf{L}_{\mathbf{x}}$ that in turn lead to desirable approximation results.

Definition 2.1. Let I be an subinterval of $[0, \infty)$ with non-empty interior. A *Roy $(n + 1)$ -system* on I is a continuous piecewise linear map $\mathbf{P} = (P_1, \dots, P_{n+1}) : I \rightarrow \mathbb{R}^{n+1}$ with the following properties:

- For each $t \in I$, we have $0 \leq P_1(t) \leq \dots \leq P_{n+1}(t)$ and $P_1(t) + \dots + P_{n+1}(t) = t$.
- If $I' \subset I$ is a nonempty open subinterval on which \mathbf{P} is differentiable, then there are integers r_1, r_2 with $1 \leq r_1 \leq r_2 \leq n + 1$ such that the functions $P_{r_1}, P_{r_1+1}, \dots, P_{r_2}$ coincide on the whole interval I' and have slope $1/(r_2 - r_1 + 1)$ on I' , while all other components P_i of \mathbf{P} have slope 0 on I' .
- If t is an interior point of I at which \mathbf{P} is not differentiable and if r_1, r_2, s_1, s_2 are integers for which

$$P'_i(t^-) = \frac{1}{r_2 - r_1 + 1} \quad (r_1 \leq i \leq r_2) \quad \text{and} \quad P'_i(t^+) = \frac{1}{s_2 - s_1 + 1} \quad (s_1 \leq i \leq s_2),$$

and if $r_1 \leq s_2$, then we have that $P_{r_1}(t) = P_{r_1+1}(t) = \dots = P_{s_2}(t)$.

Note that, for any piecewise linear function $F : \mathbb{R} \rightarrow \mathbb{R}$, the left derivative $F'(t^-)$ and the right derivative $F'(t^+)$ always exist and the points at which F is not differentiable are just the points with different left and right derivatives.

Remark 2.1. The $(n + 1)$ -systems of Schmidt & Summerer correspond to taking $r_1 = r_2$ and $s_1 = s_2$ in Definition 2.1.

A Roy $(n + 1)$ -system has the following useful approximation property. It essentially represents an amalgamation of [22, Theorems 1.3 & 1.8] and [23, Corollary 4.7] adapted for our purposes.

Theorem 2.1. *Let $n \in \mathbb{N}$ and $t_0 \geq 0$. For each $\mathbf{x} \in \mathbb{R}^n$, there exists a Roy $(n + 1)$ -system $\mathbf{P} : [t_0, \infty) \rightarrow \mathbb{R}^{n+1}$ such that the function $\mathbf{L}_{\mathbf{x}} - \mathbf{P}$ is bounded on $[t_0, \infty)$. Conversely, for each Roy $(n + 1)$ -system $\mathbf{P} : [t_0, \infty) \rightarrow \mathbb{R}^{n+1}$, there exists $\mathbf{x} \in \mathbb{R}^n$ such that the function $\mathbf{L}_{\mathbf{x}} - \mathbf{P}$ is bounded on $[t_0, \infty)$ with*

$$\|\mathbf{L}_{\mathbf{x}}(t) - \mathbf{P}(t)\| \leq 5(n + 1)^2(n + 10). \tag{2.5}$$

Proof of Theorem 2.1. As already mentioned, Theorem 2.1 draws upon the works [22, 23] of Roy and it is important to note that there, the convex body is defined slightly differently from the one given by (2.1). Indeed, for a fixed $\mathbf{u} \in \mathbb{R}^{n+1} \setminus \{0\}$ and each real number $t \geq 0$, Roy works with the convex body

$$\tilde{\mathcal{C}}_{\mathbf{u}}(e^t) := \{\mathbf{y} \in \mathbb{R}^{n+1} : \|\mathbf{y}\|_2 \leq 1, \quad |\mathbf{y} \cdot \mathbf{u}| \leq e^{-t}\}.$$

Here and elsewhere $\|\mathbf{y}\|_2$ is the Euclidean norm of \mathbf{y} . Now for any fixed $\mathbf{x} \in \mathbb{R}^n$, let $\mathbf{x}' := (\mathbf{x}, 1) \in \mathbb{R}^{n+1}$ and so by definition

$$\tilde{\mathcal{C}}_{\mathbf{x}'}(e^t) = \left\{ \mathbf{y} \in \mathbb{R}^{n+1} : \|\mathbf{y}\|_2 \leq 1, \quad \left| \sum_{i=1}^n y_i x_i + y_{n+1} \right| \leq e^{-t} \right\}.$$

Furthermore, let $\tilde{\mathbf{L}}_{\mathbf{x}'}$ denote the function corresponding to (2.4) with $\mathcal{C}_{\mathbf{x}}(e^t)$ replaced by $\tilde{\mathcal{C}}_{\mathbf{x}'}(e^t)$ within (2.3). It is not difficult to see that our convex body and the associated map, which are convenient for what we have in mind, are closely related to those of Roy and indeed Schmidt & Summerer: for any fixed $\mathbf{x} \in \mathbb{R}^n$ and any $t \geq 0$

$$\tilde{\mathcal{C}}_{\mathbf{x}'}(e^t) \subset \mathcal{C}_{\mathbf{x}}(e^t) \subset (\sqrt{n}\|\mathbf{x}\| + \sqrt{n} + 1)\tilde{\mathcal{C}}_{\mathbf{x}'}(e^t)$$

and thus it follows that

$$\|\mathbf{L}_{\mathbf{x}}(t) - \tilde{\mathbf{L}}_{\mathbf{x}'}(t)\| \leq \log((\sqrt{n}\|\mathbf{x}\| + \sqrt{n} + 1)). \tag{2.6}$$

We now proceed with establishing the theorem. On combining [22, Theorem 1.3] and [22, Lemma 2.10], we find that for any $\mathbf{x} \in \mathbb{R}^n$ there exists a $(n + 1)$ -system (see Remark 2.1) $\tilde{\mathbf{P}}$ such that the function $\tilde{\mathbf{L}}_{\mathbf{u}_{\mathbf{x}'}} - \tilde{\mathbf{P}}$ is bounded for all $t \geq t_0$. Here $\mathbf{u}_{\mathbf{x}'}$ denotes the unit vector with respect to the Euclidean norm associated to $\mathbf{x}' \in \mathbb{R}^{n+1}$. Note that $\tilde{\mathbf{L}}_{\mathbf{u}_{\mathbf{x}'}}$ and $\tilde{\mathbf{L}}_{\mathbf{x}'}$ only differ by a bounded function and so the first part of the theorem follows on using (2.6) and the fact that by definition any $(n + 1)$ -system is a Roy $(n + 1)$ -system. Regarding the converse part, it follows via [23, Corollary 4.7] that for any given Roy $(n + 1)$ -system $\mathbf{P} : [t_0, \infty) \rightarrow \mathbb{R}^{n+1}$ and any $\varepsilon > 0$, there is a $(n + 1)$ -system $\tilde{\mathbf{P}}$ such that

$$\|\mathbf{P}(t) - \tilde{\mathbf{P}}(t)\| \leq \varepsilon \quad \text{for all } t \geq t_0. \tag{2.7}$$

In view of [22, Theorem 8.1], there exists a unit vector (with respect to the Euclidean norm) $\mathbf{u} \in \mathbb{R}^{n+1}$ such that

$$\|\tilde{\mathbf{P}}(t) - \tilde{\mathbf{L}}_{\mathbf{u}}(t)\| \leq 3(n + 1)^2(n + 10) \quad \text{for all } t \geq t_0. \tag{2.8}$$

As a reordering of the coordinates of \mathbf{u} does not affect the function $\tilde{\mathbf{L}}_{\mathbf{u}}$, we can assume that $|u_{n+1}| = \|\mathbf{u}\| := \max\{|u_1|, \dots, |u_{n+1}|\}$ and so $(n + 1)^{-1/2} \leq |u_{n+1}| \leq 1$. Now let

$$\mathbf{x} := (u_1 u_{n+1}^{-1}, \dots, u_n u_{n+1}^{-1}) \in \mathbb{R}^n.$$

Then, $\|\mathbf{x}\| \leq 1$ and

$$\|\tilde{\mathbf{L}}_{\mathbf{x}'}(t) - \tilde{\mathbf{L}}_{\mathbf{u}}(t)\| \leq \frac{1}{2} \log(n + 1) \quad \text{for all } t \geq t_0. \tag{2.9}$$

The upshot is that on using (2.7) with $\varepsilon := \log(n + 1)/2$, (2.6), (2.9) and (2.8) in that order, we obtain that

$$\|\mathbf{L}_{\mathbf{x}}(t) - \mathbf{P}(t)\| \leq \frac{1}{2} \log(n + 1) + \log(2\sqrt{n} + 1) + \frac{1}{2} \log(n + 1) + 3(n + 1)^2(n + 10)$$

$$< 5(n + 1)^2(n + 10) \quad \text{for all } t \geq t_0. \tag{2.10}$$

This completes the proof of Theorem 2.1. \square

The following notion of non-equivalent systems will prove to be useful.

Definition 2.2. Two Roy $(n + 1)$ -systems \mathbf{P}_1 and \mathbf{P}_2 defined on the same subinterval I of $[0, \infty)$ are said to be *non-equivalent* if there exists some $t \in I$ such that

$$|\mathbf{P}_1(t) - \mathbf{P}_2(t)| > 10(n + 1)^2(n + 10).$$

By definition, it follows that no point in \mathbb{R}^n can be close (in the sense of Theorem 2.1) to two non-equivalent Roy $(n + 1)$ -systems defined on $[t_0, \infty)$ at the same time.

2.2. Expressing Diophantine sets via successive minima

We give a reformulation of the Diophantine sets associated with Theorem 1.5 in terms of the function \mathbf{L}_x . This is at the heart of its proof – it brings into play the parametric geometry of numbers. Also, we shall see that the equivalence of the d -badly approximable sets \mathbf{Bad}_n^d (resp. the d -singular sets \mathbf{Sing}_n^d) is in essence a direct consequence of the reformulation. Indeed, we start with this in mind.

Let $n \geq 2$ and $0 \leq d \leq n - 1$. It can be verified, by using the lemma appearing in [5, Section 4] and appropriately adapting the proof of the proposition in [5, Section 4], that

- $\mathbf{x} \in \mathbf{Bad}_n^d$ if and only if there exists a constant $\delta > 0$ such that for all sufficiently large t

$$\frac{(n - d)t}{n + 1} - (L_{\mathbf{x},1}(t) + \dots + L_{\mathbf{x},n-d}(t)) \leq \delta. \tag{2.11}$$

- $\mathbf{x} \in \mathbf{Sing}_n^d$ if and only if for any $\delta > 0$ there exists a constant $t_0 = t_0(\delta) > 0$ such that for all $t \geq t_0$

$$\frac{(n - d)t}{n + 1} - (L_{\mathbf{x},1}(t) + \dots + L_{\mathbf{x},n-d}(t)) \geq \delta. \tag{2.12}$$

For the sake of completeness, in §4, we will provide the details of how these equivalences follow from [5, Section 4]. We can now swiftly show that

$$\mathbf{Bad}_n^d = \mathbf{Bad}_n^{n-1} \quad \text{and} \quad \mathbf{Sing}_n^d = \mathbf{Sing}_n^{n-1} \quad (0 \leq d \leq n - 2);$$

that is to say that any d -badly approximable set (resp. d -singular set) is equivalent to the dual set. This will of course establish Proposition 1.1.

Proof of Proposition 1.1. For simplicity, given $\mathbf{x} \in \mathbb{R}^n$ we let

$$g_{\mathbf{x},i}(t) := \frac{t}{n+1} - L_{\mathbf{x},i}(t) \quad (0 \leq i \leq n+1).$$

By definition the quantity $L_{\mathbf{x},i}$ is increasing with i and so it follows that

$$g_{\mathbf{x},1}(t) \geq g_{\mathbf{x},2}(t) \geq \dots \geq g_{\mathbf{x},n+1}(t). \tag{2.13}$$

In view of Minkowski’s second convex body theorem, for any $\mathbf{x} \in \mathbb{R}^n$ we have that

$$L_{\mathbf{x},1}(t) + L_{\mathbf{x},2}(t) + \dots + L_{\mathbf{x},n+1}(t) = t + O(1).$$

Thus, there exists a positive constant $c = c(n) > 0$ depending only on n such that

$$g_{\mathbf{x},1}(t) + g_{\mathbf{x},2}(t) + \dots + g_{\mathbf{x},n+1}(t) \geq -c. \tag{2.14}$$

Now suppose $\mathbf{x} \in \mathbf{Bad}_n^d$. Then in view of (2.11) and (2.14), it follows that

$$\sum_{i=n-d+1}^{n+1} g_{\mathbf{x},i}(t) \geq -\delta - c$$

which together with (2.13) implies that

$$g_{\mathbf{x},i}(t) \geq \frac{1}{d+1} \sum_{j=n-d+1}^{d+1} g_{\mathbf{x},j}(t) \geq -\frac{\delta+c}{d+1} \quad (1 \leq i \leq n-d).$$

In turn, on using (2.13) again, we find that

$$\begin{aligned} g_{\mathbf{x},1}(t) &= \sum_{i=1}^{n-d} g_{\mathbf{x},i}(t) - \sum_{i=2}^{n-d} g_{\mathbf{x},i}(t) \leq \delta + (n-d-1) \frac{\delta+c}{d+1} \\ &< \frac{n}{d+1}(\delta+c). \end{aligned}$$

In other words, (2.11) holds with $d = n-1$ and so $\mathbf{x} \in \mathbf{Bad}_n^{n-1}$. For the converse, simply observe that if (2.11) holds with $d = n-1$ then for any other $0 \leq d \leq n-2$

$$\sum_{i=1}^{n-d} g_{\mathbf{x},i}(t) \stackrel{(2.13)}{\leq} (n-d) g_{\mathbf{x},1}(t) \leq (n-d)\delta.$$

In other words, $\mathbf{x} \in \mathbf{Bad}_n^d$ and this thereby completes the proof of the badly approximable part of the proposition. The proof of the singular part is similar with the most obvious modifications (namely, using (2.12) instead of (2.11)) and will be left for the reader. \square

Remark 2.2. In §4, apart from providing details of the statements associated with (2.11) and (2.12), we give a ‘dynamical’ proof of Proposition 1.1. In addition to providing an alternative insight, it has the advantage of being self-contained in that it avoids appealing to (2.11) and (2.12) which rely on the lemma and the arguments appearing in [5, Section 4].

The following statement summarises the above findings concerning the badly approximable and singular sets and deals with the other remaining Diophantine sets associated with Theorem 1.5.

Lemma 2.1. *Let $\mathbf{x} \in \mathbb{R}^n$. Then*

1. $\mathbf{x} \in \mathbf{DI}_n(\varepsilon)$ if and only if for all sufficiently large t

$$\frac{t}{n+1} - L_{\mathbf{x},1}(t) \geq -\frac{\log \varepsilon}{n+1}. \tag{2.15}$$

2. $\mathbf{x} \in \mathbf{Bad}_n$ if and only if there exists $\delta > 0$ such that

$$\limsup_{t \rightarrow \infty} \left(\frac{t}{n+1} - L_{\mathbf{x},1}(t) \right) \leq \delta.$$

3. $\mathbf{x} \in \mathbf{Sing}_n$ if and only if for any $\delta > 0$

$$\liminf_{t \rightarrow \infty} \left(\frac{t}{n+1} - L_{\mathbf{x},1}(t) \right) \geq \delta.$$

4. $\mathbf{x} \in \mathbf{W}_n^d(\tau)$ if and only if

$$\liminf_{t \rightarrow \infty} \frac{L_{\mathbf{x},1}(t) + \dots + L_{\mathbf{x},n-d}(t)}{t} \leq \frac{1}{1+\tau} \quad (0 \leq d \leq n-1).$$

5. $\mathbf{x} \in \mathbf{E}_n^d(\tau)$ if and only if

$$\liminf_{t \rightarrow \infty} \frac{L_{\mathbf{x},1}(t) + \dots + L_{\mathbf{x},n-d}(t)}{t} = \frac{1}{1+\tau} \quad (0 \leq d \leq n-1).$$

Proof of Lemma 2.1. Parts 4) and 5) are a direct consequence of [23, Proposition 3.1]. The proof of parts 2) and 3) are a direct consequence of (2.11) and (2.12) respectively together with Proposition 1.1. It remains to prove part 1). Thus, let $\mathbf{x} \in \mathbf{DI}_n(\varepsilon)$ for some $\varepsilon \in (0, 1)$. Then by definition, for all sufficiently large t'

$$|\mathbf{x} \cdot \mathbf{q} - p| \leq \varepsilon e^{-nt'} \quad \text{and} \quad \|\mathbf{q}\| \leq e^{t'}$$

always has a solution $(p, \mathbf{q}) \in \mathbb{Z} \times (\mathbb{Z}^n \setminus \{\mathbf{0}\})$. This is equivalent to saying that for all sufficiently large t'

$$\lambda_1(\mathbb{Z}^{n+1}, \mathcal{C}_{\mathbf{x}}(e^t)) \leq e^{t'}, \quad \text{where } t = (n + 1)t' - \log \varepsilon.$$

The latter is equivalent to

$$\frac{t}{n + 1} - L_{\mathbf{x},1}(t) \geq -\frac{\log \varepsilon}{n + 1}$$

for all sufficiently large t , as desired. \square

Remark 2.3. It is relatively straightforward to see that the proof of part 1) given above can be easily adapted to establish (2.11) and (2.12) when $d = n - 1$.

Remark 2.4. For the sake of completeness, it worth mentioning that in §4.2 we formulate the notion of d -Dirichlet improvable sets via successive minima. The approach taken is in line with that of this section in which the d -badly approximable and d -singular sets are expressed via (2.11) and (2.12).

3. Proof of Theorem 1.5

Let $n \geq 2$, $\varepsilon \in (0, 1)$ and $\tau_0, \dots, \tau_{n-1}$ be as in Theorem 1.5 and let

$$\gamma := -\frac{\log \varepsilon}{n + 1} + 2C_n \quad \text{where} \quad C_n := 5(n + 1)^2(n + 10). \tag{3.1}$$

Thus, C_n is simply the right hand side of inequality appearing in Theorem 2.1. Then, on making use of Theorem 2.1 and Lemma 2.1, it is easily verified that the proof of Theorem 1.5 is reduced to constructing appropriate Roy $(n + 1)$ -systems given by the following statement.

Lemma 3.1. *There exists continuum many mutually non-equivalent Roy $(n + 1)$ -systems $\mathbf{P} : [0, \infty) \rightarrow \mathbb{R}^{n+1}$, such that*

$$\liminf_{t \rightarrow \infty} \left(\frac{t}{n + 1} - P_1(t) \right) = \gamma, \tag{3.2}$$

$$\limsup_{t \rightarrow \infty} \left(\frac{t}{n + 1} - P_1(t) \right) = \infty, \tag{3.3}$$

and

$$\liminf_{t \rightarrow \infty} \frac{P_1(t) + \dots + P_d(t)}{t} = \frac{1}{1 + \tau_{n-d}} \quad (1 \leq d \leq n). \tag{3.4}$$

Proof of Theorem 1.5 modulo Lemma 3.1. Let us assume Lemma 3.1 and let $\mathbf{P} : [0, \infty) \rightarrow \mathbb{R}^{n+1}$ be a Roy $(n + 1)$ -system coming from the lemma. In view of the converse part of Theorem 2.1, there exists a point $\mathbf{x} \in \mathbb{R}^n$ such that

$$\|\mathbf{L}_x(t) - \mathbf{P}(t)\| \leq C_n.$$

Then this together with Lemma 2.1 and

- (3.2) implies $\mathbf{x} \in \mathbf{DI}_n(\varepsilon) \setminus \mathbf{DI}_n(\varepsilon e^{-4(n+1)C_n})$ and $\mathbf{x} \notin \mathbf{Sing}_n$,
- (3.3) implies $\mathbf{x} \notin \mathbf{Bad}_n$,
- (3.4) implies $\mathbf{x} \in \bigcap_{d=0}^{n-1} \mathbf{E}_n^d(\tau_d)$.

The upshot of this is that

$$\mathbf{x} \in \left(\bigcap_{d=0}^{n-1} \mathbf{E}_n^d(\tau_d) \cap \left(\mathbf{DI}_n(\varepsilon) \setminus \mathbf{DI}_n(\varepsilon e^{-4(n+1)C_n}) \right) \right) \setminus (\mathbf{Bad}_n \cup \mathbf{Sing}_n).$$

Furthermore, Lemma 3.1 implies the existence of continuum many such Roy $(n + 1)$ -systems that are mutually non-equivalent. Thus, in view of the latter (see Definition 2.2) each such system gives rise to a different point $\mathbf{x} \in \mathbb{R}^n$ and this completes the proof of Theorem 1.5. \square

The proof of Lemma 3.1 will occupy the rest of this section. It will comprise of three steps. We start by constructing Roy $(n + 1)$ -systems on certain finite intervals which will serve as building blocks for the construction of the desired systems associated with Lemma 3.1.

3.1. Building blocks

Let $[T_-, T_+]$ be a subinterval of $[0, \infty)$ with non-empty interior and let $(a_-^j)_{1 \leq j \leq n+1}$, $(a_+^j)_{1 \leq j \leq n+1}$ be sequences of increasing positive numbers satisfying:

$$a_+^1 T_+ = a_-^{n+1} T_- - (n + 1)\gamma \tag{3.5}$$

$$\sum_{1 \leq j \leq n+1} a_*^j = 1 \tag{3.6}$$

$$(a_*^{j+1} - a_*^j) T_* \geq 4n^2 \gamma \quad \forall \quad 1 \leq j \leq n, \tag{3.7}$$

where γ is as in (3.1) and throughout

$$* := - \text{ or } +.$$

We now construct a Roy $(n + 1)$ -system

$$\mathbf{P} = (P_1, \dots, P_{n+1}) : [T_-, T_+] \rightarrow \mathbb{R}^{n+1}$$

on $[T_-, T_+]$ associated with the sequences (a_-^j) and (a_+^j) . With this in mind, let

$$R_d := (a_-^{n+1} + \dots + a_-^{d+1} + da_-^d) T_- \quad \forall \quad 1 \leq d \leq n$$

$$R_{n+1} := (n + 1)a_-^{n+1}T_- - n(n + 1)\gamma$$

$$R_{n+2} := (n + 1)a_-^{n+1}T_- - (n + 1)\gamma$$

$$S_0 := (n + 1)a_+^1T_+ + (n^2 + n)\gamma$$

$$S_d := (a_+^1 + \dots + a_+^d + (n + 1 - d)a_+^{d+1}) T_+ \quad \forall \quad 1 \leq d \leq n.$$

In view of (3.5), it is easily seen that $S_0 = R_{n+2}$. Also, (3.6) ensures that $T_- = R_1$ and $S_n = T_+$ while (3.7) gives that $R_{n+1} \geq R_n$ and $S_1 \geq S_0$. Since (a_*^j) is strictly increasing, it thus follows that

$$T_- = R_1 < R_2 < \dots < R_{n+2} = S_0 < S_1 < \dots < S_n = T_+.$$

Now set

$$P_j(T_-) := a_-^j T_- \quad \forall \quad 1 \leq j \leq n + 1.$$

For $1 \leq d \leq n - 1$, on the interval $[R_d, R_{d+1}]$, let the d components P_1, \dots, P_d coincide and have slope $1/d$ while the components P_{d+1}, \dots, P_{n+1} have slope 0 and

$$P_j(R_{d+1}) := \begin{cases} a_-^{d+1}T_- & \text{if } 1 \leq j \leq d \\ a_-^j T_- & \text{if } d + 1 \leq j \leq n + 1. \end{cases}$$

On the interval $[R_n, R_{n+1}]$, let the n components P_1, \dots, P_n coincide and have slope $1/n$ while the component P_{n+1} has slope 0 and

$$P_j(R_{n+1}) := \begin{cases} a_+^1 T_+ & \text{if } 1 \leq j \leq n \\ a_-^{n+1} T_- & \text{if } j = n + 1. \end{cases}$$

On the interval $[R_{n+1}, R_{n+2}]$, let the $n - 1$ components P_2, \dots, P_n coincide and have slope $1/(n - 1)$ while the components P_1, P_{n+1} have slope 0 and

$$P_j(R_{n+2}) := \begin{cases} a_+^1 T_+ & \text{if } j = 1 \\ a_-^{n+1} T_- & \text{if } 2 \leq j \leq n + 1. \end{cases}$$

On the interval $[S_0, S_1]$, let the n components P_2, \dots, P_{n+1} coincide and have slope $1/n$ while the component P_1 has slope 0 and

$$P_j(S_1) := \begin{cases} a_+^1 T_+ & \text{if } j = 1 \\ a_+^2 T_+ & \text{if } 2 \leq j \leq n + 1. \end{cases}$$

Finally, for $1 \leq d \leq n - 1$, on the interval $[S_d, S_{d+1}]$, let the $n - d$ components P_{d+2}, \dots, P_{n+1} coincide and have slope $1/(n - d)$ while the components $P_1 \dots, P_{d+1}$ have slope 0 and

$$P_j(S_{d+1}) := \begin{cases} a_+^j T_+ & \text{if } 1 \leq j \leq d + 1 \\ a_+^{d+2} T_+ & \text{if } d + 2 \leq j \leq n + 1. \end{cases}$$

In particular, since $S_n = T_+$ it follows that

$$P_j(T_+) = a_+^j T_+ \quad \forall \quad 1 \leq j \leq n + 1.$$

Fig. 1 below represents the combined graph of the functions P_1, \dots, P_{n+1} over the interval $[T_-, T_+]$. The lower graph represents P_1 while the upper most graph represents P_{n+1} . Note that if we set $\gamma = 0$ and $a_+^j = a_-^j$ for $j = 1, \dots, n + 1$, our construction reduces to Roy’s construction in [23, Section 5] – in particular, see [23, Figure 5].

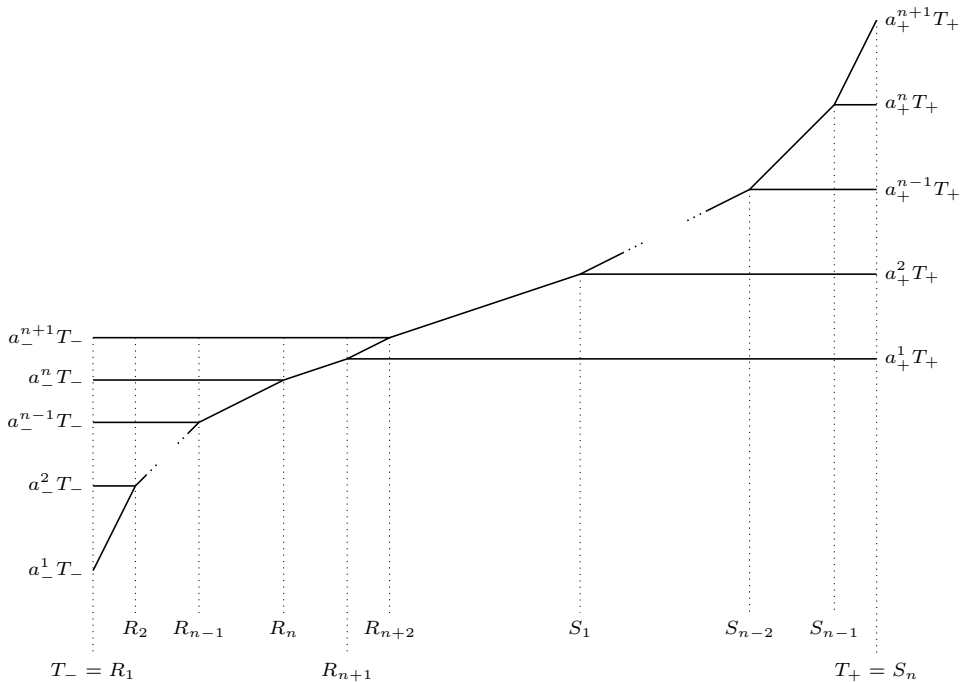


Fig. 1. The constructed Roy $(n + 1)$ -system on $[T_-, T_+]$.

We conclude this section with the following statement. It provides keys estimates for $\mathbf{P}(t)$ with $t \in [T_-, T_+]$.

Lemma 3.2. *The Roy $(n + 1)$ -system $\mathbf{P} : [T_-, T_+] \rightarrow \mathbb{R}^{n+1}$ constructed above satisfies:*

$$\min_{t \in [T_-, T_+]} \left(\frac{t}{n + 1} - P_1(t) \right) = \gamma \tag{3.8}$$

$$\max_{t \in [T_-, T_+]} \left(\frac{t}{n + 1} - P_1(t) \right) \geq \left(\frac{1}{n + 1} - a_+^1 \right) T_+ \tag{3.9}$$

$$\min_{t \in [T_-, T_+]} \frac{P_1(t) + \dots + P_d(t)}{t} = \min \left\{ \sum_{j=1}^d a_-^j, \sum_{j=1}^d a_+^j \right\} \quad (1 \leq d \leq n). \tag{3.10}$$

Proof. By construction the derivative of the function P_1 is strictly greater than $1/(n + 1)$ on the interval $[T_-, R_{n+1}]$ and is 0 on the interval $[R_{n+1}, T_+]$. Here and throughout, by the derivative of a piecewise linear function on a given interval, we mean the derivative on the union of subintervals on which the derivative exists. It follows that on the interval $[T_-, T_+]$, the local minimum of the function $f : t \rightarrow f(t) := t/(n + 1) - P_1(t)$ is achieved at $t = R_{n+1}$. In other words, the minimum of $f(t)$ on $[T_-, T_+]$ is equal to

$$\frac{R_{n+1}}{n + 1} - P_1(R_{n+1}) = a_-^{n+1} T_- - n\gamma - a_+^1 T_+ \stackrel{(3.5)}{=} \gamma.$$

This shows that \mathbf{P} satisfies (3.8). On the other hand, it is easily seen that the maximum of $f(t)$ on $[T_-, T_+]$ is achieved at either $t = T_-$ or $t = T_+$. Thus,

$$\text{l.h.s. of (3.9)} \geq \frac{T_+}{n + 1} - P_1(T_+) = \frac{T_+}{n + 1} - a_+^1 T_+$$

and this shows that \mathbf{P} satisfies (3.9). It remains to prove (3.10). For simplicity, we let $Q_d := P_1 + \dots + P_d$ ($1 \leq d \leq n$) and note that to determine when $Q_d(t)/t$ attains its minimum on $[T_-, T_+]$, it suffices to study the function

$$D : t \rightarrow D(t) := Q'_d(t)t - Q_d(t).$$

The point is that on each connected open interval I where \mathbf{P} is differentiable, the derivative of $Q_d(t)/t$ equals $D(t)/t^2$. Now, it can be verified that on any such interval I the derivative $D'(t) = 0$ and so $D(t)$ is constant which in turn implies that the function $Q_d(t)/t$ is monotonic on I and thus it obtains its minimum at the end points.

Hence, it suffices to study the quantities $D^*(R_j)$ and $D^*(S_j)$ for $1 \leq j \leq n$. This we now do systematically. Recall that for a piecewise continuous function D ,

$$D^+(t) := \lim_{\substack{s \rightarrow t, \\ s > t}} D(s), \quad \text{and} \quad D^-(t) := \lim_{\substack{s \rightarrow t, \\ s < t}} D(s).$$

- For $1 \leq j \leq d$, on the interval (R_j, R_{j+1}) , the derivative of Q_d equals 1. Hence $D(t) = t - Q_d(t) = P_{d+1}(t) + \dots + P_{n+1}(t)$ is positive on this interval.
- For $d + 1 \leq j \leq n$, on the interval (R_j, R_{j+1}) , the derivative of Q_d equals d/j . A direct computation shows that

$$\begin{aligned} jD^+(R_j) &= dR_j - jQ_d(R_j) \\ &= d \left(a_-^{n+1} + \dots + a_-^{j+1} + ja_-^j \right) T_- - jda_-^j T_- \\ &> 0 \end{aligned}$$

Hence $D(t)$ is positive on this interval.

- For $0 \leq j \leq d - 1$, on the interval (S_j, S_{j+1}) , the derivative of Q_d equals $(d - j - 1)/(n - j)$. A direct computation shows that

$$\begin{aligned} (n - j)D^-(S_{j+1}) &= (d - j - 1)S_{j+1} - (n - j)Q_d(S_{j+1}) \\ &= (d - j - 1) \left(a_+^1 + \dots + a_+^{j+1} + (n - j)a_+^{j+2} \right) T_+ \\ &\quad - (n - j) \left(a_+^1 + \dots + a_+^{j+1} + (d - j - 1)a_+^{j+2} \right) T_+ \\ &= (d - n - 1) \left(a_+^1 + \dots + a_+^{j+1} \right) T_+ \\ &< 0 \end{aligned}$$

Hence $D(t)$ is negative on this interval.

- For $d \leq j \leq n - 1$, on the interval (S_j, S_{j+1}) , the derivative of Q_d equals 0. Hence $D(t)$ is negative on this interval.

In conclusion, the function $Q_d(t)/t$ increases on the interval (R_1, R_{n+1}) and decreases on the interval (S_0, S_n) . As $Q_d(t)/t$ is monotonic on (R_{n+1}, S_0) , the minimum is thus attained at either $t = T_-$ or $t = T_+$. Hence,

$$\text{l.h.s. of (3.10)} = \min \left\{ \frac{Q_d(T_-)}{T_-}, \frac{Q_d(T_+)}{T_+} \right\} = \min \left\{ \sum_{j=1}^d a_-^j, \sum_{j=1}^d a_+^j \right\}$$

and this shows that \mathbf{P} satisfies (3.10) which in turn completes the proof of the lemma. \square

3.2. The local construction on blocks

In this section, we will exploit the generic construction presented in §3.1 to essentially prove a ‘local’ version of Lemma 3.1. More precisely, we will construct a family of Roy $(n + 1)$ -systems \mathbf{P}^δ on certain subintervals I of $[0, \infty)$ all satisfying the properties of Lemma 3.2. Here $\delta \in [0, 1/(40n^2))$ is a parameter and for $\delta' \neq \delta$, we show that the intervals I can be chosen so that the Roy $(n + 1)$ -systems \mathbf{P}^δ and $\mathbf{P}^{\delta'}$ on I are mutually non-equivalent. The construction consists of five short steps. Throughout, $n \geq 2$ and $\tau_0, \dots, \tau_{n-1} \in [0, \infty]$ are the quantities appearing in Theorem 1.5 satisfying (1.17).

Step 1. For $1 \leq i \leq n - 1$ and $1 \leq j \leq n + 1$, let

$$\alpha^{i,j} := \begin{cases} i^{-1}(1 + \tau_{n-i})^{-1} & \text{if } j \leq i \\ (1 + \tau_{n-i-1})^{-1} - (1 + \tau_{n-i})^{-1} & \text{if } j = i + 1 \\ (n - i)^{-1}\tau_{n-i-1}(1 + \tau_{n-i-1})^{-1} & \text{if } j > i + 1 \end{cases}$$

where $(1 + \tau_j)^{-1} = 0$ and $\tau_j(1 + \tau_j)^{-1} = 1$ if $\tau_j = \infty$. The following statement summarises useful properties of the associated sequence $(\alpha^{i,j})$ that we shall later exploit.

Lemma 3.3. *Let $(\alpha^{i,j})$ be given as above, then*

- (a) for any $1 \leq i \leq n - 1$, $\sum_{1 \leq j \leq n+1} \alpha^{i,j} = 1$,
- (b) for any $1 \leq i \leq n - 1$ and $1 \leq j \leq j' \leq n + 1$, $\alpha^{i,j} \leq \alpha^{i,j'}$,
- (c) $\alpha^{i,1} + \dots + \alpha^{i,j} \geq (1 + \tau_{n-j})^{-1}$ with equality holds when $j = i, i + 1$.

Proof. Part (a) follows directly from the definition. To prove the other parts, for $1 \leq i \leq n$ let

$$\theta_i = (1 + \tau_{n-i})^{-1}.$$

Then it follows that (1.17) is equivalent to

$$\frac{(n - d + 1)\theta_{n-d}}{n - d} \leq \theta_{n-d+1} \leq \frac{1 + d\theta_{n-d}}{d + 1} \quad \forall \quad 1 \leq d \leq n - 1,$$

which in turn is equivalent to

$$\frac{\theta_i}{i} \leq \frac{\theta_{i+1}}{i + 1} \quad \text{and} \quad \frac{1 - \theta_i}{n + 1 - i} \leq \frac{1 - \theta_{i+1}}{n - i} \quad \forall \quad 1 \leq i \leq n - 1. \tag{3.11}$$

To prove part (b), it suffices to show that

$$\frac{\theta_i}{i} \leq \theta_{i+1} - \theta_i \leq \frac{1 - \theta_{i+1}}{n - i} \quad \forall \quad 1 \leq i \leq n - 1.$$

This follows directly from (3.11). It remains to part (c). When $j \leq i$, on appropriately iterating the first inequality of (3.11), it follows that

$$\alpha^{i,1} + \dots + \alpha^{i,j} = \frac{j\theta_i}{i} \geq \theta_j.$$

When $j = i$ or $j = i + 1$, the statement with equality is easily checked. When $j > i + 1$, on appropriately iterating the second inequality of (3.11), it follows that

$$\begin{aligned} \alpha^{i,1} + \dots + \alpha^{i,j} &= \theta_{i+1} + \frac{(j-i-1)(1-\theta_{i+1})}{n-i} = 1 - \frac{(n+1-j)(1-\theta_{i+1})}{n-i} \\ &\geq 1 - (1-\theta_j) = \theta_j. \end{aligned}$$

This completes the proof of the lemma. \square

Step 2. Having chosen the sequence $(\alpha^{i,j})$ as above, the second step involves choosing a sequence of positive real numbers

$$\left\{ \beta_k^{i,j} : 1 \leq i \leq n-1, 1 \leq j \leq n+1, k \geq 1 \right\} \tag{3.12}$$

such that for any $k \geq 1$ and $1 \leq i \leq n-1$:

$$\sum_{1 \leq j \leq n+1} \beta_k^{i,j} = 1 \tag{3.13}$$

$$\beta_k^{i,j+1} - \beta_k^{i,j} \geq \frac{1}{4n^2k} \quad \text{for all } 1 \leq j \leq n, \tag{3.14}$$

$$\beta_k^{i,1} \in \left[\frac{1}{(k+1)(n+1)}, \frac{k}{(k+1)(n+1)} \right], \tag{3.15}$$

$$\beta_k^{i,n+1} \in \left[\frac{k+3}{(k+1)(n+1)}, 1 - \frac{1}{(k+1)(n+1)} \right] \tag{3.16}$$

$$\lim_{k \rightarrow \infty} \beta_k^{i,j} = \alpha^{i,j} \quad \forall \quad 1 \leq j \leq n+1. \tag{3.17}$$

Note that parts (a) and (b) of Lemma 3.3 guarantees the existence of such a sequence. For instance, they imply that for any $1 \leq i \leq n-1$

$$\alpha^{i,1} \leq 1/n \quad \text{and} \quad \alpha^{i,n+1} \geq 1/n.$$

Thus the conditions (3.15), (3.16) and (3.17) are compatible.

Step 3. Now, the third step is to let

$$T_1 := 128n^4\gamma \tag{3.18}$$

and then define inductively T_k^i for $k \geq 1$ and $1 \leq i \leq n$ as follows:

$$T_k^1 := T_k, \quad \beta_k^{i+1,1}T_k^{i+1} := \beta_k^{i,n+1}T_k^i - (n+1)\gamma, \quad T_k^n := T_{k+1}, \tag{3.19}$$

where we set

$$\beta_k^{n,j} := \beta_{k+1}^{1,j} \quad \forall \quad k \geq 1 \quad \text{and} \quad 1 \leq j \leq n+1. \tag{3.20}$$

We claim that for any $k \geq 1$:

$$T_k \geq 32n^4(k+1)^2\gamma \tag{3.21}$$

and

$$T_k^{i+1} > T_k^i \tag{3.22}$$

for all $1 \leq i \leq n-1$. To prove the claim we argue by induction. However, first note that given $k \geq 1$ and $1 \leq i \leq n-1$, if we know that (3.21) holds and furthermore that if $i \geq 2$ then (3.22) also holds (with i replaced by l) for all $2 \leq l \leq i$, then by (3.16) it follows that

$$\beta_k^{i,n+1}T_k^i - (n+1)\gamma \geq \beta_k^{i,n+1}T_k - (n+1)\gamma > 0. \tag{3.23}$$

Now by (3.15) and (3.20), for any $k \geq 1$

$$\min_{1 \leq i \leq n-1} (\beta_k^{i+1,1})^{-1} = (\beta_k^{n,1})^{-1} = (\beta_{k+1}^{1,1})^{-1} \geq (k+2)(n+1)/(k+1)$$

which together with (3.23) implies that

$$\begin{aligned} T_k^{i+1} &= (\beta_k^{i+1,1})^{-1} \left(\beta_k^{i,n+1}T_k^i - (n+1)\gamma \right) \\ &\geq \frac{(k+2)(k+3)}{(k+1)^2} T_k^i - \frac{(n+1)^2(k+2)\gamma}{k+1} \\ &> \frac{(k+2)^2}{(k+1)^2} T_k^i + \left(\frac{T_k^i - (n+1)^2(k+2)\gamma}{k+1} \right). \end{aligned} \tag{3.24}$$

It is important to stress that (3.23) and thus (3.24) hold for $i = 1$ without appealing to (3.22). We are now in the position to start the induction argument.

Let $k = 1$. Then (3.21) holds in view of (3.18). To prove (3.22) for all $1 \leq i \leq n - 1$ we use induction on i . With this and (3.24) in mind, when $i = 1$ it follows via (3.21) and the fact that $T_1^1 := T_1$, that

$$T_1^1 - (n + 1)^2 3\gamma \geq 128n^4\gamma - 3(n + 1)^2\gamma > 0. \tag{3.25}$$

Hence, (3.24) which recall does not rely on (3.22) for $i = 1$ implies that $T_1^2 > T_1^1$. In other words, (3.22) holds for $i = 1$. So suppose (3.22) holds for i with $i \leq n - 2$. Then, it follows via (3.25) that

$$T_1^{i+1} - (n + 1)^2 3\gamma > T_1^1 - (n + 1)^2 3\gamma > 0$$

and so (3.24) implies that $T_1^{i+2} > T_1^{i+1}$. This shows that (3.22) holds for $k = 1$ and all $1 \leq i \leq n - 1$. Now assume that (3.21) and (3.22) holds for some k and all $1 \leq i \leq n - 1$. Then, it follows via (3.24) with $i = n - 1$ and the fact that $T_k^n := T_{k+1}$ and $T_k^1 := T_k$, that

$$\begin{aligned} T_{k+1} &> \frac{(k + 2)^2}{(k + 1)^2} T_k^{n-1} + \left(\frac{T_k^{n-1} - (n + 1)^2(k + 2)\gamma}{k + 1} \right) \\ &> \frac{(k + 2)^2}{(k + 1)^2} T_k + \left(\frac{T_k - (n + 1)^2(k + 2)\gamma}{k + 1} \right) \\ &> 32n^4(k + 2)^2\gamma. \end{aligned}$$

This shows that (3.21) holds for $k + 1$ and we now use this to show that (3.22) holds for $k + 1$ and all $1 \leq i \leq n - 1$. With this in mind, when $i = 1$ it follows that

$$T_{k+1}^1 - (n + 1)^2(k + 3)\gamma \geq 32n^4(k + 2)^2\gamma - (n + 1)^2(k + 3)\gamma > 0 \tag{3.26}$$

and so (3.24) implies that $T_{k+1}^2 > T_{k+1}^1 := T_{k+1}$. In other words, (3.22) holds for $k + 1$ with $i = 1$. So suppose (3.22) holds for $k + 1$ with $i \leq n - 2$. Then it follows via (3.26) that

$$T_{k+1}^{i+1} - (n + 1)^2 3\gamma > T_{k+1}^1 - (n + 1)^2(k + 3)\gamma > 0$$

and so (3.24) implies that $T_{k+1}^{i+2} > T_{k+1}^{i+1}$. This thereby completes the inductive step and hence establishes (3.21) and (3.22) for all $k \geq 1$ and all $1 \leq i \leq n - 1$.

Now, with the generic construction of §3.1 in mind, for each $k \geq 1$, $1 \leq i \leq n - 1$, let

$$T_- = T_k^i \quad \text{and} \quad T_+ = T_k^{i+1},$$

and for $1 \leq j \leq n + 1$, let

$$a_-^j = \beta_k^{i,j} \quad \text{and} \quad a_+^j = \beta_k^{i+1,j} .$$

Then, it is readily verified on using (3.13), (3.14), (3.17), (3.19), (3.22) and (3.21) that conditions (3.5), (3.6) and (3.7) are satisfied. The upshot is that the construction described within §3.1 is applicable and gives rise to a Roy $(n + 1)$ -system $\mathbf{P} : [T_k^i, T_k^{i+1}] \rightarrow \mathbb{R}^{n+1}$ associated with the sequences $(\beta_k^{i,j})$ defined via (3.12) and (T_k^i) defined via (3.19). Moreover, for each $k \geq 1, 1 \leq i \leq n - 1$, the Roy $(n + 1)$ -system \mathbf{P} on the interval $[T_k^i, T_k^{i+1}]$ satisfies Lemma 3.2.

Remark 3.1. As we shall see in the next section, it is not difficult to extend this local statement to a Roy $(n + 1)$ -system \mathbf{P} on the interval $[0, \infty)$ that satisfies Lemma 3.1. Note that this would suffice if all we wanted to show was that the sets appearing in Theorem 1.5 are non-empty rather than continuum. In the remaining two steps, we introduce a new parameter δ and focus on the construction of mutually non-equivalent Roy $(n + 1)$ -systems on the interval $[T_k^i, T_k^{i+1}]$.

Step 4. The fourth step involves perturbing the above construction of the Roy $(n + 1)$ -system \mathbf{P} on $[T_k^i, T_k^{i+1}]$ by a parameter δ in such a way that:

- the properties of Lemma 3.2 are satisfied for the perturbed Roy $(n + 1)$ -system $\mathbf{P}^\delta : [T_k^i, T_k^{i+1}] \rightarrow \mathbb{R}^{n+1}$, and
- for $\delta' \neq \delta$, the perturbed Roy $(n + 1)$ -systems \mathbf{P}^δ and $\mathbf{P}^{\delta'}$ are mutually non-equivalent (see Definition 2.2).

With this in mind, let $(\beta_k^{i,j})$ and (T_k^i) be the sequences given by (3.12) and (3.19) respectively, and let

$$\delta \in [0, 1/(40n^2)) . \tag{3.27}$$

Now define the new sequence

$$\left\{ \beta_k^{i,j}(\delta) : 1 \leq i \leq n - 1, 1 \leq j \leq n + 1, k \geq 1 \right\} \tag{3.28}$$

by setting, for any $k \geq 1$ and $1 \leq i \leq n - 1$:

$$\beta_1^{1,1}(\delta) := \beta_1^{1,1}, \tag{3.29}$$

$$\beta_k^{i,n+1}(\delta) := \beta_k^{i,n+1} + \frac{\delta}{k}, \tag{3.30}$$

$$\beta_k^{i+1,1}(\delta)T_k^{i+1} := \beta_k^{i,n+1}(\delta)T_k^i - (n + 1)\gamma, \tag{3.31}$$

$$\beta_k^{i,1}(\delta) + \beta_k^{i,2}(\delta) + \beta_k^{i,n+1}(\delta) := \beta_k^{i,1} + \beta_k^{i,2} + \beta_k^{i,n+1}, \tag{3.32}$$

$$\beta_k^{i,j}(\delta) := \beta_k^{i,j} \quad \forall \quad 3 \leq j \leq n. \tag{3.33}$$

Also, in line with (3.20), we let

$$\beta_k^{n,j}(\delta) := \beta_{k+1}^{1,j}(\delta) \quad \forall \quad k \geq 1 \quad \text{and} \quad 1 \leq j \leq n + 1.$$

Clearly, the sequences $(\beta_k^{i,j}(\delta))$ and $(\beta_k^{i,j})$ coincide when $\delta = 0$. An immediate consequence of (3.13), (3.32) and (3.33) is that

$$\sum_{j=1}^{n+1} \beta_k^{i,j}(\delta) = 1. \tag{3.34}$$

Also note that in view of (3.19), (3.30) and (3.31), we have that

$$\left(\beta_k^{i+1,1}(\delta) - \beta_k^{i+1,1}\right) T_k^{i+1} = \left(\beta_k^{i,n+1}(\delta) - \beta_k^{i,n+1}\right) T_k^i = \frac{\delta}{k} T_k^i,$$

from which it follows that

$$\beta_k^{i,1} \leq \beta_k^{i,1}(\delta) \leq \beta_k^{i,1} + \frac{\delta}{k} \quad \forall \quad 2 \leq i \leq n - 1$$

and

$$\beta_{k+1}^{1,1} \leq \beta_{k+1}^{1,1}(\delta) \leq \beta_{k+1}^{1,1} + \frac{\delta}{k}. \tag{3.35}$$

To sum up, for all $k \geq 1$ and $1 \leq i \leq n - 1$ we have that

$$\beta_k^{i,1} \leq \beta_k^{i,1}(\delta) \leq \beta_k^{i,1} + \frac{2\delta}{k}. \tag{3.36}$$

Combining (3.36) and (3.32), we get for all $k \geq 1$ and $1 \leq i \leq n - 1$,

$$\beta_k^{i,2} - \frac{3\delta}{k} \leq \beta_k^{i,2}(\delta) \leq \beta_k^{i,2}. \tag{3.37}$$

Now, with the generic construction of §3.1 in mind, for $k \geq 1, 1 \leq i \leq n - 1$, let

$$T_- = T_k^i \quad \text{and} \quad T_+ = T_k^{i+1},$$

and for $1 \leq j \leq n + 1$, let

$$a_-^j = \beta_k^{i,j}(\delta) \quad \text{and} \quad a_+^j = \beta_k^{i+1,j}(\delta).$$

Then, it is readily verified that condition (3.5) follows from (3.31) and that condition (3.6) follows from (3.17) and (3.34). To show (3.7), first note that for all $k \geq 1, 1 \leq i \leq n - 1$ and $j \neq 1$, on using (3.30), (3.33) and (3.37), it follows that

$$\left(\beta_k^{i,j+1}(\delta) - \beta_k^{i,j}(\delta)\right) T_k \geq \left(\beta_k^{i,j+1} - \beta_k^{i,j}\right) T_k.$$

We have already shown that the right hand side satisfies (3.7). When $j = 1$, it is readily verified, on using (3.14), (3.21), (3.27), (3.36) and (3.37), that for all $k \geq 1$ and $1 \leq i \leq n - 1$

$$\left(\beta_k^{i,2}(\delta) - \beta_k^{i,1}(\delta)\right) T_k^i \geq \left(\beta_k^{i,2} - \beta_k^{i,1} - \frac{5\delta}{k}\right) T_k^i \geq \frac{T_k}{8n^2k} \geq 4n^2\gamma.$$

The upshot is that the construction described within §3.1 is applicable and gives rise to a Roy $(n + 1)$ -system $\mathbf{P}^\delta : [T_k^i, T_k^{i+1}] \rightarrow \mathbb{R}^{n+1}$ associated with the constant δ satisfying (3.27) and sequences $(\beta_k^{i,j}(\delta))$ defined via (3.28) and (T_k^i) defined via (3.19). Moreover, for each $k \geq 1$, $1 \leq i \leq n - 1$, the Roy $(n + 1)$ -system \mathbf{P}^δ on the interval $[T_k^i, T_k^{i+1}]$ satisfies Lemma 3.2.

Step 5. It remains to show that the Roy $(n + 1)$ -systems constructed in Step 4 are mutually non-equivalent. This is easily done. Let δ and δ' satisfy (3.27) and suppose $\delta' \neq \delta$. Then it is readily verified, that

$$\left|P_{n+1}^{\delta'}(T_k) - P_{n+1}^\delta(T_k)\right| \stackrel{(3.30)}{=} \frac{|\delta' - \delta| T_k}{k} \stackrel{(3.21)}{\geq} |\delta' - \delta| 32n^4k\gamma > 2C_n$$

for all $k > k_0$ sufficiently large. By definition, this implies that for any $k > k_0$, the Roy $(n + 1)$ -systems \mathbf{P}^δ and $\mathbf{P}^{\delta'}$ on the interval $[T_k, T_{k+1}]$ are mutually non-equivalent.

3.3. Proof of Lemma 3.1

The proof of the Lemma 3.1 will follow on extending the local construction of the Roy $(n + 1)$ -systems \mathbf{P}^δ on the intervals $[T_k^i, T_k^{i+1}]$ presented in §3.2 to the interval $[0, \infty)$. With this in mind, for δ satisfying (3.27) and $k \geq 1$, $1 \leq i \leq n - 1$, let us denote by $\mathbf{P}_{k,i}^\delta$ the Roy $(n + 1)$ -system on $[T_k^i, T_k^{i+1}]$. Now observe that

$$[T_1, \infty) = \bigcup_{k \geq 1} \bigcup_{1 \leq i \leq n-1} [T_k^i, T_k^{i+1}],$$

where T_1 is given by (3.18); that is, the intervals $[T_k^i, T_k^{i+1}]$ give a partition of $[T_1, \infty)$. It therefore follows that the continuous piecewise linear map $\mathbf{P}^\delta = (P_1^\delta, \dots, P_{n+1}^\delta) : [T_1, \infty) \rightarrow \mathbb{R}^{n+1}$ given by

$$\mathbf{P}^\delta(t) := \mathbf{P}_{k,i}^\delta(t) \quad \forall t \in [T_k^i, T_k^{i+1}] \quad (\text{for each } k \geq 1 \text{ and } 1 \leq i \leq n - 1),$$

is a Roy $(n + 1)$ -system on $[T_1, \infty)$. Indeed, by construction the first two properties within Definition 2.1 are automatically satisfied. It thus remains to check the final property at

the common boundary points T_k^i for each $k \geq 1$ and $1 \leq i \leq n - 1$. With this in mind, it is easily verified that

$$P'_{n+1}(T_k^{i-}) = 1 \quad \text{and} \quad P'_1(T_k^{i+}) = 1.$$

Hence, with reference Definition 2.1, it immediately follows that $r_1 = r_2 = n + 1 > s_2 = s_1 = 1$ and so the restraint associated with the final property is in fact vacuous.

It remains to extend \mathbf{P}^δ to the interval $[0, T_1]$. For this let

$$S_{d+1} := \left(\beta_1^{1,1}(\delta) + \dots + \beta_1^{1,d}(\delta) + (n + 1 - d)\beta_1^{1,d+1}(\delta) \right) T_1 \quad \forall \quad 0 \leq d \leq n,$$

and let

$$P_j^\delta(0) := 0 \quad \forall \quad 1 \leq j \leq n + 1.$$

On the interval $[0, S_1]$, let the $n + 1$ components $P_1^\delta, \dots, P_{n+1}^\delta$ coincide and have slope $1/(n + 1)$. It follows that

$$P_j^\delta(S_1) := \beta_1^{1,1}(\delta) T_1 \quad \forall \quad 1 \leq j \leq n + 1.$$

For $1 \leq d \leq n$, on the interval $[S_d, S_{d+1}]$, let the $n + 1 - d$ components $P_{d+1}^\delta, \dots, P_{n+1}^\delta$ coincide and have slope $1/(n + 1 - d)$ while the components $P_1^\delta, \dots, P_d^\delta$ have slope 0 and

$$P_j^\delta(S_{d+1}) := \begin{cases} \beta_1^{1,j}(\delta) T_1 & \text{if } 1 \leq j \leq d \\ \beta_1^{1,d+1}(\delta) T_1 & \text{if } d + 1 \leq j \leq n + 1. \end{cases}$$

In particular, by (3.34) we have that $S_{n+1} = T_1$ and so it follows that

$$P_j^\delta(T_1) = \beta_1^{1,j}(\delta) T_1 \quad \forall \quad 1 \leq j \leq n + 1.$$

In short, this coincides with left hand side of Fig. 1 with $T_- = T_1$ and $a_-^j = \beta_1^{1,j}(\delta)$ ($1 \leq j \leq n + 1$). The upshot is that the above construction enables us to extend in then obvious manner the Roy $(n + 1)$ -system on $[T_1, \infty)$ to $[0, \infty)$. We now show that the Roy $(n + 1)$ -system $\mathbf{P}^\delta : [0, \infty) \rightarrow \mathbb{R}^{n+1}$ satisfies the desired properties of Lemma 3.1.

- By construction, (3.2) follows directly from (3.8).
- In view of (3.9), (3.15), (3.21) and (3.35), it follows that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \left(\frac{t}{n + 1} - P_1^\delta(t) \right) &= \limsup_{k \rightarrow \infty} \max_{1 \leq i \leq n-1} \max_{t \in [T_k^i, T_k^{i+1}]} \left(\frac{t}{n + 1} - P_1^\delta(t) \right) \\ &\geq \limsup_{k \rightarrow \infty} \max_{1 \leq i \leq n-1} \left(\frac{1}{n + 1} - \beta_k^{i,1}(\delta) \right) T_k^{i+1} \end{aligned}$$

$$\begin{aligned}
 &\geq \limsup_{k \rightarrow \infty} \max_{1 \leq i \leq n-1} \left(\frac{1}{n+1} - \beta_k^{i,1} - \frac{\delta}{k} \right) T_k^{i+1} \\
 &\geq \limsup_{k \rightarrow \infty} \left(\frac{1}{n+1} - \frac{k}{(k+1)(n+1)} - \frac{1}{40n^2k} \right) T_k \\
 &\geq \limsup_{k \rightarrow \infty} n^3(k+2)\gamma = \infty.
 \end{aligned}$$

This shows that \mathbf{P}^δ satisfies (3.3).

- Let $1 \leq d \leq n$. Then, in view of (3.10), (3.17) and part (c) of Lemma 3.3, it follows that

$$\begin{aligned}
 \liminf_{t \rightarrow \infty} \frac{P_1^\delta(t) + \dots + P_d^\delta(t)}{t} &= \liminf_{k \rightarrow \infty} \min_{1 \leq i \leq n} \min_{t \in [T_k^i, T_k^{i+1}]} \left(\frac{P_1^\delta(t) + \dots + P_d^\delta(t)}{t} \right) \\
 &= \liminf_{k \rightarrow \infty} \min_{1 \leq i \leq n-1} \min \left\{ \sum_{j=1}^d \beta_k^{i,j}(\delta), \sum_{j=1}^d \beta_k^{i+1,j}(\delta) \right\} \\
 &= \liminf_{k \rightarrow \infty} \min_{1 \leq i \leq n-1} \min \left\{ \sum_{j=1}^d \beta_k^{i,j}, \sum_{j=1}^d \beta_k^{i+1,j} \right\} \\
 &= \min_{1 \leq i \leq n} \alpha^{i,1} + \dots + \alpha^{i,d} \\
 &= \frac{1}{1 + \tau_{n-d}}.
 \end{aligned}$$

This shows \mathbf{P}^δ satisfies (3.4).

In the previous section (see Step 5), we have already seen that for any distinct δ and δ' satisfying (3.27), the Roy $(n+1)$ -systems \mathbf{P}^δ and $\mathbf{P}^{\delta'}$ on $[T_k, T_{k+1}]$ are mutually non-equivalent for all k sufficiently large. Consequently, there are continuum many non-equivalent Roy $(n+1)$ -systems \mathbf{P}^δ on $[0, \infty)$ that satisfies the conditions of Lemma 3.1. This completes the proof. \square

3.4. Further comment

Recall, that once we have constructed in §3.3 the collection of non-equivalent Roy $(n+1)$ -systems \mathbf{P}^δ satisfying Lemma 3.1, the key ingredient towards establishing our main result is Theorem 2.1. In short, the latter guarantees that each system in our collection gives rise to a distinct point $\mathbf{x} \in \mathbb{R}^n$ such that

$$\|\mathbf{L}_x - \mathbf{P}^\delta\| \leq C_n$$

on $[0, \infty)$. In turn, it is not difficult to show that such x satisfies the desired Diophantine properties associated with Theorem 1.5. Now with this in mind, let $n \in \mathbb{N}$ and \mathcal{W} be a collection of Roy $(n + 1)$ -systems $\mathbf{P} : [0, \infty) \rightarrow \mathbb{R}^{n+1}$. In [10, Theorem 2.3], it is shown that if \mathcal{W} satisfies the so called ‘closed under finite perturbations’ hypothesis, then one is able to compute the Hausdorff dimension of the set

$$\{x \in \mathbb{R}^n : \|\mathbf{L}_x - \mathbf{P}\| < \infty \text{ for some } \mathbf{P} \in \mathcal{W}\}.$$

However, it is not clear to us whether the methods in [10] can be adapted to give a result on the Hausdorff dimension of the set

$$\{x \in \mathbb{R}^n : \|\mathbf{L}_x - \mathbf{P}\| < C \text{ for some } \mathbf{P} \in \mathcal{W}\}$$

for any fixed constant $C > 0$. Such a result would potentially enable us to replace ‘continuum’ by ‘full Hausdorff dimension’ in the statement of Theorem 1.2. We state this highly desired strengthening formally as an open problem.

Problem 3.1. *Suppose that $n \geq 2$. Prove that*

$$\dim \mathbf{FS}_n = n.$$

In §4.3 below we consider the natural generalisation of the above problem to the setting of weighted Diophantine approximation for systems of linear forms.

A more subtle version of Problem 3.1 would be to understand how the removal of $\mathbf{Bad}_n \cup \mathbf{Sing}_n$ affects the quantitative form of the set \mathbf{DI}_n , that is to estimate from below the Hausdorff dimension of $\mathbf{FS}(\varepsilon)$, where for $0 < \varepsilon < 1$

$$\mathbf{FS}_n(\varepsilon) := \mathbf{DI}_n(\varepsilon) \setminus (\mathbf{Bad}_n \cup \mathbf{Sing}_n).$$

Clearly, by definition, $\dim \mathbf{FS}_n(\varepsilon)$ cannot be larger than $\dim \mathbf{DI}_n(\varepsilon)$ and by a recent result due to Kleinbock & Mirzadeh [19], the latter is strictly smaller than n for all $\varepsilon \in (0, 1)$. Note that Theorem 1.5 guarantees that

$$\mathbf{FS}_n(\varepsilon) \neq \emptyset \quad \forall \varepsilon \in (0, 1).$$

4. Intermediate Diophantine sets revisited

The main goal of this section is to define the notion of d -Dirichlet improvable points in \mathbb{R}^n and investigate the relationship between them and the classical notions of simultaneous ($d = 0$) and dual ($d = n - 1$) Dirichlet improvable points. We will explore two different approaches:

- (i) an algebraic approach using multilinear algebra and thus developing the ideas of Laurent [20], and
- (ii) a geometric approach using Minkowski minima and thus developing the ideas of Schmidt & Summerer [25,26] and Roy [22] on the parametric geometry of numbers leading to Lemma 2.1 of §2.2.

The key ingredient required for achieving this lies in being able to state an appropriate optimal Dirichlet type theorem (see Remark 1.3 in §1.2). This will be accomplished in §4.1 via the framework of multilinear algebra and in §4.2 via the framework of the parametric geometry of numbers. In the process of describing the setup leading to the Dirichlet type theorem of §4.1.2, we will take the opportunity to first revisit the intermediate badly approximable and singular sets in order to fill in the details of the arguments (cf. Remark 2.2) leading to (2.11) and (2.12). Recall, that these equivalences are used in the “classical” proof of Proposition 1.1 given in §2.2 showing that the intermediate d -badly approximable sets \mathbf{Bad}_n^d (resp. the d -singular sets \mathbf{Sing}_n^d) are equivalent. Moreover, we will provide an alternative “dynamical” proof of the proposition that avoids appealing to (2.11) and (2.12).

4.1. The algebraic approach

To proceed, we recall notions and results from multilinear algebra. Let $n \geq 2$ and $0 \leq d \leq n - 1$. First, we endow the linear space \mathbb{R}^{n+1} with the usual inner product and let $\{\mathbf{e}_i\}_{1 \leq i \leq n+1}$ be the standard orthonormal basis. Then the wedge product $\wedge^{d+1} \mathbb{R}^{n+1}$ is also equipped with an inner product with an orthonormal basis given by

$$\{\mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_{d+1}} : 1 \leq i_1 \leq \dots \leq i_{d+1} \leq n + 1\}.$$

Note that, under this basis, we identify $\wedge^{d+1} \mathbb{R}^{n+1}$ with $\mathbb{R}^{\binom{n+1}{d+1}}$. For any $\mathbf{X} \in \wedge^{d+1} \mathbb{R}^{n+1}$, we set $\|\mathbf{X}\|_2$ and $\|\mathbf{X}\|$ to be the Euclidean norm and maximal norm (with respect to the basis given above), respectively. $\mathbf{X} \in \wedge^{d+1} \mathbb{R}^{n+1}$ is called *decomposable* if and only if there exists $\mathbf{v}_1, \dots, \mathbf{v}_{d+1} \in \mathbb{R}^{n+1}$ such that $\mathbf{X} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{d+1}$. It is worth highlighting that up to an homothety there is a one to one correspondence between decomposable $\mathbf{X} \in \wedge^{d+1} \mathbb{R}^{n+1}$ and d -dimensional rational affine subspaces of \mathbb{R}^n . Indeed, by expressing a d -dimensional affine subspace $L \subset \mathbb{R}^n \subset \mathbb{P}^n(\mathbb{R})$ using homogeneous coordinates, we obtain a unique $(d + 1)$ -dimensional subspace V_L of \mathbb{R}^{n+1} satisfying that $L = \mathbb{P}(V_L)$. Clearly, L is rational if and only if V_L has a integer basis $\{\mathbf{v}_1, \dots, \mathbf{v}_{d+1}\} \subset \mathbb{Z}^{n+1}$. By using the Plücker embedding

$$\text{Gr}(d, \mathbb{P}^n(\mathbb{R})) \hookrightarrow \mathbb{P}(\wedge^{d+1} \mathbb{R}^{n+1}), \quad L \mapsto \mathbf{X}_L := \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{d+1} \tag{4.1}$$

we obtain the Plücker coordinates for L . Conversely, given a decomposable multivector $\mathbf{X} \in \wedge^{d+1} \mathbb{Z}^{n+1} \setminus \{0\}$, the associated linear subspace V_L can be expressed as $V_L :=$

$\{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x} \wedge \mathbf{X} = \mathbf{0}\}$ and then L can be obtained by projecting the intersection $V_L \cap \{\mathbf{x} \in \mathbb{R}^{n+1} : x_{n+1} = 1\}$ onto the first n coordinates. The height $H(L)$ of L will be the Weil height, that is

$$H(L) = \|\mathbf{X}_L\|_2,$$

where \mathbf{X}_L is as in (4.1) and $\mathbf{v}_1, \dots, \mathbf{v}_{d+1}$ is a basis of $V_L \cap \mathbb{Z}^{n+1}$. In other words, $H(L)$ is the covolume of the lattice $V_L \cap \mathbb{Z}^{n+1}$ in V_L . Note that $\mathbf{X}_L \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$ has coprime coordinates. Given $\mathbf{x} \in \mathbb{R}^n$, we define the projective distance between \mathbf{x} and L by

$$d_p(\mathbf{x}, L) := \frac{\|\mathbf{x}' \wedge \mathbf{X}_L\|_2}{\|\mathbf{x}'\|_2 \|\mathbf{X}_L\|_2}, \tag{4.2}$$

where $\mathbf{x}' := (\mathbf{x}, 1) \in \mathbb{R}^{n+1}$. Geometrically, $d_p(\mathbf{x}, L)$ equals the sine of the smallest angle between \mathbf{x}' and non-zero vectors in V_L . The projective distance $d_p(\mathbf{x}, L)$ is easily seen to be locally (depending on $\|\mathbf{x}\|_2$) comparable to the distance $d(\mathbf{x}, L)$ defined by (1.10) in §1.2 and indeed the distance of \mathbf{x} from L induced by any norm on \mathbb{R}^n , see for instance, [4, Eq (3.5)].

4.1.1. Showing $\mathbf{Sing}_n^d \equiv (2.11)$ and $\mathbf{Bad}_n^d \equiv (2.12)$

The following statement provides an algebraic formulation of the sets \mathbf{Sing}_n^d and \mathbf{Bad}_n^d as defined via (1.13) and (1.14) in §1.2. Recall, given $n \in \mathbb{N}$ and $d \in \{0, 1, \dots, n - 1\}$, we let

$$\omega_d := \frac{d + 1}{n - d}.$$

Lemma 4.1. *Let $n \geq 2$ and $0 \leq d \leq n - 1$.*

- (i) $\mathbf{x} \in \mathbf{Sing}_n^d$ if and only if for any given $\varepsilon \in (0, 1)$ and $N > N_0(\mathbf{x}, \varepsilon)$ sufficiently large, there exists a $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$, such that

$$\|\mathbf{X}\| \leq N \quad \text{and} \quad \|\mathbf{x}' \wedge \mathbf{X}\|_2 \leq \varepsilon N^{-\omega_d}. \tag{4.3}$$

- (ii) $\mathbf{x} \in \mathbf{Bad}_n^d$ if and only if there exists a constant $\varepsilon := \varepsilon(\mathbf{x}) \in (0, 1)$ such that for any $N > N_0(\mathbf{x})$ sufficiently large, there are no $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$ satisfying (4.3).

Remark 4.1. In view of the discussion at the start of this section, it is straightforward to establish the above reformulation of the sets \mathbf{Sing}_n^d and \mathbf{Bad}_n^d under the extra assumption that $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$ is decomposable. In view of this, the key feature of Lemma 4.1 is that it enables us to remove the decomposable assumption.

Proof. The ‘only if part’ of both (i) and (ii) follows directly from Remark 4.1. Now concentrating on the ‘if part’ of (i), in view of Remark 4.1, it suffices to show that

for any given $\varepsilon \in (0, 1)$ and $N > N_0(\mathbf{x}, \varepsilon)$ sufficiently large, there exists a decomposable $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$ such that (4.3) holds. With this in mind, let $\varepsilon \in (0, 1)$, $N > N_0(\mathbf{x}, \varepsilon)$ and $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$ satisfying (4.3) be as given. This implies that the first successive minima of the convex body \mathcal{C} defined by

$$\{\mathbf{X} \in \wedge^{d+1}\mathbb{R}^{n+1} : \|\mathbf{X}\|_2 \leq N, \quad \|\mathbf{x}' \wedge \mathbf{X}\|_2 \leq \varepsilon N^{-\omega_d}\} \tag{4.4}$$

is less than 1. Let

$$U := \varepsilon^{-d/(d+1)}N^{n/(n-d)} \quad \text{and} \quad V := \varepsilon^{(n+d)/(nd+n)}U^{1/n} = \varepsilon^{1/(d+1)}N^{-1/(n-d)}.$$

Thus, $N = UV^d$ and $\varepsilon N^{-\omega_d} = V^{d+1}$. Then, in view of [5, Lemma 3], there exists $\beta_1 = \beta_1(n, d) > 1$ such that

$$\beta_1^{-1}\tilde{\mathcal{C}}_{d+1} \subset \mathcal{C} \subset \beta_1\tilde{\mathcal{C}}_{d+1}, \tag{4.5}$$

where $\tilde{\mathcal{C}}_{d+1}$ is the $(d + 1)$ -th compound of the convex body $\tilde{\mathcal{C}} \subset \mathbb{R}^{n+1}$ defined as

$$\tilde{\mathcal{C}} := \left\{ \mathbf{y} \in \mathbb{R}^{n+1} : |y_{n+1}| \leq U, \quad \max_{1 \leq i \leq n} |y_{n+1}x_i - y_i| \leq V \right\}.$$

In turn, it follows via Mahler’s theory of compound convex bodies that there exists $\beta_2 = \beta_2(n, d) \geq 1$ such that

$$\mathbf{X} := \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_{d+1} \in \beta_2 \lambda_1(\wedge^{d+1}\mathbb{Z}^{n+1}, \mathcal{C}) \tilde{\mathcal{C}}_{d+1} \tag{4.6}$$

where $\mathbf{x}_i \in \mathbb{Z}^{n+1}$ is the integer point at which $\tilde{\mathcal{C}}$ attains its i -th successive minima. Recall, for a given convex body $\mathcal{C} \subset \wedge^{d+1}\mathbb{R}^{n+1}$ and $i = 1, \dots, d + 1$, we write $\lambda_i(\wedge^{d+1}\mathbb{Z}^{n+1}, \mathcal{C})$ for the i -successive minima of \mathcal{C} with respect to the lattice $\wedge^{d+1}\mathbb{Z}^{n+1}$. The upshot of (4.6) is that \mathbf{X} is decomposable and this proves the ‘if part’ of (i). The proof of the ‘if part’ of (ii) is similar and we leave the details to the reader. \square

Armed with Lemma 4.1, it is relatively straightforward to obtain the sought after statement.

Lemma 4.2. *Let $n \geq 2$ and $0 \leq d \leq n - 1$.*

- (i) $\mathbf{x} \in \text{Sing}_n^d$ if and only if for any $\delta > 0$ there exists a constant $t_0 = t_0(\delta) > 0$ such that for all $t \geq t_0$ inequality (2.12) holds; that is

$$\frac{(n - d)t}{n + 1} - (L_{\mathbf{x},1}(t) + \cdots + L_{\mathbf{x},n-d}(t)) \geq \delta.$$

(ii) $\mathbf{x} \in \mathbf{Bad}_n^d$ if and only if there exists a constant $\delta > 0$ such that for all sufficiently large t inequality (2.11) holds; that is

$$\frac{(n-d)t}{n+1} - (L_{\mathbf{x},1}(t) + \dots + L_{\mathbf{x},n-d}(t)) \leq \delta.$$

Proof. Lemma 4.1 implies that $\mathbf{x} \in \mathbf{Sing}_n^d$ if and only if for any $\delta > 0$ there exists a constant $t_0 = t_0(\delta) > 0$ such that for all $t \geq t_0$,

$$\lambda_1(\wedge^{d+1}\mathbb{Z}^{n+1}, \mathcal{K}_{\mathbf{x}}^{d+1}(e^t)) \leq e^{-\delta}$$

where

$$\mathcal{K}_{\mathbf{x}}^{d+1}(e^t) := \left\{ \mathbf{X} \in \wedge^{d+1}\mathbb{R}^{n+1} : \|\mathbf{X}\| \leq e^{\frac{(n-d)t}{n+1}}, \quad \|\mathbf{x}' \wedge \mathbf{X}\| \leq e^{\frac{-(d+1)t}{n+1}} \right\}.$$

In view of (4.5), the convex body $\mathcal{K}_{\mathbf{x}}^{d+1}(e^t)$ is comparable (with implied constants depending on n and d only) to the $(d+1)$ -th compound of the convex body

$$\mathcal{K}_{\mathbf{x}}(e^t) := \left\{ \mathbf{y} \in \mathbb{R}^{n+1} : |y_{n+1}| \leq e^{\frac{nt}{n+1}}, \quad \max_{1 \leq i \leq n} |y_{n+1}x_i - y_i| \leq e^{-\frac{t}{n+1}} \right\}.$$

Note that

$$\mathcal{C}_{\mathbf{x}}(e^t) = e^{-\frac{t}{n+1}} \mathcal{K}_{\mathbf{x}}(e^t)^\vee,$$

where $\mathcal{C}_{\mathbf{x}}(e^t)$ is given by (2.1) and $\mathcal{K}_{\mathbf{x}}(e^t)^\vee$ denotes the dual of $\mathcal{K}_{\mathbf{x}}(e^t)$. Hence, by Minkowski’s second convex body theorem, it follows that

$$\begin{aligned} \log \lambda_1(\wedge^{d+1}\mathbb{Z}^{n+1}, \mathcal{K}_{\mathbf{x}}^{d+1}(e^t)) &= \sum_{i=1}^{d+1} \log \lambda_i(\mathbb{Z}^{n+1}, \mathcal{K}_{\mathbf{x}}(e^t)) + O(1) \\ &= \sum_{i=1}^{d+1} -\log \lambda_{n+2-i}(\mathbb{Z}^{n+1}, \mathcal{K}_{\mathbf{x}}(e^t)^\vee) + O(1) \\ &= \sum_{i=1}^{n-d} \log \lambda_i(\mathbb{Z}^{n+1}, \mathcal{K}_{\mathbf{x}}(e^t)^\vee) + O(1) \\ &= -\frac{(n-d)t}{n+1} + \sum_{i=1}^{n-d} L_{\mathbf{x},i}(t) + O(1), \end{aligned}$$

where the quantity $L_{\mathbf{x},i}(t)$ is given by (2.3) and the implied constants in the ‘big O ’ term depend on n and d only. This thereby completes the proof of first claim made in the lemma. The proof of (ii) is similar and we leave the details to the reader. \square

4.1.2. An optimal Dirichlet type theorem via multilinear algebra

For obvious reasons, as discussed in Remark 1.3, in order to define Dirichet improvable sets it is paramount to start with an optimal Dirichlet type theorem. Any such theorem should naturally not only imply Theorem 1.3 concerning the approximation of points by rational subspaces but also coincide with the classical simultaneous and dual forms of Dirichlet theorem. The multilinear algebra framework exploited in the previous section yields the following optimal statement.

Theorem 4.1. *Let $n \in \mathbb{N}$ and d be an integer satisfying $0 \leq d \leq n - 1$. Then for any $\mathbf{x} \in \mathbb{R}^n$ and $N > 1$, there exist $\mathbf{Z} \in \wedge^d \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{Y} \in \wedge^{d+1} \mathbb{Z}^n$ such that*

$$\|\mathbf{Z}\| \leq N \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}\| \leq N^{-\omega_d}. \tag{4.7}$$

Proof. Consider the linear space $V := \wedge^d \mathbb{R}^n \oplus \wedge^{d+1} \mathbb{R}^n$ with the lattice $L := \wedge^d \mathbb{Z}^n \oplus \wedge^{d+1} \mathbb{Z}^n$. On observing that $\omega_d = \binom{n}{d} / \binom{n}{d+1}$, it is easily verified that for any given $\mathbf{x} \in \mathbb{R}^n$ the volume of the convex body given by (4.7) is equal to $2^{\dim V}$. Hence, the statement of the theorem follows as a direct consequence of Minkowski’s convex body theorem. \square

By taking $d = 0$ and $n - 1$ in Theorem 4.1, we immediately recover the classical simultaneous and dual forms of Dirichlet’s theorem. We now show that for general d we recover Theorem 1.3.

Step 1. We show that Theorem 1.3 has the following equivalent algebraic formulation. Recall, given $\mathbf{x} \in \mathbb{R}^n$ we let $\mathbf{x}' := (\mathbf{x}, 1) \in \mathbb{R}^{n+1}$.

Lemma 4.3. *Let $n \in \mathbb{N}$ and d be integer satisfying $0 \leq d \leq n - 1$. Then for any $\mathbf{x} \in \mathbb{R}^n$ there exists a constant $c = c(n, d, \mathbf{x}) > 0$, such that for any $N \geq 1$ there exists a $\mathbf{X} \in \wedge^{d+1} \mathbb{Z}^{n+1} \setminus \{0\}$, such that*

$$\|\mathbf{X}\|_2 \leq N \quad \text{and} \quad \|\mathbf{x}' \wedge \mathbf{X}\|_2 \leq cN^{-\omega_d}. \tag{4.8}$$

Proof of equivalence of Theorem 1.3 and Lemma 4.3. For the same reasons as outlined in Remark 4.1, it is easy to deduce Lemma 4.3 from Theorem 1.3. For the converse, we adapt the proof of Lemma 4.1. This simply amounts to putting $\varepsilon = c(n, d, \mathbf{x})$ when defining the convex body \mathcal{C} given by (4.4). Apart from this the given proof remains unchanged. \square

Step 2. We show that Theorem 4.1 implies Lemma 4.3.

Proof that Theorem 4.1 implies Lemma 4.3. With reference to Theorem 4.1, given $\mathbf{x} \in \mathbb{R}^n$ and $N > \max\{\|\mathbf{x}\|_2^{-1/\omega_d}, 1\}$ let $\mathbf{Z} \in \wedge^d \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{Y} \in \wedge^{d+1} \mathbb{Z}^n$ be a solution to (4.7). Then, it follows that

$$\|\mathbf{Y}\| \leq \|\mathbf{x} \wedge \mathbf{Z}\| + N^{-\omega_d} \leq \|\mathbf{x} \wedge \mathbf{Z}\|_2 + \|\mathbf{x}\|_2 \leq 2\|\mathbf{x}\|_2 \|\mathbf{Z}\|_2. \tag{4.9}$$

Now let $\mathbf{e}_{n+1} := (0, \dots, 0, 1) \in \mathbb{Z}^{n+1}$ and identify the set $\{\mathbf{y} = (y_1, \dots, y_{n+1}) \in \mathbb{Z}^{n+1} : y_{n+1} = 0\}$ (resp. $\{\mathbf{y} = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} : y_{n+1} = 0\}$) with \mathbb{Z}^n (resp. \mathbb{R}^n). Then, we have that

$$\wedge^{d+1} \mathbb{Z}^{n+1} = \mathbf{e}_{n+1} \wedge (\wedge^d \mathbb{Z}^n) \oplus \wedge^{d+1} \mathbb{Z}^n.$$

Next, let

$$\mathbf{X} := \mathbf{e}_{n+1} \wedge \mathbf{Z} - \mathbf{Y} \in \wedge^{d+1} \mathbb{Z}^{n+1}.$$

Then, it follows that

$$\begin{aligned} \|\mathbf{X}\| &\leq \|\mathbf{Z}\| + \|\mathbf{Y}\| \stackrel{(4.9)}{\leq} (2|\mathbf{x}| + 1)\|\mathbf{Z}\|_2 \leq (2|\mathbf{x}| + 1)\sqrt{\binom{n}{d}} \|\mathbf{Z}\| \\ &\stackrel{(4.7)}{\leq} 2^{\frac{n}{2}}(2|\mathbf{x}| + 1)N, \end{aligned}$$

and so

$$\|\mathbf{X}\|_2 \leq n^{\frac{1}{2}} 2^{\frac{n}{2}}(2|\mathbf{x}| + 1)N. \tag{4.10}$$

On the other hand, we have that

$$\begin{aligned} \mathbf{x}' \wedge \mathbf{X} &= (\mathbf{e}_{n+1} + \mathbf{x}) \wedge (\mathbf{e}_{n+1} \wedge \mathbf{Z} - \mathbf{Y}) = -\mathbf{e}_{n+1} \wedge \mathbf{Y} + \mathbf{x} \wedge \mathbf{e}_{n+1} \wedge \mathbf{Z} - \mathbf{x} \wedge \mathbf{Y} \\ &= -(\mathbf{e}_{n+1} + \mathbf{x}) \wedge (\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}). \end{aligned}$$

Since \mathbf{e}_{n+1} is orthogonal to \mathbf{x} , $\mathbf{e}_{n+1} \wedge (\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y})$ is orthogonal to $\mathbf{x} \wedge (\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y})$. Thus,

$$\|\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}\|_2 \leq \|\mathbf{x}' \wedge \mathbf{X}\|_2 \leq \|\mathbf{x}'\|_2 \|\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}\|_2 \tag{4.11}$$

and on using the above right hand side inequality and (4.7), it follows that

$$\|\mathbf{x}' \wedge \mathbf{X}\|_2 \leq 2^{\frac{n}{2}} |\mathbf{x}'| \|\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}\| \leq 2^{\frac{n}{2}} |\mathbf{x}'| N^{-\omega_d}. \tag{4.12}$$

Together, (4.10) and (4.12) imply the statement of the lemma. \square

The upshot of Steps 1 & 2 is the following desired statement.

Proposition 4.1. *Theorem 4.1 \implies Theorem 1.3.*

4.1.3. *Bad, singular and Dirichlet improvable in light of Theorem 4.1*

Theorem 4.1 enables us to define an associated notion of d -Dirichlet improvable vectors as well as alternative “algebraic” notions of d -singular and d -badly approximable vectors.

Definition 4.1. Let $n \in \mathbb{N}$ and d be an integer satisfying $0 \leq d \leq n - 1$. Then for $\varepsilon \in (0, 1)$, let

- $\mathbf{DI}_n^d(\varepsilon)$ be the set of $\mathbf{x} \in \mathbb{R}^n$, such that for all $N > N_0(\mathbf{x}, \varepsilon)$ sufficiently large, there exist $\mathbf{Z} \in \wedge^d \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{Y} \in \wedge^{d+1} \mathbb{Z}^n$ such that

$$\|\mathbf{Z}\| \leq N \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}\| \leq \varepsilon N^{-\omega_d}. \tag{4.13}$$

- $\mathbf{Bad}_n^d(\varepsilon)$ be the set of $\mathbf{x} \in \mathbb{R}^n$, such that for all $N > N_0(\mathbf{x}, \varepsilon)$ sufficiently large, there are no solutions $\mathbf{Z} \in \wedge^d \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{Y} \in \wedge^{d+1} \mathbb{Z}^n$ to (4.13).

Moreover, let

$$\mathbf{DI}_n^d := \bigcup_{\varepsilon \in (0,1)} \mathbf{DI}_n^d(\varepsilon), \quad \mathbf{Sing}_n^d := \bigcap_{\varepsilon \in (0,1)} \mathbf{DI}_n^d(\varepsilon), \quad \mathbf{Bad}_n^d := \bigcup_{\varepsilon \in (0,1)} \mathbf{Bad}_n^d(\varepsilon).$$

Recall, that on taking $d = 0$ (resp. $d = n - 1$) in Theorem 4.1, we immediately recover the classical simultaneous (resp. dual) form of Dirichlet’s theorem. Thus, in these cases it is clear that the sets \mathbf{DI}_n^d , \mathbf{Sing}_n^d and \mathbf{Bad}_n^d given by Definition 4.1 coincide with those defined via the classical forms of Dirichlet theorem. In particular, for Dirichlet improvable points in \mathbb{R}^n , this means that the set \mathbf{DI}_n^0 (resp. \mathbf{DI}_n^{n-1}) is equivalent to set defined via the classical inequality (1.3) (resp. (1.9)) and in view of Davenport & Schmidt [12, Theorem 2] we have that

$$\mathbf{DI}_n^0 = \mathbf{DI}_n^{n-1} := \mathbf{DI}_n; \tag{4.14}$$

i.e. the simultaneous form of Dirichlet’s theorem is improvable if and only if the dual form of Dirichlet’s theorem is improvable.

For obvious reasons, including consistency, it is vitally important that the following ‘equivalence’ statement is true for previously defined intermediate Diophantine sets.

Lemma 4.4. *The definition of \mathbf{Sing}_n^d (resp. \mathbf{Bad}_n^d) given by Definition 4.1 is equivalent to that given by (1.13) (resp. (1.14)) in §1.2.*

Proof. Let us write $\widetilde{\mathbf{Sing}}_n^d$ (resp. $\widetilde{\mathbf{Bad}}_n^d$) to denote the set of d -singular vectors (resp. d -badly approximable vectors) given by (1.13) (resp. (1.14)). Then, Lemma 4.1 states that

- $\mathbf{x} \in \widetilde{\mathbf{Sing}}_n^d$ if and only if for any given $\varepsilon \in (0, 1)$ and $N > N_0(\mathbf{x}, \varepsilon)$ sufficiently large, there exists $\mathbf{X} \in \wedge^{d+1} \mathbb{Z}^{n+1} \setminus \{0\}$, such that

$$\|\mathbf{X}\|_2 \leq N \quad \text{and} \quad \|\mathbf{x}' \wedge \mathbf{X}\|_2 \leq \varepsilon N^{-\omega_d}. \tag{4.15}$$

- $\mathbf{x} \in \widetilde{\mathbf{Bad}}_n^d$ if and only if there exists a constant $\varepsilon := \varepsilon(\mathbf{x}) \in (0, 1)$ such that for any $N > N_0(\mathbf{x})$ sufficiently large, there are no solutions $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$ to (4.15).

As in the proof of Step 2 in §4.1.2, let $\mathbf{e}_{n+1} = (0, \dots, 0, 1) \in \mathbb{Z}^{n+1}$ and identify the set $\{\mathbf{y} = (y_1, \dots, y_{n+1}) \in \mathbb{Z}^{n+1} : y_{n+1} = 0\}$ (resp. $\{\mathbf{y} = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1} : y_{n+1} = 0\}$) with \mathbb{Z}^n (resp. \mathbb{R}^n). Then, we have that

$$\wedge^{d+1}\mathbb{Z}^{n+1} = \mathbf{e}_{n+1} \wedge (\wedge^d\mathbb{Z}^n) \oplus \wedge^{d+1}\mathbb{Z}^n.$$

Hence, we can write any $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$ uniquely as $\mathbf{e}_{n+1} \wedge \mathbf{Z} - \mathbf{Y}$ with $\mathbf{Z} \in \wedge^d\mathbb{Z}^n$ and $\mathbf{Y} \in \wedge^{d+1}\mathbb{Z}^n$ such that either \mathbf{Z} or \mathbf{Y} is non-zero. Then, the same argument leading to (4.11), enables us to conclude that

$$\|\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}\|_2 \leq \|\mathbf{x}' \wedge \mathbf{X}\|_2 \leq \|\mathbf{x}'\|_2 \|\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}\|_2. \tag{4.16}$$

Note that we also have that

$$\max\{\|\mathbf{Z}\|_2, \|\mathbf{Y}\|_2\} \leq \|\mathbf{X}\|_2 \leq 2 \max\{\|\mathbf{Z}\|_2, \|\mathbf{Y}\|_2\}. \tag{4.17}$$

Now fix $\varepsilon \in (0, 1)$ and $\mathbf{x} \in \mathbb{R}^n$. Let $N > \max\{\|\mathbf{x}\|_2^{-1/\omega_d}, 1\}$ and suppose that $\mathbf{Z} \in \wedge^d\mathbb{Z}^n \setminus \{0\}$ and $\mathbf{Y} \in \wedge^{d+1}\mathbb{Z}^n$ is a solution to (4.13). Let $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1}$ be the corresponding unique point satisfying (4.16) and (4.17). Then, the same argument leading to (4.10) and (4.12), shows that \mathbf{X} satisfies

$$\|\mathbf{X}\|_2 \leq n^{\frac{1}{2}} 2^{\frac{n}{2}} (2\|\mathbf{x}\|_2 + 1)N \quad \text{and} \quad \|\mathbf{x}' \wedge \mathbf{X}\|_2 \leq \varepsilon 2^{\frac{n}{2}} \|\mathbf{x}'\|_2 N^{-\omega_d}.$$

Conversely, let $N > \max\{\|\mathbf{x}\|_2^{-1/\omega_d}, 1\}$ and suppose that $\mathbf{X} \in \wedge^{d+1}\mathbb{Z}^{n+1} \setminus \{0\}$ is a solution to (4.15). Let $\mathbf{Z} \in \wedge^d\mathbb{Z}^n$ and $\mathbf{Y} \in \wedge^{d+1}\mathbb{Z}^n$ be the corresponding unique points (not both zero) satisfying (4.16) and (4.17). Then it follows that using the left hand side of (4.16) and (4.17), that \mathbf{Z} and \mathbf{Y} satisfy

$$\|\mathbf{Z}\| \leq N \quad \text{and} \quad \|\mathbf{x} \wedge \mathbf{Z} + \mathbf{Y}\| \leq \varepsilon N^{-\omega_d}.$$

It remains to show that $\mathbf{Z} \neq 0$. Suppose it is. Then since the right hand side of the second inequality is strictly less than one, we must have that $\mathbf{Y} = 0$. This contradicts the assumption that both are not zero. Hence we must have $\mathbf{Z} \in \wedge^d\mathbb{Z}^n \setminus \{0\}$. This completes the proof. \square

We now proceed by describing a dynamical reformulation of the intermediate Diophantine sets associated with Definition 4.1. This will enable us to provide an alternative proof of Proposition 1.1 that is self-contained in that it avoids appealing to [5, Lemma

3]. Moreover, the dynamical reformulation will enable us to extend the Davenport & Schmidt result [12, Theorem 2] concerning the equivalence of the simultaneous and dual Dirichlet improvable sets to intermediate Dirichlet improvable sets.

For simplicity, given $n \geq 2$ and $0 \leq d \leq n - 1$, we write

$$A_d := \binom{n}{d}, \quad B_d := \binom{n}{d+1} \quad \text{and} \quad N_d := A_d + B_d = \binom{n+1}{d+1}.$$

For $\mathbf{x} \in \mathbb{R}^n$, let $H_{d,\mathbf{x}}$ to be the linear transformation defined as

$$H_{d,\mathbf{x}} : \wedge^d \mathbb{R}^n \rightarrow \wedge^{d+1} \mathbb{R}^n : \mathbf{Z} \mapsto \mathbf{x} \wedge \mathbf{Z}.$$

Under the standard basis, $H_{d,\mathbf{x}}$ is given as a matrix $M_{d,\mathbf{x}} \in M_{A_d \times B_d}(\mathbb{R})$. Recall, a matrix $M \in M_{A \times B}(\mathbb{R})$ of A rows and B columns, is said to be

- *Dirichlet improvable* if and only if there exists $\varepsilon = \varepsilon(M) \in (0, 1)$ such that for all $N \geq N_0(M, \varepsilon)$ sufficiently large, there exists $\mathbf{Y} \in \mathbb{Z}^A$ and $\mathbf{Z} \in \mathbb{Z}^B \setminus \{0\}$ such that

$$\|\mathbf{Z}\| \leq N \quad \text{and} \quad \|M\mathbf{Z} + \mathbf{Y}\| \leq \varepsilon N^{-\frac{B}{A}}. \tag{4.18}$$

- *singular* if and only if for any given $\varepsilon \in (0, 1)$ and $N \geq N_0(M, \varepsilon)$ sufficiently large, there exists $\mathbf{Y} \in \mathbb{Z}^A$ and $\mathbf{Z} \in \mathbb{Z}^B \setminus \{0\}$ such that (4.18) holds.
- *badly approximable* if and only if there exists $\varepsilon = \varepsilon(M) \in (0, 1)$ such that for all $N \geq N_0(M, \varepsilon)$ sufficiently large, there are no solutions $\mathbf{Y} \in \mathbb{Z}^A$ and $\mathbf{Z} \in \mathbb{Z}^B \setminus \{0\}$ to (4.18).

On comparing the notions as given in Definition 4.1 with the corresponding matrix notions above we immediately obtain the following statement.

Lemma 4.5. *Let $\mathbf{x} \in \mathbb{R}^n$, $0 \leq d \leq n - 1$ be an integer, $A = A_d$ and $B = B_d$. Then \mathbf{x} is d -Dirichlet improvable (resp. d -badly approximable, d -singular) if and only if the matrix $M = M_{d,\mathbf{x}}$ is Dirichlet improvable (resp. badly approximable, singular).*

Using this lemma, we can provide another proof of Proposition 1.1.

Alternative proof of Proposition 1.1. Let

$$\rho_d : \text{SL}_{n+1}(\mathbb{R}) \rightarrow \text{SL}_{N_d}(\mathbb{R}), \tag{4.19}$$

be the homomorphism induced by the natural representation on $\wedge^{d+1} \mathbb{R}^{n+1}$. It is easily seen that

$$\rho_d(\text{SL}_{n+1}(\mathbb{Z})) \subset \text{SL}_{N_d}(\mathbb{Z}) \quad \text{and} \quad \rho_d(g_t) = g_t^d,$$

where

$$g_t := \begin{pmatrix} e^{-t}I_n & 0 \\ 0 & e^{nt} \end{pmatrix} \quad \text{and} \quad g_t^d := \begin{pmatrix} e^{-B_d t}I_{A_d} & 0 \\ 0 & e^{A_d t}I_{B_d} \end{pmatrix}.$$

According to [21, Theorem 1.13], the induced map

$$\phi_d : X_{n+1} \rightarrow X_{N_d} \quad \text{where} \quad X_m := \text{SL}_m(\mathbb{R})/\text{SL}_m(\mathbb{Z}) \tag{4.20}$$

is proper. This together with Dani’s correspondence [9] and Lemma 4.5, implies that

$$\begin{aligned} \mathbf{x} \text{ is } d\text{-badly approximable} &\iff \text{the orbit } \{g_t^d \Lambda_{M_d, \mathbf{x}} : t \geq 0\} \text{ is bounded} \\ &\iff \text{the orbit } \{g_t \Lambda_{\mathbf{x}} : t \geq 0\} \text{ is bounded} \\ &\iff \mathbf{x} \text{ is badly approximable.} \end{aligned}$$

Here and elsewhere,

$$\Lambda_{\mathbf{x}} := \begin{pmatrix} I_n & 0 \\ \mathbf{x} & 1 \end{pmatrix} \mathbb{Z}^{n+1} \quad \text{and} \quad \Lambda_{M_d, \mathbf{x}} := \begin{pmatrix} I_{A_d} & 0 \\ M_{d, \mathbf{x}} & I_{B_d} \end{pmatrix} \mathbb{Z}^{N_d}.$$

The proof of equivalence in the singular case is similar and we leave the details to the reader. \square

Concerning the set of intermediate Dirichlet improvable vectors, we are able to prove the following statement. It shows that the sets \mathbf{DI}_n^d are all the same. This is of course in perfect harmony with the situation concerning the intermediate badly approximable and singular sets.

Proposition 4.2. *Let $n \in \mathbb{N}$ and d be an integer satisfying $0 \leq d \leq n - 1$. Then*

$$\mathbf{DI}_n^d = \mathbf{DI}_n.$$

Recall, that \mathbf{DI}_n is the set of Dirichlet improvable points in \mathbb{R}^n defined via the classical simultaneous or (equivalently) dual form of Dirichlet’s theorem. Note that the proposition trivially implies the Davenport & Schmidt result [12, Theorem 2] conveniently summed up by (4.14). The proof of Proposition 4.2 is based on a dynamical reformulation of \mathbf{DI}_n^d , which in turn relies on the following theorem of Hajós [15].

Theorem 4.2 (Hajós). *Let $k \in \mathbb{N}$ and $L \subset \mathbb{R}^k$ be a lattice of covolume 1. Then*

$$L \cap \Pi_k = \emptyset \iff L \in \bigcup_{w \in \text{Sym}_k} w^{-1}U_k w \mathbb{Z}^k,$$

where $\Pi_k := (-1, 1)^k$ denotes the open unit cube in \mathbb{R}^k , Sym_k represents the Weyl group and $U_k \subset \text{SL}_k(\mathbb{R})$ denotes the subgroup of upper triangular matrices with all diagonal entries equal to 1.

By Dani’s correspondence, $\mathbf{x} \in \mathbf{DI}_n$ if and only if the ω -limit set

$$\left\{ \Lambda \in X_{n+1} : \text{there exists } (t_k)_{k \geq 0} \text{ with } \lim_{k \rightarrow \infty} t_k = \infty \text{ and } g_{t_k} \Lambda_{\mathbf{x}} = \Lambda \right\}$$

does not intersect the set

$$E := \{ \Lambda \in X_{n+1} : \Lambda \cap \Pi_{n+1} = \emptyset \}.$$

It follows from Theorem 4.2, that

$$E = \bigcup_{w \in \text{Sym}_{n+1}} w^{-1} U w \mathbb{Z}^{n+1}.$$

Let ϕ_d be the map given by (4.20), then, in view of Lemma 4.5, \mathbf{x} is d -Dirichlet improvable if and only if the ω -limit set

$$\left\{ \Lambda \in X_{N_d} : \text{there exists } (t_k)_{k \geq 0} \text{ with } \lim_{k \rightarrow \infty} t_k = \infty \text{ and } \phi_d(g_{t_k} \Lambda_{\mathbf{x}}) = \Lambda \right\}$$

does not intersect the set

$$E_d := \bigcup_{w \in \text{Sym}_{N_d}} w^{-1} U_{N_d} w \mathbb{Z}^{N_d}.$$

Thus, \mathbf{x} is d -Dirichlet improvable if and only if the ω -limit set

$$\left\{ \Lambda \in X_{n+1} : \text{there exists } (t_k)_{k \geq 0} \text{ with } \lim_{k \rightarrow \infty} t_k = \infty \text{ and } g_{t_k} \Lambda_{\mathbf{x}} = \Lambda \right\}$$

does not intersect the set

$$E'_d := X_{n+1} \cap \phi_d^{-1}(E_d).$$

The upshot of the above is that Proposition 4.2 is a direct consequence of the following statement.

Lemma 4.6. *Let $n \in \mathbb{N}$ and d be an integer satisfying $0 \leq d \leq n - 1$. Then, $E = E'_d$.*

We now proceed with the proof of Lemma 4.6. This will be done in two steps.

- *Step 1:* $E \subseteq E'_d$. This inclusion is straightforward. Indeed, for any $w \in \text{Sym}_{n+1}$, we have that

$$\rho_d(w^{-1}U_{n+1}w) \subseteq \psi_d(w)^{-1}U_{N_d}\psi_d(w),$$

where ρ_d is given by (4.19) and ψ_d is the natural map from Sym_{n+1} to Sym_{N_d} . Hence, it follows that $E \subseteq E'_d$.

- *Step 2:* $E \supseteq E'_d$. To prove this inclusion, it suffices to show that for any $w \in \text{Sym}_{N_d}$, there exists $w_1 \in \text{Sym}_{n+1}$ such that

$$X_{n+1} \cap \phi_d^{-1}(w^{-1}U_{N_d}w\mathbb{Z}^{N_d}) \subseteq w_1^{-1}U_{w_1}\mathbb{Z}^{n+1}. \tag{4.21}$$

Without loss of generality, let us fix $w \in \text{Sym}_{N_d}$ and write

$$H_1 := \phi_d(\text{SL}_{n+1}(\mathbb{R})) \quad \text{and} \quad H_2 := w^{-1}U_{N_d}w. \tag{4.22}$$

The proof of the inclusion (4.21) will make use of the following technical lemma.

Lemma 4.7. *Let $Y = H_1H_2 \subset \text{SL}_{N_d}(\mathbb{R})$. Then Y is a variety defined over \mathbb{Z} and*

$$Y(\mathbb{Z}) = H_1(\mathbb{Z})H_2(\mathbb{Z}).$$

For the moment, let us assume the truth of the lemma and continue with the proof of Step 2. With this in mind, let $\Gamma = \text{SL}_{N_d}(\mathbb{Z})$ and note that

$$H_1\Gamma \cap H_2\Gamma = \bigcup_{\gamma \in \Gamma} (H_1\gamma \cap H_2)\Gamma.$$

Now if $H_1\gamma \cap H_2 \neq \emptyset$, then $\gamma \in Y(\mathbb{Z})$ and in view of Lemma 4.7, it follows that $\gamma \in H_1(\mathbb{Z})H_2(\mathbb{Z})$. Thus,

$$(H_1\gamma \cap H_2)\Gamma = (H_1 \cap H_2)\Gamma,$$

and so

$$H_1\Gamma \cap H_2\Gamma = (H_1 \cap H_2)\Gamma. \tag{4.23}$$

Next, let T be the diagonal subgroup of $\text{SL}_{n+1}(\mathbb{R})$ with positive entries. Then $D := \phi_d(T)$ is also a connected diagonal subgroup of $\text{SL}_{N_d}(\mathbb{R})$. Note that both H_1 and H_2 are invariant under the conjugation of D and so it follows that their intersection $H_1 \cap H_2$ is also invariant under the conjugation of D . Note that the group $H_1 \cap H_2$ is clearly unipotent, thus connected. Hence $D(H_1 \cap H_2)$ is a connected solvable subgroup of H_1 . Then there exists a Borel subgroup B such that $D(H_1 \cap H_2) \subset B$. Thus, there exists $w_1 \in \text{Sym}_{n+1}$ such that

$$\phi_d^{-1}(H_1 \cap H_2) \subset w_1^{-1}Uw_1. \tag{4.24}$$

Now (4.21) follows from (4.23) and (4.24). This completes the proof of Step 2 and thereby the proof of Lemma 4.6 modulo Lemma 4.7. \square

Remark 4.2. With n and d as in Lemma 4.6, it is relatively straightforward to prove the weaker statement that

$$\mathbf{DI}_n^d = \mathbf{DI}_n^{n-1-d} \tag{4.25}$$

without appealing to Lemma 4.7. Note that when $d = 0$ or $n - 1$, this implies the Davenport & Schmidt result summed up by (4.14). To prove (4.2), let $\tau_d : X_{N_d} \rightarrow X_{N_{n-d-1}}$ be the isomorphism induced by the Hodge star operator, which is a natural isomorphism between $\wedge^{d+1}(\mathbb{R}^{n+1})$ and $\wedge^{n-d}(\mathbb{R}^{n+1})$, see §4.2.1 for its definition. It is easily verified that $\tau_d(E_d) = E_{n-d-1}$ and $\tau_d \circ \phi_d = \phi_{n-d-1}$. Hence, it follows that $E'_d = E'_{n-d-1}$ which in turn this implies the desired result.

The proof of Lemma 4.7 will make use of tools from non-abelian cohomology. For a general reference on non-abelian cohomology, we refer the reader to Giraud’s book [14]. On considering the action of $H := H_1 \times H_2$ on Y defined by $(h_1, h_2) \times y \mapsto \psi_d(h_1)yh_2^{-1}$ we can identify Y with the quotient space $H/(H_1 \cap H_2)$. It follows that Lemma 4.7 is equivalent to saying that $Y(\mathbb{Z})$ consists of a single orbit under the action of $H(\mathbb{Z})$. The space of orbits $Y(\mathbb{Z})/H(\mathbb{Z})$ is controlled by the non-abelian cohomology of \mathbb{Z} with coefficients in $H_1 \cap H_2$. The non-abelian cohomology that we are going to use is the fppf-cohomology and we will always work over $\mathbf{Spec}(\mathbb{Z})$ – the spectrum of the commutative ring \mathbb{Z} of integers. For any group scheme \mathbf{H} defined over $\mathbf{Spec}(\mathbb{Z})$, we let

$$H^1(\mathbb{Z}, \mathbf{H}) := H_{\text{fppf}}^1(\mathbf{Spec}(\mathbb{Z}), \mathbf{H})$$

denote the fppf-cohomology of $\mathbf{Spec}(\mathbb{Z})$ with coefficients in \mathbf{H} . With this in mind, are now in the position to prove the lemma.

Proof of Lemma 4.7. For the sake of simplicity, we let $\mathbf{G} := \text{SL}_{N_d}$ and $\mathbf{H}_1 := \text{SL}_{n+1}$. Let $\mathbf{T} \subset \mathbf{H}_1$ be the subgroup scheme of diagonal matrices. Let \mathbf{H}_2 be the group scheme over \mathbb{Z} given by H_2 endowed with their \mathbb{Z} -structure, where H_2 is given by (4.22). Let \mathbf{Y} be the scheme defined over \mathbb{Z} given by Y endowed with its \mathbb{Z} -structure, where Y is given in Lemma 4.7. It is clear that, all these schemes are flat (indeed smooth) over \mathbb{Z} .

There is an action of $\mathbf{H} := \mathbf{H}_1 \times \mathbf{H}_2$ on \mathbf{Y} such that for any ring R , we have

$$\mathbf{H}(R) \times \mathbf{Y}(R) \rightarrow \mathbf{Y}(R), \quad (h_1, h_2) \times y \mapsto \phi_d(h_1)yh_2^{-1}.$$

Let $\mathbf{H}_0 = \mathbf{H}_1 \times_{\mathbf{G}} \mathbf{H}_2$; that is, for any ring R

$$\mathbf{H}_0(R) = \mathbf{H}_1(R) \times_{\mathbf{G}(R)} \mathbf{H}_2(R).$$

It follows that \mathbf{H}_0 is a unipotent subgroup scheme of \mathbf{H}_1 smooth over \mathbb{Z} and is normalized by \mathbf{T} . Consequently, \mathbf{H}_0 is split over \mathbb{Z} . In particular, there is a sequence of smooth unipotent \mathbb{Z} -group schemes

$$\{1\} = \mathbf{H}_0^{\dim \mathbf{H}_0} \subset \mathbf{H}_0^{\dim \mathbf{H}_0 - 1} \subset \dots \subset \mathbf{H}_0^2 \subset \mathbf{H}_0^1 = \mathbf{H}_0$$

such that \mathbf{H}_0^{i+1} is normal in \mathbf{H}_0^i and $\mathbf{H}_0^i/\mathbf{H}_0^{i+1} \cong \mathbf{G}_a$. Now observe that $H^1(\mathbb{Z}, \mathbf{G}_a) = \{1\}$. Then on arguing by induction on the dimension and making use of [14, Proposition 3.3.1], it follows that

$$H^1(\mathbb{Z}, \mathbf{H}_0) = \{1\}. \tag{4.26}$$

Furthermore, it is easily checked that \mathbf{Y} is isomorphism to the quotient \mathbf{H}/\mathbf{H}_0 . In view of [14, Corollary 3.2.3], there is an injection

$$\mathbf{Y}(\mathbb{Z})/\mathbf{H}(\mathbb{Z}) \hookrightarrow H^1(\mathbb{Z}, \mathbf{H}_0),$$

where $\mathbf{Y}(\mathbb{Z})/\mathbf{H}(\mathbb{Z})$ denotes the orbit space of $\mathbf{Y}(\mathbb{Z})$ under the action of $\mathbf{H}(\mathbb{Z})$. It follows from (4.26) that $\mathbf{Y}(\mathbb{Z})$ consists of a single orbit under the action of $\mathbf{H}(\mathbb{Z})$. This completes the proof of Lemma 4.7. \square

4.2. The geometric approach

In this section we explore a geometric approach to defining d -Dirichlet improvable sets. The approach taken is in line with that of §2.2 in which the d -badly approximable and d -singular sets are expressed via successive minima; namely via (2.11) and (2.12).

4.2.1. An optimal Dirichlet type theorem via successive minima

Fix $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^n$. Then, as in §2.1, for each $i = 1, \dots, n + 1$ and $t > 0$, let $\lambda_{\mathbf{x},i}(t) := \lambda_i(\mathbb{Z}^{n+1}, \mathcal{C}_{\mathbf{x}}(e^t))$ be the i -th successive minima of the convex body $\mathcal{C}_{\mathbf{x}}(e^t)$ given by (2.1) with respect to the lattice \mathbb{Z}^{n+1} . In turn, let

$$L_{\mathbf{x},i}(t) := \log \lambda_{\mathbf{x},i}(t) \tag{4.27}$$

be the natural logarithm of $\lambda_{\mathbf{x},i}(t)$. It follows, via Minkowski’s second convex body theorem, that

$$\frac{e^t}{(n + 1)!} \leq \prod_{i=1}^{n+1} \lambda_{\mathbf{x},i}(t) \leq e^t \tag{4.28}$$

or equivalently

$$t - \log(n + 1)! \leq \sum_{i=1}^{n+1} L_{\mathbf{x},i}(t) \leq t.$$

Now, (4.28) together with the fact that $\lambda_{\mathbf{x},1}(t) \leq \lambda_{\mathbf{x},2}(t) \leq \dots \leq \lambda_{\mathbf{x},n+1}(t)$ gives rise to the following statement.

Theorem 4.3. *Let $n \in \mathbb{N}$ and d be an integer satisfying $0 \leq d \leq n - 1$. Then for any $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$*

$$\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}(t) \leq e^{\frac{(n-d)t}{n+1}}, \tag{4.29}$$

or equivalently

$$\sum_{i=1}^{n-d} L_{\mathbf{x},i}(t) \leq \frac{(n-d)t}{n+1}.$$

Theorem 4.3 can be viewed as an optimal Dirichlet type theorem for approximating points $\mathbf{x} \in \mathbb{R}^n$ by d -dimensional rational subspaces. In the dual case ($d = n - 1$) when we approximate by hyperplanes, this realisation is relatively straightforward to see. For arbitrary $d \in \{0, \dots, n - 1\}$, the following statement provides the link. Recall that, by the definition of successive minima, given \mathbf{x} and $t > 0$, there exist linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in \mathbb{Z}^{n+1}$, depending on \mathbf{x} and t , such that

$$\mathbf{v}_i \in \lambda_{\mathbf{x},i}(t) \mathcal{C}_{\mathbf{x}}(e^t) \quad (1 \leq i \leq n + 1). \tag{4.30}$$

As we shall soon see, the vectors play a key role in detecting the sought after d -dimensional rational subspaces. As usual, given $\mathbf{x} \in \mathbb{R}^n$, we let $\mathbf{x}' := (\mathbf{x}, 1) \in \mathbb{R}^{n+1}$ and we also let

$$\|\mathbf{x}\|_1 := |x_1| + \dots + |x_n|.$$

Lemma 4.8. *Let $n \in \mathbb{N}$ and d be an integer satisfying $0 \leq d \leq n - 1$. Then for any $\mathbf{x} \in \mathbb{R}^n$ and $t > 0$, let $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in \mathbb{Z}^{n+1}$ be as in (4.30). In turn, let $\mathbf{u}_1, \dots, \mathbf{u}_{d+1}$ be any basis of $V(\mathbf{v}_1, \dots, \mathbf{v}_{n-d})^\perp \cap \mathbb{Z}^{n+1}$, where $V(\mathbf{v}_1, \dots, \mathbf{v}_{n-d})^\perp$ is the linear subspace of \mathbb{R}^{n+1} orthogonal to the vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n-d}$. Finally, let $\mathbf{X} := \mathbf{u}_1 \wedge \dots \wedge \mathbf{u}_{d+1} \in \bigwedge^{d+1} \mathbb{Z}^{n+1}$. Then*

$$\|\mathbf{x}' \wedge \mathbf{X}\| \leq (n - d)!(1 + \|\mathbf{x}\|_1) \left(\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}(t) \right) e^{-t} \tag{4.31}$$

and

$$0 < \|\mathbf{X}\| \leq (n - d)!(1 + \|\mathbf{x}\|_1) \left(\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}(t) \right). \tag{4.32}$$

To prove the lemma, it will be useful to make use of the notion of the Hodge dual – the image of a multivector under the Hodge star operator. In relation to multivectors over \mathbb{R}^{n+1} , the Hodge star operator relates any basis $\mathbf{a}_1, \dots, \mathbf{a}_\ell$ of a vector subspace of \mathbb{R}^{n+1} to a basis $\mathbf{b}_1, \dots, \mathbf{b}_{n+1-\ell}$ of the orthogonal subspace. Naturally, we will use the symbol $^\perp$ to denote the Hodge dual within the context of multivectors over \mathbb{R}^{n+1} . In particular,

$$\mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_\ell = \pm (\mathbf{b}_1 \wedge \dots \wedge \mathbf{b}_{n+1-\ell})^\perp$$

provided the basis vectors under consideration span parallelepipeds of the same volume. We refer the reader to [4, §3] for the definition of the Hodge star operator and its useful properties that we shall exploit. In particular, we have that for any indices $1 \leq i_1 < \dots < i_\ell \leq n + 1$

$$(\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_\ell})^\perp = \pm \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_{n+1-\ell}}, \tag{4.33}$$

where $\{j_1, \dots, j_{n+1-\ell}\} = \{1, \dots, n + 1\} \setminus \{i_1, \dots, i_\ell\}$, and so the Hodge star operator preserves both Euclidean and maximum norms (in the standard basis).

Proof. To start with, simply note that $\mathbf{X} = \pm(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{n-d})^\perp \in \bigwedge^{d+1} \mathbb{Z}^{n+1}$ and so \mathbf{X} is decomposable and non-zero. Without loss of generality, we assume that the sign is positive. By the properties of the Hodge star operator, see [4, Eq (3.8)&(3.9)], we have that

$$(\mathbf{X} \wedge \mathbf{x}')^\perp = ((\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{n-d})^\perp \wedge \mathbf{x}')^\perp = (-1)^{(d+1)(n-d)} (\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{n-d}) \cdot \mathbf{x}', \tag{4.34}$$

where $(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{n-d}) \cdot \mathbf{x}' \in \bigwedge^{n-d-1} \mathbb{R}^{n+1}$ is the interior product of multivectors under consideration. By [4, Eq (3.6)], the $\mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_{n-d-1}}$ coordinate of $(\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{n-d}) \cdot \mathbf{x}'$ is equal to

$$((\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{n-d}) \cdot \mathbf{x}') \cdot (\mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_{n-d-1}}) = (\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{n-d}) \cdot (\mathbf{x}' \wedge \mathbf{e}_{j_1} \wedge \dots \wedge \mathbf{e}_{j_{n-d-1}}).$$

By Laplace’s identity, see [4, Eq (3.3)], this coordinate is equal to

$$\det \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{x}' & \mathbf{v}_1 \cdot \mathbf{e}_{j_1} & \dots & \mathbf{v}_1 \cdot \mathbf{e}_{j_{n-d-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_{n-d} \cdot \mathbf{x}' & \mathbf{v}_{n-d} \cdot \mathbf{e}_{j_1} & \dots & \mathbf{v}_{n-d} \cdot \mathbf{e}_{j_{n-d-1}} \end{pmatrix}. \tag{4.35}$$

By (4.30), we have that $|\mathbf{v}_i \cdot \mathbf{x}'| \leq \lambda_{\mathbf{x},i}(t)e^{-t}$ and $|\mathbf{v}_i \cdot \mathbf{e}_j| \leq \lambda_{\mathbf{x},i}(t)$ for $1 \leq j \leq n$. In turn, since $\mathbf{e}_{n+1} = \mathbf{x}' - \sum_{j=1}^n x_j \mathbf{e}_j$, we have that

$$|\mathbf{v}_i \cdot \mathbf{e}_{n+1}| = \left| \mathbf{v}_i \cdot \mathbf{x}' - \sum_{j=1}^n x_j \mathbf{v}_i \cdot \mathbf{e}_j \right| \leq \lambda_{\mathbf{x},i}(t)e^{-t} + \lambda_{\mathbf{x},i}(t)\|\mathbf{x}\|_1 \leq \lambda_{\mathbf{x},i}(t)(1 + \|\mathbf{x}\|_1).$$

Hence, by (4.35), we get that

$$\|(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-d}) \cdot \mathbf{x}'\| \leq (n-d)!(1 + \|\mathbf{x}\|_1) \left(\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}(t) \right) e^{-t}.$$

By (4.33), up to sign and order, the Hodge dual of a multivector has the same coordinates in the standard basis. Therefore, by (4.34) it follows that

$$\|\mathbf{x}' \wedge \mathbf{X}\| = \|(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-d}) \cdot \mathbf{x}'\| \leq (n-d)!(1 + \|\mathbf{x}\|_1) \left(\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}(t) \right) e^{-t}.$$

This establishes (4.31). Next, calculating the $\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{n-d}}$ coordinate of $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-d}$ gives

$$(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{n-d}) \cdot (\mathbf{e}_{j_1} \wedge \cdots \wedge \mathbf{e}_{j_{n-d}}) = \det \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{e}_{j_1} & \cdots & \mathbf{v}_1 \cdot \mathbf{e}_{j_{n-d}} \\ \vdots & \ddots & \vdots \\ \mathbf{v}_{n-d} \cdot \mathbf{e}_{j_1} & \cdots & \mathbf{v}_{n-d} \cdot \mathbf{e}_{j_{n-d}} \end{pmatrix}. \tag{4.36}$$

By the already established bounds on $|\mathbf{v}_i \cdot \mathbf{e}_j|$, the absolute value of (4.36) is bounded by

$$(n-d)!(1 + \|\mathbf{x}\|_1) \left(\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}(t) \right).$$

This implies (4.32) and thereby completes the proof of the lemma. \square

Given $\mathbf{x} \in \mathbb{R}^n$, $t > 0$ and $i \in \{1, \dots, n+1\}$, it will be convenient to define the following ‘normalised’ value of the i -th successive minima:

$$\lambda_{\mathbf{x},i}^*(t) := e^{-\frac{t}{n+1}} \lambda_{\mathbf{x},i}(t). \tag{4.37}$$

Then, (4.29) becomes

$$\prod_{i=1}^{n-d} \lambda_i^*(t, \mathbf{x}) \leq 1$$

and (4.31) and (4.32) respectively can be rewritten as

$$\|\mathbf{x}' \wedge \mathbf{X}\| \leq (n-d)!(1 + \|\mathbf{x}\|_1) \underbrace{\left(\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}^*(t) \right)}_{\leq 1} e^{-\frac{(d+1)t}{n+1}}, \tag{4.38}$$

$$\|\mathbf{X}\| \leq (n-d)!(1 + \|\mathbf{x}\|_1) \underbrace{\left(\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}^*(t) \right)}_{\leq 1} e^{\frac{(n-d)t}{n+1}}. \tag{4.39}$$

Now, given $N \in \mathbb{N}$ there exists $t \in \mathbb{R}$ such that

$$(n-d)!(1 + \|\mathbf{x}\|_1) e^{\frac{(n-d)t}{n+1}} = N. \tag{4.40}$$

Clearly, if N is sufficiently large we can guarantee that $t > 0$ and the following statement is a direct consequence of Lemma 4.8.

Proposition 4.3. *Let $n \in \mathbb{N}$ and d be an integer satisfying $0 \leq d \leq n - 1$. Then for any $\mathbf{x} \in \mathbb{R}^n$ and any*

$$N > (n-d)!(1 + \|\mathbf{x}\|_1),$$

the decomposable vector $\mathbf{X} \in \wedge^{d+1} \mathbb{Z}^{n+1} \setminus \{0\}$ associated with Lemma 4.8 with t given by (4.40), satisfies

$$\|\mathbf{x}' \wedge \mathbf{X}\| \leq c(n, d, \mathbf{x}) N^{-\omega_d} \quad \text{and} \quad \|\mathbf{X}\| \leq N,$$

where

$$c(n, d, \mathbf{x}) := \left((n-d)!(1 + \|\mathbf{x}\|_1) \right)^{\frac{n+1}{n-d}}.$$

Clearly, the proposition is a version of Lemma 4.3 obtained via the algebraic approach of §4.1. We have already seen that the latter leads to Theorem 1.3 – a Dirichlet type statement concerning the approximation of points $\mathbf{x} \in \mathbb{R}^n$ by d -dimensional rational subspaces. The upshot of this is that Theorem 4.3, which implies Proposition 4.3 can thus be viewed as an optimal Dirichlet type theorem for approximating points by rational subspaces.

4.2.2. *Bad, singular and Dirichlet improvable in light of Theorem 4.3*

For any given $\mathbf{x} \in \mathbb{R}^n$, observe that during the course of establishing Proposition 4.3 we only used the trivial fact that $\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}^*(t) \leq 1$. Clearly if the product under consideration gets significantly smaller than 1, then we will obtain a stronger version of Proposition 4.3. Specifically, if for some $\varepsilon > 0$

$$\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}^*(t) \leq \varepsilon \tag{4.41}$$

for all sufficiently large t , then for any sufficiently large N there exists a decomposable $\mathbf{X} \in \wedge^{d+1} \mathbb{Z}^{n+1}$ such that

$$\|\mathbf{x}' \wedge \mathbf{X}\| \leq c(n, d, \mathbf{x}) \varepsilon^{\frac{n+1}{n-d}} N^{-\omega_d} \quad \text{and} \quad 0 < \|\mathbf{X}\| \leq N.$$

This is obtained in exactly the same way as Proposition 4.3 is obtained from Theorem 4.3 but now we are able to insert the factor ε into the left hand side of (4.40). Now observe that in view of (4.27) and (4.37), inequality (4.41) is equivalent to the statement that

$$\frac{(n-d)t}{n+1} - \sum_{i=1}^{n-d} L_{\mathbf{x},i}(t) \geq \log(1/\varepsilon) =: \delta$$

which is precisely inequality (2.12). We have already seen that the validity of (2.12) for arbitrarily large δ for all sufficiently large t is equivalent to \mathbf{x} being d -singular (see Lemma 4.2). As a consequence, we have that:

- $\mathbf{x} \in \mathbf{Sing}_n^d$ if and only if for any $\varepsilon > 0$ there exists a constant $t_0 = t_0(\varepsilon) > 0$ such that (4.41) holds for all $t \geq t_0$.

Now suppose that for some $\varepsilon > 0$

$$\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}^*(t) \geq \varepsilon \tag{4.42}$$

for all sufficiently large t . This is easily seen to be equivalent to the statement that

$$\frac{(n-d)t}{n+1} - \sum_{i=1}^{n-d} L_{\mathbf{x},i}(t) \leq \log(1/\varepsilon) =: \delta$$

which is precisely inequality (2.11). This together with Lemma 4.2 implies that

- $\mathbf{x} \in \mathbf{Bad}_n^d$ if and only if there exists $\varepsilon > 0$ such that (4.42) holds for all $t > t_0(\mathbf{x}, \varepsilon)$ sufficiently large.

Remark 4.3. Note that the above discussion together with Proposition 1.1 and Lemma 4.4, implies that the d -singular sets (respectively the d -badly approximable sets) defined via the algebraic approach of §4.1 or the geometric approach of this section coincide with the set \mathbf{Sing}_n (respectively \mathbf{Bad}_n); that is, the set of singular (respectively, badly approximable) points in \mathbb{R}^n defined via either the classical simultaneous or dual form of Dirichlet’s Theorem.

The above observations naturally lead us to the following notion of d -Dirichlet improvable points in \mathbb{R}^n .

Definition 4.2. Let $n \in \mathbb{N}$ and d be an integer satisfying $0 \leq d \leq n - 1$. Then for $0 < \varepsilon < 1$, let

- $\widehat{\mathbf{DI}}_n^d(\varepsilon)$ to be the set of $\mathbf{x} \in \mathbb{R}^n$, such that inequality (4.41), or equivalently

$$\frac{(n-d)t}{n+1} - \sum_{i=1}^{n-d} L_{\mathbf{x},i}(t) \geq \log(1/\varepsilon), \tag{4.43}$$

holds for all $t > t_0(\mathbf{x}, \varepsilon)$ sufficiently large.

Moreover, let

$$\widehat{\mathbf{DI}}_n^d := \bigcup_{0 < \varepsilon < 1} \widehat{\mathbf{DI}}_n^d(\varepsilon).$$

Note that by definition, $\mathbf{x} \in \widehat{\mathbf{DI}}_n^d$ if and only if there exists $\varepsilon \in (0, 1)$ such that (4.41), or equivalently (4.43), holds for all sufficiently large t . The following result shows that these sets are in fact all equivalent and coincide with the set \mathbf{DI}_n ; that is, the set of Dirichlet improvable points in \mathbb{R}^n defined via either the classical simultaneous or dual form of Dirichlet’s Theorem (both forms give rise to the same set). In turn, this together with Proposition 4.2 implies that the set of d -Dirichlet improvable points defined via the algebraic approach (cf. Definition 4.1) and geometric approach also coincide.

Lemma 4.9.

- (i) For any $\varepsilon \in (0, 1)$, we have that

$$\widehat{\mathbf{DI}}_n^{n-1}(\varepsilon) = \mathbf{DI}_n(\varepsilon^{n+1}). \tag{4.44}$$

Therefore, $\widehat{\mathbf{DI}}_n^{n-1} = \mathbf{DI}_n$.

- (ii) For any $\varepsilon \in (0, 1)$ we have that

$$\widehat{\mathbf{DI}}_n^d(\varepsilon) \subset \widehat{\mathbf{DI}}_n^{d+1}\left(\varepsilon^{\frac{n-d-1}{n-d}}\right) \quad \text{if } 0 \leq d \leq n-2 \tag{4.45}$$

and

$$\widehat{\mathbf{DI}}_n^d(\varepsilon) \subset \widehat{\mathbf{DI}}_n^{d-1}\left(\varepsilon^{\frac{d}{d+1}}\right) \quad \text{if } 1 \leq d \leq n-1. \tag{4.46}$$

Therefore, the sets $\widehat{\mathbf{DI}}_n^d$ are the same for all $d \in \{0, 1, \dots, n-1\}$.

Proof. Part (i) follows immediately on comparing (4.43) in Definition 4.2 and (2.15) in Lemma 2.1. Regarding Part (ii), first suppose that $\varepsilon \in (0, 1)$, $0 \leq d \leq n-2$ and $\mathbf{x} \in \widehat{\mathbf{DI}}_n^d(\varepsilon)$. Then, by definition, for all sufficiently large t we have (4.41). Now on raising (4.41) to the power $n-d-1$ and using the inequalities $\lambda_{\mathbf{x},1}^*(t) \leq \dots \leq \lambda_{\mathbf{x},n+1}^*(t)$ we obtain that

$$\varepsilon^{n-d-1} \geq (\lambda_{\mathbf{x},1}^*(t) \cdots \lambda_{\mathbf{x},n-d}^*(t))^{n-d-1} \geq (\lambda_{\mathbf{x},1}^*(t) \cdots \lambda_{\mathbf{x},n-d-1}^*(t))^{n-d}.$$

Hence

$$\lambda_{\mathbf{x},1}^*(t) \cdots \lambda_{\mathbf{x},n-d-1}^*(t) \leq \varepsilon^{\frac{n-d-1}{n-d}}$$

and we conclude that $\mathbf{x} \in \widehat{\mathbf{DI}}_n^{d+1}(\varepsilon^{\frac{n-d-1}{n-d}})$. This proves the ‘going-up’ inclusion (4.45).

To prove the ‘going-down’ inclusion (4.46), let $\varepsilon \in (0, 1)$ and $1 \leq d \leq n - 1$ and $\mathbf{x} \in \widehat{\mathbf{DI}}_n^d(\varepsilon)$. Then, for all sufficiently large t inequality (4.41) holds and since $\lambda_{\mathbf{x},1}^*(t) \leq \cdots \leq \lambda_{\mathbf{x},n+1}^*(t)$, it follows via Minkowski’s second convex body theorem that

$$\lambda_{\mathbf{x},n-d+1}^*(t)^{d+1} \leq \lambda_{\mathbf{x},n-d+1}^*(t) \cdots \lambda_{\mathbf{x},n+1}^*(t) \leq \frac{1}{\lambda_{\mathbf{x},1}^*(t) \cdots \lambda_{\mathbf{x},n-d}^*(t)}.$$

Hence, on using (4.41) we find that

$$\prod_{i=1}^{n-d+1} \lambda_{\mathbf{x},i}^*(t) \leq \left(\prod_{i=1}^{n-d} \lambda_{\mathbf{x},i}^*(t) \right)^{1-\frac{1}{d+1}} \leq \varepsilon^{\frac{d}{d+1}}.$$

Since this holds for all sufficiently large t , we conclude that $\mathbf{x} \in \widehat{\mathbf{DI}}_n^{d-1}(\varepsilon^{\frac{d}{d+1}})$ and this thereby establishes (4.46). The ‘therefore’ statements in both parts of the lemma follow on taking the union over all $\varepsilon \in (0, 1)$ in (4.44), (4.45) and (4.46). \square

4.3. Final remarks: linear forms and weighted approximation

The geometric approach of §4.2 can be readily adapted to extend the definition of badly, singular and Dirichlet improvable points to systems of linear forms and thereby consider the “matrix” analogue of the main problem considered in this paper. We will now develop/describe this “matrix” theory within the more general setup of *weighted* Diophantine approximation. The parametric geometry of numbers, the underlying tool exploited in proving our main result, for weighted approximation has recently been initiated by Schmidt [27], albeit it remains in its ‘infancy’ to cope with the weighted analogue of the main question considered in this paper.

Let $m, n \in \mathbb{N}$ and fix a pair of weights $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}_{\geq 0}^n$ and $\mathbf{s} = (s_1, \dots, s_m) \in \mathbb{R}_{\geq 0}^m$ satisfying

$$r_1 + \cdots + r_n = 1 \quad \text{and} \quad s_1 + \cdots + s_m = 1.$$

For $t > 0$, let

$$g_t = \text{diag}\{e^{-r_1 t}, \dots, e^{-r_n t}, e^{s_1 t}, \dots, e^{s_m t}\} \quad \text{and} \quad u_X = \begin{pmatrix} I_n & 0 \\ X & I_m \end{pmatrix},$$

where X is an $m \times n$ real matrix; i.e. $X \in M_{m \times n}(\mathbb{R})$. For each $i = 1, \dots, m + n$ and $t > 0$, let

$$\lambda_{X,i}^*(t) := \lambda_i(g_t u_X \mathbb{Z}^{m+n}, B_1)$$

denote the i -th successive minima of the convex body $B_1 = [-1, 1]^{m+n}$ with respect to the lattice $g_t u_X \mathbb{Z}^{m+n}$. The following provides a natural generalisation of the Diophantine sets \mathbf{Bad}_n^d , \mathbf{Sing}_n^d and \mathbf{DI}_n^d to the weighted matrix setup.

Definition 4.3. Let $X \in M_{m \times n}(\mathbb{R})$ and d be an integer satisfying $0 \leq d \leq m + n - 2$. Then for any pair of weights $(\mathbf{r}, \mathbf{s}) \in \mathbb{R}^n \times \mathbb{R}^m$, we say that

$$\begin{aligned} X \in \mathbf{Bad}^d(\mathbf{r}, \mathbf{s}) &\iff \liminf_{t \rightarrow \infty} \prod_{i=1}^{n+m-1-d} \lambda_{X,i}^*(t) > 0; \\ X \in \mathbf{Sing}^d(\mathbf{r}, \mathbf{s}) &\iff \lim_{t \rightarrow \infty} \prod_{i=1}^{n+m-1-d} \lambda_{X,i}^*(t) = 0; \\ X \in \mathbf{DI}^d(\mathbf{r}, \mathbf{s}) &\iff \limsup_{t \rightarrow \infty} \prod_{i=1}^{n+m-1-d} \lambda_{X,i}^*(t) < 1. \end{aligned}$$

It is easily seen that up to the linear change of the variable

$$t \mapsto \frac{n}{n+1}t,$$

the definition of $\lambda_{X,i}^*(t)$ for matrices is consistent with that of $\lambda_{\mathbf{x},i}^*(t)$ for vectors $\mathbf{x} \in \mathbb{R}^n$ given by (4.37). So if we put $m = 1$ and $r_1 = \dots = r_n = 1/n$, we immediately see, in view of the discussion in §4.2.2, that sets $\mathbf{Bad}^d(\mathbf{r}, \mathbf{s})$, $\mathbf{Sing}^d(\mathbf{r}, \mathbf{s})$ and $\mathbf{DI}^d(\mathbf{r}, \mathbf{s})$ coincide with \mathbf{Bad}_n^d , \mathbf{Sing}_n^d and \mathbf{DI}_n^d respectively. Moreover, on using the same techniques as in the proof of Lemma 4.9, it is relatively straightforward to show that for any fixed pair of weights (\mathbf{r}, \mathbf{s}) the badly approximable sets $\mathbf{Bad}^d(\mathbf{r}, \mathbf{s})$ are all the same; the singular sets $\mathbf{Sing}^d(\mathbf{r}, \mathbf{s})$ are all the same, and the Dirichlet improvable sets $\mathbf{DI}^d(\mathbf{r}, \mathbf{s})$ are all the same. It is therefore natural to drop the dependence on ‘ d ’ from the defining notation. With this in mind, the following is the natural ‘matrix’ extension of the main question considered in this paper.

Problem 4.1. Let $m, n \in \mathbb{N}$ such that $mn > 1$. Prove that for any pair of weights $(\mathbf{r}, \mathbf{s}) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$\mathbf{FS}(\mathbf{r}, \mathbf{s}) := \mathbf{DI}(\mathbf{r}, \mathbf{s}) \setminus (\mathbf{Bad}(\mathbf{r}, \mathbf{s}) \cup \mathbf{Sing}(\mathbf{r}, \mathbf{s})) \neq \emptyset.$$

We suspect that a lot more is true and the set under consideration has full dimension; that is to say that $\dim \mathbf{FS}(\mathbf{r}, \mathbf{s}) = nm$. This is in line with the content of Problem 3.1 appearing in §3.4 of the main body.

Remark 4.4. Potentially, the non-weighted case of Problem 4.1 can be resolved using the generalisation of Theorem 2.1 to systems of linear forms due to Das, Fishman, Simmons

and Urbański [10, Theorem 5.2]. However, there is currently a natural barrier in using their generalisation for this purpose. The point is that if we apply [10, Theorem 5.2], then the analogue of the constant appearing in the right hand side of (2.5) becomes dependent on the matrix X under consideration. The upshot of this is that we are not able to simply implement the machinery used to address Problem 1.1 in order to deal with the analogous problem for systems of linear forms.

Acknowledgments

SV would like to thank Dorothy and the Tin Man for eighteen glorious years. It has been a privilege watching you turn into feisty, loving, young people with firm moral compasses. May you have many many wonderful adventures somewhere over the rainbow! SV would also like to take this opportunity to thank Prafula and Chimán for their love and for always being around for their little brother! Finally, many thanks to Bridget who, like me, sees our differences (oh and there are many!) as our great strength.

We would like to thank the anonymous referees for their extremely detailed reports. They have greatly improved the paper both in terms of accuracy and exposition. Thank you!

References

- [1] R.K. Akhunzhanov, N. Moshchevitin, A note on Dirichlet spectrum, Pre-print: arXiv:2111.12259, 2021.
- [2] R.K. Akhunzhanov, D.O. Shatskov, On Dirichlet spectrum for two-dimensional simultaneous Diophantine approximation, *Mosc. J. Comb. Number Theory* 3 (3–4) (2013) 5–23.
- [3] V. Beresnevich, D. Dickinson, S. Velani, Sets of exact logarithmic order in the theory of Diophantine approximation, *Math. Ann.* 321 (2) (2001) 253–273.
- [4] V. Beresnevich, Rational points near manifolds and metric Diophantine approximation, *Ann. Math.* (2) 175 (2012) 187–235.
- [5] Y. Bugeaud, M. Laurent, On transfer inequalities in Diophantine approximation II, *Math. Z.* 265 (2010) 249–262.
- [6] Y. Cheung, Hausdorff dimension of the set of singular pairs, *Ann. Math.* 173 (2011) 127–167.
- [7] Y. Cheung, N. Chevallier, Hausdorff dimension of singular vectors, *Duke Math. J.* 165 (2016) 2273–2329.
- [8] S. Chow, A. Ghosh, L. Guan, A. Marnat, D. Simmons, Diophantine transference inequalities: weighted, inhomogeneous, and intermediate exponents, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* (5) XXI (2020) 643–671, arXiv:1808.07184.
- [9] S.G. Dani, Divergent trajectories of flows on homogeneous spaces and Diophantine approximation, *J. Reine Angew. Math.* 359 (1985) 55–89.
- [10] T. Das, L. Fishman, D. Simmons, M. Urbański, A variational principle in the parametric geometry of numbers, ArXiv preprint arXiv:1901.06602.
- [11] H. Davenport, W.M. Schmidt, Dirichlet’s Theorem on Diophantine Approximation, *Symposia Mathematica*, vol. 4, Istituto Nazionale di Alta Matematica, Academic Press, London, 1970, pp. 113–132, II, 829.
- [12] H. Davenport, W.M. Schmidt, Dirichlet’s theorem on Diophantine approximation. II, *Acta Arith.* 16 (1969/1970) 413–424.
- [13] O.N. German, Intermediate Diophantine exponents and parametric geometry of numbers, *Acta Arith.* 154 (2012) 79–101.
- [14] J. Giraud, Cohomologie non abélienne, *Die Grundlehren der Mathematischen Wissenschaften*, vol. 179, Springer-Verlag, Berlin-New York, 1971, ix+467 pp.

- [15] G. Hajós, Über einfache und mehrfache Bedeckung des n -dimensionalen Raumes mit einem Würfelgitter, *Math. Z.* 47 (1941) 427–467.
- [16] V. Jarník, Zum khintchineschen Übertragungssatz, *Trav. Inst. Math. Tbilissi* 3 (1938) 193–212.
- [17] A.Ya. Khinchine, Über eine Klasse linearer diophantischer Approximationen, *Rend. Circ. Mat. Palermo* 50 (1926) 170–195.
- [18] A.Ya. Khinchine, Zur metrischen Theorie der diophantischen Approximationen, *Math. Z.* 24 (1926) 706–714.
- [19] D. Kleinbock, S. Mirzadeh, On the dimension drop conjecture for diagonal flows on the space of lattices, Pre-print, arXiv:2010.14065, 2020.
- [20] M. Laurent, On transfer inequalities in Diophantine approximation, in: *Analytic Number Theory*, 2009, pp. 306–314.
- [21] M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer, 1972.
- [22] D. Roy, On Schmidt and Summerer parametric geometry of numbers, *Ann. Math.* 182 (2015) 739–786.
- [23] D. Roy, Spectrum of the exponents of best rational approximation, *Math. Z.* 283 (2016) 143–155.
- [24] W.M. Schmidt, On heights of algebraic subspaces and Diophantine approximations, *Ann. Math.* (2) 85 (1967) 430–472.
- [25] W.M. Schmidt, L. Summerer, Parametric geometry of numbers and applications, *Acta Arith.* 140 (1) (2009) 67–91.
- [26] W.M. Schmidt, L. Summerer, Diophantine approximation and parametric geometry of numbers, *Monatshefte Math.* 169 (1) (2013) 51–104.
- [27] W.M. Schmidt, On parametric geometry of numbers, *Acta Arith.* 195 (2020) 383–414.
- [28] F. Süess, Simultaneous Diophantine approximation on affine subspaces and Dirichlet improbability, PhD Thesis, University of York, 2017, 113 pages, arXiv:1711.08288.