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Zero-queue traffic control, using green-times and prices together[☆]

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ABSTRACT

Reducing stationary or very slowly moving queues is one way of reducing congestion, pollution, inefficient stop–start travel and carbon emissions in cities. This paper considers traffic signal control and road pricing together; aiming to eliminate queueing in at least a subnetwork. Link-exit green-times and link-exit bottleneck delays are considered first in some detail; largely using a simple network. The paper then shows that policy P_0, specified in Smith (1980, 2015), is capacity-maximising for a general network with vertical queueing delays. Then link exit prices, co-ordinated with green-times and also red times, are considered. It is shown that using prices (instead of delays) in the P_0 control policy maximises the capacity of a general steady state network, *with zero queues*. This steady state capacity-maximisation + zero-queue result is then extended to dynamic networks in two ways; an equilibrium extension and a day-to-day stability extension. The equilibrium extension shows that P_0-with-prices maximises network capacity with zero queues in a dynamic network and the stability extension shows that a smoothed version of the P_0-with-prices policy, called $P_0^{\mathbf{f}}$, is able to deliver some stability as well as zero queue capacity maximisation. A simple example network has been given to illustrate several of the control-with-prices policies. It is shown that a biased version of P_0-with-prices, $P_{\mathbf{h}}$-with-prices, yields, *for this simple network*, higher utility than P_0-with-prices itself.

1. Introduction

Reducing vehicle queues is one way of reducing congestion, pollution and carbon emissions in cities, because these are typically significantly correlated with vehicle queue volumes.

This paper seeks combined traffic signal control and road pricing strategies which allow a maximum network throughput, at a user-equilibrium, while maintaining zero queues. It is envisaged that this may be achieved within an entire given network or within selected subsets of a given network. Both general steady state and dynamic network models are considered. The dynamic model utilises a continuum model of traffic signal control.

A simple example network is also considered. For this network several zero-queue throughput maximising policies are stated, including one policy which is, for this simple network, not only zero-queue and throughput maximising but also utility maximising, at a user equilibrium, with a natural definition of utility.

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Maximising throughput is a well-known objective for signal-control strategies and requires little introduction or justification. There are many advantages of having throughput maximising control strategies available. For example, compared to other strategies, a throughput maximising control strategy allows more network road capacity to be allocated to pedestrians while maintaining a given vehicular throughput.

1.1. A brief justification of the user-equilibrium constraint

Smith et al. (2019b) specifies signal control strategies which maximise network throughput when there is no route choice and at a user-equilibrium when there is route choice even if demand exceeds network capacity, *but only for certain simple networks*. Seeking to maximise throughput at a user equilibrium flow pattern is natural: if the throughput-maximising flow pattern is not a user equilibrium then travellers will naturally swap routes as time passes, so the flow pattern will not be stable. The purpose of the user equilibrium constraint is to avoid the possibility that drivers are encouraged to switch routes tomorrow to a set of route choices with less throughput.

There are many other stability issues which arise in this paper, and including the user equilibrium constraint is one natural response to the need for stability with respect to routes choices.

1.2. A justification of the zero-queue constraint

This paper seeks control policies, using signal green-times and prices together, which eliminate queueing in at least a subnetwork at a user equilibrium, while still maximising throughput. There are several reasons for seeking zero queues or small queues. Reducing queue volumes in a network to zero, or even to close to zero, may be expected to have a substantial impact on congestion, pollution and carbon emissions in that network. Reducing congestion should of course reduce public transport running times and costs.

To further justify aiming to reduce queues to zero or close to zero, we now consider the negative impacts of queues in more detail. We consider six aspects. Firstly, queues typically represent delays and stop–start travel which are both inefficient for travellers and also more energy consuming than non-queueing travel. Secondly, queueing vehicles pollute the environment by producing micro-size particulates when vehicles brake and, in the case of internal combustion powered vehicles, exhaust emissions when vehicles accelerate. Thirdly, long queues may block upstream junctions and such blockages may, and often will, severely reduce network efficiency. Fourthly, in certain sensitive locations, usually in towns like the City of York, UK, but also elsewhere (for example Stonehenge), queues of motorised vehicles obstruct and detract from the visual prospect of inspiring architecture. Fifthly, queues take up road space, so removing queues will on many occasions release some roadspace for other uses, perhaps for active travel including pedestrians and cyclists, as extra convivial space outside popular restaurants or as locations for cycle racks. Sixthly, public transport is currently often impeded by general traffic queues, so in these cases removing queues will reduce running time for public transport vehicles. In this last instance the greatest benefits to public transport will arise from reductions of queues on those parts of the network where there are no bus lanes which allow public transport vehicles to bypass queues.

Much research has been devoted to all of the six aspects of queueing above. Here we mention just two. The following studies each consider one of the aspects above, but in quite different ways. Gao et al. (2020) demonstrate empirically the link between [speed changes and acceleration] and [fuel consumption and emissions]. This study shows the disbenefits of stop–start travel associated with queueing, which is our first aspect in the previous paragraph. This study concentrates on diesels, but presumably petrol results will be similar. Ngoduy et al. (2016) demonstrate, theoretically and computationally, that distributing queues more evenly across a network helps to avoid spillbacks, reducing large queues in damaging locations. This study addresses the third aspect discussed in the previous paragraph.

This paper seeks controls which allow a maximum network throughput subject to (i) zero queueing and (ii) user equilibrium. The paper may be regarded as an initial study of this combination and leaves open many opportunities for further study.

1.3. A short technical background

This subsection gives a short technical traffic control background.

There is a large literature on traffic signal control and on road user charging; but much less on their combination. Bell (1992) suggested using traffic signal control and automatic debiting and Smith et al. (1994a) is one of the few papers which has considered this combination of network controls in detailed network models.

Many local responsive traffic signal control policies have been considered, by Wongpiromsarn et al. (2012), Varaiya (2013a,b) (the "maxpressure" or MP policy), Gregoire et al. (2014), Kouvelas et al. (2014) and Le et al. (2015). This work has been motivated by research in telecommunication networks; including especially the classic paper by Tassiulas and Ephremides (1992). The main aim of all these policies is to make the most of the capacity of a given network; or to maximise network throughput when network inflows are within the network capacity and route choices or turning proportions are fixed and known. In this work the user equilibrium condition is not imposed.

Wada (2013), Wada and Akamatsu (2012) show how control and tradeable permits might work together. Road pricing has also been studied in Tan et al. (2015) and Zhou et al. (2015). Yang et al. (2019) consider congestion pricing and perimeter control. Cantarella et al. (2019) study deterministic and stochastic traffic assignment. It would be interesting to combine some of the ideas presented in these papers with the *capacity-maximising* control approach taken here.

Recently there have been considerations of how newly available data, sometimes including data obtained from connected vehicles, may be best utilised within urban traffic control. This is a large subject and here we just mention three studies.

Mohebifard et al. (2019) present a formulation and a distributed solution technique for cooperative signal control and perimeter traffic metring in urban street networks, allowing for varying numbers of connected vehicles. In case studies using implementation in Vissim the cooperative signal timing and perimeter control yielded significant improvements in traffic operations, compared to independent signal control and perimeter control.

Ma et al. (2020) consider how vehicle trajectory data may be utilised to optimise fixed time signal timings at isolated intersections. Approximately uniform arrivals during a time period are assumed, but the method might also be applied to different time periods. The paper suggests that, instead of total delay or total travel time as the control objective to be minimised by a control system, the number of oversaturated stages might be utilised instead. The control design in this paper aims at minimising the number of oversaturated stages.

Mercader et al. (2020) envisage using travel times recorded by mobile phones as inputs to distributed traffic control systems, including especially backpressure control systems. The paper suggests theoretical work to help design suitable control responses to this increasingly available data. The data used here might also be used to implement control/pricing strategies such as those suggested in this paper.

Zhu et al. (2020) considers a combined equilibrium model with users of cars and electric bicycles; it would be interesting to extend the control ideas in this paper to that two-mode situation.

This paper has strong connections to work on the control of telecommunication networks. See for example Eryilmaz and Srikant (2006) who describe and analyse a joint scheduling, routing and congestion control mechanism for wireless networks, that asymptotically guarantees stability of the buffers and fair allocation of the network resources. The authors prove the asymptotic optimality of a primal–dual congestion controller, in which queue-lengths serve as common information. The connections between the primal dual algorithm here and the work in this paper merits further study.

1.4. Outline and the main contributions of this paper

Initially this paper considers signal-controlled networks with a steady demand and vertical queueing delays. Such networks are simple ‘‘quasi-dynamic networks", where queueing is represented but demand is constant. See Bliemer et al. (2012), Nesterov and de Palma (2003) and Smith et al. (2019a). This paper gives control policies using just signal green-time proportions, and not involving prices, which, when there is vertical queueing, maximise network capacity at a user-equilibrium in these quasi-dynamic cases.

It is shown that under certain conditions prices may be used to replace the equilibrium delays arising from the vertical queues, yielding a user equilibrium *with zero queueing*, provided a particular policy for determining green-time proportions in terms of prices, instead of delays, is followed.

The paper then considers dynamic networks and again gives (green-time)-price policies which maximise network capacity at a user equilibrium *with zero queueing*. Some of the ideas utilised here are simple, special, versions of those utilised by Mounce (2006, 2009).

Finally the paper shows that there is a spectrum of responsive (green-time)-price control policies which provide stability of the zero-queue equilibrium in a dynamic case. In each case considered here the (green-time)-price policy depends on a suitable ‘‘barrier function’’ $\mathbf{f}$, and there are many of these.

1.5. Main conditions and assumptions in the paper, and some complicating issues

It is assumed that on each day travellers face locally determined variable link-exit prices as well as locally determined variable link-exit green-times.

The main additional conditions or assumptions in the paper are:

(a) that on each day the same unknown demand, within the network capacity *with zero queues,* occurs;

(b) that travellers swap to cheaper routes as day succeeds day; and

(c) that over time green-time proportions and prices change according to the relevant P_0-with-prices policy. Sometimes red-time proportions are utilised instead of green-time proportions.

The motivation here is that day-to-day travel is often repetitive with the same or similar journeys undertaken day after day.

The dynamic modelling utilises a continuum model of signal control where green-time proportions and flows are defined at each time instant. Such models have often been used in signal control studies; see, for example, Han and Gayah (2015).

Given the results obtained in this paper it would be natural to seek to relax the conditions used to obtain our results. For example, it would be natural to consider on-and-off, instead of continuum, models of signal control and then either (i) aim for zero overflow queues rather than zero queues or (ii) allow the link traversal times, which are constant here, to vary; aiming now to adjust platoon trajectories so that they pass smoothly through a signal-controlled junction with zero queueing. Here the *overflow queue* on an approach to a signal is the queue at the end of the green-period.

To operate the control system here fully *the vehicles, or possibly the vehicle occupants, must be connected to the traffic signal control system and must be automatically debited when a congested signal controlled link is exited*.

When practical implementation is considered there are many complicating aspects of this *zero queue* control/pricing scheme. These complicating aspects include:

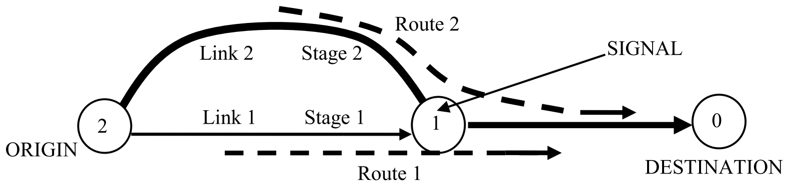

Fig. 1. A simple two route signal controlled network; link 2 is wider and longer than link 1.

(i) What happens if an incident causes queues to arise?
(ii) How should very large input flows be treated?
(iii) How do we ensure that location information is anonymous but still useful?

These and other questions will be the subject of further research; the paper will leave many opportunities open for further consideration.

2. Network control with variable green-times and vertical queueing delays or prices

A simple network model is used in this section to illustrate the main themes in the paper, including especially (a) "network capacity-maximisation" and (b) the need for a supply-feasibility constraint, ensuring that demand is not too large. (a) and (b) are illustrated in Sections 2.2.2 and 2.2.3.

Aside from Sections 2.2.2 and 2.2.3, this section contains only a simple example and related definitions and concepts.

2.1. A simple network example

Consider the network in Fig. 1 in a steady state equilibrium with vertical queues. Let

s_1 = the saturation flow at the link 1 exit (in vehicles per second);
s_2 = the saturation flow at the link 2 exit (in vehicles per second);
C_1 = the freeflow time of travel via route 1 (in seconds; constant);
C_2 = the freeflow time of travel via route 2 (in seconds; constant);
b_1 = the bottleneck delay at the link 1 exit (in seconds);
b_2 = the bottleneck delay at the link 2 exit (in seconds);
$\mathbf{b}$ = the vector $[b_1, b_2]$;
p_1 = the price paid at the link 1 exit (in seconds);
p_2 = the price paid at the link 2 exit (in seconds);
Q_1 = the queue volume on link 1 (in vehicles);
Q_2 = the queue volume on link 2 (in vehicles);
X_1 = the steady flow on route 1 (in vehicles per second);
X_2 = the steady flow on route 2 (in vehicles per second);
$\mathbf{X}$ = the vector $[X_1, X_2]$;
G_1 = the duration of stage 1 (dimensionless) as a proportion;
G_2 = the duration of stage 2 (dimensionless) as a proportion;
$\mathbf{G}$ = the vector $[G_1, G_2]$;
g_1 = the proportion of time that link 1 is green (dimensionless);
g_2 = the proportion of time that link 2 is green (dimensionless);
$\mathbf{g}$ = the vector $[g_1, g_2]$;
T = a given steady flow from the origin to the destination (vehicles per second).

Prices are measured in seconds. Here we assume that all travellers have the same value of time.

In the equilibrium model described here the bottleneck delays on unsaturated link exits will be zero and the bottleneck delays on saturated link exits will be determined by a steady state equilibrium condition (that more costly routes are not used); they are not determined by a cost-flow function. Thus at equilibrium queueing delays balance the travel costs along different routes. In this case there are just two routes.

With vertical queueing the cost of traversing route 1 is $C_1 + b_1$ (seconds) and the cost of traversing route 2 is $C_2 + b_2$ (seconds).

Suppose given a steady demand T (vehicles per second) and that

$$X_1 + X_2 = T, X_1 \geq 0, X_2 \geq 0. \tag{1}$$

Suppose green-times satisfy:

$$G_1 + G_2 = 1, G_1 \geq 0, G_2 \geq 0. \tag{2}$$

These suppositions apply to this small network and similar assumptions apply throughout the paper. Suppose for definiteness here that route 2 is longer and wider than route 1; so that

$$s_1 < s_2 \quad \text{and} \quad C_1 < C_2.$$

Suppose finally that

$$s_1 < T \le s_2.$$

The question then arises: for given T which stage green time vector $\mathbf{G}$ should be chosen?

It is also natural to seek responsive control policies which have the chance of responding favourably using only local data; and also, hopefully, simple calculations. However it is not always clear that locally determined signal control policies achieve sound network-wide consequences. One policy which has been designed to achieve sound network-wide effects, under certain conditions, is policy P_0. In this paper we look at how the responsive policy P_0 may be used and then show how P_0 may be used effectively with prices rather than delays. The following sections outline the original form of P_0 using this simple network and show how prices may be introduced in this simple case.

2.2. The P_0 responsive control policy for this simple network, capacity-maximisation and supply-feasibility

2.2.1. The original P_0 control policy

For this network the original P_0 signal control policy may be stated as follows: choose stage green-times G_1 and G_2 so that $s_1 b_1 = s_2 b_2$. If for example delays b_1, b_2 are given by $b_1 = Q_1/(s_1 G_1)$ and $b_2 = Q_2/(s_2 G_2)$ then P_0 green-times G_1, G_2 are determined by:

$$Q_1/G_1 = s_1 b_1 = s_2 b_2 = Q_2/G_2$$

or (provided queues are positive)

$$G_1 = Q_1/(Q_1 + Q_2) \text{ and } G_2 = Q_2/(Q_1 + Q_2).$$

This definition is given in Smith (1980) and Smith (1979b). For further information see Smith (2015), Smith et al. (2015) and Smith et al. (2019a).

We now use this simple network model to demonstrate, in Section 2.2.2, a capacity maximising property of P_0. We also demonstrate, in Section 2.2.3, the need for a supply-feasibility constraint. Both capacity-maximisation and supply-feasibility constraints are relevant throughout the paper.

2.2.2. A capacity-maximising effect of P_0 on this network

In this simple network model, described in Section 2.1, the demand T satisfies $s_1 < T \le s_2$ and, with vertical queueing, the travel time via route i is here defined to be $C_i + b_i$ $(i = 1, 2)$. Here we agree that $(\mathbf{X}, \mathbf{G}, \mathbf{b})$ is *a feasible equilibrium* if

(1) and (2) hold, $X_1 \le s_1 G_1$ and $X_2 \le s_2 G_2$, and $C_1 + b_1 = C_2 + b_2$.

Consider the following values of $G_1, G_2, X_1, X_2, b_1, b_2$, where $\Delta = C_2 - C_1 > 0$:

$$G_1 = (s_2 - T)/(s_2 - s_1), G_2 = (T - s_1)/(s_2 - s_1), X_1 = s_1 G_1, X_2 = s_2 G_2, b_1 = s_2\Delta/(s_2 - s_1), b_2 = s_1\Delta/(s_2 - s_1).$$

It is easy to check that these values constitute a feasible equilibrium as defined above and *also* satisfy policy P_0 because $s_1 b_1 = s_2 b_2$. So policy P_0 holds too. Thus the extra condition imposed by policy P_0 ($s_1 b_1 = s_2 b_2$ here) *does not prevent* the existence of an equilibrium $(\mathbf{X}, \mathbf{G}, \mathbf{b})$ and, for this reason, we say in this paper that P_0 *maximises the capacity of this simple network*.

More generally applicable definitions of capacity maximisation are given in Section 3 below.

2.2.3. A more generally applicable feasibility constraint

Section 2.2.2 above shows that if $T \le s_2$ then there is a feasible equilibrium consistent with P_0; such a feasible equilibrium is specified in 2.2.2 above. To obtain such capacity-maximisation results in more general networks, we now write the feasibility constraints here, including $T \le s_2$, in a more generally applicable way, as follows. We let:

$$D = \{\mathbf{X}; X_1 + X_2 = T, X_1 \ge 0, X_2 \ge 0\}, F = \{\mathbf{G}; G_1 + G_2 = 1, G_1 \ge 0, G_2 \ge 0\},$$

$$S = \{(\mathbf{X}, \mathbf{G}); X_1 \le s_1 G_1 \text{ and } X_2 \le s_2 G_2\},$$

and then consider the feasibility condition:

$$(D \times F) \cap S \text{ is non-empty.}$$

This will be the constraint we use throughout the paper, with more general specifications of D, F and S. This constraint will always ensure that the demand is not too large for the network being considered. In the case of this simple network it is easy to see that this constraint implies that $T \le s_2$.

2.3. A dynamical extension of the P_0 policy

Here we utilise a dynamical version of the P_0 policy as stated in Section 2.2.1 above. In this dynamical version only the stage green-time dynamics are specified, assuming given delays. Delay changes will be considered later.

For given delays b_1 and b_2 and given stage green-times G_1 and G_2:

1. if $s_1b_1 > s_2b_2$ and $G_2 > 0$, swap green time from stage 2 to stage 1,
2. if $s_1b_1 < s_2b_2$ and $G_1 > 0$. swap green time from stage 1 to stage 2, and
3. otherwise, keep the green times constant.

Here, in this section, we say that the stage green-time vector **G** *satisfies* the P_0 policy for this network if and only if no green-time motion is possible fitting in with 1–3 above. This is the case if and only if:

1. $s_1b_1 > s_2b_2$ and $G_2 = g_2 = 0$, or
2. $s_1b_1 < s_2b_2$ and $G_1 = g_1 = 0$, or
3. $s_1b_1 = s_2b_2$.

This condition is the original, *equilibrium*, version of the P_0 policy (with queues) for this simple network and may be written equivalently as shown below, as a variational inequality, in Section 4.2.

In the next subsection we introduce P_0-with-prices.

2.4. The P_0-with-prices responsive control policy for the simple network in Fig. 1

Here only the green-time dynamics are described; the changes in the prices will be discussed below. For given prices p_1 and p_2 and given stage green-times G_1 and G_2:

1. if $s_1p_1 > s_2p_2$ and $G_2 > 0$, swap green time from stage 2 to stage 1,
2. if $s_1p_1 < s_2p_2$ and $G_1 > 0$. swap green time from stage 1 to stage 2, and
3. otherwise, keep the green times constant.

Here, in this section, we say that the stage green-time vector **G** *satisfies* the P_0-with-prices policy for this network if and only if no green-time motion is possible fitting in with 1–3 above. This is the case if and only if:

1. $s_1p_1 > s_2p_2$ and $G_2 = g_2 = 0$, or
2. $s_1p_1 < s_2p_2$ and $G_1 = g_1 = 0$, or
3. $s_1p_1 = s_2p_2$.

This condition is the *equilibrium* version of the P_0-with-prices policy for this simple network and may be written equivalently as shown below, as a variational inequality, in Section 4.3.

2.5. Contribution

This Section 2 has given definitions of policy P_0, illustrated a capacity-maximising effect of this control policy on a simple network and also defined the P_0-with-prices policy. In Section 2.2.3 we have illustrated the feasibility constraint: $(D \times F) \cap S$ *is non-empty*. This feasibility constraint is utilised in the proofs of Theorems 1–4.

3. Capacity maximising traffic signal control policies

Capacity maximising traffic signal control policies are central elements of this paper. These policies typically are not delay-minimising. In this section we define this capacity-maximising concept in three scenarios: the first uses vertical queueing, route-flows and stage green-times, the second uses vertical queueing, link-flows and link green-times and the third uses prices, link flows and link green-times. These three definitions apply to the simple network in Section 2 and we later show how to utilise these definitions with more general networks. The policies shown to be capacity-maximising in Theorem 1–4 below are fairly simple and take reasonable account of traffic re-routeing, as illustrated in Section 2

3.1. Three definitions

Suppose given a capacitated signal-controlled network with specified routes and signal stages, and a control policy P_{stage} determining the stage green-time vector **G** in terms of the vertical queueing delay vector **b**.

Definition 1. For a given network with traffic signals, the control policy P_{stage} "*maximises network capacity with vertical queueing delays*" will here mean: *if an inelastic demand is such that there is a route-inflow vector* **X**, *stage green-time vector* **G** *which together meet the inelastic demand and are within the capacity limitations of the given network then there is a route-inflow vector* $\mathbf{X}^*$, *a vertical delay vector* $\mathbf{b}^*$ *and a stage green-time vector* $\mathbf{G}^*$ *which meet the given inelastic demand, are within the capacity limitations of the given network, and also satisfy:*

(a) $\mathbf{X}^*$ is a Wardrop equilibrium with vertical queueing delay when the delay vector is $\mathbf{b}^*$ and
(b) $\mathbf{G}^*$ satisfies the control policy P_{stage} when the delay vector is $\mathbf{b}^*$.

There is an equivalent link formulation as follows. Suppose given a control policy P determining the link green-time vector $\mathbf{g}$ in terms of vertical queueing delay $\mathbf{b}$. This definition also applies to the network in Fig. 1 and to a general network.

Definition 2. The control policy P "*maximises network capacity with vertical queueing delays*" will here mean: *if an inelastic demand is such that there is a link-flow vector* $\mathbf{x}$*, link green-time vector* $\mathbf{g}$ *which together meet the inelastic demand and are within the capacity limitations of the given network then there is a link flow vector* $\mathbf{x}^*$*, a vertical queueing delay vector* $\mathbf{b}^*$ *and a link green-time vector* $\mathbf{g}^*$ *which meet the given inelastic demand, are within the capacity limitations of the given network, and also satisfy:*

(a) $\mathbf{x}^*$ is a Wardrop equilibrium with vertical queueing delay when the delay vector is $\mathbf{b}^*$ and
(b) $\mathbf{g}^*$ satisfies the control policy P when the delay vector is $\mathbf{b}^*$.

It is obvious here, with our simple network, that these two definitions are equivalent since the stage green-times = the link green-times and there is also a similar simple connection between route flows and link flows. Under certain conditions it is easy to generalise this to a more general network where each link green-time equals the sum of certain stage green-times and each link flow equals the sum of certain route-flows. In this case too the two definitions are equivalent. See for example Smith (2015).

In this paper we also seek to use prices and green-times; aiming to reduce queueing delays. Thus we are led to the following variation of Definition 2, where prices, instead of delays, control green-times. This allows the possibility of reducing queueing delays to zero or to small values, while still allowing equilibrium. To consider this possibility we now consider the following Definition 3.

In Definition 3 we suppose given a control policy P determining the link green-time vector $\mathbf{g}$ in terms of the vector $\mathbf{p}$ of prices charged to exit the approaches at each junction. This definition also applies to the network in Fig. 1 and to a general network.

Definition 3. The control policy P "*maximises network capacity with prices (and zero queueing delays)*" will here mean: *if an inelastic demand is such that there is a link-outflow vector* $\mathbf{x}$ *and a link green-time vector* $\mathbf{g}$ *which together meet the inelastic demand and are within the capacity limitations of the given network with zero queues then there is a link outflow vector* $\mathbf{x}^*$*, a price vector* $\mathbf{p}^*$ *and a link green-time vector* $\mathbf{g}^*$ *which meet the given inelastic demand, are within the capacity limitations of the given network with zero queues, and also satisfy:*

(a) $\mathbf{x}^*$ *is a Wardrop equilibrium with prices when the price vector is* $\mathbf{p}^*$ *and*
(b) $\mathbf{g}^*$ *satisfies the control policy* P *when the price vector is* $\mathbf{p}^*$.

This Definition 3 may be thought of with reference to steady state and dynamic demand. In the dynamic case the given demand varies with time and $\mathbf{x}$, $\mathbf{g}$, $\mathbf{p}$ are vector functions of time. An explicitly dynamic definition is given as Definition 13 below in Section 11.

3.2. Contribution

Capacity maximising traffic signal control policies, utilising queueing delays and prices, have been defined. The three definitions here apply to the simple network in Section 2 and we later show how to utilise these definitions with more general networks. The control policies shown to be capacity-maximising in this paper, in Theorem 1- 4 below, are fairly simple and take reasonable account of traffic re-routeing, as illustrated in Section 2. The feasibility constraint: "$(D\times F)\cap S$ *is non-empty*" (introduced in Section 2) ensures that the given demand is "within the capacity limitations of the given network", and is utilised in the proofs of Theorems 1–4.

4. Normality, variational inequalities and more general statements of P_0 and P_0-with-prices

This section gives some geometrical definitions which are the basis of the central results in this paper, and then uses these to specify various signal control policies developed from P_0*. Throughout this section* D *is to be a closed bounded convex set.*

4.1. Normality and variational inequalities

Let D be a closed bounded convex subset of some Euclidean space. Let $\mathbf{v}$ and $\mathbf{x}^*$ be points (vectors) in the Euclidean space. Let $\mathbf{x}^* \in D$.

Definition 4. The statement that

$\mathbf{v}$ is normal at $\mathbf{x}^*$ to the set D

means that

$$\mathbf{v}\cdot(\mathbf{x}-\mathbf{x}^*) \le 0 \text{ for all } \mathbf{x}\in D.$$

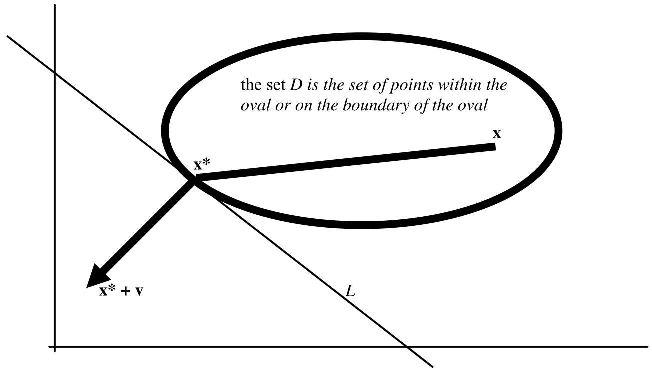

Fig. 2. The diagonal faint line L makes a right angle with $\mathbf{v}$ and all D is in the upper half plane determined by the line L.

Fig. 2 illustrates this statement in two dimensions. The vector $\mathbf{v}$ is based at $\mathbf{x}^*$ and joins $\mathbf{x}^*$ to $\mathbf{x}^* + \mathbf{v}$: the dot product of $\mathbf{v}$ and $\mathbf{x} - \mathbf{x}^*$ is never positive for any choice of $\mathbf{x} \in D$.

Each of the following statements is a *variational inequality*:

$\mathbf{x}^* \in D$ and $\mathbf{v}$ is normal at $\mathbf{x}^*$ to the set D;

$\mathbf{x}^* \in D$ and $\mathbf{v} \cdot \left(\mathbf{x} - \mathbf{x}^*\right) \leq 0$ for all $\mathbf{x} \in D$.

These both say that if you stand at $\mathbf{x}^*$ and look in direction $\mathbf{v}$ there will be no point in D in front of you. Usually in a variational inequality like this the vector $\mathbf{v}$ will be a function of $\mathbf{x}^*$.

4.2. The P_0 policy redefined as a variational inequality for the simple network in Fig. 1

For our simple network let $F = \{\mathbf{g}; g_1 + g_2 = 1, g_1 \geq 0, g_2 \geq 0\}$.

Definition 5. We shall say that "$\mathbf{g}$ satisfies the P_0 control policy when the vertical queueing delay vector is $\mathbf{b}$" for our simple network if and only if

$\mathbf{g} \in F$ and $(s_1 b_1, s_2 b_2)$ is normal at $\mathbf{g}$ to F.

See the previous section for a definition of "the vector $\mathbf{v}$ is normal at $\mathbf{x}^*$ to the set D". Here of course

$\mathbf{v} = (s_1 b_1, s_2 b_2), \mathbf{x}^* = \mathbf{g}$ and $D = F$.

Thus the equilibrium form of P_0 in the second paragraph of Section 2.3 has been expressed as a variational inequality; this variational inequality summarises the three conditions specifying the equilibrium form of the P_0 policy in the third paragraph of Section 2.3.

4.3. The P_0-with-prices policy redefined as a variational inequality for the simple network in Fig. 1

Definition 6. We shall say that "$\mathbf{g}$ satisfies the P_0-with-prices control policy when the price vector is $\mathbf{p}$" for our simple network if and only if

$\mathbf{g} \in F$ and $(s_1 p_1, s_2 p_2)$ is normal at $\mathbf{g}$ to F.

See the previous section for a definition of "the vector $\mathbf{v}$ is normal at $\mathbf{x}^*$ to the set D". Here of course

$\mathbf{v} = (s_1 p_1, s_2 p_2), \mathbf{x}^* = \mathbf{g}$ and $D = F$.

Thus the equilibrium form of P_0-with-prices in Section 2.4 has been expressed as a variational inequality; this variational inequality summarises the three conditions specifying the equilibrium form of the P_0-with-prices policy in Section 2.4.

4.4. Contribution

This section has given some geometrical definitions, involving variational inequalities, which are the basis of the central results in this paper. This section has also redefined, for the simple network, the equilibrium forms of (i) policy P_0 with vertical queueing delay and (ii) policy P_0-with-prices; using variational inequalities.

5. Definition of equilibrium with vertical queueing delay vector b in a more general network

This section introduces a general network with traffic signals and using this gives a more general definition of a Wardrop equilibrium with vertical queueing delay.

5.1. A more general network with delays and prices

In this section we suppose given a more general single mode network with capacitated links and signal controlled junctions, each with a specified set of signal stages. We will exploit the geometry described in Section 4 to generalise the results in Section 2 so that they apply to this more general network. We consider a steady state and follow the line adopted with the simple network above.

We suppose that our given more general single-mode network has:

(i) for each OD pair, given routes (these are loop-free sets of contiguous links) joining that OD pair; and

(ii) for each junction, given sets of links which are shown green simultaneously during each signal stage at that junction.

We let:

s_i = the saturation flow at the link i exit (in vehicles per second);
c_i = the freeflow time of travel via link i (in seconds; constant);
X_r =the steady flow of vehicles on route r;
x_i = the steady flow of vehicles on link i (in vehicles per second);
G_k =proportion of time that stage k is green (dimensionless);
g_i = proportion of time that link i is green (dimensionless);
b_i = queueing delay at the link i exit (secs);
p_i = price paid at the link i exit (secs); and
d_{ij} =the steady demand for travel from node i to node j (vehicles per second).

We suppose that there are K^1 routes and K^2 stages. We also suppose that there are defined:

(iii) a set D^+ of non-negative route flow vectors $\mathbf{X}$ with K^1 co-ordinates $X_r \geq 0$ meeting all the given origin–destination demands d_{ij}; and

(iv) a set F^+ of stage green-time vectors $\mathbf{G}$ with K^2 co-ordinates $G_k \geq 0$, where the stage green-times at each junction add to 1.

These sets are defined as follows: Given the demands d_{ij} the set D^+ is to be given by:

$$D^+ = \{\mathbf{X} \geq \mathbf{0};\ \text{for each OD pair } ij, \sum_{\{r;\ \text{route } r \text{ joins node } i \text{ to node } j\}} X_r = d_{ij}\} \subset R^{K^1},$$

and the set F^+ is to be given by:

$$F^+ = \{\mathbf{G} \geq \mathbf{0};\ \text{for each junction } j, \sum_{\{k;\ \text{stage } k \text{ is at junction } j\}} G_k = 1\} \subset R^{K^2}.$$

Given D^+ above we now specify a set D of link flow vectors $\mathbf{x}$ with co-ordinates $x_i \geq 0$ meeting all the given origin–destination demands d_{ij}, and given F^+ above we now specify a set F of feasible link green-time vectors $\mathbf{g}$ with co-ordinates $g_i \geq 0$ (arising by adding certain stage green-times G_k at each junction, as specified below and illustrated below in Section 5.2).

Given D^+, for any route-flow vector $\mathbf{X}$ of route flows in D^+, the link i flow is defined by:

$$x_i(\mathbf{X}) = \sum_{\{r;\ \text{route } r \text{ traverses link } i\}} X_r,$$

and then the set D is defined by:

$$D = \{\mathbf{x}(\mathbf{X}); \mathbf{X} \in D^+\}.$$

Also, given F^+, for any vector $\mathbf{G}$ of stage green-times in F^+, the link i green-time is defined by:

$$g_i(\mathbf{G}) = \sum_{\{k;\ \text{stage } k \text{ shows green to link } i\}} G_k,$$

and then the set F is defined by:

$$F = \{\mathbf{g}(\mathbf{G}); \mathbf{G} \in F^+\}.$$

Given the above variables we now specify the set S of supply feasible (link flow vector, link green-time vector) $[\mathbf{x}, \mathbf{g}]$ pairs to be those pairs for which each link flow x_i is no greater than the saturation flow s_i multiplied by the link green-time proportion g_i. Thus we put:

$$S = \{(\mathbf{x}, \mathbf{g}); x_i - s_i g_i \leq 0 \text{ for all } i\}.$$

Here, there are no non-negativity constraints such as $x_i \geq 0$, $g_i \geq 0$.

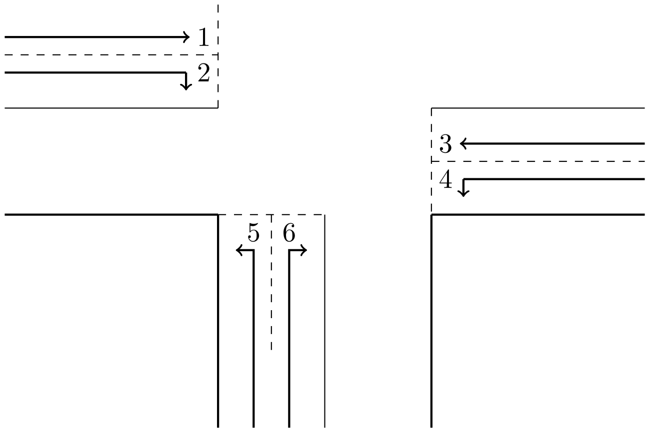

Fig. 3. A signal-controlled T-junction with 3 stages, 6 approaches and 6 turning movements.

5.2. *An example of a signal-controlled junction; showing how link green-times g_i are related to stage green-times G_k*

Fig. 3 shows a junction with approach links or lanes 1, 2, 3, 4, 5 and 6. Each approach lane has an arrow at the lane exit showing the turn which is permitted when exiting the lane. The signal has 3 stages: stage 1, stage 2 and stage 3; the approach links given green during the three stages are as follows:

Thus, for example, the green-time awarded to approach link 1 = stage 1 green-time + stage 2 green-time. All six link green times are shown below, in terms of the three stage green times:

$g_1 = G_1 + G_2$ $\qquad$ $g_4 = G_3$

$g_2 = G_2$ $\qquad$ $g_5 = G_2 + G_3$

$g_3 = G_1$ $\qquad$ $g_6 = G_3$

We suppose that the sum of stage green-times at each junction = 1 and that stage green-times are non-negative.

5.3. *Wardrop equilibrium with queueing delays*

Suppose given a general network with sets D, F and S as specified above in Section 5.1. Suppose that

$(D \times F) \cap S$ is non-empty.

This condition is a more generally applicable version of the same statement in Section 2.2.2, because the specifications of D, F and S are here more general. Then in this steady state case we have the following generally applicable definition of Wardrop equilibrium with vertical queueing delays.

Definition 7. Following Smith (1987) we here agree that $\mathbf{x}^*$ is a Wardrop equilibrium with vertical queueing delay vector $\mathbf{b}$ if

$\mathbf{x}^* \in D$ and $-(\mathbf{c} + \mathbf{b})$ is normal at $\mathbf{x}^*$ to D.

This is a Variational Inequality or a VI, and is illustrated in Fig. 2 by letting

$\mathbf{v} = -(\mathbf{c} + \mathbf{b}.)$

This definition is equivalent to the usual definition of a Wardrop equilibrium: “no traveller has a quicker route", provided that all travellers have the same value of time and the link i travel time t_i is given by:

$t_i = c_i + b_i$

seconds. The, standard, proof of this is essentially given in Smith (1979a) and elsewhere.

Some account may be taken of spatial as opposed to vertical queuing by using the link travel time formula:

$t_i = c_i + k_i b_i$ instead of $t_i = c_i + b_i$

where $k_i < 1$. See Smith et al. (2019a). We do not do this in this paper.

5.4. *Contribution*

This section has introduced a general network with traffic signals and either queueing delays or prices. Using this network we give a more general definition of a Wardrop equilibrium with vertical queueing delay.

6. Proof that P_0 is capacity-maximising in the general network above, with vertical queueing

In this section (i) we extend the equilibrium definition of P_0 in Section 2.2.1 to allow for the general network specified in Section 5.1. Then (ii) we show that P_0 is capacity maximising with vertical queueing. The variational inequalities defined in Section 4 are central to both (i) and (ii).

Suppose that the general network has n links with link saturation flows s_i, vertical bottleneck delays b_i and link green-times g_i. Let:

$$\mathbf{s} = [s_1, s_2, s_3, \dots, s_{n-1}, s_n], \mathbf{b} = [b_1, b_2, b_3, \dots, b_{n-1}, b_n] \text{ and } \mathbf{g} = [g_1, g_2, g_3, \dots, g_{n-1}, g_n].$$

A general definition of the P_0 policy is then as follows.

Definition 8. Given the general network above in Section 5.1, and given $(\mathbf{s}, \mathbf{g}, \mathbf{b})$,

the P_0 control policy is satisfied at $(\mathbf{g}, \mathbf{b})$

if and only if:

$$\mathbf{g} \in F \text{ and } \mathbf{s} \circ \mathbf{b} = [s_1 b_1, s_2 b_2 \dots, s_{n-1} b_{n-1}, s_n b_n] \text{ is normal at } \mathbf{g} \text{ to } F. \quad (3)$$

Here F is the set of feasible link green-time vectors specified in Section 5.1. Condition (3) is a Variational Inequality and is illustrated in Fig. 2 by letting

$$\mathbf{v} = \mathbf{s} \circ \mathbf{b}, \mathbf{x}^* = \mathbf{g} \text{ and } D = F.$$

Theorem 1. *The P_0 control policy specified in Definition 8 is capacity-maximising for the general single-mode network with vertical queueing in Section 5.1.*

Proof. Suppose that

$$(D \times F) \cap S \text{ is non-empty.}$$

To prove the theorem we need to show that, on this assumption, there is an equilibrium with vertical queueing delay consistent with policy P_0. *Equilibrium with vertical queueing delay* and *policy P_0* are defined in Definitions 7 and 8.

To do this, for $(\mathbf{x}, \mathbf{g}) \in (D \times F) \cap S$, let

$$Z(\mathbf{x}, \mathbf{g}) = \sum_i c_i x_i.$$

Consider the following Variational Inequality Problem.

VI problem 1*:*

Find $(\mathbf{x}^*, \mathbf{g}^*) \in (D \times F) \cap S$ such that

$$-(\mathbf{c}, \mathbf{0}) = -grad\, Z(\mathbf{x}^*, \mathbf{g}^*) \text{ is normal at } (\mathbf{x}^*, \mathbf{g}^*) \text{ to } (D \times F) \cap S.$$

This is an LP. VI problem 1 has a solution since Z is continuous and $(D \times F) \cap S$ is non-empty and compact and so Z has a minimum over the set $(D \times F) \cap S$. If $(\mathbf{x}^*, \mathbf{g}^*)$ is any such Z-minimiser then $(\mathbf{x}^*, \mathbf{g}^*)$ must satisfy variational inequality 1; if this were not true then there would be a descent direction for Z at $(\mathbf{x}^*, \mathbf{g}^*)$ and this cannot be so since $(\mathbf{x}^*, \mathbf{g}^*)$ is a minimiser of Z.

Let $(\mathbf{x}^*, \mathbf{g}^*)$ solve variational inequality problem 1. Then

$$-(\mathbf{c}, \mathbf{0}) \ (= -grad\, Z(\mathbf{x}^*, \mathbf{g}^*)) \text{ is normal at } (\mathbf{x}^*, \mathbf{g}^*) \text{ to } (D \times F) \cap S.$$

So

$$-(\mathbf{c}, \mathbf{0}) = (\mathbf{n}_D, \mathbf{n}_F) + \mathbf{n}_S \quad (4)$$

where

$$\mathbf{n}_D \text{ is normal at } \mathbf{x}^* \text{ to } D, \mathbf{n}_F \text{ is normal at } \mathbf{g}^* \text{ to } F \text{ and } \mathbf{n}_S \text{ is normal at } (\mathbf{x}^*, \mathbf{g}^*) \text{ to } S.$$

Now

$$S = S_1 \times S_2 \times S_3 \times \cdots \times S_{n-1} \times S_n \text{ where } S_i = \{(x_i, g_i); x_i - s_i g_i \le 0\}.$$

S_i has no non-negativity constraints such as $x_i \ge 0, g_i \ge 0$. Therefore any normal to S_i is of the form $b_i(1, -s_i) = (b_i, -s_i b_i)$, where $b_i \ge 0$, and so any normal to S is of the form

$$\mathbf{n}_S = (\mathbf{b}, -\mathbf{s} \circ \mathbf{b})$$

for some vector $\mathbf{b}$ where all the co-ordinates $b_i \ge 0$. Therefore, using (4),

$$-(\mathbf{c}, \mathbf{0}) = (\mathbf{n}_D, \mathbf{n}_F) + \mathbf{n}_S = (\mathbf{n}_D, \mathbf{n}_F) + (\mathbf{b}, -\mathbf{s} \circ \mathbf{b}) \quad (5)$$

Subtracting $\mathbf{n}_S = (\mathbf{b}, -\mathbf{s}\circ\mathbf{b})$ from both sides of (5) yields:

$$(-(\mathbf{c}+\mathbf{b}), \mathbf{s}\circ\mathbf{b}) = (\mathbf{n}_D, \mathbf{n}_F) \text{ is normal at } (\mathbf{x}^*, \mathbf{g}^*) \text{ to } D \times F,$$

and so

$$-(\mathbf{c}+\mathbf{b}) = \mathbf{n}_D \text{ is normal at } \mathbf{x}^* \text{ to } D \tag{6}$$

and

$$\mathbf{s}\circ\mathbf{b} = \mathbf{n}_F \text{ is normal at } \mathbf{g}^* \text{ to } F. \tag{7}$$

Thus:

by (6), $\mathbf{x}^*$ is a Wardrop equilibrium with delay vector $\mathbf{b}$ as defined in definition 7 in section 5.3

and

by (7), $\mathbf{g}^*$ satisfies policy P_0 with delay vector $\mathbf{b}$ as defined in definition 8 early in this section 6.

We have now proved Theorem 1, that policy P_0 is capacity maximising with vertical queueing delays; see Definition 2 above in Section 3. This theorem may be extended somewhat (i) by allowing for spatial queueing as in Smith et al. (2019a) and (ii) by redefining b_i as the excess of the actual link travel time t_i over c_i; that is by re-defining $\mathbf{b}$ to be that vector $\mathbf{b}$ satisfying $\mathbf{b} = \mathbf{t} - \mathbf{c}$, where the t_i are measured.

6.1. Contribution

In this section (i) we extended the equilibrium definition of P_0 in Section 2.3 to allow for the general network specified in Section 5.1. Then (ii) we have shown that P_0 is capacity maximising if queueing is vertical. The variational inequalities defined in Section 4 are central to both (i) and (ii). In the next sections we show how queueing delays may be replaced by prices under certain conditions.

7. Definition of wardrop equilibrium with a price vector p

In this section we define Wardrop equilibria with prices using the variational inequalities introduced in Section 4.

Consider the general network described in Section 5.1. Here we use prices p_i instead of vertical delays b_i, supposing that all $b_i = 0$ and

$$(D \times F) \cap S \text{ is non-empty.}$$

We follow Definition 7 above in Section 5.3,

Definition 9. Given the general network above in Section 5.1, and given $\mathbf{p}$, we here agree that

$\mathbf{x}^*$ is a Wardrop equilibrium with price vector $\mathbf{p}$

if and only if:

$$\mathbf{x}^* \in D \text{ and } -(\mathbf{c}+\mathbf{p}) \text{ is normal at } \mathbf{x}^* \text{ to } D. \tag{8}$$

Condition (8) is a Variational Inequality or a VI, as illustrated in Fig. 2 with

$$\mathbf{v} = -(\mathbf{c}+\mathbf{p}).$$

Again this definition is equivalent to the usual “more costly routes are not used".

8. Proof that the P_0-with-prices policy is capacity-maximising (with zero queueing delays) in the general single-mode network with prices in Section 5.1

In this section (i) we extend the equilibrium definition of P_0-with-prices in Section 2.4 to allow for the general network specified in Section 5.1. Then (ii) we show that P_0-with-prices is capacity maximising. The variational inequalities defined in Section 4 are central to both (i) and (ii).

8.1. The P_0-with-prices control policy, in link form, as a solution to a VI

Here we extend the equilibrium definition of P_0-with-prices in Section 2.4 to allow for the general network specified in Section 5.1 above. Suppose that the general network has n links with link saturation flows s_i, link bottleneck prices p_i and link green-times g_i. Let:

$$\mathbf{s} = [s_1, s_2, s_3, \ldots, s_{n-1}, s_n], \mathbf{p} = [p_1, p_2, p_3, \ldots, p_{n-1}, p_n] \text{ and } \mathbf{g} = [g_1, g_2, g_3, \ldots, g_{n-1}, g_n].$$

A general definition of the P_0-with-prices policy is then as follows.

Definition 10. Given the general network above in Section 5.1, and given $(\mathbf{s}, \mathbf{g}, \mathbf{p})$,

the P_0-with-prices control policy is satisfied at $(\mathbf{g}, \mathbf{p})$

if and only if:

$$\mathbf{g} \in F \text{ and } \mathbf{s} \circ \mathbf{p} = [s_1 p_1, s_2 p_2 \ldots, s_{n-1} p_{n-1}, s_n p_n] \text{ is normal at } \mathbf{g} \text{ to } F. \quad (9)$$

This definition may be compared with Definition 8; the vertical bottleneck delays in Definition 8 become prices in Definition 10 here.

8.2. The P_0-with-prices control policy is capacity-maximising

Theorem 2. *The P_0-with-prices control policy is capacity-maximising for the general single-mode network with prices and zero delays in Section 5.1. Here capacity-maximising is defined in Definition 3.*
Outline Proof. *Replace the vertical bottleneck delay vector* **b** *in the proof of Theorem 1 with the price vector* **p**; *associated parallel changes must be made to certain words. Or apply theorem 3 with* $\mathbf{h} = \mathbf{0}$.

8.3. Contribution

In this section (i) we extended the equilibrium definition of P_0-with-prices in Section 2.4 to allow for the general network specified in Section 5.1. Then (ii) we have shown that P_0-with-prices is capacity maximising. The variational inequalities defined in Section 4 are central to both (i) and (ii).

9. A generalisation of the P_0-with-prices control policy for a general single-mode network

With the same general single-mode network in Section 5.1, we now generalise policy P_0-with-prices slightly and prove that each of the more general policies is also capacity-maximising.

Let

$$\mathbf{h} = [h_1, h_2, h_3 \ldots, h_{n-1}, h_n]$$

be any constant n-vector **h**. Then, for any price vector **p**,

$$\mathbf{s} \circ \mathbf{p} + \mathbf{h} = [s_1 p_1 + h_1, s_2 p_2 + h_2, s_3 p_3 + h_3, \ldots, s_{n-1} p_{n-1} + h_{n-1}, s_n p_n + h_n].$$

We follow previous definitions.

Definition 11. Given the general network above in Section 5.1, and given $(\mathbf{s}, \mathbf{g}, \mathbf{p})$,

the $P_{\mathbf{h}}$-with-prices control policy is satisfied at $(\mathbf{g}, \mathbf{p})$

if and only if:

$$\mathbf{g} \in F \text{ and } \mathbf{s} \circ \mathbf{p} + \mathbf{h} \text{ is normal at } \mathbf{g} \text{ to } F. \quad (10)$$

Theorem 3. *For any n-vector* **h** *the $P_{\mathbf{h}}$-with-prices control policy is capacity-maximising, with zero queueing delays, in the general single-mode network with prices in Section 5.1. The definition of capacity-maximising here is given in Definition 3.*

Proof. This is given in Appendix A.

9.1. Contribution

We have generalised policy P_0-with-prices to the policies $P_{\mathbf{h}}$-with-prices, and proved that each of the $P_{\mathbf{h}}$-with-prices policies is also capacity-maximising.

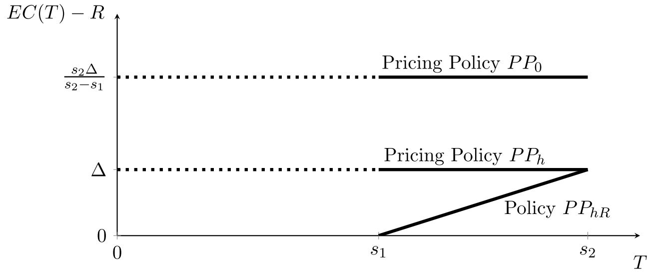

Fig. 4. Comparative performances of the pricing control policies PP_0, PP_h, PP_{hR}.

10. Comparative equilibrium evaluation of three price control policies on a small network

This section compares the equilibrium performances of several P_h-with-prices policies on the simple network.

Most of the calculations in this section are provided explicitly in detail in Smith et al. (2019a). Consider the network in Fig. 1; suppose that prices p_1, p_2 are charged at the exits of links 1 and 2. We here consider the pricing control policies $P_\mathbf{0}$-with-prices, $P_\mathbf{h}$-with-prices and a third policy stated below.

Let $EC(T)$ denote the "excess travel cost" (in seconds) experienced at equilibrium by each traveller when T travellers per second are entering the network and leaving the network. This is the excess over C_1 (seconds) and includes any prices paid on links 1 and 2. C_1 (seconds) is the least possible travel cost on this network.

We suppose that all travellers have the same value of time and we express all travel costs in seconds. Thus (in seconds):

$$
\begin{aligned}
EC(T) &= \text{travel cost felt by each traveller at equilibrium minus } C_1 \\
&= C_1 + p_1 - C_1 = p_1 \\
&= C_2 + p_2 - C_1 = C_2 - C_1 + p_2 = \Delta + p_2.
\end{aligned}
$$

where $\Delta = C_2 - C_1$. We focus on only those T which satisfy: $s_1 < T \le s_2$.

Now suppose that for policy $P_\mathbf{h}$-with-prices there is the possibility of modifying the policy by returning the financial equivalent of R seconds to each traveller; whether that traveller used the narrow road, link 1, or the longer wider road, link 2. Here we only consider returning R seconds where R = the average of all prices paid = $[x_1p_1 + x_2p_2]/[x_1 + x_2]$. Other values of R may be considered.

10.1. Notation

We will use the following notation:

Policy $P_\mathbf{0}$-with-prices will be written $PP_\mathbf{0}$.

Policy $P_\mathbf{h}$-with-prices will be written $PP_\mathbf{h}$.

Policy $P_{\mathbf{h}R}$-with-prices will be written $PP_{\mathbf{h}R}$.

Here R stands for "RETURN": Pricing policy $PP_{\mathbf{h}R}$ is the same as $PP_\mathbf{h}$ except that the total of all prices paid is re-distributed equally to each traveller. So, with $P_{\mathbf{h}R}$,

in the evaluation below each traveller receives a rebate equal to the financial equivalent of R seconds

where R seconds = the average of all prices paid.

We now suppose that $\mathbf{h} = [s_1C_1, s_1C_2]$.

10.2. Comparative equilibrium evaluation of the policies $PP_\mathbf{0}, PP_\mathbf{h}, PP_{\mathbf{h}R}$

We are now in this section supposing that $\mathbf{h} = [s_1C_1, s_1C_2]$.

To allow for the return R in the comparative evaluation of the original policies $PP_\mathbf{0}$, $PP_\mathbf{h}$ and the modified policy $PP_{\mathbf{h}R}$ we now consider the objective function $EC(T) - R$ instead of $EC(T)$ itself. Of course with $PP_\mathbf{0}$, $PP_\mathbf{h}$ there is no returning, $R = 0$, and for these policies $EC(T) - R = EC(T)$. Fig. 4 illustrates the comparative equilibrium performances of both the original policies $PP_\mathbf{0}, PP_\mathbf{h}$ and the modified policy $PP_{\mathbf{h}R}$ using this performance measure $EC(T) - R$, which may be thought of as the disutility felt by each traveller.

In Fig. 4 we suppose that $s_1 < T \le s_2$. The solid lines show how the equilibrium value of $EC(T) - R$ varies with T for all three policies. For the original pricing policies $PP_\mathbf{0}$ and $PP_\mathbf{h}$, $EC(T) - R$ stays constant as T varies (and $R = 0$ as prices are not returned to travellers under these policies). These results are proved in Smith et al. (2019a).

For the modified policy, if $T > s_1$ but only slightly greater than s_1, since we return just sufficient to make the policy revenue neutral and almost all travellers use the lower route and pay a toll = Δ secs, nearly $T\Delta$ seconds is returned and so nearly Δ seconds is returned to each traveller. So $EC(T) - R$ must be very close to zero if T is just slightly larger than s_1, as illustrated in Fig. 4.

On the other hand if T is very close to, but less than, s_2 very few travellers use the lower short route and pay; thus in this case there is very little returning, R is close to 0 and $EC(T) - R$ must be very close to Δ, as illustrated in Fig. 4. (In this latter case almost all travellers are on the long upper route at equilibrium.)

The return of prices must not affect route choices, which is why we have returned the same R seconds to all travellers whether they used the longer or the shorter route.

Returning all the prices paid in this way makes the new policy $PP_{\mathbf{h}R}$ less expensive for travellers; which should enhance the attractiveness of the pricing/control policy. Returning all the prices paid makes $PP_{\mathbf{h}R}$ revenue neutral, of course.

Fig. 4 illustrates a wide performance disparity between the different policies.

10.3. Utility maximisation

For this network, define the utility of an individual traveller by:

$$\text{traveller utility} = U = R - EC(T).$$

Then plainly, the value of U depends on the policy adopted and from the graphs in Fig. 3 it is plain that, for every T satisfying $s_1 < T < s_2$,

$$U[PP_{\mathbf{0}}] < U[PP_{\mathbf{h}}] < U[PP_{\mathbf{h}R}].$$

Thus, of the three policies considered, $PP_{\mathbf{h}R}$ yields the maximum traveller utility for this network. A little thought shows that no other pricing and green-time policy, with non-negative total revenue, can yield a higher traveller utility and so $PP_{\mathbf{h}R}$ is utility maximising *for this particular network and this particular definition of utility*.

For this network only, this result is easily generalised to cover elastic demand, where T depends on the cost of travel; by adding the graph of an inverse demand function to Fig. 4. The standard utility is then maximised *for this particular network* by control policy $PP_{\mathbf{h}R}$.

If returning prices paid is not permitted then $PP_{\mathbf{h}}$ yields, for this network, the maximum traveller utility of the two no-return policies; since the price on the longer route is zero with this policy and so the total of the prices paid is least. Again this result may be generalised to include a standard elastic demand: for this network, if there is a standard elastic demand then the standard utility is maximised by policy $PP_{\mathbf{h}}$.

10.4. Contribution

This section has compared the equilibrium performances of several green-time-pricing policies $PP_{\mathbf{h}}$ and a policy $PP_{\mathbf{h}R}$ which returns all of the prices paid, on the simple network. By Theorem 3, all the $PP_{\mathbf{h}}$ policies are capacity-maximising; however there is a wide disparity in the results in this section. For example, if $\mathbf{h} = [s_1C_1, s_1C_2]$ then the policy $PP_{\mathbf{h}R}$ gives typically much greater utility than some of the other $PP_{\mathbf{h}}$ policies, where prices are not returned and $\mathbf{h} \neq [s_1C_1, s_1C_2]$.

11. A zero queue window-constrained dynamic control model with green-times and prices

In this section we extend one of the previous steady state results to certain dynamic networks, using similar methods including variational inequalities as central elements. We show that P_0-with-prices maximises the capacity of a dynamic network with zero queues.

11.1. Introduction and notation

The previous green-time/pricing results may be generalised in various different ways to design controls for dynamic networks. There are very many possibilities and here we consider one dynamic development which is surely one of the most simple; in part because it involves finite dimensional Euclidean space and avoids functional analysis. The model is a time-slice model.

We suppose given a network, a positive δ and M time-slices of duration δ seconds:

$$(0\delta, 1\delta], (1\delta, 2\delta], \dots, ((m-1)\delta, m\delta], \dots, ((M-1)\delta, M\delta]. \tag{11}$$

In the model below all journeys must be started and completed within the overall time window $(0, T] = (0, \delta M]$.

The overall time window $(0, T] = (0, \delta M]$ might be the whole day and might be a time period in which motor vehicles are permitted to service a large City Centre. A third example might be where a major event has specified latest times of arrival, say 10 min before kick-off, and travellers departure times from home are also restricted. Future extensions of the setup here might include commuters with fixed windows in which to travel to work; with both their departure times and their arrival times subject to hard constraints; and a further extension might then include soft constraints or targets with penalties for early and late arrival at work.

Suppose that the given network has N links, a given set of K^2 signal stages or phases and a given set of K^1 routes; each route is a sequence of contiguous links and for simplicity we suppose here that each route in this route set traverses no link twice. The route-inflow rate to a single route r during time-slice m is constant over the time-slice and $= X_{rm}$. The green-time proportion awarded to stage k during time-slice m is constant over the time-slice and $= G_{km}$.

A complete list of the notation in this section is as follows.*Note: In the expressions below for the link travel times $c_i\delta$, the partial route travel times $C_{ir}\delta$ and the whole route travel times $C_r\delta$; c_i, C_{ir} and C_r are all fixed positive integers.*

$s_{im} =$ the saturation flow at the link i exit during time-slice m (in vehicles per second);
$\mathbf{s} =$ $[s_{im}]$ = the NM vector with co-ordinates s_{im};
$c_i\delta =$ the freeflow time of travel via link i (in seconds; each c_i *is a constant positive integer*);
$C_{ir}\delta =$ the freeflow time of travel from the entry point of route r to the exit of link i (seconds);
$C_r\delta =$ the freeflow time of travel from the entry point of route r to the exit of route r (seconds);
$x_{im} =$ the out-flow rate of vehicles from link i during time interval m (in vehicles per second);
$\mathbf{x} =$ $[x_{im}]$ = the NM vector with co-ordinates x_{im};
$X_{rm} =$ the in-flow rate of vehicles to route r during time-slice m (in vehicles per second);
$\mathbf{X} =$ $[X_{rm}]$ = the K^1M vector with co-ordinates X_{rm};
$g_{im} =$ proportion of time that link i is green during time interval m (dimensionless);
$\mathbf{g} =$ $[g_{im}]$ = the NM-vector with co-ordinates g_{im};
$G_{um} =$ the proportion of time stage u is green in time-slice m (dimensionless)
$\mathbf{G} =$ $[G_{um}]$ = the K^2M-vector with co-ordinates G_{um};
$p_{im} =$ price paid at the link i exit during time interval m; and
$d_{a_i,a_j,m} =$outflow rate from node a_i during time-slice m heading toward node a_j.

Here c_i is a constant positive integer, C_{ir} is the sum of the c_j over all links j on route r up to and including link i, and C_r is the sum of all the c_j over all links j in route r.

11.2. Zero queue time-constrained demand in link flow form

We suppose given a demand $d_{a_i,a_j,m}$ for travel from each node a_i to each node a_j which sets out during time-slice m. These given demands give rise to an initial set of route-inflow vectors as follows:

$$D^{++} = \{\mathbf{X} \geq \mathbf{0}; \sum_{r \text{ joins node } a_i \text{ to node } a_j} X_{rm} = d_{a_i,a_j,m}\} \subset R^{K^1 M}. \tag{12}$$

Consider X_{rm}, a particular constant route-inflow rate over time-slice $m = (\delta(m-1), \delta m]$. For each link i in route r; the constant route-inflow rate X_{rm} over time-slice m gives rise to a constant contribution to certain link i outflow rates during certain future time-slices. These contributions are here calculated by assuming there is no queueing; and, further, that the link j traversal time is always given by the constant δc_j (where c_j is a positive integer). Thus, for example, if route r comprises the 4 links numbered $1, 2, 3, 4$, the contributions of the time-slice 1 inflow rate X_{r1} to the outflow rates from these links 1, 2, 3, 4 during certain subsequent time-slices are as follows:

the contribution of inflow rate X_{r1} to the link 1 outflow rate $x_{1(c_1+1)} = X_{r1}$,
the contribution of inflow rate X_{r1} to the link 2 outflow rate $x_{2(c_1+c_2+1)} = X_{r1}$,
the contribution of inflow rate X_{r1} to the link 3 outflow rate $x_{3(c_1+c_2+c_3+1)} = X_{r1}$,
the contribution of inflow rate X_{r1} to the link 4 outflow rate $x_{4(c_1+c_2+c_3+c_4+1)} = X_{r1}$.

The route-r-time-slice-1 inflow rate, or the $r1$ inflow rate, X_{r1} enters link 1 during time-slice 1. The first line here says that the leading edge of X_{r1} reaches the exit of link 1 after c_1 time-slices of duration δ, and then all of the length and volume of the route-inflow rate X_{r1} exits link 1 in the next time-slice $(\delta c_1, \delta(c_1+1)]$ of duration δ. This is time-slice $c_1 + 1$. Similarly the contributions of the route-r-timeslice-1 inflow rate X_{r1} to the exit flows from links 2, 3 and 4 during the time-slices $c_1 + c_2 + 1$, $c_1 + c_2 + c_3 + 1$ and $c_1 + c_2 + c_3 + c_4 + 1$ are also X_{r1} as shown above. For any other [link i / time-slice m] the contribution of X_{r1} to outflow rate $x_{im} = 0$. Thus (the contribution of X_{r1} to outflow rate x_{im}) = 0 unless

$$(i, m) \in \{(1, c_1+1), (2, c_1+c_2+1), (3, c_1+c_2+c_3+1), (4, c_1+c_2+c_3+c_4+1)\}. \tag{13}$$

In this case (where route r comprises links $1, 2, 3, 4$), since link 4 is the last link in route r, the time taken for the whole of the inflow X_{r1} to leave the whole route also $= (c_1 + c_2 + c_3 + c_4 + 1)\delta$. The leading edge of X_{r1} takes $\delta(c_1 + c_2 + c_3 + c_4)$ seconds to reach the end of the route and then this $r1$ entry flow takes a further δ seconds to leave the route. We now agree that, here, with our four link route r and just looking at the $r1$ inflow (the route-r inflow rate during time-slice 1),

the time at which the destination is reached is the time at which all the $r1$ inflow has exited the route r.

Here this time at which the destination is reached = the time for all inflow X_{r1} to leave the route $r = (c_1 + c_2 + c_3 + c_4 + 1)\delta$.

The above detailed analysis applies also to the route-r-timeslice-m inflow X_{rm} where $m > 1$. In this case the time to reach the destination is still $(c_1 + c_2 + c_3 + c_4 + 1)\delta$ and so the time at which the destination is reached is now $(m - 1 + c_1 + c_2 + c_3 + c_4 + 1)\delta$. This answer agrees with that given above when $m = 1$.

The analysis also applies to a general route r with $n(r)$ links and with link traversal times

$$\{c_{i1}\delta, c_{i2}\delta, \dots, c_{in(r)}\delta\}, \tag{14}$$

entered during time-slice m. In this case the time to reach the destination is

$$(c_{i1} + c_{i2} + c_{i3} + \dots, +c_{in(r)} + 1)\delta \tag{15}$$

and so the time at which the destination is reached is now

$$(m - 1 + c_{i1} + c_{i2} + c_{i3} + \dots + c_{in(r)} + 1)\delta.$$

(This answer agrees with that given above when $m = 1$ and $n(r) = 4$.) For this route r, define the route r traversal time to be $C_r\delta$ where:

$$C_r = c_{i1} + c_{i2} + c_{i3} + \dots + c_{in(r)}. \tag{16}$$

Then the rm inflow X_{rm} reaches the destination at time

$$(m - 1 + c_{i1} + c_{i2} + c_{i3} + \dots + c_{in(r)} + 1)\delta = (m + C_r)\delta.$$

We now impose a time-window constraint on the set of route-inflow vectors. If the (route r, time-slice m) combination is such that the whole rm route-inflow X_{rm} reaches the destination *after time* $T = M\delta$ then in this case we impose the condition that $X_{rm} = 0$. Thus this route-inflow rm is in this case here essentially prohibited. We now define demand sets which explicitly ensure that such late-arriving route r time-slice m inflows are zero. Let D^+ be the subset of D^{++} containing all route inflow vectors $\mathbf{X} = [X_{rm}] \in D^{++}$ which also satisfy:

$$X_{rm} = 0 \text{ whenever } (m + C_r)\delta > T. \tag{17}$$

For each single route-inflow K^1M-vector $\mathbf{X} \in D^+$, corresponding link i outflow $x_{im}(\mathbf{X})$ in time-slice $m \le M$ is then obtained by addition over all the co-ordinates $X_{rm'}$ of $\mathbf{X}$ for those rm' inflows which exit link i in time-slice m using the equation:

$$x_{im}(\mathbf{X}) = \sum_{\{rm';1\le r\le K^1,\text{ route } r \text{ traverses link } i \text{ and } m'=m-C_{ir}\}} X_{rm'}. \tag{18}$$

If $\mathbf{X} \in D$ and x_{im} is given by (18), then $x_{im} = 0$ if $m > M$. Because, for all routes r passing through link i,

$$m > M \text{ implies } m' = m - C_{ir} > M - C_{ir} \text{ implies } m' + C_r \ge m' + C_{ir} > M \text{ implies } (m' + C_r)\delta > M\delta = T,$$

which ensures, by the definition of D^+ above, that $X_{rm'} = 0$. Thus the link-outflow created by (18) contains no positive late-arriving elements x_{im}, or X_{rm}, and so no part of the X_{rm} inflow fails to reach its destination by time $T = M\delta$.

These co-ordinates $x_{im}(\mathbf{X})$ defined by (18) fit together to create the link outflow NM-vector $\mathbf{x}(\mathbf{X}) = [x_{im}(\mathbf{X})]$ generated by $\mathbf{X}$. Note that $\mathbf{x(X)}$ is a linear function of the K^1M-vector $\mathbf{X}$.

We may now define the demand set D, in a link outflow form, by putting:

$$D = \{\mathbf{x}(\mathbf{X}); \mathbf{X} \in D^+\}.$$

D is the set of demand-feasible link outflow vectors $\mathbf{x}$, with co-ordinates $x_{im} \ge 0$, meeting the given inelastic origin–destination demands $d_{a_i,a_j,m}$, *within the time interval* $(0,T] = (0, M\delta]$, *with zero queues*.

As shown above, the co-ordinates $x_{im}(\mathbf{X})$ specified in (18) are automatically zero if $m > M$, by the definition of D^+, and so all inflows reach their destinations by time T.

11.3. Green-time constraints and red-time constraints

Suppose also given a set F of feasible link green-time vectors $\mathbf{g}$ with co-ordinates $g_{im} \ge 0$ (arising from stage green-times at each junction which add to 1 for each time-slice m and are non-negative). Given F we define the set F_{red} of feasible red-time vectors $\mathbf{r}$ by the equation:

$$F_{red} = \{\mathbf{r} = [r_{im}]; r_{im} = 1 - g_{im} \text{ where } \mathbf{g} \in F\}.$$

11.4. All the link variables

Suppose that prices $p_{im} \ge 0$ may be charged for exiting link i during time-slice m. For any link outflow rates x_{im}, saturation flows s_{im}, prices p_{im} and green-times g_{im} (for $1 \le i \le N$ and $1 \le m \le M$), define the NM−vectors $\mathbf{x}, \mathbf{s}, \mathbf{p}, \mathbf{g}$ as follows:

$$\mathbf{x} = [x_{im}] = [x_{1,1}, x_{1,2}, \dots, x_{1,M-1}, x_{1,M}, \dots, x_{i,1}, x_{i,2}, \dots, x_{i,M-1}, x_{i,M}, \dots, x_{N,1}, x_{N,2}, \dots, x_{N,M-1}, x_{NM}],$$
$$\mathbf{s} = [s_{im}] = [s_{1,1}, s_{1,2}, \dots, s_{1,M-1}, s_{1,M}, \dots, s_{i,1}, s_{i,2}, \dots, s_{i,M-1}, s_{i,M}, \dots, s_{N,1}, s_{N,2}, \dots, s_{N,M-1}, s_{NM}],$$
$$\mathbf{p} = [p_{im}] = [p_{1,1}, p_{1,2}, \dots, p_{1,M-1}, p_{1,M}, \dots, p_{i,1}, p_{i,2}, \dots, p_{i,M-1}, p_{i,M}, \dots, p_{N,1}, p_{N,2}, \dots, p_{N,M-1}, p_{NM}],$$

and

$$\mathbf{g} = \left[g_{im}\right] = \left[g_{1,1}, g_{1,2}, \cdots, g_{1,M-1}, g_{1,M}, \cdots, g_{i,1}, g_{i,2}, \cdots, g_{i,M-1}, g_{i,M}, \cdots, g_{N,1}, g_{N,2}, \cdots, g_{N,M-1}, s_{NM}\right].$$

(Here: $x_{i,m} = x_{im}$, $s_{i,m} = s_{im}$, $p_{i,m} = p_{im}$, and $g_{i,m} = g_{im}$.)

Also define the Hadamard or Schur products of $\mathbf{s} = \left[s_{im}\right]$ and $\mathbf{p} = \left[s_{im}\right]$ by

$$\mathbf{s}\circ\mathbf{p} = \left[s_{im}\right]\circ\left[p_{im}\right] = \left[s_{im}p_{im}\right].$$

(Multiplication here is co-ordinate-wise.) Similarly let

$$\mathbf{s}\circ\mathbf{g} = \left[s_{im}\right]\circ\left[g_{im}\right] = \left[s_{im}g_{im}\right].$$

Given the above variables x_{im}, s_{im}, g_{im} define the set S of supply feasible (link outflow vector, link green-time vector) $\left[\mathbf{x}, \mathbf{g}\right]$ pairs; for which each link flow x_{im} is no greater than the saturation flow s_{im} multiplied by the link green-time proportion g_{im}, as follows:

$$S = \{(\mathbf{x}, \mathbf{g}); x_{im} - s_{im}g_{im} \leq 0 \text{ for all } i, m \text{ such that } 1 \leq i \leq N \text{ and } 1 \leq m \leq M\}. \tag{19}$$

NOTE: S has no non-negativity constraints such as $x_{im} \geq 0, g_{im} \geq 0$.

11.5. Zero queue window-constrained wardrop equilibrium with prices

Definition 12. We here agree that $\mathbf{x}^*$ is a dynamic window-constrained Wardrop equilibrium with price vector $\mathbf{p}$ if

$$\mathbf{x}^* \in D \text{ and } -(\delta\mathbf{c} + \mathbf{p}) \text{ is normal at } \mathbf{x}^* \text{ to } D. \tag{20}$$

Condition (20) is a Variational Inequality or a VI, as illustrated in Fig. 2 with

$$\mathbf{v} = -(\delta\mathbf{c} + \mathbf{p}).$$

This definition is equivalent to "at each time-slice m, if $(m + C_r)\delta > T$ then $X_{rm} = 0$ and more costly permitted route-inflows are zero". This is an extension of the usual Wardrop equilibrium condition "more costly routes are not used" to take account of the time window here.

11.6. Capacity-maximising policies in a dynamic framework with time-windows and zero queues

Suppose given a network as above, with N links and M time slices of duration δ seconds where $\delta > 0$. Suppose also given a control policy P determining a link green-time vector $\mathbf{g}$ in terms of the vector $\mathbf{p}$ of prices charged to exit the approaches at each junction. In this time-window setting we have the following definition.

Definition 13. The control policy P with prices "*maximises the dynamic network capacity with zero queues*" will mean the following. *If an inelastic dynamic demand D is such that there is a link out-flow vector $\mathbf{x}\in D$ and a link green-time vector $\mathbf{g} \in F$ such that $(\mathbf{x}, \mathbf{g}) \in S$ (with zero queues)*; then there is a link out-flow vector $\mathbf{x}^* \in D$, a link-exit price vector $\mathbf{p}^* \geq 0$, and a link green-time vector $\mathbf{g}^* \in \mathbf{F}$ such that $(\mathbf{x}^*, \mathbf{g}^*) \in S$ and also:

(a) $\mathbf{x}^*$ is a dynamic window-constrained Wardrop equilibrium with prices when the link-exit price vector is $\mathbf{p}^*$ and
(b) $\mathbf{g}^*$ satisfies the control policy P when the link-exit price vector is $\mathbf{p}^*$.

11.7. The P_0-with-prices control policy, in link form in a dynamic context, as a solution to a VI

Definition 14. Given $\mathbf{s}$, the P_0-with-prices control policy here is satisfied at $(\mathbf{g}, \mathbf{p})$ if and only if:

$$\mathbf{g} \in F \text{ and } \mathbf{s}\circ\mathbf{p} = \left[s_{im}p_{im}\right] \text{ is normal at } \mathbf{g} \text{ to } F. \tag{21}$$

Compare with Definition 10 of P_0-with-prices in the steady state.

Theorem 4. *The P_0-with-pricing control policy is capacity-maximising with zero queues and a time-window for the dynamic network with time-slices described above.*

Proof. Suppose that

$$(D \times F) \cap S \text{ is non-empty.}$$

To prove the theorem we need to show that, in the model described above and under this condition, there is an equilibrium (with prices) consistent with P_0-with-prices.

To do this, for **(x, g)** $\in (D \times F) \cap S$, let

$$Z(\mathbf{x}, \mathbf{g}) = \sum_{1\leq i\leq N \text{ and } 1\leq m\leq M} \delta c_i x_{im}.$$

Consider the following Variational Inequality Problem.

VI problem 3*:*

Find $(\mathbf{x}^*, \mathbf{g}^*) \in (D \times F) \cap S$ such that

$-(\delta\mathbf{c}, \mathbf{0}) = -grad\, Z(\mathbf{x}^*, \mathbf{g}^*)$ is normal at $(\mathbf{x}^*, \mathbf{g}^*)$ to $(D \times F) \cap S$.

VI problem 3 has a solution since Z is continuous and $(D \times F) \cap S$ is non-empty and compact and so Z has a minimum over the set $(D \times F) \cap S$. Let $(\mathbf{x}^*, \mathbf{g}^*)$ solve variational inequality problem 3. Then

$-(\delta\mathbf{c}, \mathbf{0})$ $(= -grad\, Z(\mathbf{x}^*, \mathbf{g}^*))$ is normal at $(\mathbf{x}^*, \mathbf{g}^*)$ to $(D \times F) \cap S$.

So

$$-(\delta\mathbf{c}, \mathbf{0}) = (\mathbf{n}_D, \mathbf{n}_F) + \mathbf{n}_S \tag{22}$$

where

$\mathbf{n}_D$ is normal at $\mathbf{x}^*$ to D, $\mathbf{n}_F$ is normal at $\mathbf{g}^*$ to F and $\mathbf{n}_S$ is normal at $(\mathbf{x}^*, \mathbf{g}^*)$ to S.

Now, by (19),

$$S = S_{1,1} \times S_{1,2} \times \cdots \times S_{1,M-1} \times S_{1,M} \times S_{2,1} \times S_{2,2} \times \cdots S_{2,M} \times \cdots \times S_{N,1} \times S_{N,2} \times \cdots \times S_{N,M-1} \times S_{N,M}$$

where $S_{i,m} = \{(x_{im}, g_{im}); x_{im} - s_{im}g_{im} \le 0\}$. We will often now put $S_{i,m} = S_{im}$.

Any normal to S_{im} is of the form $p_{im}(1, -s_{im}) = (p_{im}, -s_{im}p_{im})$, where $p_{im} \ge 0$, and so any normal to S is of the form

$$\mathbf{n}_S = (\mathbf{p}, -\mathbf{s}\circ\mathbf{p})$$

for some vector $\mathbf{p}$ where all the co-ordinates $p_{im} \ge 0$. Therefore, using (22),

$$-(\delta\mathbf{c}, \mathbf{0}) = (\mathbf{n}_D, \mathbf{n}_F) + \mathbf{n}_S = (\mathbf{n}_D, \mathbf{n}_F) + (\mathbf{p}, -\mathbf{s}\circ\mathbf{p}). \tag{23}$$

Subtracting $\mathbf{n}_S = (\mathbf{p}, -\mathbf{s}\circ\mathbf{p})$ from both sides of (23) yields:

$(-(\delta\mathbf{c} + \mathbf{p}), \mathbf{s}\circ\mathbf{p}) = (\mathbf{n}_D, \mathbf{n}_F)$ is normal at $(\mathbf{x}^*, \mathbf{g}^*)$ to $D \times F$,

and so

$$-(\delta\mathbf{c} + \mathbf{p}) = \mathbf{n}_D \text{ is normal at } \mathbf{x}^* \text{ to } D \tag{24}$$

and

$$\mathbf{s}\circ\mathbf{p} = \mathbf{n}_F \text{ is normal at } \mathbf{g}^* \text{ to } F. \tag{25}$$

Thus:

by (24), $\mathbf{x}^*$ is a window-constrained dynamic Wardrop equilibrium with price vector $\mathbf{p}$
as defined in the definition (12),
and,
by (25), $\mathbf{g}^*$ satisfies policy P_0-with-prices when the price vector is $\mathbf{p}$
as defined in definition (14).

We have shown that policy P_0-with-prices is capacity maximising with zero queues in this dynamic context with time windows, as defined in definition (13). We have proved Theorem 4.

11.8. Contribution

In this section we have extended one previous steady state capacity-maximising result to certain dynamic networks, using similar methods including variational inequalities as central elements. Further extensions may be easily stated and proved.

12. The dynamic control policy $P_0^{\mathbf{f}}$, a P_0-with-pricing policy: a stability result

The previous sections do not consider stability. Now we consider stability in one particular form: stability with respect to route choices when the responsive control/pricing system outlined below is followed.

12.1. The model

Throughout this section we use the dynamic continuum model of traffic signal control and pricing in the previous Section 11, *including the notation*. But here the control policy is different. To obtain this new control policy we extend Section 11 model a little, adding a price-flow function to determine prices in terms of link out-flows and red-times. The effect of this is, as shown below, to ensure that link-exit capacities are not approached too closely at or near equilibrium, and that there is some stability.

As in Section 11 we here suppose given a network, a fixed positive δ and M time-slices of duration δ seconds:

$$(0\delta, 1\delta], (1\delta, 2\delta], \ldots, ((m-1)\delta, m\delta], \ldots, ((M-1)\delta, M\delta].$$

Also as in Section 11, in the model below all route-inflow rates, all link outflow rates, all link green-time proportions and all link red-time proportions are constant over each time interval $((m-1)\delta, m\delta]$. As in Section 10, all journeys must be started and completed within the overall time window $(0, T] = (0, \delta M]$.

Using the notation of Section 11, given a signal stage u at a junction, *antistage* u is here defined to be the set of all links, terminating at the same junction, NOT in stage u. In all timeslices m the (constant) red-time proportion allocated to antistage u equals the proportion of green-time allocated to stage u. (Anti-stage red-times are utilised in Smith et al. (2015).) Link red times are sums of antistage red times, as link green times are sums of stage green times.

Now in this section we modify the definition of the supply-feasible set of link outflow, green-time pairs in (19) to the following definition of the supply-feasible set of link outflow, red-time pairs:

$$S_{red} = \{(\mathbf{x}, \mathbf{r}); x_{im} + s_{im} r_{im} < s_{im} \text{ for all } i, m \text{ such that } 1 \le i \le N \text{ and } 1 \le m \le M\}. \tag{26}$$

Of course (19) and (26) are consistent since $r_{im} = 1 - g_{im}$ for all relevant im;*but note that* (26) *has a strict inequality and so* S_{red} *does not include its boundary.* As before, S_{red}, as given in Eq. (26), has no non-negativity constraints, such as $x_{im} \ge 0$.

12.2. Determining link-exit prices on day k in the modified P_0 with prices policy

Let $(\mathbf{X}, \mathbf{R})$ be the route-inflow, antistage red-time vector on day k, giving rise to link-exit flows x_{im} and link red-times r_{im} on day k.

First we now insist that there is given a positive number $a < 1$ and that for each link i and time-slice m there is given a non-decreasing continuous function f_{im}, defined for all u such that $0 \le u < s_{im}$, taking the value 0 on $[0, as_{im}]$, strictly increasing over the set $[as_{im}, s_{im}) = \{u; as_{im} \le u < s_{im}\}$ and satisfying:

$$\int_0^{s_{im}} f_{im}(u)du = +\infty. \tag{27}$$

Such a function may be thought of as a barrier function; and a graph of such a function is given in Fig. 5. One example of such a function is given in Eq. (28).

$$f_{im}(u) = \begin{cases} 0 & \text{if } \; 0 \le u < as_{im} \\ \dfrac{1}{1 - u/s_{im}} - \dfrac{1}{1-a} & \text{if } \; as_{im} \le u < s_{im}. \end{cases} \tag{28}$$

Now let $\mathbf{f}$ have co-ordinate functions f_{im} satisfying (27). Suppose that on day k:

$$(D \times F_{red}) \cap S_{red} \neq \phi$$

and

$$(\mathbf{x}, \mathbf{r}) \in (D \times F_{red}) \cap S_{red}.$$

Here S_{red} is given by (26), D is defined in Section 11.2, and F_{red} is defined in Section 11.3. Given the feasible $(\mathbf{x}, \mathbf{r})$ on day k, determine the day k link-exit price vector $\mathbf{p}$ by putting

$$p_{im} = f_{im}(x_{im} + s_{im} r_{im}) \tag{29}$$

for all im.

12.3. Determining antistage red-time costs RC_{wm} and route-entry costs C_{rm} on day k

We are here supposing that on day k:

$$(\mathbf{x}, \mathbf{r}) \in (D \times F_{red}) \cap S_{red}.$$

For each (anti-stage, time-slice) pair wm define the antistage w red-time cost RC_{wm}, in link flow terms, as follows:

$$RC_{wm}(\mathbf{x}, \mathbf{r}) = \sum_{\{i;\, i \in \text{anti-stage } w\}} s_{im} p_{im} = \sum_{\{i;\, i \in \text{anti-stage } w\}} s_{im} f_{im}(x_{im} + s_{im} r_{im}) \tag{30}$$

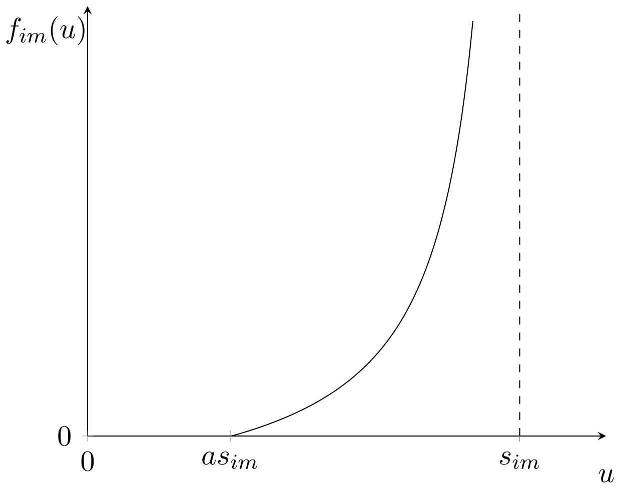

Fig. 5. A barrier function for link exit i during time-slice m.

using (29). The anti-stage w red-time cost at time-slice m is the sum of the terms $s_{im} f_{im}(x_{im} + s_{im} r_{im})$ over all links in anti-stage w. Here the antistage wm red-time feels a redtime cost given by (30) (and (31) below) on day k.

To write Eq. (30) in terms of the route-inflow vector $\mathbf{X}$ and anti-stage red-time vector $\mathbf{R}$ on day k, let matrices $\mathbf{A}$ and $\mathbf{B}$ be defined as follows.

$$A_{rm,im'} = 1 \text{ if route entry flow } X_{rm} \text{ exits link } i \text{ during time-slice } m',$$
$$A_{rm,im'} = 0 \text{ otherwise},$$
$$B_{wm,im'} = 1 \text{ if antistage } w \text{ contains link } i \text{ and } m' = m, \text{ and}$$
$$B_{wm,im'} = 0 \text{ otherwise}.$$

For each (anti-stage, time-slice) pair wm define the antistage w red-time cost RC^{+}_{wm} in terms of route inflows and antistage red times, instead of link outflows and link red green-times, as follows:

$$RC^{+}_{wm}(\mathbf{X}, \mathbf{R}) = RC_{wm}(\mathbf{AX}, \mathbf{BR}). \tag{31}$$

where RC_{wm} is given by (30). Again, the anti-stage w red-time cost at time-slice m is the sum of the terms $s_{im} f_{im}(x_{im} + s_{im} r_{im})$ over all links in anti-stage w and the antistage wm red-time feels a redtime cost given by (30) and (31) on day k.

Now we turn to route costs. For each (route, time-slice) pair rm define the rm route-cost C^{+}_{rm} by putting:

$$C^{+}_{rm} = C^{+}_{rm}(\mathbf{X}, \mathbf{R}) = C_r \delta + \sum_{\{im'; X_{rm} \text{ exits link } i \text{ during time-slice } m'\}} f_{im'}(x_{im'} + s_{im'} r_{im'}) \tag{32}$$

where $\mathbf{x} = \mathbf{AX}$ and $\mathbf{r} = \mathbf{BR}$. Travellers on route rm feel a route cost C^{+}_{rm}, given by (32), on day k.

Putting all route costs and anti-stage red-time costs together, we define $\mathbf{C}^{+}(\mathbf{X}, \mathbf{R})$ and $\mathbf{RC}^{+}(\mathbf{X}, \mathbf{R})$, using Eqs. (31) and (32), by putting:

$$\mathbf{C}^{+}(\mathbf{X}, \mathbf{R}) = \text{the vector with } K^1 M \text{ components } C^{+}_{rm}(\mathbf{X}, \mathbf{R}) \text{ and} \tag{33}$$

$$\mathbf{RC}^{+}(\mathbf{X}, \mathbf{R}) = \text{the vector with } K^2 M \text{ components } RC^{+}_{wm}(\mathbf{X}, \mathbf{R}). \tag{34}$$

12.4. Swapping permitted route-inflows yielding permitted route-inflows on day $k+1$ and swapping antistage red-times yielding anti-stage red-times on day $k+1$

In this paper we suppose that the route-entry flows on day $k+1$ result from route-entry flows X_{rm} on day k by natural route-entry flow swapping *toward less costly permitted routes only*. This re-routeing rule is designed as a simple model of rational day to day behaviour. Of course, route-inflow swaps from a particular route can only occur toward a less costly permitted route *joining the same OD pair*.

Also, under control policy $P_0^{\mathbf{f}}$, we suppose that the anti-stage red-times on day $k+1$ are to be obtained from anti-stage red-times R_{wm} on day k by natural anti-stage red-time swapping *toward less costly antistages only*. Of course, antistage red-time swaps from a particular antistage can only occur toward a less costly antistage *at the same junction*.

We further suppose that these swaps toward less-costly permitted routes and anti-stages, from day k to day $k+1$, depend only on the day k (route-inflow vector, antistage red-time vector) $(\mathbf{X}, \mathbf{R})$. Let the total effect of all these swaps be to change $(\mathbf{X}, \mathbf{R})$ on day

k to $(\mathbf{X},\mathbf{R})+(\Delta_1(\mathbf{X},\mathbf{R}),\Delta_2(\mathbf{X},\mathbf{R}))$ on day $k+1$. Here, for any $(\mathbf{X},\mathbf{R})$ on day k, $\Delta_1(\mathbf{X},\mathbf{R})$ only moves $\mathbf{X}$ in D^+ and $\Delta_2(\mathbf{X},\mathbf{R})$ only moves $\mathbf{R}$ in F^+_{red}.

The total change vector $(\Delta_1(\mathbf{X},\mathbf{R}),\Delta_2(\mathbf{X},\mathbf{R}))$ is to be a sum of swap vectors satisfying certain conditions including those described above. The main condition has already been stated and is that all swaps must be toward less costly permitted routes and antistages. This condition may be expressed more precisely as follows:

For each time-slice m, each OD pair od and each route r joining od, if $X_{rm}>0$ and there is a less costly permitted route joining od, then on day $k+1$,

$$\text{some of the route entry flow } X_{rm} \text{ swaps but only to less costly permitted routes joining } od, \tag{35}$$

and, also, for each time-slice m, each junction J and each antistage w at that junction, if $R_{wm}>0$ and there is a less costly antistage at the junction J, then on day $k+1$,

$$\text{some of the antistage red-time } R_{wm} \text{ swaps but only to less costly antistages at junction } J. \tag{36}$$

Also if no swaps according to (35) are possible (perhaps because on day k, for each OD pair and each within-day time m, all permitted routes have the same route cost), then the route-inflow vector $\mathbf{X}$ remains the same on day $k+1$ *and* if no swaps according to (36) are possible at any junction (perhaps because on day k, at all junctions and at all within-day time-slices m, all antistages have the same antistage red-time cost), then $P_0^{\mathbf{f}}$ insists that the anti-stage red-time vector $\mathbf{R}$ remains the same on day $k+1$.

(35) and (36) imply that $(\Delta_1(\mathbf{X},\mathbf{R}),\Delta_2(\mathbf{X},\mathbf{R}))$ satisfies:

$$(\Delta_1(\mathbf{X},\mathbf{R}))\cdot[\mathbf{C}(\mathbf{X},\mathbf{R})]<0 \text{ if } \Delta_1(\mathbf{X},\mathbf{R})\neq\mathbf{0} \text{ and} \tag{37}$$

$$(\Delta_2(\mathbf{X},\mathbf{R}))\cdot[\mathbf{RC}(\mathbf{X},\mathbf{R})]<0 \text{ if } \Delta_2(\mathbf{X},\mathbf{R})\neq\mathbf{0}. \tag{38}$$

We also require that

$$(\mathbf{X},\mathbf{R})\in(D^+\times F^+_{red})\cap S^+_{red} \text{ implies } (\mathbf{X},\mathbf{R})+(\Delta_1(\mathbf{X},\mathbf{R}),\Delta_2(\mathbf{X},\mathbf{R}))\in(D^+\times F^+_{red})\cap S^+_{red}. \tag{39}$$

This condition (39) ensures that origin–destination constraints, non-negativity and capacity constraints remain satisfied as days pass. Our next vital condition on $\Delta_1(\mathbf{X},\mathbf{R})$ and $\Delta_2(\mathbf{X},\mathbf{R})$ is as follows:

$$(\Delta_1(\mathbf{X},\mathbf{R}),\Delta_2(\mathbf{X},\mathbf{R})) \text{ is a continuous function of } (\mathbf{X},\mathbf{R})\in(D^+\times F^+_{red})\cap S^+_{red}. \tag{40}$$

Below we give an example of a possible $(\Delta_1(\mathbf{X},\mathbf{R}),\Delta_2(\mathbf{X},\mathbf{R}))$.

12.4.1. Wardrop equilibria and $P_0^{\mathbf{f}}$ equilibria

Definition 15. Here, given **(X, R)** on day k, $\mathbf{X}$ is called a Wardrop equilibrium if and only if for each timeslice m and each route joining OD pair od with $X_{rm}>0$ there is no less costly permitted route joining od, using costs given by (32) and (33).

Definition 16. Here, given **(X, R)** on day k, $\mathbf{R}$ is called a $P_0^{\mathbf{f}}$ equilibrium if and only if for each time-slice m and each antistage w at junction J with $R_{wm}>0$ there is no less costly antistage at junction J, using costs given by (31) and (34).

12.4.2. An example $(\Delta_1(\mathbf{X},\mathbf{R}),\Delta_2(\mathbf{X},\mathbf{R}))$.

An example of a change vector $(\Delta_1(\mathbf{X},\mathbf{R}),\Delta_2(\mathbf{X},\mathbf{R}))$, satisfying the above conditions (37)– (40), may be obtained by supposing that route-inflow swaps away from route r increase as route cost differences increase and that red-time swaps away from antistage w increase also as red-time cost differences increase. The example follows Smith (1984) which was in a simpler no-control context. In that paper it is supposed that, at each time-slice m, the swaps away from a more costly route r are proportional to day k route inflows X_{rm}. Here we extend that principle to antistage red-time swaps.

The example is as follows:

$$\Delta_1(\mathbf{X},\mathbf{R})=\alpha^1(\mathbf{X},\mathbf{R})\sum_{\{(rm,r'm);1\le r,r'\le K^1,rm\sim r'm,1\le m\le M\}} X_{rm}[C^+_{rm}(\mathbf{X},\mathbf{R})-C^+_{r'm}(\mathbf{X},\mathbf{R})]_+\Delta^1_{rm,r'm} \text{ and} \tag{41}$$

$$\Delta_2(\mathbf{X},\mathbf{R})=\alpha^2(\mathbf{X},\mathbf{R})\sum_{\{(wm,w'm);1\le w,w'\le K^2,w\sim w',1\le m\le M\}} R_{wm}[RC^+_{wm}(\mathbf{X},\mathbf{R})-RC^+_{w'm}(\mathbf{X},\mathbf{R})]_+\Delta^2_{wm,w'm}. \tag{42}$$

where

$K^1=$ the number of routes traversing no link twice and $K^2=$ the number of stages;

$[C^+_{rm}(\mathbf{X},\mathbf{R})-C^+_{r'm}(\mathbf{X},\mathbf{R})]_+=\max\{[C^+_{rm}(\mathbf{X},\mathbf{R})-C^+_{r'm}(\mathbf{X},\mathbf{R})],0\}$;

$[RC^+_{wm}(\mathbf{X},\mathbf{R})-RC^+_{w'm}(\mathbf{X},\mathbf{R})]_+=\max\{[RC^+_{wm}(\mathbf{X},\mathbf{R})-RC^+_{w'm}(\mathbf{X},\mathbf{R})],0\}$;

RC^+_{wm} is given by (31) and C^+_{rm} is given by (32);

$rm\sim r'm$ means that route r and route r' are both permitted routes at m joining the same OD pair;

$w\sim w'$ means that antistage w and antistage w' are at the same junction;

$\Delta^1_{rm,r'm}$ = the K^1M-vector with −1 in the rm^{th} place, +1 in the $r'm^{th}$ place and zero elsewhere;

$\Delta^2_{wm,w'm}$ = the K^2M-vector with −1 in the wm^{th} place, +1 in the $w'm^{th}$ place and zero elsewhere;

$\alpha^1(\mathbf{X},\mathbf{R}) > 0$ is continuous and defined for all $(\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}$ and

$\alpha^2(\mathbf{X},\mathbf{R}) > 0$ is continuous and defined for all $(\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}$.

$\Delta^1_{rm,r'm}$ is called a swap vector (swapping inflow from permitted route r to permitted route r' in time-slice m).

$\Delta^2_{wm,w'm}$ is called a swap vector (swapping red-time from antistage w to antistage w' in time-slice m).

12.5. Stability

To consider stability we utilise Beckmann-like Lyapunov functions V, V^+.

V is defined as follows. For all $(\mathbf{x},\mathbf{r}) \geq (\mathbf{0},\mathbf{0})$ such that $(\mathbf{x},\mathbf{r}) \in S_{red}$:

$$V(\mathbf{x},\mathbf{r}) = \sum_{m=1}^{m=M}\sum_{i=1}^{i=N}\int_0^{x_{im}+s_{im}r_{im}} f_{im}(u)du + \sum_{m=1}^{m=M}\sum_{i=1}^{i=N} c_{im}x_{im}. \tag{43}$$

(43) gives rise to the following formula for V^+. For all $(\mathbf{X},\mathbf{R}) \geq (\mathbf{0},\mathbf{0})$ such that $(\mathbf{X},\mathbf{R}) \in S^+_{red}$, let

$$V^+(\mathbf{X},\mathbf{R}) = V(\mathbf{AX},\mathbf{BR}). \tag{44}$$

V is defined for all $(\mathbf{x},\mathbf{r}) \in (D \times F_{red}) \cap S_{red}$ and V^+ is defined for all $(\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}$. Clearly

$$V^+(\mathbf{X},\mathbf{R}) \geq 0 \text{ for all } (\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}.$$

Further, since V^+ is continuous and (27) holds, there is $(\mathbf{X}^*,\mathbf{R}^*) \in (D^+ \times F^+_{red}) \cap S^+_{red}$ and $V_{min} \geq 0$ such that

$$V^+(\mathbf{X}^*,\mathbf{R}^*) = V_{min} \leq V(\mathbf{X},\mathbf{R}) \text{ for all } (\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}.$$

$(\mathbf{X}^*,\mathbf{R}^*)$ is not necessarily unique.

Let

$$MIN = \{(\mathbf{X}^*,\mathbf{R}^*) \in (D^+ \times F^+_{red}) \cap S^+_{red} \text{ such that } V^+(\mathbf{X}^*,\mathbf{R}^*) = V_{min}\}.$$

Since each f_{im} is non-decreasing, V^+ is convex and so the set MIN of V^+-minimising $(\mathbf{X}^*,\mathbf{R}^*)$ is convex. Let $(\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}$, which fixes $(\Delta_1(\mathbf{X},\mathbf{R}), \Delta_2(\mathbf{X},\mathbf{R}))$. Define

$$T(\mathbf{X},\mathbf{R}) = (\mathbf{X},\mathbf{R}) + (\Delta_1(\mathbf{X},\mathbf{R}), \Delta_2(\mathbf{X},\mathbf{R})). \tag{45}$$

We assume that (37)–(40) hold. It therefore follows that T is continuous and

$$T(\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red} \text{ for all } (\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}. \tag{46}$$

Now

$$V^+[T(\mathbf{X},\mathbf{R})] = V^+[(\mathbf{X},\mathbf{R}) + (\Delta_1(\mathbf{X},\mathbf{R}), \Delta_2(\mathbf{X},\mathbf{R}))]. \tag{47}$$

for all $(\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}$. We now make the following added assumption.

$$\text{If } T(\mathbf{X},\mathbf{R}) \neq (\mathbf{X},\mathbf{R}) \text{ then } V^+[T(\mathbf{X},\mathbf{R})] = V^+[(\mathbf{X},\mathbf{R}) + (\Delta_1(\mathbf{X},\mathbf{R}), \Delta_2(\mathbf{X},\mathbf{R}))] < V^+(\mathbf{X},\mathbf{R}). \tag{48}$$

Bearing in mind (37) and (38), which together ensure that $(\Delta_1(\mathbf{X},\mathbf{R}), \Delta_2(\mathbf{X},\mathbf{R}))$ is a descent direction for V, this condition (48) is saying that the vector $(\Delta_1(\mathbf{X},\mathbf{R}), \Delta_2(\mathbf{X},\mathbf{R}))$ is a vector with small length. For any feasible start point $(\mathbf{X}_0,\mathbf{R}_0) \in (D^+ \times F^+_{red}) \cap S^+_{red}$, on day 0, we now consider the day to day dynamical system

$$(\mathbf{X}_0,\mathbf{R}_0), (\mathbf{X}_1,\mathbf{R}_1) = T(\mathbf{X}_0,\mathbf{R}_0), (\mathbf{X}_2,\mathbf{R}_2) = T(T(\mathbf{X}_0,\mathbf{R}_0)), \ldots, (\mathbf{X}_k,\mathbf{R}_k) = T^k(\mathbf{X}_0,\mathbf{R}_0), \ldots. \tag{49}$$

Condition (46) ensures that this sequence (49) does in fact go on for ever: it is an infinite sequence.

We are now able to state our stability result.

Theorem 5. *Let D^+ be the set of demand feasible route flow vectors defined in Section 11. Let F^+_{red} be the set of feasible antistage red-time vectors and let S^+ be the set of supply feasible (route-inflow vector, antistage redtime vector) pairs. Suppose that*

$$(D^+ \times F^+_{red}) \cap S^+_{red} \neq \phi.$$

Suppose that the following definitions, conditions and rules above hold: definition (26), condition (27), price specification (29), route-cost and antistage red-time cost definitions (30)–(34), swapping rules (35), (36) and direction conditions (37), (38), (39) and *(40). Suppose that V and V^+ are specified by (43)* and *(44). Suppose that T is defined by (45) and satisfies (46), (47)* and *(48). Then, under these conditions, for any start point $(\mathbf{X}_0,\mathbf{R}_0) \in (D^+ \times F^+_{red}) \cap S^+_{red}$, the sequence (49) converges to a non-empty set E^+ of equilibria consistent with control policy $P_0^{\mathbf{f}}$. Here:*

$$E^+ = \{(\mathbf{X},\mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}; \mathbf{X} \text{ is a Wardrop equilibrium and } \mathbf{R} \text{ is a } P_0^{\mathbf{f}} \text{ equilibrium.}\}$$

***Proof*.** This is given in Appendix B.

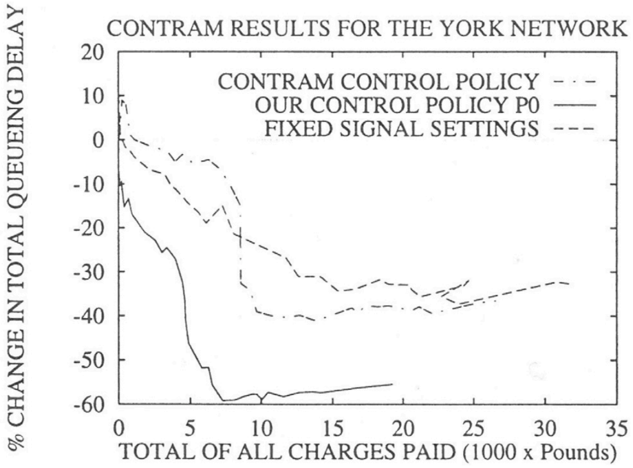

Fig. 6. CONTRAM results. Change in total queueing delay with delay-based pricing combined with three signal control policies.

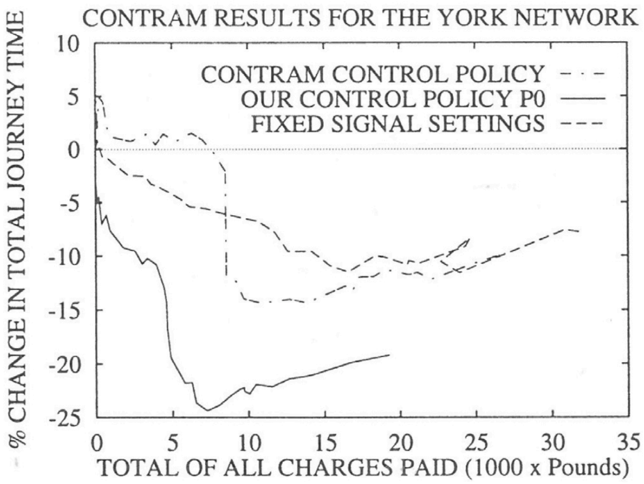

Fig. 7. CONTRAM results. Change in total travel time with delay-based pricing combined with three signal control policies.

12.6. Contribution

We have shown that a responsive price-green-time policy and natural route choices yields a stable evolutionary system.

13. Related simulation results

Smith et al. (1994b) studies the impact on York and Cambridge of different combinations of pricing and control strategies. The results in this paper show that delay-based road pricing combined with control policy P_0 is likely to be much the best, of the alternatives tested, for York.

Two simulation results are shown in Figs. 6 and 7. In both figures, delay-based pricing is utilised as the pricing system. Both figures then compare the performances of three different traffic signal control policies when these are combined with delay-based road pricing. Fig. 6 shows that control policy P_0 (combined with delay-based pricing) yields very substantial decreases in total queueing delay (of about 60 per cent) and Fig. 7 shows that total travel time is reduced by rather more than 20 per cent.

The $P0$ control policy combined with delay-based road pricing system, which gives rise to these simulation results, has been extended in this paper to a price-control policy which reduces queues to zero or to small values.

14. Conclusion

This paper has considered signal control and prices together, aiming to design a control system which maximises network capacity and also eliminates queueing (in at least a subnetwork). It has been shown that using prices (instead of delays) in the P_0 control policy specified in Smith (1980, 1987) maximises the capacity of a general steady state network, *with zero queues*.

This steady state capacity-maximisation + zero-queue result has been extended to dynamic networks, in two ways, giving an equilibrium extension and a stability extension.

The equilibrium extension shows that P_0-with-prices maximises network capacity with zero queues in a dynamic network and the stability extension shows that $P_0^{\mathbf{f}}$ is able to deliver some stability as well as zero queue capacity maximisation.

A simple example has been given to illustrate several of the policies. It is shown that a biased version, $P_{\mathbf{h}}$-with-prices, yields, *for a very simple network* higher utility than P_0-with-prices itself.

There are many open areas for research; including for example optimising access prices and controls to make sure the network is not flooded by external inputs. Combining the strategies here with perimeter control is likely to be very interesting and effective in practice (we believe). Equity is an important area which has not been addressed at all in this paper. It would be interesting to see whether the change to $P_{\mathbf{h}}$ increases utility in more general networks than the network in Section 10 here. Stability requires much more detailed study.

It is also important to seek to weaken the assumptions here, (i) to allow on-and-off signal control, perhaps by allowing the c_i to vary, (ii) to allow a spectrum of travellers, including non-repetitive, random elements and travellers with varying valuations of travel time, (iii) to consider discrete versions which consider individual travellers and the values they ascribe to different forms of mobility and (iv) to include the need to decarbonise travel networks.

The control + pricing strategies here must also be compared with more standard strategies such as cordon charging or distance charging.

Acknowledgements

We acknowledge here the substantial inputs by four referees. These inputs have substantially improved the paper. We also acknowledge inputs to Appendix C. We are also grateful to Mustapha Ghali, for producing the simulation results for York.

Appendix A. Proof of Theorem 3

Theorem 3. *The $P_{\mathbf{h}}$-with-prices control policy is capacity-maximising, with zero queueing delays, in the general single-mode network with prices in Section 5.1.*

Proof. Suppose that, in the general network in Section 5.1,

$(D \times F) \cap S$ is non-empty.

To prove the theorem we need to show that, on this assumption, there is an equilibrium with prices consistent with the control policy $P_{\mathbf{h}}$-with-prices; the definition of the $P_{\mathbf{h}}$-with-prices policy is given in Definition 11. Here we show that there exists $(\mathbf{x}^*, \mathbf{g}^*) \in (D \times F) \cap S$ and a price vector $\mathbf{p}$ such that:

$\mathbf{x}^*$ is a Wardrop equilibrium with price vector $\mathbf{p}$, as defined in definition 9, and

$\mathbf{g}^*$ satisfies policy $P_{\mathbf{h}}$-with-prices with price vector $\mathbf{p}$, as defined in definition 11.

To do this, for $(\mathbf{x}, \mathbf{g}) \in (D \times F) \cap S$, let

$$Z_{\mathbf{h}}(\mathbf{x}, \mathbf{g}) = \sum_i [c_i x_i - h_i g_i].$$

Generalise variational inequality problem 1 above to obtain variational inequality 2 below.

VI problem 2*:*

Find $(\mathbf{x}^*, \mathbf{g}^*) \in (D \times F) \cap S$ such that

$(-\mathbf{c}, \mathbf{h})$ $(= -grad\, Z_{\mathbf{h}}(\mathbf{x}^*, \mathbf{g}^*))$ is normal at $(\mathbf{x}^*, \mathbf{g}^*)$ to $(D \times F) \cap S$.

$Z_{\mathbf{h}}$ is continuous and $(D \times F) \cap S$ is non-empty and compact and so $Z_{\mathbf{h}}$ has a minimum over the set $(D \times F) \cap S$. Let $(\mathbf{x}^*, \mathbf{g}^*)$ be the point which attains the minimum of $Z_{\mathbf{h}}$ over the set $(D \times F) \cap S$. Then $(\mathbf{x}^*, \mathbf{g}^*)$ solves VI problem 2; if this were not so then there would be a feasible descent direction for $Z_{\mathbf{h}}$ at $(\mathbf{x}^*, \mathbf{g}^*)$ and $(\mathbf{x}^*, \mathbf{g}^*)$ would not be a minimum of $Z_{\mathbf{h}}$ over the set $(D \times F) \cap S$.

Since $(\mathbf{x}^*, \mathbf{g}^*)$ solves variational inequality 2,

$(-\mathbf{c}, \mathbf{h})$ $(= -grad\, Z(\mathbf{x}^*, \mathbf{g}^*))$ is normal at $(\mathbf{x}^*, \mathbf{g}^*)$ to $(D \times F) \cap S$.

So

$$(-\mathbf{c}, \mathbf{h}) = (\mathbf{n}_D, \mathbf{n}_F) + \mathbf{n}_S \tag{50}$$

where

$\mathbf{n}_D$ is normal at $\mathbf{x}^*$ to $D, \mathbf{n}_F$ is normal at $\mathbf{g}^*$ to F and $\mathbf{n}_S$ is normal at $(\mathbf{x}^*, \mathbf{g}^*)$ to S.

Now

$S = S_1 \times S_2 \times S_3 \times \cdots \times S_{n-1} \times S_n$ where $S_i = \{(x_i, g_i); x_i - s_i g_i \le 0\}$.

S_i has no non-negativity constraints such as $x_i \ge 0, g_i \ge 0$. Therefore any normal to S_i is of the form $p_i(1, -s_i) = (p_i, -s_i p_i)$, where $p_i \ge 0$, and so any normal to S is of the form

$\mathbf{n}_S = (\mathbf{p}, -\mathbf{s}\circ\mathbf{p})$

for some vector $\mathbf{p}$ where all the co-ordinates $p_i \ge 0$. Therefore, using (50),

$$(-\mathbf{c}, \mathbf{h}) = (\mathbf{n}_D, \mathbf{n}_F) + \mathbf{n}_S = (\mathbf{n}_D, \mathbf{n}_F) + (\mathbf{p}, -\mathbf{s}\circ\mathbf{p}) \quad (51)$$

Subtracting $\mathbf{n}_S = (\mathbf{p}, -\mathbf{s}\circ\mathbf{p})$ from both sides of (51) yields:

$(-(\mathbf{c} + \mathbf{p}), \mathbf{h} + \mathbf{s}\circ\mathbf{p}) = (\mathbf{n}_D, \mathbf{n}_F)$ is normal at $(\mathbf{x}^*, \mathbf{g}^*)$ to $D \times F$,

and so

$-(\mathbf{c} + \mathbf{p}) = \mathbf{n}_D$ is normal at $\mathbf{x}^*$ to D (52)

and

$\mathbf{h} + \mathbf{s}\circ\mathbf{p} = \mathbf{n}_F$ is normal at $\mathbf{g}^*$ to F. (53)

Thus:

by (52), $\mathbf{x}^*$ is a Wardrop equilibrium with price vector $\mathbf{p}$ (see definition 9) and

by (53), $\mathbf{g}^*$ satisfies policy $P_{\mathbf{h}}$-with-prices with price vector $\mathbf{p}$ (see definition 11).

We have now proved Theorem 3, that policy $P_{\mathbf{h}}$-with-prices is capacity maximising; see Definition 3 in Section 3 above.

Appendix B. Proof of Theorem 5

Theorem 5. *Let D^+ be the set of demand feasible route flow vectors defined in Section 11. Let F^+_{red} be the set of feasible antistage red-time vectors and let S^+ be the set of supply feasible (route-inflow vector, antistage redtime vector) pairs. Suppose that*

$(D^+ \times F^+_{red}) \cap S^+_{red} \ne \phi.$

Suppose that the following definitions, conditions and rules above hold: definition (26), condition (27), price specification (29), route-cost and antistage red-time cost definitions (30)–(34), swapping rules (35), (36) and direction conditions (37), (38), (39) and *(40). Suppose that V and V^+ are specified by (43)* and *(44). Suppose that T is defined by (45) and satisfies (46), (47)* and *(48). Then, under these conditions, for any start point $(\mathbf{X}_0, \mathbf{R}_0) \in (D^+ \times F^+_{red}) \cap S^+_{red}$, the sequence (49) converges to a non-empty set E^+ of equilibria consistent with control policy $P_0^{\mathbf{f}}$. Here:*

$E^+ = \{(\mathbf{X}, \mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}; \mathbf{X}$ *is a Wardrop equilibrium and* $\mathbf{R}$ *is a* $P_0^{\mathbf{f}}$ *equilibrium.*$\}$

Proof of Theorem 5. We assume that policy $P_0^{\mathbf{f}}$ holds; and that all the conditions in the theorem statement also hold.

Let $(\mathbf{X}_0, \mathbf{R}_0) \in (D^+ \times F^+_{red}) \cap S^+_{red}$ and consider sequence (49). We show first that sequence (49) converges, under the above conditions, to the set

$\text{MIN} = \{(\mathbf{X}^*, \mathbf{R}^*) \in (D^+ \times F^+_{red}) \cap S^+_{red}; V(\mathbf{X}^*, \mathbf{R}^*) \le V(\mathbf{X}, \mathbf{R})$ for all $(\mathbf{X}, \mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}\}$.

MIN is the set of V-minimising $(\mathbf{X}, \mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}$. Then we show that

$\text{MIN} \subset E^+ = \{(\mathbf{X}, \mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}; \mathbf{X}$ is a Wardrop equilibrium and $\mathbf{R}$ satisfies $P_0^{\mathbf{f}}\}$,

so that (49), in converging to MIN, must also converge to the set E^+ of equilibria consistent with $P_0^{\mathbf{f}}$ which proves the theorem.

To show that (49) converges to MIN, we suppose that (49) does not converge to MIN and show that this leads to a contradiction. If (49) does converge to MIN then for all $\epsilon > 0$

there is k^{**} such that $V(T^k(\mathbf{X}_0, \mathbf{R}_0)) \le V^+_{min} + \epsilon$ for all $k > k^{**}$ (54)

where V^+_{min} is the minimum value of $V^+(\mathbf{X}, \mathbf{R})$ as $(\mathbf{X}, \mathbf{R})$ varies over $(D^+ \times F^+_{red}) \cap S^+_{red}$.

Suppose now that (49) does not converge to MIN. Then for some $\epsilon > 0$

there is no k^{**} such that $V(T^k(\mathbf{X}_0, \mathbf{R}_0)) \le V^+_{min} + \epsilon$ for all $k > k^{**}$. (55)

Now $V(T^k(\mathbf{X}_0,\mathbf{R}_0))$ is non-increasing and so, for any ϵ satisfying (55),

$$V(T^k(\mathbf{X}_0,\mathbf{R}_0)) > V^+_{min} + \epsilon \text{ for all } k. \tag{56}$$

It then follows, using (48), that, for any ϵ satisfying (55) (and hence (56)), and for all $k = 1, 2, 3, \ldots.$,

$$V^+(\mathbf{X}_0,\mathbf{R}_0) > V^+(T(\mathbf{X}_0,\mathbf{R}_0)) > V^+(T(T(\mathbf{X}_0,\mathbf{R}_0))) > \ldots > V^+(T^k(\mathbf{X}_0,\mathbf{R}_0)) > \ldots > V^+_{min} + \epsilon. \tag{57}$$

We now show *also* that, for any ϵ satisfying (55) (and hence (56) and (57)),

$$\text{there is } k^{**} \text{ such that } V(T^k(\mathbf{X}_0,\mathbf{R}_0)) < V^+_{min} + \epsilon \text{ for all } k > k^{**}. \tag{58}$$

To do this, for any ϵ satisfying (55) (and hence (56) and (57)), let

$$H = [(D^+ \times F^+_{red}) \cap S^+_{red}] \cap \{(\mathbf{X},\mathbf{R}); V^+(\mathbf{X}_0,\mathbf{R}_0) \geq V^+(\mathbf{X},\mathbf{R}) \geq V^+_{min} + \epsilon\}. \tag{59}$$

By (57),

$$T^k(\mathbf{X}_0,\mathbf{R}_0) \in H \tag{60}$$

for all k.

Now H is a closed bounded set and $(\Delta_1(\mathbf{X},\mathbf{R}), \Delta_2(\mathbf{X},\mathbf{R}))$ is continuous and non-zero over this set. Therefore, by (57),

$$V^+(\mathbf{X},\mathbf{R}) - V^+[\mathbf{X} + \Delta_1(\mathbf{X},\mathbf{R}), \mathbf{R} + \Delta_2(\mathbf{X},\mathbf{R})]$$

is continuous and positive over H. It follows that there is $h > 0$ such that

$$V^+(\mathbf{X},\mathbf{R}) - V^+[\mathbf{X} + \Delta_1(\mathbf{X},\mathbf{R}), \mathbf{R} + \Delta_2(\mathbf{X},\mathbf{R})] > h > 0$$

for all $(\mathbf{X},\mathbf{R}) \in H$ and so, using (57),

$$V^+(T^k(\mathbf{X}_0,\mathbf{R}_0)) - V^+(T^{k+1}(\mathbf{X}_0,\mathbf{R}_0)) > h$$

for all k. Hence, for all k,

$$\begin{aligned}&[V^+(T^0(\mathbf{X}_0,\mathbf{R}_0)) - V^+(T^k(\mathbf{X}_0,\mathbf{R}_0))]\\ &\quad = [V^+(T^0(\mathbf{X}_0,\mathbf{R}_0)) - V^+(T^1(\mathbf{X}_0,\mathbf{R}_0))] + [V^+(T^1(\mathbf{X}_0,\mathbf{R}_0)) - V^+(T^2(\mathbf{X}_0,\mathbf{R}_0))]\\ &\qquad + \;\ldots\ldots\; + [V^+(T^{k-1}(\mathbf{X}_0,\mathbf{R}_0)) - V^+(T^k(\mathbf{X}_0,\mathbf{R}_0))] > kh.\end{aligned}$$

(The sum above collapses.) Therefore:

$$[V^+(\mathbf{X}_0,\mathbf{R}_0) - V^+(T^k(\mathbf{X}_0,\mathbf{R}_0))] = [V^+(T^0(\mathbf{X}_0,\mathbf{R}_0)) - V^+(T^k(\mathbf{X}_0,\mathbf{R}_0))] > kh$$

for all k. Rearranging, it then follows that

$$V^+(T^k(\mathbf{X}_0,\mathbf{R}_0)) < V^+(\mathbf{X}_0,\mathbf{R}_0) - kh < 0 \text{ if } k > V^+(\mathbf{X}_0,\mathbf{R}_0)/h. \tag{61}$$

Suppose now that $k > V^+(\mathbf{X}_0,\mathbf{R}_0)/h$. Then

$$V^+(T^k(\mathbf{X}_0,\mathbf{R}_0)) < 0 \text{ by (61) and } V^+(T^k(\mathbf{X}_0,\mathbf{R}_0)) \geq 0 \text{ by definitions (43) and (44).} \tag{62}$$

(62) is our contradiction, created by assuming that sequence (49) does not converge to MIN. It follows that sequence (49) must converge to MIN.

We now show that MIN is the set of (**X, R**) where **X** is a Wardrop equilibrium when the red-time is **R** and **R** satisfies $P_0^{\mathbf{f}}$ when the flow is **X**.

Let $(\mathbf{X},\mathbf{R}) \in$ MIN and so be a V^+- minimiser over the set $(D^+ \times F^+_{red}) \cap S^+_{red}$. Thus at $(\mathbf{X},\mathbf{R})$ there can be no descent direction of V^+ remaining within the set $(D^+ \times F^+_{red}) \cap S^+_{red}$, and hence

$$-[grad\, V^+(\mathbf{X},\mathbf{R})] = -[\mathbf{C}, \mathbf{RC}](\mathbf{X},\mathbf{R}) \text{ is normal at } (\mathbf{X},\mathbf{R}) \text{ to } (D^+ \times F^+_{red}) \cap S^+_{red}. \tag{63}$$

By (27), V^+ tends to $+\infty$ as $(\mathbf{X},\mathbf{R})$ tends to the boundary of S^+_{red}. Therefore a V^+-minimising $(\mathbf{X},\mathbf{R})$ is a positive distance from the boundary of S^+_{red}. Hence $(\mathbf{X},\mathbf{R}) \in$ the interior of S^+_{red} and it follows from (63) that

$$-[grad\, V^+(\mathbf{X},\mathbf{R})] = -[\mathbf{C}, \mathbf{RC}](\mathbf{X},\mathbf{R}) \text{ is normal at } (\mathbf{X},\mathbf{R}) \text{ to } (D^+ \times F^+_{red}).$$

Hence:

$$-\mathbf{C}(\mathbf{X},\mathbf{R}) \text{ is normal at } \mathbf{X} \text{ to } D^+ \tag{64}$$

and

$$-\mathbf{RC}(\mathbf{X},\mathbf{R}) \text{ is normal at } \mathbf{R} \text{ to } F^+_{red}. \tag{65}$$

Now:

condition (64) implies that $\mathbf{X}$ is a Wardrop equilibrium when the red-time vector is $\mathbf{R}$

and

condition (65) implies that $\mathbf{R}$ satisfies $P_0^{\mathbf{f}}$ when the route-entry vector is $\mathbf{X}$.

Thus sequence (49) converges to the non-empty set E^+ of equilibria consistent with control policy $P_0^{\mathbf{f}}$. Here:

$$E^+ = \{(\mathbf{X}, \mathbf{R}) \in (D^+ \times F^+_{red}) \cap S^+_{red}; \mathbf{X} \text{ is a Wardrop equilibrium and } \mathbf{R} \text{ satisfies } P_0^{\mathbf{f}}\}.$$

We have proved Theorem 5.

Appendix C. A traffic signal control background

C.1. Traffic control with fixed routes

Webster (1958) was one of the first to seek to model signal timings and their effect on traffic flow at a single junction; assuming that average flows are essentially fixed, not changed by signal timing changes. He considered mathematical and simulation models of traffic signal control at a single junction, to design fixed time or time of day signal timings. He proposed the well-known "equisaturation" policy; at a signal controlled junction with just two conflicting approaches the equisaturation policy chooses green-time proportions which equalise the degrees of saturation of the two conflicting approaches. This is the same as choosing green-time proportions which minimise the maximum of the degrees of saturation of the two approaches.

If a junction has many approaches and the signal cycle is divided into non-overlapping periods called stages in each of which a certain set of approaches have green and the rest have red, and these stages are separated by specified interstage periods and arranged in a specified sequence so that each approach has green for just one period consisting of one or more stages (and if more than one the intervening interstages), then the equisaturation policy may again be written: choose stage green time proportions to minimise the maximum of the degrees of saturation of all the approaches. This type of rule has often been utilised in adaptive traffic control systems.

Allsop extended Webster's model to enable calculation of practicable signal timings for a junction of any of a wide range of layouts to be expressed in terms of optimisation subject to linear constraints: this includes maximisation of capacity for known ratios of arrival rates on the various approaches by linear programming (Allsop, 1972) and minimisation of estimated delay for arrival rates within that capacity by convex programming (Allsop, 1971). This approach was adopted internationally and later for about 20 years in the software package OSCADY (Burrow, 1987). Through the work of his colleagues and of Tully (1976) and Kimber and Hollis (1979), the requirements for the sequence of stages to be specified and the arrival rates to be within capacity to enable estimated delay to be minimised were relaxed (Allsop, 1992).

Robertson (1969) gives a model of a whole network (TRANSYT) allowing whole network optimisation of traffic signals (for known Origin–Destination (OD) inputs and known route flows). Hunt et al. (1982) developed the real time control system SCOOT; essentially from the TRANSYT model.

Heydecker (2004) and Heydecker et al. (2007) propose an adaptive dynamic control system for traffic signals and also consider possible future objectives for traffic signal control. Aboudolas et al. (2009) outline a control designed to optimise a store and forward network model.

C.2. Traffic control with variable routes

It is striking that route choices are regarded as fixed in the design of most traffic control strategies; this is true of all the strategies mentioned above in Appendix C.1 and by far the majority of all strategies mentioned in this appendix.

Wardrop (1952) specified the two basic routeing principles: selfish routeing or user equilibrium routeing where travellers utilise their own best or quickest routes, and system optimal routeing where the routes are those which minimise total travel time.

Allsop (1974) pointed out the importance of allowing for route choices, and other choices, when considering the impacts of signal control changes; quoting Beckmann who emphasised that travellers are playing their own games while the signal-setter is playing his. Allsop (1974) was one of the first to specify a network model with coherent traffic signal control variables and route choice variables, seeking to allow the interaction between traffic signal control policies and user-equilibrium, selfish, route choice to be sensibly considered within a single network model.

To study the problem of optimising signal timings subject to reasonable estimates of route-choice behaviour, many researchers have used traffic assignment models where travel times depend on both traffic flows and green-times. Gartner (1976) studied the interaction between area traffic control and user equilibrium routeing.

Dickson (1981) showed that signal timings which minimise travel time for fixed routes do not necessarily minimise travel time when routeing is variable.

Optimisation of signal timings in signal-controlled networks, allowing route choices to vary, is considered by Fisk (1984) and Sheffi and Powell (1983). In this work queues are not explicitly represented; it is assumed that there is a cost function depending on flow and green-time.

C.3. Representing queues in traffic assignment and traffic control models

Seeking a simple representation of queues, within traffic assignment or routeing models and also within control models, vertical queueing assignment models were introduced by Thompson and Payne (1975). Vertical queue models have since been utilised, in an assignment-and-control context by Smith (1987) and in an assignment context by Larsson and Patriksson (1995).

Nesterov and de Palma (2003) introduce steady state solutions of dynamic assignment models, and these are very close to the original model of Thompson and Payne (1975). Bliemer et al. (2012) extends this steady state vertical queueing approach so as to allow for a time-varying input demand and hence time-varying queues; this makes the solution of the traffic assignment problem much less straightforward.

The need for transport models with more realistic explicit queues has been emphasised by Bliemer et al. (2012), Daganzo (1998) and others.

A transfer of the vertical queueing traffic assignment theory introduced by Thompson and Payne to embrace some limited spatial queueing was developed in Smith et al. (2013) and further developed in Smith et al. (2019a).

C.4. Traffic control with variable routes, queues and capacity maximising policies

Smith (1980) proposed the P_0 distributed traffic signal control policy. This control policy is designed to encourage efficient routeing and to maximise the capacity of many networks, while requiring very little on-line information and very little computation. The policy has been developed in Smith (1979a,b, 1984, 2010, 2011, 2015) and Smith et al. (2015, 1987).

Smith and van Vuren (1993), Smith and Mounce (2011) and Liu et al. (2015) have considered the stability of route choice and signal control, taken together; both with standard delay-minimising policies and with capacity-maximising policies such as P_0. Much of this work utilises vertical queueing.

Representing spatial queueing within control models affects the design and evaluation of variants of the P_0 policy. Smith et al. (2019a) gives the results of applying different variants of P_0, including more spatially aware variants, on a very simple two route network. While the network is very simple, quite complicated phenomena arise: for example if P_0 is utilised and the assignment of routes utilises spatial queueing, and not vertical queueing, then the equilibrium travel time is not a non-decreasing function of the network input load. This implies that many traffic assignment and control models using P_0 have multiple equilibria. Certain variants of the P_0 control policy avoid this problem by taking better account of horizontal queueing.

C.5. Traffic control with variable routes; bilevel approaches

Chiou (1999, 2003) consider the interaction between traffic control and route choice, using the TRANSYT model of traffic flow. Van Vuren and Van Vliet (1992) includes the additional consideration of route guidance.

Yang and Yagar (1995) and Yang (1996) have considered in detail the interaction between signal control and routeing in saturated road networks. Hu and Mahmassani (1997) have studied (within a model) day to day evolution of network flows under real-time information and reactive signal control.

Historically, assignment- or routeing-only models (at least those models designed for transportation planning purposes) have been fixed point models without queueing; see for example Cantarella and Cascetta (1995) and Cantarella (1997). Cantarella et al. (1991) and Cantarella (2010) have considered signal setting as part of a dynamical assignment process involving re-routeing. They discuss the interaction between route choices and signal control in both a dynamical and an equilibrium setting. Cascetta et al. (2006) have designed models and algorithms for optimising signals with SUE routeing. Meneguzzer (1996) has conducted computational experiments with a combined traffic assignment and control model with asymmetric cost functions. Maher et al. (2001) propose a bi-level programming approach for traffic control problems with SUE routeing and Lam and Zhang (2000) have considered capacity-constrained traffic assignment in networks with residual queues.

C.6. Traffic control within operational models

LINSIG (2010) software generates signal timings for given flows; this software is often used in real life for junctions and small networks, and may involve small scale routeing considerations. Schlaich and Haupt (2012) describe a large scale implementation of the equisaturation policy within the VISUM assignment model; the aim is to generate fixed time signal timings over a wide area taking some account of route choice. This has now been installed as part of the VISUM assignment model by Gentile and Meschini (2013). There is scope for using other traffic control policies such as the P_0 policy.

C.7. Traffic control from a dynamical systems viewpoint

Taale (2008) has considered integrated anticipatory control of road networks, in a dynamic setting, using game theory. Mahmassani et al. (2013) has considered the performance of urban traffic networks including gridlock and the dynamics of gridlock.

Xiao and Lo (2015) consider the combined route choice and adaptive traffic control as a day-to-day dynamical system and Hajiahmadi et al. (2015) have considered how to optimally control perimeter and network control plans for urban traffic networks.

Han and Gayah (2015) have considered a "continuum" signalised junction model and utilised this model to help design signal controls in dynamic traffic networks. This "continuum" model does not represent signal cycles but instead considers green-time proportions as being defined at each time t. This paper utilises such a continuum control model.

Huang et al. (2016) has designed iterative optimisation procedures for estimating optimal anticipatory network traffic control; sometimes utilising a dual approach and also real-time network estimation.

Keyvan-Ekbatani et al. (2016) has studied combining the traffic-responsive control policy P_0 and perimeter gating control in urban networks.

van den Berg et al. (2008) consider signal timings which aim to control routeing. Lammer and Helbing (2008) have considered the possibility of self-control of traffic lights and vehicle flows in urban road networks, without an explicit re-routeing model. It would be of interest to understand how the ideas in Lammer and Helbing (2008) connect to other parts of the literature on the distributed control of signal-controlled networks and also whether the mixed integer linear programming method outlined in van den Berg et al. (2008) can, at least for certain objectives, be written as a distributed control policy.

C.8. Traffic control reviews

There have been a number of reviews. Wood (1993) and Meneguzzer (1997) reviewed models combining traffic assignment and signal control up to 1997. Taale and van Zuylen (2001) give a review of research on the combined traffic assignment and control problem over the previous 25 years. Papageorgiou et al. (2003), give interesting reviews of road traffic control systems and strategies actually utilised on-street. Yang and Bell (1998) give a review of models and algorithms for road network design.

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