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KRONECKER POSITIVITY AND 2-MODULAR REPRESENTATION THEORY

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Abstract. This paper consists of two prongs. Firstly, we prove that any Specht module labelled by a 2-separated partition is semisimple and we completely determine its decomposition as a direct sum of graded simple modules. Secondly, we apply these results and other modular representation theoretic techniques on the study of Kronecker coefficients and hence verify Saxl's conjecture for several large new families of partitions. In particular, we verify Saxl's conjecture for all height zero characters of $\mathfrak{S}_n$ for $p = 2$.

Introduction

This paper brings together, for the first time, the two oldest open problems in the representation theory of the symmetric groups and their quiver Hecke algebras. The first problem is to understand the structure of Specht modules and the second is to describe the decomposition of a tensor product of two Specht modules — the infamous *Kronecker problem.*

Kronecker positivity: The Kronecker problem is not only one of the central open problems in the classical representation theory of the symmetric groups, but it is also one of the definitive open problems in algebraic combinatorics as identified by Richard Stanley in [Sta00]. The problem of deciding the positivity of Kronecker coefficients arose in recent times also in quantum information theory [Kly04, CM06, CHM07, CDW12] and Kronecker coefficients have subsequently been used to study entanglement entropy [CSW18].

A new benchmark for the Kronecker positivity problem is a conjecture of Heide, Saxl, Tiep and Zalesskii [HSTZ13] that was inspired by their investigation of the square of the Steinberg character for simple groups of Lie type. It says that for any $n \neq 2, 4, 9$ there is always a complex irreducible character of $\mathfrak{S}_n$ whose square contains all irreducible characters of $\mathfrak{S}_n$ as constituents. For n a triangular number, an explicit candidate was suggested by Saxl in 2012: Let $\rho := \rho(k) = (k, k-1, k-2, k-3, \dots, 2, 1)$ denote the kth staircase partition. Phrased in terms of modules, Saxl's conjecture states that all simple modules appear in the tensor square of the simple $\mathbb{C}\mathfrak{S}_n$-module $\mathbf{D}^{\mathbb{C}}(\rho)$. In other words, we have that

$$\mathbf{D}^{\mathbb{C}}(\rho) \otimes \mathbf{D}^{\mathbb{C}}(\rho) = \bigoplus_{\lambda} g(\rho, \rho, \lambda)\mathbf{D}^{\mathbb{C}}(\lambda)$$

with $g(\rho, \rho, \lambda) \neq 0$ for all partitions λ of n. This conjecture has been attacked by algebraists, probabilists, and complexity theorists [Bes18, Ike15, LS17, PPV16] but remains stubbornly elusive. Positivity of the Kronecker coefficient $g(\rho, \rho, \lambda)$ has been verified for hooks and two-row partitions when n is sufficiently large in [PPV16], and then for arbitrary n and λ a hook in [Ike15, Bes18] or a double-hook partition (i.e., when the Durfee size is 2) in [Bes18], and for any λ comparable to ρ in dominance order in [Ike15].

This paper begins with the observation that the $\Bbbk\mathfrak{S}_n$-module $\mathbf{D}^{\Bbbk}(\rho)$ is projective over a field $\Bbbk$ of characteristic $p = 2$, or equivalently, that the character to the Specht module $\mathbf{D}^{\mathbb{C}}(\rho) = \mathbf{S}^{\mathbb{C}}(\rho)$ is the character ξ^ρ associated to a projective indecomposable $\Bbbk\mathfrak{S}_n$-module (via its integral lift to characteristic 0). Therefore, the tensor square of $\mathbf{D}^{\Bbbk}(\rho)$ is again a projective module, and the square of ξ^ρ is the character to a projective module. This allows us to bring to bear the tools of modular and graded representation theory on the study of the Kronecker coefficients. In particular, we deduce that if $\mathbf{D}^{\Bbbk}(\lambda) = \mathbf{S}^{\Bbbk}(\lambda)$ is a simple Specht module, then all constituents of the projective cover of $\mathbf{D}^{\Bbbk}(\lambda)$ must also appear in Saxl's tensor-square. For example, using this property for the trivial simple module $\mathbf{D}^{\Bbbk}((n))$ of $\mathfrak{S}_n$ at characteristic 2 gives all characters of odd degree as constituents in the Saxl square; more generally, we will detect all irreducible characters of height 0 as constituents. Our aim is to understand the columns of the 2-modular graded decomposition matrix which are labelled by *simple Specht modules* and to utilise these results towards Saxl's conjecture.

Modular representation theory: The classification of simple Specht modules for symmetric groups and their Hecke algebras was a massive undertaking involving over 30 years work [Jam78, JLM06, JM99, Fay05, JM96, Lyl07, JM97, Fay04, FL09, FL13, Fay10]. The pursuit of a description of semisimple and decomposable Specht modules is similarly old [Jam78] and yet has proven a much more difficult nut to crack. The decomposable Specht modules labelled by hook partitions were characterised by Murphy and Speyer [Mur80, Spe14]; the graded decomposition numbers of these Specht modules were calculated by Chuang, Miyachi, and Tan [CMT04]; the first examples of decomposable Specht modules labelled by non-hook partitions were given by Dodge and Fayers [DF12]; Donkin and Geranios very recently unified and extended these results to certain "framed staircase" partitions [DG18] which we will discuss (within the wider context of "2-separated" partitions) below.

We show that any Specht module labelled by a 2-separated partition is semisimple and we completely determine its decomposition as a sum of graded simple modules. Our proof makes heavy use of recent results in the *graded* representation theory of Hecke and rational Cherednik algebras. We shall denote the quantisations of the Specht and simple modules by $\mathbf{S}^{\Bbbk}_q(\lambda)$ and $\mathbf{D}^{\Bbbk}_q(\lambda)$ respectively over $\Bbbk$. We completely determine the rows of the graded decomposition matrix of $H^{\mathbb{C}}_q(n)$ labelled by 2-separated partitions; this serves as a first approximation to our goal and subsumes and generalises the results on decomposability and decomposition numbers of Specht modules for hook partitions (belonging to blocks of small 2-core) [Spe14, CMT04], and results on decomposition numbers of Specht modules in blocks of enormous 2-cores [JM96].

Graded decomposition numbers of semisimple Specht modules: The partitions of interest to us (for both Saxl's conjecture and our decomposability classification) are the 2-separated partitions. Such partitions are obtained by taking a staircase partition, τ, and adding 2 copies of a partition λ to the right of τ and 2 copies of a partition μ to the bottom of τ in such a way that λ and μ do not touch (except perhaps diagonally). Such partitions, denoted τ^λ_μ can be pictured as in Figure 1.

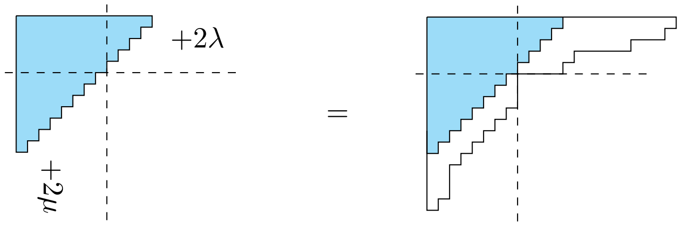

FIGURE 1. A 2-separated partition τ^λ_μ (see Definition 1.6).

Notice that if the weight of a block is small compared to the size of the core, then all partitions in that block are 2-separated. We emphasise that the size of the staircase $\rho(k)$ in the following statement is immaterial (provided that $k \geqslant \ell(\lambda) + \ell(\mu^T)$, where $\ell(\lambda)$ denotes the length of the partition λ), and so we simply write $\tau := \rho(k)$. For those interested in the extra graded structure, we refer the reader to the full statement in Corollary 4.3.

Theorem A. *Let $e = 2$ and let τ^λ_μ denote a 2-separated partition of n. We have that the $H^{\mathbb{C}}_{-1}(n)$-module $\mathbf{S}^{\mathbb{C}}_{-1}(\tau^\lambda_\mu)$ is semisimple and decomposes as a direct sum of simples as follows*

$$\mathbf{S}^{\mathbb{C}}_{-1}(\tau^\lambda_\mu) = \bigoplus_{\nu} c(\nu^T, \lambda^T, \mu)\mathbf{D}^{\mathbb{C}}_{-1}(\tau^\nu_\varnothing)$$

where $c(\nu^T, \lambda^T, \mu)$ is the Littlewood–Richardson coefficient labelled by this triple of partitions.

In particular, there exist many blocks of $H^{\mathbb{C}}_{-1}(n)$ (those with large cores) for which all Specht modules in the block are semisimple. In [DF12] Dodge and Fayers remark that *"every known example of a decomposable Specht module is labelled by a 2-separated partition"* and *"it is interesting to speculate whether the 2-separated condition is necessary for a Specht module to be decomposable".* In fact in Section 6 we show that their speculation is *not* true by exhibiting two infinite families of decomposable Specht modules obtained by "inflating" the smallest decomposable Specht module (indexed by $(3, 1^2)$).

Theorem A implies that all known examples of decomposable Specht modules for $\mathfrak{S}_n$ are obtained by reduction modulo $p = 2$ from decomposable semisimple Specht modules for $H^{\mathbb{C}}_{-1}(n)$.

Applications to Kronecker coefficients: We now discuss the results and insights which 2-modular representation theory affords us in the study of Kronecker coefficients. We verify the positivity of the Kronecker coefficients in Saxl's conjecture for a large new class of partitions, and propose conjectural strengthened and generalised versions of Saxl's original conjecture. Our first main theorem on Kronecker coefficients is as follows:

Theorem B. *Let $\lambda \vdash n = k(k+1)/2$ such that χ^λ is of height 0. Then $g(\rho, \rho, \lambda) > 0$. In particular, all χ^λ of odd degree are constituents of the Saxl square.*

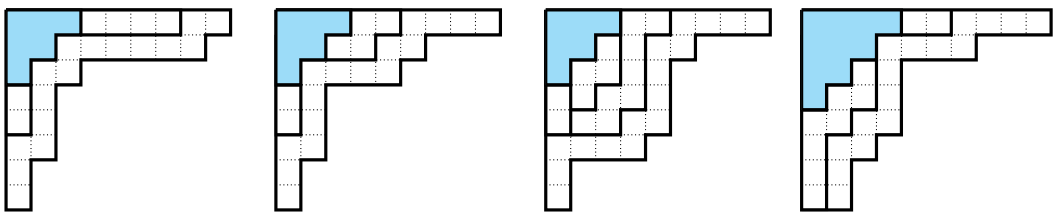

FIGURE 2. Examples of partitions which label height 0 characters for $\mathfrak{S}_{28}$ and $\mathfrak{S}_{36}$ (and therefore label constituents of Saxl's tensor square by Theorem B). There are 672 and 1417 such characters for these groups, respectively. A combinatorial construction of all such partitions (for arbitrary $n \in \mathbb{N}$) is given in Subsection 1.4.

We now shift focus to the Kronecker coefficients labelled by 2-separated partitions. In what follows, we shall write $g(\rho, \rho, \tau^\lambda_\mu)$ for the Kronecker coefficient labelled by a staircase ρ of size $n = k(k+1)/2$ for some $k \in \mathbb{N}$ and some 2-separated partition τ^λ_μ of n; in other words, we do not encumber the notation by explicitly recording the size of the staircases involved.

Theorem C. *For (α, β) a k-Carter–Saxl pair (as in Theorem 5.10) we have that $g(\rho, \rho, \alpha) \geqslant k$. In particular, all framed staircase partitions $\tau^{(a)}_{(1^b)}$ appear in the Saxl square.*

We do not recall the definition of a k-Carter–Saxl pair here, but rather discuss some examples and consequences of Theorem B. In particular, Theorem B implies that every 2-block contains a wealth of constituents of the Saxl square $\mathbf{S}^{\mathbb{C}}(\rho) \otimes \mathbf{S}^{\mathbb{C}}(\rho)$ which can be deduced using our techniques. Carter–Saxl pairs cut across hook partitions, partitions of arbitrarily large Durfee size, symmetric and non-symmetric partitions, partitions from arbitrary blocks, and across the full range of the dominance order. (In fact, the only common trait of these partitions is that they label semisimple Specht modules for $H^{\mathbb{C}}_{-1}(n)$.) We shall illustrate below that the property of being a Carter–Saxl pair is actually very easy to work with diagrammatically. For example, the above theorem includes the infinite family of "framed staircases" as some of the simplest examples: these are partitions which interpolate hooks and staircases. More explicitly, these are the partitions of the form $\alpha = \tau^{(a)}_{(1^b)}$. These can be pictured as in Figure 3 below.

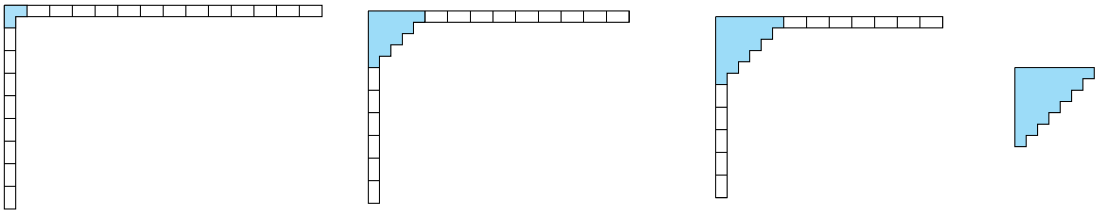

FIGURE 3. Some examples of framed staircases: $\alpha = \tau^{(13)}_{(1^8)}$, $\tau^{(9)}_{(1^6)}$, $\tau^{(7)}_{(1^5)}$, and $\tau = \rho(9)$ are all partitions of $n = 45$. Up to conjugation, there are 35 framed staircase partitions of $n = 45$. The classification of decomposable Specht modules labelled by framed staircases is the main result of [DG18]. In Theorem 5.5 we prove Saxl's conjecture for all framed staircase partitions. The key ingredient in our proof is that $(\tau^{(a)}_{(1^b)}, \tau^{(a+b)}_{\varnothing})$ is a 1-Carter–Saxl pair for $a, b \in \mathbb{N}$.

We wish to provide bounds on the Kronecker coefficients: the maximal possible values obtained by Kronecker products are studied in [PPV16], and the Kronecker products whose coefficients are all as small as possible (namely all 0 or 1) are classified in [BB17]. For constituents to partitions of depth at most 4, explicit formulae for their multiplicity in squares were provided by Saxl in 1987, and later work by Zisser and Vallejo, respectively. For the Kronecker coefficients studied here, the easiest (and well known) non-trivial case is $g(\rho(k), \rho(k), (n-1, 1)) = k - 1$, so the Kronecker coefficients are even unbounded; this also holds for the other families corresponding to partitions of small depth. Lower bounds coming from character values on a specific class were obtained by Pak and Panova in [PP17], where also the asymptotic behaviour of the multiplicity of special constituents is studied. Theorem B allows us to provide explicit lower bounds on the Kronecker coefficients $g(\rho(k), \rho(k), \lambda)$ for new infinite families of Saxl constituents, where again the multiplicities are unbounded.

We now provide some examples of more complicated Carter–Saxl pairs. For $n = 78$, if we first focus on the (unique) block of weight $w = 6$ we find 6 constituents in this block labelled by framed staircases as well as the Carter–Saxl pairs given (up to conjugation) in Figure 4 below.

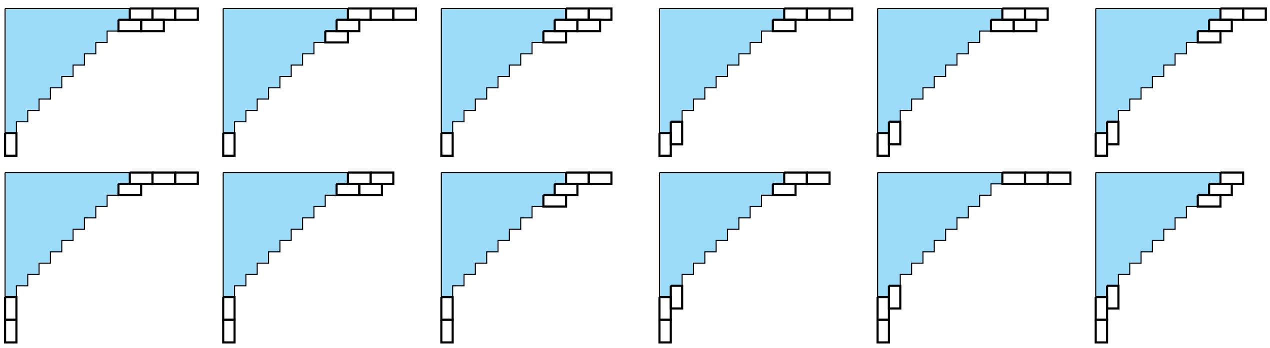

FIGURE 4. More examples of coefficients $g(\rho, \rho, \tau^{\lambda}_{\mu}) > 0$. These belong to the block of weight 6 for the symmetric group of rank 78. We have that each of τ^{λ}_{μ} belongs to a Carter–Saxl pair of the form $(\tau^{\lambda}_{\mu}, \tau^{(3,2,1)}_{\varnothing})$.

Finally, we propose two extensions of Saxl's conjecture based on its modular representation theoretic interpretation. The first conjecture reduces the problem to the case of 2-regular partitions, but at the expense of working in the more difficult modular setting. We remark that towards Saxl's conjecture over $\mathbb{C}$, it has already been verified that for any 2-regular partition λ of $n = k(k+1)/2$ the Kronecker coefficient $g(\rho(k), \rho(k), \lambda)$ is positive [Ike15], and so it is natural to hope that this can be extended to positive characteristic.

Strengthened Saxl Conjecture. *Let $\Bbbk$ be a field of characteristic 2. We have that*

$$\dim_{\Bbbk}\left(\mathrm{Hom}_{\mathfrak{S}_n}\left(\mathbf{D}^{\Bbbk}(\rho(k))\otimes\mathbf{D}^{\Bbbk}(\rho(k)),\mathbf{D}^{\Bbbk}(\lambda)\right)\right)>0.$$

for any 2-regular partition λ of $n=k(k+1)/2$. Equivalently: Saxl's 2-modular tensor square contains all indecomposable projective modules as direct summands with positive multiplicity.

What could be a suitable candidate for arbitrary n, not just triangular numbers?

Generalised Saxl Conjecture. *For $n\in\mathbb{N}$ there exists a symmetric p-core λ for some $p\le n$ such that $\mathbf{D}^{\mathbb{C}}(\lambda)\otimes\mathbf{D}^{\mathbb{C}}(\lambda)$ contains all simple $\mathbb{C}\mathfrak{S}_n$-modules with positive multiplicity.*

While this sounds reasonable, in fact, for larger n it hardly restricts the search for a good candidate as almost any partition of n is then a p-core for some $p\le n$. So as a guide towards finding a simple module $\mathbf{D}^{\mathbb{C}}(\lambda)$ whose tensor square contains all simples, one would try to find a suitable symmetric p-core for a small prime p.

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1. The Hecke algebra

Let $\Bbbk$ be a commutative integral domain. We let $\mathfrak{S}_n$ denote the symmetric group on n letters, with presentation

$$\mathfrak{S}_n=\langle s_1,\dots,s_{n-1}\mid s_is_{i+1}s_i=s_{i+1}s_is_{i+1}, s_is_j=s_js_i \text{ for } |i-j|>1\rangle.$$

We are interested in the representation theory (over $\Bbbk$) of symmetric groups and their deformations. Given $q\in\Bbbk$, we define the Hecke algebra $H_q^{\Bbbk}(n)$ to be the unital associative $\Bbbk$-algebra with generators $T_1,T_2,,\dots,T_{n-1}$ and relations

$$(T_i-q)(T_i+1)=0\quad T_iT_j=T_jT_i,\quad T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$$

for $i=1,\dots,n-1$ and $i\neq j$. We let $e\in\mathbb{N}$ be the smallest integer such that $1+q+q^2+\dots+q^{e-1}=0$ or set $e=\infty$ if no such integer exists. If $\Bbbk$ is a field of characteristic p and $p=e$, then $H_q^{\Bbbk}(n)$ is isomorphic to $\Bbbk\mathfrak{S}_n$.

We define a composition, λ, of n to be a finite sequence of non-negative integers $(\lambda_1,\lambda_2,\dots)$ whose sum, $|\lambda|=\lambda_1+\lambda_2+\dots$, equals n. If the sequence $(\lambda_1,\lambda_2,\dots)$ is weakly decreasing, we say that λ is a partition. The number of non-zero parts of a partition, λ, is called its length, $\ell(\lambda)$; the size of the largest part is called the width, $w(\lambda)=\lambda_1$. Given $\lambda\in\mathscr{P}_n$, its Young diagram $[\lambda]$ is defined to be the configuration of nodes,

$$[\lambda]=\{(r,c)\mid 1\le r\le\ell(\lambda), 1\leqslant c\leqslant\lambda_r\}.$$

The conjugate partition, λ^T, is the partition obtained by interchanging the rows and columns of λ. Given $\lambda\in\mathscr{P}_n$, we define a tableau of shape λ to be a filling of the nodes of the Young diagram of λ with the numbers $\{1,\dots,n\}$. We define a standard tableau to be a tableau in which the entries increase along both the rows and columns of each component. We let $\mathrm{Std}(\lambda)$ denote the set of all standard tableaux of shape $\lambda\in\mathscr{P}_n$. Given $\mathsf{t}\in\mathrm{Std}(\lambda)$, we set $\mathrm{Shape}(\mathsf{t})=\lambda$. Given $1\leqslant k\leqslant n$, we let $\mathsf{t}{\downarrow}_{\{1,\dots,k\}}$ be the subtableau of t whose entries belong to the set $\{1,\dots,k\}$. We write $\mathsf{t}\trianglerighteq\mathsf{s}$ if $\mathsf{t}(k)\trianglerighteq\mathsf{s}(k)$ for all $1\leqslant k\leqslant s$ and refer to this as the dominance order on $\mathrm{Std}(\lambda)$.

We let t^λ and t_λ denote the most and least dominant tableaux respectively. We let $w_\lambda \in \mathfrak{S}_n$ be the permutation such that $w_\lambda \mathsf{t}^\lambda = \mathsf{t}_\lambda$. For example, $w_{(3,2,1)} = (2,4)(3,6)$ and

$$\mathsf{t}^{(3,2,1)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \qquad \mathsf{t}_{(3,2,1)} = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \quad .$$

Definition 1.1. Given λ a partition of n, we set $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \dots \leqslant \mathfrak{S}_n$ and we set

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \qquad y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-q)^{\ell(w)} T_w$$

and we define the Specht module, $\mathbf{S}_q^{\Bbbk}(\lambda)$, to be the left $H_q^{\Bbbk}(n)$-module

$$\mathbf{S}_q^{\Bbbk}(\lambda) := H_q^{\Bbbk}(n) y_\lambda T_{w_\lambda} x_\lambda.$$

Remark 1.2. Letting $\Bbbk = \mathbb{C}$ and specialising $q = 1$ we have that $H_q^{\Bbbk}(n)$ is isomorphic to $\mathbb{C}\mathfrak{S}_n$. In this case, we drop the subscript on the Specht modules and we have that

$$\{\mathbf{S}^{\mathbb{C}}(\lambda) \mid \lambda \in \mathscr{P}_n\}$$

provide a complete set of non-isomorphic simple $\mathbb{C}\mathfrak{S}_n$-modules. We let χ^λ denote the character of the complex irreducible module $\mathbf{S}^{\mathbb{C}}(\lambda)$.

1.1. **Modular representation theory.** Let $\Bbbk$ be a field and $q \in \Bbbk$. The group algebra of the symmetric group $\Bbbk\mathfrak{S}_n$ is a semisimple algebra if and only if $\Bbbk$ is a field of characteristic $p > n$. The Hecke algebra of the symmetric group is a non-semisimple algebra if and only if q is a primitive eth root of unity for some $e \leqslant n$ or $\Bbbk$ is a field of characteristic $p \leqslant n$. We shall now recall the basics of the non-semisimple representation theory of these algebras.

Modular representation theory seeks to deconstruct the non-semisimple representations of an algebra in terms of their simple constituents. To this end, we define the *radical* of a finite-dimensional A-module M, denoted $\mathrm{rad}(M)$, to be the smallest submodule of M such that the corresponding quotient is semisimple. We then let $\mathrm{rad}^2 M = \mathrm{rad}(\mathrm{rad} M)$ and inductively define the *radical series*, $\mathrm{rad}^i M$, of M by $\mathrm{rad}^{i+1} M = \mathrm{rad}(\mathrm{rad}^i M)$. We have a finite chain

$$M \supset \mathrm{rad}(M) \supset \mathrm{rad}^2(M) \supset \dots \supset \mathrm{rad}^i(M) \supset \mathrm{rad}^{i+1}(M) \supset \dots \supset \mathrm{rad}^s(M) = 0.$$

In the non-semisimple case, the Specht modules are no longer simple but they continue to play an important role in the representation theory of $H_q^{\Bbbk}(n)$ as we shall now see. We say that a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ is e-regular if there is no $1 \leqslant i \leqslant \ell$ such that $\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+e-1} > 0$. We let $\mathscr{R}_n^e$ denote the set of all e-regular partition of n. Occasionally, we will also use the notation $\lambda \vdash_e n$ in place of $\lambda \in \mathscr{R}_n^e$. For $\Bbbk$ an arbitrary field, we have that

$$\{\mathbf{D}_q^{\Bbbk}(\mu) \mid \mathbf{D}_q^{\Bbbk}(\mu) = \mathbf{S}_q^{\Bbbk}(\mu)/\mathrm{rad}(\mathbf{S}_q^{\Bbbk}(\mu)), \mu \in \mathscr{R}_n^e\} \tag{1.1}$$

provides a full set of non-isomorphic simple $H_q^{\Bbbk}(n)$-modules. Of course, the radical of a Specht module is not easy to compute! The passage between the Specht and simple modules is recorded in the decomposition matrix,

$$(d_{\lambda\mu}^{\Bbbk})_{\substack{\lambda \in \mathscr{P}_n \\ \mu \in \mathscr{R}_n^e}} \qquad d_{\lambda\mu}^{\Bbbk} = [\mathbf{S}_q^{\Bbbk}(\lambda) : \mathbf{D}_q^{\Bbbk}(\mu)]$$

where $[\mathbf{S}_q^{\Bbbk}(\lambda) : \mathbf{D}_q^{\Bbbk}(\mu)]$ denotes the multiplicity of $\mathbf{D}_q^{\Bbbk}(\mu)$ as a composition factor of $\mathbf{S}_q^{\Bbbk}(\lambda)$. This matrix is uni-triangular with respect to the dominance ordering on $\mathscr{P}_n$. We have already seen in equation (1.1) that every column of the decomposition matrix contains an entry equal to 1; namely if $\mu \in \mathscr{R}_n^e$ then $d_{\mu,\mu} = 1$. We now recall James' regularisation theorem, which states that every row of the decomposition matrix contains an entry equal to 1 (and identifies this entry).

Example 1.3. We picture a partition λ and its 2-regularisation $\mathsf{R}(\lambda)$ in Figure 5. We have highlighted which nodes are moved and to where they have been moved.

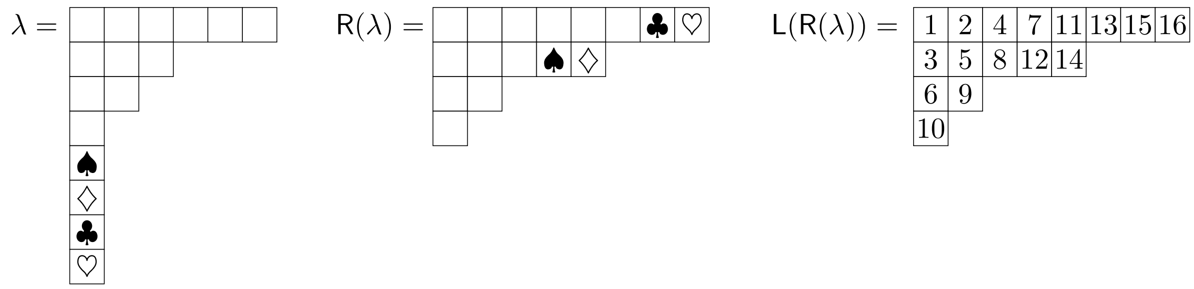

FIGURE 5. The partition $\lambda = \tau^{(1)}_{(1^2)}(4)$, its 2-regularisation $\mathsf{R}(\lambda)$, and the ladder tableau of shape $\mathsf{R}(\lambda)$.

First we require some combinatorics. We define the (e-)ladder number of a node $(r, c) \in [\lambda]$ to be $\mathfrak{l}(r, c) = r + c(e - 1)$. The ith ladder of λ is defined to be the set

$$\mathscr{L}_i = \{(r, c) \in \mathbb{N}^2 \mid \mathfrak{l}(r, c) = i\} \cap [\lambda].$$

Given a node $(r, c) \in [\lambda]$ we define the (e-)residue to be $\mathfrak{l}(r, c)$ modulo e. The e-regularisation of λ is the partition $\mathsf{R}(\lambda)$ obtained by moving all of the nodes of λ as high along their ladders as possible. When $q = -1$, each ladder of λ is a complete north-east to south-westerly diagonal in $[\lambda]$. In particular, when $e = 2$ the partition $\mathsf{R}(\lambda)$ is obtained from λ by sliding nodes as high along their south-west to north-easterly diagonals as possible.

Theorem 1.4 (James' regularisation theorem)**.** *Let λ be a partition of n and $\Bbbk$ be an arbitrary field. We have that $[\mathbf{S}^{\Bbbk}_q(\lambda) : \mathbf{D}^{\Bbbk}_q(\mu)]$ is equal to 1 if $\mu = \lambda^R$ and is zero if $\mu \lhd \lambda^R$.*

1.2. **Brauer–Humphrey's reciprocity.** Given λ an e-regular partition and $\mathbf{D}^{\Bbbk}_q(\lambda)$ the corresponding simple $H^{\Bbbk}_q(n)$-module, we let $\mathbf{P}^{\Bbbk}_q(\lambda)$ denote its projective cover. Brauer–Humphrey's reciprocity states that $\mathbf{P}^{\Bbbk}_q(\mu)$ has a Specht filtration, in other words

$$0 = S_1 \subset S_2 \subset \cdots \subset S_z = \mathbf{P}^{\Bbbk}_q(\mu)$$

such that for each $1 \leqslant r \leqslant z$, we have $S_r/S_{r-1} \cong \mathbf{S}^{\Bbbk}_q(\mu)$ for some $\lambda \in \mathscr{P}_n$ dependent on $1 \leqslant r \leqslant z$. Furthermore Brauer–Humphrey's reciprocity states that the multiplicity, $[\mathbf{P}^{\Bbbk}_q(\mu) : \mathbf{S}^{\Bbbk}_q(\lambda)]$, of $\mathbf{S}^{\Bbbk}_q(\lambda)$ in such a filtration is well-defined and that

$$[\mathbf{P}^{\Bbbk}_q(\mu) : \mathbf{S}^{\Bbbk}_q(\lambda)] = [\mathbf{S}^{\Bbbk}_q(\lambda) : \mathbf{D}^{\Bbbk}_q(\mu)]. \tag{1.2}$$

In other words, the λth column of the decomposition matrix determines the Specht filtration multiplicities for $\mathbf{P}^{\Bbbk}_q(\lambda)$. This will be a key observation for our applications to Kronecker coefficients in Section 5.

1.3. **2-blocks.** We first recall the block-structure of Hecke algebras in (quantum) characteristic $e = 2$ (which will be the main case of interest in this paper). Throughout this section $e = 2$ and $\Bbbk$ can be taken to be an arbitrary field (although we are mainly interested in the cases when $\Bbbk = \mathbb{C}$ or $\Bbbk$ is of characteristic $p = 2$). The algebra $H^{\Bbbk}_{-1}(n)$ decomposes as a direct sum of primitive 2-sided ideals, called blocks. All questions concerning modular representation theory break-down block-by-block according to this decomposition: in particular each simple/Specht module belongs to a unique block.

The rim of the Young diagram of $\lambda \vdash n$ is the collection of nodes $R[\lambda] = \{(r, c) \in [\lambda] \mid (r + 1, c + 1) \notin [\lambda]\}$. Given $(r, c) \in [\lambda]$, we define the associated rim-hook to be the set of nodes $h(r, c) = \{(i, j) \in R[\lambda] \mid r \leqslant i, c \leqslant j\}$. If $|h(r, c)| = e \in \mathbb{N}$, then we refer to h as a removable e-hook; if $e = 2$ we refer to $h(r, c)$ as a removable domino. Removing $h(r, c)$ from $[\lambda]$ gives the Young diagram $[\lambda] \setminus h(r, c)$ of a partition of $n - e$. It is easy to see that a partition has no removable dominoes if and only if it is of the form $\rho(k) = (k, k-1, k-2, \ldots, 2, 1)$ for some $k \geqslant 0$;

in which case we say that it is a 2-core. We let $\mathsf{core}(\lambda)$ denote the 2-core partition obtained by successively removing all removable dominoes from λ (this defines a unique partition). The number of dominoes removed from λ is referred to as the weight of the partition λ and is denoted $w(\lambda)$. Given $k, n \in \mathbb{N}_0$, we define $B_k(n) = \{\lambda \in \mathscr{P}_n \mid \mathsf{core}(\lambda) = \rho(k)\}$ to be the corresponding combinatorial 2-block. The set $\mathscr{P}_n$ decomposes as the disjoint union of the non-empty $B_k(n)$. We note that it makes sense to speak of the weight of a 2-block since any two partitions in the same 2-block necessarily have the same weight. Two simple $H^{\Bbbk}_{-1}(n)$-modules (or irreducible characters of $\mathbb{C}\mathfrak{S}_n$) belong to the same 2-block if and only if their labelling partitions belong to the same combinatorial 2-block.

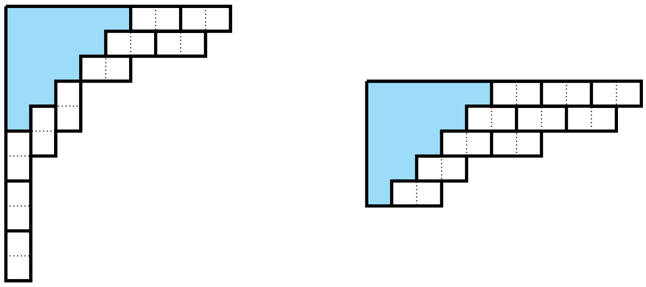

FIGURE 6. The partitions $\tau^{(2,2,1)}_{(3,1,1)}$ and $\tau^{(3^2,2,1^2)}_{\varnothing}$ for $\rho = \rho(5)$.

Example 1.5. The partition $(9,8,5,3^2,2,1^5)$ has 4 removable dominoes: two (2)-dominoes $\{(2,7),(2,8)\}$ and $\{(3,4),(3,5)\}$ and two (1^2)-dominoes $\{(3,3),(4,3)\}$ and $\{(1,10),(1,11)\}$. One can continue to successively remove such dominoes until one is left with the 2-core $\rho(5) = (5,4,3,2,1)$ as depicted on the lefthand-side of Figure 6.

Definition 1.6. Let $w_1, w_2 \in \mathbb{N}_0$ be arbitrary and $\lambda \in \mathscr{P}_{w_1}$ and $\mu \in \mathscr{P}_{w_2}$ such that $\lambda^T_1 + \mu_1 \leqslant k+1$. We let τ^λ_μ denote the partition

$$\tau^\lambda_\mu = (\rho(k) + 2\mu^T)^T + 2\lambda.$$

We say that any partition, τ^λ_μ, of this form is 2-separated.

Remark 1.7. We note that 2-separated partitions appear across all 2-blocks of the Hecke algebra. If the weight of a block is small compared to the size of the core, then all partitions in that block are 2-separated.

Remark 1.8. While the name "2-separated" may seem odd to some readers, it is motivated by the form this partition takes on a 2-abacus. In [JM96] these partitions are referred to as "2-quotient separated".

1.4. **Characters of height zero.** We now wish to discuss the defect groups of 2-blocks of symmetric groups and their characters of height zero (see [JK] for background and more details). Write $n = 2^{a_1} + \ldots + 2^{a_s}$ where $a_1 > \ldots > a_s \geqslant 0$; we set $s(n) = s$. For $m \in \mathbf{N}$, let m_2 be the largest 2-power dividing m. Then $(n!)_2 = 2^{n-s(n)}$ is the size of a Sylow 2-subgroup of $\mathfrak{S}_n$. Let B be a 2-block of $\mathfrak{S}_n$ of weight w; then a defect group of B is isomorphic to a Sylow 2-subgroup of $\mathfrak{S}_{2w}$, and thus is of cardinality $2^{2w-s(w)}$; the number $d(B) = 2w - s(w)$ is called the *defect* of B.

Example 1.9. The five 2-blocks of $\mathfrak{S}_{36}$ are indexed by the 2-cores $\varnothing, \rho(3), \rho(4), \rho(7)$ and $\rho(8)$. These blocks are of weight $18, 12, 10, 4$, and 0 respectively. Since $18 = 2^4 + 2$, the 2-block B of weight 18 has defect $d(B) = 34$.

We now recall the important notion of height 0 characters and simple modules. First we recall the fact that the dimension of any simple module $\mathbf{D}^{\Bbbk}(\lambda)$ or $\mathbf{S}^{\mathbb{C}}(\mu)$ belonging to a 2-block

B of the symmetric group $\mathfrak{S}_n$ is divisible by $2^{n-s(n)-d(B)}$. Such a module is said to be of height 0 if this 2-power is the largest 2-power dividing its dimension. We also say that the character χ^μ associated to the Specht module $\mathbf{S}^{\mathbb{C}}(\mu)$ is a height 0 character. For the 2-block B, we set

$$\mathrm{Irr}_0^{\mathbb{C}}(B) = \{\chi^\lambda \mid \chi^\lambda \text{ is a height zero character of } B\}.$$

Generalising an earlier result of Macdonald on characters of odd degree, a combinatorial description for the partitions labels of height 0 characters was given in terms of the so-called 2-core tower by Olsson (see [O76, O93]). A new characterisation was recently given in [GMT18, Section 3.2], again generalising an earlier version for the principal 2-block. This says that a partition λ in a 2-block $B = B_k(n)$ of weight $w = 2^{w_1} + \ldots + 2^{w_{s(w)}}$, where $w_1 > \ldots > w_{s(w)} \geq 0$, labels a height 0 character if and only if there is a sequence

$$\lambda = \lambda(0) \supset \lambda(1) \supset \cdots \supset \lambda(s(w) - 1) \supset \lambda(s(w)) = \rho_k$$

of partitions such that $\lambda(i-1) \setminus \lambda(i)$ is a 2^{w_i} rim-hook for $i = 1, \ldots, s(w)$.

A formula for the number $k_0(B)$ of height 0 characters in a 2-block of weight w was already given by Olsson [O76], and was also deduced from the description above in [GMT18]. With B and w given as above, we have

$$k_0(B) = \prod_{j=1}^{s(w)} 2^{w_j+1}.$$

Since we get this number for each 2-block, the set of height 0 characters χ^μ for $\mathfrak{S}_n$ constitutes quite a large class of irreducible characters.

Example 1.10. The irreducible characters of height 0 belonging to the principal 2-block of $\mathfrak{S}_n$ are precisely the characters of odd degree.

Example 1.11. The partitions $(9, 8, 3, 2^3, 1^2), (9, 6, 5, 2^3, 1^2) \vdash 28$ and the partitions $(9, 6, 5^3, 4, 1^2)$ $(10, 7, 4^3, 3, 2^2) \vdash 36$ all label height zero characters. These partitions are depicted in Figure 2 in such a manner as to illustrate their combinatorial construction via adding rim hooks (detailed above).

Example 1.12. The five 2-blocks of $\mathfrak{S}_{36}$, their weights w, the 2-adic expansions of $2w$, and the number of height 0 characters in the 2-block are recorded in the table below.

2-core	weight w	$2w$	$k_0(B)$
$\varnothing$	18	$2^5 + 2^2$	2^7
$\rho(3)$	15	$2^4 + 2^3 + 2^2 + 2$	2^{10}
$\rho(4)$	13	$2^4 + 2^3 + 2$	2^8
$\rho(7)$	4	2^3	2^3
$\rho(8)$	0	–	1

In particular, there are in total 1417 height 0 characters in 2-blocks of $\mathfrak{S}_{36}$, amongst which there are 128 of odd degree.

The following theorem will be one of the key results we use later on. It says that while there are many complex characters of height 0, there is only one simple module $\mathbf{D}^{\Bbbk}(\lambda)$ of height 0 in each 2-block.

Theorem 1.13 ([KOW12, Theorem 1.4])**.** *Let $\Bbbk$ be a field of characteristic 2. For any 2-block B of weight w of the symmetric group $\mathfrak{S}_n$, the module $\mathbf{D}^{\Bbbk}(\lambda)$ to the most dominant partition $\lambda = \tau_{\varnothing}^{(w)}$ is the unique simple $\Bbbk\mathfrak{S}_n$-module in B of height 0.*

2. KLR algebras and coloured tableaux

Given $n \in \mathbb{N}$ and an indeterminate t we define the quantum integers and quantum factorials

$$[n]_t = \frac{1 + t^2 + t^4 \cdots + t^{2n-2}}{t^{n-1}} \qquad [n]_t! = [1]_t[2]_t \ldots [n]_t$$

and given $\mu \in \mathscr{P}_n$ a partition of length ℓ, we set

$$[\mu]_t! = [\mu_1]_t![\mu_2]_t! \dots [\mu_\ell]_t!$$

We now define the **quantum binomial coefficients** to be

$$\begin{bmatrix} a \\ b \end{bmatrix}_t := \frac{[a]_t!}{[b]_t![a-b]_t!},$$

for all $a \geqslant b \geqslant 0$. The motivating observation for studying Hecke algebras is the following. Let $\Bbbk$ be a field of characteristic p and let $q \in \Bbbk$ be an element of order $e = p$: then $H_q^{\Bbbk}(n)$ is isomorphic to $\Bbbk\mathfrak{S}_n$. This gives us a way of factorising representation theoretic questions into two steps: firstly *specialise the quantum parameter* q to be a pth root of unity (in $\mathbb{C}$ and compatibly in $\Bbbk$) and study the non-semisimple algebra $H_q^{\mathbb{C}}(n)$; then *reduce modulo* p by studying $H_q^{\Bbbk}(n) = H_q^{\mathbb{Z}}(n) \otimes_{\mathbb{Z}} \Bbbk$. This allows us to factorise the problem of understanding decomposition matrices as follows,

$$[\mathbf{S}_q^{\Bbbk}(\lambda) : \mathbf{D}_q^{\Bbbk}(\mu)] = [\mathbf{S}_q^{\mathbb{C}}(\lambda) : \mathbf{D}_q^{\mathbb{C}}(\nu)] \times [\mathbf{D}_q^{\mathbb{C}}(\nu) \otimes_{\mathbb{Z}} \Bbbk : \mathbf{D}_q^{\Bbbk}(\mu)]. \tag{2.1}$$

On the right-hand side of the equality we have two matrices: the first is the **decomposition matrix for** $H_q^{\mathbb{C}}(n)$ and the second is known as "**James' adjustment matrix**". Therefore understanding the decomposition matrix of $H_q^{\mathbb{C}}(n)$ serves as a first step toward understanding the decomposition matrix of $\Bbbk\mathfrak{S}_n$.

We now recall the manner in which the grading can be incorporated into the picture and its immense power in understanding the decomposition matrix for $H_q^{\mathbb{C}}(n)$ (and hence, by equation (2.1) give us a method for attacking the problem of calculating decomposition numbers for symmetric groups). Let t be an indeterminate over $\mathbb{Z}$. The following theorem provides us with a $\mathbb{Z}$-graded presentation (which we record with respect to the indeterminate t) of the Hecke algebra.

Theorem 2.1 ([BK09a, KL09, Rou08a])**.** *The Hecke algebra $H_q^{\Bbbk}(n)$ admits a graded presentation with generators*

$$\{e(\underline{i}) \mid \underline{i} = (i_1, \dots, i_n) \in (\mathbb{Z}/e\mathbb{Z})^n\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}, \tag{2.2}$$

subject to a list of relations given in [BK09a, Main Theorem]*. The $\mathbb{Z}$-grading on $H_q^{\Bbbk}(n)$ is given by*

$$\deg(e(\underline{i})) = 0, \quad \deg(y_r) = 2, \quad \deg(\psi_r e(\underline{i})) = \begin{cases} -2 & \text{if } i_r = i_{r+1}, \\ 1 & \text{if } i_r = i_{r+1} \pm 1 \ \&\ e \neq 2, \\ 2 & \text{if } i_r = i_{r+1} \pm 1 \ \&\ e = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The importance of Theorem 2.1 is that it allows to consider an extra, richer graded structure on the Specht modules. We now recall the definition of this grading on the tableau basis of the Specht module. Let $\lambda \in \mathscr{P}_n$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$. We let $\mathsf{t}^{-1}(k)$ denote the node in t containing the integer $k \in \{1, \dots, n\}$. Given $1 \leqslant k \leqslant n$, we let $\mathcal{A}_{\mathsf{t}}(k)$, (respectively $\mathcal{R}_{\mathsf{t}}(k)$) denote the set of all addable $\mathrm{res}(\mathsf{t}^{-1}(k))$-nodes (respectively all removable $\mathrm{res}(\mathsf{t}^{-1}(k))$-nodes) of the partition $\mathrm{Shape}(\mathsf{t}{\downarrow}_{\{1,\dots,k\}})$ which are above $\mathsf{t}^{-1}(k)$, i.e. those in an earlier row. Let $\lambda \in \mathscr{P}_n$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$. We define the degree of t as follows:

$$\deg(\mathsf{t}) = \sum_{k=1}^{n} \left(|\mathcal{A}_{\mathsf{t}}(k)| - |\mathcal{R}_{\mathsf{t}}(k)|\right).$$

Given $\mathsf{t} \in \mathrm{Std}(\lambda)$ we define the **residue sequence** of t as follows:

$$\mathrm{res}(\mathsf{t}) = (\mathrm{res}(\mathsf{t}^{-1}(1)), \mathrm{res}(\mathsf{t}^{-1}(2)), \dots, \mathrm{res}(\mathsf{t}^{-1}(n))) \in (\mathbb{Z}/e\mathbb{Z})^n.$$

Let t be an indeterminate over $\mathbb{N}_0$. If $M = \oplus_{z\in\mathbb{Z}} M_z$ is a free graded $\Bbbk$-module, then its *graded dimension* is the Laurent polynomial

$$\mathrm{Dim}_t(M) = \sum_{k\in\mathbb{Z}} (\dim_{\Bbbk} M_k) t^k.$$

If M is a graded $H_q^{\Bbbk}(n)$-module and $k \in \mathbb{Z}$, define $M\langle k\rangle$ to be the same module with $(M\langle k\rangle)_i = M_{i-k}$ for all $i \in \mathbb{Z}$. We call this a *degree shift* by k. The graded dimensions of Specht modules admit a combinatorial description as follows:

Theorem 2.2 ([BKW11]). *The Specht module* $\mathbf{S}_q^{\Bbbk}(\lambda)$ *is a free* $\mathbb{Z}$*-graded* $\Bbbk$*-module with basis* $\{\psi^{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}(\lambda)\}$ *and where* $\deg(\psi^{\mathsf{t}}) = t^{\deg(\mathsf{t})}$.

Of course, this theorem gives us an added level of graded structure to consider: the graded decomposition numbers of symmetric groups and their Hecke algebras. By Theorem 2.2, we obtain a grading on the module $\mathbf{D}_q(\mu) = \mathbf{S}_q(\lambda)/\mathrm{rad}(\mathbf{S}_q(\lambda))$. We define the graded decomposition number to be the polynomial

$$d^{\Bbbk}_{\lambda,\mu}(t) = \sum_{k\in\mathbb{Z}} [\mathbf{S}_q^{\Bbbk}(\lambda) : \mathbf{D}_q^{\Bbbk}(\mu)\langle k\rangle] t^k \tag{2.3}$$

which records the composition multiplicity of each simple module and its relevant degree shift. In particular upon specialisation $t \to 1$ the polynomials of equation (2.3) specialise to be the usual decomposition numbers. While one might expect this grading to *increase* the level of difficulty of our question, we find that by keeping track of this extra grading information we are rewarded with an incredibly powerful algorithm for understanding the decomposition numbers of $H_q^{\mathbb{C}}(n)$.

Equation (2.1) hints that we could first study the decomposition numbers of $H_q^{\mathbb{C}}(n)$ as an intermediary first step toward understanding the decomposition numbers of symmetric groups in positive characteristic. In fact, this approach has been incredibly successful: Lascoux, Leclerc and Thibon provided an iterative algorithm for understanding the *graded* decomposition numbers of $H_q^{\mathbb{C}}(n)$ in [LLT96]. We now provide an elementary tableau-theoretic re-interpretation of this algorithm (using the work of Kleshchev and Nash [KN10]).

2.1. **Coloured tableaux.** We now recast ideas from [KN10] in terms of orbits of tableaux which we encode as "coloured tableaux". Let λ a partition of n and μ a composition of n. We define a Young tableau of shape λ and weight μ to be a filling of the nodes of λ with the entries

$$\underbrace{1,\dots,1}_{\mu_1},\underbrace{2,\dots,2}_{\mu_2},\dots,\underbrace{\ell,\dots,\ell}_{\mu_\ell}.$$

We say that a tableau is row standard if the entries are weakly increasing along the rows of λ; we denote the set of such tableaux by $\mathrm{RStd}(\lambda,\mu)$. We say that the Young tableau is semistandard if the entries are weakly increasing along the rows and are strictly increasing along the columns of λ; we denote the set of such tableaux by $\mathrm{SStd}(\lambda,\mu)$.

Definition 2.3. Given $\mu \vdash_e n$, we let $\mathrm{Lad}(\mu)$ denote the composition ν such that

$$\nu_i = \sharp\{(r,c) \in \mu \mid \mathfrak{l}(r,c) = i\}.$$

where we have that $\nu_1 = 0$ by definition. We define a semistandard coloured tableaux, S, to be a semistandard tableau of weight $\mathrm{Lad}(\mu)$ such that the entry of any node is congruent to its residue. We denote the set of all such tableaux of shape λ by $\mathrm{CStd}(\lambda,\mu) \subseteq \mathrm{SStd}(\lambda,\mathrm{Lad}(\mu))$. We let L^μ denote the unique element of $\mathrm{CStd}(\mu,\mu)$. We set $e(\mu) = e(\mathrm{res}(\mathsf{L}^\mu)) \in H_q^{\Bbbk}(n)$.

Example 2.4. For $q = -1$, $e = 2$ and $\mu = (6), (5,1), (4,2)$, we have that $\mathrm{Lad}(\mu)$ is equal to $(0,1,1,1,1,1,1)$, $(0,1,2,1,1,1)$ and $(0,1,2,2,1)$ respectively. All semistandard coloured tableaux (up to conjugation) for the principal 2-block of $H_{-1}^{\Bbbk}(6)$ are listed in the table below.

$\mathrm{CStd}(\lambda,\mu)$	6	$5,1$	$4,2$
6	$\begin{array}{llllll} 2&3&4&5&6&7 \end{array}$	$*$	$*$
$5,1$	$\begin{array}{lllll} 2&3&4&5&6\\ 7 \end{array}$	$\begin{array}{lllll} 2&3&4&5&6\\ 3 \end{array}$	$*$
$4,2$	$\begin{array}{llll} 2&3&4&7\\ 5&6 \end{array}$	$\begin{array}{llll} 2&3&4&5\\ 3&6 \end{array}$	$\begin{array}{llll} 2&3&4&5\\ 3&4 \end{array}$
$4,1^2$	$\begin{array}{llll} 2&5&6&7\\ 3\\ 4 \end{array}$ $\begin{array}{llll} 2&3&4&7\\ 5\\ 6 \end{array}$	$\begin{array}{llll} 2&3&4&5\\ 3\\ 6 \end{array}$	$\begin{array}{llll} 2&3&4&5\\ 3\\ 4 \end{array}$
3^2	$*$	$*$	$\begin{array}{lll} 2&3&4\\ 3&4&5 \end{array}$

The importance of coloured semistandard tableaux of weight μ is that they encode an $\mathfrak{S}_{\mathrm{Lad}(\mu)}$-orbit of standard Young tableaux; we shall now make this idea more precise. Given a composition ν and $c \geqslant 1$, we set $[\nu]_c = \nu_1 + \nu_2 + \cdots + \nu_c \in \mathbb{N}$ and we set $\nu_0 = 0$. Let μ be an e-regular partition and let s be a standard Young tableau of shape λ such that the residue sequence of s is given by

$$0, \underbrace{-1, -1, \dots, -1}_{\nu_3 \text{ times}}, \underbrace{-2, -2, \dots, -2}_{\nu_4 \text{ times}}, \underbrace{-3, -3, \dots, -3}_{\nu_5 \text{ times}}, \dots$$

for $\nu = (0, 1, \nu_3, \dots, \nu_\ell) = \mathrm{Lad}(\mu)$; we refer to such an s as a ladder tableau of ladder weight μ. Then define $\mu(\mathsf{s})$ to be the coloured tableau obtained from s by replacing each entry i for $[\mathrm{Lad}(\mu)]_{c-1} < i \leqslant [\mathrm{Lad}(\mu)]_c$ in s by the entry c for $c \geqslant 1$.

We identify a coloured semistandard Young tableau, S, of weight μ with the set of standard Young tableaux, $[\mathsf{S}]_\mu = \{\mathsf{s} \mid \mu(\mathsf{s}) = \mathsf{S}\}$. Given $\mathsf{S} \in \mathrm{SStd}(\lambda, \mu)$ we let $\mathsf{s}^\lambda \in [\mathsf{S}]_\mu$ denote the unique most dominant tableau in $[\mathsf{S}]_\mu$.

Example 2.5. Continuing with Example 2.4, we let $\mathsf{S} \in \mathrm{CStd}((4,1^2),(4,2))$ depicted above. We have that

$$\left[\begin{array}{llll} 2&3&4&5\\ 3\\ 4 \end{array}\right]_\mu = \left\{ \begin{array}{llll} 1&2&4&6\\ 3\\ 5 \end{array}, \begin{array}{llll} 1&3&4&6\\ 2\\ 5 \end{array}, \begin{array}{llll} 1&3&5&6\\ 2\\ 4 \end{array}, \begin{array}{llll} 1&2&5&6\\ 3\\ 4 \end{array} \right\}$$

as an orbit of standard tableaux (which we have coloured in order to facilitate comparison).

For $\mu \in \mathscr{R}^e_n$, we let T^μ be the unique element of $\mathrm{CStd}(\mu,\mu)$. We set $\deg(\mathsf{T}^\mu) = 0$ so that

$$\mathrm{Dim}_t(e(\mu)\mathbf{S}_q(\mu)) = \sum_{\mathsf{t} \in \mathsf{T}^\mu} t^{\deg(\mathsf{t})} = [\mathrm{Lad}(\mu)]_t! = [\mathrm{Lad}(\mu)]_t! \times t^{\deg(\mathsf{T}^\mu)}$$

which is invariant under the bar map interchanging $t \leftrightarrow t^{-1}$ (see also [KN10, Lemma 3.4]). We now provide a general definition of the degree of a coloured tableau which allows us to calculate the graded characters of weight spaces of Specht modules in terms of coloured tableaux. Let $(a,b) \in \lambda \in \mathscr{P}_n$ be a node of residue $i \in \mathbb{Z}/e\mathbb{Z}$ and $\mu \in \mathscr{R}^e_n$, $\mathsf{S} \in \mathrm{CStd}(\lambda,\mu)$. We let $\mathcal{A}_{\mathsf{S}}(a,b)$ denote the set of all addable i-nodes of the partition

$$\lambda \cap \{(r,c) \mid \mathsf{S}(r,c) \leqslant \mathfrak{l}(a,b)\}$$

which are above $(a,b) \in \lambda$. We let $\mathcal{R}_{\mathsf{S}}(a,b)$ denote the set of all removable i-nodes of the partition

$$\lambda \cap \{(r,c) \mid \mathsf{S}(r,c) < \mathfrak{l}(a,b)\}$$

which are above $(a,b) \in \lambda$. We then define the degree of the node $(a,b) \in \lambda$ to be $|\mathcal{A}_{\mathsf{S}}(a,b)| - |\mathcal{R}_{\mathsf{S}}(a,b)|$. We define $\deg(\mathsf{S})$ to be the sum over the degrees of all nodes $(a,b) \in \lambda$.

We have seen that the tableaux of $\mathrm{CStd}(\lambda,\mu)$ are simply the orbits of tableaux from $\mathrm{Std}(\lambda)$ with a given residue sequence. Therefore, by comparing the degree function for coloured tableaux with that of standard tableaux we obtain

$$\mathrm{Dim}_t(e(\mu)\mathbf{S}_q(\lambda)) = [\mathrm{Lad}(\mu)]_t! \sum_{\mathsf{S} \in \mathrm{CStd}(\lambda,\mu)} t^{\deg(\mathsf{S})}. \tag{2.4}$$

And so coloured standard tableaux provide a combinatorial description of the ladder-weight multiplicity as defined in [KN10, Section 3.3].

Example 2.6. Continuing with $\mathsf{S} \in \mathrm{CStd}((4,1^2),(4,2))$ in Example 2.5, we have that

$$\mathcal{R}_{\mathsf{S}}(a,b) = \emptyset \text{ for all } (a,b) \in (4,1^2) \quad \text{and} \quad \mathcal{A}_{\mathsf{S}}(a,b) = \begin{cases} \{(2,2)\} & \text{if } (a,b) = (3,1) \\ \emptyset & \text{otherwise.} \end{cases}$$

and therefore

$$\deg_{\mathsf{S}}(a,b) = \begin{cases} 1 & \text{if } (a,b) = (3,1) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\deg(\mathsf{S}) = 1$. The four distinct standard tableaux $\mathsf{s} \in [\mathsf{S}]_{(4,2)}$ are depicted in Example 2.5; these four tableaux are obtained from each other by permuting the pairs $2,3$ in the third ladder and the pairs $4,5$ in the fourth ladder. We have that

$$\sum_{\mathsf{s} \in \mathsf{S}} t^{\deg(\mathsf{s})} = t^3 + 2t + t^{-1} = t \times (t + t^{-1})^2 = \deg(\mathsf{S}) \times [2]_t![2]_t! = \deg(\mathsf{S}) \times [\mathsf{Lad}(\mu)]_t!$$

With our new tableaux theoretic combinatorics in place, we can recast the (LLT) algorithm from [KN10, Section 4] in this combinatorial setting.

Example 2.7. We record the graded degrees of the coloured tableaux appearing in Example 2.4 and their conjugates (which are not pictured). Notice that conjugation does not preserve the degrees of tableaux.

	6	$5,1$	$4,2$
6	1	$*$	$*$
$5,1$	t	1	$*$
$4,2$	1	t	1
$4,1^2$	$2t$	t^2	t
3^2	$*$	$*$	t
2^3	$*$	$*$	t^2
$3,1^3$	$2t^2$	t	t^2
$2^2,1^2$	t^3	t^2	t^3
$2,1^4$	t^2	t^3	$*$
1^6	t^3	$*$	$*$

We are almost ready to restate the LLT algorithm in terms of our combinatorics, we simply require two observations about the graded structure of the Hecke algebra. The first is almost trivial, but the proof of the latter depends on incredibly deep geometric or categorical insights.

Theorem 2.8 ([BK09b, Theorem 4.18])**.** *For $\lambda \in \mathscr{P}_n$ and $\mu \in \mathscr{R}^e_n$, the polynomial $\mathrm{Dim}_t(e(\mu)\mathbf{D}^{\Bbbk}_q(\lambda))$ is bar-invariant (i.e., fixed under interchanging t and t^{-1}).*

Theorem 2.9 ([VV99])**.** *Let $\Bbbk = \mathbb{C}$. For $\lambda \in \mathscr{P}_n$ and $\mu \in \mathscr{R}^e_n$ with $\mu \neq \lambda$, $d_{\lambda,\mu}(t) \in t\mathbb{N}_0[t]$.*

Rearranging [KN10, Theorem 3.8] in terms of our coloured tableaux, we obtain the following relationships between coloured tableaux, simple characters, and graded decomposition numbers:

Proposition 2.10. *For $\lambda \in \mathscr{P}_n$ and $\mu \in \mathscr{R}_n^e$ we have that*

$$\mathrm{Dim}_t(e(\mu)\mathbf{S}^{\mathbb{C}}(\lambda)) = \sum_{\mathsf{S} \in \mathrm{CStd}(\lambda,\mu)} t^{\deg(\mathsf{S})}[\mathrm{Lad}(\mu)]_t! \in \mathbb{N}_0[t,t^{-1}] \quad \text{and} \quad \mathrm{Dim}_t(e(\mu)\mathbf{D}_q^{\Bbbk}(\lambda)) \in \mathbb{N}_0[t+t^{-1}].$$

Moreover, the following hold:

(i) if $\mathrm{CStd}(\lambda,\mu) = \emptyset$, *then* $d_{\lambda,\mu}(t) = 0$ *and* $\mathrm{Dim}_t(e(\mu)\mathbf{D}_q^{\Bbbk}(\lambda)) = 0$;

(ii) we have $\mathrm{Dim}_t(e(\mu)\mathbf{S}_q^{\Bbbk}(\mu))) = \mathrm{Dim}_t(e(\mu)\mathbf{D}_q^{\Bbbk}(\mu)) = [\mathrm{Lad}(\mu)]_t!$;

(iii) we have that

$$\mathrm{Dim}_t(e(\mu)\mathbf{D}_q^{\Bbbk}(\lambda)) + d_{\lambda,\mu}(t)[\mathrm{Lad}(\mu)]_t! = \sum_{\mathsf{S} \in \mathrm{CStd}(\lambda,\mu)} t^{\deg(\mathsf{S})}[\mathrm{Lad}(\mu)]_t! - \sum_{\lambda \lhd \nu \lhd \mu} \mathrm{Dim}_t(e(\mu)\mathbf{D}_q^{\Bbbk}(\nu))d_{\lambda,\nu}(t)$$

Now we set $\Bbbk = \mathbb{C}$. The right-hand side of the equation in Proposition 2.10(*iii*) is calculated by induction along the dominance ordering. Any polynomial in $\mathbb{N}_0[t,t^{-1}]$ can be written *uniquely* as the sum of a bar-invariant polynomial from $\mathbb{N}_0[t,t^{-1}]$ and a polynomial from $t\mathbb{N}_0[t]$. Putting together Theorems 2.8 and 2.9 we deduce that the lefthand-side is uniquely determined by the righthand-side and induction on the dominance order.

Example 2.11. We continue with Example 2.4. Using the equation in Proposition 2.10(*iii*), we obtain the first 5 rows of the graded decomposition matrix of the principal block of $H_{-1}^{\mathbb{C}}(6)$ and $\frac{1}{[\mathrm{Lad}(\mu)]_t!}\mathrm{Dim}_t(e(\mu)\mathbf{D}_q(\lambda))$ as follows:

	6	$5,1$	$4,2$
6	1	$*$	$*$
$5,1$	t	1	$*$
$4,2$	$*$	t	1
$4,1^2$	t	t^2	t
3^2	$*$	$*$	t
2^3	$*$	$*$	t^2
$3,1^3$	t^2	t	t^2
$2^2,1^2$	$*$	t^2	t^3
$2,1^4$	t^2	t^3	$*$
1^6	t^3	$*$	$*$

	6	$5,1$	$4,2$
6	1	$*$	$*$
$5,1$	$*$	1	$*$
$4,2$	1	$*$	1

Notice that if we multiply these two matrices together we obtain the matrix from Example 2.7. The remaining entries of the table can be deduced by applying the sign automorphism to the Specht modules (although this automorphism is not of degree zero and so the entries will differ by a degree shift). Comparing with the table in Example 2.7, we observe that the entry in the row labelled by $(4,2)$ and column labelled by (6) is bar-invariant in Example 2.7 and so does not contribute to the decomposition matrix, but instead contributes a vector in the simple module $\mathbf{D}_{-1}(4,2)$. Then in the row labelled by $(4,1^2)$ we see another discrepancy between the two tables: this is because $\mathbf{D}_{-1}(4,2)$ is a composition factor of $\mathbf{S}_{-1}(4,1^2)$ and so it contributes to the sum in Proposition 2.10(*iii*).

We are now ready to provide new upper bounds for (graded) decomposition numbers in terms of our coloured tableaux.

Theorem 2.12. *For $\lambda \in \mathscr{P}_n$ and $\mu \in \mathcal{R}_n^e$ and $\Bbbk$ an arbitrary field, we have that*

$$[\mathbf{S}_q^{\Bbbk}(\lambda) : \mathbf{D}_q^{\Bbbk}(\mu)\langle k\rangle] \leqslant |\{\mathsf{S} \mid \mathsf{S} \in \mathrm{CStd}(\lambda,\mu), \deg(\mathsf{S}) = k\}|$$

for $k \in \mathbb{Z}$ and in particular, $d_{\lambda,\mu}^{\Bbbk} \leqslant |\mathrm{CStd}(\lambda,\mu)|$.

Proof. It is immediate from equation (2.4) that

$$[\mathbf{S}_q^{\Bbbk}(\lambda) : \mathbf{D}_q^{\Bbbk}(\mu)\langle k\rangle] \leqslant [\mathrm{Lad}(\mu)]_t! \times |\{\mathsf{S} \mid \mathsf{S} \in \mathrm{CStd}(\lambda,\mu), \deg(\mathsf{S}) = k\}|$$

and indeed this is just rephrasing a classical observation due to Gordon James. The new observation is that by Proposition 2.10, we know that $[\mathrm{Lad}(\mu)]_t!$ divides both

$$\mathrm{Dim}_t(e(\mu)(\mathbf{S}_q^{\Bbbk}(\lambda)) \qquad \mathrm{Dim}_t(e(\mu)(\mathbf{D}_q^{\Bbbk}(\mu)))$$

and the result follows by induction on the dominance ordering and the equation in Proposition 2.10(iii). In more detail, our base case for induction is when $\mu = \lambda$ mentioned above. Now, by Proposition 2.10(iii) and our inductive assumption, the result holds for all ν such that $\mu \rhd \nu \rhd \lambda$. Putting this together with Proposition 2.10(ii), we deduce that $[\mathrm{Lad}(\mu)]_t!$ divides $\mathrm{Dim}_t(e(\mu)\mathbf{D}_q^{\Bbbk}(\lambda))$ as required. □

Example 2.13. If $p = 2$ then the graded decomposition matrix of $\Bbbk\mathfrak{S}_6$ is given by the table in Example 2.7. In other words, the bounds of Theorem 2.12 are sharp.

Remark 2.14. The inductive approach to calculating decomposition numbers of $H_q^{\mathbb{C}}(n)$ highlighted in the equation in Proposition 2.10(iii) above is used in the arXiv appendix to this paper to prove decomposability of an infinite family of Specht modules. In Section 4 the above algorithm will not work (as the set of 2-separated partitions is not saturated in the dominance order). However, we provide an analogous algorithm for calculating 2-separated decomposition numbers using "2-dilated" coloured tableaux.

3. The Cherednik algebra and a simple criterion for semisimplicity of a Specht module

The group $\mathfrak{S}_n$ acts on the algebra, $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots y_n\rangle$, of polynomials in $2n$ non-commuting variables. The rational Cherednik algebra $\mathscr{H}_q(\mathfrak{S}_n)$ is a quotient of the semidirect product algebra $\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle \rtimes \mathfrak{S}_n$ by commutation relations in the x's and y's that are similar to those of the Weyl algebra but involve an error term in $\mathbb{C}\mathfrak{S}_n$ (see [EG02, Section 1] for the full list of relations). In particular, these relations tell us that the x's commute with each other and so do the y's. The algebra $\mathscr{H}_q(\mathfrak{S}_n)$ has three distinguished subalgebras: $\mathbb{C}[\underline{y}] := \mathbb{C}[y_1, \dots, y_n]$, $\mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \dots, x_n]$, and the group algebra $\mathbb{C}\mathfrak{S}_n$. The *PBW theorem* [EG02, Theorem 1.3] asserts that multiplication gives a vector space isomorphism

$$\mathbb{C}[\underline{x}] \otimes \mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}[\underline{y}] \xrightarrow{\cong} \mathscr{H}_q(\mathfrak{S}_n)$$

called the *triangular decomposition* of $\mathscr{H}_q(\mathfrak{S}_n)$, by analogy with the triangular decomposition of the universal enveloping algebra of a semisimple Lie algebra. We define the category $\mathcal{O}_q(\mathfrak{S}_n)$ to be the full subcategory consisting of all finitely generated $\mathscr{H}_q(\mathfrak{S}_n)$-modules on which $y_1, \dots, y_n$ act locally nilpotently. The category $\mathcal{O}_q(\mathfrak{S}_n)$ is a highest weight category with respect to the poset $(\mathscr{P}_n, \rhd)$. The standard modules are constructed as follows. Extend the action of $\mathfrak{S}_n$ on $\mathbf{S}^{\mathbb{C}}(\lambda)$ to an action of $\mathbb{C}[\underline{y}] \rtimes \mathfrak{S}_n$ by letting $y_1, \dots, y_n$ act by 0. The algebra $\mathbb{C}[\underline{y}] \rtimes \mathfrak{S}_n$ is a subalgebra of $\mathscr{H}_q(\mathfrak{S}_n)$ and we define the Weyl modules,

$$\Delta(\lambda) := \mathrm{Ind}_{\mathbb{C}[\underline{y}]\rtimes\mathfrak{S}_n}^{\mathscr{H}_q(\mathfrak{S}_n)} \mathbf{S}^{\mathbb{C}}(\lambda) := \mathscr{H}_q(\mathfrak{S}_n) \otimes_{\mathbb{C}[\underline{y}]\rtimes\mathfrak{S}_n} \mathbf{S}^{\mathbb{C}}(\lambda) = \mathbb{C}[\underline{x}] \otimes \mathbf{S}^{\mathbb{C}}(\lambda)$$

where the last equality is *only as* $\mathbb{C}[\underline{x}]$*-modules* and follows from the triangular decomposition. We let $L(\lambda)$ denote the unique irreducible quotient of $\Delta(\lambda)$. In [RSVV16, Theorem 7.4.] (see also [Los16, Web13]) it is shown that $\mathcal{O}_q(\mathfrak{S}_n)$ is standard Koszul. We do not recall the definition of a standard Koszul algebra here, but merely the following useful proposition. The following proposition is proven in [BGS96, Proposition 2.4.1] in the generality of all Koszul algebras.

Proposition 3.1. *For* $\lambda, \mu \in \mathscr{P}_n$ *we have that*

$$[\Delta(\lambda) : L(\mu)\langle i\rangle] = \dim_{\Bbbk} \mathrm{Hom}_{\mathscr{H}_q(\mathfrak{S}_n)}(\mathrm{rad}_i(\Delta^{\mathbb{C}}(\lambda)), L^{\mathbb{C}}(\mu)).$$

Proof. By [BGS96, Corollary 2.3.3], any Koszul algebra is quadratic. Therefore, since $\Delta(\lambda)/\mathrm{rad}(\Delta(\lambda)) = L(\lambda)$ is simple and concentrated in degree zero, the radical filtration of $\Delta(\lambda)$ coincides with the grading filtration of $\Delta(\lambda)$ by [BGS96, Proposition 2.4.1]. □

Now, there exists an exact functor (the Knizhnik–Zamolodchikov functor) relating the module categories of Cherednik and Hecke algebras,

$$\mathsf{KZ} : \mathscr{O}_q(\mathfrak{S}_n) \longrightarrow H_q^{\mathbb{C}}(n)\text{-mod}.$$

A construction of this functor is given in [GGOR03], here we will only need the fact (from [GGOR03, Section 6]) that

$$\mathsf{KZ}(\Delta(\lambda)) = \mathbf{S}_q^{\mathbb{C}}(\lambda), \quad \mathsf{KZ}(L(\lambda)) = \begin{cases} \mathbf{D}_q^{\mathbb{C}}(\lambda) & \text{if } \lambda \text{ is } e\text{-regular}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

This allows us to prove the following criterion for decomposability of Specht modules, denoted $\mathbf{S}_q^{\mathbb{C}}(\lambda)$, for the Hecke algebra $H_q^{\mathbb{C}}(n)$.

Theorem 3.2. *Fix $\lambda \in \mathscr{P}_n$. Suppose that for all e-regular partitions μ, we have that*

$$[\mathbf{S}_q^{\mathbb{C}}(\lambda) : \mathbf{D}_q^{\mathbb{C}}(\mu)] = a_\mu t^{p(\lambda)} \tag{3.2}$$

for some fixed $p(\lambda) = z \in \mathbb{N}$ (independent of μ) and some scalars $a_\mu \in \mathbb{N}$. It follows that the Specht module $\mathbf{S}_q^{\mathbb{C}}(\lambda)$ is semisimple.

Proof. Throughout the proof, we let $\lambda \in \mathscr{P}_n$ be an arbitrary partition. For an e-regular partition $\mu \in \mathscr{P}_n$, we have that

$$[\Delta(\lambda) : L(\mu)\langle k\rangle] = [\mathsf{KZ}(\Delta(\lambda)) : \mathsf{KZ}(L(\mu)\langle k\rangle)] = [\mathbf{S}_q^{\mathbb{C}}(\lambda) : \mathbf{D}_q^{\mathbb{C}}(\mu)\langle k\rangle].$$

by equation (3.1). Putting together Proposition 3.1 and our assumption in equation (3.2), we have that

$$[\mathrm{rad}_i(\Delta(\lambda)) : L(\mu)] = 0$$

for μ any e-regular partition and any $i \neq z$. Therefore

$$\mathsf{KZ}(\mathrm{rad}_i(\Delta(\lambda)) = \begin{cases} \bigoplus_\mu a_\mu \mathbf{D}_q^{\mathbb{C}}(\mu)\langle z\rangle & \text{for } i = z, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\mathsf{KZ}(\Delta(\lambda)) = \mathsf{KZ}(\mathrm{rad}_z(\Delta(\lambda))) = \bigoplus_\mu a_\mu \mathbf{D}_q^{\mathbb{C}}(\mu)\langle z\rangle,$$

and the result follows. ☐

Remark 3.3. We have seen the grading and radical structure of standard $\mathscr{H}_q(\mathfrak{S}_n)$-modules are intimately related. It is unknown as to whether or not the Schur functor preserves this property. Thus Theorem 3.2 represents all that is currently known about the relationship between the grading and radical structure on Specht modules for $H_q^{\mathbb{C}}(n)$.

4. The Hecke algebra and 2-separated partitions

Throughout this section, we shall consider the representation theory of the Hecke algebra $H_{-1}^{\mathbb{C}}(n)$ as a first approximation to the 2-modular representation theory of symmetric groups. We shall focus on the Specht modules labelled by 2-separated partitions. We shall prove that these modules are semisimple and calculate their decomposition as a direct sum of graded simple modules. Throughout this section our character formulas will be given in terms of binomial coefficients that have been "dilated" by a factor of 2. We shall write $[\![m]\!]_t = [m]_{t^2}$ and extend this notation to the quantum factorials and binomials in the obvious fashion.

Theorem 4.1. *We have that*

$$\mathrm{Dim}_t\left(e\left(\mathsf{L}^{\tau^\alpha_\varnothing}\right)\mathbf{S}_{-1}^{\mathbb{C}}\left(\tau^\lambda_\mu\right)\right) = \left|\mathrm{SStd}(\lambda^T \cup \mu, \alpha^T)\right| \times [\![\alpha^T]\!]_t!^2 \times t^{|\mu|}.$$

Before embarking on the proof, we note the following immediate corollaries.

Corollary 4.2. *We have that*

$$e\left(\mathsf{L}^{\tau^{\alpha}_{\varnothing}}\right)\mathbf{S}^{\mathbb{C}}_{-1}\left(\tau^{\lambda}_{\mu}\right)=t^{|\mu|}\sum_{\nu}c\left(\nu^{T},\lambda^{T},\mu\right)e\left(\mathsf{L}^{\tau^{\alpha}_{\varnothing}}\right)\mathbf{S}^{\mathbb{C}}_{-1}\left(\tau^{\nu}_{\varnothing}\right)$$

and therefore

$$\left[\mathbf{S}^{\mathbb{C}}_{-1}\left(\tau^{\lambda}_{\mu}\right)\right]=\sum_{\nu}t^{|\mu|}\times c\left(\nu^{T},\lambda^{T},\mu\right)\times\left[\mathbf{S}^{\mathbb{C}}_{-1}\left(\tau^{\nu}_{\varnothing}\right)\right].$$

Therefore the Specht modules $\mathbf{S}^{\mathbb{C}}_{-1}(\tau^{\nu}_{\varnothing})$ are simple (as their characters are bar-invariant) and by Section 3 we deduce the following corollary.

Corollary 4.3. *We have that, as an $H^{\mathbb{C}}_{-1}(n)$-module, any Specht module labelled by a 2-separated partition is semisimple and decomposes as follows*

$$\mathbf{S}^{\mathbb{C}}_{-1}\left(\tau^{\lambda}_{\mu}\right)=\bigoplus_{\nu}c\left(\nu^{T},\lambda^{T},\mu\right)\mathbf{D}^{\mathbb{C}}_{-1}\left(\tau^{\nu}_{\varnothing}\right)\langle|\mu|\rangle.$$

In particular, the Specht $H^{\mathbb{C}}_{-1}(n)$-module $\mathbf{S}^{\mathbb{C}}_{-1}(\tau^{\lambda}_{\mu})$ is simple if and only if λ or μ is equal to $\varnothing$.

We now turn to the proof of the main result.

Proof of Theorem 4.1. We let $\rho:=\rho(k)$ and we assume for notational purposes that k is even; the k odd case is identical except that the residues 0 and 1 must be transposed. We let $w=|\alpha|=|\lambda|+|\mu|$. Consider the ladder tableau of the partition $\tau^{\alpha}_{\varnothing}$. We have that

$$\mathrm{res}(\mathsf{L}^{\tau^{\alpha}_{\varnothing}})=\mathrm{res}(\rho)\circ(\underbrace{0,0,\dots,0}_{\alpha^{T}_{1}\text{ times}},\underbrace{1,1,\dots,1}_{\alpha^{T}_{1}\text{ times}})\circ(\underbrace{0,0,\dots,0}_{\alpha^{T}_{2}\text{ times}},\underbrace{1,1,\dots,1}_{\alpha^{T}_{2}\text{ times}})\circ\cdots.$$

Given $\mathsf{t}\in\mathrm{Std}(\tau^{\lambda}_{\mu})$, it is clear that $\mathrm{res}(\mathsf{t})=\mathrm{res}(\mathsf{L}^{\tau^{\alpha}_{\varnothing}})$ if and only if $\mathsf{t}=\mathsf{L}^{\rho}\circ\mathsf{s}$ for some $\mathsf{s}\in\mathrm{Std}(\tau^{\lambda}_{\mu}\backslash\rho)$ with

$$\mathrm{res}(\mathsf{s})=(\underbrace{0,0,\dots,0}_{\alpha^{T}_{1}\text{ times}},\underbrace{1,1,\dots,1}_{\alpha^{T}_{1}\text{ times}})\circ(\underbrace{0,0,\dots,0}_{\alpha^{T}_{2}\text{ times}},\underbrace{1,1,\dots,1}_{\alpha^{T}_{2}\text{ times}})\circ\cdots$$

and we let $\mathrm{Std}_{\alpha}(\tau^{\lambda}_{\mu}\setminus\rho)$ denote the set of all such tableaux. All that remains is to show that

$$\sum_{\mathsf{s}\in\mathrm{Std}_{\alpha}(\tau^{\lambda}_{\mu}\backslash\rho)}t^{\deg(\mathsf{s})}=\left|\mathrm{SStd}\left(\lambda^{T}\cup\mu,\alpha^{T}\right)\right|\times[\![\alpha^{T}]\!]^{2}_{t}\times t^{|\mu|}. \tag{4.1}$$

Given an integer $j\in\{1,2,\dots,2w\}$ we have that there exists a unique corresponding integer $a(j)\in\{1,\dots,w\}$ such that

$$1+\sum_{i=1}^{a(j)}|\alpha^{T}_{i}|\leqslant j\leqslant 1+\sum_{i=1}^{a(j)+1}|\alpha^{T}_{i}|;$$

these integers will record the weight of the semistandard tableaux in the statement of equation (4.1). Namely, we record a skew-tableau s by placing both the usual entry $j\in\{1,2,\dots,2w\}$ but we also add a subscript $a(j)$. An example is depicted on the left-hand side of Figure 7. Recall that we can think of the partition τ^{λ}_{μ} as being obtained by adding (2)-dominoes to the right of ρ and (1^2)-dominoes to the bottom of ρ in an intuitive fashion demonstrated in Figure 6. Take the partition ρ and add a total of α^{T}_{1} nodes of residue 0; the resulting partition has precisely α^{T}_{1} addable 1-nodes $X_1,\dots,X_{\alpha_1}$: namely, those which belong to the (2)- and (1^2)-dominoes containing the nodes $X_1,\dots,X_{\alpha_1}$. Repeating this observation as necessary, we deduce that any two nodes in the same domino of a tableau $\mathsf{s}\in\mathrm{Std}_{\alpha}(\tau^{\lambda}_{\mu}\setminus\rho)$ have the same subscript. Furthermore, we note that the fact that the residue sequence is of the form

$$(\underbrace{0,0,\dots,0}_{\alpha^{T}_{1}\text{ times}},\underbrace{1,1,\dots,1}_{\alpha^{T}_{1}\text{ times}})\circ(\underbrace{0,0,\dots,0}_{\alpha^{T}_{2}\text{ times}},\underbrace{1,1,\dots,1}_{\alpha^{T}_{2}\text{ times}})\circ\cdots$$

implies that no two (2)-dominoes of the same subscript can be added in the same row and no two (1^2)-dominoes of the same subscript can be added in the same column. Therefore we obtain a well-defined map

$$\varphi : \mathrm{Std}_\alpha\left(\tau^\lambda_\mu \setminus \rho\right) \longmapsto \mathrm{SStd}\left(\lambda^T \cup \mu, \alpha^T\right)$$

given by scaling the sizes of all the dominoes by $1/2$, conjugating λ, and recording *only* the subscripts (i.e., deleting the integers $\{1, \dots, 2w\}$). An example is depicted in Figure 7.

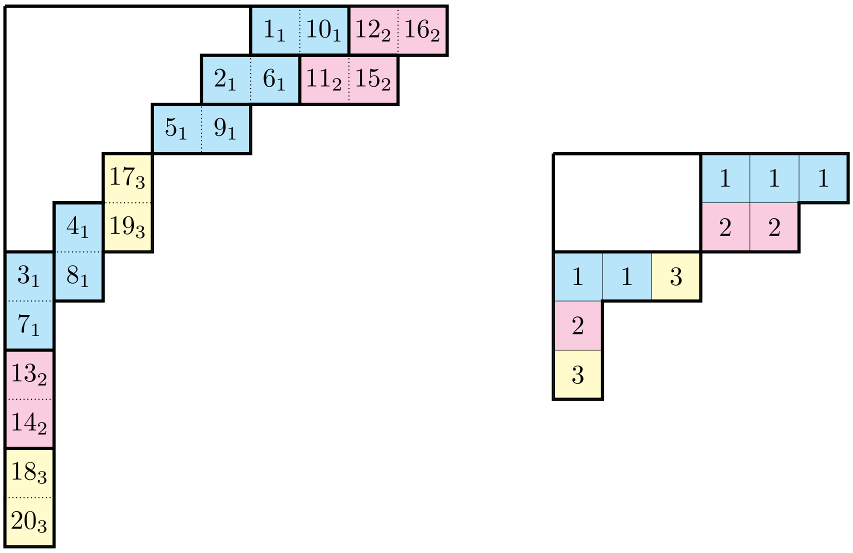

FIGURE 7. A $(3^2, 2, 1^2)$-decorated standard tableau of shape $\tau^{(2,2,1)}_{(3,1,1)} \setminus \rho$ and the corresponding element of $\mathrm{SStd}((3,2)\cup(3,1,1),(5,3,2))$. The associated sequence of partitions (as in equation (4.3)) is $\tau \subset \tau^{(1^3)}_{(2)} \subset \tau^{(2^2,1)}_{(2,1)} \subset \tau^{(2^2,1)}_{(3,1^2)}$.

All that remains to show is that

$$\sum_{\{\mathsf{s} \mid \varphi(\mathsf{s})=\mathsf{S}\}} t^{\deg(\mathsf{s})} = [\![\alpha^T]\!]_t!^2 \times t^{|\mu|} \tag{4.2}$$

for any $\mathsf{S} \in \mathrm{SStd}(\lambda^T \cup \mu, \alpha^T)$. The set $\{\mathsf{s} \mid \varphi(\mathsf{s}) = \mathsf{S}\}$ consists of an orbit

$$\mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \mathfrak{S}_{\alpha_2} \times \dots$$

of standard tableaux. In other words, s, t are such $\varphi(\mathsf{s}) = \varphi(\mathsf{t})$ if and only if they differ by permuting nodes whose subscripts and residues are both matching. Therefore, we have that

$$\sum_{\{\mathsf{s} \mid \varphi(\mathsf{s})=\mathsf{S}\}} 1^{\deg(\mathsf{s})} = \left(\alpha^T_1!\alpha^T_2!\dots\right)^2 = \left([\![\alpha^T]\!]_t!^2\right)|_{t=1},$$

and so the ungraded version of equation (4.2) follows.

It remains to consider the grading. We first cut the diagram of any 2-separated partition τ^λ_μ into four regions by drawing a vertical line immediately after the μ_1th column of τ^λ_μ and a horizontal line immediately below the λ^T_1th row. An example is depicted in Figure 8. We label the three of the four quarters of the diagram $X := X(\tau^\lambda_\mu)$, $Y := Y(\tau^\lambda_\mu)$, and $Z := Z(\tau^\lambda_\mu)$ as suggested in Figure 8. The intersection of τ^λ_μ with the region Y is equal to the staircase partition of width $\rho_1 - \mu_1 - \lambda^T_1$; we set $\rho_Y := [\tau^\lambda_\mu] \cap Y$.

We shall calculate the lefthand-side of equation (4.2) by peeling off a row of α at a time, in a manner which we now make precise. Fix $\alpha \in \mathscr{P}_n$ a partition and set $\ell = \ell(\alpha^T)$ and $\mathsf{S} \in \mathrm{SStd}(\lambda^T \cup \mu, \alpha^T)$. We define

$$\mu^{(i)}_j = |\{(j, c) \in \mu \mid \mathsf{S}(j, c) \leqslant i\}| \qquad \lambda^{(i)}_j = |\{(r, j) \in \lambda \mid \mathsf{S}(r, j) \leqslant i\}|$$

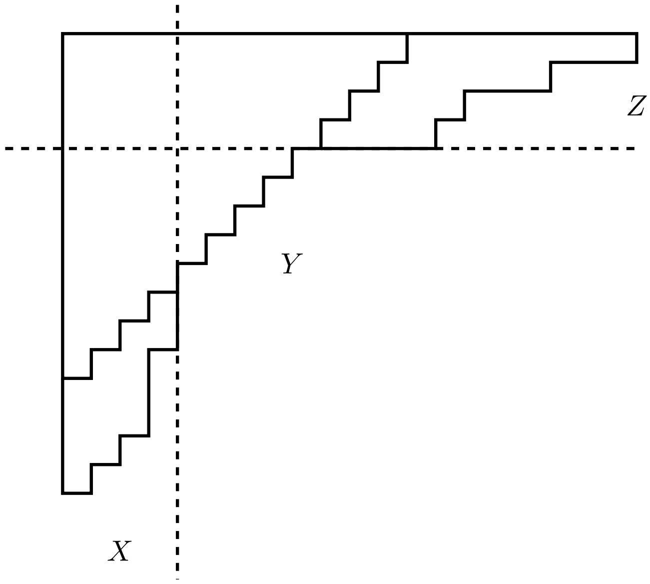

FIGURE 8. Dividing the partition τ^λ_μ into regions $X := X(\tau^\lambda_\mu)$, $Y := Y(\tau^\lambda_\mu)$ and $Z := Z(\tau^\lambda_\mu)$. In this case ρ_Y is the copy of the partition $(4, 3, 2, 1)$ in region Y.

for $1 \leqslant i \leqslant \ell(\alpha)$. We set $\mu(i) = (\mu_1^{(i)}, \dots, \mu_\ell^{(i)})$ and $\lambda(i) = (\lambda_1^{(i)}, \dots, \lambda_\ell^{(i)})$ We consider the associated sequence of partitions

$$\tau = \tau^{\lambda(0)}_{\mu(0)} \subseteq \tau^{\lambda(1)}_{\mu(1)} \subseteq \tau^{\lambda(2)}_{\mu(2)} \cdots \subseteq \tau^{\lambda(\ell)}_{\mu(\ell)} = \tau^\lambda_\mu, \tag{4.3}$$

an example is given in Figure 7. Setting $\alpha^T = (a_1, \dots, a_\ell)$, we will show that

$$\sum_{\mathsf{s} \in \mathrm{Std}_{(a_m)}(\tau^{\lambda(m)}_{\mu(m)} \setminus \tau^{\lambda(m-1)}_{\mu(m-1)})} t^{\deg(\mathsf{s})} = [\![a_m]\!]_t!^2 \times t^{|\mu(m)| - |\mu(m-1)|},$$

and hence deduce the result. Clearly we can calculate the degree contribution of the node (r, c) to the tableau

$$\mathsf{s} \in \mathrm{Std}_{(a_m)}(\tau^{\lambda(m)}_{\mu(m)} \setminus \tau^{\lambda(m-1)}_{\mu(m-1)})$$

by considering the addable/removable nodes above (r, c) from the regions

$$X_m := X(\tau^{\lambda(m)}_{\mu(m)}) \quad Y_m := Y(\tau^{\lambda(m)}_{\mu(m)}) \quad Z_m := Z(\tau^{\lambda(m)}_{\mu(m)})$$

separately. We let $\deg_{X_m}(r, c)$, $\deg_{Y_m}(r, c)$ and $\deg_{Z_m}(r, c)$ denote these respective contributions. We first consider the contribution from Y_m. By definition, the sum total

$$\deg_{Y_m}(\mathsf{s}) = \sum_{1 \leqslant k \leqslant 2m} \deg_{Y_m} \mathsf{s}^{-1}(k)$$

is independent of the tableau $\mathsf{s} \in \mathrm{Std}(\tau^{\lambda(m)}_{\mu(m)} \setminus \tau^{\lambda(m-1)}_{\mu(m-1)})$. For ρ_{Y_m} a staircase partition of $p(p+1)/2$, we have that

$$|\mathrm{Rem}_0(\rho_{Y_m})| = 0 \quad |\mathrm{Rem}_1(\rho_{Y_m})| = p \quad |\mathrm{Add}_0(\rho_{Y_m})| = p + 1 \quad |\mathrm{Add}_1(\rho_{Y_m})| = 0.$$

Therefore $\deg_{Y_m}(\mathsf{s})$ is equal to the number of (1^2)-bricks in region $X_m \cap \tau^{\lambda(m)}_{\mu(m)} \setminus \tau^{\lambda(m-1)}_{\mu(m-1)}$. In other words $\deg_{Y_m}(\mathsf{s}) = |\mu| - |\mu'|$. Finally, it remains to prove that

$$\sum_{\mathsf{s} \in \mathrm{Std}_{(a_m)}(\tau^{\lambda(m)}_{\mu(m)} \setminus \tau^{\lambda(m-1)}_{\mu(m-1)})} t^{\deg_{X_m \cup Z_m}(\mathsf{s})} = [\![a_m]\!]_t!^2 \tag{4.4}$$

which we shall do by induction. We set $\mathrm{Std}_m := \mathrm{Std}(\rho(a_m) \setminus \rho(a_m - 1))$. It is easy to see that

$$\sum_{\mathsf{s} \in \mathrm{Std}_{(a_m)}(\tau^{\lambda(m)}_{\mu(m)} \setminus \tau^{\lambda(m-1)}_{\mu(m-1)})} t^{\deg_{X_m \cup Z_m}(\mathsf{s})} = \left(\sum_{\mathsf{s} \in \mathrm{Std}_m} t^{\deg(\mathsf{s})} \right) \times \left(\sum_{\mathsf{s} \in \mathrm{Std}_m} t^{\deg(\mathsf{s})} \right)$$

where the first/second multiplicand on the righthand-side counts the contribution of all the residue 0-boxes/1-boxes respectively. We set $\square_j = (j, a_m + 1 - j) \in \rho(a_m)$ for $1 \leqslant j \leqslant a_m$. We have that

$$\sum_{\substack{1 \leqslant j \leqslant a_m \\ \{\mathsf{s} \in \mathrm{Std}_m \mid \mathsf{s}^{-1}(a_m) = \square_j\}}} t^{\deg(\mathsf{s})} = \sum_{\substack{\mathsf{t} \in \mathrm{Std}_{m-1} \\ 1 \leqslant j \leqslant a_m}} t^{a_m - j} \times t^{1-j} \times t^{\deg(\mathsf{t})} = \sum_{1 \leqslant j \leqslant a_m} t^{a_m + 1 - 2j} [\![a_m - 1]\!]_t!$$

which is equal to $[\![a_m]\!]_t!$. Here the first equality follows by considering both the degree of the node $\mathsf{s}^{-1}(a_m)$ (which is equal to $1-j$) and the resulting shift to the degrees of each of the $(a_m - j)$ nodes below $\mathsf{s}^{-1}(a_m)$. The second equality holds by induction. Therefore equation (4.4) holds and the result follows. $\square$

5. Kronecker coefficients and Saxl's conjecture

Let λ, μ, ν be partitions of n. We define the Kronecker coefficients $g(\lambda, \mu, \nu)$ to be the coefficients in the expansion

$$\mathbf{D}^{\mathbb{C}}(\lambda) \otimes \mathbf{D}^{\mathbb{C}}(\mu) = \bigoplus_{\nu \vdash n} g(\lambda, \mu, \nu)\, \mathbf{D}^{\mathbb{C}}(\nu)\,.$$

We now recall Saxl's conjecture concerning the positivity of these coefficients. We let χ^λ denote the complex irreducible $\mathfrak{S}_n$-character to the partition λ of n, i.e., the character of the Specht module $\mathbf{S}^{\mathbb{C}}(\lambda)$.

Saxl's conjecture. *Let $n = k(k+1)/2$ and $\rho = (k, k-1, \dots, 2, 1)$. For all $\lambda \vdash n$, the multiplicity of χ^λ in the Kronecker product $\chi^\rho \cdot \chi^\rho$ is strictly positive.*

In [HSTZ13], Heide, Saxl, Tiep and Zalesski verified that for almost all finite simple groups of Lie type the square of the Steinberg character contains all irreducible characters as constituents. They also conjectured that for all alternating groups there is some irreducible character with this property. For symmetric groups $\mathfrak{S}_n$ to triangular numbers n, Saxl then suggested the candidate χ^ρ as stated above. Saxl's conjecture has been attacked by algebraists and complexity theorists using a variety of combinatorial and probabilistic methods [Bes18, Ike15, LS17, PPV16]. From our perspective, a particularly useful result is the following.

Theorem 5.1 ([Ike15, Theorem 2.1])**.** *Let $n = k(k+1)/2$ and $\rho = (k, k-1, \dots, 2, 1)$. If λ is a partition of n such that $\lambda \trianglerighteq \rho$ or $\lambda \trianglelefteq \rho$, then $g(\rho, \rho, \lambda) > 0$.*

We let $\Bbbk$ be a field of characteristic 2. We keep the notation $\rho = \rho(k)$ and $n = k(k+1)/2$, and we note that $\mathbf{D}^{\Bbbk}(\rho) = \mathbf{S}^{\Bbbk}(\rho) = \mathbf{P}^{\Bbbk}(\rho)$ is a simple projective $\Bbbk\mathfrak{S}_n$-module. Therefore its tensor square is also projective and decomposes as a direct sum of indecomposable projective modules labelled by 2-regular partitions; we let $G(\rho, \rho, \nu)$ denote the corresponding coefficients as follows

$$\mathbf{D}^{\Bbbk}(\rho) \otimes \mathbf{D}^{\Bbbk}(\rho) = \bigoplus_{\nu \vdash_2 n} G(\rho, \rho, \nu)\, \mathbf{P}^{\Bbbk}(\nu)\,.$$

Equivalently, on the level of complex characters we have for the irreducible character χ^ρ the decomposition of its square into characters ξ^ν to projective indecomposable modules (i.e., to integral lifts of the projective modules at characteristic 2):

$$(\chi^\rho)^2 = \bigoplus_{\nu \vdash_2 n} G(\rho, \rho, \nu)\, \xi^\nu\,.$$

We wish to pass information back and forth between the 2-modular coefficients $G(\lambda, \mu, \nu)$ and the Kronecker coefficients $g(\lambda, \mu, \nu)$ in order to make headway on Saxl's conjecture. The following observation is immediate

$$g(\rho, \rho, \lambda) = \sum_{\nu} G(\rho, \rho, \nu) d_{\lambda,\nu}. \tag{5.1}$$

We first want to explain how to apply this to obtain positivity for new classes of Kronecker coefficients. Recall from Subsection 1.3 that the 2-blocks of $\mathfrak{S}_n$ are parameterized by the common 2-core of the partitions labelling the simple modules in characteristic 2 and the Specht modules in the block, together with the weight. Further recall from Subsection 1.1 that the decomposition matrix for each block is unitriangular with respect to the dominance ordering. In particular, the most dominant partition, $\tau_{\varnothing}^{(w)}$, in a given 2-block of weight w labels a Specht module with simple reduction mod 2. Hence, applying Ikenmeyer's result we deduce that any non-zero entry of the first column (labelled by $\tau_{\varnothing}^{(w)}$) of the 2-decomposition matrix of any 2-block corresponds to a non-zero Kronecker coefficient. This allows us to verify Kronecker positivity in Saxl's conjecture for two new infinite families of partitions:

Theorem 5.2. *Let $n = k(k+1)/2$, $\rho = \rho(k)$ and $\lambda \vdash n$ such that χ^λ is of height 0. Then $g(\rho, \rho, \lambda) > 0$. In particular, all χ^λ of odd degree are constituents of the Saxl square.*

Proof. Let B be the 2-block of $\mathfrak{S}_n$ to which χ^λ belongs. Because χ^λ is a character of height 0, the modulo 2 reduction $\mathbf{S}^{\Bbbk}(\lambda)$ must have a composition factor of height zero and this composition factor must appear with odd multiplicity (simply by comparing the dimensions, see Subsection 1.4). By Theorem 1.13, the 2-block B contains a unique simple module $D = \mathbf{D}^{\Bbbk}(\mu)$ of height 0, with $\mu = \tau_{\varnothing}^{(w)} = \tau + (2w)$ the most dominant partition belonging to the block. The discussion preceding the theorem now implies

$$g(\rho, \rho, \lambda) \geq d_{\lambda, \tau+(2w)} > 0. \quad \square$$

Since we get the large number $k_0(B)$ of height 0 irreducible characters for each 2-block B, this constitutes quite a large class of constituents in the Saxl square.

Example 5.3. We have already seen (in Example 1.12) that the block $B_3(36)$ contains 1024 height 0 characters, which all appear in Saxl's tensor square.

Example 5.4. We consider the partition $\lambda = (9, 8, 3, 2, 2, 2, 1, 1) \vdash 28$. This is the first of the four partitions pictured in Figure 2. In this case, the desired positivity of $g(\rho(7), \rho(7), \lambda)$ cannot be deduced using the available non-vanishing criteria in the literature [Bes18, PPV16], and λ is incomparable to ρ in the dominance order, so [Ike15] does not apply. The character χ^λ belongs to the 2-block of weight $w = 11$ and 2-core $\rho(3)$, and it is of height 0. Instead of computing the degree explicitly, this can also be seen by applying one of the combinatorial descriptions for labels of height 0 characters, e.g., the one due to [GMT18] recalled in Subsection 1.4. Finally, referring forward in this paper: we remark that $(9, 8, 3, 2, 2, 2, 1, 1)$ is not 2-separated. Thus Theorem 5.2 provides us with constituents of the Saxl square that cannot be deduced using any other results in the literature.

For the second new family; we will also need to apply our new results on Specht modules for the Hecke algebra.

Theorem 5.5. *For $\tau_{(1^\ell)}^{(m)}$ any framed staircase partition of $n = k(k+1)/2$, we have that $g(\rho, \rho, \tau_{(1^\ell)}^{(m)}) > 0$.*

Proof. Let $w = \ell + m$ be the weight of the 2-block B to which $\tau_{(1^\ell)}^{(m)}$ belongs. We have that $\tau_{\varnothing}^{(w)}$ is the most dominant partition in B and so the corresponding Specht module is simple. Now

$$[\mathbf{S}_{-1}^{\Bbbk}(\tau_{(1^\ell)}^{(m)}) : \mathbf{D}_{-1}^{\Bbbk}(\tau_{\varnothing}^{(w)})] \geqslant [\mathbf{S}_{-1}^{\mathbb{C}}(\tau_{(1^\ell)}^{(m)}) : \mathbf{D}_{-1}^{\mathbb{C}}(\tau_{\varnothing}^{(w)})] = c((1^w), (1^m), (1^\ell)) = 1 > 0$$

and so the result follows by the discussion above. ☐

For reasons that will soon become apparent, we now recall Carter's criterion explicitly.

Theorem 5.6 ([JM99])**.** *We let $\Bbbk$ be a field of characteristic 2. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ be a partition. Then the Specht module $\mathbf{S}^{\Bbbk}(\lambda)$ is simple if and only if one of the following conditions holds:*

(i) $\lambda_i - \lambda_{i+1} \equiv -1$ *modulo* $2^{\ell_2(\lambda_{i+1}-\lambda_{i+2})}$ *for all* $i \geqslant 1$*;*
(ii) the transpose partition, λ^T*, satisfies (i);*
(iii) $\lambda = (2,2)$*,*

where here $\ell_2(k)$ is the least non-negative integer such that $k < 2^{\ell_2(k)}$. We say that any partition as in (i) satisfies Carter's criterion.

Example 5.7. The most dominant partition, $\tau_{\varnothing}^{(w)}$, in a 2-block of weight w satisfies Carter's criterion.

Example 5.8. In a 2-block of weight $w = m(m+1)/2$, we find the partition $\tau_{\varnothing}^{\rho(m)}$ that satisfies Carter's criterion.

If λ is a 2-regular partition, then all the rows of λ are of distinct length. It immediately follows that $\lambda \trianglerighteq \rho$ and therefore $g(\rho, \rho, \lambda) > 0$. If furthermore the partition λ satisfies Carter's criterion, then by equation (1.2) we have that $\mathbf{P}^{\Bbbk}(\lambda)$ is the unique projective module in which $\mathbf{S}^{\Bbbk}(\lambda)$ appears as a composition factor of a Specht filtration. Putting these two statements together (in light of equation (5.1)) we obtain the following.

Proposition 5.9. *Let $n = k(k+1)/2$ and $\alpha \vdash n$. If α satisfies Carter's criterion, then we have that $G(\rho(k), \rho(k), \alpha) > 0$.*

The following result is immediate by equation (5.1). It is the key to all of our results on Kronecker positivity (as it relates this problem to that of determining the positivity of modular decomposition numbers) and vastly generalises Theorem 5.5.

Theorem 5.10. *Let $n = k(k+1)/2$ and $\alpha \vdash n$. If there exists some β satisfying Carter's criterion such that $d_{\alpha,\beta} = m > 0$, then $g(\rho(k), \rho(k), \alpha) \geqslant m$. We refer to such a pair (α, β) as an m-*Carter–Saxl pair*.*

We are now ready to use the results of Sections 3 and 4 toward the Kronecker problem.

Theorem 5.11. *Let $w = k(k+1)/2$, $n = w(2w+1)$ and $\tau = \rho(2w-1)$. Then for λ, μ any pair such that $c(\rho(k), \lambda, \mu^T) > 0$ we have that*

$$g(\rho(2w), \rho(2w), \tau_\mu^\lambda) \geqslant c(\rho(k), \lambda, \mu^T) > 0.$$

Proof. Clearly, $\rho(2w)$ is a partition of $n = w(2w+1)$. We restrict our attention to the block B of weight w in $\Bbbk\mathfrak{S}_n$ with 2-core $\tau = \rho(2w-1)$. For $\nu = \rho(k)$, the partition $\tau_{\varnothing}^{\nu}$ belongs to this block, and it satisfies Carter's criterion. Note that $c(\rho(k), \lambda, \mu^T) > 0$ implies that $|\lambda| + |\mu| = w$, and τ_μ^λ also belongs to B. Finally, we have that

$$[\mathbf{S}_{-1}^{\Bbbk}(\tau_\mu^\lambda) : \mathbf{D}_{-1}^{\Bbbk}(\tau_{\varnothing}^{\nu})] \geqslant [\mathbf{S}_{-1}^{\mathbb{C}}(\tau_\mu^\lambda) : \mathbf{D}_{-1}^{\mathbb{C}}(\tau_{\varnothing}^{\nu})] = c(\rho(k), \lambda, \mu^T) > 0,$$

and the result follows from Theorem 5.10. ☐

We remark that none of the partitions above are covered by existing results in the literature. It is clear that they are not hooks or double-hooks and providing that neither λ or μ is the empty partition, then these partitions are not comparable with $\rho(2w)$ in the dominance order (as τ_μ^λ is both wider and longer that $\rho(2w)$).

An explicit new infinite family with unbounded Kronecker coefficients is given in the following corollary.

Corollary 5.12. *For $w = k(k+1)/2$, $n = w(2w+1)$ and $\tau = \rho(2w-1)$, we have that*

$$g(\rho(2w), \rho(2w), \tau^{\rho(k-1)}_{(k-1,1)}) \geqslant k-1.$$

Proof. We have

$$c(\rho(k), \rho(k-1), (k-1,1)^T) = k-1,$$

from which the result follows from Theorem 5.11. ☐

Example 5.13. In particular, for $w = k(k+1)/2$, $g(\rho(2w), \rho(2w), \tau^\lambda_\mu) > 0$ for λ, μ any pair such that $\lambda + \mu^T = \rho(k)$. Examples of such partitions τ^λ_μ are pictured in Figure 4.

Example 5.14. Let $n = 210$ and consider the 2-block of weight 20 with 2-core $\rho(19)$. There exist 35 Carter–Saxl pairs belonging to pairs (λ, μ) such that $\lambda + \mu^T$ is equal to either (10) or $(4,3,2,1)$. There are many more Carter–Saxl pairs in this block.

Finally, we conclude this section by remarking that we have only used positivity of decomposition numbers for the Hecke algebra over $\mathbb{C}$. These are the easiest decomposition numbers to calculate, but only provide lower bounds for the decomposition numbers of symmetric groups.

6. More semisimple decomposable Specht modules

Semisimplicity and decomposability of Specht modules has long been a subject of major interest: the highlight being the recent progress on the classification of *simple* Specht modules for symmetric groups and their Hecke algebras [FL13, Jam78, JLM06, JM99, Fay05, JM96, Lyl07, JM97, Fay04, FL09, Fay10]. Progress towards understanding the wider family of *semisimple* and decomposable modules has been snail-like in comparison [Mur80, Spe14, CMT04, DF12, Rou08b, FS16] and reserved solely to near-hook partitions. All examples of decomposable Specht modules discovered to date have been labelled by 2-separated partitions. Our Theorem A proves that for any 2-separated partition, the corresponding Specht module for the algebra $H^{\mathbb{C}}_{-1}(n)$ is decomposable. It is natural to ask whether the converse is true: *are all decomposable Specht modules for $H^{\mathbb{C}}_{-1}(n)$ and more generally $H^{\Bbbk}_{-1}(n)$ indexed by 2-separated partitions?*. In [DF12, Section 8.2], Dodge and Fayers asked exactly this question for the symmetric group with $\operatorname{char} \Bbbk = 2$. In this section, we provide counterexamples to this question for the Hecke algebra.

6.1. **Two new infinite families of decomposable Specht modules.** Given $k \in \mathbb{N}$ and $l \in 2\mathbb{N}+1$, we define $\alpha_k \vdash (k+2)^2 - 4$ and $\beta_l \vdash (l+2)^2 - 2$, respectively, to be the partitions

$$\alpha_k = ((k+2)^k, k^2), \qquad \beta_l = (l+3, (l+2)^{l-1}, l^2, 1).$$

For example, the Young diagrams of the partitions α_5 and β_5, along with their residues, are drawn as follows:

$$[\alpha_5] = \begin{array}{|c|c|c|c|c|c|c|} \hline 0&1&0&1&0&1&0 \\ \hline 1&0&1&0&1&0&1 \\ \hline 0&1&0&1&0&1&0 \\ \hline 1&0&1&0&1&0&1 \\ \hline 0&1&0&1&0&1&0 \\ \hline 1&0&1&0&1 \\ \cline{1-5} 0&1&0&1&0 \\ \cline{1-5} \end{array} \qquad [\beta_5] = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0&1&0&1&0&1&0&1 \\ \hline 1&0&1&0&1&0&1 \\ \cline{1-7} 0&1&0&1&0&1&0 \\ \cline{1-7} 1&0&1&0&1&0&1 \\ \cline{1-7} 0&1&0&1&0&1&0 \\ \cline{1-7} 1&0&1&0&1 \\ \cline{1-5} 0&1&0&1&0 \\ \cline{1-5} 1 \\ \cline{1-1} \end{array}$$

In the arXiv appendix to this paper, we prove that

$$\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_{[k]}) \quad \text{and} \quad \mathbf{S}^{\mathbb{C}}_{-1}(\beta_{[l]})$$

are decomposable for all $k \geqslant 1$ and odd $l \geq 1$. In fact, we show that both Specht modules have a direct summand equal to a (different) simple Specht module. Namely,

$$\mathbf{S}^{\mathbb{C}}_{-1}((k+4)^k) \cong \mathbf{D}^{\mathbb{C}}_{-1}(2k+3, 2k+1, 2k-1, \ldots, 9, 7, 5) \text{ is a direct summand of } \mathbf{S}^{\mathbb{C}}_{-1}(\alpha_{[k]}).$$

These provide the first examples of decomposable Specht modules indexed by partitions which are **not** 2-separated. The proof of decomposability is not difficult, but it does involve twenty pages of extensive calculations. The basic idea is to (1) show that

$$\mathrm{Hom}_{H^{\mathbb{C}}_{-1}(n)}(\mathbf{S}^{\mathbb{C}}_{-1}((k+4)^k), \mathbf{S}^{\mathbb{C}}_{-1}(\alpha_{[k]})) \neq 0$$

using results on semistandard homomorphisms and (2) prove that

$$[\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_{[k]}) : \mathbf{D}^{\mathbb{C}}_{-1}(2k+3, 2k+1, 2k-1, \dots, 9, 7, 5)] \leqslant 1$$

by counting corresponding coloured tableaux. One hence deduces that this simple composition factor occurs exactly once as a composition factor but in both the head and the socle of $\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_{[k]})$, and thus is a direct summand. We refer the reader to the arXiv appendix for more details.

Conjecture 6.1. *For $k \in \mathbb{N}$, we set $\alpha_k^C = (2k+3, 2k+1, 2k-1, \dots, 9, 7, 5)$. Then we expect that*

$$\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k) = \mathbf{D}^{\mathbb{C}}_{-1}\left(\alpha_k^R\right)\langle w(\alpha_k)/2\rangle \oplus \mathbf{D}^{\mathbb{C}}_{-1}\left(\alpha_k^C\right)\langle w(\alpha_k)/2\rangle.$$

By 1-induction, we conjecture the direct sum decomposition of $\mathbf{S}^{\mathbb{C}}_{-1}(\beta_{[l]})$.

Conjecture 6.2. *For $l \in 2\mathbb{N}+1$, we set $\beta_l^C = (2l+3, 2l+1, 2l-1, \dots, 9, 7, 6, 1)$. Then we expect that*

$$\mathbf{S}^{\mathbb{C}}_{-1}(\beta_l) = \mathbf{D}^{\mathbb{C}}_{-1}\left(\beta_l^R\right)\langle w(\beta_l)/2\rangle \oplus \mathbf{D}^{\mathbb{C}}_{-1}\left(\beta_l^C\right)\langle w(\beta_l)/2\rangle.$$

6.2. **Other decomposable Specht modules.** We are indebted to Matt Fayers for sharing the following examples (which he discovered by computer) after we posited that the two families in Subsection 6.1 might be the only counterexamples to the quantised version of his question [DF12, Section 8.2].

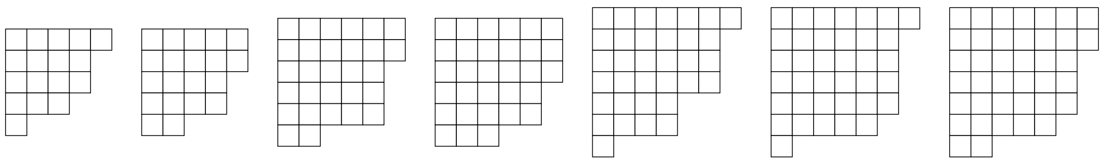

FIGURE 9. More partitions labelling semisimple Specht modules.

We hope that the examples in Figure 9 serve as inspiration for further work towards a classification of semisimple Specht modules. Several more examples can be obtained from those in Figure 9 by i-induction for $i = 0, 1$ (analogously to Subsection 6.1) namely: $(8, 7, 6^2, 4^2, 2, 1)$, $(7^3, 6^3, 3)$ and $(8, 7^3, 6^2, 4, 1)$. Finally we have one partition which breaks the mould: $(6, 5^3, 3, 1)$ which is the only partition in this section not equal to its own conjugate.

6.3. **Patterns.** The examples of Subsections 6.1 and 6.2 do share several striking similarities. Firstly, all the examples in Subsections 6.1 and 6.2 have a direct summand which is isomorphic to a simple Specht module. Secondly, all those of Subsection 6.2 decompose as a direct sum of simples concentrated in one degree and so are semisimple by Theorem 3.2. We conjecture this is also true of the infinite families in Theorem 3.2. It is interesting to speculate whether the converse of Theorem 3.2 is also true: is semisimplicity of a Specht module equivalent to its composition factors being focussed in one degree?

Appendix A. Two new infinite families of decomposable Specht modules of Hecke algebras over $\mathbb{C}$

We now prove that the two infinite families of decomposable Specht modules announced in Section 6 are indeed decomposable.

A.1. **Semistandard homomorphisms.**

We now recall the construction of Young permutation modules for Hecke algebras and how they can be used to calculate homomorphisms between Specht modules. Given λ a partition of n, we define the $H_q^{\Bbbk}(n)$-module

$$\mathbf{M}_q^{\Bbbk}(\lambda) = H_q^{\Bbbk}(n)x_\lambda$$

which we refer to as the permutation module labelled by λ. For $\mathsf{S} \in \mathrm{RStd}(\lambda, \mu)$, there exists a homomorphism $\Theta_{\mathsf{S}} : \mathbf{M}_q^{\Bbbk}(\lambda) \to \mathbf{M}_q^{\Bbbk}(\mu)$ (see [Mat99, Section 4] for explicit details) and the set of all row standard tableau homomorphisms $\{\Theta_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{RStd}(\lambda, \mu)\}$ forms a basis of $\mathrm{Hom}_{H_q^{\Bbbk}(n)}\left(\mathbf{M}_q^{\Bbbk}(\lambda), \mathbf{M}_q^{\Bbbk}(\mu)\right)$. We let $\widehat{\Theta}_{\mathsf{S}}$ denote the restriction of Θ_{S} to $\mathbf{S}_q^{\Bbbk}(\lambda)$.

Theorem A.1. [DJ91, Corollary 8.7] *The set $\{\widehat{\Theta}_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu)\}$ is a linearly independent subset of $\mathrm{Hom}_{H_q^{\Bbbk}(n)}\left(\mathbf{S}_q^{\Bbbk}(\lambda), \mathbf{M}_q^{\Bbbk}(\mu)\right)$. If either $q \neq -1$ or λ is 2-regular, then $\{\widehat{\Theta}_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu)\}$ provides a basis of this space.*

This paper focuses on the case $q = -1$. While the homomorphisms $\widehat{\Theta}_{\mathsf{S}}$ for $\mathsf{S} \in \mathrm{RStd}(\lambda, \mu)$ do not necessarily provide a basis of $\mathrm{Hom}_{H_{-1}^{\Bbbk}(n)}\left(\mathbf{S}_{-1}^{\Bbbk}(\lambda), \mathbf{S}_{-1}^{\Bbbk}(\mu)\right)$, we shall see that tableau-theoretic homomorphisms are a useful tool for analysing the structure of Specht modules in appendix A. Let $\lambda \vdash n$. Fix $d \in \mathbb{N}$ and t such that $0 \leqslant t < \lambda_{d+1}$. We set

$$\lambda_i^{d,t} = \begin{cases} \lambda_i + \lambda_{i+1} - t & \text{if } i = d, \\ t & \text{if } i = d+1, \\ \lambda_i & \text{otherwise.} \end{cases}$$

Let $\mathsf{S}_{d,t} \in \mathrm{RStd}(\lambda, \lambda^{d,t})$ be such that all of its entries in row i are equal to i, except for row $d+1$ which contains $\mu_{d+1} - t$ entries equal to d and t entries equal to $d+1$. We now write $\varphi_{d,t}$ for the homomorphism $\Theta_{\mathsf{S}_{d,t}} : \mathbf{M}_q^{\Bbbk}(\lambda) \to \mathbf{M}_q^{\Bbbk}(\lambda^{d,t})$. The homomorphism $\varphi_{d,t}$ enables us to give the following alternative definition of a Specht module.

Theorem A.2 ([DJ86, Theorem 7.5]). *Let $\lambda \vdash n$. Then the Specht module $\mathbf{S}_q^{\Bbbk}(\lambda)$ can be constructed as the intersection of the kernels of these homomorphisms as follows,*

$$\mathbf{S}_q^{\Bbbk}(\lambda) = \bigcap_{\substack{d \geqslant 1 \\ 1 \leqslant t \leqslant \lambda_{d+1}}} \ker \varphi_{d,t}.$$

Therefore we can use the homomorphisms of Theorem A.2 to determine whether or not the image of a tableau homomorphism $\widehat{\Theta}_{\mathsf{S}}$ lies in the Specht module $\mathbf{S}_q^{\Bbbk}(\lambda)$.

Corollary A.3. *Let $\lambda, \mu \vdash n$ and suppose that $\widehat{\Theta}_{\mathsf{T}} \in \mathrm{Hom}_{H_q^{\Bbbk}(n)}\left(\mathbf{S}_q^{\Bbbk}(\lambda), \mathbf{M}_q^{\Bbbk}(\mu)\right)$ for some $\mathsf{T} \in \mathrm{RStd}(\lambda, \mu)$. Then $\mathrm{im}(\widehat{\Theta}_{\mathsf{T}}) \subseteq \mathbf{S}_q^{\Bbbk}(\mu)$ if and only if $\varphi_{d,t} \circ \widehat{\Theta}_{\mathsf{T}} = 0$ for all $d \geqslant 1$ and $0 \leqslant t < \mu_{d+1}$.*

A.2. **Statement and outline of proof.**

Definition A.4. Given $k \in \mathbb{N}$, we define $\alpha_k \vdash (k+2)^2 - 4$ to be the partition $\alpha_k = ((k+2)^k, k^2)$. Given $k \in 2\mathbb{N}+1$, we define $\beta_k \vdash (k+2)^2 - 2$ to be the partition $\beta_k = (k+3, (k+2)^{k-1}, k^2, 1)$.

Example A.5. The Young diagrams of the partitions α_5 and β_5, along with their residues, are drawn as follows:

$$[\alpha_5] = \begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&0&1&0&1&0\\ \hline 1&0&1&0&1&0&1\\ \hline 0&1&0&1&0&1&0\\ \hline 1&0&1&0&1&0&1\\ \hline 0&1&0&1&0&1&0\\ \hline 1&0&1&0&1\\ \cline{1-5} 0&1&0&1&0\\ \cline{1-5}\end{array} \qquad [\beta_5] = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0&1&0&1&0&1&0&1\\ \hline 1&0&1&0&1&0&1\\ \cline{1-7} 0&1&0&1&0&1&0\\ \cline{1-7} 1&0&1&0&1&0&1\\ \cline{1-7} 0&1&0&1&0&1&0\\ \cline{1-7} 1&0&1&0&1\\ \cline{1-5} 0&1&0&1&0\\ \cline{1-5} 1\\ \cline{1-1}\end{array}.$$

We now state our main result in this section (which we later split into two results).

Theorem A.6. *The Specht modules $\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k)$ and $\mathbf{S}^{\mathbb{C}}_{-1}(\beta_k)$ are decomposable.*

We remark that $\alpha_k = ((k+2)^k, k^2)$ is 2-separated if and only if $k = 1$, in which case, we already know from Section 4 that $\mathbf{S}^{\mathbb{C}}_{-1}(3, 1^2)$ is semisimple (see also [Spe14]). We shall focus on the decomposability of $\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k)$; we shall see later that it is relatively straightforward to determine the decomposability of $\mathbf{S}^{\mathbb{C}}_{-1}(\beta_k)$ from $\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k)$; this is because β_k is obtained from α_k by adding all possible addable nodes of residue 1 (the observant reader will notice that this is why k being odd is a necessary condition in the definition of β_k).

Proposition A.7. [JM96, Theorem 4.7] *Let $\alpha, \mu \vdash n$ with $\alpha_r \geqslant r$ for some $r \in \mathbb{N}$ and μ a 2-regular partition. If $[\mathbf{S}^{\mathbb{C}}_{-1}(\alpha) : D^{\mathbb{C}}_{-1}(\mu)] > 0$, then $\ell(\mu) \geqslant r$.*

Fayers–Lyle [FL13] provide a conjectural classfication of all irreducible Specht $H^{\mathbb{C}}_{-1}(n)$-modules. Mathas has provided a proof of the Fayers–Lyle conjecture for rectangular partitions, using the above result. *The following result and its proof is unpublished and conveyed to us by Fayers.*

Theorem A.8. *Let $\nu = (m^k)$ for some $m, k \in \mathbb{N}$. Then $\mathbf{S}^{\mathbb{C}}_{-1}(\lambda)$ irreducible. In particular, $\mathbf{S}^{\mathbb{C}}_{-1}(\lambda) \cong \mathbf{D}^{\mathbb{C}}_{-1}(\lambda^R)$.*

Proof. We suppose without loss of generality (as $\mathbf{S}^{\mathbb{C}}_{-1}(\lambda)$ and $\mathbf{S}^{\mathbb{C}}_{-1}(\lambda^T)$ have the same decomposition multiplicities) that $m \geqslant k$ and so $\nu_k \geqslant k$. Now, by applying Proposition A.7, we know that if $D^{\mathbb{C}}_{-1}(\mu)$ is a composition factor of $\mathbf{S}^{\mathbb{C}}_{-1}(\lambda)$ for some 2-regular $\mu \vdash n$, then μ has at least k non-zero parts. On the other hand, we know from Theorem 1.4 that $[\mathbf{S}^{\mathbb{C}}_{-1}(\lambda) : \mathbf{D}^{\mathbb{C}}_{-1}(\mu)] \neq 0$ implies that $\mu \trianglerighteq \nu^R$, and hence μ has at most k parts.

We can thus assume that μ has exactly k parts, and write $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$. We now apply induction on the statement that $\mathbf{S}^{\mathbb{C}}_{-1}(\lambda)$ is simple for all $m \in \mathbb{N}$. If $m = 1$, then $\mathbf{S}^{\mathbb{C}}_{-1}(m) \cong \mathbf{D}^{\mathbb{C}}_{-1}(m)$ and the result holds. Now suppose that $m > 1$, we obtain

$$\left[\mathbf{S}^{\mathbb{C}}_{-1}(\lambda) : \mathbf{D}^{\mathbb{C}}_{-1}(\mu)\right] = \begin{cases} \left[\mathbf{S}^{\mathbb{C}}_{-1}\left((m-1)^k\right) : \mathbf{D}^{\mathbb{C}}_{-1}(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_k - 1)\right] & \text{if } \ell(\mu) = k \\ 0 & \text{otherwise,} \end{cases}$$

by applying the column removal result [Jam90, Rule 5.8]. By induction, we have that

$$\mathbf{S}^{\mathbb{C}}_{-1}\left((m-1)^k\right) \cong \mathbf{S}^{\mathbb{C}}_{-1}\left(k^{m-1}\right) \cong \mathbf{D}^{\mathbb{C}}_{-1}(\mu_1 - 1, \mu_2 - 1, \ldots, \mu_k - 1),$$

and the result follows. $\square$

Corollary A.9. *Let $\lambda = ((k+4)^k)$ for all $k \in \mathbb{N}$. Then $\mathbf{S}^{\mathbb{C}}_{-1}(\lambda)$ is irreducible and*

$$\mathbf{S}^{\mathbb{C}}_{-1}(\lambda) \cong \mathbf{D}^{\mathbb{C}}_{-1}(2k+3, 2k+1, 2k-1, \ldots, 9, 7, 5).$$

In what follows, we shall denote

$$\alpha^C_k = ((k+4)^k)^R = (2k+3, 2k+1, 2k-1, \ldots, 9, 7, 5).$$

While we are able to calculate both the non-zero decomposition numbers in the above and prove decomposability, we do not have a proof that all other decomposition numbers are zero (and hence we do not prove semisimplicity either). Our proof of Theorem A.6 proceeds in two steps as follows:

(i) Prove that $\mathbf{D}^{\mathbb{C}}_{-1}(\alpha^{\mathrm{C}}_k)$ belongs to the socle of $\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k)$;
(ii) Prove that $[\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k) : \mathbf{D}^{\mathbb{C}}_{-1}(\alpha^{\mathrm{C}}_k)] \leqslant 1$ and thus deduce (by self-duality) that $\mathbf{D}^{\mathbb{C}}_{-1}(\alpha^{\mathrm{C}}_k)$ is a direct summand of $\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k)$.

Using the previous corollary, it is enough to show that

$$\mathrm{Hom}_{H^{\mathbb{C}}_{-1}(k^2-2)}\left(\mathbf{S}^{\mathbb{C}}_{-1}\left((k+4)^k\right), \mathbf{D}^{\mathbb{C}}_{-1}\left(\alpha_k\right)\right) \neq 0$$

to immediately deduce part (i). To deduce part (ii), we shall prove that

$$\dim_t\left(e_{\mathsf{L}(\alpha^{\mathrm{C}}_k)}\mathbf{S}^{\mathbb{C}}_{-1}\left(\alpha_k\right)\right) = 2t^{\lfloor w(\alpha_k)/2\rfloor}\dim_t\left(e_{\mathsf{L}(\alpha^{\mathrm{C}}_k)}\mathbf{D}^{\mathbb{C}}_{-1}(\alpha^{\mathrm{C}}_k)\right) \tag{A.1}$$

$$\dim_t\left(e_{\mathsf{L}(\alpha^{\mathrm{C}}_k)}\mathbf{D}^{\mathbb{C}}_{-1}(\alpha^{\mathrm{R}}_k)\right) = \dim_t\left(e_{\mathsf{L}(\alpha^{\mathrm{C}}_k)}\mathbf{D}^{\mathbb{C}}_{-1}(\alpha^{\mathrm{C}}_k)\right) + \dots \tag{A.2}$$

where the other terms in equation (A.2) belong to $t\mathbb{N}_0[t]$. We deduce from equation (A.1) that $[\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k) : \mathbf{D}^{\mathbb{C}}_{-1}(\alpha^{\mathrm{C}}_k)] \leqslant 2$ and then refine this to the sharp upper bound of 1 using equation (A.2) and part (i) along with the results of Kleshchev–Nash from Section 2.

A.3. **Proof part (i): the homomorphism.** We employ the method set out in appendix A.1 to describe a Specht module homomorphism between $\mathbf{S}^{\mathbb{C}}_{-1}((k+4)^k)$ and $\mathbf{S}^{\mathbb{C}}_{-1}((k+2)^k, k^2)$ from the following definition of a semistandard tableau.

Let $\mu = ((k+4)^k)$ and $\lambda = ((k+2)^k, k^2)$ for all $k > 1$. We define the tableau $\mathsf{A} \in \mathrm{SStd}(\mu, \lambda)$ as follows:

$\mathsf{A} =$

1	1	1	1	1	1	1	1	1	1	1	1	$\cdots$	1	2	3
2	2	2	2	2	2	2	2	2	2	2	$\cdots$	2	3	4	4
3	3	3	3	3	3	3	3	3	3	$\cdots$	3	4	5	5	5
4	4	4	4	4	4	4	4	4	$\cdots$	4	5	6	6	6	6
5	5	5	5	5	5	5	5	$\cdots$	5	6	7	7	7	7	7
6	6	6	6	6	6	6	$\cdots$	6	7	8	8	8	8	8	8
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\iddots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-4$	$k-4$	$k-4$	$k-4$	$k-4$	$k-4$	$k-4$	$k-3$	$k-2$	$\cdots$	$k-2$	$k-2$	$k-2$	$k-2$	$k-2$	$k-2$
$k-3$	$k-3$	$k-3$	$k-3$	$k-3$	$k-3$	$k-2$	$k-1$	$\cdots$	$k-1$	$k-1$	$k-1$	$k-1$	$k-1$	$k-1$	$k-1$
$k-2$	$k-2$	$k-2$	$k-2$	$k-2$	$k-1$	k	$\cdots$	k	k	k	k	k	k	k	k
$k-1$	$k-1$	$k-1$	$k-1$	k	$k+1$	$\cdots$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$	$k+1$
k	k	k	$k+1$	$k+2$	$\cdots$	$k+2$	$k+2$	$k+2$	$k+2$	$k+2$	$k+2$	$k+2$	$k+2$	$k+2$	$k+2$

Since A is semistandard, we know that $\widehat{\Theta}_{\mathsf{A}} : \mathbf{S}^{\mathbb{C}}_{-1}((k+4)^k) \to M^{\mathbb{C}}_{-1}((k+2)^k, k^2)$ is a non-zero homomorphism; we claim that the image of $\widehat{\Theta}_{\mathsf{A}}$ lies in $\mathbf{S}^{\mathbb{C}}_{-1}((k+2)^k, k^2)$. We will heavily rely on Fayers' algorithm [Fay12, Theorem 3.1], which semistandardises tableau homomorphisms solely by manipulating row-standard tableaux.

Theorem A.10. *Suppose that λ and μ are partitions of n, and T is a row-standard λ-tableau of type μ. Suppose $i \geqslant 1$, and A, B, C are multisets of positive integers such that $|B| > \lambda_i$ and*

$$(A \sqcup B \sqcup C) \cap \{(r,c) \mid r \in \{i, i+1\}, c \geqslant 1\} = \mathsf{T} \cap \{(r,c) \mid r \in \{i, i+1\}, c \geqslant 1\}.$$

Let $\mathscr{B}$ be the set of all pairs (D, E) of multisets such that $|D| = \lambda_i - |A|$ and $B = D \sqcup E$. For each such pair (D, E), define $\mathsf{T}_{D,E}$ to be the row-standard tableau with

$$\mathsf{T}_{D,E} \cap \{(r,c) \mid c \geqslant 1\} = \begin{cases} A \sqcup D & (r = i) \\ C \sqcup E & (r = i+1) \\ \mathsf{T} \cap \{(r,c) \mid c \geqslant 1\} & (\textit{otherwise}). \end{cases}$$

Then

$$\sum_{(D,E)\in\mathscr{B}} \prod_{i\geqslant 1} \begin{bmatrix} A_i + D_i \\ A_i \end{bmatrix} \begin{bmatrix} C_i + E_i \\ C_i \end{bmatrix} \prod_{i<j} q^{(A_j D_i + C_i E_j)} \widehat{\Theta}_{\mathsf{T}_{D,E}} = 0. \tag{A.3}$$

Given T as in Theorem A.10, we know that there exists some $c \geqslant 1$ such that the entries in the boxes (i, c) and $(i+1, c)$ of T are equal, that is $\mathsf{T}(i, c) = \mathsf{T}(i+1, c) = j$ for some $j \geqslant 1$. We let $\mathsf{T}^{(i)} = \{(i, c) \mid c \geqslant 1\}$ for all $i \geqslant 1$. We now explicitly define the multisets A, B and C as follows:

$$\begin{aligned} A &= \{k \in \mathsf{T}^{(i)} \mid k < j\} \\ B &= \{k \in \mathsf{T}^{(i)} \mid k \geqslant j\} \sqcup \{k \in \mathsf{T}^{(i+1)} \mid k \leqslant j\} \\ C &= \{k \in \mathsf{T}^{(i+1)} \mid k > j\}. \end{aligned}$$

Recall from appendix A.1 that $\widehat{\Theta}_{\mathsf{A}} \subseteq \mathrm{im}(\mathbf{S}^{\mathbb{C}}_{-1}((k+2)^k, k^2))$ if the map composition of $\widehat{\Theta_{\mathsf{A}}}$ with $\varphi_{d,t}$ is zero for all d and t. In order to do so, we now make extensive use of Fayers' semistandardising algorithm to show that certain row-standard tableau homomorphisms are zero; we invite the general reader to skip to the main result of this Subsection on Page 29 to avoid the following technical results, should they wish to do so. We note that when $q = -1$, the tableau homomorphism $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ appears in the above sum of equation (A.3) with coefficient -1, 0 or 1; we shade diagrams throughout for clarity.

Lemma A.11. *Let $\lambda = ((k+4)^k)$ and $\mu = ((k+2)^k, k^2)$ for some $k > 1$, and let $d \in \{2, \ldots, k-1\}$.*

(1) Suppose that $t \geqslant u \geqslant 1$, $w \geqslant 3$ such that w is odd, and $v \geqslant 0$. We have that $\widehat{\Theta}_{\mathsf{S}} = 0$ for any tableau $\mathsf{S} \in \mathrm{RStd}(\lambda, \mu)$ whose dth and $(d+1)$th rows have the following form:

$$\begin{array}{c}\overbrace{\hphantom{d \;\cdots\; d \;\; d \;\cdots\; d \;\; d \;\cdots\; d \;\; d \;\; d}}^{t \text{ times}} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline d & \cdots & d & d & \cdots & d & d & \cdots & d & d & d & d{+}2 & \cdots & d{+}2 \\ \hline d & \cdots & d & d{+}1 & \cdots & d{+}1 & d{+}2 & \cdots & d{+}2 & d{+}3 & d{+}3 & d{+}3 & \cdots & d{+}3 \\ \hline\end{array} \\ \underbrace{\hphantom{d \cdots d}}_{u \text{ times}} \; \underbrace{\hphantom{d{+}1 \cdots d{+}1}}_{v \text{ times}} \; \underbrace{\hphantom{d{+}2 \cdots d{+}2}}_{w \text{ times}} \end{array}$$

(2) Suppose that $t \geqslant u \geqslant 2$, $w \geqslant 3$ such that w is odd, and $v \geqslant 0$. We have that $\widehat{\Theta}_{\mathsf{T}} = 0$ for any tableau $\mathsf{T} \in \mathrm{RStd}(\lambda, \mu)$ whose dth and $(d+1)$th rows have the following form:

$$\begin{array}{c}\overbrace{\hphantom{d \;\cdots\; d \;\; d \;\cdots\; d \;\; d \;\cdots\; d \;\; d}}^{t \text{ times}} \\ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline d & \cdots & d & d & \cdots & d & d & \cdots & d & d & d{+}1 & d{+}2 & \cdots & d{+}2 \\ \hline d & \cdots & d & d{+}1 & \cdots & d{+}1 & d{+}2 & \cdots & d{+}2 & d{+}3 & d{+}3 & d{+}3 & \cdots & d{+}3 \\ \hline\end{array} \\ \underbrace{\hphantom{d \cdots d}}_{u \text{ times}} \; \underbrace{\hphantom{d{+}1 \cdots d{+}1}}_{v \text{ times}} \; \underbrace{\hphantom{d{+}2 \cdots d{+}2}}_{w \text{ times}} \end{array}$$

Proof. We let $i = d$, and apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{S}}$ and $\widehat{\Theta}_{\mathsf{T}}$, respectively. In both cases, we set $A = \emptyset$ and $C = \{(d+1)^v, (d+2)^w, (d+3)^{k+4-u-v-w}\}$, by which we mean

$$C = \left\{ \underbrace{d+1, \ldots, d+1}_{v \text{ times}}, \underbrace{d+2, \ldots, d+2}_{w \text{ times}}, \underbrace{d+3, \ldots, d+3}_{k+4-u-v-w \text{ times}} \right\}.$$

Thus every morphism $\widehat{\Theta}_{\mathsf{S}_{D,E}}$ and $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ in equation (A.3) appears with some scalar multiple of

$$\begin{bmatrix} E_{d+2}+w \\ w \end{bmatrix} = \begin{cases} 1 & \text{if } E_{d+2}=0 \\ 0 & \text{if } E_{d+2} \geqslant 1. \end{cases}$$

Therefore we need now only consider the case $E_{d+2}=0$, which we now do in both cases:

Case 1: Observe that $B=\{d^{t+u},(d+2)^{k-t+4}\}$ and hence when we set $E_{d+2}=0$ we obtain only the original tableau S (which always appears with coefficient 1) in equation (A.3); hence $\widehat{\Theta}_{\mathsf{S}}=0$.

Case 2: Observe that $B=\{d^{t+u},d+1,(d+2)^{k-t+3}\}$. There are two tableaux obtained from setting $E_{d+2}=0$, namely T and the tableau R that is identical to T except for its dth and $(d+1)$th rows, which have the following form

$$\begin{array}{cccccccccccccc}
\multicolumn{11}{c}{\overbrace{\hphantom{d \;\cdots\; d \;\; d \;\; d \;\cdots\; d \;\; d \;\cdots\; d \;\; d}}^{t+1\text{ times}}} & & & \\
d & \cdots & d & d & d & \cdots & d & d & \cdots & d & d & d{+}2 & \cdots & d{+}2 \\
d & \cdots & d & d{+}1 & d{+}1 & \cdots & d{+}1 & d{+}2 & \cdots & d{+}2 & d{+}3 & d{+}3 & \cdots & d{+}3 \\
\multicolumn{3}{c}{\underbrace{\hphantom{d \cdots d}}_{u-1\text{ times}}} & \multicolumn{4}{c}{\underbrace{\hphantom{d{+}1 \; d{+}1 \cdots d{+}1}}_{v+1\text{ times}}} & \multicolumn{3}{c}{\underbrace{\hphantom{d{+}2 \cdots d{+}2}}_{w\text{ times}}} & & & &
\end{array}$$

We can thus write $\alpha_{\mathsf{R}}\widehat{\Theta}_{\mathsf{R}}+\widehat{\Theta}_{\mathsf{T}}=0$ for some $\alpha_{\mathsf{R}}\in\{-1,1\}$. It follows from part (1) that $\widehat{\Theta}_{\mathsf{R}}=0$, and hence $\widehat{\Theta}_{\mathsf{T}}=0$, as required. □

Lemma A.12. *Let $\lambda=((k+4)^k)$ and $\mu=((k+2)^k,k^2)$ for some $k>1$, and let $d\in\{2,\dots,k-1\}$. Suppose that either*

(1) $u>s\geqslant 0$, $v\geqslant 3$ such that v is odd, and $t\geqslant 0$, or
(2) $u>s+1>0$, $v\geqslant 2$ such that v is even, and $t\geqslant 0$.

We have that $\widehat{\Theta}_{\mathsf{T}}=0$ for any $\mathsf{T}\in\mathrm{RStd}(\lambda,\mu)$ whose $(d-1)$th and dth rows have the following form

$$\begin{array}{cccccccccccc}
\multicolumn{3}{c}{\overbrace{\hphantom{d{-}1 \cdots d{-}1}}^{s\text{ times}}} & \multicolumn{5}{c}{\overbrace{\hphantom{d \;\; d \;\; d \cdots d}}^{t\text{ times}}} & & & & \\
d{-}1 & \cdots & d{-}1 & d & d & d & \cdots & d & d{+}1 & d{+}1 & \cdots & d{+}1 \\
d & \cdots & d & d & d & d{+}1 & \cdots & d{+}1 & d{+}2 & d{+}2 & \cdots & d{+}2 \\
\multicolumn{5}{c}{\underbrace{\hphantom{d \cdots d \;\; d \;\; d}}_{u\text{ times}}} & \multicolumn{3}{c}{\underbrace{\hphantom{d{+}1 \cdots d{+}1}}_{v\text{ times}}} & & & &
\end{array}$$

Proof. Fix $i=d-1$ and set

$$A=\{(d-1)^s\},\quad B=\{d^{t+u},(d+1)^{k+4-s-t}\},\quad C=\{(d+1)^v,(d+2)^{k+4-u-v}\}.$$

Case 1:

Observe that every homomorphism $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ in Theorem A.10 appears with some scalar multiple of

$$\begin{bmatrix} E_{d+1}+v \\ v \end{bmatrix} = \begin{cases} 1 & \text{if } E_{d+1}=0 \\ 0 & \text{if } E_{d+1} \geqslant 1. \end{cases}$$

The only tableau $\mathsf{T}_{D,E}$ with $E_{d+1}=0$ is T itself, and hence $\widehat{\Theta}_{\mathsf{T}}=0$.

Case 2: In this case, every homomorphism $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ in Theorem A.10 appears with some scalar multiple of

$$\begin{bmatrix} E_{d+1}+v \\ v \end{bmatrix} = \begin{cases} 1 & \text{if } E_{d+1}=0 \text{ or } 1 \\ 0 & \text{if } E_{d+1} \geqslant 2. \end{cases}$$

There are two possible tableaux $\mathsf{T}_{D,E}$ such that the above quantum binomial coefficient is non-zero, namely T (with $E_{d+1} = 0$) and the tableau S (with $E_{d+1} = 1$) that is identical to T except for its $(d-1)$th and dth rows, which have the following form

$$\begin{array}{cccccccccccc}
\multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{s\text{ times}}} & \multicolumn{6}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{t+1\text{ times}}} & & & \\
\hline
\multicolumn{1}{|c|}{d-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} \\
\hline
\multicolumn{1}{|c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+2} \\
\hline
\multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{u-1\text{ times}}} & \multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{v+1\text{ times}}} & & & &
\end{array}$$

We thus have $\alpha_{\mathsf{S}}\widehat{\Theta}_{\mathsf{S}} + \widehat{\Theta}_{\mathsf{T}} = 0$ for some $\alpha_{\mathsf{S}} \in \{-1, 1\}$. It follows from part (1) that $\widehat{\Theta}_{\mathsf{S}} = 0$, and hence $\widehat{\Theta}_{\mathsf{T}} = 0$. $\square$

The following result will be the most instrumental in proving that there exists a non-zero homomorphism between $\mathbf{S}^{\mathbb{C}}_{-1}((k+4)^k)$ and $\mathbf{S}^{\mathbb{C}}_{-1}((k+2)^k, k^2)$; in particular, we cancel out most pairs of relevant tableau homomorphisms (two exceptional cases will be deferred to the next lemma).

Lemma A.13. *Let $\lambda = ((k+4)^k)$ and $\mu = ((k+2)^k, k^2)$ for some $k > 1$ and $d \in \{2, \ldots, k-1\}$. The set of tableaux in $\mathrm{RStd}(\lambda, \mu)$ obtained from A by replacing $\bar{t} \in \{1, \ldots, k+2\}$ of the entries equal to $d+1$ with the entry d fall into two families:*

The first family is denoted by S^l. The $(d-1)$th, dth and $(d+1)$th rows of S^l are

$$\begin{array}{cccccccccccccccccc}
\multicolumn{9}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{k-d+4\text{ times}}} & & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{\bar{t}-l\text{ times}}} & & & & & \\
\hline
\multicolumn{1}{|c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} \\
\hline
\multicolumn{1}{|c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+2} \\
\hline
\multicolumn{1}{|c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+3} \\
\hline
\multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{l\text{ times}}} & & & & & \multicolumn{8}{c}{\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{d+1\text{ times}}}
\end{array}$$

with $l \leqslant \bar{t}$, and all other rows agree with those of A. The second family is T^l. The $(d-1)$th, dth and $(d+1)$th rows of T^l are

$$\begin{array}{cccccccccccccccccc}
\multicolumn{9}{c}{\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{k-d+4\text{ times}}} & & \multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{\bar{t}-l\text{ times}}} & & & & & \\
\hline
\multicolumn{1}{|c|}{d-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} \\
\hline
\multicolumn{1}{|c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+2} \\
\hline
\multicolumn{1}{|c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{d+3} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+3} \\
\hline
\multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad}_{l-1\text{ times}}} & & & & & & \multicolumn{8}{c}{\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{d+1\text{ times}}}
\end{array}$$

with $l \leqslant \bar{t}$, and all other rows agree with those of A.

(1) If $k-d$ is odd and $l \in \{1, \ldots, k-d+2\}$, then $\widehat{\Theta}_{\mathsf{S}^l} = -\widehat{\Theta}_{\mathsf{T}^l}$.
(2) If $k-d$ is even and $l \in \{1, \ldots, k-d+2\}$, then $\widehat{\Theta}_{\mathsf{S}^l} = 0$.

Proof. We fix $i = d$ and apply Theorem A.10 to S^l only. We set

$$A = \emptyset, \quad B = \{d^{k-d+l+3}, d+1, (d+2)^d\}, \quad C = \{(d+1)^{k-d-l+2}, d+2, (d+3)^{d+1}\},$$

so $|D| = k+4$, $|E| = l$. Observe that

$$\alpha^l_{\mathsf{S}_{D,E}} := \prod_{i \geqslant 1} \begin{bmatrix} A_i + D_i \\ A_i \end{bmatrix} \begin{bmatrix} C_i + E_i \\ C_i \end{bmatrix} = \begin{bmatrix} E_{d+1} + k - d - l + 2 \\ k - d - l + 2 \end{bmatrix} [E_{d+2} + 1] \in \{0, 1\}.$$

If E_{d+2} is odd then this product is zero since $[E_{d+2}+1]=0$. So suppose that E_{d+2} is even. In particular, we note that if $E_{d+2}=0$, then

$$\mathsf{S}^l = \mathsf{S}^l_{D,E} \text{ where } E=\{d^l\} \qquad \text{and} \qquad \mathsf{T}^l = \mathsf{S}^l_{D,E} \text{ where } E=\{d^{l-1}, d+1\}. \tag{A.4}$$

Let $n_{D,E} = \sum_{1\leqslant i<j}(A_jD_i + C_iE_j)$. Then substituting equation (A.4) into equation (A.3) we obtain

$$\widehat{\Theta}_{\mathsf{S}^l} + \widehat{\Theta}_{\mathsf{T}^l} + \sum_{\{(D,E)\mid E_{d+2}\geqslant 1\}} (-1)^{n_{D,E}} \alpha^l_{\mathsf{S}_{D,E}} \widehat{\Theta}_{\mathsf{S}^l_{D,E}} = 0.$$

Case 1: Suppose that $k-d$ is odd. We proceed to show that $\alpha^l_{\mathsf{S}_{D,E}}\widehat{\Theta}_{\mathsf{S}^l_{D,E}}$ in equation (A.3) equals zero for all pairs (D,E) except in the cases of equation (A.4).

(1) Assume that $E_{d+1}=1$. In other words, we first consider tableaux $\mathsf{S}^l_{D,E}$ such that $E=\{d^{l-r}, d+1, (d+2)^{r-1}\}$ with $1<r\leqslant l$. Then

$$\begin{bmatrix} E_{d+1}+k-d-l+2 \\ k-d-l+2 \end{bmatrix} = [k-d-l+3] = \begin{cases} 1 & \text{if } l \text{ is odd,} \\ 0 & \text{if } l \text{ is even.} \end{cases}$$

We now suppose that l is odd and recall that $l\leqslant \bar{t}$; it thus follows that $\bar{t}\geqslant 3$. Since $E_{d+2}=r-1$, and we assume above that E_{d+2} is even, it suffices to consider $\mathsf{S}^l_{D,E}$ such that r is odd. Under this assumption, we can further suppose that $d\geqslant 3$. If $r<l$ (so that $E_d>0$), then rows d and $d+1$ of $\mathsf{S}^l_{D,E}$ are of the form

$\overbrace{\qquad\qquad\qquad}^{k-d+r+3 \text{ times}}$ (first 10 columns), $\overbrace{\qquad}^{d-r+1 \text{ times}}$ (last 4 columns)

d	d	$\cdots$	d	d	d	d	$\cdots$	d	d	$d{+}2$	$d{+}2$	$\cdots$	$d{+}2$
d	$\cdots$	d	$d{+}1$	$\cdots$	$d{+}1$	$d{+}2$	$d{+}2$	$\cdots$	$d{+}2$	$d{+}3$	$d{+}3$	$\cdots$	$d{+}3$

$\underbrace{\qquad}_{l-r \text{ times}}$ (first 3 columns), $\underbrace{\qquad}_{r \text{ times}}$ (columns 7–10), $\underbrace{\qquad}_{d+1 \text{ times}}$ (last 4 columns)

It follows from part (1) of Lemma A.11 that $\widehat{\Theta}_{\mathsf{S}^l_{D,E}}=0$.

We now suppose that $r=l$, and consider the exceptional tableau homomorphism $\mathsf{R} := \mathsf{S}^l_{D,E}$ with $E=\{d+1,(d+2)^{l-1}\}$. We now proceed to show that $\widehat{\Theta}_{\mathsf{R}}=0$ by repeated application of Theorem A.10. Observe that the $(d-1)$th and dth rows in R have the following form

$\overbrace{\qquad\qquad}^{k-d+4 \text{ times}}$ (columns 1–6), $\overbrace{\qquad}^{\bar{t}-l \text{ times}}$ (columns 8–11)

$d{-}1$	$\cdots$	$d{-}1$	$d{-}1$	$\cdots$	$d{-}1$	d	d	d	$\cdots$	d	$d{+}1$	$d{+}1$	$\cdots$	$d{+}1$
d	$\cdots$	d	d	$\cdots$	d	d	d	$d{+}2$	$\cdots$	$d{+}2$	$d{+}2$	$d{+}2$	$\cdots$	$d{+}2$

$\underbrace{\qquad\qquad\qquad}_{k-d+l+3 \text{ times}}$ (columns 1–8), $\underbrace{\qquad\qquad}_{d-l+1 \text{ times}}$ (columns 9–15)

We now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{R}}$ with $i=d-1$. We set

$$A=\{(d-1)^{k-d+4}\}, \quad B=\{d^{k-d+\bar{t}+4}, (d+1)^{d-\bar{t}+l-1}\}, \quad C=\{(d+2)^{d-l+1}\},$$

so that $|D|=d$, $|E|=k+l-d+3$. We write $\sum_{(D,E)} \alpha_{\mathsf{R}_{D,E}}\widehat{\Theta}_{\mathsf{R}_{D,E}}=0$, and observe that $0\neq \alpha_{\mathsf{R}_{D,E}}\in\{-1,1\}$ for all possible pairs (D,E) with A,B,C as above.

Observe that

$$D=\{d^{\bar{t}-l+u+1}, (d+1)^{d-\bar{t}+l-u-1}\}, \quad E=\{d^{k+l-d+3-u}, (d+1)^u\}$$

with $u\geqslant 0$. If $u=0$, we observe that $\mathsf{R}_{D,E}=\mathsf{R}$. So we can write

$$\widehat{\Theta}_{\mathsf{R}} + \sum_{\{(D,E)\mid E_{d+1}>0\}} \alpha_{\mathsf{R}_{D,E}}\widehat{\Theta}_{\mathsf{R}_{D,E}} = 0,$$

where $\alpha_{\mathsf{R}_{D,E}} \in \{-1,0,1\}$. We now suppose that $u > 0$. Then rows $d-1$ and d of $\mathsf{R}_{D,E}$ are as follows

$$\begin{array}{l}
\overbrace{d{-}1\ \cdots\ d{-}1\ d{-}1\ \cdots\ d{-}1}^{k-d+4\text{ times}}\ \overbrace{d\ d\ \cdots\ d\ d}^{\bar{t}-l+u+1\text{ times}}\ d{+}1\ d{+}1\ \cdots\ d{+}1\\
\underbrace{d\ \cdots\ d\ d\ d\ \cdots\ d}_{k-d+l+3-u\text{ times}}\ \underbrace{d{+}1\ d{+}1\ \cdots\ d{+}1}_{u\text{ times}}\ \underbrace{d{+}2\ d{+}2\ \cdots\ d{+}2}_{d-l+1\text{ times}}
\end{array}$$

We now show that $\widehat{\Theta}_{\mathsf{R}_{D,E}} = 0$ for all $u > 0$, treating the cases $u = 1$ and $u > 1$ separately, and hence $\widehat{\Theta}_{\mathsf{R}} = 0$.

We first suppose that $u > 1$ and write $\mathsf{R}' = \mathsf{R}_{D,E}$ in this case. If $l > u+1$ and $u > 1$, then we can observe from above that $\mathsf{R}'(d-1,c) = \mathsf{R}'(d,c) = d$ for some column c, and hence $\widehat{\Theta}_{\mathsf{R}'} = 0$ by Lemma A.12. We now suppose that $l \leqslant u+1$; we observe that rows $d-1$ and d in R' above are standard. Since $d \geqslant 3$, we see that the $(d-2)$th and $(d-1)$th rows in R' have the following form

$$\begin{array}{l}
\overbrace{d{-}2\ d{-}2\ \cdots\ d{-}2\ d{-}2}^{k-d+5\text{ times}}\ d{-}1\ \overbrace{d\ d\ \cdots\ d\ d}^{d-2\text{ times}}\\
\underbrace{d{-}1\ d{-}1\ \cdots\ d{-}1}_{k-d+4\text{ times}}\ \underbrace{d\ d\ \cdots\ d}_{\bar{t}-l+u+1\text{ times}}\ d{+}1\ \cdots\ d{+}1
\end{array}$$

We now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{R}'}$ with $i = d-2$. We set

$$A = \{(d-2)^{k-d+5}, d-1\}, \quad B = \{(d-1)^{k-d+4}, d^{\bar{t}+d-l+u-1}\}, \quad C = \{(d+1)^{d-\bar{t}+l-u-1}\},$$

so that $|D| = d-2$ and $|E| = k+\bar{t}-l-d+u+5$. Observe that each homomorphism $\widehat{\Theta}_{\mathsf{R}'_{D,E}}$ in Theorem A.10 appears with some scalar multiple of $[D_{d-1}+1]$, which equals zero if and only if D_{d-1} is odd. So suppose that D_{d-1} is even. If $D_{d-1} = 0$, then $\mathsf{R}'_{D,E} = \mathsf{R}'$. Now suppose that $D_{d-1} > 0$ and consider $\mathsf{R}'_{D,E}$ with

$$D = \{(d-1)^{2j}, d^{d-2j-2}\}, \quad E = \{(d-1)^{k-d-2j+4}, d^{\bar{t}-l+u+2j+1}\},$$

where $j \geqslant 1$; we abuse notation and let $\mathsf{R}'' = \mathsf{R}'_{D,E}$ in this case. Now observe that the $(d-2)$th and $(d-1)$th rows of $\mathsf{R}'_{D,E}$ are

$$\begin{array}{l}
\overbrace{d{-}2\ d{-}2\ \cdots\ d{-}2\ d{-}2}^{k-d+5\text{ times}}\ d{-}1\ \cdots\ d{-}1\ \overbrace{d\ \cdots\ d}^{d-2j-2\text{ times}}\\
\underbrace{d{-}1\ d{-}1\ \cdots\ d{-}1}_{k-d-2j+4\text{ times}}\ d\ d\ \cdots\ d\ \underbrace{d{+}1\ \cdots\ d{+}1}_{d-\bar{t}+l-u-1\text{ times}}
\end{array}$$

Suppose that $\bar{t}+u-2j > l+1$. Then we can apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{R}''}$ with $i = d-2$. We observe that in this case, we have $A_{d-1} = 2j+1$. Hence the only tableau homomorphism $\widehat{\Theta}_{\mathsf{R}''_{D,E}}$ that appears in Theorem A.10 with a non-zero coefficient is $\widehat{\Theta}_{\mathsf{R}''}$ itself, and hence $\widehat{\Theta}_{\mathsf{R}''} = 0$. Instead suppose that $l \geqslant \bar{t}+u-2j-1$, and observe rows $d-1$ and d of R'' are of the form

$$\begin{array}{l}
\overbrace{d{-}1\ d{-}1\ \cdots\ d{-}1}^{k-d-2j+4\text{ times}}\ d\ d\ \cdots\ d\ \overbrace{d{+}1\ \cdots\ d{+}1}^{d-\bar{t}+l-u-1\text{ times}}\\
d\ d\ \cdots\ d\ d\ \underbrace{d{+}1\ \cdots\ d{+}1}_{u\text{ times}}\ \underbrace{d{+}2\ \cdots\ d{+}2}_{d-l+1\text{ times}}
\end{array}$$

Since $\bar{t} \geqslant 3$, we have that $l > u - 2j + 1$. It thus follows that there exists a column c such that $\mathsf{R}''(d-1,c) = \mathsf{R}''(d,c) = d$, and hence $\widehat{\Theta}_{\mathsf{R}''} = 0$ by Lemma A.12.

We now suppose that $u = 1$; we write $\mathsf{U} = \mathsf{R}_{D,E}$ with $D = \{d^{\bar{t}-l+2}, (d+1)^{d-\bar{t}+l-2}\}$, $E = \{d^{k+l-d+2}, d+1\}$. We now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{U}}$ with $i = d-1$. We set

$$A = \{(d-1)^{k-d+4}\}, \quad B = \{d^{k+\bar{t}-d+4}, (d+1)^{d-\bar{t}+l-2}\}, \quad C = \{d+1, (d+2)^{d-l+1}\},$$

so that $|D| = d$ and $|E| = k+l-d+2$. Thus each homomorphism $\widehat{\Theta}_{\mathsf{U}_{D,E}}$ appears in Theorem A.10 with some scalar multiple of $[E_{d+1}+1]$, which equals zero if and only if E_{d+1} is odd. So suppose that E_{d+1} is even. If $E_{d+1} = 0$, then $\mathsf{U}_{D,E} = \mathsf{U}$. We can thus write

$$\widehat{\Theta}_{\mathsf{R}} + \widehat{\Theta}_{\mathsf{U}} + \sum_{\{(D,E) \mid E_{d+1} \text{ even},\ E_{d+1}>1\}} \alpha_{\mathsf{R}_{D,E}} \widehat{\Theta}_{\mathsf{R}_{D,E}} = 0,$$

where $\alpha_{\mathsf{R}_{D,E}} \in \{-1,1\}$. For $E_{d+1} > 0$, we let

$$D = \{d^{\bar{t}-l+2m+2}, (d+1)^{d-\bar{t}+l-2m-2}\}, \quad E = \{d^{k+l-d-2m+2}, (d+1)^{2m}\}$$

with $m \geqslant 1$, and we write $\mathsf{U}' = \mathsf{U}_{D,E}$ in this case.

By following the above proof to show that $\widehat{\Theta}_{\mathsf{R}_{D,E}} = 0$ when $u > 1$, we can similarly show that $\widehat{\Theta}_{\mathsf{U}'} = 0$ by applying the substitution $u = 2m+1$.

(2) Assume that $E_{d+1} = 0$. In other words, consider the tableaux $\mathsf{S}^l_{D,E}$ with

$$D = \{d^{k-d+r+3}, d+1, (d+2)^{d-r}\}, \quad E = \{d^{l-r}, (d+2)^r\},$$

where $1 \leqslant r \leqslant l$. Since $E_{d+2} = r$, and E_{d+2} is even (by assumption), it thus suffices to consider $\mathsf{S}^l_{D,E}$ with r even. If $l > r$, then it follows from part (2) of Lemma A.11 that $\widehat{\Theta}_{\mathsf{S}^l_{D,E}} = 0$. We now suppose that $r = l$ is even and consider the tableau homomorphism $\mathsf{R} := \widehat{\Theta}_{\mathsf{S}^l_{D,E}}$ with

$$D = \{d^{k-d+l+3}, d+1, (d+2)^{d-l}\}, \quad E = \{(d+2)^l\}.$$

Observe that the $(d-1)$th and dth rows of R are of the form

$$\begin{array}{ccccccccccccc}
\multicolumn{4}{c}{\overbrace{\hphantom{d-1\ d-1\ \cdots\ d-1}}^{k-d+4 \text{ times}}} & & \multicolumn{4}{c}{\overbrace{\hphantom{d\ d\ \cdots\ d}}^{\bar{t}-l \text{ times}}} & & & & \\
\hline
\multicolumn{1}{|c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} \\
\hline
\multicolumn{1}{|c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+2} \\
\hline
\multicolumn{8}{c}{\underbrace{\hphantom{d\ d\ \cdots\ d\ d\ d\ \cdots\ d}}_{k-d+l+3 \text{ times}}} & & \multicolumn{4}{c}{\underbrace{\hphantom{d+2\ d+2\ \cdots\ d+2}}_{d-l \text{ times}}}
\end{array}$$

We now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{R}}$ with $i = d-1$. We set

$$A = \{(d-1)^{k-d+4}\}, \quad B = \{d^{k-d+\bar{t}+4}, (d+1)^{d-\bar{t}+l-1}\}, \quad C = \{d+1, (d+2)^{d-l}\},$$

so that $|D| = d$ and $|E| = k+l-d+3$. Observe that $\widehat{\Theta}_{\mathsf{R}_{D,E}}$ appears in Theorem A.10 with some scalar multiple of $[E_{d+1}+1]$, which is zero if and only if E_{d+1} is odd. So suppose that E_{d+1} is even, and let

$$D = \{d^{k-d+\bar{t}-2m+3}, (d+1)^{2d-k-\bar{t}+2m-3}\}, \quad E = \{d^{2m+1}, (d+1)^{k+l-d-2m+2}\}$$

with $m \geqslant 1$. Observe that $\mathsf{R}_{D,E} = \mathsf{R}$ if $2m = k-d+l+2$. So suppose that $2m < k-d+l$, and write $\mathsf{R}' = \mathsf{R}_{D,E}$ in this case. Then rows $d-1$ and d of $\mathsf{R}'_{D,E}$ are of the form

$$\begin{array}{ccccccccccccc}
\multicolumn{4}{c}{\overbrace{\hphantom{d-1\ d-1\ \cdots\ d-1}}^{k-d+4 \text{ times}}} & \multicolumn{5}{c}{\overbrace{\hphantom{d\ d\ d\ \cdots\ d}}^{k-d+\bar{t}-2m+3 \text{ times}}} & & & & \\
\hline
\multicolumn{1}{|c|}{d-1} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d-1} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} \\
\hline
\multicolumn{1}{|c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+1} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{d+2} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{d+2} \\
\hline
\multicolumn{5}{c}{\underbrace{\hphantom{d\ d\ \cdots\ d\ d}}_{2m+1 \text{ times}}} & & & & & \multicolumn{4}{c}{\underbrace{\hphantom{d+2\ d+2\ \cdots\ d+2}}_{d-l \text{ times}}}
\end{array}$$

Observe that if $2m > k-d+3$, then there exists a column c such that $\mathsf{R}'(d-1,c) = \mathsf{R}'(d,c) = d$. Hence $\widehat{\Theta}_{\mathsf{R}''} = 0$ by part two of Lemma A.12. We now suppose that $2m \leqslant k-d+2$ and observe that the $(d-2)$th and $(d-1)$th rows of R' are of the form

$\overbrace{\qquad}^{k-d+5 \text{ times}}$ (first five columns); $\overbrace{\qquad}^{d-2 \text{ times}}$ (last seven columns)

$d-2$	$d-2$	$\cdots$	$d-2$	$d-2$	$d-1$	d	$\cdots$	d	d	d	$\cdots$	d
$d-1$	$d-1$	$\cdots$	$d-1$	d	d	d	$\cdots$	d	$d+1$	$d+1$	$\cdots$	$d+1$

$\underbrace{\qquad}_{k-d+4 \text{ times}}$ (first four columns); $\underbrace{\qquad}_{k-d+\bar{t}-2m+3 \text{ times}}$ (columns six to nine)

Since $k-d \geqslant 2m-2$ and $\bar{t} \geqslant 3$, we have $k-d+\bar{t}-2m \geqslant 0$. Hence $\mathsf{R}'(d-2,c) = \mathsf{R}'(d-1,c) = d$ for some column c. We can thus apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{R}'}$ with $i = d-2$. We set

$$A = \{(d-2)^{k-d+5}, d-1\}, \quad B = \{(d-1)^{k-d+4}, d^{k+\bar{t}-2m+1}\}, \quad C = \{(d+1)^{2d-k-\bar{t}+2m-3}\},$$

so that $|D| = d-2$ and $|E| = 2k-2d+\bar{t}-2m+7$. We observe that $\widehat{\Theta}_{\mathsf{R}'_{D,E}}$ appears in Theorem A.10 with some scalar multiple of $[D_{d-1}+1]$, which is zero if and only if D_{d-1} is odd. So suppose that D_{d-1} is even, and write

$$D = \{(d-1)^{2j}, d^{d-2j-2}\}, \quad E = \{(d-1)^{k-d-2j+4}, d^{k+\bar{t}-d-2m+2j+3}\}$$

with $j \geqslant 0$. We have $\mathsf{R}'_{D,E} = \mathsf{R}'$ when $j = 0$. So suppose that $j > 0$, and write $\mathsf{R}'' = \mathsf{R}'_{D,E}$ in this case. Then the $(d-2)$th and $(d-1)$th rows of R'' are of the form

$\overbrace{\qquad}^{k-d+5 \text{ times}}$ (first five columns); $\overbrace{\qquad}^{d-2j-2 \text{ times}}$ (last four columns)

$d-2$	$d-2$	$\cdots$	$d-2$	$d-2$	$d-1$	$d-1$	$\cdots$	$d-1$	d	d	$\cdots$	d
$d-1$	$d-1$	$\cdots$	$d-1$	d	d	d	$\cdots$	d	$d+1$	$d+1$	$\cdots$	$d+1$

$\underbrace{\qquad}_{k-d-2j+4 \text{ times}}$ (first four columns); $\underbrace{\qquad}_{2d-k-\bar{t}+2m-3 \text{ times}}$ (last four columns)

If $k+\bar{t}-d-2m-2j \geqslant 0$, then there exists a column c such that $\mathsf{R}''(d-2,c) = \mathsf{R}''(d-1,c) = d$. We can thus apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{R}''}$ with $i = d-2$. We observe that $\widehat{\Theta}_{\mathsf{R}''_{D,E}}$ appears in Theorem A.10 with some scalar multiple of

$$\begin{bmatrix} A_{d-1} + D_{d-1} \\ A_{d-1} \end{bmatrix} = \begin{bmatrix} D_{d-1} + 2j + 1 \\ 2j+1 \end{bmatrix},$$

which equals zero if and only if $D_{d-1} > 0$. If $D_{d-1} = 0$, then $\mathsf{R}''_{D,E} = \mathsf{R}''$. Hence $\widehat{\Theta}_{\mathsf{R}_{D,E}}$ is the only homomorphism in the sum of Theorem A.10, and thus must be zero. We now suppose that $k+\bar{t}-d-2m-2j+1 \leqslant 0$, and observe that the $(d-1)$th and dth rows of R'' have the form

$\overbrace{\qquad}^{k-d-2j+4 \text{ times}}$ (first four columns); $\overbrace{\qquad}^{2d-k-\bar{t}+2m-3 \text{ times}}$ (last four columns)

$d-1$	$d-1$	$\cdots$	$d-1$	d	d	$\cdots$	d	$d+1$	$d+1$	$\cdots$	$d+1$
d	d	$\cdots$	d	$d+1$	$d+1$	$\cdots$	$d+1$	$d+2$	$d+2$	$\cdots$	$d+2$

$\underbrace{\qquad}_{2m+1 \text{ times}}$ (first four columns); $\underbrace{\qquad}_{d-l \text{ times}}$ (last three columns)

Since $\bar{t} \geqslant 3$, it follows that $\mathsf{R}''(d-1,c) = \mathsf{R}''(d,c) = d$ for some column c. Moreover, since $2m \leqslant k-d+3$, there are $k+l-d-2m+3 \geqslant l+1 \geqslant 3$ entries equal to $d+1$ in row d of R''. Hence $\widehat{\Theta}_{\mathsf{R}''} = 0$ by Lemma A.12.

Hence the sum in Theorem A.10 is $\widehat{\Theta}_{\mathsf{S}^l} + \widehat{\Theta}_{\mathsf{T}^l} = 0$ for $k-d$ odd.

Case 2: Suppose that $k-d$ is even. Arguing as in the first case, we also deduce that $\widehat{\Theta}_{\mathsf{S}^l}+\widehat{\Theta}_{\mathsf{T}^l}=0$. We shall now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{T}^l}$ with $i=d$ and $k-d$ even in order to show that $\widehat{\Theta}_{\mathsf{T}^l}=0$ and thus obtain the required result. We have

$$A=\emptyset,\quad B=\{d^{k+l-d+3},(d+2)^d\},\quad C=\{(d+1)^{k-d-l+3},d+2,(d+3)^{d+1}\}$$

and $|D|=k+4$, $|E|=l-1$. It follows that $\alpha^l_{\mathsf{T}_{D,E}}$ is zero if and only if E_{d+2} is odd. We thus assume that E_{d+2} is even. We know that $\mathsf{T}^l_{D,E}=\mathsf{T}^l$ when $E_{d+2}=0$, so it suffices to only consider the pairs (D,E) with

$$D=\{d^{k-d+2m+4},(d+2)^{d-2m}\},\quad E=\{d^{l-2m-1},(d+2)^{2m}\},$$

where $m\geqslant 1$. Now observe that the tableau $\mathsf{T}^l_{D,E}$ with $E=\{d^{l-2m-1},(d+2)^{2m}\}$ is similar to $\mathsf{S}^l_{D,E}$ with $E=\{d^{l-r},(d+2)^r\}$ such that r is even. Thus following the argument as in part (2) of the first case with $k-d$ odd, we find that $\widehat{\Theta}_{\mathsf{T}^l_{D,E}}=0$ for all possible pairs (D,E) as above. Hence $\widehat{\Theta}_{\mathsf{T}}=0$ and thus $\widehat{\Theta}_{\mathsf{S}}=0$. $\square$

Remark A.14. We note that neither S^{k-d+3} nor T^0 are considered in the previous result due to the restriction on l. In fact, we consider these two cases to be *exceptional*; we show that these tableaux are well defined for certain $\bar{t}$ and that their corresponding tableau homomorphisms are zero.

Lemma A.15. *Let $\lambda=((k+4)^k)$ and $\mu=((k+2)^k,k^2)$ for some $k>1$, $d\in\{2,\dots,k-1\}$ and $\bar{t}\in\{1,\dots,k+2\}$.*

(1) Suppose that $d>2$ and $1<\bar{t}\leqslant d-1$. We have that $\widehat{\Theta}_{\mathsf{S}}=0$ for $\mathsf{S}\in\mathrm{RStd}(\lambda,\mu)$ the tableau obtained from A by replacing $\bar{t}$ entries equal to $d+1$ in the $(d-1)$th row with the entry d. The tableau S is identical to A except for row $d-1$, which is of the form

$k-d+4$ times							$\bar{t}$ times							
$d{-}1$	$d{-}1$	$\cdots$	$d{-}1$	$d{-}1$	$d{-}1$	d	d	d	$\cdots$	d	$d{+}1$	$d{+}1$	$\cdots$	$d{+}1$

(2) We suppose that $k-d+3\leqslant\bar{t}$. We have that $\widehat{\Theta}_{\mathsf{T}}=0$ for $\mathsf{T}\in\mathrm{RStd}(\lambda,\mu)$ the tableau obtained from A by replacing the leftmost $\bar{t}$ entries equal to $d+1$ with entries equal to d. The tableau T is identical to A except for in the $(d-1)$th, dth and $(d+1)$th rows which are of the form

$k-d+4$ times							$\bar{t}+d-k-3$ times							
$d{-}1$	$d{-}1$	$\cdots$	$d{-}1$	$d{-}1$	$d{-}1$	d	d	d	$\cdots$	d	$d{+}1$	$d{+}1$	$\cdots$	$d{+}1$
d	d	$\cdots$	d	d	d	$d{+}2$	$d{+}2$	$d{+}2$	$\cdots$	$d{+}2$	$d{+}2$	$d{+}2$	$\cdots$	$d{+}2$
d	d	$\cdots$	d	$d{+}2$	$d{+}3$	$d{+}3$	$d{+}3$	$d{+}3$	$\cdots$	$d{+}3$	$d{+}3$	$d{+}3$	$\cdots$	$d{+}3$
$k-d+2$ times					$d+1$ times									

Proof. We consider these two exceptional cases separately.

Since $d>2$ and $\bar{t}\geqslant 2$, we have $k\geqslant 4$. Thus S contains the rows $d-2$, $d-1$ and d for any $d\in\{3,\dots,k-1\}$.

Case 1: We fix $i=d-2$ and apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{S}}$. We set

$$A=\{(d-2)^{k-d+5},d-1\},\quad B=\{(d-1)^{k-d+4},d^{d+\bar{t}-1}\},\quad C=\{(d+1)^{d-\bar{t}-1}\},$$

and $|D|=d-2$, $|E|=k+\bar{t}-d+5$. Thus every homomorphism $\widehat{\Theta}_{\mathsf{S}_{D,E}}$ in equation (A.3) appears with some scalar multiple of $[D_{d-1}+1]$, which is zero if and only if D_{d-1} is odd. So suppose that D_{d-1} is even. If $D_{d-1}=0$ then $\mathsf{S}=\mathsf{S}_{D,E}$. Now let $\mathsf{R}^r=\mathsf{S}_{D,E}$ with $D_{d-1}=2r$ for some $r\geqslant 1$, so that the pair (D,E) are of the form

$$D=\{(d-1)^{2r},d^{d-2r-2}\},\quad E=\{(d-1)^{k-d-2r+4},d^{\bar{t}+2r+1}\}.$$

We observe that rows $d-1$ and d of $\mathsf{R}^r_{D,E}$ are of the form

$\bar{t}+2r+1$ times (columns 4–11); $d-\bar{t}-1$ times (columns 12–14)

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
d{-}1 & \cdots & d{-}1 & d & d & d & \cdots & d & d & \cdots & d & d{+}1 & \cdots & d{+}1 \\
\hline
d & \cdots & d & d & d{+}1 & d{+}2 & \cdots & d{+}2 & d{+}2 & \cdots & d{+}2 & d{+}2 & \cdots & d{+}2 \\
\hline
\end{array}$$

$k-d+3$ times (columns 1–4); d times (columns 6–14)

We can thus write

$$\widehat{\Theta}_{\mathsf{S}}+\sum_{r\geqslant 1}\alpha_{\mathsf{R}^r}\widehat{\Theta}_{\mathsf{R}^r}=0,$$

where $\alpha_{\mathsf{R}^r}\in\{-1,1\}$.

We now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{R}^r}$ with $i=d-1$. We set

$$A=\{(d-1)^{k-d-2r+4}\},\quad B=\{d^{k-d+\bar{t}+2r+4},(d+1)^{d-\bar{t}-1}\},\quad C=\{d+1,(d+2)^d\},$$

and $|D|=d+2r$, $|E|=k-d+3$. Thus every homomorphism $\widehat{\Theta}_{\mathsf{R}^r_{D,E}}$ in equation (A.3) appears with some scalar multiple of $[E_{d+1}+1]$, which is zero if and only if E_{d+1} is odd. So suppose that E_{d+1} is even. If $E_{d+1}=0$ then $\mathsf{R}^r=\mathsf{R}^r_{D,E}$. Now let $\mathsf{U}^{r,s}=\mathsf{R}^r_{D,E}$ with $E_{d+1}=2s$ for some $s\geqslant 1$, so that the pair (D,E) are of the form

$$D=\{d^{\bar{t}+2r+2s+1},(d+1)^{d-\bar{t}-2s-1}\},\quad E=\{d^{k-d-2s+3},(d+1)^{2s}\}.$$

We can thus write

$$\widehat{\Theta}_{\mathsf{R}^r}+\sum_{s\geqslant 1}\alpha_{\mathsf{U}^{r,s}}\widehat{\Theta}_{\mathsf{U}^{r,s}}=0,$$

where $\alpha_{\mathsf{U}^{r,s}}\in\{-1,1\}$.

We now observe that we lie in one of the following two cases.

(1) Suppose that $r\geqslant s+1$. Then $\mathsf{U}^{r,s}(d-1,c)=\mathsf{U}^{r,s}(d,c)=d$ for some column c. We can thus apply part one of Lemma A.12 to $\mathsf{U}^{r,s}$ with $v=2s+1$, and hence $\widehat{\Theta}_{\mathsf{U}^{r,s}}=0$.

(2) Suppose that $s\geqslant r$. Then rows $d-1$ and d in $\mathsf{U}^{r,s}$ are standard. Now observe that the $(d-2)$th and $(d-1)$th rows of $\mathsf{U}^{r,s}$ are of the form

$k-d+5$ times (columns 1–4); $2r+1$ times (columns 5–7)

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
d{-}2 & \cdots & d{-}2 & d{-}2 & d{-}1 & \cdots & d{-}1 & d & d & \cdots & d \\
\hline
d{-}1 & \cdots & d{-}1 & d & d & \cdots & d & d & d{+}1 & \cdots & d{+}1 \\
\hline
\end{array}$$

$k-d+4-2r$ times (columns 1–3); $d-\bar{t}-2s-1$ times (columns 9–11)

Observe that $\mathsf{U}^{r,s}(d-2,c)=\mathsf{U}^{r,s}(d-1,c)=d$ for some column c. We now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{U}^{r,s}}$ with $i=d-2$. We have

$$A=\{(d-2)^{k-d+5},(d-1)^{2r+1}\},\quad B=\{(d-1)^{k-d+4-2r},d^{\bar{t}+d+2s-1}\}\quad C=\{(d+1)^{d-\bar{t}-2s-1}\}.$$

Thus every homomorphism $\widehat{\Theta}_{\mathsf{U}^{r,s}_{D,E}}$ in equation (A.3) appears with some scalar multiple of

$$\begin{bmatrix}A_{d-1}+D_{d-1}\\A_{d-1}\end{bmatrix}=\begin{bmatrix}D_{d-1}+2r+1\\2r+1\end{bmatrix}=\begin{cases}1 & \text{if } D_{d-1}=0,\\ 0 & \text{otherwise,}\end{cases}$$

and hence $\widehat{\Theta}_{\mathsf{U}^{r,s}}=0$ for all $s\geqslant 1$.

Thus $\widehat{\Theta}_{\mathsf{R}^r} = 0$ for all $r \geqslant 1$, and moreover $\widehat{\Theta}_{\mathsf{S}} = 0$.

Case 2: We fix $i = d$ and apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{T}}$. We set

$$A = \emptyset, \quad B = \{d^{2k-2d+6}, (d+2)^d\}, \quad C = \{d+2, (d+3)^{d+1}\},$$

and $|D| = k+4$, $|E| = k-d+2$. Thus every homomorphism $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ in equation (A.3) appears with some scalar multiple of $[E_{d+2}+1]$, which is zero if and only if E_{d+2} is odd.

Now assume that E_{d+2} is even. Let $\mathsf{T}_{D,E}$ be the tableau that is identical to T except for its dth and $(d+1)$th rows, which have the following form

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\multicolumn{5}{c}{} & \multicolumn{6}{c}{\overbrace{\hspace{16em}}^{d-E_{d+2}\text{ times}}} \\
\hline
d & d & \cdots & d & d & d{+}2 & \cdots & d{+}2 & d{+}2 & \cdots & d{+}2 \\
\hline
d & d & \cdots & d & d{+}2 & d{+}2 & \cdots & d{+}2 & d{+}3 & \cdots & d{+}3 \\
\hline
\multicolumn{4}{c}{} & \multicolumn{4}{c}{\underbrace{\hspace{11em}}_{E_{d+2}+1\text{ times}}} & \multicolumn{3}{c}{\underbrace{\hspace{8em}}_{d+1\text{ times}}}
\end{array}$$

Observe that $\mathsf{T} = \mathsf{T}_{D,E}$ when $E_{d+2} = 0$, and we can thus write

$$\widehat{\Theta}_{\mathsf{T}} + \sum_{(D,E)|E_{d+2}>0\}} \alpha_{\mathsf{T}_{D,E}} \widehat{\Theta}_{\mathsf{T}_{D,E}},$$

where $\alpha_{\mathsf{T}_{D,E}} \in \{-1, 1\}$. If $E_{d+2} > 0$, then $\widehat{\Theta}_{\mathsf{T}_{D,E}} = 0$ by setting v=0 in part (1) of Lemma A.11. Hence the sum in equation (A.3) contains only the tableau homomorphism $\widehat{\Theta}_{\mathsf{T}}$, which must be zero. ☐

Thanks to Lyle, we know how to compute the composition of any tableau homomorphism with the map $\varphi_{d,t}$.

Proposition A.16. [Lyl07, Proposition 2.14] *Let $\mu, \alpha \vdash n$, $1 \leqslant d$, $0 \leqslant t < \alpha_{d+1}$ and $\bar{t} = \alpha_{d+1} - t$. Suppose $\mathsf{S} \in \mathrm{RStd}(\mu, \alpha)$ and let $\mathscr{S} \subseteq \mathrm{RStd}(\mu, \nu^{d,t})$ be the set of row standard tableaux obtained by replacing $\bar{t}$ entries of $d+1$ in S with d. For each $\mathsf{T} \in \mathscr{S}$, set*

- $\beta_i = |\{j \mid \mathsf{S}(i,j) = d+1\}| - |\{j \mid \mathsf{T}(i,j) = d+1\}|$;
- $x_i = |\{(k,j) \mid k > i \text{ and } \mathsf{S}(k,j) = d\}|$;
- $y_i = |\{j \mid \mathsf{T}(i,j) = d\}|$,

for all $i \geqslant 1$. Then

$$\varphi_{d,t} \circ \Theta_{\mathsf{S}} = \sum_{\mathsf{T} \in \mathscr{S}} \alpha_{\mathsf{T}} \Theta_{\mathsf{T}}, \qquad \text{where} \qquad \alpha_{\mathsf{T}} = \prod_{i \geqslant 1} q^{x_i \beta_i} \begin{bmatrix} y_i \\ \beta_i \end{bmatrix}.$$

In our case, we note that $\alpha_{\mathsf{T}} = \pm \left[\begin{smallmatrix} y_i \\ \beta_i \end{smallmatrix}\right]$ since $q = -1$.

We are now ready to determine that the semistandard tableau homomorphism $\widehat{\Theta}_{\mathsf{A}}$ belongs to the homomorphism space $\mathrm{Hom}_{H_q^{\Bbbk}(n)} \left(\mathbf{S}_q^{\Bbbk}(((k+4)^k)), \mathbf{S}_q^{\Bbbk}((k+2)^k, k^2) \right)$.

Theorem A.17. *Let $k > 1$. Then $\varphi_{d,t} \circ \widehat{\Theta}_{\mathsf{A}} = 0$ for all $d \geqslant 1$ and $0 \leqslant t < \lambda_{d+1}$.*

Proof. We apply Proposition A.16 to $\widehat{\Theta}_{\mathsf{A}}$ for all possible d and t.

Case 1: Suppose that $d = 1$. Then we have $\varphi_{1,t} \circ \widehat{\Theta}_{\mathsf{A}} = \alpha_{\mathsf{S}} \widehat{\Theta}_{\mathsf{S}} + \alpha_{\mathsf{T}} \widehat{\Theta}_{\mathsf{T}}$, where S is identical to A except for its first two rows, which are of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\multicolumn{8}{c}{\overbrace{\hspace{22em}}^{k+2\text{ times}}} & \multicolumn{2}{c}{} \\
\hline
1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 3 \\
\hline
1 & \cdots & 1 & 2 & 2 & \cdots & 2 & 3 & 4 & 4 \\
\hline
\multicolumn{3}{c}{\underbrace{\hspace{8em}}_{\bar{t}-1\text{ times}}} & \multicolumn{7}{c}{}
\end{array}$$

and where T is identical to A except for its first two rows, which are of the form

$$\begin{array}{l} \overbrace{1\quad 1\quad \cdots\quad 1\quad 1\quad \cdots\quad 1\quad 1}^{k+2\text{ times}}\quad 2\quad 3 \\ \underbrace{1\quad 1\quad \cdots\quad 1}_{\bar{t}\text{ times}}\quad 2\quad \cdots\quad 2\quad 3\quad 4\quad 4 \end{array}$$

We observe that $\alpha_{\mathsf{T}} = 1$ and $\alpha_{\mathsf{S}} = [k+3] = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases}$

Hence

$$\varphi_{1,t} \circ \widehat{\Theta}_{\mathsf{A}} = \begin{cases} \widehat{\Theta}_{\mathsf{T}} & \text{if } k \text{ is odd,} \\ \widehat{\Theta}_{\mathsf{S}} + \widehat{\Theta}_{\mathsf{T}} & \text{if } k \text{ is even.} \end{cases}$$

We first apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{T}}$ with $i = 1$. We have

$$A = \emptyset, \quad B = \{1^{k+\bar{t}+2}, 2, 3\}, \quad C = \{2^{k-\bar{t}+1}, 3, 4, 4\},$$

and we thus have the following four possible pairs of (D, E):

$$(\{1^{k+2}, 2, 3\}, \{1^{\bar{t}}\}) \quad (\{1^{k+3}, 3\}, \{1^{\bar{t}-1}, 2\}) \quad (\{1^{k+3}, 2\}, \{1^{\bar{t}-1}, 3\}) \quad (\{1^{k+4}\}, \{1^{\bar{t}-2}, 2, 3\}).$$

Observe that $\mathsf{T}_{D,E} = \mathsf{T}$ where (D, E) is the first pair above. We note that $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ appears in equation (A.3) with some scalar multiple of

$$\begin{bmatrix} E_2 + k - \bar{t} + 1 \\ k - \bar{t} + 1 \end{bmatrix} \cdot [E_3 + 1] \cdot \begin{bmatrix} E_4 + 2 \\ 2 \end{bmatrix}.$$

Hence the homomorphism $\widehat{\Theta}_{\mathsf{T}_{D,E}}$, where (D, E) is one of the last two pairs above, appears with zero coefficient in equation (A.3) since $[E_3 + 1] = [2] = 0$. Also, $\mathsf{T}_{D,E} = \mathsf{S}$ where (D, E) is the second pair above. We can thus write $\widehat{\Theta}_{\mathsf{S}} + \widehat{\Theta}_{\mathsf{T}} = 0$, and hence

$$\varphi_{1,t} \circ \widehat{\Theta}_{\mathsf{A}} = \begin{cases} \widehat{\Theta}_{\mathsf{T}} = -\widehat{\Theta}_{\mathsf{S}} & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

We now suppose that k is odd. If $\bar{t} = 1$ and we apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{T}}$ with $i = 1$ as above, we observe that $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ with (D, E) as the second pair above appears in equation (A.3) with zero coefficient since

$$\begin{bmatrix} E_2 + k - \bar{t} + 1 \\ k - \bar{t} + 1 \end{bmatrix} = \begin{bmatrix} k + 1 \\ k \end{bmatrix} = [k + 1] = 0.$$

Hence $\widehat{\Theta}_{\mathsf{T}} = 0$ when $\bar{t} = 1$.

We now let $\bar{t} > 1$ and apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{S}}$ with $i = 1$ as follows. We have

$$A = \emptyset, \quad B = \{1^{k+\bar{t}+2}, 3\}, \quad C = \{2^{k-\bar{t}+2}, 3, 4, 4\},$$

and we thus have the following two possible pairs of (D, E):

$$D = \{1^{k+3}, 3\}, \quad E = \{1^{\bar{t}-1}\} \quad \text{or} \quad D = \{1^{k+4}\}, \quad E = \{1^{\bar{t}-2}, 3\}.$$

Observe that $\mathsf{S}_{D,E} = \mathsf{S}$ where (D, E) is the first pair above. By applying Theorem A.10 to $\mathsf{S}_{D,E}$ where (D, E) is the second pair above, we see that the corresponding sum in equation (A.3) contains only a single homomorphism, and hence $\widehat{\Theta}_{\mathsf{S}_{D,E}} = 0$. It thus follows that $\widehat{\Theta}_{\mathsf{S}} = 0$, and hence $\varphi_{1,t} \circ \widehat{\Theta}_{\mathsf{A}} = 0$ when $\bar{t} > 1$.

Case 2: Suppose that $d \in \{2, 3, \ldots, k-1\}$, so that $\bar{t} \in \{1, 2, \ldots, k+2\}$. Note that the entries equal to $d+1$ lie in the $(d-1)$th, dth and $(d+1)$th rows of A, which have the following form

$d{-}1$	$d{-}1$	$\cdots$	$d{-}1$	$d{-}1$	$d{-}1$	d	$d{+}1$	$d{+}1$	$\cdots$	$d{+}1$
d	d	$\cdots$	d	d	$d{+}1$	$d{+}2$	$d{+}2$	$d{+}2$	$\cdots$	$d{+}2$
$d{+}1$	$d{+}1$	$\cdots$	$d{+}1$	$d{+}2$	$d{+}3$	$d{+}3$	$d{+}3$	$d{+}3$	$\cdots$	$d{+}3$

$\underbrace{\qquad\qquad\qquad}_{k-d+2 \text{ times}} \qquad \underbrace{\qquad\qquad\qquad\qquad}_{d+1 \text{ times}}$

The row-standard tableaux appearing in equation (A.3) split into the following three cases, depending on $\bar{t}$.

(1) Suppose that $\bar{t} \leqslant d-1$. Then there are $2\bar{t}+1$ such tableaux, namely R, $\mathsf{S}^1, \mathsf{S}^2, \ldots, \mathsf{S}^{\bar{t}}$ and $\mathsf{T}^1, \mathsf{T}^2, \ldots, \mathsf{T}^{\bar{t}}$, where $\mathsf{R}, \mathsf{S}^l, \mathsf{T}^l$ are row-standard (k^4)-tableaux of type $((k+2)^k, k^2)$ that are obtained from A by replacing $\bar{t}$ entries equal to $d+1$ with ds. We let R be identical to A except for its $(d-1)$th row, which is drawn as in part (1) of Lemma A.15; let S^l and T^l be identical to A except for their $(d-1)$th, dth and $(d+1)$th rows, which are drawn as in parts (1) and (2) of Lemma A.13, respectively. Then

$$\varphi_{d,t} \circ \widehat{\Theta}_{\mathsf{A}} = \alpha_{\mathsf{R}} \widehat{\Theta}_{\mathsf{R}} + \sum_{l=1}^{\bar{t}} \left(\alpha_{\mathsf{S}^l} \widehat{\Theta}_{\mathsf{S}^l} + \alpha_{\mathsf{T}^l} \widehat{\Theta}_{\mathsf{T}^l} \right),$$

where $\alpha_{\mathsf{R}}, \alpha_{\mathsf{S}^l}, \alpha_{\mathsf{T}^l} \in \{-1, 0, 1\}$.

(2) Suppose that $d \leqslant \bar{t} \leqslant k-d+2$. Then there are $2\bar{t}$ such tableaux, namely $\mathsf{S}^1, \mathsf{S}^2, \ldots, \mathsf{S}^{\bar{t}}$ and $\mathsf{T}^1, \mathsf{T}^2, \ldots, \mathsf{T}^{\bar{t}}$, where S^l and T^l are defined as in Lemma A.13. Then

$$\varphi_{d,t} \circ \widehat{\Theta}_{\mathsf{A}} = \sum_{l=1}^{\bar{t}} \left(\alpha_{\mathsf{S}^l} \widehat{\Theta}_{\mathsf{S}^l} + \alpha_{\mathsf{T}^l} \widehat{\Theta}_{\mathsf{T}^l} \right),$$

where $\alpha_{\mathsf{S}^l}, \alpha_{\mathsf{T}^l} \in \{-1, 0, 1\}$.

(3) Suppose that $\bar{t} \geqslant k-d+3$. Then there are $2(k-\bar{t})+5$ such tableaux, namely U, $\mathsf{S}^{\bar{t}-d+1}$, $\mathsf{S}^{\bar{t}-d+2}, \ldots, \mathsf{S}^{k-d+2}$ and $\mathsf{T}^{\bar{t}-d+1}, \mathsf{T}^{\bar{t}-d+2}, \ldots, \mathsf{T}^{k-d+2}$. We let U be defined to be the tableau that is identical to A except for its $(d-1)$th, dth and $(d+1)$th rows, which are drawn as in the second part of Lemma A.15; we let S^l and T^l be defined as in Lemma A.13. Then

$$\varphi_{d,t} \circ \widehat{\Theta}_{\mathsf{A}} = \alpha_{\mathsf{U}} \widehat{\Theta}_{\mathsf{U}} + \sum_{l=\bar{t}-d+1}^{k-d+2} \left(\alpha_{\mathsf{S}^l} \widehat{\Theta}_{\mathsf{S}^l} + \alpha_{\mathsf{T}^l} \widehat{\Theta}_{\mathsf{T}^l} \right),$$

where $\alpha_{\mathsf{U}}, \alpha_{\mathsf{S}^l}, \alpha_{\mathsf{T}^l} \in \{-1, 0, 1\}$.

We now observe the scalars of the tableau homomorphisms of R, S^l, T^l and U appearing in Proposition A.16 are as follows:

$$\alpha_{\mathsf{R}} = \pm[\bar{t}+1], \ \alpha_{\mathsf{S}^l} = [\bar{t}-l+1], \ \alpha_{\mathsf{T}^l} = [\bar{t}-l+1][k-d+4], \ \alpha_{\mathsf{U}} = \pm[\bar{t}-k+d-2][k-d+4].$$

Observe that

$$[k-d+4] = \begin{cases} 0 & \text{if } k-d \text{ is even,} \\ 1 & \text{if } k-d \text{ is odd;} \end{cases} \qquad [\bar{t}+1] = \begin{cases} 0 & \text{if } \bar{t} \text{ is odd,} \\ 1 & \text{if } \bar{t} \text{ is even.} \end{cases}$$

We thus suppose that $\bar{t}$ is even. Then we know from part (1) of Lemma A.15 that $\widehat{\Theta}_{\mathsf{R}} = 0$ when $\bar{t} \leqslant d-1$.

Suppose that $k-d$ is odd. Then $\alpha_{\mathsf{S}^l} = [\bar{t}-l+1] = \alpha_{\mathsf{T}^l}$ for all l. Moreover, we know from part (1) of Lemma A.13 that $\widehat{\Theta}_{\mathsf{S}^l} = -\widehat{\Theta}_{\mathsf{T}^l}$ for all l. Also, $\alpha_{\mathsf{U}} = [\bar{t}-k+d-2]$, and when this coefficient is non-zero, it follows from part (2) of Lemma A.15 that $\widehat{\Theta}_{\mathsf{U}} = 0$. Hence, for all $\bar{t}$,

$$\varphi_{d,t} \circ \widehat{\Theta}_{\mathsf{A}} = \sum_l \left(\alpha_{\mathsf{S}^l} \widehat{\Theta}_{\mathsf{S}^l} + \alpha_{\mathsf{T}^l} \widehat{\Theta}_{\mathsf{T}^l} \right) = \sum_l \alpha_{\mathsf{S}^l} \left(\widehat{\Theta}_{\mathsf{S}^l} - \widehat{\Theta}_{\mathsf{S}^l} \right) = 0.$$

We now suppose that $k - d$ is even, in which case both $\alpha_{\mathsf{U}} = 0$ and $\alpha_{\mathsf{T}^l} = 0$. Moreover, we know from part (2) of Lemma A.13 that $\widehat{\Theta}_{\mathsf{S}^l} = 0$ for all l, and hence $\varphi_{d,t} \circ \widehat{\Theta}_{\mathsf{A}} = 0$ for all $\bar{t}$.

Case 3: Suppose that $d = k$. Then we have $\varphi_{k,t} \circ \widehat{\Theta}_{\mathsf{A}} = \alpha_{\mathsf{S}} \widehat{\Theta}_{\mathsf{S}} + \alpha_{\mathsf{T}} \widehat{\Theta}_{\mathsf{T}}$ with $\alpha_{\mathsf{S}}, \alpha_{\mathsf{T}} \in \{-1, 0, 1\}$, where S and T are as follows: S is identical to A except for its last two rows, which are of the form

$\bar{t} - 1$ times

$k{-}1$	$k{-}1$	$k{-}1$	$k{-}1$	k	k	$\cdots$	k	$k{+}1$	$k{+}1$	$\cdots$	$k{+}1$
k	k	k	k	$k{+}2$	$k{+}2$	$\cdots$	$k{+}2$	$k{+}2$	$k{+}2$	$\cdots$	$k{+}2$

k times

and T is identical to A except for its last two rows, which are of the form

$\bar{t}$ times

$k{-}1$	$k{-}1$	$k{-}1$	$k{-}1$	k	k	k	$\cdots$	k	$k{+}1$	$\cdots$	$k{+}1$
k	k	k	$k{+}1$	$k{+}2$	$k{+}2$	$k{+}2$	$\cdots$	$k{+}2$	$k{+}2$	$\cdots$	$k{+}2$

k times

We observe that $\alpha_{\mathsf{S}} = [\bar{t}] \cdot [4] = 0$ and $\alpha_{\mathsf{T}} = [\bar{t} + 1] = \begin{cases} 0 & \text{if } \bar{t} \text{ is odd,} \\ 1 & \text{if } \bar{t} \text{ is even.} \end{cases}$

We thus suppose that $\bar{t}$ is even. We now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{T}}$ with $i = k - 2$. We set

$$A = \{(k-2)^5, k-1\}, \quad B = \{k-1, k-1, k-1, k-1, k^{k+\bar{t}-1}\}, \quad C = \{(k+1)^{k-\bar{t}-1}\},$$

and we thus have the following five possible pairs of (D, E):

$$(\{k^{k-2}\}, \{(k-1)^4, k^{\bar{t}+1}\}) \quad (\{k-1, k^{k-3}\}, \{(k-1)^3, k^{\bar{t}+2}\}) \quad (\{(k-1)^2, k^{k-4}\}, \{(k-1)^2, k^{\bar{t}+3}\})$$
$$(\{(k-1)^3, k^{k-5}\}, \{k-1, k^{\bar{t}+4}\}) \quad (\{(k-1)^4, k^{k-6}\}, \{k^{\bar{t}+5}\}.$$

Observe that $\mathsf{T}_{D,E} = \mathsf{T}$ where (D, E) is the first pair above. Each homomorphism $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ appears in equation (A.3) with coefficient $\pm[D_{k-1} + 1]$, which is zero when (D, E) is either the second or the fourth pair above. We let $\mathsf{R} := \mathsf{T}_{D,E}$ where (D, E) is the third pair above, and $\mathsf{U} := \mathsf{T}_{D,E}$ where (D, E) is the fifth pair above. We can thus write $\widehat{\Theta}_{\mathsf{T}} + \alpha_{\mathsf{R}} \widehat{\Theta}_{\mathsf{R}} + \alpha_{\mathsf{U}} \widehat{\Theta}_{\mathsf{U}} = 0$, where $\alpha_{\mathsf{R}}, \alpha_{\mathsf{U}} \in \{-1, 1\}$.

We first apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{R}}$ with $i = k - 2$. We set

$$A = \{(k-2)^5, (k-1)^3\}, \quad B = \{(k-1)^2, k^{k+\bar{t}-1}\}, \quad C = \{(k+1)^{k-\bar{t}-1}\},$$

and we thus have the following three possible pairs of (D, E):

$$(\{k^{k-4}\}, \{(k-1)^2, k^{\bar{t}+3}\}) \quad (\{k-1, k^{k-5}\}, \{k-1, k^{\bar{t}+4}\}) \quad (\{(k-1)^2, k^{k-6}\}, \{k^{\bar{t}+5}\}).$$

Observe that $\mathsf{R}_{D,E} = \mathsf{T}$ where (D, E) is the first pair above. The other two homomorphisms $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ appear in equation (A.3) with zero coefficient since $\left[\begin{smallmatrix} D_{k-1}+3 \\ 3 \end{smallmatrix}\right]$ is zero if and only if $D_{k-1} > 0$. It thus follows that $\widehat{\Theta}_{\mathsf{R}} = 0$.

We now apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{U}}$ with $i = k - 2$. The only homomorphism that appears in equation (A.3) is $\widehat{\Theta}_{\mathsf{U}}$ since $B = \{k^{\bar{t}+k-1}\}$, and hence $\widehat{\Theta}_{\mathsf{U}}$ must be zero.

Finally, since both $\widehat{\Theta}_{\mathsf{R}}$ and $\widehat{\Theta}_{\mathsf{U}}$ are zero homomorphisms, so is $\widehat{\Theta}_{\mathsf{T}}$.

Case 4: Suppose that $d = k+1$. Then we have $\varphi_{k+1,t} \circ \widehat{\Theta}_{\mathsf{A}} = \alpha_{\mathsf{T}} \widehat{\Theta}_{\mathsf{T}}$ with $\alpha_{\mathsf{T}} \in \{-1, 0, 1\}$, and where T is identical to A except for its last row. Observe that the last two rows of T are of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline k{-}1 & k{-}1 & k{-}1 & k{-}1 & k & k{+}1 & \cdots & k{+}1 & k{+}1 & k{+}1 & \cdots & k{+}1 \\ \hline k & k & k & k{+}1 & k{+}1 & k{+}1 & \cdots & k{+}1 & k{+}2 & k{+}2 & \cdots & k{+}2 \\ \hline \end{array}$$
$$\underbrace{\qquad\qquad\qquad}_{\bar{t} \text{ times}}$$

We have $\alpha_{\mathsf{T}} = [\bar{t}+1] = \begin{cases} 0 & \text{if } \bar{t} \text{ is odd,} \\ 1 & \text{if } \bar{t} \text{ is even.} \end{cases}$

So suppose that $\bar{t}$ is even and apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{T}}$ with $i = k-1$. We set

$$A = \{k-1^4, k\}, \quad B = \{k^3, (k+1)^{k+\bar{t}}\}, \quad C = \{(k+2)^{k-\bar{t}}\},$$

and we thus have the following four possible pairs of (D, E):

$$(\{(k+1)^{k-1}\}, \{k^3, (k+1)^{\bar{t}+1}\}) \quad (\{k, (k+1)^{k-2}\}), \{k^2, (k+1)^{\bar{t}+2}\})$$
$$(\{k^2, (k+1\})^{k-3}\}), \{k, (k+1)^{\bar{t}+3}\}) \quad (\{k^3, (k+1)^{k-4}\}, \{(k+1)^{\bar{t}+4}\}).$$

Observe that $\mathsf{T} = \mathsf{T}_{D,E}$ when (D, E) is the first pair above. Each homomorphism $\widehat{\Theta}_{\mathsf{T}_{D,E}}$ appears in equation (A.3) with coefficient $\pm[D_k + 1]$, which is zero when (D, E) is either the second or the fourth pair above. We now let $\mathsf{S} = \mathsf{T}_{(D,E)}$, where (D, E) is the third pair above, whose last two rows are of the form

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline k{-}1 & k{-}1 & k{-}1 & k{-}1 & k & k & k & k{+}1 & k{+}1 & k{+}1 & \cdots & k{+}1 \\ \hline k & k{+}1 & k{+}1 & k{+}1 & k{+}1 & k{+}1 & \cdots & k{+}1 & k{+}2 & k{+}2 & \cdots & k{+}2 \\ \hline \end{array}$$
$$\underbrace{\qquad\qquad\qquad}_{\bar{t} \text{ times}}$$

We can thus write $\alpha_{\mathsf{S}} \widehat{\Theta}_{\mathsf{S}} + \widehat{\Theta}_{\mathsf{T}} = 0$, where $\alpha_{\mathsf{S}} \in \{-1, 1\}$.

Case 4a: Assume that $\bar{t} = 2$. Let $r \geqslant k$, and suppose that T' (respectively, S') is the $((k+4)^r)$-tableau whose first k rows agree with T (respectively, S) and the remaining rows are arbitrary. Now observe that the $(k-1)$th and kth rows of both T and T' in this case are of the form

$$\overbrace{\qquad\qquad\qquad}^{k-1 \text{ times}}$$
$$\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline k{-}1 & k{-}1 & k{-}1 & k{-}1 & k & k{+}1 & k{+}1 & \cdots & k{+}1 \\ \hline k & k & k & k{+}1 & k{+}1 & k{+}1 & k{+}2 & \cdots & k{+}2 \\ \hline \end{array}$$
$$\underbrace{\qquad\qquad\qquad}_{k-2 \text{ times}}$$

We proceed to show that $\widehat{\Theta}_{\mathsf{T}'} = 0$ by induction on k. Note that $\widehat{\Theta}_{\mathsf{T}'}$ is the only homomorphism appearing in equation (A.3) when $k = 2$, and hence $\widehat{\Theta}_{\mathsf{T}'} = 0$. Now suppose that $k > 2$, and observe that the $(k-2)$th, $(k-1)$th and kth rows of S' are of the form

$$\overbrace{\qquad\qquad\qquad}^{k-2 \text{ times}}$$
$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline k{-}2 & k{-}2 & k{-}2 & k{-}2 & k{-}2 & k{-}1 & k & k & \cdots & k \\ \hline k{-}1 & k{-}1 & k{-}1 & k{-}1 & k & k & k & k{+}1 & \cdots & k{+}1 \\ \hline k & k{+}1 & k{+}1 & k{+}1 & k{+}1 & k{+}1 & k{+}2 & k{+}2 & \cdots & k{+}2 \\ \hline \end{array}$$

By the inductive hypothesis, $\widehat{\Theta}_{\mathsf{S}'} = 0$ and hence $\widehat{\Theta}_{\mathsf{T}} = 0$. It thus follows that $\widehat{\Theta}_{\mathsf{T}}$ is also zero.

Case 4b: We now assume that $\bar{t} \geqslant 4$. We apply Theorem A.10 to $\widehat{\Theta}_{\mathsf{S}}$ with $i = k-1$ as follows. We set

$$A = \{(k-1)^4, k^3\}, \quad B = \{k, (k+1)^{k+\bar{t}}\}, \quad C = \{(k+2)^{k-\bar{t}}\},$$

and we thus have the following two possible pairs of (D, E):

$$(\{(k+1)^{k-3}\}, \{k, (k+1)^{\bar{t}+3}\}) \qquad (\{k, (k+1)^{k-4}\}, \{(k+1)^{\bar{t}+4}\}).$$

Observe that $\mathsf{S} = \mathsf{S}_{D,E}$ where (D, E) is the first pair. Also, the homomorphism $\widehat{\Theta}_{\mathsf{S}_{D,E}}$, where (D, E) is the second pair above, appears in the sum of Theorem A.10 with zero coefficient since $\left[\begin{smallmatrix} D_k+3 \\ 3 \end{smallmatrix}\right]$ is zero if and only if $D_k > 0$. Hence $\widehat{\Theta}_{\mathsf{S}}$ is zero, and moreover, so is $\widehat{\Theta}_{\mathsf{T}}$. $\square$

A.4. **Proof part** (ii)**: the decomposition numbers.** We now calculate the graded decomposition multiplicity $[\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k) : \mathbf{D}^{\mathbb{C}}_{-1}(\alpha^C_k)]$ and hence prove that $\mathbf{S}^{\mathbb{C}}_{-1}(\alpha_k)$ is decomposable.

Proposition A.18. *For $k \geqslant 1$, we have that $\sharp\{\mathrm{CStd}(\alpha_k, \alpha^C_k)\} = 2$. The first tableau is simply given by adding the nodes in weakly increasing order in the most dominant possible position in α_k. The second tableau is the conjugate of the first.*

Proof. The entries from the set $\{2, \dots, k+1\}$ are all forced and the only tableau is that of shape $(k, k-1, \dots, 2, 1)$ and weight $(k, k-1, \dots, 2, 1)$. There are now k possible choices for how to add the nodes with entry $k+2$ (this is because we have k such nodes and $k+1$ addable nodes all of the correct residue). Regardless of how these nodes are added, there are no choices for how to add the nodes with entry $k+3$. Having added these nodes , we now have a tableau which is one of k distinct possible shapes: namely those obtained from $(k, k-1, \dots, 1)$ by adding k 2-dominoes either of shape (2) or of shape (1^2) (this is precisely the set of 2-separated partitions from this block). Of these partitions, only two of them have a total of k distinct addable nodes of the form $\{(r, c) \mid (r, c) \in \alpha_k\}$ of the required residue (namely those of the form $(k, k-1, \dots, 1) + 2(1^k)$ and its transpose). We now proceed to add the nodes with entry $k+5$ to these two possible partitions. There are now *no* choices for where to add these nodes; continuing in this fashion there are no more choices to be made and the result follows. $\square$

Example A.19. Let $k = 3$. The two elements of $\mathrm{CStd}((5^3, 3^2), (9, 7, 5))$ are as follows

$$\mathsf{S}_1 = \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 4 & 5 & 6 \\ \hline 3 & 4 & 5 & 6 & 7 \\ \hline 4 & 5 & 6 & 7 & 8 \\ \hline 7 & 8 & 9 \\ \cline{1-3} 8 & 9 & 10 \\ \cline{1-3} \end{array} \qquad \mathsf{S}_2 = \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 4 & 7 & 8 \\ \hline 3 & 4 & 5 & 8 & 9 \\ \hline 4 & 5 & 6 & 9 & 10 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 6 & 7 & 8 \\ \cline{1-3} \end{array}.$$

Proposition A.20. *Let $\mathsf{T} \in \mathrm{CStd}(\alpha^R_k, \alpha^C_k)$ be the tableau constructed as follows:*

- (i) *Add the nodes with ladder number $\ell \in \{2, \dots, k+3\}$ in weakly increasing order in the most dominant possible position.*
- (ii) *Add the first $k-1$ of the nodes of ladder number $k+4$ in the most dominant possible position.*
- (iii) *Add the final node of ladder number $k+4$ in the least dominant position: namely $(k+1, 1)$.*
- (iv) *At this point, there remain k distinct ladders numbers $\{k+4+i \mid 1 \leqslant i \leqslant k\}$ to add; which we add in weakly increasing order. For $1 \leqslant i \leqslant k-2$, we add the first $k-1-i$ nodes in the most dominant positions possible and the final two nodes in the least dominant positions possible: namely $(k+2-i, 2i-1)$ and $(k+1-i, 2i)$.*
- (v) *For ladder numbers $\ell = 2k+3$ (respectively $\ell = 2k+4$) there are precisely 2 nodes (respectively 1 node) of this ladder number, which we add in the only available positions: namely $(4, 2k-2)$ and $(4, 2k-3)$ (respectively $(2, 2k)$).*

For $\mathsf{T} \in \mathrm{CStd}(\alpha^C_k, \alpha^R_k)$ constructed as above, we have that $\deg(\mathsf{T}) = 0$.

Proof. For (a, b) a node added in steps (i) and (ii), we have that $\mathcal{A}_{\mathsf{S}}(a, b) = \varnothing = \mathcal{R}_{\mathsf{S}}(a, b)$.

We have that the one node added in step (iii) has $\mathcal{A}_{\mathsf{S}}(a, b) = \{(k, k)\}$ and $\mathcal{R}_{\mathsf{S}}(a, b) = \varnothing$.

Fix $1 \leqslant i \leqslant k$ in step (iv). The most dominant nodes of ladder number $k+4+i$ each have $\mathcal{A}_{\mathsf{S}}(a, b) = \varnothing = \mathcal{R}_{\mathsf{S}}(a, b)$. The two least dominant nodes added in step (iv) have $\mathcal{A}_{\mathsf{S}}(a, b) = \{(k-i-1, 2i+4)\}$ and $\varnothing = \mathcal{R}_{\mathsf{S}}(a, b) = \{(k-i, 2i+1)\}$.

In step (v), for the two nodes (a,b) of ladder number $2k+3$ we have that $\mathcal{A}_{\mathsf{S}}(a,b) = \{(2,2k)\}$ and $\varnothing = \mathcal{R}_{\mathsf{S}}(a,b) = \{(1,2k+1)\}$. The node (a,b) of ladder number $2k+4$ has $\mathcal{A}_{\mathsf{S}}(a,b) = \varnothing$ and $\mathcal{R}_{\mathsf{S}}(a,b) = \{(1,2k+1)\}$.

The only nodes with non-zero degree contribution are the nodes $(k+1,1)$ (added in step (iii)) and the node $(2,2k)$ (added in step (v)). The degree contributions of these nodes are $+1$ and -1 respectively. The result follows. □

Example A.21. Let $k = 2,3,4$. The semistandard coloured tableau described in the Proposition A.20 is as follows. We have coloured the nodes for which the degree contribution is 1 and -1, all other nodes contribute nothing to the relative degree.

2	3	4	5	6
3	4	5	8	
6	7			
7				

2	3	4	5	6	7	8
3	4	5	6	7	10	
4	5	6	9			
7	8	9				
8						

2	3	4	5	6	7	8	9	10
3	4	5	6	7	8	9	12	
4	5	6	7	8	11			
5	6	7	10	11				
8	9	10						
9								

.

Proposition A.22. *For $k \geqslant 1$ we have that $\sharp\{\mathrm{CStd}(\alpha_k, \alpha_k^R)\} = 1$ and we let T denote the unique tableau in this set. We have that $\deg(\mathsf{T}) = w(\alpha_k)/2$.*

Proof. This follows simply by counting nodes moved under James' regularisation map. For a proof (for all e-singular partitions) in the language of Fock-spaces, see [Fay07, Theorem 2.2]. □

Now, with the combinatorics in place we are ready to calculate decomposition numbers. By Proposition A.18, we know that

$$[\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k) : \mathbf{D}_{-1}^{\mathbb{C}}(\alpha_k^C)] = at^{w(\alpha_k)/2}$$

for $a = 0, 1$ or 2. By appendix A.3, we know that $a \neq 0$. Now, we know that

$$[\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k) : \mathbf{D}_{-1}^{\mathbb{C}}(\alpha_k^R)] = t^{w(\alpha_k)/2}$$

by Proposition A.22 and that

$$e(\alpha^C)(\mathbf{D}_{-1}^{\mathbb{C}}(\alpha_k^R)) \neq 0$$

by Proposition A.20 and the fact that $d_{\alpha_k^C, \alpha_k^R} \in t\mathbb{N}_0[t]$. Therefore $a = 1$ (as the other vector of weight α_k^R is in the span of the composition factor $\mathbf{D}_{-1}^{\mathbb{C}}(\alpha_k^R)$ of $\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k)$). We thus obtain:

Theorem A.23. *Let $k \in \mathbb{N}$. Then the Specht module $\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k)$ is decomposable with $\mathbf{D}_{-1}^{\mathbb{C}}(\alpha_k^C)$ appearing as a direct summand of $\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k)$.*

Proof. Since $\mathbf{D}_{-1}^{\mathbb{C}}(\alpha_k^C)$ appears in the socle of $\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k)$ and as a decomposition number with multiplicity 1, the result follows as $\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k)$ is self-dual (as the partition α_k is self-conjugate). □

A.5. **The decomposability of Specht modules labelled by β_k for $k \in 2\mathbb{N}$.** We now proceed to prove that the second exceptional family of partitions $\{\beta_k \mid k \in 2\mathbb{N}+1\}$ also label decomposable Specht modules. For $k \in 2\mathbb{N}+1$, we let β_k denote the partition

$$\beta_k^C = (2k+3, 2k+1, 2k-1, \ldots, 9, 7, 6, 1).$$

Example A.24. The Young diagrams of the partitions α_5^C and β_5^C, along with their residues, are drawn as follows:

$[\alpha_5^C] =$

0	1	0	1	0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1	0	1		
0	1	0	1	0	1	0	1	0				
1	0	1	0	1	0	1						
0	1	0	1	0								

$[\beta_5^C] =$

0	1	0	1	0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1	0	1		
0	1	0	1	0	1	0	1	0				
1	0	1	0	1	0	1						
0	1	0	1	0	1							
1												

Notice that β_5^C is obtained from α_5^C by adding certain nodes of residue 1 to $[\alpha_5^C]$; we will see that these nodes are predetermined by the *Modular Branching Rule*. There exist *cyclotomic divided power i-restriction and i-induction functors*, $e_i^{(r)}$ and $f_i^{(r)}$ respectively (we refer the reader to [BK09b] for details), that act on $H_q^{\mathbb{C}}(n)$-modules and enable us to move between different blocks of $H_q^{\mathbb{C}}(n)$. We will describe the action of $f_i^{(r)}$ on both Specht modules and irreducible $H_q^{\mathbb{C}}(n)$-modules using the following combinatorics (the action of $e_i^{(r)}$ can be analogously described).

Let $i \in \mathbb{Z}/e\mathbb{Z}$ and $\lambda \vdash n$. We define the *i-signature of* λ by reading the Young digram $[\lambda]$ from the top of the first component down to the bottom of the last component, writing a $+$ for each addable node and writing a $-$ for each removable node, where the leftmost $+$ corresponds to the highest addable node of λ. We obtain the *reduced i-signature of* λ by successively deleting all adjacent pairs $+-$ from the i-signature of λ, always of the form $-\cdots-+\cdots+$. The addable nodes of residue i corresponding to the $+$ signs in the reduced i-signature of λ are called the *conormal* nodes of residue i of λ. We now define

- $\lambda^{\triangle_i}$ to be the partition obtained from λ by adding all addable nodes of residue i,
- $\lambda^{\blacktriangle_i}$ to be the partition obtained from λ by adding all conormal nodes of residue i.

Denote the total number of addable *(respectively conormal)* nodes of residue i of λ by $\mathrm{add}_i(\lambda)$ *(respectively* $\mathrm{conor}_i(\lambda)$*)*.

Example A.25. The 1-signature of α_5^C in Example A.24 is $+-+-++$, and removing all pairs $+-$, we obtain the reduced 1-signature $++$. Observe that $(6,1)$ and $(5,6)$ are the conormal nodes of residue 1 of α_5^C, and upon adding these to α_5^C we have $(\alpha_5^C)^{\blacktriangle_1} = \beta_5^C$.

We now introduce versions of the *Branching Rules*, which will allow us to determine decomposability of $\mathbf{S}_{-1}^{\mathbb{C}}(\beta_k)$ from $\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k)$; we first observe how the functor $f_i^{(r)}$ acts on Specht modules. The following is a generalisation of [Spe14, Theorem 3.2].

Proposition A.26. *Let $i \in \mathbb{Z}/e\mathbb{Z}$ and $\lambda \vdash n$ and let $r = |\mathrm{add}_i(\lambda)|$ and suppose that $\mathrm{Rem}_i(\lambda) = \emptyset$ and $f_i^r \mathbf{S}_q^{\mathbb{C}}(\lambda) \cong \mathbf{S}_q^{\mathbb{C}}(\lambda^{\triangle_i})$. Then $\mathbf{S}_q^{\mathbb{C}}(\lambda)$ is indecomposable (respectively simple) if and only if $\mathbf{S}_q^{\mathbb{C}}(\lambda^{\triangle_i})$ is indecomposable (respectively simple).*

Proof. This follows by exactness of the functors $e_i^{(r)}$ and $f_i^{(r)}$. □

The following result is a natural corollary of Theorem A.23.

Theorem A.27. *Let $k \in 2\mathbb{N}+1$. Then the Specht module $\mathbf{S}_{-1}^{\mathbb{C}}(\beta_k)$ is decomposable with $\mathbf{D}_{-1}^{\mathbb{C}}(\beta_k^C)$ appearing as a direct summand of $\mathbf{S}_{-1}^{\mathbb{C}}(\beta_k)$.*

Proof. First observe that $(\alpha_k)^{\triangle_1} = \beta_k$, and we thus have $f_1^{|\mathrm{add}_i(\lambda)|}\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k) \cong \mathbf{S}_{-1}^{\mathbb{C}}(\beta_k)$ by applying Proposition A.26. It follows from Theorem A.23 that $\mathbf{S}_{-1}^{\mathbb{C}}(\beta_k)$ is decomposable.

Moreover, we know from Theorem A.23 that $\mathbf{D}_{-1}^{\mathbb{C}}(\alpha_k^C)$ is a direct summand of $\mathbf{S}_{-1}^{\mathbb{C}}(\alpha_k)$. Now observe that α_k^C has k (odd) non-zero parts, where the rth row of α_k^C has:

- an addable node of residue 1 when $r \in \{1,3,5,\dots,k\} \cup \{k+1\}$,
- a removable node of residue 1 when $r \in \{2,4,6,\dots,k-1\}$.

Hence α_k^C has 1-signature

$$\underbrace{(+-)(+-)\dots(+-)}_{(k-1)/2 \text{ times}}++,$$

and thus reduced 1-signature $++$, corresponding to the conormal nodes of residue 1 $(k+1,1)$ and $(k,6)$. Hence $(\alpha_k^C)^{\blacktriangle_1} = \beta_k^C$. Now, we observe that

$$f_i^{\mathrm{conor}_i(\lambda)}\mathbf{D}_q^{\mathbb{C}}(\lambda) \cong \mathbf{D}_q^{\mathbb{C}}(\lambda^{\blacktriangle_i})$$

follows trivially from Kleshchev's branching rule; therefore $f_1^{\mathrm{conor}_1(\alpha_k^C)}\mathbf{D}^{\mathbb{C}}_{-1}(\alpha_k^C) \cong \mathbf{D}^{\mathbb{C}}_{-1}(\beta_k^C)$ is a direct summand of $\mathbf{S}^{\mathbb{C}}_{-1}(\beta_k)$. □

Analogous to Conjecture 6.1, we predict that the Specht modules $\mathbf{S}^{\mathbb{C}}_{-1}(\beta_k)$ are semisimple.

Conjecture A.28. *For all $k \in 2\mathbb{N}+1$, we expect that*

$$\mathbf{S}^{\mathbb{C}}_{-1}(\beta_k) = \mathbf{D}^{\mathbb{C}}_{-1}(\beta_k^R)\langle w(\beta_k)/2\rangle \oplus \mathbf{D}^{\mathbb{C}}_{-1}(\beta_k^C)\langle w(\beta_k)/2\rangle.$$

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