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# Patterns and quasipatterns from the superposition of two hexagonal lattices\*

G erard Iooss<sup>†</sup> and Alastair M. Rucklidge<sup>‡</sup>

**Abstract.** When two-dimensional pattern-forming problems are posed on a periodic domain, classical techniques (Lyapunov–Schmidt, equivariant bifurcation theory) give considerable information about what periodic patterns are formed in the transition where the featureless state loses stability. When the problem is posed on the whole plane, these periodic patterns are still present. Recent work on the Swift–Hohenberg equation (an archetypal pattern-forming partial differential equation) has proved the existence of quasipatterns, which are not spatially periodic and yet still have long-range order. Quasipatterns may have 8-fold, 10-fold, 12-fold and higher rotational symmetry, which preclude periodicity. There are also quasipatterns with 6-fold rotational symmetry made up from the superposition of two equal-amplitude hexagonal patterns rotated by almost any angle  $\alpha$  with respect to each other. Here, we revisit the Swift–Hohenberg equation (with quadratic as well as cubic nonlinearities) and prove existence of several new quasipatterns. The most surprising are *hexa-rolls*: periodic and quasiperiodic patterns made from the superposition of hexagons and rolls (stripes) oriented in almost any direction with respect to each other and with any relative translation; these bifurcate directly from the featureless solution. In addition, we find quasipatterns made from the superposition of hexagons with unequal amplitude (provided the coefficient of the quadratic nonlinearity is small). We consider the periodic case as well, and extend the class of known solutions, including the superposition of hexagons and rolls. While we have focused on the Swift–Hohenberg equation, our work contributes to the general question of what periodic or quasiperiodic patterns should be found generically in pattern-forming problems on the plane.

**Key words.** Quasipatterns, superlattice patterns, Swift–Hohenberg equation.

**AMS subject classifications.** 35B36, 37L10, 52C23

**1. Introduction.** Regular patterns are ubiquitous in nature, and carefully controlled laboratory experiments are capable of producing patterns, in the form of rolls (stripes), squares or hexagons, with an astonishingly high degree of symmetry. One particular example is the Faraday wave experiment, in which a layer of viscous fluid is subjected to sinusoidal vertical vibrations. Without the forcing, the surface of the fluid is flat and featureless, but as the strength of the forcing increases beyond a critical value, the flat surface loses stability to two-dimensional patterns of standing waves, which in simple cases take the form of roll, square or hexagonal patterns [2]. But, with more elaborate forcing, more complex patterns can be found. Figure 1 shows examples of (a,b) superlattice patterns and (c,d) quasipatterns [2, 29]. The images in (a,c) show the pattern of standing waves on the surface of the fluid, while (b,d) show the Fourier power spectra. In both cases, the patterns are dominated by twelve waves, indicated by twelve small circles in Figure 1(b) and by twelve blobs lying on a circle in Figure 1(d). The distance from the origin to the twelve peaks gives the wavenumber that

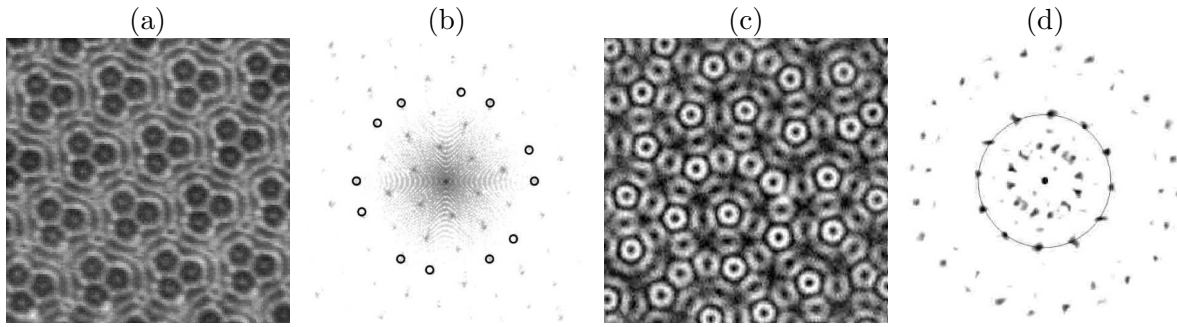
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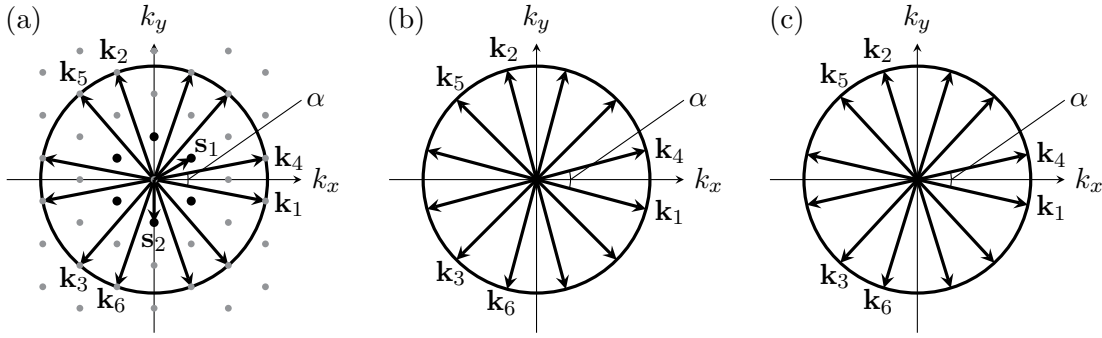


**Figure 1.** Examples of (a,b) superlattice patterns (reproduced with permission from [29]) and (c,d) quasipatterns (reproduced with permission from [2]). (a,c) show images representing the surface height of the fluid in Faraday wave experiments, with thin layers of viscous liquids subjected to large-amplitude multi-frequency forcing; (b,d) are Fourier power spectra of the images in (a,c), and indicate the twelve peaks that dominate the patterns in each case.

38 dominates the pattern. In the superlattice example, the twelve peaks are unevenly spaced,  
 39 but the basic structure is still hexagonal, and it is spatially periodic with a periodicity equal to  
 40  $\sqrt{7}$  times the wavelength of the instability [29]. In the quasipattern example, spatial period-  
 41 icity has been lost. Instead, the quasipattern has (on average) twelve-fold rotation symmetry,  
 42 as seen in the repeating motif of twelve pentagons arranged in a circle and in the twelve evenly  
 43 spaced peaks in the Fourier power spectrum in Figure 1(d). The lack of spatial periodicity is  
 44 apparent in Figure 1(c), while the point nature of the power spectrum in Figure 1(d) indicates  
 45 that the pattern has long-range order. These two features, the lack of periodicity (implicit  
 46 in this case from twelve-fold rotational symmetry) and the presence of long-range order, are  
 47 characteristics of quasicrystals in metallic alloys [44] and soft matter [23], and in quasipatterns  
 48 in fluid dynamics [18], reaction–diffusion systems [12] and optical systems [6].

49 The discovery of twelve-fold quasipatterns in the Faraday wave experiment [18] inspired  
 50 a sequence of papers investigating this phenomenon [31, 35, 38, 41, 42, 46, 47, 55]. One of the  
 51 main outcomes of this body of work is an understanding of the mechanism for stabilizing  
 52 quasipatterns in Faraday waves. Twelve-fold quasicrystals have also been found in block  
 53 copolymer and dendrimer systems [23, 54], in turn inspiring a considerable volume of work [1,  
 54 4, 8, 27, 48]. It turns out that the same stabilization mechanism operates in the Faraday  
 55 wave and the polymer crystallization systems [30, 39]. In both cases, and indeed in other  
 56 systems [12, 20], a common feature is that a second unstable or weakly damped length scale  
 57 plays a key role in stabilizing the pattern. See [43] for a recent review.

58 However, as well the question of how superlattice patterns and quasipatterns are stabi-  
 59 lized, there is the question of their existence as solutions of pattern-forming partial differential  
 60 equations (PDEs) posed on the plane, without lateral boundaries [5, 9, 10, 26]. Superlattice  
 61 patterns, which have spatial periodicity (as in Figure 1a) can be analysed in finite domains  
 62 with periodic boundary conditions. In this case, and near the bifurcation point, spatially  
 63 periodic patterns have Fourier expansions with wave vectors that live on a lattice, and the  
 64 infinite-dimensional PDE can be reduced rigorously to a finite-dimensional set of equations  
 65 for the amplitudes of the primary modes [11, 51]. In the finite dimensional setting, ampli-



**Figure 2.** (a) Two sets of six equally spaced wave vectors ( $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$  and their opposites, and  $\mathbf{k}_4, \mathbf{k}_5, \mathbf{k}_6$  and their opposites) rotated an angle  $\alpha$  with respect to each other so as to produce spatially periodic patterns:  $\alpha \approx 21.79^\circ$ , with  $\cos \alpha = \frac{13}{14}$  and  $\sqrt{3} \sin \alpha = \frac{9}{14}$ . The gray dots indicate that the twelve vectors lie on an underlying hexagonal lattice, generated by the vectors  $\mathbf{s}_1$  and  $\mathbf{s}_2$ . Compare with [Figure 1\(b\)](#). (b) 12-fold quasiperiodic patterns are generated by twelve equally spaced vectors:  $\alpha = \frac{\pi}{6} = 30^\circ$ , with  $\cos \alpha = \frac{1}{2}\sqrt{3}$ . Compare with [Figure 1\(d\)](#). (c) 6-fold quasiperiodic case:  $\alpha \approx 25.66^\circ$ , with  $\cos \alpha = \frac{1}{4}\sqrt{13}$  and  $\sqrt{3} \sin \alpha = \frac{3}{4}$ . Quasipatterns generated by equal combinations of the twelve waves have six-fold rotation symmetry but lack spatial periodicity.

66 tude equations can be written down, bifurcating equilibrium points found and their stability  
 67 analysed [15]. Equivariant bifurcation theory [21] is a powerful tool that uses symmetry tech-  
 68 niques to prove existence of certain classes of symmetric periodic patterns without recourse  
 69 to amplitude equations.

70 But quasipatterns pose a particular challenge for proving existence, in that the formal  
 71 power series that describes small amplitude solutions may diverge [26, 40] owing to the ap-  
 72 pearance of small divisors. Nonetheless, existence of quasipatterns with  $Q$ -fold rotation sym-  
 73 metry ( $Q = 8, 10, 12, \dots$ ) as solutions of the steady Swift–Hohenberg equation (see below)  
 74 has been proved using methods based on the Nash–Moser theorem [10]. The same approach  
 75 has been applied to other pattern-forming PDEs, such as those for steady Bénard–Rayleigh  
 76 convection [9]. Throughout, the existence proofs show that as the amplitude of the quasi-  
 77 pattern solution goes to zero, the solution from the truncated formal expansion approaches  
 78 a quasipattern solution of the PDE in a union of disjoint parameter intervals, going to full  
 79 measure as the amplitude goes to zero.

80 Most previous work on quasipatterns has concentrated on Fourier spectra that exhibit  
 81 “prohibited” symmetries: eight-, ten-, twelve-fold and higher rotation symmetries, as in [Fig-](#)  
 82 [ure 1\(c\)](#), or icosahedral symmetry in three dimensions [48]. There is, however, a class of  
 83 quasipatterns with six-fold rotation symmetry, related to the superlattice patterns already  
 84 discussed. These patterns can be described in terms of the superposition of twelve waves with  
 85 twelve wavevectors, grouped into two sets of six as in [Figure 2](#), with the six vectors within  
 86 each set spaced evenly around the circle, and with the two sets rotated by an angle  $\alpha$  with  
 87 respect to each other, with  $0 < \alpha < \frac{\pi}{3}$ . In the quasiperiodic case, we can choose  $\alpha$  to be the  
 88 smallest angle between the vectors, so  $0 < \alpha \leq \frac{\pi}{6}$ .

89 The discovery, in the Faraday wave experiment and elsewhere, of these elaborate superlat-  
 90 tice patterns and quasipatterns, with and without spatial periodicity, motivated investigations  
 91 into the bifurcation structure of pattern formation problems posed both in periodic domains

92 and on the whole plane, without lateral boundaries. We focus on an example of such a  
 93 problem, the steady Swift–Hohenberg equation, which is:

$$94 \quad (1.1) \quad (1 + \Delta)^2 u - \mu u + \chi u^2 + u^3 = 0,$$

95 where  $u(\mathbf{x})$  is a real function of  $\mathbf{x} = (x, y) \in \mathbb{R}^2$ ,  $\Delta$  is the Laplace operator,  $\mu$  is a real  
 96 bifurcation parameter and  $\chi$  is a real parameter. The time-dependent version of this PDE  
 97 was proposed originally as a model of small-amplitude fluctuations near the onset of convec-  
 98 tion [50], but is now considered an archetypal model of pattern formation [24].

99 The trivial state  $u = 0$  is always a solution of (1.1), and as  $\mu$  increases through zero, many  
 100 branches of small-amplitude solutions of (1.1) are created. These include periodic patterns  
 101 such as rolls, squares, hexagons and superlattice patterns, quasipatterns with the prohibited  
 102 rotation symmetries of eight-, ten-, twelve-fold and higher (proved in [10] with  $\chi = 0$ ), as  
 103 well as (again with  $\chi = 0$ ) two families of six-fold quasipatterns with equal sums of the twelve  
 104 Fourier modes illustrated in Figure 2(c) [19,25]. In this paper, we extend the analysis in [25] by  
 105 allowing  $\chi \neq 0$  and including quasipatterns with unequal combinations of the twelve Fourier  
 106 modes, discovering several new classes of solutions.

107 We approach this problem by deriving nonlinear amplitude equations for the twelve Fourier  
 108 modes on the unit circle. One important requirement on the twelve selected modes illustrated  
 109 in Figure 2 is therefore that nonlinear combinations of these modes should generate no further  
 110 modes with wavevectors on the unit circle. If they did, additional amplitude equations would  
 111 have to be included, a problem we leave for another day. We call the (full measure, as proved  
 112 in [25] in Lemma 5) set of  $\alpha$  that satisfy this condition  $\mathcal{E}_0$ , defined more precisely in [25] and  
 113 in Definition 2.4 below. Throughout, we use the names of the sets of values of  $\alpha$  from [25].

114 There are three possible situations as  $\alpha$  is varied: the (zero measure) periodic case, the  
 115 (full measure) quasiperiodic case where the results of [25] can be used, and other quasiperiodic  
 116 values of  $\alpha$  (zero measure). See the definitions below and in Appendix A for more detail.

- 117 1. The lattice is *periodic*, and  $\alpha \in \mathcal{E}_p$ , as in Figure 2(a) (see Definition 2.1). For these  
 118 angles, restricted to  $0 < \alpha < \frac{\pi}{3}$ , both  $\cos \alpha$  and  $\sqrt{3} \sin \alpha$  must be rational, and the wave  
 119 vectors generate a lattice (see Definition 2.1 and Lemma 2.2 below). This is the case  
 120 examined by [15], and  $\alpha \approx 21.79^\circ$  ( $\cos \alpha = \frac{13}{14}$  and  $\sqrt{3} \sin \alpha = \frac{9}{14}$ ) is an example. For  
 121 reasons explained below, for some values of  $\alpha \in \mathcal{E}_p$ , is it more convenient to consider  
 122  $\frac{\pi}{3} - \alpha$  instead, relabelling the vectors. This set is dense but of measure zero. Not all  
 123 values of  $\alpha \in \mathcal{E}_p$  are also in  $\mathcal{E}_0$ .
- 124 2. The angle  $\alpha$  is not in  $\mathcal{E}_p$  but it *satisfies all three of the requirements* for the existence  
 125 proofs in [25]. The first requirement is that  $\alpha \in \mathcal{E}_0$  (see Definition 2.4 below): no  
 126 integer combination of the twelve vectors already chosen should lie on the unit circle  
 127 apart from the twelve. The second and third requirements are that the numbers  $\cos \alpha$   
 128 and  $\sqrt{3} \sin \alpha$  should satisfy two “good” Diophantine properties. We define  $\mathcal{E}_1$  and  $\mathcal{E}_2$   
 129 to be the set of such angles, restricted to  $0 < \alpha \leq \frac{\pi}{6}$  (see definitions in Appendix A).  
 130 Then, the set  $\mathcal{E}_2$ , which itself requires  $\mathcal{E}_0$  and  $\mathcal{E}_1$ , is the set of angles that satisfy all  
 131 three requirements. All rational multiples of  $\pi$  (restricted to  $0 < \alpha \leq \frac{\pi}{6}$ ) are in  $\mathcal{E}_2$ , for  
 132 example,  $\alpha = \frac{\pi}{6} = 30^\circ$  as in Figure 2(b). The angle  $\alpha \approx 25.66^\circ$  is another example,  
 133 ( $\cos \alpha = \frac{1}{4}\sqrt{13}$  and  $\sqrt{3} \sin \alpha = \frac{3}{4}$ , see Figure 2(c) and Appendix B). This set is of full  
 134 measure.

135 3. The angle  $\alpha$ , still restricted to  $0 < \alpha \leq \frac{\pi}{6}$ , is not in  $\mathcal{E}_p$  or  $\mathcal{E}_2$ , and although pat-  
 136 terns made from these modes may be quasiperiodic, the existence proofs based on  
 137 the approach of [25] do not work, at least not without further extension. The angle  
 138  $\alpha \approx 26.44^\circ$  ( $\cos \alpha = \frac{1}{12}(5 + \sqrt{33})$  and  $\sqrt{3} \sin \alpha = \frac{1}{12}(15 - \sqrt{33})$ ) is an example (see  
 139 [Appendix B](#)) since it is not in  $\mathcal{E}_0$ . This set is dense but of measure zero.

140 For  $\alpha \in \mathcal{E}_p \cap \mathcal{E}_0$ , the resulting superlattice patterns are spatially periodic, and their bifur-  
 141 cation structure is determined at finite order when the small amplitude pattern is expressed as  
 142 a formal power series [15]. The wavevectors for these spatially periodic superlattice patterns  
 143 lie on a finer hexagonal lattice (as in [Figure 2a](#)).

144 We define  $\mathcal{E}_{qp}$  to be the complement of  $\mathcal{E}_p$  restricted to  $0 < \alpha \leq \frac{\pi}{6}$ . For  $\alpha \in \mathcal{E}_{qp}$ , linear  
 145 combinations of waves are typically quasiperiodic, but only for  $\alpha \in \mathcal{E}_2 \subset \mathcal{E}_{qp}$  can the techniques  
 146 of [25] be used to prove existence of quasipatterns with these modes as nonlinear solutions  
 147 of the PDE (1.1). For the special case  $\alpha = \frac{\pi}{6} \in \mathcal{E}_2$ , as in [Figure 2\(b\)](#), the quasipattern has  
 148 twelve-fold rotation symmetry, but more generally, as in [Figure 2\(c\)](#), there can be six-fold  
 149 rotation symmetry, more usually associated with hexagons. The proof in [25] makes use of  
 150 the properties of  $\mathcal{E}_2$ ; at this time, no existence result is known about  $\alpha \notin \mathcal{E}_2 \cup \mathcal{E}_p$ .

151 The periodic case has been analysed by [15, 45]. They write the small-amplitude pattern  
 152  $u(\mathbf{x})$  as the sum of six complex amplitudes  $z_1, \dots, z_6$  times the six waves  $e^{i\mathbf{k}_1 \cdot \mathbf{x}}, \dots, e^{i\mathbf{k}_6 \cdot \mathbf{x}}$ :

$$153 \quad (1.2) \quad u(\mathbf{x}) = \sum_{j=1}^6 z_j e^{i\mathbf{k}_j \cdot \mathbf{x}} + c.c. + \text{high-order terms},$$

154 where *c.c.* refers to the complex conjugate, and the six wavevectors  $\mathbf{k}_1, \dots, \mathbf{k}_6$  are as illustrated  
 155 in [Figure 2\(a\)](#). They then derive, using symmetry considerations, the amplitude equations:

$$156 \quad (1.3) \quad 0 = z_1 f_1(u_1, \dots, u_6, q_1, q_4, \bar{q}_4) + \bar{z}_2 \bar{z}_3 f_2(u_1, \dots, u_6, \bar{q}_1, q_4, \bar{q}_4) + \\ + \text{high-order resonant terms},$$

157 where  $u_1 = |z_1|^2, \dots, u_6 = |z_6|^2, q_1 = z_1 z_2 z_3$ , and  $q_4 = z_4 z_5 z_6$ . Here,  $f_1$  and  $f_2$  are  
 158 smooth functions of their nine arguments. Five additional equations can be deduced from  
 159 permutation symmetry. The high-order resonant terms, present only in the periodic case, are  
 160 at least fifth order polynomial functions of the six amplitudes and their complex conjugates,  
 161 and depend on the choice of  $\alpha \in \mathcal{E}_p$ . Even without the amplitude equations (1.3), equivariant  
 162 bifurcation theory can be used [15, 21] to deduce the existence of various hexagonal and  
 163 triangular superlattice patterns, and, within the amplitude equations, the stability of these  
 164 patterns can be computed.

165 The approach we take does not use equivariant bifurcation theory. Instead, we derive am-  
 166 plitude equations of the form (1.3) in the quasiperiodic and periodic cases. In the quasiperiodic  
 167 case, the equation is a formal power series, but in both cases, the cubic truncation of the first  
 168 component of amplitude equations is of the form

$$169 \quad (1.4) \quad 0 = \mu z_1 - \alpha_0 \bar{z}_2 \bar{z}_3 - z_1 (\alpha_1 u_1 + \alpha_2 u_2 + \alpha_2 u_3 + \alpha_4 u_4 + \alpha_5 u_5 + \alpha_6 u_6),$$

170 where  $\alpha_0, \dots, \alpha_6$  are coefficients that can be computed from the PDE (1.1). We find small  
 171 amplitude solutions of the cubic truncation (1.4) then verify that these correspond to small

Name	Section Figure	Periodic or QP	Example amplitudes	$\chi$	Earlier results
QP-super-hexagons	§4.2.1 Fig. 4	QP	$z_1 = \dots = z_6 \in \mathbb{R}$	Any	[25]
Unequal QP-super-hexagons	§4.2.1 Fig. 4	QP	$z_1 = z_2 = z_3 \neq z_4 = z_5 = z_6 \in \mathbb{R}$	$ \chi  \ll 1$	New
QP-anti-hexagons, QP-triangles etc.	§4.2.1 Fig. 5	QP	Various: see (4.2)	$\chi = 0$	New
Super-hexagons	§4.2.2 Fig. 6	Periodic	$z_1 = \dots = z_6 \in \mathbb{R}$	Any	[15]
Triangular superlattice	§4.2.2 Fig. 6	Periodic	Equal amplitudes Phases $\approx \frac{\pi}{3}, \frac{2\pi}{3}$	Any	[45]
Hexa-rolls (rolls dominant)	§4.3.1 Fig. 7	QP and periodic	$z_1 \approx z_2 \approx z_3 \ll z_4,$ $z_5 = z_6 = 0$	$\chi$ neither too small nor too large	New
Hexa-rolls (balanced)	§4.3.2 Fig. 7	QP and periodic	$z_1 \approx z_2 \approx z_3 \sim z_4,$ $z_5 = z_6 = 0$	$ \chi  \ll 1$	New

Table 1

Summary of the different solutions we consider. “Periodic” and “QP” refer to periodic ( $\alpha \in \mathcal{E}_p \cap \mathcal{E}_0$ ) and quasiperiodic ( $\alpha \in \mathcal{E}_2$ ) respectively. We give examples of the six  $z_j$  amplitudes as well as restrictions on the values of  $\chi$ . The term “super-hexagon” refers to the superposition of two hexagonal patterns, which can be equal or unequal amplitude. The last column gives references to relevant earlier results or indicates whether the solutions are new.

172 amplitude solutions of the untruncated amplitude equations (1.3). One remarkable result is  
 173 that the formal expansion in powers of the amplitude (and parameter  $\chi$  in the cases when  $\chi$  is  
 174 close to 0) of the bifurcating patterns is given at leading order by the same formulae in both the  
 175 quasiperiodic and the periodic cases. From solutions of the amplitude equations, the mathe-  
 176 matical proof of existence of the periodic patterns is given by the classical Lyapunov–Schmidt  
 177 method, while for quasipatterns the proof follows the same lines as in [25]. The truncated  
 178 expansion of the formal power series provides the first approximation to the quasipattern  
 179 solution, which is a starting point for the Newton iteration process, using the Nash–Moser  
 180 method for dealing with the small divisor problem [25] (for more details see §4.2.1).

181 We find several new types of solution, in the quasiperiodic and in the periodic cases, and  
 182 in the  $\chi \neq 0$  and  $|\chi| \ll 1$  cases. These are summarized in Table 1. The most significant  
 183 new class of solutions is the superposition of hexagons and roll patterns (*hexa-rolls*), with the  
 184 rolls arranged at almost any orientation with respect to the hexagons ( $\alpha \in (\mathcal{E}_p \cup \mathcal{E}_2) \cap \mathcal{E}_0$ ) and  
 185 translated with respect to each other by arbitrary amounts. These bifurcate directly from the  
 186 featureless pattern even when  $\chi$  is not small (provided  $\chi$  is not too large, see §4.3.1), in both  
 187 the periodic and the quasiperiodic cases. In the quasiperiodic case, the phason symmetry [17]  
 188 characteristic of quasipatterns leads to the freedom to have arbitrary relative translations of

189 the hexagons and rolls; finding this same freedom in the periodic case was a surprise.

190 We also show that the particular example of periodic triangular superlattice patterns  
 191 reported experimentally in [29] (see Figure 1a) and explored theoretically in [45] can also be  
 192 found in a much wider class of periodic lattices. Moreover, for nearby angles  $\alpha \in \mathcal{E}_2$ , we find  
 193 that the quasiperiodic super-hexagons can be thought of as long-range modulations between  
 194 the periodic super-hexagons and two types of periodic superlattice triangles (see Figure 6).

195 Our work extends the periodic results of [15] to the quasiperiodic case, including quasiperi-  
 196 odic versions of the anti-hexagon, super-triangle and anti-triangle patterns that occur with  
 197  $\chi = 0$ . We also extend the previous quasiperiodic work of [19, 25], which took  $\chi = 0$ : we  
 198 find small-amplitude bifurcating solutions in (1.3) for any  $\chi \neq 0$ , including new quasiperiodic  
 199 superposed hexagon patterns with unequal amplitudes for  $0 < |\chi| \ll 1$ , and show that there  
 200 are corresponding quasiperiodic (and periodic) solutions of the Swift–Hohenberg equation.

201 Amongst the solutions we find in the quasiperiodic case are combinations of two hexagonal  
 202 patterns, as well as the hexa-roll patterns mentioned above. In both the periodic and the  
 203 quasiperiodic cases, the superposed hexagon and roll patterns are new, and would not be  
 204 found using the equivariant bifurcation lemma as they have no symmetries (beyond periodic  
 205 in that case). Also in both cases, we consider the possibility that  $\chi$  is small, and use the  
 206 method of [25] on power series in two small parameters to find new superposed hexagon  
 207 patterns with unequal amplitudes, again out of range of the equivariant bifurcation lemma.

208 We open the paper with a statement of the problem in section 2 and develop the formal  
 209 power series for the amplitude equations in section 3. We solve these equations in section 4,  
 210 focusing on the new solutions, and conclude in section 5. Some details of the definitions,  
 211 examples and proofs are in the six appendices.

212 **2. Statement of the problem.** We begin by explaining how we describe functions on  
 213 lattices and quasilattices, and how the symmetries of the problem act on these functions.

214 **2.1. Lattices and quasilattices.** In the Fourier plane, we have two sets of six basic wave  
 215 vectors as illustrated in Figure 2:  $\{\mathbf{k}_j, -\mathbf{k}_j : j = 1, 2, 3\}$  and  $\{\mathbf{k}_j, -\mathbf{k}_j : j = 4, 5, 6\}$ , both  
 216 equally spaced on the unit circle, with angle  $\frac{2\pi}{3}$  between  $\mathbf{k}_1, \mathbf{k}_2$  and  $\mathbf{k}_3$  and between  $\mathbf{k}_4, \mathbf{k}_5$   
 217 and  $\mathbf{k}_6$ , such that  $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$  and  $\mathbf{k}_4 + \mathbf{k}_5 + \mathbf{k}_6 = 0$ . The two sets of six vectors are  
 218 rotated by an angle  $\alpha$  ( $0 < \alpha < \frac{\pi}{3}$ ) with respect to each other, so that  $\mathbf{k}_1$  makes an angle  $-\alpha/2$   
 219 with the  $x$  axis, while  $\mathbf{k}_4$  makes an angle  $\alpha/2$  with the  $x$  axis. The case  $\alpha = \frac{\pi}{6}$  corresponds to  
 220 the situation 12-fold quasipattern treated in [10], though with  $\chi = 0$ .

221 The lattice (in the periodic case) or quasilattice  $\Gamma$  are made up of integer sums of the six  
 222 basic wave vectors:

$$223 \quad (2.1) \quad \Gamma = \left\{ \mathbf{k} \in \mathbb{R}^2 : \mathbf{k} = \sum_{j=1}^6 m_j \mathbf{k}_j, \quad \text{with } m_j \in \mathbb{Z} \right\}.$$

224 Notice that if  $\mathbf{k} \in \Gamma$  then  $-\mathbf{k} \in \Gamma$ . In the periodic case, the lattice is not dense, as in  
 225 Figure 2(a), while in the quasiperiodic case, the points in  $\Gamma$  are dense in the plane.

226 The periodic case occurs whenever the two sets of six wave vectors are not rationally  
 227 independent, meaning that, for example,  $\mathbf{k}_4, \mathbf{k}_5$  and  $\mathbf{k}_6$  can all be written as rational sums of

228  $\mathbf{k}_1$  and  $\mathbf{k}_2$ . This happens whenever  $\cos \alpha$  and  $\cos(\alpha + \frac{\pi}{3})$  are both rational, and in this case,  
 229 patterns defined by (1.2) are periodic in space. We define the set  $\mathcal{E}_p$  to be these angles.

230 **Definition 2.1.** *Periodic case: the set  $\mathcal{E}_p$  of angles is defined as*

$$231 \quad \mathcal{E}_p := \left\{ \alpha \in \left(0, \frac{\pi}{3}\right) : \cos \alpha \in \mathbb{Q} \quad \text{and} \quad \cos \left(\alpha + \frac{\pi}{3}\right) \in \mathbb{Q} \right\}.$$

232 In this case,  $\Gamma$  is a lattice with hexagonal symmetry. We can replace  $\cos(\alpha + \frac{\pi}{3})$  in this  
 233 definition by  $\sqrt{3} \sin \alpha$ . The set  $\mathcal{E}_p$  has the following properties:

234 **Lemma 2.2.** (i) *The set  $\mathcal{E}_p$  is dense and has zero measure in  $(0, \frac{\pi}{3})$ .*

235 (ii) *If the wave vectors  $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4$  and  $\mathbf{k}_5$  are not independent on  $\mathbb{Q}$ , then  $\alpha \in \mathcal{E}_p$ .*

236 (iii) *If  $\alpha \in \mathcal{E}_p$  then there exist co-prime integers  $a, b$  such that*

$$237 \quad a > b > \frac{a}{2} > 0, \quad a \geq 3, \quad a + b \text{ not a multiple of } 3,$$

$$238 \quad (2.2) \quad \cos \alpha = \frac{a^2 + 2ab - 2b^2}{2(a^2 - ab + b^2)}, \quad \sqrt{3} \sin \alpha = \frac{3a(2b - a)}{2(a^2 - ab + b^2)}.$$

240 *Then the wave vectors  $\mathbf{k}_j$  are integer combinations of two smaller vectors  $\mathbf{s}_1$  and  $\mathbf{s}_2$ , of equal  
 241 length  $\lambda = (a^2 - ab + b^2)^{-1/2}$ , making an angle of  $\frac{2\pi}{3}$ , with*

$$242 \quad (2.3) \quad \begin{aligned} \mathbf{k}_1 &= a\mathbf{s}_1 + b\mathbf{s}_2, & \mathbf{k}_2 &= (b - a)\mathbf{s}_1 - a\mathbf{s}_2, & \mathbf{k}_3 &= -b\mathbf{s}_1 + (a - b)\mathbf{s}_2, \\ \mathbf{k}_4 &= a\mathbf{s}_1 + (a - b)\mathbf{s}_2, & \mathbf{k}_5 &= -b\mathbf{s}_1 - a\mathbf{s}_2, & \mathbf{k}_6 &= (b - a)\mathbf{s}_1 + b\mathbf{s}_2. \end{aligned}$$

245 Part (ii) of the Lemma is proved in [25], and parts (i) and (iii) are proved in [Appendix C](#).  
 246 The vectors  $\mathbf{s}_1$  and  $\mathbf{s}_2$  are illustrated in [Figure 2](#) in the case  $(a, b) = (3, 2)$  with  $\lambda = 1/\sqrt{7}$ .  
 247 Requiring  $a + b$  not to be a multiple of 3 means that we need to allow  $0 < \alpha < \frac{\pi}{3}$  in the  
 248 periodic case. In the quasiperiodic case ( $\alpha \in \mathcal{E}_{qp}$ ), we can always take  $\alpha$  to be the smallest  
 249 of the angles between the vectors, which is why we define the set  $\mathcal{E}_{qp}$  to be the complement  
 250 of  $\mathcal{E}_p$  within the interval  $(0, \frac{\pi}{6}]$ .

251 In (2.1), vectors  $\mathbf{k} \in \Gamma$  are indexed by six integers  $\mathbf{m} = (m_1, \dots, m_6) \in \mathbb{Z}^6$ . However,  
 252 using the fact that  $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$  and  $\mathbf{k}_4 + \mathbf{k}_5 + \mathbf{k}_6 = 0$ , the set  $\Gamma$  can be indexed by fewer  
 253 than six integers, and any  $\mathbf{k} \in \Gamma$  may be written, in both the periodic and the quasiperiodic  
 254 cases, as

$$255 \quad (2.4) \quad \mathbf{k}(\mathbf{m}) = m_1\mathbf{k}_1 + m_2\mathbf{k}_2 + m_4\mathbf{k}_4 + m_5\mathbf{k}_5, \quad (m_1, m_2, m_4, m_5) \in \mathbb{Z}^4,$$

256 though in fact  $\Gamma$  is indexed by two integers in the periodic case  $\alpha \in \mathcal{E}_p$ .

257 **2.2. Functions on the (quasi)lattice.** We are now in a position to specify more precisely  
 258 the form of the sum in (1.2). The function  $u(\mathbf{x})$  is a real function that we write in the form  
 259 of a Fourier expansion with Fourier coefficients  $u^{(\mathbf{k})}$ :

$$260 \quad (2.5) \quad u(\mathbf{x}) = \sum_{\mathbf{k} \in \Gamma} u^{(\mathbf{k})} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad u^{(\mathbf{k})} = \bar{u}^{(-\mathbf{k})} \in \mathbb{C}.$$

261 With  $\mathbf{k} \in \Gamma$  written as in (2.4), in the quasiperiodic case ( $\alpha \in \mathcal{E}_{qp}$ ) four indices are needed in  
 262 the sum since the four vectors in (2.4) are rationally independent. In the periodic case, two  
 263 indices are needed. A norm  $N_{\mathbf{k}}$  for  $\alpha \in \mathcal{E}_{qp}$  is defined by

$$264 \quad N_{\mathbf{k}(\mathbf{m})} = |m_1| + |m_2| + |m_4| + |m_5| = |\mathbf{m}|,$$

265 where the coefficients  $m_j$  are uniquely defined for a given vector  $\mathbf{k} \in \Gamma$ . To give a meaning to  
 266 the above Fourier expansion we need to introduce Hilbert spaces  $\mathcal{H}_s$ ,  $s \geq 0$ :

$$267 \quad \mathcal{H}_s = \left\{ u = \sum_{\mathbf{k} \in \Gamma} u^{(\mathbf{k})} e^{i\mathbf{k} \cdot \mathbf{x}}; \quad u^{(\mathbf{k})} = \bar{u}^{(-\mathbf{k})} \in \mathbb{C}, \quad \sum_{\mathbf{k} \in \Gamma} |u^{(\mathbf{k})}|^2 (1 + N_{\mathbf{k}}^2)^s < \infty \right\},$$

268 It is known that  $\mathcal{H}_s$  is a Hilbert space with the scalar product

$$269 \quad \langle u, v \rangle_s = \sum_{\mathbf{k} \in \Gamma} (1 + N_{\mathbf{k}}^2)^s u^{(\mathbf{k})} \bar{v}^{(\mathbf{k})},$$

270 and that  $\mathcal{H}_s$  is an algebra for  $s > 2$  (see [10]), and possesses properties of Sobolev spaces  $H_s$   
 271 in dimension 4, for example  $u$  is of class  $C^l$  for  $s > l + 2$ . For  $\alpha \in \mathcal{E}_{qp}$ , a function in  $\mathcal{H}_s$ ,  
 272 defined by a convergent Fourier series as in (2.5), represents in general a quasipattern, i.e., a  
 273 function that is quasiperiodic in all directions. It is possible of course for such functions still  
 274 to be periodic (e.g., rolls or hexagons) if subsets of the Fourier amplitudes are zero. With this  
 275 definition of the scalar product, the twelve basic modes are orthogonal in  $\mathcal{H}_s$  and orthonormal  
 276 in  $\mathcal{H}_0$ :

$$277 \quad \left\langle e^{i\mathbf{k}_j \cdot \mathbf{x}}, e^{i\mathbf{k}_l \cdot \mathbf{x}} \right\rangle_0 = \left\langle e^{-i\mathbf{k}_j \cdot \mathbf{x}}, e^{-i\mathbf{k}_l \cdot \mathbf{x}} \right\rangle_0 = \delta_{j,l} \quad \text{and} \quad \left\langle e^{\pm i\mathbf{k}_j \cdot \mathbf{x}}, e^{\mp i\mathbf{k}_l \cdot \mathbf{x}} \right\rangle_0 = 0,$$

278 where  $\delta_{j,l}$  is the Kronecker delta.

279 The following useful Lemma is proven in [25]:

280 **Lemma 2.3.** *For nearly all  $\alpha \in (0, \frac{\pi}{6}]$ , and in particular for  $\alpha \in \mathbb{Q}\pi \cap (0, \frac{\pi}{6}]$ , the only*  
 281 *solutions of  $|\mathbf{k}(\mathbf{m})| = 1$  are  $\pm \mathbf{k}_j$ ,  $j = 1, \dots, 6$ . These vectors can be expressed with four*  
 282 *integers as in (2.4):*

$$283 \quad \mathbf{m} = (\pm 1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1), \pm(1, 1, 0, 0), \pm(0, 0, 1, 1).$$

284 For these values of  $\alpha$ , the only vectors in  $\Gamma$  that are on the unit circle are the original twelve  
 285 vectors, defining the set  $\mathcal{E}_0$ :

286 **Definition 2.4.**  $\mathcal{E}_0$  is the set of  $\alpha$ 's such that Lemma 2.3 applies: the set of  $\alpha \in (0, \frac{\pi}{6}]$  such  
 287 that the only solutions of  $|\mathbf{k}(\mathbf{m})| = 1$  are  $\pm \mathbf{k}_j$ ,  $j = 1, \dots, 6$ .

288 The set  $\mathcal{E}_0$  is dense and of full measure in  $(0, \frac{\pi}{6}]$  (see [25], proof of Lemma 5), and contains  
 289 angles  $\alpha \in \mathcal{E}_p$  and  $\alpha \in \mathcal{E}_{qp}$ . Not every  $\alpha \in \mathcal{E}_p$  is also in  $\mathcal{E}_0$ ; for example, if  $(a, b) = (8, 5)$ ,  
 290 we have  $3\mathbf{k}_1 + \mathbf{k}_2 - 2\mathbf{k}_4 + \mathbf{k}_5 = (5b - 4a)\mathbf{s}_2 = (0, 1)$ , which is a vector on the unit circle but  
 291 not in the original twelve. For  $\alpha \in \mathcal{E}_{qp}$ , it is possible to show, for example, that  $\alpha \approx 25.66^\circ$   
 292 ( $\cos \alpha = \frac{1}{4}\sqrt{13}$ ) is in  $\mathcal{E}_0$ , while  $\alpha \approx 26.44^\circ$  ( $\cos \alpha = \frac{1}{12}(5 + \sqrt{33})$ ) is not (neither of these  
 293 examples is a rational multiple of  $\pi$ ). See Appendix B for details of these two examples.

294 **2.3. Symmetries and actions.** Our problem possesses important symmetries. First, the  
 295 system (1.1) is invariant under the Euclidean group  $E(2)$  of rotations, reflections and trans-  
 296 lations of the plane. We denote by  $\mathbf{R}_\theta u$  the pattern  $u$  rotated by an angle  $\theta$  centered at the  
 297 origin, so  $(\mathbf{R}_\theta u)(\mathbf{x}) = u(\mathbf{R}_{-\theta}\mathbf{x})$ , where  $\mathbf{R}_{-\theta}\mathbf{x}$  is  $\mathbf{x}$  rotated by an angle  $-\theta$ . We define similarly  
 298 the reflection  $\tau$  in the  $x$  axis, and the translation  $\mathbf{T}_\delta$  by an amount  $\delta$ , so  $(\tau u)(x, y) = u(x, -y)$   
 299 and  $(\mathbf{T}_\delta u)(\mathbf{x}) = u(\mathbf{x} - \delta)$ . Finally, in the case  $\chi = 0$ , equation (1.1) is odd in  $u$  and so  
 300 commutes with the symmetry  $\mathbf{S}$  defined by  $\mathbf{S}u = -u$ . If  $\chi \neq 0$ , then in addition to the change  
 301  $u \rightarrow -u$ , we need to change  $\chi \rightarrow -\chi$ .

302 The leading order part  $v_1(\mathbf{x})$  of our solution will be as in (1.2):

$$303 \quad (2.6) \quad v_1(\mathbf{x}) = \sum_{j=1}^6 z_j e^{i\mathbf{k}_j \cdot \mathbf{x}} + \bar{z}_j e^{-i\mathbf{k}_j \cdot \mathbf{x}}, \quad \text{with } z_j \in \mathbb{C}.$$

304 With Fourier modes restricted to those with wavevectors in  $\Gamma$ , not all symmetries in  $E(2)$  are  
 305 possible, in particular, only rotations that preserve the (quasi)lattice  $\Gamma$  are permitted. Those  
 306 that are allowed act on the basic Fourier functions as follows:

$$307 \quad \mathbf{T}_\delta(e^{i\mathbf{k}_j \cdot \mathbf{x}}) = e^{i\mathbf{k}_j \cdot (\mathbf{x} - \delta)},$$

$$308 \quad \mathbf{R}_{\frac{\pi}{3}}(e^{i\mathbf{k}_1 \cdot \mathbf{x}}, \dots, e^{i\mathbf{k}_6 \cdot \mathbf{x}}) = (e^{-i\mathbf{k}_3 \cdot \mathbf{x}}, e^{-i\mathbf{k}_1 \cdot \mathbf{x}}, e^{-i\mathbf{k}_2 \cdot \mathbf{x}}, e^{-i\mathbf{k}_6 \cdot \mathbf{x}}, e^{-i\mathbf{k}_4 \cdot \mathbf{x}}, e^{-i\mathbf{k}_5 \cdot \mathbf{x}}),$$

$$309 \quad \tau(e^{i\mathbf{k}_1 \cdot \mathbf{x}}, \dots, e^{i\mathbf{k}_6 \cdot \mathbf{x}}) = (e^{i\mathbf{k}_4 \cdot \mathbf{x}}, e^{i\mathbf{k}_6 \cdot \mathbf{x}}, e^{i\mathbf{k}_5 \cdot \mathbf{x}}, e^{i\mathbf{k}_1 \cdot \mathbf{x}}, e^{i\mathbf{k}_3 \cdot \mathbf{x}}, e^{i\mathbf{k}_2 \cdot \mathbf{x}}).$$

311 This leads to a representation of the symmetries acting on the six complex amplitudes  $z_j$  as

$$312 \quad \mathbf{T}_\delta : (z_1, \dots, z_6) \mapsto (z_1 e^{-i\mathbf{k}_1 \cdot \delta}, z_2 e^{-i\mathbf{k}_2 \cdot \delta}, z_3 e^{-i\mathbf{k}_3 \cdot \delta}, z_4 e^{-i\mathbf{k}_4 \cdot \delta}, z_5 e^{-i\mathbf{k}_5 \cdot \delta}, z_6 e^{-i\mathbf{k}_6 \cdot \delta}),$$

$$313 \quad (2.7) \quad \mathbf{R}_{\frac{\pi}{3}} : (z_1, \dots, z_6) \mapsto (\bar{z}_2, \bar{z}_3, \bar{z}_1, \bar{z}_5, \bar{z}_6, \bar{z}_4),$$

$$314 \quad \tau : (z_1, \dots, z_6) \mapsto (z_4, z_6, z_5, z_1, z_3, z_2).$$

316 We will use these symmetries, as well as the ‘‘hidden symmetries’’ in  $E(2)$  [13–15], to restrict  
 317 the form of the formal power series for the amplitudes  $z_j$ .

318 **3. Formal power series for solutions.** In this section, we look for amplitude equations  
 319 for solutions of (1.1), expressed in the form of a formal power series of the following type

$$320 \quad (3.1) \quad u(\mathbf{x}) = \sum_{n \geq 1} v_n(\mathbf{x}), \quad \mu = \sum_{n \geq 1} \mu_n,$$

321 where  $v_n$  and  $\mu_n$  are real. As in [25], the leading order part  $v_1$  of a solution  $u$  satisfies

$$322 \quad \mathbf{L}_0 v_1 = 0,$$

323 where the linear operator  $\mathbf{L}_0$  is defined by

$$324 \quad \mathbf{L}_0 = (1 + \Delta)^2,$$

325 so that  $v_1$  lies in the kernel of  $\mathbf{L}_0$ . Our twelve chosen wavevectors  $\pm\mathbf{k}_j$  all have length 1, so  
 326  $\mathbf{L}_0 e^{\pm i\mathbf{k}_j \cdot \mathbf{x}} = 0$ , and we can write  $v_1$  as a linear combination of these waves as in (2.6).

327 Higher order terms are written concisely using multi-index notation: let  $\mathbf{p} = (p_1, \dots, p_6)$   
 328 and  $\mathbf{p}' = (p'_1, \dots, p'_6)$ , where  $p_j$  and  $p'_j$  are non-negative integers, and define

$$329 \quad \mathbf{z}^{\mathbf{p}} = z_1^{p_1} z_2^{p_2} z_3^{p_3} z_4^{p_4} z_5^{p_5} z_6^{p_6} \quad \text{and} \quad \bar{\mathbf{z}}^{\mathbf{p}'} = \bar{z}_1^{p'_1} \bar{z}_2^{p'_2} \bar{z}_3^{p'_3} \bar{z}_4^{p'_4} \bar{z}_5^{p'_5} \bar{z}_6^{p'_6}.$$

330 We also take  $|\mathbf{p}| = p_1 + \dots + p_6$  and  $|\mathbf{p}'| = p'_1 + \dots + p'_6$ . Each order  $n$  means a corresponding  
 331 degree in monomials  $\mathbf{z}^{\mathbf{p}} \bar{\mathbf{z}}^{\mathbf{p}'}$  with  $n = |\mathbf{p}| + |\mathbf{p}'|$ , so we look for  $v_n$  and  $\mu_n$  of the form

$$332 \quad (3.2) \quad v_n(\mathbf{x}) = \sum_{|\mathbf{p}|+|\mathbf{p}'|=n} \mathbf{z}^{\mathbf{p}} \bar{\mathbf{z}}^{\mathbf{p}'} v_{\mathbf{p},\mathbf{p}'}(\mathbf{x}) \quad \text{and} \quad \mu_n = \sum_{|\mathbf{p}|+|\mathbf{p}'|=n} \mathbf{z}^{\mathbf{p}} \bar{\mathbf{z}}^{\mathbf{p}'} \mu_{\mathbf{p},\mathbf{p}'}$$

333 Here,  $\mu_{\mathbf{p},\mathbf{p}'}$  are constants and  $v_{\mathbf{p},\mathbf{p}'}(\mathbf{x})$  are functions made up of sums of modes of order  
 334  $n = |\mathbf{p}| + |\mathbf{p}'|$ , such that

$$335 \quad \left\langle v_{\mathbf{p},\mathbf{p}'}, e^{\pm i\mathbf{k}_j \cdot \mathbf{x}} \right\rangle_0 = 0, \quad \text{for } n > 1 \text{ and } j = 1, \dots, 6.$$

336 Writing (1.1) as

$$337 \quad (3.3) \quad \mathbf{L}_0 u = \mu u - \chi u^2 - u^3$$

338 and replacing  $u$  and  $\mu$  by their expansions (3.1) and (3.2), we project the PDE (1.1) onto  
 339 the kernel and the range of  $\mathbf{L}_0$ . Solving (3.3) is equivalent to solving the projection of (3.3)  
 340 onto the kernel together with the projection of (3.3) onto the orthogonal complement of the  
 341 kernel. Notice that for the quasipattern case the range is not closed, so that the projection  
 342 on the range is in fact a projection onto the orthogonal complement of the kernel. The  
 343 operator  $\mathbf{L}_0$  is self adjoint, so the left hand side of (3.3) is orthogonal to the kernel of  $\mathbf{L}_0$ :  
 344  $\langle \mathbf{L}_0 u, e^{\pm i\mathbf{k}_j \cdot \mathbf{x}} \rangle_0 = \langle u, \mathbf{L}_0 e^{\pm i\mathbf{k}_j \cdot \mathbf{x}} \rangle_0 = 0$  for any  $u$ . In fact, for any given degree  $n > 1$ , the right  
 345 hand side of (3.3) is a finite Fourier series, and eliminating the part lying in the kernel gives  
 346 a remaining series with Fourier modes  $e^{i\mathbf{k} \cdot \mathbf{x}}$ , with  $\mathbf{k} \in \Gamma$  apart from  $\{\pm\mathbf{k}_j, j = 1, \dots, 6\}$ . For  
 347 these modes we have  $|\mathbf{k}| \neq 1$  since  $\alpha \in \mathcal{E}_0$ . Then, the operator  $\mathbf{L}_0$  has a formal pseudo-inverse  
 348 on its range that is orthogonal to the kernel of  $\mathbf{L}_0$ . This pseudo-inverse is a bounded operator  
 349 in any  $\mathcal{H}_s$  when  $\alpha \in \mathcal{E}_0 \cap \mathcal{E}_p$ , since in the periodic case, nonlinear modes are on a lattice  $\Gamma$   
 350 and are bounded away from the unit circle. However, the pseudo-inverse is unbounded when  
 351  $\alpha \in \mathcal{E}_{qp}$  as a result of the presence of small divisors (see [25]). But, for a formal computation of  
 352 the power series (3.2), we only need at each order to pseudo-invert a *finite* Fourier series, which  
 353 is always possible provided that  $\alpha \in \mathcal{E}_0$ . Solving the range equation allows us to get  $\mathbf{Q}_0 u$ ,  
 354 which is the part of  $u$  orthogonal to the kernel, as functions of  $(v_1, \mu)$ , with  $v_1$  given by (2.6).  
 355 Taking the series obtained by solving the range equation (formally in the quasipattern case),  
 356 and replacing them in the kernel equation (6 complex components), leads to

$$357 \quad (3.4) \quad 0 = \mu z_j - P_j(\chi, \mu, z_1, \dots, z_6, \bar{z}_1, \dots, \bar{z}_6),$$

358 where  $j = 1, \dots, 6$  and

$$359 \quad P_j(\chi, \mu, z_1, \dots, z_6, \bar{z}_1, \dots, \bar{z}_6) = \left\langle \chi u^2 + u^3, e^{i\mathbf{k}_j \cdot \mathbf{x}} \right\rangle_0,$$

where  $u$  here is thought of as a function of  $\mathbf{z}$  and  $\bar{\mathbf{z}}$  through the formal power series (3.1) and the expansion (3.2). The dependency in  $\mu$  of  $P_j$  occurs at orders at least  $\mu|z_j|^3$ .

Expanding  $P_j$  in powers of  $(\mu, z_1, \dots, z_6, \bar{z}_1, \dots, \bar{z}_6)$  results in a convergent power series in the periodic case (the  $P_j$  functions are analytic in some ball around the origin), but in general these power series are not convergent in the quasiperiodic case. Nonetheless, the formal power series are useful in the proof of existence of the corresponding quasipatterns.

We can now use the symmetries of the problem to investigate the structure of the bifurcation equation (3.4). The equivariance of (3.3) under the translations  $\mathbf{T}_\delta$  and its propagation onto the bifurcation equation, using (2.7), leads to

$$(3.5) \quad e^{i\mathbf{k}_1 \cdot \delta} P_1(\chi, \mu, z_1 e^{-i\mathbf{k}_1 \cdot \delta}, \dots, \bar{z}_6 e^{i\mathbf{k}_6 \cdot \delta}) = P_1(\chi, \mu, z_1, \dots, \bar{z}_6).$$

A typical monomial in  $P_1$  has the form  $\mathbf{z}^{\mathbf{P}} \bar{\mathbf{z}}^{\mathbf{P}'}$ , so let us define

$$\begin{aligned} 371 \quad n_1 &= p_1 - p'_1 - 1, & n_2 &= p_2 - p'_2, & n_3 &= p_3 - p'_3, \\ 372 \quad n_4 &= p_4 - p'_4, & n_5 &= p_5 - p'_5, & n_6 &= p_6 - p'_6. \end{aligned}$$

Then, a monomial appearing in  $P_1$  should satisfy (3.5), which leads to

$$375 \quad n_1 \mathbf{k}_1 + n_2 \mathbf{k}_2 + n_3 \mathbf{k}_3 + n_4 \mathbf{k}_4 + n_5 \mathbf{k}_5 + n_6 \mathbf{k}_6 = 0,$$

and, since  $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$  and  $\mathbf{k}_4 + \mathbf{k}_5 + \mathbf{k}_6 = 0$ , we obtain

$$377 \quad (3.6) \quad (n_1 - n_3) \mathbf{k}_1 + (n_2 - n_3) \mathbf{k}_2 + (n_4 - n_6) \mathbf{k}_4 + (n_5 - n_6) \mathbf{k}_5 = 0,$$

which is valid in all cases (periodic or not).

In the quasilattice case, the wave vectors  $\mathbf{k}_1$ ,  $\mathbf{k}_2$ ,  $\mathbf{k}_4$  and  $\mathbf{k}_5$  are rationally independent, so (3.6) implies  $n_1 = n_2 = n_3$  and  $n_4 = n_5 = n_6$ , which leads to monomials of the form

$$\begin{aligned} 381 \quad & z_1 u_1^{p'_1} u_2^{p'_2} u_3^{p'_3} u_4^{p'_4} u_5^{p'_5} u_6^{p'_6} q_1^{n_1} q_4^{n_4} && \text{for } n_1 \geq 0 \text{ and } n_4 \geq 0, \\ 382 \quad & z_1 u_1^{p'_1} u_2^{p'_2} u_3^{p'_3} u_4^{p_4} u_5^{p_5} u_6^{p_6} q_1^{n_1} \bar{q}_4^{|n_4|} && \text{for } n_1 \geq 0 \text{ and } n_4 < 0, \\ 383 \quad & \bar{z}_2 \bar{z}_3 u_1^{p_1} u_2^{p_2} u_3^{p_3} u_4^{p'_4} u_5^{p'_5} u_6^{p'_6} \bar{q}_1^{|n_1|-1} q_4^{n_4} && \text{for } n_1 < 0 \text{ and } n_4 \geq 0, \\ 384 \quad & \bar{z}_2 \bar{z}_3 u_1^{p_1} u_2^{p_2} u_3^{p_3} u_4^{p_4} u_5^{p_5} u_6^{p_6} \bar{q}_1^{|n_1|-1} \bar{q}_4^{|n_4|} && \text{for } n_1 < 0 \text{ and } n_4 < 0, \end{aligned}$$

where we define

$$387 \quad u_j = z_j \bar{z}_j, \quad q_1 = z_1 z_2 z_3 \quad \text{and} \quad q_4 = z_4 z_5 z_6.$$

Then, the quasilattice case gives the following structure for  $P_1$ :

$$389 \quad (3.7) \quad P_1(\chi, \mu, z_1, \dots, \bar{z}_6) = z_1 f_1(\chi, \mu, u_1, \dots, u_6, q_1, q_4, \bar{q}_4) + \bar{z}_2 \bar{z}_3 f_2(\chi, \mu, u_1, \dots, u_6, \bar{q}_1, q_4, \bar{q}_4),$$

where  $f_1$  and  $f_2$  are power series in their arguments. We deduce the five other components of the bifurcation equation by using the equivariance under symmetries  $\mathbf{R}_{\frac{\pi}{3}}$ ,  $\tau$ , and  $\mathbf{S}$  (changing

392  $\chi$  to  $-\chi$ ), observing that

$$\begin{aligned}
393 \quad \mathbf{R}_{\frac{\pi}{3}} &: (u_1, u_2, u_3, u_4, u_5, u_6, q_1, q_4) \mapsto (u_2, u_3, u_1, u_5, u_6, u_4, \bar{q}_1, \bar{q}_4), \\
394 \quad \tau &: (u_1, u_2, u_3, u_4, u_5, u_6, q_1, q_4) \mapsto (u_4, u_6, u_5, u_1, u_3, u_2, q_4, q_1), \\
395 \quad \mathbf{S} &: (\chi, u_1, u_2, u_3, u_4, u_5, u_6, q_1, q_4) \mapsto (-\chi, u_1, u_2, u_3, u_4, u_5, u_6, -q_1, -q_4).
\end{aligned}$$

397 Equivariance under symmetry  $\mathbf{R}_{\pi}$ , which changes  $z_j$  into  $\bar{z}_j$ , gives the following property of  
398 functions  $f_j$  in (3.7)

$$\begin{aligned}
399 \quad f_1(\chi, \mu, u_1, \dots, u_6, \bar{q}_1, \bar{q}_4, q_4) &= \bar{f}_1(\chi, \mu, u_1, \dots, u_6, q_1, q_4, \bar{q}_4), \\
400 \quad f_2(\chi, \mu, u_1, \dots, u_6, q_1, \bar{q}_4, q_4) &= \bar{f}_2(\chi, \mu, u_1, \dots, u_6, \bar{q}_1, q_4, \bar{q}_4).
\end{aligned}$$

401 It follows that the coefficients in  $f_1$  and in  $f_2$  are *real*. Equivariance under symmetry  $\mathbf{S}$  leads  
402 to the property that in (3.7)  $f_1$  and  $f_2$  are respectively even and odd in  $(\chi, q_1, q_4)$ .

403 In the periodic case, when  $\alpha \in \mathcal{E}_p$ , we deduce from Appendix D that  $P_1(\chi, z_1, \dots, \bar{z}_6)$  may  
404 be written as

$$\begin{aligned}
405 \quad & z_1 f_3(\chi, \mu, u_1, \dots, u_6, q_1, q_4, \bar{q}_4, q_{l,k}, \bar{q}_{l,k}) + \\
406 \quad (3.8) \quad & + \bar{z}_2 \bar{z}_3 f_4(\chi, \mu, u_1, \dots, u_6, \bar{q}_1, q_4, \bar{q}_4, q_{l,k}, \bar{q}_{l,k}) + \\
407 \quad & + \sum_{s,t} q'_{s,t} f_{s,t}(\chi, \mu, u_1, \dots, u_6, q_1, \bar{q}_1, q_4, \bar{q}_4, q_{l,k}, \bar{q}_{l,k}), \\
408 \quad &
\end{aligned}$$

409 where the monomials  $q_{l,k}$ ,  $l = I, II, III, IV, V, VI, VII, VIII, IX$ , and  $k = 1, 2, 3$ , are defined  
410 in Appendix D, the functions  $f_j$  depend on all arguments  $q_{l,k}$  and  $\bar{q}_{l,k}$ , and the monomials  
411  $q'_{s,t}$ ,  $s = IV, V, VI, VII, VIII, IX$ ,  $t = 1, 2, 3$ , are defined by

$$412 \quad q'_{s,t} = \frac{\bar{q}_{s,t}}{\bar{z}_1}.$$

413 We observe that the “exotic” terms with lowest degree in (3.8) have degree  $2a - 1$ , which is  
414 at least of 5th order, since  $a \geq 3$ . Moreover, the symmetries act as indicated in Appendix D.

415 **4. Solutions of the bifurcation equations.** The strategy for proving existence of solutions  
416 of the PDE (1.1) is first to find solutions of the amplitude equations  $P_j(\chi, z_1, \dots, \bar{z}_6) = \mu z_j$   
417 truncated at some order, and then to use an appropriate implicit function theorem to show  
418 that there is a corresponding solution to the PDE, using the results of [25] in the quasiperiodic  
419 case. We refer the reader to Table 1 for a summary of the solutions we find. The main ones  
420 are periodic and quasiperiodic versions of equal amplitude superpositions of hexagons (*super-*  
421 *hexagons*, for any  $\chi$ ), unequal amplitude superpositions of hexagons (unequal super-hexagons,  
422  $|\chi| \ll 1$  only), and superpositions of hexagons and rolls (*hexa-rolls*,  $\chi$  not too large).

423 **4.1. Truncation to cubic order.** Let us first consider the terms up to cubic order for  $P_1$ .  
424 In the periodic case, where we notice that  $a \geq 3$ , and in the quasiperiodic case, we find the  
425 same equation:

$$426 \quad P_1^{(3)} = \alpha_0 \bar{z}_2 \bar{z}_3 + z_1 \sum_{j=1}^6 \alpha_j u_j.$$

427 We compute coefficients  $\alpha_j$ ,  $j = 0, \dots, 6$  from (see [Appendix E](#))

$$428 \quad \mu z_1 = P_1^{(3)} = \chi \langle v_1^2, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle + \langle v_1^3, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle - 2\chi^2 \langle v_1 \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle$$

429 where  $u = v_1$  (2.6) at leading order, the scalar product is the one of  $\mathcal{H}_0$ ,  $\mathbf{Q}_0$  is the orthogonal  
 430 projection on the range of  $\mathbf{L}_0$ ,  $\widetilde{\mathbf{L}}_0$  being the restriction of  $\mathbf{L}_0$  on its range, the inverse of which  
 431 is the pseudo-inverse of  $\mathbf{L}_0$ , as explained above in [section 3](#), as  $\mathbf{Q}_0 v_1^2$  has a finite Fourier series.  
 432 The higher orders (at increasing orders) are uniquely determined from the infinite dimensional  
 433 part of the problem, provided that  $\alpha \in \mathcal{E}_0$ , they start from order at least  $|v_1|^4$ .

434 It is straightforward to check that

$$\begin{aligned} 435 \quad \alpha_0 &= 2\chi, \\ 436 \quad \alpha_1 &= 3 - \chi^2 c_1, \\ 437 \quad \alpha_2 &= \alpha_3 = 6 - \chi^2 c_2, \\ 438 \quad \alpha_4 &= 6 - \chi^2 c_\alpha, \\ 439 \quad \alpha_5 &= 6 - \chi^2 c_{\alpha+}, \\ 440 \quad \alpha_6 &= 6 - \chi^2 c_{\alpha-}, \end{aligned}$$

442 where  $c_1$ ,  $c_2$  are constants and  $c_\alpha$ ,  $c_{\alpha+}$  and  $c_{\alpha-}$  are *real functions* of  $\alpha$  (real because of the  
 443 equivariance under  $\mathbf{R}_\pi$ , see the detailed computation in [Appendix E](#)). Hence we have the  
 444 bifurcation system, written up to cubic order in  $z_j$

$$\begin{aligned} 445 \quad 2\chi \overline{z_2 z_3} &= z_1 [\mu - \alpha_1 u_1 - \alpha_2 (u_2 + u_3) - \alpha_4 u_4 - \alpha_5 u_5 - \alpha_6 u_6] \\ 446 \quad 2\chi \overline{z_1 z_3} &= z_2 [\mu - \alpha_1 u_2 - \alpha_2 (u_1 + u_3) - \alpha_4 u_5 - \alpha_5 u_6 - \alpha_6 u_4] \\ 447 \quad (4.1) \quad 2\chi \overline{z_1 z_2} &= z_3 [\mu - \alpha_1 u_3 - \alpha_2 (u_1 + u_2) - \alpha_4 u_6 - \alpha_5 u_4 - \alpha_6 u_5] \\ 448 \quad 2\chi \overline{z_5 z_6} &= z_4 [\mu - \alpha_1 u_4 - \alpha_2 (u_5 + u_6) - \alpha_4 u_1 - \alpha_5 u_3 - \alpha_6 u_2] \\ 449 \quad 2\chi \overline{z_4 z_6} &= z_5 [\mu - \alpha_1 u_5 - \alpha_2 (u_4 + u_6) - \alpha_4 u_2 - \alpha_5 u_1 - \alpha_6 u_3] \\ 450 \quad 2\chi \overline{z_4 z_5} &= z_6 [\mu - \alpha_1 u_6 - \alpha_2 (u_4 + u_5) - \alpha_4 u_3 - \alpha_5 u_2 - \alpha_6 u_1]. \end{aligned}$$

452 It remains to find all small solutions of these six equations and check whether they are affected  
 453 by including further higher order terms.

454 Before proceeding, we note that in the periodic case ( $\alpha \in \mathcal{E}_p \cap \mathcal{E}_0$ ), the equivariant branching  
 455 lemma can be used to find some bifurcating branches of patterns [15]. In the case  $\chi \neq 0$ , where  
 456 there is no  $\mathbf{S}$  symmetry, these branches are called:

$$\begin{aligned} 457 \quad \text{Super-hexagons:} \quad & z_1 = z_2 = z_3 = z_4 = z_5 = z_6 \in \mathbb{R}, \\ 458 \quad \text{Simple hexagons:} \quad & z_1 = z_2 = z_3 \in \mathbb{R}, \quad z_4 = z_5 = z_6 = 0, \\ 459 \quad \text{Rolls (stripes):} \quad & z_1 \in \mathbb{R}, \quad z_2 = z_3 = z_4 = z_5 = z_6 = 0, \\ 460 \quad \text{Rhombs}_{1,4}: \quad & z_1 = z_4 \in \mathbb{R}, \quad z_2 = z_3 = z_5 = z_6 = 0, \\ 461 \quad \text{Rhombs}_{1,5}: \quad & z_1 = z_5 \in \mathbb{R}, \quad z_2 = z_3 = z_4 = z_6 = 0, \\ 462 \quad \text{Rhombs}_{1,6}: \quad & z_1 = z_6 \in \mathbb{R}, \quad z_2 = z_3 = z_4 = z_5 = 0, \end{aligned}$$

464 where the conditions on the  $z_j$ 's give examples of each type of solution. When  $\chi = 0$  and  
 465 there is **S** symmetry, there are additional branches:

$$\begin{aligned}
 466 & \text{Anti-hexagons: } z_1 = z_2 = z_3 = -z_4 = -z_5 = -z_6 \in \mathbb{R}, \\
 467 & \text{Super-triangles: } z_1 = z_2 = z_3 = z_4 = z_5 = z_6 \in \mathbb{R}i, \\
 468 \quad (4.2) & \text{Anti-triangles: } z_1 = z_2 = z_3 = -z_4 = -z_5 = -z_6 \in \mathbb{R}i, \\
 469 & \text{Simple triangles: } z_1 = z_2 = z_3 \in \mathbb{R}i, \quad z_4 = z_5 = z_6 = 0, \\
 470 & \text{Rhomb}_{1,2}: \quad z_1 = z_2 \in \mathbb{R}, \quad z_3 = z_4 = z_5 = z_6 = 0.
 \end{aligned}$$

472 For  $(a, b) = (3, 2)$ , it is known that there are additional branches of the form  $|z_1| = \dots = |z_6|$ ,  
 473 with  $\arg(z_1) = \dots = \arg(z_6) \approx \pm \frac{\pi}{3}$  and  $\arg(z_1) = \dots = \arg(z_6) \approx \pm \frac{2\pi}{3}$ , where the amplitude  
 474 and phases of the modes are determined at fifth order [45]. We recover all these solutions  
 475 below for all  $\alpha \in \mathcal{E}_p \cap \mathcal{E}_0$ , with the addition of a new branch, consisting of a superposition  
 476 of hexagons and rolls, for example with  $z_1, z_2, z_3, z_4 \neq 0$  and  $z_5 = z_6 = 0$ . This new kind of  
 477 solution exists in both the periodic and quasiperiodic cases, but only exists if  $\alpha_1, \alpha_2, \alpha_4, \alpha_5$ ,  
 478 and  $\alpha_6$  satisfy certain inequalities (true if  $\chi$  is not too large). This new solution cannot be  
 479 found using the equivariant branching lemma since it does not live in a one-dimensional space  
 480 fixed by a symmetry subgroup (though see also [33]).

481 We will focus below primarily on the new types of solutions: superposition of two hexagon  
 482 patterns and superposition of hexagons and rolls, but even in the quasiperiodic case, there  
 483 are branches of periodic patterns. These include rolls, simple hexagons, rhombs etc., and can  
 484 be found even with  $\alpha \in \mathcal{E}_{qp}$ . But, since they involve only a reduced set of wavevectors that  
 485 can be accommodated in periodic domains, there is no need for the quasiperiodic techniques  
 486 of [25] in these cases.

487 **4.2. Super-hexagons: superposition of two hexagonal patterns.** In the case  $q_1 q_4 \neq 0$   
 488 (all six amplitudes are non-zero), we multiply each equation in (4.1) by the appropriate  $\bar{z}_j$  to  
 489 obtain at cubic order

$$\begin{aligned}
 490 & 2\chi\bar{q}_1 = u_1[\mu - \alpha_1 u_1 - \alpha_2(u_2 + u_3) - \alpha_4 u_4 - \alpha_5 u_5 - \alpha_6 u_6] \\
 491 & 2\chi\bar{q}_1 = u_2[\mu - \alpha_1 u_2 - \alpha_2(u_1 + u_3) - \alpha_4 u_5 - \alpha_5 u_6 - \alpha_6 u_4] \\
 492 \quad (4.3) & 2\chi\bar{q}_1 = u_3[\mu - \alpha_1 u_3 - \alpha_2(u_1 + u_2) - \alpha_4 u_6 - \alpha_5 u_4 - \alpha_6 u_5] \\
 493 & 2\chi\bar{q}_4 = u_4[\mu - \alpha_1 u_4 - \alpha_2(u_5 + u_6) - \alpha_4 u_1 - \alpha_5 u_3 - \alpha_6 u_2] \\
 494 & 2\chi\bar{q}_4 = u_5[\mu - \alpha_1 u_5 - \alpha_2(u_4 + u_6) - \alpha_4 u_2 - \alpha_5 u_1 - \alpha_6 u_3] \\
 495 & 2\chi\bar{q}_4 = u_6[\mu - \alpha_1 u_6 - \alpha_2(u_4 + u_5) - \alpha_4 u_3 - \alpha_5 u_2 - \alpha_6 u_1].
 \end{aligned}$$

497 This implies that  $q_1$  and  $q_4$  are real since the  $u_j$ 's and the coefficients are real, and shows that

$$498 \quad u_1 = u_2 = u_3 \quad \text{and} \quad u_4 = u_5 = u_6$$

499 is always a possible solution.

500 There are other possible solutions, particularly when  $\chi$  is close to zero. Such solutions are  
 501 difficult to find in general as they involve solving six coupled cubic equations. Furthermore,

502 other solutions at cubic order might not give solutions when we consider higher order terms  
 503 in the bifurcation system (3.4). Considering these further is beyond the scope of this paper.

504 To solve (4.3) with  $u_1 = u_2 = u_3$  and  $u_4 = u_5 = u_6$ , and with  $q_1$  and  $q_4$  real, let us set

$$505 \quad z_j = \varepsilon e^{i\theta_j} \text{ for } j = 1, 2, 3, \quad \varepsilon > 0, \quad \Theta_1 = \theta_1 + \theta_2 + \theta_3 = k\pi,$$

$$506 \quad (4.4) \quad z_j = \delta e^{i\theta_j} \text{ for } j = 4, 5, 6, \quad \delta > 0, \quad \Theta_4 = \theta_4 + \theta_5 + \theta_6 = k'\pi,$$

508 where  $k$  and  $k'$  are integers, so  $u_1 = u_2 = u_3 = \varepsilon^2$ ,  $u_4 = u_5 = u_6 = \delta^2$ ,  $\overline{q_1} = \varepsilon^3 e^{-i\Theta_1} = \varepsilon^3 (-1)^k$   
 509 and  $\overline{q_4} = \delta^3 e^{-i\Theta_4} = \delta^3 (-1)^{k'}$ . Then, for  $\varepsilon\delta > 0$  we have only 2 equations

$$510 \quad 2\chi\varepsilon(-1)^k = \mu - (\alpha_1 + 2\alpha_2)\varepsilon^2 - (\alpha_4 + \alpha_5 + \alpha_6)\delta^2,$$

$$511 \quad 2\chi\delta(-1)^{k'} = \mu - (\alpha_1 + 2\alpha_2)\delta^2 - (\alpha_4 + \alpha_5 + \alpha_6)\varepsilon^2.$$

512 It follows that

$$513 \quad (4.5) \quad 2\chi \left( \varepsilon(-1)^k - \delta(-1)^{k'} \right) = [(\alpha_4 + \alpha_5 + \alpha_6) - (\alpha_1 + 2\alpha_2)](\varepsilon^2 - \delta^2).$$

514 Hence  $\left( \varepsilon(-1)^k - \delta(-1)^{k'} \right)$  is a factor in (4.5), and there are two types of solutions, depending  
 515 on whether this factor is zero or not.

516 *Equal amplitude super-hexagons.* We first consider the case where the factor is zero; it  
 517 follows that

$$518 \quad \delta = \varepsilon > 0 \quad \text{and} \quad k = k' = 0 \text{ or } 1,$$

519 and

$$520 \quad (4.6) \quad \mu = 2\chi\varepsilon(-1)^k + (\alpha_1 + 2\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)\varepsilon^2,$$

521 or equivalently,

$$522 \quad \mu = 2\chi\varepsilon(-1)^k + (33 - \chi^2(c_0 + 2c_1 + c_\alpha + c_{\alpha+} + c_{\alpha-}))\varepsilon^2.$$

523 We call these solutions super-hexagons in the periodic case, as in [15], and *QP-super-hexagons*  
 524 in the quasiperiodic case. Notice that when  $|\chi|$  is not too large, the coefficient of  $\varepsilon^2$  is positive,  
 525 and  $k$  is set by the relative signs of  $\mu$  and  $\chi$ . For  $|\chi| \ll \varepsilon$ , the bifurcation is supercritical  
 526 ( $\mu > 0$ ).

527 *Unequal amplitude super-hexagons.* If the factor is non-zero, this implies

$$528 \quad \varepsilon(-1)^k \neq \delta(-1)^{k'}, \text{ i.e., } \delta \neq \varepsilon, \text{ or } (-1)^k \neq (-1)^{k'}.$$

529 Dividing (4.5) by the non-zero factor leads to

$$530 \quad 2\chi = C \left( \varepsilon(-1)^k + \delta(-1)^{k'} \right),$$

531 with

$$532 \quad C \stackrel{def}{=} (\alpha_4 + \alpha_5 + \alpha_6) - (\alpha_1 + 2\alpha_2).$$

533 This leads to the non-degeneracy condition  $C \neq 0$ , and to the fact that this unequal amplitude  
 534 solution is valid only for  $|\chi|$  close to 0. The assumption on  $C$  is satisfied for most values of  $\chi$   
 535 since

$$536 \quad C = 3 - \chi^2(c_\alpha + c_{\alpha+} + c_{\alpha-} - c_1 - 2c_2).$$

537 Hence, for  $|\chi|$  close enough to 0, we find new solutions parameterized by  $\varepsilon > 0$  and  $k$ :

$$538 \quad (4.7) \quad \delta = \left[ \frac{2\chi}{3} - \varepsilon(-1)^k \right] (-1)^{k'} + \mathcal{O}(\chi^3).$$

539 Here  $k$  may be 0 or 1 and  $k'$  is chosen so that  $\delta > 0$ . At leading order in  $(\varepsilon, \chi)$ , we have

$$540 \quad (4.8) \quad \mu = 33\varepsilon^2 - 22\chi\varepsilon(-1)^k + 8\chi^2.$$

541 The solutions are unequal ( $\delta \neq \varepsilon$ , with  $\chi \neq 0$ ) superpositions of hexagons, so we call them  
 542 unequal super-hexagons and unequal QP-super-hexagons in the periodic and quasiperiodic  
 543 cases.

544 The next step is to show that these solutions to the cubic amplitude equations persist as  
 545 solutions of the bifurcation equations (3.4) once higher order terms are considered. This is  
 546 simpler in the quasiperiodic case as there are no resonant higher order terms to consider.

547 **4.2.1. Quasipattern cases – higher orders.** In this case wave vectors  $\mathbf{k}_1$ ,  $\mathbf{k}_2$ ,  $\mathbf{k}_4$  and  $\mathbf{k}_5$   
 548 are rationally independent. Using the symmetries, the general form of the six-dimensional bi-  
 549 furcation equation is deduced from (3.7) and (4.4), which gives two real bifurcation equations,  
 550 where functions  $f_j$  are formal power series in their arguments:

$$551 \quad \begin{aligned} \mu &= f_1(\chi, \mu, \varepsilon^2, \varepsilon^2, \varepsilon^2, \delta^2, \delta^2, \delta^2, \varepsilon^3(-1)^k, \delta^3(-1)^{k'}) + \\ 552 \quad (4.9) \quad &+ \varepsilon(-1)^k f_2(\chi, \mu, \varepsilon^2, \varepsilon^2, \varepsilon^2, \delta^2, \delta^2, \delta^2, \varepsilon^3(-1)^k, \delta^3(-1)^{k'}), \\ 553 \quad \mu &= f_1(\chi, \mu, \delta^2, \delta^2, \delta^2, \varepsilon^2, \varepsilon^2, \varepsilon^2, \delta^3(-1)^{k'}, \varepsilon^3(-1)^k) + \\ 554 \quad &+ \delta(-1)^{k'} f_2(\chi, \mu, \delta^2, \delta^2, \delta^2, \varepsilon^2, \varepsilon^2, \varepsilon^2, \delta^3(-1)^{k'}, \varepsilon^3(-1)^k). \end{aligned}$$

556 **Equal amplitude QP-super-hexagons.** It is clear that we still have solutions with

$$557 \quad \varepsilon(-1)^k = \delta(-1)^{k'}, \quad \text{i.e., } \varepsilon = \delta > 0, k = k',$$

558 which leads to a single equation

$$559 \quad \begin{aligned} \mu &= f_1(\chi, \mu, \varepsilon^2, \varepsilon^2, \varepsilon^2, \varepsilon^2, \varepsilon^2, \varepsilon^2, \varepsilon^3(-1)^k, \varepsilon^3(-1)^k) + \\ 560 \quad (4.10) \quad &+ \varepsilon(-1)^k f_2(\chi, \mu, \varepsilon^2, \varepsilon^2, \varepsilon^2, \varepsilon^2, \varepsilon^2, \varepsilon^2, \varepsilon^3(-1)^k, \varepsilon^3(-1)^k), \end{aligned}$$

562 which may be solved with respect to  $\mu$  by the implicit function theorem adapted for use with  
 563 formal power series: we use the implicit function theorem for analytic functions, suppressing  
 564 the proof of convergence for the series. This gives a formal power series in  $\varepsilon$ , the leading order  
 565 terms being (4.6).

566 Following the process used in section 3 of [25] for solving the range equation (the projection  
 567 of (1.1) on the orthogonal complement of  $\ker \mathbf{L}_0$ , with  $z_j = \varepsilon e^{i\theta_j}$ ,  $\Theta_1 = \Theta_4 = 0$ ), we need  
 568 typically to take  $(\varepsilon, \mu, \chi)$  in a “good set” of parameters, where the Diophantine conditions  
 569 of Appendix A are useful. Then the bifurcation equation (4.10) may be solved by the usual  
 570 implicit function theorem. Checking that at the end the parameters lie in the “good set”  
 571 needs a “transversality condition,” which is the same as in [25]. The solution finally is proved  
 572 to exist in a union of disjoint intervals for  $\varepsilon$ , going to full measure as  $\varepsilon$  goes to 0.

573 *Remark 4.1.* In the case of a quasiperiodic lattice, for all formal solutions found below in  
 574 the form of a power series of some amplitudes, the proof of existence of a true solution follows  
 575 the same lines as above. So we shall not repeat the argument.

576 *Unequal amplitude QP-super-hexagons.* Now, assuming that  $\varepsilon(-1)^k \neq \delta(-1)^{k'}$ , and taking  
 577 the difference between the two equations in (4.9), we find (simplifying the notation):

$$578 \quad 0 = f_1(\chi, \mu, \varepsilon^2, \delta^2, \varepsilon^3(-1)^k, \delta^3(-1)^{k'}) - f_1(\chi, \mu, \delta^2, \varepsilon^2, \delta^3(-1)^{k'}, \varepsilon^3(-1)^k) + \\
 579 \quad + \varepsilon(-1)^k f_2(\chi, \mu, \varepsilon^2, \delta^2, \varepsilon^3(-1)^k, \delta^3(-1)^{k'}) - \delta(-1)^{k'} f_2(\chi, \mu, \delta^2, \varepsilon^2, \delta^3(-1)^{k'}, \varepsilon^3(-1)^k)$$

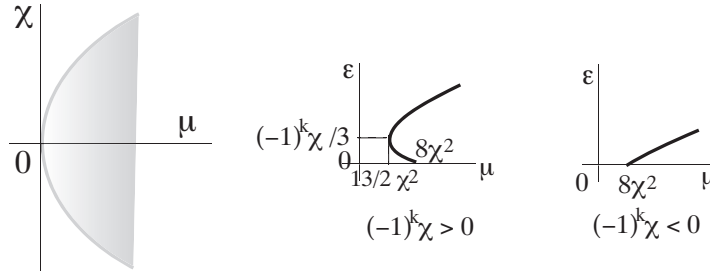
580 where we can simplify by the factor  $\varepsilon(-1)^k - \delta(-1)^{k'}$ . The leading terms are

$$581 \quad 0 = 2\chi - C(\varepsilon(-1)^k + \delta(-1)^{k'}),$$

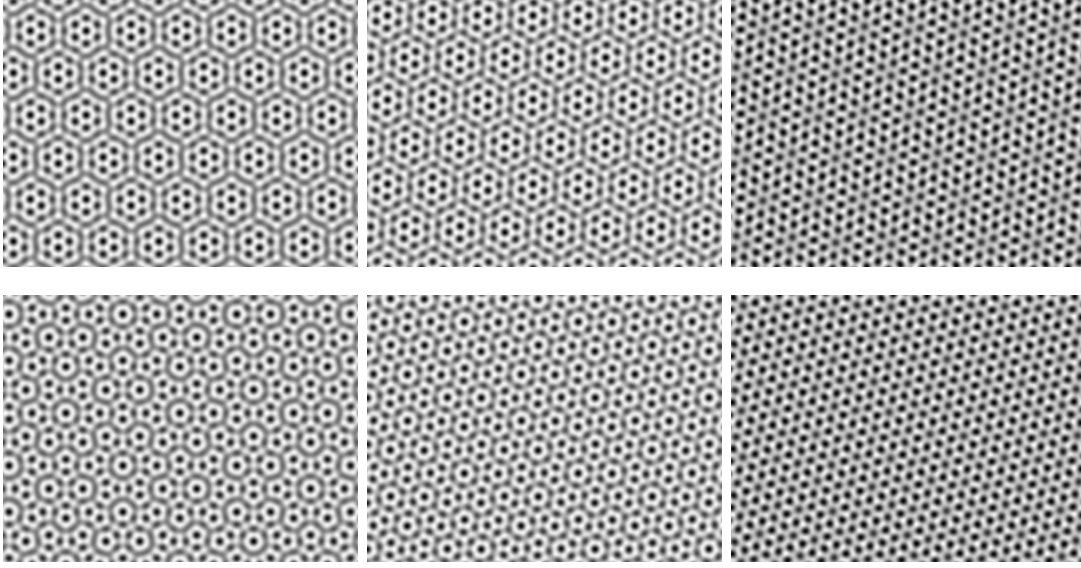
582 as in the cubic truncation, showing again that these solutions are only valid for  $\chi$  close to 0. It  
 583 is then clear that provided that  $C \neq 0$ , which holds for  $\chi$  close to zero, the system formed by  
 584 this last equation, with the first one of (4.9), may be solved with respect to  $\delta$  and  $\mu$  using the  
 585 formal implicit function theorem (as above, since the solution given by the principal part is  
 586 not degenerate) to obtain a formal power series in  $(\varepsilon, \chi)$ , their leading order terms being given  
 587 in (4.7), (4.8). We notice that there are four degrees of freedom, with the values of  $\theta_1, \theta_2, \theta_4$   
 588 and  $\theta_5$  being arbitrary. We also notice that we have two possible amplitudes depending on  
 589 the parity of  $k$ . All these bifurcating solutions correspond to the superposition of hexagonal  
 590 patterns of unequal amplitude, where the change in  $\theta_j$ ,  $j = 1, 2, 4, 5$  correspond to a shift of  
 591 each pattern in the plane.

592 For both types of solution, we have thus proved that there are formal power series solutions  
 593 of (3.3), unique up to the allowed indeterminacy on the  $\theta_j$ , of the form (4.4). This does not  
 594 prove that all solutions take the form (4.4). We can state

595 **Theorem 4.2 (Quasiperiodic superposed hexagons).** *Assume  $\alpha \in \mathcal{E}_0 \cap \mathcal{E}_{qp}$ , then for  $\varepsilon, \chi$*



**Figure 3.** Domain of existence (shaded) of bifurcating unequal amplitude QP-super-hexagons, for small  $|\chi|$ . These solutions only bifurcate from  $\mu = 0$  when  $\chi = 0$ .



**Figure 4.** Examples of quasipatterns: superposition of hexagons. Top row:  $\alpha = \frac{\pi}{12} = 15^\circ$ ; bottom row:  $\alpha = 25.66^\circ$  ( $\cos \alpha = \frac{1}{4}\sqrt{13}$ ). Left: equal amplitude QP-super-hexagons; center and right: unequal amplitude QP-super-hexagons, with  $k = k'$  (center) and  $k = k' + 1$  (right).

596 fixed, we can build a four-parameter formal power series solution of (3.3) of the form

$$597 \quad (4.11) \quad u(\varepsilon, \chi, k, \Theta) = \varepsilon u_1 + \sum_{n \geq 2} \varepsilon^n u_n(\chi, k, \Theta), \quad \varepsilon > 0, \quad u_n \perp e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad j = 1, \dots, 6,$$

$$598 \quad \mu(\varepsilon, \chi, k) = (-1)^k 2\chi\varepsilon + \mu_2(\chi)\varepsilon^2 + \sum_{n \geq 3} \varepsilon^n \mu_n(\chi, k), \quad k = 0, 1,$$

$$599 \quad \text{with } u_1 = \sum_{j=1, \dots, 6} e^{i(\mathbf{k}_j \cdot \mathbf{x} + \theta_j)} + c.c., \quad \Theta = (\theta_1, \dots, \theta_6),$$

$$600 \quad \mu_2(\chi) = 33 - \chi^2(c_1 + 2c_2 + c_\alpha + c_{\alpha+} + c_{\alpha-})$$

$$601 \quad \theta_1 + \theta_2 + \theta_3 = k\pi, \quad \theta_4 + \theta_5 + \theta_6 = k'\pi, \quad k = k' = 0, 1$$

$$602 \quad u_n(-\chi, k, \Theta) = (-1)^{n+1} u_n(\chi, k, \Theta), \quad \mu_n(-\chi, k) = (-1)^n \mu_n(\chi, k).$$

604 These are the equal amplitude QP-super-hexagons. Moreover, for a range of  $(\mu, \chi)$  close to 0  
 605 (see [Figure 3](#)), there are in addition two unequal amplitude QP-super-hexagon solutions (for  
 606  $k = 0, 1$ ), given by

$$607 \quad u(\varepsilon, \chi, k, \Theta) = \varepsilon u_{10} + \delta u_{11} + \sum_{m+p \geq 2} \varepsilon^m \chi^p u_{mp}(k, \Theta), \quad \varepsilon > 0, \delta > 0,$$

$$608 \quad u_{mp} \perp e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad j = 1, \dots, 6, \quad u \text{ odd in } (\varepsilon, \chi),$$

$$609 \quad (4.12) \quad u_{10} = \sum_{j=1,2,3} e^{i(\mathbf{k}_j \cdot \mathbf{x} + \theta_j)} + c.c., \quad u_{11} = \sum_{j=4,5,6} e^{i(\mathbf{k}_j \cdot \mathbf{x} + \theta_j)} + c.c.,$$

$$610 \quad \theta_1 + \theta_2 + \theta_3 = k\pi, \quad k = 0, 1, \quad \theta_4 + \theta_5 + \theta_6 = k'\pi, \quad k' = 0, 1 \text{ determined below},$$

$$611 \quad \delta(\varepsilon, \chi, k) = (-1)^{k'} \left\{ \frac{2\chi}{3} - (-1)^k \varepsilon + \sum_{m+p \geq 2} \varepsilon^m \chi^p \delta_{mp}(k) \right\}, \quad (-1)^{k'} \delta \text{ odd in } ((-1)^k \varepsilon, \chi),$$

$$612 \quad \mu(\varepsilon, \chi, k) = 33\varepsilon^2 - 22(-1)^k \varepsilon \chi + 8\chi^2 + \sum_{m+p \geq 3} \varepsilon^m \chi^p \mu_{mp}(k), \quad \mu \text{ even in } ((-1)^k \varepsilon, \chi).$$

613

614 In the expression for  $\delta$ ,  $k'$  is chosen so that  $\delta > 0$ . For either type of solution, changing  
 615  $\theta_1, \theta_2, \theta_4, \theta_5$  corresponds to translating each hexagonal pattern arbitrarily. [Figure 4](#) shows  
 616 examples of  $u_1$  for the two types of superposed hexagon quasipatterns, for two values of  $\alpha$ .

617 Then, for  $\alpha \in \mathcal{E}_2$ , which is included in  $\mathcal{E}_0 \cap \mathcal{E}_{qp}$ , and using the same proof as in [\[25\]](#), both  
 618 types of bifurcating quasipattern solutions of [\(1.1\)](#) are proved to exist. The equal amplitude  
 619 QP-super-hexagons have asymptotic expansion [\(4.11\)](#), provided that  $\varepsilon$  is small enough, and  
 620 the unequal amplitude QP-super-hexagons have asymptotic expansion [\(4.12\)](#), provided that  $\varepsilon, \chi$   
 621 are small enough.

622 **Remark 4.3.** Symmetries of quasipatterns are hard to write down precisely [\[7\]](#) since the  
 623 arbitrary relative position of the two hexagonal patterns may mean that there is no point of  
 624 rotation symmetry or line of reflection symmetry. Nonetheless, with  $\varepsilon = \delta$ , the first type of  
 625 solution is symmetric ‘on average’ under rotations by  $\frac{\pi}{3}$  and reflections conjugate to  $\tau$ . In  
 626 fact the 4 parameter family of solutions is globally invariant under symmetries  $\mathbf{R}_{\pi/3}$  and  $\tau$ .  
 627 Notice that, for the unequal amplitude QP-super-hexagon solutions, the reflection symmetry  
 628  $\tau$  exchanges  $(k, \varepsilon)$  with  $(k', \delta)$ .

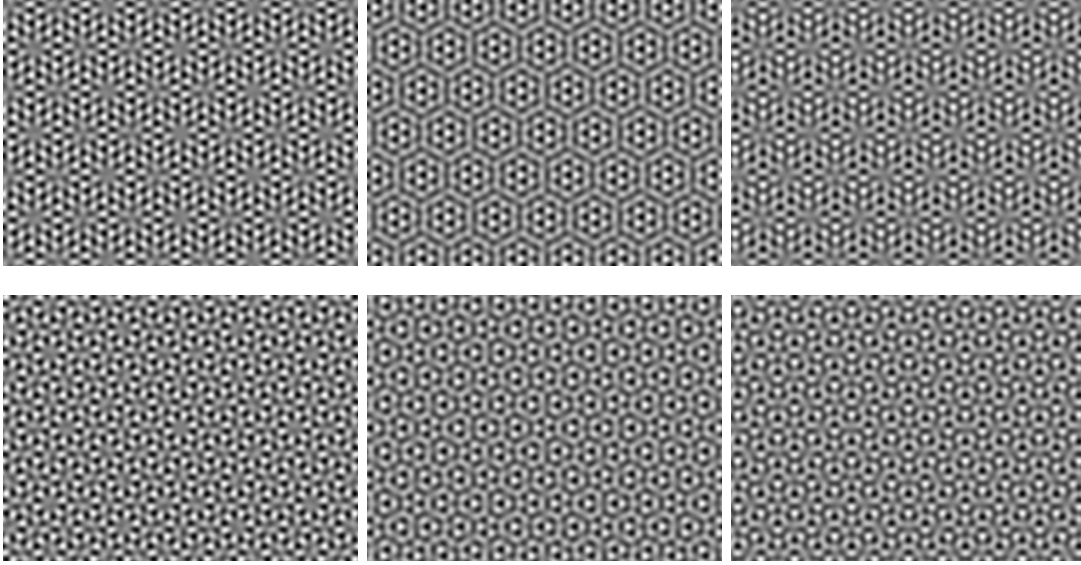
629 **Remark 4.4.** Let us observe that equal amplitude QP-super-hexagons for  $\theta_j = 0$ ,  $j =$   
 630  $1, \dots, 6$  were already obtained for  $\chi = 0$  in [\[25\]](#).

631 In the case  $\chi = 0$ , the unequal amplitude solutions do not exist. The original system [\(1.1\)](#)  
 632 is equivariant under the symmetry  $\mathbf{S}$ , which implies that in [\(3.7\)](#),  $f_1$  and  $f_2$  are respectively  
 633 even and odd in  $(q_1, q_4)$ . For  $\varepsilon = \delta$  the bifurcation system reduces to two equations of the  
 634 form

$$635 \quad \mu = f_1(\mu, \varepsilon^2, q_1, q_4) + \varepsilon e^{-i\Theta_1} f_2(\mu, \varepsilon^2, q_1, q_4)$$

$$636 \quad \mu = f_1(\mu, \varepsilon^2, q_4, q_1) + \varepsilon e^{-i\Theta_4} f_2(\mu, \varepsilon^2, q_4, q_1),$$

637 and we may observe new quasipattern solutions, illustrated in [Figure 5](#). The names here are  
 638 analogous to the related periodic patterns [\[15\]](#).



**Figure 5.** Examples of quasipatterns: superposition of hexagons with  $\chi = 0$ . Top row:  $\alpha = \frac{\pi}{12} = 15^\circ$ ; bottom row:  $\alpha = 25.66^\circ$  ( $\cos \alpha = \frac{1}{4}\sqrt{13}$ ). Left: QP-anti-hexagons; center: QP-super-triangles; right: QP-anti-triangles.

639 QP-anti-hexagons are obtained for (also obtained in [25])

$$\begin{aligned} 640 \quad & \theta_j = 0, \quad j = 1, 2, 3, \\ 641 \quad & \theta_j = \pi, \quad j = 4, 5, 6, \end{aligned}$$

642 which leads to

$$\begin{aligned} 643 \quad & e^{-i\Theta_1} = 1, \quad e^{-i\Theta_4} = -1, \\ 644 \quad & q_1 = \varepsilon^3 = -q_4, \end{aligned}$$

645 and the parity properties of  $f_j$  give only one bifurcation equation

$$646 \quad \mu = f_1(\mu, \varepsilon^2, \varepsilon^3, -\varepsilon^3) + \varepsilon f_2(\mu, \varepsilon^2, \varepsilon^3, -\varepsilon^3).$$

647 QP-super-triangles are obtained for

$$648 \quad \theta_j = \pi/2, \quad j = 1, \dots, 6,$$

649 which leads to

$$\begin{aligned} 650 \quad & e^{-i\Theta_1} = e^{-i\Theta_4} = i, \\ 651 \quad & q_1 = -i\varepsilon^3 = q_4, \end{aligned}$$

652 and it is clear that we have only one real bifurcation equation, with evenness (resp. oddness)  
653 with respect to the two last arguments of  $f_1$  (resp.  $f_2$ ) leading to

$$654 \quad \mu = f_1(\mu, \varepsilon^2, -i\varepsilon^3, -i\varepsilon^3) + i\varepsilon f_2(\mu, \varepsilon^2, -i\varepsilon^3, -i\varepsilon^3).$$

655 *QP-anti-triangles* are obtained for

$$\begin{aligned} 656 \quad \theta_j &= \pi/2 & j &= 1, 2, 3, \\ 657 \quad \theta_j &= -\pi/2, & j &= 4, 5, 6, \end{aligned}$$

658 which leads to

$$\begin{aligned} 659 \quad e^{-i\Theta_1} &= i, \quad e^{-i\Theta_4} = -i, \\ 660 \quad q_1 &= -i\varepsilon^3 = -q_4, \end{aligned}$$

661 and the parity properties of  $f_j$  give only one real bifurcation equation

$$662 \quad \mu = f_1(\mu, \varepsilon^2, -i\varepsilon^3, i\varepsilon^3) + i\varepsilon f_2(\mu, \varepsilon^2, -i\varepsilon^3, i\varepsilon^3).$$

663 All these cases lead to series for  $u$  and  $\mu$ , respectively odd and even in  $\varepsilon$ , and hence quasiperi-  
664 odic anti-hexagons, super-triangles and anti-triangles in (1.1) for  $\alpha \in \mathcal{E}_2$  and for  $\chi = 0$ . Using  
665 the same arguments as above, we can say that these QP-anti-hexagons etc. are solutions of  
666 the PDE with  $\chi = 0$ .

667 **4.2.2. Periodic case – higher orders.** In this case we have more resonant terms in the  
668 bifurcation equation, as seen in (3.8). These resonant terms introduce relations between  
669 the phases of the complex amplitudes, so the periodic superposed hexagon solutions come  
670 in two-parameter, rather than four-parameter, families. We consider here only the equal  
671 amplitude solutions, with  $\varepsilon = \delta$ , but even in this case there are two sub-types of solutions:  
672 super-hexagon solutions, and *triangular superlattice* solutions, where the phase relationships  
673 depend on amplitude. The triangular superlattice solutions we find are generalizations of those  
674 found by [45]; the name comes from the triangular appearance of the  $(a, b) = (3, 2)$  version of  
675 this periodic pattern (see Figure 1a and [29]).

676 *Super-hexagons.* We notice that, in setting

$$677 \quad z_j = \varepsilon e^{i\theta_j}, \quad \varepsilon > 0, \quad j = 1, \dots, 6$$

678 and taking

$$679 \quad (4.13) \quad \theta_1 = \theta_2 = \theta_3 = -\theta_4 = -\theta_5 = -\theta_6 = k\frac{\pi}{3}$$

680 we have  $q_1 = q_4 = (-1)^k \varepsilon^3$  and we can check that the nine sets  $G_j$  of invariant monomials  
681 satisfy (see Appendix D)

$$\begin{aligned} 682 \quad G_1 &= \varepsilon^{2a}, & G_2 &= G'_2 = \varepsilon^{3a-b} e^{i(a+b)k\pi}, & G_3 &= G'_3 = \varepsilon^{2a+b} e^{ibk\pi}, \\ 683 \quad G_4 &= \varepsilon^{4a-2b}, & G_5 &= G'_5 = \varepsilon^{3a} e^{iak\pi}, & G_6 &= \varepsilon^{2a+2b}, \end{aligned}$$

685 all these monomials being real. In Appendix D we show that each group on the same line above  
686 is invariant under the actions of  $\mathbf{R}_{\pi/3}$  and  $\tau$ . It then follows that the system of bifurcation

687 equations reduces to only one equation with real coefficients, as in the quasiperiodic case for  
 688 the first solutions. We have now a solution of the form

$$689 \quad z_1 = z_2 = z_3 = \varepsilon e^{i\theta},$$

$$690 \quad z_4 = z_5 = z_6 = \varepsilon e^{-i\theta}, \quad \theta = k \frac{\pi}{3}, \quad k = 0, \dots, 5.$$

691 The conclusion is that the power series starting as in (4.6) for  $\mu$  in terms of  $\varepsilon$  is still valid for  
 692 the periodic case (the modifications occurring at high order), provided we restrict the choice  
 693 of arguments  $\theta_j$  as (4.13). We show in Appendix F that solutions with  $k = 0, 2, 4$  or with  
 694  $k = 1, 3, 5$  may be obtained from one of them, in acting a suitable translation  $\mathbf{T}_\delta$ . It follows  
 695 that we only find two different bifurcating patterns, corresponding to opposite signs of  $\mu$ .  
 696 Moreover, we notice that the solution obtained for  $k = 0$  is changed into the solution obtained  
 697 for  $k = 3$  by acting the symmetry  $\mathbf{S}$  on it, and changing  $\chi$  into  $-\chi$ . Finally, notice that since  
 698 the Lyapunov–Schmidt method applies in this case, the series converges, for  $\varepsilon$  small enough.  
 699 The above solutions have arguments  $\theta_j = 0$  or  $\pi$  that do not depend on parameters  $(\mu, \chi)$ ;  
 700 these solutions correspond to super-hexagons.

701 *Triangular superlattice solutions.* Now, in [45] other solutions were found for  $(a, b) = (3, 2)$ ,  
 702 just taking into account of terms of order five in the bifurcation system. Let us show that  
 703 these solutions exist indeed for any  $(a, b)$  and taking into account of all resonant terms.

704 Let us consider the particular cases with

$$705 \quad z_j = \varepsilon e^{i\theta},$$

706 then the nine sets  $G_j$  of monomials defined in Appendix D satisfy

$$707 \quad G_1 = \varepsilon^{2a} e^{i(4b-2a)\theta}, \quad \mathbf{R}_{\pi/3} G_1 = \overline{G_1}, \quad \tau G_1 = G_1,$$

$$708 \quad G_2 = G_2' = \varepsilon^{3a-b} e^{i(a+b)\theta}, \quad \mathbf{R}_{\pi/3} G_2 = \overline{G_2}, \quad \tau G_2 = G_2,$$

$$709 \quad G_3 = \overline{G_3}' = \varepsilon^{2a+b} e^{i(2a-b)\theta}, \quad \mathbf{R}_{\pi/3} G_3 = \overline{G_3}, \quad \tau G_3 = G_3,$$

$$710 \quad G_4 = \varepsilon^{4a-2b} e^{i(4a-2b)\theta}, \quad \mathbf{R}_{\pi/3} G_4 = \overline{G_4}, \quad \tau G_4 = G_4,$$

$$711 \quad G_5 = \overline{G_5}' = \varepsilon^{3a} e^{i(2b-a)\theta}, \quad \mathbf{R}_{\pi/3} G_5 = \overline{G_5}, \quad \tau G_5 = G_5,$$

$$712 \quad G_6 = \varepsilon^{2a+2b} e^{i(2a+2b)\theta}, \quad \mathbf{R}_{\pi/3} G_6 = \overline{G_6}, \quad \tau G_6 = G_6.$$

713 Then the first bifurcation equation becomes

$$714 \quad (4.14) \quad \mu = f_3 + \varepsilon e^{-3i\theta} f_4 + \frac{G_1}{\varepsilon^2} f_{G_1} + \frac{G_2}{\varepsilon^2} f_{G_2} + \frac{\overline{G_4}}{\varepsilon^2} f_{G_4} + \frac{\overline{G_5}}{\varepsilon^2} f_{G_5} + \frac{\overline{G_6}}{\varepsilon^2} f_{G_6},$$

715 with all  $f_j$  functions of  $(\chi, \mu, \varepsilon^2, \varepsilon^3 e^{3i\theta}, \varepsilon^3 e^{-3i\theta}, G_1, \overline{G_1}, G_2, \overline{G_2}, G_3, \overline{G_3}, G_4, \overline{G_4}, G_5, \overline{G_5}, G_6, \overline{G_6})$ .  
 716 They have real coefficients, and are invariant under symmetry  $\tau$ , while the arguments are  
 717 changed into their complex conjugate by symmetry  $\mathbf{R}_{\pi/3}$ . It follows that the bifurcation  
 718 system reduces to only one complex (because of the occurrence of  $\theta$ ) equation, where we can  
 719 express the unknowns  $(\mu, \theta)$  as functions of  $\varepsilon$ . Then truncated at cubic order in  $(\mu, \varepsilon)$  this  
 720 equation reads

$$721 \quad \mu = f_3^{(0)}(\chi, \varepsilon^2, \varepsilon^3 e^{3i\theta}, \varepsilon^3 e^{-3i\theta}) + \varepsilon e^{-3i\theta} f_4^{(0)}(\chi, \varepsilon^2),$$

722 which is a nice perturbation at order  $\varepsilon^3$  of the known equation

$$723 \quad \mu = (\alpha_1 + 2\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6)\varepsilon^2 + 2\chi\varepsilon e^{-3i\theta}.$$

724 This leads to the two types of solutions:

$$725 \quad e^{3i\theta} = \pm 1,$$

$$726 \quad \mu = f_3^{(0)}(\chi, \varepsilon^2, \pm\varepsilon^3, \pm\varepsilon^3) \pm \varepsilon f_4^{(0)}(\chi, \varepsilon^2).$$

727 These solutions are not degenerate, so that, if we consider the complex equation (4.14), the  
728 implicit function theorem applies for solving with respect to  $(\mu, \theta)$  in convergent powers series  
729 of  $\varepsilon$ . This gives solutions of the form

$$730 \quad \theta_l(\varepsilon) = l\frac{\pi}{3} + \mathcal{O}(\varepsilon), \quad l = 0, 1, 2, 3, 4, 5$$

$$731 \quad \mu = f_3^{(0)}(\chi, \varepsilon^2, (-1)^l\varepsilon^3, (-1)^l\varepsilon^3) + (-1)^l\varepsilon f_4^{(0)}(\chi, \varepsilon^2) + \mathcal{O}(\varepsilon^4).$$

732 Now, we observe that the cases  $l = 0, 3$  lead to a real bifurcation equation, which fixes the  
733 argument  $\theta = 0$  or  $\pi$ . This recovers the super-hexagon solutions, already found. The remaining  
734 cases are the solutions suggested by [45] (for  $(a, b) = (3, 2)$ , not including all resonant terms).  
735 Let us sum up the results in the following

736 **Theorem 4.5 (Periodic equal amplitude superposed hexagons).** *Assume  $\alpha \in \mathcal{E}_0 \cap \mathcal{E}_p$ , then*  
737 *for  $\varepsilon$  small enough, and  $\chi$  fixed, we can build convergent power series solutions of (3.3), of*  
738 *the form*

$$739 \quad u(\varepsilon, \chi, k) = \varepsilon u_1 + \sum_{n \geq 2} \varepsilon^n u_n(\chi, k), \quad u_n \perp e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad j = 1, \dots, 6, \quad n \geq 2$$

$$740 \quad (4.15) \quad \mu(\varepsilon, \chi, k) = (-1)^k 2\chi\varepsilon + \mu_2(\chi)\varepsilon^2 + \sum_{n \geq 3} \varepsilon^n \mu_n(\chi, k),$$

$$741 \quad u_n(-\chi, k) = (-1)^n u_n(\chi, k), \quad \mu_n(-\chi, k) = (-1)^n \mu_n(\chi, k);$$

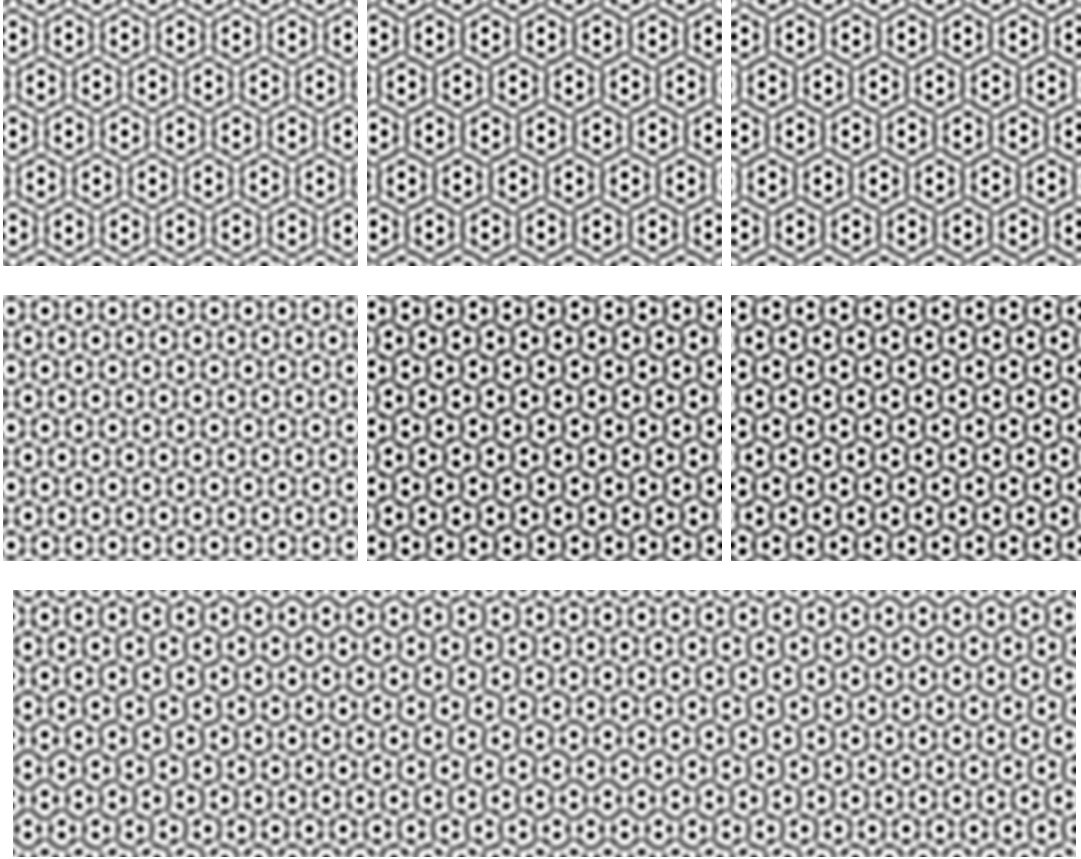
743 where  $\mu$  is even in  $((-1)^k\varepsilon, \chi)$  and  $\mu_2(\chi)$  is defined at Theorem 4.2 and such that, for super-  
744 hexagon solutions

$$745 \quad u_1 = \sum_{j=1, \dots, 6} e^{i(\mathbf{k}_j \cdot \mathbf{x} + \theta_j)} + c.c., \quad \theta_1 = \theta_2 = \theta_3 = -\theta_4 = -\theta_5 = -\theta_6 = k\pi, \quad k = 0, \text{ or } 1.$$

746 For triangular superlattice solutions, we have

$$747 \quad u_1 = \sum_{j=1, \dots, 6} e^{i(\mathbf{k}_j \cdot \mathbf{x} + \theta)} + c.c., \quad \theta(\varepsilon, \chi, k) = k\frac{\pi}{3} + \sum_{n \geq 1} \varepsilon^n \theta_n(\chi, k), \quad k = 1, 2, 4, 5.$$

748 **Remark 4.6.** For triangular superlattice solutions, the phases of the amplitudes are not  
749 independent of the parameters, in contrast to the super-hexagon solutions. These patterns are  
750 illustrated in Figure 6. The figure includes (middle row) periodic patterns with  $\alpha = 21.79^\circ$  and



**Figure 6.** Examples of periodic patterns: superposition of hexagons. Top row:  $\alpha = 13.17^\circ$  ( $\cos \alpha = \frac{37}{38}$ ,  $(a, b) = (5, 3)$ ); middle row:  $\alpha = 21.79^\circ$  ( $\cos \alpha = \frac{13}{14}$ ,  $(a, b) = (3, 2)$  – see also [Figure 1a](#)). For these, the left column (super-hexagons) has  $\theta_j = 0$  for  $j = 1, \dots, 6$ . The middle and right (superlattice triangles) have  $\theta_j = \frac{2\pi}{3}$  and  $\theta_j = \frac{4\pi}{3}$  respectively. The bottom row shows a related quasiperiodic example with  $\alpha = 21.00^\circ$ , close to  $21.79^\circ$ , showing long-range modulation between the three periodic patterns in the middle row.

751 (bottom row) a quasiperiodic pattern with  $\alpha = 21^\circ$ , showing how, with a slightly different  
 752 value of  $\alpha$ , the quasiperiodic pattern modulates between the three periodic solutions with  
 753  $l = 0, 2, 4$ .

754 *Remark 4.7.* In the  $\chi = 0$  case, we can recover all the solutions found by [\[15\]](#) using these  
 755 ideas.

756 **4.3. Hexa-rolls: superposition of hexagons and rolls.** As in [§4.2](#), we start with the cubic  
 757 truncation of the quasiperiodic and periodic cases together, then consider the effect of higher  
 758 order terms. Here we consider the case where  $q_1 \neq 0$  and  $q_4 = 0$  in [\(4.1\)](#), so that we assume  
 759 now

760 
$$q_1 \neq 0, \quad z_4 \neq 0, \quad z_5 = z_6 = 0.$$

761 Then the system (4.1) reduces to 4 equations

$$\begin{aligned}
762 \quad & 2\chi\bar{q}_1 = u_1[\mu - \alpha_1 u_1 - \alpha_2(u_2 + u_3) - \alpha_4 u_4], \\
763 \quad & 2\chi\bar{q}_1 = u_2[\mu - \alpha_1 u_2 - \alpha_2(u_1 + u_3) - \alpha_6 u_4], \\
764 \quad (4.16) \quad & 2\chi\bar{q}_1 = u_3[\mu - \alpha_1 u_3 - \alpha_2(u_1 + u_2) - \alpha_5 u_4], \\
765 \quad & 0 = \mu - \alpha_1 u_4 - \alpha_4 u_1 - \alpha_5 u_3 - \alpha_6 u_2,
\end{aligned}$$

767 where again this implies that  $q_1$  is real. Below, we study solutions of the bifurcation problem,  
768 built on a lattice spanned by the four wave vectors  $\mathbf{k}_1$ ,  $\mathbf{k}_2$ ,  $\mathbf{k}_3$ , and  $\mathbf{k}_4$ , and so we find solutions  
769 composed of a superposition of hexagons and rolls. Unlike in the super-hexagon cases above,  
770 the three amplitudes ( $|z_1|$ ,  $|z_2|$  and  $|z_3|$ ) of the hexagonal part of the pattern are of similar  
771 size but will not be exactly equal. We find two different types of solution distinguished by  
772 the relative magnitudes of the hexagonal and roll parts of the pattern. The first type occurs  
773 when  $|\chi|$  is neither too small nor too large and is such that rolls dominate the hexagons. The  
774 second type occurs only for small  $|\chi|$  and is such that rolls and hexagons are more balanced.

775 **4.3.1. Hexa-rolls: rolls dominate hexagons.** A consistent balance of terms in (4.16) is  
776 to have  $u_1$ ,  $u_2$  and  $u_3$  be  $\mathcal{O}(\mu^2)$ , so that  $q_1$  is  $\mathcal{O}(\mu^3)$ , while  $u_4$  is  $\mathcal{O}(\mu)$ . With this balance, at  
777 leading order we have the reduced system

$$\begin{aligned}
778 \quad & 2\chi\bar{q}_1 = u_1[\mu - \alpha_4 u_4], \\
779 \quad (4.17) \quad & 2\chi\bar{q}_1 = u_2[\mu - \alpha_6 u_4], \\
780 \quad & 2\chi\bar{q}_1 = u_3[\mu - \alpha_5 u_4], \\
781 \quad & 0 = \mu - \alpha_1 u_4,
\end{aligned}$$

783 which leads to

$$\begin{aligned}
784 \quad & z_j = \sqrt{u_j} e^{i\theta_j}, \quad j = 1, 2, 3, \\
785 \quad & u_j = \mu^2 u_j^{(0)}, \quad u_4 = \frac{\mu}{a_1}, \\
786 \quad & \Theta_1 = \theta_1 + \theta_2 + \theta_3 = k\pi,
\end{aligned}$$

787 with

$$\begin{aligned}
788 \quad & u_1^{(0)} = \frac{(\alpha_5 - \alpha_1)(\alpha_6 - \alpha_1)}{4\chi^2 a_1^2}, \\
789 \quad (4.18) \quad & u_2^{(0)} = \frac{(\alpha_5 - \alpha_1)(\alpha_4 - \alpha_1)}{4\chi^2 a_1^2}, \\
790 \quad & u_3^{(0)} = \frac{(\alpha_4 - \alpha_1)(\alpha_6 - \alpha_1)}{4\chi^2 a_1^2}, \\
791 \quad & (-1)^k = \text{sign}[\chi(\alpha_1 - \alpha_4)].
\end{aligned}$$

793 The condition for the existence of this solution is that  $(\alpha_4 - \alpha_1)$ ,  $(\alpha_5 - \alpha_1)$ ,  $(\alpha_6 - \alpha_1)$  should  
794 be nonzero and have the same sign. This condition is realized in (1.1) provided that

$$795 \quad 3 + \chi^2(c_1 - c_\alpha), \quad 3 + \chi^2(c_1 - c_{\alpha+}), \quad 3 + \chi^2(c_1 - c_{\alpha-}),$$

796 have the same sign, which holds at least for  $|\chi|$  not too large. For applying later the implicit  
 797 function theorem, we typically need  $|\mu| \ll \min(1, |\chi|)$ , so  $|\chi|$  should also be not too small.  
 798 Here, for  $|\chi|$  not too large,  $\alpha_1 > 0$ , so the bifurcation is supercritical in this case.

799 Now let us consider the full bifurcation system. Setting

$$800 \quad (4.19) \quad u_j = \mu^2 u_j^{(0)}(1 + x_j), \quad j = 1, 2, 3, \quad u_4 = \frac{\mu}{a_1}(1 + x_4),$$

801 we replace these expressions in (4.16) plus higher order terms appearing in (3.7) or (3.8), and  
 802 noticing that we obtain a real system of four equations in all periodic and quasiperiodic cases  
 803 except in the periodic case when  $a - b = 1$ , as defined in Lemma 2.2.

804 *Remark 4.8.* In the case  $\alpha = \frac{\pi}{6}$ , this combination of hexagons and rolls was reported  
 805 by [28, 32, 49].

806 *Remark 4.9.* In the periodic case when  $a - b = 1$ , a careful examination of high order  
 807 resonant terms (as defined in Appendix D) shows that there remains six equations, instead  
 808 of four. We might compute some new solution looking like the superposed hexagons and  
 809 rolls (but with small  $|z_5|$  and  $|z_6|$ ), however there are not strictly of the required form since  
 810  $q_1 q_4 \neq 0$ . We do not pursue these solutions further here.

811 Then, dividing the first three equations in (4.17) (with (4.19)) by  $\mu^3$ , dividing the fourth one  
 812 by  $\mu$ , and computing the linear part in  $x_j$ , we obtain

$$\begin{aligned} 813 \quad & a(x_1 + x_2 + x_3) - u_1^{(0)} \left( \left(1 - \frac{\alpha_4}{\alpha_1}\right)x_1 - \frac{\alpha_4}{\alpha_1}x_4 \right) = h_1, \\ 814 \quad (4.20) \quad & a(x_1 + x_2 + x_3) - u_2^{(0)} \left( \left(1 - \frac{\alpha_6}{\alpha_1}\right)x_2 - \frac{\alpha_6}{\alpha_1}x_4 \right) = h_2, \\ 815 \quad & a(x_1 + x_2 + x_3) - u_3^{(0)} \left( \left(1 - \frac{\alpha_5}{\alpha_1}\right)x_3 - \frac{\alpha_5}{\alpha_1}x_4 \right) = h_3, \\ 816 \quad & x_4 = h_4, \end{aligned}$$

818 with

$$819 \quad a = (-1)^k \chi \sqrt{u_1^{(0)} u_2^{(0)} u_3^{(0)}},$$

820 and all  $h_j$  have  $\mu$  in factor. The left hand side of the system (4.20) represents the differential  
 821 at the origin with respect to  $(x_1, x_2, x_3, x_4)$ , defining a matrix  $M'$  that needs to be inverted in  
 822 order to use the implicit function theorem. The determinant of matrix  $M'$  can be computed  
 823 and it is

$$824 \quad \frac{[3(-1)^k \text{sign}(\chi) - 2]}{128\chi^6 \alpha_1^9} [(\alpha_1 - \alpha_4)(\alpha_1 - \alpha_5)(\alpha_1 - \alpha_6)]^3,$$

825 which is not zero. Therefore the implicit function theorem applies, so we can find series in  
 826 powers of  $\mu$  for  $(x_1, x_2, x_3, x_4)$  solving the full bifurcation system in both the quasiperiodic  
 827 case (3.7) and the periodic case (3.8). We can state the following

828 **Theorem 4.10 (Hexa-rolls: superposed hexagons and rolls with rolls dominant).** Assume that  
 829  $\alpha \in \mathcal{E}_0$ , and in case of a periodic lattice assume  $a - b > 1$ . Then for fixed values of  $\chi$  such  
 830 that

$$831 \quad (\alpha_4 - \alpha_1), (\alpha_5 - \alpha_1), (\alpha_6 - \alpha_1)$$

832 are nonzero and have the same sign, and for  $\mu$  close enough to 0, we can build a three-  
 833 parameter formal power series in  $\varepsilon$  solution of (1.1) of the form

$$834 \quad u(\varepsilon, \Theta, \chi, j) = u_1(\varepsilon, \Theta, \chi, j) + \sum_{n \geq 3} \varepsilon^n u_n(\chi, \Theta, j), \quad u_{2p+1} \perp e^{i\mathbf{k}_j \cdot \mathbf{x}}, \quad j = 4, \text{ or } 5 \text{ or } 6,$$

$$835 \quad u_1(\varepsilon, \Theta, \chi, j) = \varepsilon e^{i(\mathbf{k}_j \cdot \mathbf{x} + \theta_j)} + \alpha_1 \varepsilon^2 \sum_{m=1,2,3} \sqrt{u_m^{(0)}} e^{i(\mathbf{k}_m \cdot \mathbf{x} + \theta_m)} + c.c.$$

$$836 \quad \Theta = (\theta_1, \theta_2, \theta_3, \theta_j), \quad \theta_1 + \theta_2 + \theta_3 = k\pi, \quad k = 0 \text{ or } 1,$$

$$837 \quad \mu(\varepsilon, \chi, j) = \alpha_1 \varepsilon^2 + \sum_{n \geq 2} \mu_{2n}(\chi, j) \varepsilon^{2n}, \text{ even in } \varepsilon,$$

838 where  $u_m^{(0)}$  and  $k$  are determined in (4.18). For  $\alpha_1 > 0$  the bifurcation is supercritical with  
 839  $\mu > 0$ . In the case  $\alpha_1 < 0$ , subcritical patterns can be found with  $\mu < 0$ . In the quasiperiodic  
 840 case ( $\alpha \in \mathcal{E}_2$ ), these solutions give quasipatterns using the techniques of [25]. In the periodic  
 841 case ( $\alpha \in \mathcal{E}_p \cap \mathcal{E}_0$ ), the classical Lyapunov–Schmidt method give periodic pattern solutions of  
 842 the PDE (1.1). In both cases, the freedom left for  $\Theta$  corresponds to an arbitrary choice for  
 843 translations  $\mathbf{T}_\delta$  of the hexagons, and the arbitrary choice of  $\theta_j$  ( $j = 4, 5, 6$ ) allows an arbitrary  
 844 relative translation of the rolls. Figure 7 shows quasiperiodic examples of  $u_1$  (QP-hexa-rolls).

845 **Remark 4.11.** These hexa-roll solutions are new, even in the case of a periodic lattice.  
 846 They have the surprising feature in the periodic case of allowing arbitrary relative translations  
 847 between the hexagons and rolls. Unlike the super-hexagon solutions, these solutions require  
 848 a condition on the cubic coefficients to be satisfied in order to exist. They were not found  
 849 by [15] since there the equivariant branching lemma was used, which finds only solutions  
 850 that are characterized by a single amplitude (these solutions have two) and that exist for  
 851 all non-degenerate values of the cubic coefficients (here the cubic coefficients must satisfy an  
 852 inequality).

853 **4.3.2. Hexa-rolls: rolls and hexagons balance.** With small  $|\chi|$ , solutions can be found  
 854 where the rolls and hexagons are of similar size. Let us consider the system (4.16), without  
 855 the terms with  $\chi^2$  in coefficients, and set

$$856 \quad z_1 = \varepsilon e^{i\theta_1}, \quad z_2 = \varepsilon e^{i\theta_2}, \quad z_3 = \varepsilon \zeta_3 e^{i\theta_3}, \quad \theta_1 + \theta_2 + \theta_3 = k\pi, \quad \varepsilon > 0,$$

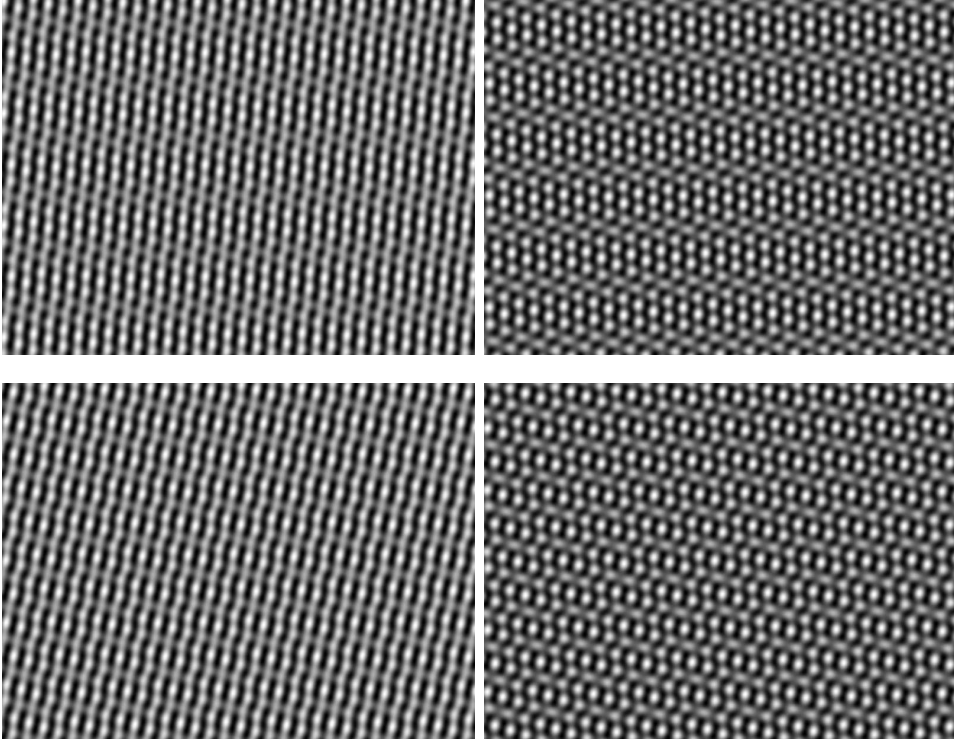
$$857 \quad u_4 = |z_4|^2 = \varepsilon^2 u_4^{(0)}, \quad z_5 = z_6 = 0, \quad \mu = \varepsilon^2 \mu^{(0)}, \quad \chi = \varepsilon \kappa,$$

859 then, after division by  $\varepsilon^4$  the first equations, and by  $\varepsilon^2$  the fourth one, this gives

$$860 \quad 2\kappa(-1)^k \zeta_3 = \mu^{(0)} - 9 - 6\zeta_3^2 - 6u_4^{(0)},$$

$$861 \quad 2\kappa(-1)^k \zeta_3 = \zeta_3^2 [\mu^{(0)} - 3\zeta_3^2 - 12 - 6u_4^{(0)}],$$

$$862 \quad 0 = \mu^{(0)} - 3u_4^{(0)} - 12 - 6\zeta_3^2.$$



**Figure 7.** Examples of quasipatterns: superposition of hexagons and rolls. Top row:  $\alpha = \frac{\pi}{12} = 15^\circ$ ; bottom row:  $\alpha = 25.66^\circ$  ( $\cos \alpha = \frac{1}{4}\sqrt{13}$ ). Left: QP-hexa-rolls with rolls dominating hexagons; right: QP-hexa-rolls with rolls and hexagons in balance.

863 Eliminating  $\mu^{(0)}$  and  $u_4^{(0)}$  leads to

864 
$$u_4^{(0)} = 1 - \frac{2\kappa}{3}\zeta_3(-1)^k,$$

865 and

866 
$$(3\zeta_3 + 2\kappa(-1)^k)(\zeta_3^2 - 1) = 0.$$

867 **Balanced hexa-rolls type 1.** For the solution  $\zeta_3 = 1$ , we obtain

868 (4.21) 
$$z_3 = \varepsilon e^{i\theta_3}, \quad u_4^{(0)} = 1 + \frac{2\kappa}{3}(-1)^{k+1}, \quad \mu^{(0)} = 21 + 2\kappa(-1)^{k+1},$$

869 for which we need to satisfy  $u_4^{(0)} > 0$ , i.e.,

870 (4.22) 
$$\kappa(-1)^k < \frac{3}{2},$$

871 and we observe that  $\mu^{(0)} > 0$  (supercritical bifurcation). These solutions have the three  
872 hexagon amplitudes equal at leading order.

873 Now, we observe that the solution  $\zeta_3 = -1$  may be obtained from (4.21) in adding  $\pi$  to  
874  $\theta_3$  and change  $k$  into  $k + 1$ . It follows that this does not give a new solution.

875 *Balanced hexa-rolls type 2.* For the solution  $\zeta_3 = \frac{2}{3}\kappa(-1)^{k+1}$ , we obtain

$$876 \quad (4.23) \quad z_3 = \frac{2}{3}\kappa(-1)^{k+1}\varepsilon, \quad u_4^{(0)} = 1 + \frac{4}{9}\kappa^2, \quad \mu^{(0)} = 15 + 4\kappa^2,$$

877 where there is no restriction on  $\kappa$ , and we observe that  $\mu^{(0)} > 0$  (supercritical bifurcation).  
878 These solutions have one of the three hexagon amplitudes different from the other two at  
879 leading order.

880 For proving that these balanced hexa-roll solutions at leading order provide solutions for  
881 the full system at all orders, let us define

$$882 \quad (4.24) \quad z_1 = \varepsilon e^{i\theta_1}(1 + x_1), \quad z_2 = \varepsilon e^{i\theta_2}(1 + x_2), \quad z_3 = \varepsilon \zeta_3 e^{i\theta_3}(1 + x_3),$$

$$883 \quad u_4 = \varepsilon^2(u_4^{(0)} + v_4), \quad \mu = \varepsilon^2(\mu^{(0)} + \nu), \quad z_5 = z_6 = 0, \quad \theta_1 + \theta_2 + \theta_3 = k\pi,$$

885 where  $u_4^{(0)}$ ,  $\mu^{(0)}$ , and  $\zeta_3$  are those computed above in (4.21), (4.23). Replacing these expressions  
886 in (4.16), it is clear that the previously neglected terms play the role of a perturbation of higher  
887 order. Higher orders of the bifurcation equation are given by (3.7) or (3.8). We notice that  
888 the system is real because in setting (4.24), the monomials  $q_4$ ,  $q_{j,k}$ ,  $q'_{st}$  cancel for all  $j, k, s, t$ .  
889 Hence there are only four remaining equations in the bifurcation system, with the same form  
890 in the quasiperiodic and in the periodic cases.

891 Dividing by the suitable power of  $\varepsilon$ , the linear terms in  $(x_1, x_2, x_3, v_4, \nu)$  are, at leading  
892 order (replacing  $\mu^{(0)}$  and  $u_4^{(0)}$  by their values)

$$893 \quad \nu - 6v_4 + 2(3 + 2\kappa\zeta_3(-1)^k)x_1 - 12(\zeta_3^2 - 1)x_3 - [2\kappa(-1)^k\zeta_3 + 12](x_1 + x_2 + x_3)$$

$$894 \quad \nu - 6v_4 + 2(3 + 2\kappa\zeta_3(-1)^k)x_2 - 12(\zeta_3^2 - 1)x_3 - [2\kappa(-1)^k\zeta_3 + 12](x_1 + x_2 + x_3)$$

$$895 \quad \nu - 6v_4 + 2(3 + 2\kappa\zeta_3(-1)^k)x_3 - [2\kappa(-1)^k(\zeta_3)^{-1} + 12](x_1 + x_2 + x_3)$$

$$896 \quad \nu - 3v_4 - 12(\zeta_3^2 - 1)x_3 - 12(x_1 + x_2 + x_3).$$

897 The fact that we have a freedom for the choice of the scale  $\varepsilon$  allows us to take  $x_1 = 0$ . So, if  
898 we are able to invert the matrix  $M$  defined above, acting on  $(x_2, x_3, v_4, \nu)$ , i.e., solving

$$899 \quad M(x_2, x_3, v_4, \nu)^t = (h_1, h_2, h_3, h_4)^t,$$

900 with an inverse with a norm of order 1, then this would mean that we can invert the differential  
901 at the origin for  $\varepsilon = 0$ , for the full system in  $(x_2, x_3, v_4, \nu)$ , hence we can use the implicit  
902 function theorem to solve the full system, including all orders.

903 Now, we obtain

$$904 \quad h_2 - h_1 = 2x_2(3 + 2\kappa\zeta_3(-1)^k),$$

$$905 \quad h_3 - h_1 = 2x_3(3 + 2\kappa\zeta_3(-1)^k) + 12(\zeta_3^2 - 1)x_3 + 2\kappa(-1)^k[\zeta_3 - (\zeta_3)^{-1}](x_2 + x_3),$$

906 which gives  $x_2$  and  $x_3$  provided that

$$907 \quad (4.25) \quad (3 + 2\kappa\zeta_3(-1)^k) \neq 0,$$

908 and

$$909 \quad (4.26) \quad -6 + 6\kappa\zeta_3(-1)^k + 12\zeta_3^2 - 2\kappa(\zeta_3)^{-1}(-1)^k \neq 0.$$

910 It appears that condition (4.26) is the same as (4.25) in the cases when  $\zeta_3 = \pm 1$ . In the third  
911 case, when  $\zeta_3 = \frac{2}{3}\kappa(-1)^{k+1}$ , both conditions (4.25) and (4.26) give

$$912 \quad (4.27) \quad \kappa^2 \neq \frac{9}{4}.$$

913 Once these conditions are realized, it is clear that we can invert the matrix  $M$  (solving with  
914 respect to  $(\nu, v_4)$  is straightforward, once  $x_2, x_3$  is computed). The solution is obtained under  
915 the form of a power series in  $\varepsilon$ , with coefficients depending on  $\kappa$ . The series is formal in  
916 the quasiperiodic case, while it is convergent for  $\varepsilon$  small enough in the periodic case. In all  
917 cases, the bifurcation is supercritical ( $\mu > 0$ ). Finally, the solutions (4.21) and (4.23) are the  
918 principal parts of superposed rolls and hexagons. Notice that we can shift the hexagons in  
919 the plane using  $\theta_1$  and  $\theta_2$ , and independently shift the rolls using the phase  $\theta_4$ . Notice that a  
920 similar result holds by replacing  $z_4$  by  $z_5$  or  $z_6$ .

921 For understanding in the plane  $(\mu, \chi)$  where the solutions bifurcate, we first look at  $\mu > 0$   
922 and solve at leading order the second degree equation for  $\varepsilon$ . For the solution (4.21) this gives

$$923 \quad 21\varepsilon^2 + 2\chi\varepsilon(-1)^{k+1} - \mu = 0$$

924 i.e., (since  $\varepsilon > 0$ )

$$925 \quad \varepsilon = \frac{(-1)^k\chi + \sqrt{\chi^2 + 21\mu}}{21}.$$

926 Hence the conditions (4.22) and (4.25) lead to

$$927 \quad 13(-1)^k\chi < \sqrt{\chi^2 + 21\mu},$$

$$928 \quad 15\chi(-1)^{k+1} \neq \sqrt{\chi^2 + 21\mu}.$$

929 This gives the conditions (see Figure 8 left side)

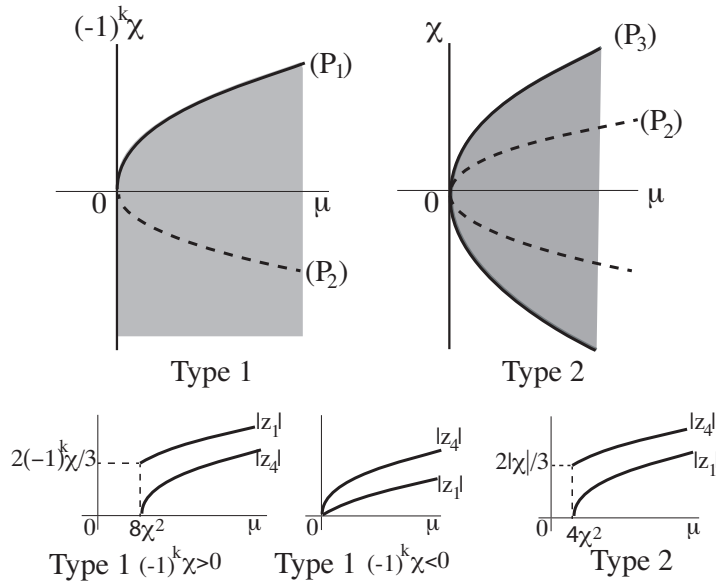
$$930 \quad \mu > 8\chi^2, \text{ for } (-1)^k\chi > 0, \text{ Parabola } (P_1)$$

$$931 \quad \mu \neq \frac{32}{3}\chi^2 \text{ for } (-1)^k\chi < 0, \text{ Parabola } (P_2)$$

932 For the solution (4.23) we have, from the expression of  $\mu$  and from (4.27), the conditions (see  
933 Figure 8 right side)

$$934 \quad \mu > 4\chi^2, \quad \mu \neq \frac{32}{3}\chi^2, \quad \text{Parabolas } (P_3) \text{ and } (P_2).$$

935 Finally, we state the following



**Figure 8.** Domain of existence of bifurcating superposition of hexagons and rolls (balanced hexa-rolls types 1 and 2) for small  $|\chi|$ . Solutions of type 1 (three hexagon amplitudes equal at leading order) are on the left side, solutions of type 2 (two of the three hexagon amplitudes equal at leading order) are on the right side. The parabola  $(P_2)$  (dashed line) is a forbidden place.

936 **Theorem 4.12 (Hexa-rolls: superposed hexagons and rolls in balance).** Assume that  $\alpha \in \mathcal{E}_0$ .  
 937 Then, for  $\chi = \varepsilon\kappa$ ,  $\varepsilon > 0$  close enough to 0, we can build a series in powers of  $\varepsilon$ , solution  
 938 of (3.3), of the form

$$939 \quad u(\varepsilon, \kappa, \Theta, k, j) = \varepsilon u_1(\Theta) + \sum_{n \geq 1} \varepsilon^{2n+1} u_{2n+1}(\kappa, \Theta, k, j), \quad u_{2n+1} \perp e^{i\mathbf{k}_1 \cdot \mathbf{x}}, \quad n \geq 1,$$

$$940 \quad u_1(\Theta, \kappa, k, j) = \sum_{m=1,2} e^{i(\mathbf{k}_m \cdot \mathbf{x} + \theta_m)} + \zeta_3 e^{i(\mathbf{k}_3 \cdot \mathbf{x} + \theta_3)} + \sqrt{u_4^{(0)}} e^{i(\mathbf{k}_j \cdot \mathbf{x} + \theta_j)} + c.c.,$$

$$941 \quad \Theta = (\theta_1, \theta_2, \theta_3, \theta_j), \quad j = 4 \text{ or } 5 \text{ or } 6, \quad \theta_1 + \theta_2 + \theta_3 = k\pi, \quad k = 0 \text{ or } 1$$

$$942 \quad \mu(\varepsilon, \kappa, k, j) = \varepsilon^2 \mu^{(0)}(\kappa, k) + \sum_{n \geq 2} \varepsilon^{2n} \mu_{2n}(\kappa, k, j),$$

943 *Balanced hexa-rolls type 1 (three hexagon amplitudes equal at leading order):*

$$944 \quad \zeta_3 = 1, \quad \mu^{(0)}(\kappa, k) = (-1)^{k+1} 2\kappa + 21, \quad u_4^{(0)} = (-1)^{k+1} \frac{2}{3} \kappa + 1, \quad (-1)^k \kappa < 3/2, \quad (-1)^k \kappa \neq -3/2.$$

945 *Balanced hexa-rolls type 2 (two of the three hexagon amplitudes equal at leading order):*

$$946 \quad \zeta_3 = \frac{2}{3} \kappa (-1)^{k+1}, \quad \mu^{(0)}(\kappa) = 15 + 4\kappa^2, \quad u_4^{(0)} = 1 + \frac{4}{9} \kappa^2, \quad \kappa \neq \pm 3/2.$$

947 The freedom left for  $\Theta$  corresponds to an arbitrary choice for translations  $\mathbf{T}_\delta$ , as well for  
 948 hexagons as for rolls (for  $\theta_j$ ). In the quasiperiodic case ( $\alpha \in \mathcal{E}_2$ ), these solutions give quasi-

949 patterns using the methods of [25]. See Figure 8 for understanding the domain of bifurcating  
 950 solutions in the plane  $(\mu, \chi)$ . Figure 7 shows quasiperiodic examples of  $u_1$ .

951 *Remark 4.13.* As for hexa-rolls with rolls dominating, these solutions are new, even in the  
 952 periodic case. Moreover, notice that in this case also we have the surprising freedom on shifts  
 953 for the roll part, even in the periodic case. This follows from the reality of the 4-dimensional  
 954 system.

955 **5. Conclusion.** We have shown the existence of new quasipattern solutions of the Swift–  
 956 Hohenberg equation with quadratic as well as cubic nonlinearity: superposed hexagons with  
 957 unequal amplitudes (valid only for small  $\mu, \chi$ ). The existence of superposed hexagons with  
 958 equal amplitudes ( $\varepsilon = \pm\delta$ ) had already been established in [19, 25]. We have also found  
 959 (provided the cubic coefficients satisfy an inequality) a new class of solutions, superposed  
 960 hexagons and rolls: the roll amplitude dominates if the quadratic coefficient  $\chi$  is not small,  
 961 but for small  $\chi = \mathcal{O}(\sqrt{|\mu|})$ , the rolls and hexagons can have similar amplitudes. For small  $\chi$ ,  
 962 we have also found superposed symmetry-broken hexagons and rolls. Our approach relies on  
 963 the small-divisor techniques from [25] for solutions of the amplitude equations to be translated  
 964 into quasipattern solutions of the PDE (1.1). The end result is that for a full measure set  
 965 of angles ( $\alpha \in \mathcal{E}_2$ ), two hexagonal patterns with essentially arbitrary relative orientation  
 966 and position can be superposed to produce quasipattern solutions of the Swift–Hohenberg  
 967 equation. Similarly, superposed hexagons and rolls, again with essentially arbitrary relative  
 968 orientation and position, also give quasipattern solutions.

969 In the periodic case we recover the superposed hexagon solutions already known from [15].  
 970 We have shown that the additional triangular superlattice solutions identified by [45] in the  
 971 case  $(a, b) = (3, 2)$  also arise for general  $(a, b)$ . We find a new class of periodic superposed  
 972 hexagon and roll solutions, provided the cubic coefficients satisfy an inequality and  $a > b + 1$ .  
 973 Surprisingly, even in the periodic case, the hexagons and rolls can be translated arbitrarily  
 974 with respect to each other.

975 The approach we have taken differs from that familiar from equivariant bifurcation theory  
 976 (which applies only in the periodic case). When the amplitude equations reduce to a single  
 977 equation, the results are of course the same. The new solutions arise in cases where there  
 978 is more than one equation to solve, and in some cases, these solutions have no symmetry.  
 979 Our approach indicates how a wider class of pattern solutions can be investigated in pattern  
 980 formation problems posed on the whole plane. It is likely that there are many other solutions  
 981 still to be found: hexagons with superposed rhombuses dominating (see [49]), three sets of  
 982 rolls at different angles to each other, superpositions of hexagons and squares, or squares and  
 983 rolls at different angles, . . . . In all of these cases, careful consideration will have to be given  
 984 to the Diophantine condition and to the behavior of high-order nonlinear modes.

985 We have not discussed stability of these quasipatterns: that is an important and diffi-  
 986 cult problem. However, the reason for including a quadratic term in the Swift–Hohenberg  
 987 equation (1.1) is that three-wave interactions generated by quadratic terms, particularly in  
 988 problems in which patterns on two length scales are simultaneously unstable, are known to  
 989 play a key role in stabilizing quasipatterns in a variety of contexts [4, 5, 12, 18, 31, 34, 37, 39,  
 990 41, 42, 47, 48, 56]. Despite this, we do not expect any of the new solutions to be stable in the  
 991 Swift–Hohenberg equation, but they (or related solutions) may be stable in other situations.

992 The recently discovered “bronze-mean hexagonal quasicrystals” described in [3, 16, 36]  
 993 fall into the class of superposed hexagons. These quasicrystals are not solutions of a PDE,  
 994 but rather are constructed from assemblies of three tiles: small equilateral triangles, large  
 995 equilateral triangles, and rectangles. The Fourier transform of a six-fold aperiodic tiling made  
 996 from these tiles has prominent peaks arranged as in Figure 2(c), with  $\alpha = 25.66^\circ$ , and the  
 997 ideas presented here may be relevant to existence of this type of quasipattern in a pattern-  
 998 forming PDE.

999 Finally, we mention a potential application of this body of work to bi-layer graphene, where  
 1000 two layers of hexagonally connected carbon atoms are superposed with a small orientation  
 1001 difference [53]: for  $\alpha$  about  $1^\circ$ , these bi-layer structures can be superconducting [52]. Our  
 1002 work may be relevant for finding quasiperiodic structures in models of this system.

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1008 **Appendix A. Definitions of all the sets of angles.** Here we first recall definitions given  
 1009 in main text, and supplement these with descriptions of  $\mathcal{E}_1$  and  $\mathcal{E}_2$ .

1010 The set  $\mathcal{E}_p$  (periodic case) is given in Definition 2.1, and has  $\cos \alpha$  and  $\sqrt{3} \sin \alpha$  both  
 1011 rational, with  $\alpha \in (0, \frac{\pi}{3})$ . The complement of  $\mathcal{E}_p$ , restricted to  $(0, \frac{\pi}{6}]$ , is  $\mathcal{E}_{qp}$  (quasiperiodic  
 1012 case). The set  $\mathcal{E}_0$ , given in Definition 2.4, is the set of angles  $\alpha$  such that the only solutions  
 1013 of  $|\mathbf{k}(\mathbf{m})| = 1$  are  $\pm \mathbf{k}_j$ ,  $j = 1, \dots, 6$ .

1014 The two sets  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are defined in detail in [25] and described below: these are angles  
 1015  $\alpha \in \mathcal{E}_{qp}$  where additional Diophantine conditions are satisfied. The final set is  $\mathcal{E}_2$ .

1016 Lemma 7 of [25] states that for nearly all  $\alpha \in \mathcal{E}_{qp} \cap (0, \frac{\pi}{6}]$ , and for any  $\varepsilon > 0$ , there exists  
 1017  $c > 0$  such that for all  $\mathbf{m} \neq 0$  with  $|\mathbf{k}(\mathbf{m})| \neq 1$ ,

$$1018 \quad (|\mathbf{k}(\mathbf{m})|^2 - 1)^2 \geq \frac{c}{|N_{\mathbf{k}}|^{12+\varepsilon}}$$

1019 holds. The set  $\mathcal{E}_1$  is the set of all  $\alpha$ 's such that this inequality holds, and  $\mathcal{E}_1$  is of full measure.

1020 Let us now choose an integer  $1 \leq d \leq 4$  and consider an expression of the form

$$1021 \quad (\text{A.1}) \quad P = a_0 + \sum_{1 \leq n \leq d} a_{n0} \cos^n \alpha + \sqrt{3} a_{n-1,1} \sin \alpha \cos^{n-1} \alpha,$$

1022 where the coefficients  $\mathbf{a} = (a_0, a_{n0}, a_{n-1,1}, n = 1, \dots, d)$  are integers:  $\mathbf{a} \in \mathbb{Z}^{(2d+1)}$ . The  
 1023 following proposition is proved in [25] (see Proposition 21):

1024 **Proposition A.1.** *For nearly all  $\alpha \in \mathcal{E}_0 \cap \mathcal{E}_1 \cap (0, \frac{\pi}{6}]$ , there exists  $c > 0$  such that for all*  
 1025  *$\mathbf{a} \in \mathbb{Z}^{(2d+1)} \setminus \{0\}$  and for  $l = 2d(2d+1)$ ,*

$$1026 \quad P \geq \frac{c}{|\mathbf{a}|^l},$$

1027 where  $\mathbf{a} = (a_0, a_{n0}, a_{n-1,1}, n = 1, \dots, d)$ ,  $1 \leq d \leq 4$ , and

$$1028 \quad |\mathbf{a}| = |a_0| + \sum_{1 \leq n \leq d} |a_{n0}| + |a_{n-1,1}|.$$

1029 The set  $\mathcal{E}_2$  is the set of all  $\alpha \in (0, \frac{\pi}{6}]$  such that this inequality holds for any  $d \leq 4$ , provided  
1030 that  $|\mathbf{a}| \neq 0$ . The set  $\mathcal{E}_2$  is a subset of  $\mathcal{E}_0 \cap \mathcal{E}_1$ , and  $\mathcal{E}_2$  is of full measure [25].

1031 **Appendix B. Proof of the properties of two example angles.** While the set  $\mathcal{E}_2$  is of full  
1032 measure [25], in practice it can be difficult to determine whether any particular angle is or is  
1033 not in the set. Here we take two examples and prove that  $\alpha \approx 25.66^\circ$  ( $\cos \alpha = \frac{1}{4}\sqrt{13}$ ) is in  $\mathcal{E}_2$ ,  
1034 while  $\alpha \approx 26.44^\circ$  ( $\cos \alpha = \frac{1}{12}(5 + \sqrt{33})$ ) is not.

1035 **B.1. First example.** Let us consider  $\alpha \in \mathcal{E}_{qp}$  such that

$$1036 \quad \cos \alpha = \frac{\sqrt{13}}{4}, \quad \sqrt{3} \sin \alpha = \frac{3}{4},$$

1037 with  $\alpha \approx 25.66^\circ$ . In order to show that  $\alpha \in \mathcal{E}_2$ , we must first prove that  $\alpha \in \mathcal{E}_0$ , which  
1038 means that the points of the lattice  $\Gamma$  on the unit circle are only the twelve basic points  $\pm \mathbf{k}_j$ ,  
1039  $j = 1, \dots, 6$ . For

$$1040 \quad \mathbf{k} = n_1 \mathbf{k}_1 + n_2 \mathbf{k}_2 + n_4 \mathbf{k}_4 + n_5 \mathbf{k}_5, \quad n_j \in \mathbb{Z},$$

1041 the condition  $|\mathbf{k}|^2 = 1$  becomes

$$1042 \quad 1 = n_1^2 + n_2^2 + n_4^2 + n_5^2 - n_1 n_2 - n_4 n_5 +$$

$$1043 \quad + \cos \alpha (2n_1 n_4 + 2n_2 n_5 - n_1 n_5 - n_2 n_4) +$$

$$1044 \quad + \sqrt{3} \sin \alpha (n_2 n_4 - n_1 n_5),$$

1045 which, separating the rational and irrational parts, and with the given value of  $\alpha$ , leads to

$$1046 \quad (\text{B.1}) \quad 2n_1 n_4 + 2n_2 n_5 - n_1 n_5 - n_2 n_4 = 0,$$

$$1047 \quad 3(n_2 n_4 - n_1 n_5) + 4(n_1^2 + n_2^2 + n_4^2 + n_5^2 - n_1 n_2 - n_4 n_5) = 4.$$

1049 Solving with respect to  $n_5$  leads to

$$1050 \quad n_5 = n_4 \frac{n_2 - 2n_1}{2n_2 - n_1},$$

1051 provided that  $n_1 \neq 2n_2$ ,

$$1052 \quad 0 = 4n_4^2 \left( 1 + \left( \frac{n_2 - 2n_1}{2n_2 - n_1} \right)^2 - \frac{n_2 - 2n_1}{2n_2 - n_1} \right) +$$

$$1053 \quad + 3n_4 \left( n_2 - n_1 \frac{n_2 - 2n_1}{2n_2 - n_1} \right) + 4(n_1^2 + n_2^2 - n_1 n_2 - 1),$$

1054 i.e.,

$$1055 \quad 6n_4^2(n_1^2 + n_2^2 - n_1 n_2) + 3n_4(n_1^2 + n_2^2 - n_1 n_2)(2n_2 - n_1) + 2(n_1^2 + n_2^2 - n_1 n_2 - 1)(2n_2 - n_1)^2 = 0.$$

1056 The discriminant of this quadratic equation for  $n_4$  reads

$$1057 \quad \Delta = 9(n_1^2 + n_2^2 - n_1n_2)^2(2n_2 - n_1)^2 - 48(n_1^2 + n_2^2 - n_1n_2 - 1)(2n_2 - n_1)^2(n_1^2 + n_2^2 - n_1n_2)$$

$$1058 \quad = 3(n_1^2 + n_2^2 - n_1n_2)(2n_2 - n_1)^2 [16 - 13(n_1^2 + n_2^2 - n_1n_2)].$$

1059 We observe that  $\Delta$  should be  $\geq 0$ , and since  $(n_1^2 + n_2^2 - n_1n_2)(2n_2 - n_1)^2 \geq 0$ , this implies

$$1060 \quad 16 \geq 13(n_1^2 + n_2^2 - n_1n_2).$$

1061 This in turn implies that

$$1062 \quad n_1^2 + n_2^2 - n_1n_2 = 1 \text{ or } 0.$$

1063 The only solutions are

$$1064 \quad (n_1, n_2) = (0, 0), (0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1),$$

1065 leading to

$$1066 \quad \Delta = 9 \text{ for } (n_1, n_2) = (\pm 1, 0), (\pm 1, \pm 1),$$

$$1067 \quad \Delta = 36 \text{ for } (n_1, n_2) = (0, \pm 1).$$

1068 The case  $(n_1, n_2) = (0, 0)$  in (B.1), leads to  $n_4^2 + n_5^2 - n_4n_5 = 1$ , which correspond to  $\pm \mathbf{k}_4$ ,  
1069  $\pm \mathbf{k}_5$  and  $\pm \mathbf{k}_6$ . The case  $(n_1, n_2) = (\pm 1, 0), (\pm 1, \pm 1)$  leads to  $n_4 = 0$  or  $\mp \frac{1}{2}$  (which is not  
1070 acceptable). Finally the case is  $(n_1, n_2) = (0, \pm 1)$  gives

$$1071 \quad n_4 = 0 \text{ or } \mp 1,$$

1072 and  $n_5 = 0$  or  $\pm \frac{1}{2}$ , and the only good possibility is  $n_4 = n_5 = 0$  and this corresponds to  
1073  $\pm \mathbf{k}_1, \pm \mathbf{k}_2, \pm \mathbf{k}_3$ . It remains to study the case  $n_1 = 2n_2, n_4 = 0$ . Replacing this in (B.1), we  
1074 obtain

$$1075 \quad 6n_2^2 - 3n_2n_5 + 2n_5^2 - 2 = 0$$

1076 and it is easy to conclude that there are no other solutions of (B.1). The conclusion is that  
1077  $\alpha \in \mathcal{E}_0$ .

1078 Let us now prove that  $\alpha$  satisfies the two Diophantine conditions required in [25] and  
1079 described in Appendix A. We observe that

$$1080 \quad 4(|\mathbf{k}|^2 - 1) = q_0\sqrt{13} + q_1,$$

$$1081 \quad q_0 = 2n_1n_4 + 2n_2n_5 - n_1n_5 - n_2n_4,$$

$$1082 \quad q_1 = 3(n_2n_4 - n_1n_5) + 4(n_1^2 + n_2^2 + n_4^2 + n_5^2 - n_1n_2 - n_4n_5) - 4.$$

1083 Since  $\sqrt{13}$  is a quadratic irrational (the solution of a quadratic equation with integer coeffi-  
1084 cients), it is known [22] that there exists  $C > 0$  such that

$$1085 \quad |q_0\sqrt{13} + q_1| \geq \frac{C}{|q_0| + |q_1|}, \quad (q_0, q_1) \in \mathbb{Z}^2 \setminus \{0\}.$$

1086 Since we have

$$\begin{aligned}
1087 \quad & |q_0| \leq \frac{3}{2}(n_1^2 + n_2^2 + n_4^2 + n_5^2), \\
1088 \quad & |q_1| \leq \frac{15}{2}(n_1^2 + n_2^2 + n_4^2 + n_5^2) + 4 \\
1089 \quad & |q_0| + |q_1| \leq 11(n_1^2 + n_2^2 + n_4^2 + n_5^2),
\end{aligned}$$

1090 hence

$$1091 \quad (|\mathbf{k}|^2 - 1)^2 \geq \frac{C'}{(n_1^2 + n_2^2 + n_4^2 + n_5^2)^2},$$

1092 which means that  $\alpha \in \mathcal{E}_1$  as defined in [25] and described in [Appendix A](#).

1093 Now for  $\mathcal{E}_2$ , let us follow the lines of [Appendix A](#). For this choice of  $\alpha$ , and for any integer  
1094  $d \leq 4$ , the expression (A.1) takes the form

$$1095 \quad P = \frac{b_0 + b_1\sqrt{13}}{b_2}, \quad b_0, b_1, b_2 \in \mathbb{Z},$$

1096 where the integer denominator depends on  $\alpha$  and  $d$  but not on the integers  $\mathbf{a}$  in (A.1). Then,  
1097 as soon as  $|b_0| + |b_1| \neq 0$  we again have a Diophantine estimate

$$1098 \quad P > \frac{C'}{|b_0| + |b_1|},$$

1099 where  $b_2$  is absorbed into  $C'$ . This is the required property for  $\alpha \in \mathcal{E}_2$  in [25] (see also  
1100 [Appendix A](#)), and so the proof that  $\alpha \in \mathcal{E}_2$  is complete. More generally if  $\cos \alpha$  is rational  
1101 and  $\sqrt{3}\sin \alpha$  is a quadratic irrational, or vice versa,  $\mathcal{E}_1$  should be satisfied, as should the  
1102 Diophantine requirement of  $\mathcal{E}_2$ .

1103 **B.2. Second example.** Let us consider  $\alpha \in \mathcal{E}_{qp}$  such that

$$1104 \quad \cos \alpha = \frac{5 + \sqrt{33}}{12}, \quad \sqrt{3}\sin \alpha = \frac{15 - \sqrt{33}}{12},$$

1105 with  $\alpha \approx 26.44^\circ$ . We wish to prove that  $\alpha \notin \mathcal{E}_2$ . We have

$$1106 \quad \mathbf{k} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2 + n_4\mathbf{k}_4 + n_5\mathbf{k}_5, \quad n_j \in \mathbb{Z},$$

1107 and, again separating rational and irrational parts, the condition  $|\mathbf{k}|^2 = 1$  leads to

$$1108 \quad (\text{B.2}) \quad 0 = 3(n_1^2 + n_2^2 + n_4^2 + n_5^2 - n_1n_2 - n_4n_5 - 1) + 5(n_2n_4 - n_1n_5)$$

1109 and

$$1110 \quad (\text{B.3}) \quad n_1n_4 + n_2n_5 - n_2n_4 = 0.$$

1111 Then we observe that

$$1112 \quad (n_1, n_2, n_4, n_5) = (2, 1, -1, 1)$$

1113 is solution of (B.2), (B.3). This means that the following wave vectors lie on the unit circle

$$1114 \quad \pm(\mathbf{k}_1 - \mathbf{k}_3 - \mathbf{k}_4 + \mathbf{k}_5)$$

$$1115 \quad \pm(\mathbf{k}_2 - \mathbf{k}_1 - \mathbf{k}_5 + \mathbf{k}_6)$$

$$1116 \quad \pm(\mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_6 + \mathbf{k}_4)$$

1117 and it is clear that  $\pm\mathbf{k}_j$ ,  $j = 1, \dots, 6$  are not the only elements of  $\Gamma$  on the unit circle, so  
1118  $\alpha \notin \mathcal{E}_0$  and  $\alpha \notin \mathcal{E}_2$ .

1119 **Appendix C. Proof of Lemma 2.2.** Let us show the following

1120 **Lemma C.1.** *Let  $\alpha \in \mathcal{E}_p \cap (0, \frac{\pi}{6})$ , with  $\cos \alpha$  and  $\sqrt{3} \sin \alpha$  both rational, and define positive*  
1121 *integers  $p, q, p'$  such that*

$$1122 \quad (\text{C.1}) \quad \cos \alpha = \frac{p}{q}, \quad \sqrt{3} \sin \alpha = \frac{p'}{q}, \quad 3p^2 + p'^2 = 3q^2,$$

1123 where  $(p, q, p')$  have no common divisor. We define  $d$  to be the greatest common divisor of  
1124  $2(p+q)$  and  $(p+q+p')$ . Then,  $(a, b)$  defined by

$$1125 \quad (\text{C.2}) \quad a = \frac{2(p+q)}{d}, \quad b = \frac{p+q+p'}{d}$$

1126 are relatively prime integers that satisfy (2.2) and  $a > b > \frac{1}{2}a > 0$ .

1127 *Proof.* Let us assume that (C.1) holds, and we seek integers  $(a, b)$  such that (2.2) holds.  
1128 If  $(a, b)$  are integers given by (C.2), then (using  $3p^2 + p'^2 = 3q^2$ ) this leads to

$$1129 \quad a^2 + 2ab - 2b^2 = p \times \frac{12(p+q)}{d^2},$$

$$1130 \quad 3a(2b - a) = p' \times \frac{12(p+q)}{d^2},$$

$$1131 \quad 2(a^2 - ab + b^2) = q \times \frac{12(p+q)}{d^2}.$$

1132 Dividing the first and second lines by the third leads to (2.2). Now since  $\alpha \in (0, \frac{\pi}{3})$  we have

$$1133 \quad p' < \frac{3}{2}q < 3p < 3q,$$

1134 which leads to

$$1135 \quad a > b > \frac{1}{2}a > 0. \quad \blacksquare$$

1136 It remains to check that we can assume  $a + b$  not multiple of 3. Suppose that this is not  
1137 the case, then we define

$$1138 \quad a' = \frac{1}{3}(a + b), \quad b' = \frac{1}{3}(2a - b),$$

1139 then it is easy to check that

$$1140 \quad \cos\left(\frac{\pi}{3} - \alpha\right) = \frac{a'^2 + 2a'b' - 2b'^2}{2(a'^2 - a'b' + b'^2)}, \quad \sqrt{3} \sin\left(\frac{\pi}{3} - \alpha\right) = \frac{3a'(2b' - a')}{2(a'^2 - a'b' + b'^2)},$$

1141 hence we have for  $\frac{\pi}{3} - \alpha$  the same formulas as for  $\alpha$  in replacing  $(a, b)$  by  $(a', b')$ . This means  
 1142 that in such a case we should choose to consider the angle  $\alpha' = \frac{\pi}{3} - \alpha$  instead of  $\alpha$ , which  
 1143 does not change the fact that  $\alpha' \in (0, \frac{\pi}{3})$ . If it appears that  $a' + b'$  is also multiple of 3,  
 1144 then we need to iterate the operation. In fact this operation means that we can choose basis  
 1145 vectors  $(s_1 - s_2, s_1 + 2s_2)$  instead of  $(s_1, s_2)$ , for the periodic lattice: these are  $\sqrt{3}$  larger. The  
 1146 property (iii) of [Lemma 2.2](#) is proved.

1147 Now, we prove the density of  $\mathcal{E}_p$ . The continuous monotonous function of  $x$

$$1148 \quad \frac{x^2 + 2x - 2}{2(x^2 - x + 1)}$$

1149 makes a homeomorphism between  $(1, 2)$  and  $(\frac{1}{2}, 1)$ , it is clear that the set of values taken by  
 1150  $\cos \alpha$  for  $x = a/b$  rational is dense on  $(\frac{1}{2}, 1)$ . It follows that the set of angles  $\alpha \in [0, \frac{\pi}{3})$   
 1151 satisfying (2.2) for  $a/b$  rational is dense. Hence the property (i) of [Lemma 2.2](#) (the density  
 1152 of  $\mathcal{E}_p$ ) is proved.

1153 *Remark C.2.* We notice that  $d$  divides  $2(p + q)$ , and  $2p'$  and that  $d^2$  divides  $12(p + q)$   
 1154 because  $p, q$  and  $p'$  have no common divisor and  $12(p + q)(q - p) = 4p'^2$

1155 **Appendix D. Proof of (3.8).** In this case the wave vectors  $\mathbf{k}_j$  are defined in (2.3), and  
 1156 (3.6) leads to

$$1157 \quad (n_1 - n_3)a + (n_2 - n_3)(b - a) + (n_4 - n_6)a - (n_5 - n_6)b = 0,$$

$$1158 \quad (n_1 - n_3)b - (n_2 - n_3)a + (n_4 - n_6)(a - b) - (n_5 - n_6)a = 0.$$

1159 Since  $a$  and  $b$  have no common factor, it follows that there exist  $(j, l) \in \mathbb{Z}^2$  such that

$$1160 \quad n_1 - n_2 + n_4 - n_6 = jb,$$

$$1161 \quad n_2 - n_3 - n_5 + n_6 = -ja,$$

$$1162 \quad n_2 - n_3 - n_4 + n_5 = lb,$$

$$1163 \quad n_1 - n_3 - n_4 + n_6 = la.$$

1164 This system leads to

$$1165 \quad n_1 - n_3 = jb + \frac{l-j}{3}(a+b),$$

$$1166 \quad n_1 - n_2 = la - \frac{l-j}{3}(a+b),$$

$$1167 \quad n_4 - n_5 = -ja - \frac{l-j}{3}(a+b),$$

$$1168 \quad n_4 - n_6 = jb - la + \frac{l-j}{3}(a+b).$$

1169 Since  $a + b$  is not a multiple of 3, this implies that there is a  $k \in \mathbb{Z}$  such that

$$1170 \quad l - j = 3k,$$

1171 and

$$\begin{aligned}
 1172 \quad & n_1 - n_3 = (j+k)b + ka, \\
 1173 \quad & n_1 - n_2 = (j+2k)a - kb, \\
 1174 \quad & n_4 - n_5 = -(j+k)a - kb, \\
 1175 \quad & n_4 - n_6 = (j+k)b - (j+2k)a.
 \end{aligned}$$

1176 We notice that the monomials invariant under  $\mathbf{T}_\delta$ , of minimal degree found in [15] correspond  
 1177 to the following choices:  $(j, k) = (1, 0), (-2, 1), (1, -1)$ , their complex conjugate being given  
 1178 by the opposite values of  $(j, k)$ . The basic invariant monomials where  $a$  and  $b$  occur are found  
 1179 by looking for the 27 monomials independent of two of the  $z_j$ :

$$1180 \quad q_{I,1} = z_2^b \bar{z}_3^{a-b} \bar{z}_5^{a-b} z_6^b, \quad q_{I,2} = \bar{z}_2^a \bar{z}_3^b z_5^a z_6^{a-b}, \quad q_{I,3} = z_2^{a-b} z_3^a \bar{z}_5^b \bar{z}_6^a,$$

1181

$$1182 \quad q_{II,1} = z_2^b \bar{z}_3^{a-b} z_4^{a-b} z_6^a, \quad q_{II,2} = z_2^{a-b} z_3^a z_4^b \bar{z}_6^{a-b}, \quad q_{II,3} = z_2^a z_3^b z_4^a z_6^b,$$

1183

$$1184 \quad q_{III,1} = z_2^a z_3^b z_4^{a-b} z_5^b, \quad q_{III,2} = z_2^b \bar{z}_3^{a-b} \bar{z}_4^b z_5^a, \quad q_{III,3} = z_2^{a-b} z_3^a z_4^a z_5^{a-b},$$

1185

$$1186 \quad q_{IV,1} = z_1^b z_3^a z_5^{a-b} z_6^b, \quad q_{IV,2} = z_1^{a-b} \bar{z}_3^b z_5^b z_6^a, \quad q_{IV,3} = z_1^a z_3^{a-b} z_5^a z_6^{a-b},$$

1187

$$1188 \quad q_{V,1} = z_1^{a-b} \bar{z}_3^b z_4^b z_6^{a-b}, \quad q_{V,2} = z_1^a z_3^{a-b} \bar{z}_4^a \bar{z}_6^b, \quad q_{V,3} = z_1^b z_3^a \bar{z}_4^{a-b} z_6^a,$$

1189

$$1190 \quad q_{VI,1} = z_1^a z_3^{a-b} \bar{z}_4^{a-b} z_5^b, \quad q_{VI,2} = z_1^{a-b} \bar{z}_3^b z_4^a \bar{z}_5^{a-b}, \quad q_{VI,3} = z_1^b z_3^a z_4^b z_5^a,$$

1191

$$1192 \quad q_{VII,1} = z_1^b \bar{z}_2^{a-b} z_5^a z_6^{a-b}, \quad q_{VII,2} = z_1^{a-b} z_2^a \bar{z}_5^{a-b} z_6^b, \quad q_{VII,3} = z_1^a z_2^b z_5^b z_6^a,$$

1193

$$1194 \quad q_{VIII,1} = z_1^b \bar{z}_2^{a-b} \bar{z}_4^a z_6^b, \quad q_{VIII,2} = z_1^a z_2^b \bar{z}_4^b z_6^{a-b}, \quad q_{VIII,3} = z_1^{a-b} z_2^a z_4^{a-b} z_6^a,$$

1195

$$1196 \quad q_{IX,1} = z_1^b \bar{z}_2^{a-b} \bar{z}_4^{a-b} z_5^b, \quad q_{IX,2} = z_1^a z_2^b \bar{z}_4^a z_5^{a-b}, \quad q_{IX,3} = z_1^{a-b} z_2^a z_4^b z_5^a.$$

1197 Notice that  $q_{I,1}, q_{V,1}, q_{IX,1}$  are mentioned in [15]. We may also notice that these invariants  
 1198 are not independent since there are relationships between them and the  $u_j$ . We may group

1199 these invariant monomials into nine sets of monomials

$$\begin{aligned}
1200 \quad G_1 &= \{q_{I,1}, \overline{q_{V,1}}, q_{IX,1}\} \text{ with degree } 2a, \\
1201 \quad G_2 &= \{q_{II,1}, \overline{q_{VI,2}}, q_{VII,1}\} \text{ with degree } 3a - b, \\
1202 \quad G'_2 &= \{q_{II,2}, q_{VI,1}, q_{VII,2}\} \text{ with degree } 3a - b, \\
1203 \quad G_3 &= \{q_{III,1}, q_{IV,1}, q_{VIII,2}\} \text{ with degree } 2a + b, \\
1204 \quad G'_3 &= \{q_{III,2}, \overline{q_{IV,2}}, q_{VIII,1}\} \text{ with degree } 2a + b, \\
1205 \quad G_4 &= \{q_{III,3}, q_{IV,3}, q_{VIII,3}\} \text{ with degree } 4a - 2b, \\
1206 \quad G_5 &= \{\overline{q_{I,2}}, q_{V,3}, q_{IX,2}\} \text{ with degree } 3a, \\
1207 \quad G'_5 &= \{q_{I,3}, q_{V,2}, q_{IX,3}\}, \text{ with degree } 3a, \\
1208 \quad G_6 &= \{q_{II,3}, q_{VI,3}, q_{VII,3}\} \text{ with degree } 2a + 2b,
\end{aligned}$$

1209 and their complex conjugates.

1210 Let us control the action of various symmetries (other than  $\mathbf{T}_\delta$ , which leaves them invari-  
1211 ant), useful for obtaining the system of 6 complex bifurcation equations. We have

$$\begin{aligned}
1212 \quad \mathbf{R}_{\pi/3}\{q_{I,1}, \overline{q_{V,1}}, q_{IX,1}\} &= \{q_{V,1}, \overline{q_{IX,1}}, \overline{q_{I,1}}\}, \\
1213 \quad \tau\{q_{I,1}, \overline{q_{V,1}}, q_{IX,1}\} &= \{q_{I,1}, q_{IX,1}, \overline{q_{V,1}}\}, \\
1214 \quad \mathbf{S}\{q_{I,1}, \overline{q_{V,1}}, q_{IX,1}\} &= \{q_{I,1}, \overline{q_{V,1}}, q_{IX,1}\}, \\
1215 \\
1216 \quad \mathbf{R}_{\pi/3}\{q_{II,1}, \overline{q_{VI,2}}, q_{VII,1}\} &= \{q_{VI,2}, \overline{q_{VII,1}}, \overline{q_{II,1}}\}, \\
1217 \quad \tau\{q_{II,1}, \overline{q_{VI,2}}, q_{VII,1}\} &= \{q_{VII,2}, q_{VI,1}, q_{II,2}\}, \\
1218 \quad \mathbf{S}\{q_{II,1}, \overline{q_{VI,2}}, q_{VII,1}\} &= (-1)^{a+b}\{q_{II,1}, \overline{q_{VI,2}}, q_{VII,1}\}, \\
1219 \\
1220 \quad \mathbf{R}_{\pi/3}\{q_{II,2}, q_{VI,1}, q_{VII,2}\} &= \{\overline{q_{VI,1}}, \overline{q_{VII,2}}, \overline{q_{II,2}}\}, \\
1221 \quad \tau\{q_{II,2}, q_{VI,1}, q_{VII,2}\} &= \{q_{VII,1}, \overline{q_{VI,2}}, q_{II,1}\}, \\
1222 \quad \mathbf{S}\{q_{II,2}, q_{VI,1}, q_{VII,2}\} &= (-1)^{a+b}\{q_{II,2}, q_{VI,1}, q_{VII,2}\}, \\
1223 \\
1224 \quad \mathbf{R}_{\pi/3}\{q_{III,1}, q_{IV,1}, q_{VIII,2}\} &= \{\overline{q_{IV,1}}, \overline{q_{VIII,2}}, \overline{q_{III,1}}\}, \\
1225 \quad \tau\{q_{III,1}, q_{IV,1}, q_{VIII,2}\} &= \{q_{IV,2}, \overline{q_{III,2}}, \overline{q_{VIII,1}}\}, \\
1226 \quad \mathbf{S}\{q_{III,1}, q_{IV,1}, q_{VIII,2}\} &= (-1)^b\{q_{III,1}, q_{IV,1}, q_{VIII,2}\}, \\
1227 \\
1228 \quad \mathbf{R}_{\pi/3}\{q_{III,2}, \overline{q_{IV,2}}, q_{VIII,1}\} &= \{q_{IV,2}, \overline{q_{VIII,1}}, \overline{q_{III,2}}\}, \\
1229 \quad \tau\{q_{III,2}, \overline{q_{IV,2}}, q_{VIII,1}\} &= \{\overline{q_{IV,1}}, \overline{q_{III,1}}, \overline{q_{VIII,2}}\}, \\
1230 \quad \mathbf{S}\{q_{III,2}, \overline{q_{IV,2}}, q_{VIII,1}\} &= (-1)^b\{q_{III,2}, \overline{q_{IV,2}}, q_{VIII,1}\}, \\
1231 \\
1232 \quad \mathbf{R}_{\pi/3}\{q_{III,3}, q_{IV,3}, q_{VIII,3}\} &= \{\overline{q_{IV,3}}, \overline{q_{VIII,3}}, \overline{q_{III,3}}\}, \\
1233 \quad \tau\{q_{III,3}, q_{IV,3}, q_{VIII,3}\} &= \{q_{IV,3}, q_{III,3}, q_{VIII,3}\}, \\
1234 \quad \mathbf{S}\{q_{III,3}, q_{IV,3}, q_{VIII,3}\} &= \{q_{III,3}, q_{IV,3}, q_{VIII,3}\},
\end{aligned}$$

1235

1236

$$\mathbf{R}_{\pi/3}\{\overline{q_{I,2}}, q_{V,3}, q_{IX,2}\} = \{\overline{q_{V,3}}, \overline{q_{IX,2}}, q_{I,2}\},$$

1237

$$\tau\{\overline{q_{I,2}}, q_{V,3}, q_{IX,2}\} = \{\overline{q_{I,3}}, \overline{q_{IX,3}}, \overline{q_{V,2}}\},$$

1238

$$\mathbf{S}\{\overline{q_{I,2}}, q_{V,3}, q_{IX,2}\} = (-1)^a \{\overline{q_{I,2}}, q_{V,3}, q_{IX,2}\},$$

1239

1240

$$\mathbf{R}_{\pi/3}\{q_{I,3}, q_{V,2}, q_{IX,3}\} = \{\overline{q_{V,2}}, \overline{q_{IX,3}}, \overline{q_{I,3}}\},$$

1241

$$\tau\{q_{I,3}, q_{V,2}, q_{IX,3}\} = \{q_{I,2}, \overline{q_{IX,2}}, \overline{q_{V,3}}\},$$

1242

$$\mathbf{S}\{q_{I,3}, q_{V,2}, q_{IX,3}\} = (-1)^a \{q_{I,3}, q_{V,2}, q_{IX,3}\},$$

1243

1244

$$\mathbf{R}_{\pi/3}\{q_{II,3}, q_{VI,3}, q_{VII,3}\} = \{\overline{q_{VI,3}}, \overline{q_{VII,3}}, \overline{q_{II,3}}\},$$

1245

$$\tau\{q_{II,3}, q_{VI,3}, q_{VII,3}\} = \{q_{VII,3}, q_{VI,3}, q_{II,3}\},$$

1246

$$\mathbf{S}\{q_{II,3}, q_{VI,3}, q_{VII,3}\} = \{q_{II,3}, q_{VI,3}, q_{VII,3}\}.$$

1247 All this leads in a straightforward way to (3.8).

1248

### Appendix E. Form of the cubic part of the bifurcation system.

Equation (3.3),

1249 projected orthogonally on the complement of  $\ker \mathbf{L}_0$ , leads to

1250 (E.1)

$$\widetilde{\mathbf{L}}_0 w = \mu w - \chi \mathbf{Q}_0 (v_1 + w)^2 - \mathbf{Q}_0 (v_1 + w)^3,$$

1251 where we set

1252

$$u = v_1 + w, \quad v_1 \in \ker \mathbf{L}_0, \quad w \in \{\ker \mathbf{L}_0\}^\perp,$$

1253

and  $\mathbf{Q}_0$  is the orthogonal projection on the complement of  $\ker \mathbf{L}_0$ ,  $\widetilde{\mathbf{L}}_0$  being the restriction of  $\mathbf{L}_0$  on its range, the inverse of which is the pseudo-inverse of  $\mathbf{L}_0$  (bounded in the periodic case, unbounded in the quasiperiodic case because of small divisors). Equation (E.1) may be solved formally with respect to  $w$  as a power series in  $v_1$  and  $\mu$ . We have at quadratic order

1254

1255

1256

1257

$$w_2 = -\chi \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2,$$

1258

and at cubic order in  $v_1, \mu$

1259

$$w_3 = -\mu \chi \widetilde{\mathbf{L}}_0^{-2} \mathbf{Q}_0 v_1^2 + 2\chi^2 \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 [v_1 \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2] - \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^3.$$

1260

Now the bifurcation equation is

1261

$$0 = \mu v_1 - \chi \mathbf{P}_0 (v_1 + w)^2 - \mathbf{P}_0 (v_1 + w)^3,$$

1262

where  $\mathbf{P}_0$  is the orthogonal projection on  $\ker \mathbf{L}_0$  and where we replace  $w$  by its formal expansion in powers of  $(\mu, v_1)$ . This leads to

1263

1264

$$\mu v_1 = \chi \mathbf{P}_0 v_1^2 + \mathbf{P}_0 v_1^3 - 2\chi^2 \mathbf{P}_0 v_1 \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2 + \mathcal{O}(v_1^4).$$

1265 It follows that, up to cubic order in  $(\mu, v_1)$ , the bifurcation system reads

$$1266 \quad \mu v_1 = \chi \mathbf{P}_0 v_1^2 + \mathbf{P}_0 v_1^3 - 2\chi^2 \mathbf{P}_0 v_1 \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2.$$

1267 The scalar product with  $e^{i\mathbf{k}_1 \cdot \mathbf{x}}$  gives

$$1268 \quad (\text{E.2}) \quad \mu z_1 = \chi \langle v_1^2, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle + \langle v_1^3, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle - 2\chi^2 \langle v_1 \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle.$$

1269 It is straightforward to check that

$$1270 \quad \langle v_1^2, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle = 2\overline{z_2 z_3},$$

1271

$$1272 \quad \langle v_1^3, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle = \langle 3z_1^2 \overline{z_1} e^{i\mathbf{k}_1 \cdot \mathbf{x}} + 6 \sum_{j=2, \dots, 6} z_1 z_j \overline{z_j} e^{i\mathbf{k}_1 \cdot \mathbf{x}}, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle$$

$$1273 \quad = 3z_1 u_1 + 6z_1 (u_2 + u_3 + u_4 + u_5 + u_6).$$

1274 The next term is more complicated:

$$1275 \quad \langle v_1 \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle = \sum_{j=1, \dots, 6} z_j \langle \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2, e^{i(\mathbf{k}_1 - \mathbf{k}_j) \cdot \mathbf{x}} \rangle + \sum_{j=1, \dots, 6} \overline{z_j} \langle \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2, e^{i(\mathbf{k}_1 + \mathbf{k}_j) \cdot \mathbf{x}} \rangle,$$

1276 and the relevant terms in  $v_1^2$  are those with an exponent

$$1277 \quad (\mathbf{k}_1 \mp \mathbf{k}_j) \cdot \mathbf{x}, \text{ such that } \mathbf{k}_1 \mp \mathbf{k}_j \neq \pm \mathbf{k}_l, \quad l = 1, \dots, 6.$$

1278 the operator  $\widetilde{\mathbf{L}}_0^{-1}$  provides a multiplication by

$$1279 \quad (1 - |\mathbf{k}_1 \mp \mathbf{k}_j|^2)^{-2}.$$

1280 We notice that

$$1281 \quad |\mathbf{k}_1 - \mathbf{k}_2| = |\mathbf{k}_1 - \mathbf{k}_3|, \text{ while } |\mathbf{k}_1 + \mathbf{k}_2|, |\mathbf{k}_1 + \mathbf{k}_3| \text{ do not appear,}$$

$$1282 \quad |\mathbf{k}_1 \pm \mathbf{k}_4|, |\mathbf{k}_1 \pm \mathbf{k}_5|, |\mathbf{k}_1 \pm \mathbf{k}_6| \text{ all different and functions of } \alpha.$$

1283 Hence

$$1284 \quad 2\chi^2 \langle v_1 \widetilde{\mathbf{L}}_0^{-1} \mathbf{Q}_0 v_1^2, e^{i\mathbf{k}_1 \cdot \mathbf{x}} \rangle = \chi^2 z_1 [c_1 u_1 + c_2 (u_2 + u_3) + c_\alpha u_4 + c_{\alpha+} u_5 + c_{\alpha-} u_6],$$

1285 with

$$1286 \quad c_1 = 2(1 + 1/9), \text{ since } |2\mathbf{k}_1| = 2,$$

$$1287 \quad c_2 = 2(1 + 1/2), \text{ since } |\mathbf{k}_1 - \mathbf{k}_2| = \sqrt{3},$$

$$1288 \quad c_\alpha = 2[1 + 2(1 - |\mathbf{k}_1 - \mathbf{k}_4|^2)^{-2} + 2(1 - |\mathbf{k}_1 + \mathbf{k}_4|^2)^{-2}],$$

$$1289 \quad c_{\alpha+} = 2[1 + 2(1 - |\mathbf{k}_1 - \mathbf{k}_5|^2)^{-2} + 2(1 - |\mathbf{k}_1 + \mathbf{k}_5|^2)^{-2}],$$

$$1290 \quad c_{\alpha-} = 2[1 + 2(1 - |\mathbf{k}_1 - \mathbf{k}_6|^2)^{-2} + 2(1 - |\mathbf{k}_1 + \mathbf{k}_6|^2)^{-2}].$$

1291 **Appendix F. Looking for translations.** Let us consider the cases with  $\alpha \in \mathcal{E}_p$ , then we  
 1292 can choose the translation operator  $\mathbf{T}_\delta$  such that

$$1293 \quad (\text{F.1}) \quad \delta \cdot \mathbf{k}_j = \begin{cases} \frac{2\pi}{3} \pmod{2\pi}, & \text{for } j = 1, 2, 3, \\ -\frac{2\pi}{3} \pmod{2\pi}, & \text{for } j = 4, 5, 6. \end{cases}$$

1294  
 1295  
 1296 Indeed, we set

$$1297 \quad \delta = \frac{2\pi}{3} \lambda^2 m \mathbf{s}_1,$$

1298 where  $\mathbf{s}_1$  and  $\lambda$  are defined at [Lemma 2.2](#) and  $m$  is an integer. Then [\(F.1\)](#) leads to

$$\begin{aligned} 1299 \quad & m(2a - b) = 2(1 + 3n_1), \\ 1300 \quad & m(2b - a) = 2(1 + 3n_2), \\ 1301 \quad & m(a + b) = 2(-1 + 3n_4), \\ 1302 \quad & m(a - 2b) = 2(-1 + 3n_5), \end{aligned}$$

1303 where  $n_1, n_2, n_4, n_5$  are integers. It follows that

$$\begin{aligned} 1304 \quad & n_2 + n_5 = 0, \\ 1305 \quad & am = 2(n_1 + n_4), \\ 1306 \quad & a(2n_4 - n_1 - 1) = b(n_1 + n_4), \\ 1307 \quad & a(n_1 + n_4 + 3n_2 + 1) = 2b(n_1 + n_4). \end{aligned}$$

1308 The last two lines give

$$1309 \quad n_2 = n_4 - n_1 - 1,$$

1310 and so

$$\begin{aligned} 1311 \quad & n_1 + n_4 = la, \\ 1312 \quad & 2n_4 - n_1 - 1 = lb, \end{aligned}$$

1313 where  $l$  is an integer, leading to

$$1314 \quad 3n_4 = 1 + l(a + b).$$

1315 Since  $a + b$  is not multiple of 3, we have to look at two cases:  $a + b = 3j + 1$  or  $a + b = 3j + 2$ .

1316 For  $a + b = 3j + 1$  we choose  $l = 2$ , hence

$$1317 \quad n_4 = 2j + 1, \quad n_1 = 2a - 2j - 1, \quad n_2 = 4j - 2a + 1, \quad n_5 = -n_2, \quad m = 4.$$

1318 For  $a + b = 3j + 2$  we choose  $l = 1$ , hence

$$1319 \quad n_4 = j + 1, \quad n_1 = a - j - 1, \quad n_2 = 2j - a + 1, \quad n_5 = -n_2, \quad m = 2.$$

1320 It follows that the solutions in [Theorem 4.5](#) obtained for  $\theta_1 = \theta_2 = \theta_3 = -\theta_4 = -\theta_5 =$   
 1321  $-\theta_6 = k\frac{\pi}{3}$ , provide *only two different patterns*, one corresponding to  $k = 0, 2, 4$ , the other for  
 1322  $k = 1, 3, 5$ .

1323

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