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Time-Domain Wideband DOA Estimation under the Convolutional Sparse Coding Framework

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***Abstract*—The wideband direction of arrival (DOA) estimation problem can be formulated into a narrowband form by applying discrete Fourier Transform (DFT) to sensor measurements; however, a large number of temporal snapshots are required in order to meet the narrowband assumption in the frequency-domain. To reduce the number of snapshots required, a convolutional sparse coding (CSC) based wideband signal model is proposed for direct time-domain DOA estimation, and a group sparsity based minimization problem is formulated. Simulation results indicate that the proposed time-domain CSC (TD-CSC) based method has a better performance than the frequency-domain method, but with a higher computational complexity.**

***Index Terms*—wideband array, DOA estimation, convolutional sparse coding, time domain.**

I. Introduction

Direction of arrival (DOA) estimation has various applications such as radar, sonar and wireless communications [1]. Many DOA estimation algorithms have been proposed based on the narrowband signal model, where array measurements can be simply expressed by direct multiplication of the steering matrix of the array and the baseband signal vector in addition to additive noise [2]–[5].

Those methods cannot be applied to wideband signals directly, since the array output for wideband signals is obtained through a convolution process [6], [7]. For such a wideband DOA estimation problem, a common approach is applying discrete Fourier transform (DFT) to those measurements, and decomposing wideband signals into different frequency bins, where each bin provides a similar model as the narrowband one, when the number of DFT points is sufficiently large [8]. Under such a framework, many subspace based methods for wideband DOA estimation have been proposed, such as incoherent subspace method (ISM) [9], [10], coherent subspace method (CSM) [11], and test of orthogonality of projected subspaces (TOPS) [12] as well as their variants [13].

Most recently, compressive sensing (CS) has been exploited for narrowband DOA estimation as the signal is sparse in the spatial domain [4], [5], [14]. These CS based algorithms can be extended to wideband by applying it to each frequency bin separately. As the spatial support of incident signals over all frequency bins is identical, joint group sparsity has been introduced for the wideband signal model [15]–[17], which performs more effectively.

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However, those aforementioned methods are based on the assumption that at each frequency bin, the signal can be treated as narrowband, which means the number of DFT points should be large enough; moreover, for many DOA estimation methods to work, especially the subspace based one, the number of snapshots at each frequency bin should be large enough too. As a result, these frequency-domain methods have to rely on a sufficiently large number of snapshots in the time-domain. With a limited number of temporal snapshots, these methods cannot work properly. Thus, in this letter, a wideband DOA estimation method still working effectively for a small number of snapshots is proposed. Unlike those popular frequency-domain methods, the proposed one works on the time-domain data directly based on the idea of convolutional sparse coding (CSC) [18]–[20], where the wideband DOA estimation problem is fomulated into an $l_{2,1}$ norm minimization problem. Compared to frequency-domain methods [15], the proposed one has a better estimation result when the number of snapshots is small. In addition, Cramer-Rao bound for the proposed time-domain model is derived.

The remaining part is structured as follows. The wideband signal moldel with the CSC framework is described in Sec. II. The proposed wideband DOA estimation method and its CRB are presented in Sec. III-A. Simulation results are provided in Sec. IV and conclusions are drawn in Sec. V.

II. Wideband signal model

Assume that there are K wideband signals $s_k(t)$ from directions θ_k, $k = 1, 2, ..., K$, respectively, impinging on a uniform linear array (ULA) of M sensors with an adjacent sensor spacing d. The corresponding received signal at the m-th sensor is expressed as

$$x_m(t) = \sum_{k=1}^{K} \delta(t - \tau_{m,\theta_k}) * s_k(t) + n_m(t), \tag{1}$$

where $n_m(t)$ is noise, c is signal propagation speed, and $\tau_{m,\theta_k} = \frac{md \sin\theta_k}{c}$ is the time delay of the k-th signal with DOA θ_k at the m-th sensor, $m \in \{0, ..., M-1\}$, with the zeroth sensor regarded as the reference one.

With a sampling frequency f_s, the received signal at the m-th sensor can be expressed in a discrete convolution form [6], given by

$$x_m[p] = \sum_{k=1}^{K} (\sum_{i=-\infty}^{+\infty} a_{i,m,\theta_k} s_k[p-i]) + n_m[p], \tag{2}$$

where $s_k[p]$ represents the p-th snapshot of the k-th source signal, and $a_{i,m,\theta_k} = \mathrm{sinc}(i - \tau_{m,\theta_k}/T_s)$. In (3), T_s is its sampling period, and $\mathrm{sinc}()$ is the normalized sinc function, defined as $\mathrm{sinc}(v) = \sin(\pi v)/(\pi v)$.

If $d = \lambda_{min}/2$, where λ_{min} is the wavelength corresponding to the maximum frequency of the signal and we sample the signal at the Nyquist rate, i.e. the sampling period $T_s = \frac{\lambda_{min}}{2c}$, then we have

$$\frac{\tau_{m,\theta_k}}{T_s} = m \sin\theta_k,\ a_{i,m,\theta_k} = \mathrm{sinc}(i - m\sin\theta_k), \tag{3}$$

For convenience of modelling in the following steps, the infinite impulse response in (2) is truncated to the range from $-I$ to I, where I is a large enough number. Then, (2) can be approximated with a small error, as

$$x_m[p] \approx \sum_{k=1}^{K} (\sum_{i=-I}^{I} a_{i,m,\theta_k} s_k[p-i]) + n_m[p]. \tag{4}$$

Considering P snapshots for the received signal $x_m[p],\ p \in \{0, ..., P-1\}$, the required snapshots range of source signal $s_k[p]$ for $x_m[0]$ is $-I < p < I$, while for $x_m[P-1]$, the required snapshots range of source signal $s_k[p]$ is $P-1-I < p < P-1+I$. As a result, to calculate all values of $x_m[p]$ for all $0 \le p \le P-1$, the required range of $s_k[p]$ is $-I < p < P-1+I$, and the total number of required different snapshots for $s_k[p]$ is $P+2I$.

Therefore, constructing source signal vectors

$$\mathbf{s}_k = [s_k[-I], ..., s_k[P-1+I]]^T, \tag{5}$$

and the measurements vector at the m-th sensor $\mathbf{x}_m$

$$\mathbf{x}_m = [x_m[0], ..., x_m[P-1]]^T, \tag{6}$$

(6) can be written in a convolutional sparse coding (CSC) form [18]–[20], as

$$\mathbf{x}_m = \sum_{k=1}^{K} \mathbf{C}_{m,\theta_k} \mathbf{s}_k + \mathbf{n}_m, \tag{7}$$

where $\mathbf{C}_{m,\theta_k}$ is a $P \times (P+2I)$ banded and circulant matrix, given by

$$\mathbf{C}_{m,\theta_k} = \begin{bmatrix} a_{I,m,\theta_k} & \cdots & a_{-I,m,\theta_k} & \cdots & 0 \\ & \ddots & & \ddots & \vdots \\ 0 & \cdots & a_{I,m,\theta_k} & \cdots & a_{-I,m,\theta_k} \end{bmatrix} \tag{8}$$

and $\mathbf{n}_m = [n_m[0], ..., n_m[P-1]]^T$ is the noise vector at the m-th sensor.

Furthermore, (7) can be reformulated in a more compact form as

$$\mathbf{x}_m = \mathbf{C}_m \mathbf{s} + \mathbf{n}_m, \tag{9}$$

where

$$\begin{aligned} \mathbf{s} &= [\mathbf{s}_1^T, ..., \mathbf{s}_K^T]^T, \\ \mathbf{C}_m &= [\mathbf{C}_{m,\theta_1}, ..., \mathbf{C}_{m,\theta_K}]. \end{aligned} \tag{10}$$

III. Proposed method and CRB

A. Proposed method

Unlike those traditional CSC model, the signal vector $\mathbf{s}_k$ in DOA estimation is not sparse in the time-domain, and correspondingly, $\mathbf{s}$ is not sparse either. However, the signals are indeed sparse in the spatial domain, i.e., they only come from a rather limited number different directions [5]. Based on this spatial sparsity concept, to construct a spatially sparse source vector, we divide the whole admissible DOA range into G grid points with $G \gg K$, and for each direction $\theta_g,\ g \in \{1, 2, ...G\}$ we can construct the corresponding matrix $\mathbf{C}_{m,\theta_g}$, and form an overcomplete matrix

$$\tilde{\mathbf{C}}_m = [\mathbf{C}_{m,\theta_1}, ..., \mathbf{C}_{m,\theta_G}]. \tag{11}$$

Then, the signal vector $\mathbf{s}$ is extended to its corresponding $G(P+2I) \times 1$ sparse vector

$$\tilde{\mathbf{s}} = [\mathbf{s}_1^T, ..., \mathbf{s}_G^T]^T, \tag{12}$$

where only K groups out of its G groups corresponding to the true incident angles are supposed to be non-zero. Finally, the array output at the m-th sensor is given by

$$\mathbf{x}_m = \tilde{\mathbf{C}}_m \tilde{\mathbf{s}} + \mathbf{n}_m. \tag{13}$$

For M sensors in total, we have

$$\begin{aligned} \mathbf{x} &= [\mathbf{x}_0^T, ..., \mathbf{x}_{M-1}^T]^T = \tilde{\mathbf{C}}\tilde{\mathbf{s}} + \mathbf{n}, \\ \tilde{\mathbf{C}} &= [\tilde{\mathbf{C}}_0^T, ..., \tilde{\mathbf{C}}_{M-1}^T]^T, \\ \mathbf{n} &= [\mathbf{n}_0^T, ..., \mathbf{n}_{M-1}^T]^T. \end{aligned} \tag{14}$$

Now, since signal $\tilde{\mathbf{s}}$ is sparse, the wideband time-domain based DOA estimation problem can be formulated as a LASSO problem [21],

$$\min_{\tilde{\mathbf{s}}} \|\tilde{\mathbf{C}}\tilde{\mathbf{s}} - \mathbf{x}|_2^2 + \gamma \|\tilde{\mathbf{s}}\|_1, \tag{15}$$

where $\|\cdot\|_1$ represents l_1 norm of its variables to enforce its sparsity and γ is the penalty term.

In addition, the signals $\tilde{\mathbf{s}}$ has a group sparsity structure, where all the entries within a group are all zeros if there is no signal coming from that direction. Thus, the DOA estimation problem can be further formulated as a group LASSO problem [22], represented by the $l_{2,1}$ norm, given by

$$\min_{\tilde{\mathbf{s}}} \|\tilde{\mathbf{C}}\tilde{\mathbf{s}} - \mathbf{x}\|_2^2 + \gamma \|\tilde{\mathbf{s}}\|_{2,1} \tag{16}$$

and the $l_{2,1}$ norm $\|\cdot\|_{2,1}$ is defined as

$$\|\tilde{\mathbf{s}}\|_{2,1} := \sum_{g=1}^{G} \|\mathbf{s}_g\|_2, \tag{17}$$

where $\mathbf{s}_g$ has been defined in (12).

Thus, the wideband DOA estimation problem can be formulated as

$$\min_{\tilde{\mathbf{s}}} \gamma \sum_{g=1}^{G} \|\mathbf{s}_g\|_2 + \|\tilde{\mathbf{C}}\tilde{\mathbf{s}} - \mathbf{x}\|_2^2, \tag{18}$$

This problem is convex and can be solved by convex optimization methods directly, such as the FISTA algorithm, which is an accelerated version of the proximal gradient method [23]–[26]. A summary of FISTA can be found in the following Algorithm Summary part.

Algorithm Summary (FISTA)

Input: $\tilde{\mathbf{C}}$, $\mathbf{x}$, ρ, λ,
Output: $\tilde{\mathbf{s}}$ (reconstructed signal).
Initialization: Set $\tilde{\mathbf{s}}^0$ as a zero vector, $\tilde{\mathbf{b}}^0 = \tilde{\mathbf{s}}^0$.
$\beta^0 = 1$. Defining G groups of $\tilde{\mathbf{s}}$,
General steps: for q=0, ..., Q
1) Calculate gradient as
Gradient:$f(\tilde{\mathbf{b}}) = \nabla F(\tilde{\mathbf{b}}^q) = 2\tilde{\mathbf{C}}^H(\tilde{\mathbf{C}}\tilde{\mathbf{b}}^q - \mathbf{x})$,
2)For $i = 1, ..., G$,
Find $\tilde{\mathbf{s}}_i^{q+1}$ as
$\tilde{\mathbf{s}}_i^{q+1} = (\mathbf{b}_i^q - \lambda f(\mathbf{b}_i^q)) \max(1 - \frac{\rho\lambda}{\|\mathbf{b}_i^q - \lambda\nabla f(\mathbf{b}_i^q)\|_2}, 0)$,
where $\mathbf{s}_i$ is subvector of $\tilde{\mathbf{s}}$ indexed by i.
3) **Update**: $\beta^{q+1} = \frac{1+\sqrt{1+4(\beta^q)^2}}{2}$.
$\mathbf{b}^{q+1} = \tilde{\mathbf{s}}^{q+1} + \frac{\beta^q - 1}{\beta^{q+1}}(\tilde{\mathbf{s}}^{q+1}) - \tilde{\mathbf{s}}^q$.
4) q=q+1.

B. Cramer-Rao Bound

Based on the new model, the CRB for time-domain wideband DOA estimation is derived. From (14), the probability density function is expressed as

$$p(\mathbf{x}; \mathbf{\Phi}) = \prod_{n=0}^{MP-1} \frac{1}{2\pi\sigma^2} e^{(x_n - \mathbf{C}_n \mathbf{s}_n)^2/2\sigma^2}, \tag{19}$$

where $\mathbf{C}_n$ and x_n represent the n-th row of $\mathbf{C}$ and $\mathbf{x}$, separately. The unknown parameter vector of arriving angles, magnitude, phase difference and noise level can be represented as

$$\mathbf{\Phi} = [\theta_1, ..., \theta_K, \boldsymbol{s}^T, \sigma^2], \tag{20}$$

For deterministic but unknown $\mathbf{Cs}$, the Fisher information matrix (FIM) $\mathbf{F}$ is defined as

$$\mathbf{F}(\mathbf{\Phi}) = \mathbb{E}\{\frac{\partial \ln^2 p(\mathbf{x}; \mathbf{\Phi})}{\partial \mathbf{\Phi} \partial \mathbf{\Phi}^T}\}, \tag{21}$$

The $\{i,j\}$-th entry of the FIM $\mathbf{F}$ is given by [27]

$$\begin{aligned} \mathbf{F}_{i,j} = & \Big[\frac{\partial \boldsymbol{\mu}(\mathbf{\Phi})}{\partial \Phi_i}\Big]^T \mathbf{\Gamma}^{-1}(\mathbf{\Phi}) \Big[\frac{\partial \boldsymbol{\mu}(\mathbf{\Phi})}{\partial \Phi_j}\Big] \\ & + \frac{1}{2}\Big[\mathbf{\Gamma}^{-1}(\boldsymbol{\beta}) \frac{\partial \mathbf{\Gamma}^{-1}(\mathbf{\Phi})}{\partial \Phi_i} \mathbf{\Gamma}^{-1}(\mathbf{\Phi}) \frac{\partial \mathbf{\Gamma}^{-1}(\mathbf{\Phi})}{\partial \Phi_j}\Big], \end{aligned} \tag{22}$$

where $\mathbf{\Gamma}^{-1}(\mathbf{\Phi}) = \frac{1}{\sigma^2}\mathbf{I}_{MP}$, $\mathbf{I}_{MP}$ is the identity matrix, $(\cdot)^{-1}$ is the matrix inverse operator, and $\boldsymbol{\mu}(\mathbf{\Phi}) = \mathbf{Cs}$. Since $\boldsymbol{\mu}(\mathbf{\Phi})$ is independent with the noise level, we have

$$\mathbf{F} = \begin{bmatrix} \tilde{\mathbf{F}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_\sigma \end{bmatrix}, \tag{23}$$

where the DOA related block is in $\tilde{\mathbf{F}}$ and its $\{i,j\}$-th entry is expressed as

$$\tilde{\mathbf{F}}_{i,j} = \Big[\frac{\partial \boldsymbol{\mu}(\mathbf{\Phi})}{\partial \Phi_i}\Big]^T \mathbf{\Gamma}^{-1}(\mathbf{\Phi}) \Big[\frac{\partial \boldsymbol{\mu}(\mathbf{\Phi})}{\partial \mathbf{\Phi}_{\boldsymbol{j}}}\Big], \tag{24}$$

with $(\cdot)^{-1}$ being the matrix inverse operator. As the FIM is block diagonal, $\mathbf{F}_\sigma$ has no effect on the CRB result of DOAs. Thus, CRB of DOAs can be determined by the inverse of $\tilde{\mathbf{F}}$. Computing the derivatives of $\boldsymbol{\mu}(\mathbf{\Phi})$ with respect to $\mathbf{\Phi}$, we have

$$\begin{aligned} \mathbf{D} &= \frac{\partial \boldsymbol{\mu}(\mathbf{\Phi})}{\partial \mathbf{\Phi}} = [\mathbf{G}, \Delta, \mathbf{0}], \\ \mathbf{G} &= [\mathbf{c}_{\theta_1}\mathbf{s}_1, ..., \mathbf{c}_{\theta_k}\mathbf{s}_K], \ \mathbf{c}_{\theta_K} = \frac{\partial \mathbf{C}_{\theta_k}}{\partial \theta_k}, \\ \frac{\partial \mathrm{sinc}(i - m\sin\theta_k)}{\partial \theta_k} &= \frac{m \sin(\pi(i - m\sin\theta_k))\cos\theta_k}{\pi(i - m\sin\theta_k)^2} \\ & \quad - \frac{m\cos(\pi(i - m\sin\theta_k))\cos\theta_k}{i - m\sin\theta_k}, \\ \mathbf{C}_{\theta_k} &= [\mathbf{C}_{0,\theta_k}^T, ..., \mathbf{C}_{M-1,\theta_k}^T]^T, \\ \Delta &= \frac{\partial \boldsymbol{\mu}(\mathbf{\Phi})}{\partial \mathbf{s}} = \mathbf{C}. \end{aligned} \tag{25}$$

Then, $\tilde{\mathbf{F}}$ can be given by

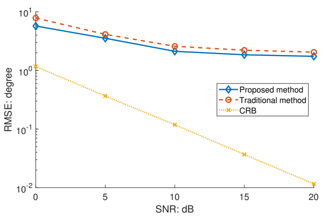
$$\tilde{\mathbf{F}} = \frac{1}{\sigma^2}\mathbf{D}^H\mathbf{D}, \tag{26}$$

The CRB associated with the DOA of signals can be obtained by the diagonal elements of the inverse $\tilde{\mathbf{F}}$. However, in the proposed signal model, it is assumed that $I >> P$ and $K(P+2I) > MP$, which leads to a singular and uninvertible FIM [28]. Thus, CRB is approximated by the Moore–Penrose pseudoinverse of FIM in steand of its inverse [29].

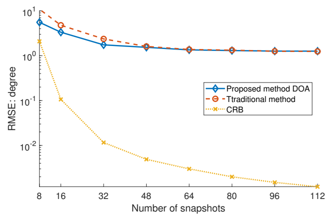
IV. Simulations

In this section, performance of the proposed time-domain CSC based method (TD-CSC) is studied and compared with the traditional frequency-domain method in [15] for wideband DOA estimation. A ULA of $M = 7$ sensors is used with $d = \lambda_{min}/2$ and sampling frequency $T_s = \lambda_{min}/(2c)$. The steering matrix is formed based on a step size of $0.5°$, and the truncated convolution filter for generating wideband signals has a value of $I = 100$. The normalized frequency of wideband signals ranges from 0.5π to π, and for the traditional method, P-point DFT is applied and the normalized frequency range of impinging signals covers the frequency bin range of $U = [P/4 + 1, P/2 - 1]$ (the same settings as in [15]). Note that with P time-domain snapshots and a P-point DFT, for the traditional method, there is only one data sample for each frequency bin. Both methods are run with FISTA [24]. The FISTA setting for both the proposed and the traditional methods are the same, with the number of iterations fixed at $R = 300$, and stepsize set as $1/(2\lambda_{max}(\tilde{\mathbf{C}}^H\tilde{\mathbf{C}}))$, where $\lambda_{max}(\cdot)$ is the maximum eigenvalue of its variable and $\tilde{\mathbf{C}}$ represents the overcomplete dictionary of sparse signals.

First, performances of the two methods are evaluated with different SNR values ranging from 0 dB to 20 dB in terms of the root mean square error (RMSE). $P = 32$ temporal measurements are collected, with two signals located at $-10°$ and $10°$ with equal signal power, and the value I to construct time-domain steering matrix $\mathbf{C}$ in the algorithm is set as 50. The results are shown in Fig. 1a, with each point obtained by averaging over 100 trials. It can be observed that, although

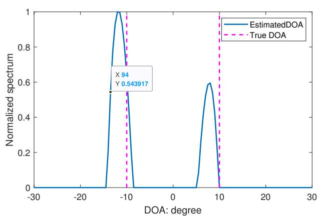


(a) RMSE vs. SNR.

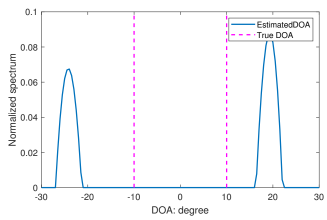


(b) RMSE vs. snapshots.

Figure 1: RMSE results versus SNR and number of snapshots.



(a) The proposed method.



(b) The traditional method.

Figure 2: Spatial spectrums of the two methods with 8 snapshots.

both methods have achieved more accurate results with increasing SNR, the proposed method consistently outperforms the traditional one.

Next, the impact of the number of snapshots P is considered. The SNR is fixed at 20 dB while the number of snapshots ranges from 8 to 112. As shown in Fig. 1b, the performance of the proposed time-domain method is acceptable even with only 8 snapshots, while the traditional frequency-domain method leads to a rather high RMSE initially (about $11°$). Moreover, increasing the number of snapshots can enhance the estimation performance of both methods, and with around 64 snapshots, both methods have reached almost the same performance.

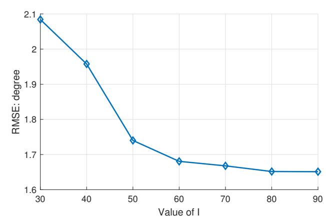
Now we examine the case with only 8 snapshots in a bit more detail. An example of estimated spatial spectrum in one run is given in Fig. 2, which shows that although the proposed method maintains some error with only 8 snapshots, the traditional method has effectively failed. Note that the spacing between the two sources is 20 degrees, while the RMSE has been about 11 degrees for each source according to Fig. 2.

Then, we evaluate the computational complexity and the number of parameters involved in both methods. As listed in Table 1, the number of parameters to be estimated for the proposed method is much more than that for the traditional method since $U \approx P/4$ in this simulation example. As a result, we can find that the running time of the proposed method is much longer. In addition, it is clear that the number of parameters to be estimated for the time-domain method is not only related to the number of snapshots, but also the value of I to construct the circulant matrix $\mathbf{C}$.

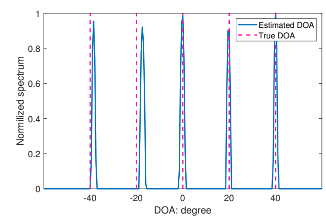
The performance of the proposed method with respect to the value of I for constructing $\mathbf{C}$ in the solution is presented in Fig. 3a. All settings are the same as in the first simulation except for I and SNR is 20 dB. As shown, although the RMSE decreases with the value of I, the effect of I on RMSE can

Table I: Running times of the proposed method and the traditional method.

Computational time (sec)		
Snapshots	The proposed method	The traditional method
112	53.52	19.35
64	25.36	4.66
16	4.64	0.64
Number of parameters to be estimated		
	The proposed method	The traditional method
	G(2I+P)	GU



(a) RMSE vs. I



(b) Five incident signals.

Figure 3: The effect of I and the underdetermined case.

be ignored for a value larger than about 80.

Finally, we examine whether the proposed method can deal with the underdetermined case or not. For underdetermined DOA estimation, a sparse array is normally employed and here we consider a minimum redundancy array (MRA) [1] and the results with $P = 64$ snapshots are shown in Fig. 3b. It can be seen that although there are only 4 sensors, the proposed method can identify all the five sources successfully. This is an interesting result, as no co-array operation is employed in the process, which is different from the frequency-domain method in [15].

V. Conclusions

A wideband DOA estimation method called TD-CSC based on the time-domain model directly has been proposed. The wideband DOA estimation problem was formulated in a CSC form first, and the $l_{2,1}$ norm was then employed to enforce spatial sparsity. Unlike those existing frequency-domain based methods, no additional Fourier transform operation is needed by the proposed method, which means that even a small number of temporal snapshots are sufficient for DOA estimation. Simulation results shows that the proposed method outperforms the frequency-domain based method in terms of RMSE when the number of snapshots is small, but at a cost of extra computational time.

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