

# THE CONSTRUCTIVE KAN–QUILLEN MODEL STRUCTURE: TWO NEW PROOFS

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## Abstract

We present two new proofs of Simon Henry’s result that the category of simplicial sets admits a constructive counterpart of the classical Kan–Quillen model structure. Our proofs are entirely self-contained and avoid complex combinatorial arguments on anodyne extensions. We also give new constructive proofs of the left and right properness of the model structure.

## 1. Introduction

The Kan–Quillen model structure on simplicial sets, that is, the model structure in which the fibrant objects are the Kan complexes and the cofibrations are the monomorphisms [29], has long been recognized as the cornerstone of modern simplicial homotopy theory [16]. Over the past decade, however, this model structure has become of great interest also in mathematical logic and theoretical computer science, since it provides inspiration for Voevodsky’s Univalent Foundations programme [36] and Homotopy Type Theory [35]. In particular, it plays an essential role in the simplicial model of Univalent Foundations [25].

While there are several proofs of the existence of the Kan–Quillen model structure [9, 16, 24, 27, 33], all of them use non-constructive principles, that is, the law of excluded middle (EM) and the axiom of choice (AC). Since these principles are not generally valid in the internal logic of a Grothendieck topos, the construction of an analogue of the Kan–Quillen model structure on simplicial sheaves is very subtle [20, 22]. This situation is also an obstacle to the definition of a constructive version of the simplicial model of Univalent Foundations, which is still an open problem. Furthermore, the results in [6] show that some results on Kan fibrations are simply not provable constructively.

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Recently, Simon Henry obtained a breakthrough result by establishing a constructive counterpart of the Kan–Quillen model structure [18], namely a model structure whose existence can be proved using constructive methods but coincides with the usual model structure once EM and AC are assumed. A key aspect of this model structure is that, in contrast to the classical case, not all objects are cofibrant, but only those in which degeneracy of simplices is decidable. The existence of this model structure has already been applied to provide a partial solution to the problem of giving a constructive simplicial model of Univalent Foundations [12] and suggests the possibility of defining a new model structure on simplicial objects. Indeed, the results in this paper have led to the construction of a new model structure on categories of simplicial objects in countably lexextensive categories, obtained in collaboration with Simon Henry [13].

The main goal of this paper is to give two new proofs of the existence of the constructive Kan–Quillen model structure. We believe these proofs to be simpler than Henry’s proof [18], formulated in clear category-theoretic terms and essentially self-contained. In contrast, Henry’s proof uses subtle combinatorial arguments on anodyne maps, including results that do not seem to have been known even in a classical setting, and relies on his earlier work on weak model structures [17]. We also provide two new proofs of the left and right properness of the constructive Kan–Quillen model structure, which were also already proved in [18].

Overall, this paper establishes all the results of constructive simplicial homotopy theory needed for [12]. Indeed, the desire to give self-contained and streamlined proofs of these results was one of the motivations for this paper, which can then be seen as contributing to the effort to define a constructive simplicial model of Univalent Foundations. We also give new proofs of some results in [12] and establish constructive versions of well-known theorems, such as Quillen’s Theorem A.

Cofibrancy considerations play a key role in both of our proofs. On the one hand, we had to check carefully that the decidability assumptions encapsulated in the notion of cofibrancy allow us to carry over some classical arguments. This is sometimes subtle, for example when extending results about the  $\text{Ex}$  functor to the  $\text{Ex}^\infty$  functor for our first proof and when proving a version of the equivalence extension property for our second proof. On the other hand, we also had to develop new arguments, necessary to extend results from the full subcategory of cofibrant simplicial sets to the category of all simplicial sets, which do not have counterparts in the classical setting. Furthermore, this situation requires us to work with more notions of weak homotopy equivalence than in the classical setting and then check that they are mutually consistent.

We should mention that we had to refine the assumptions of the arguments in [33] before we could apply them in the cofibrant fragment. For example, the category of cofibrant simplicial sets cannot constructively be shown locally Cartesian closed. However, the exponentials appearing in the proof of the equivalence extension property nevertheless preserve cofibrancy, and this verification is rather subtle. For these reasons, we hope that the methods developed in this paper are valuable not only for obtaining a new model structure on simplicial objects but also for defining other model structures in which not all objects are cofibrant. We will comment in more detail on the differences between our proofs and Simon Henry’s proof in Remarks 3.6.6, 4.3.10 and Remark 5.11, after concluding our proofs of the existence of the model structure and of its properness.

We regret that this paper is longer than we originally intended, but we hope that kind readers will appreciate that the proofs are given in some detail, hopefully making our results more widely accessible.

### 1.1. Outline of the paper

Our development begins in Section 2 with some material useful for both proofs. We begin in Subsection 2.1 with some remarks on decidable inclusions and split surjections in the category of sets. These are used in Subsection 2.2 to define the weak factorization systems on simplicial sets of cofibrations and trivial fibrations and of trivial cofibrations and fibrations. The pushout product properties for these weak factorization systems are proved in Subsection 2.3. We then identify the weak factorization system of cofibrations and trivial fibrations as the Reedy weak factorization system on simplicial sets induced by the weak factorization system of decidable inclusions and split surjections on sets in Subsection 2.4. We conclude the section by introducing weak homotopy equivalences in Subsection 2.5 and proving that, for a cofibrant simplicial set  $B$ , the full subcategory of the slice  $\mathbf{sSet} \downarrow B$  spanned by fibrations with cofibrant domain is a fibration category in Subsection 2.6.

Section 3 presents our first proof of the existence of the Kan–Quillen model structure, which is inspired by classical ideas of simplicial homotopy theory, in particular [26]. The proof is organized in five steps, each presented in a subsection. In Subsection 3.2, we show that the full subcategory of  $\mathbf{sSet}$  spanned by cofibrant simplicial sets admits the structure of a cofibration category. In Subsection 3.3, we obtain a constructive proof of the so-called diagonal lemma, asserting that if a map between cofibrant bisimplicial sets is pointwise a weak homotopy equivalence, then so is its diagonal. In Subsection 3.4, we prove counterparts of standard facts on Kan’s  $\mathrm{Ex}^\infty$  functor on cofibrant simplicial sets. In Subsection 3.5, we prove a version of Quillen’s Theorem A and use it to introduce a cofibrant replacement functor with good properties. Finally, in Subsection 3.6, we combine these results to present our first proof of the existence of the model structure.

Section 4 presents our second proof, which is based on the ideas in [14, 33]. The proof is organized in three subsections. In Subsection 4.1, we establish a restricted version of the Frobenius property, showing that trivial cofibrations are closed under pullback along fibrations with cofibrant domain. In Subsection 4.2, we prove the so-called equivalence extension property in the full subcategory of simplicial sets spanned by cofibrant objects. In Subsection 4.3, we combine these results to establish the restriction of the model structure to cofibrant objects. We then extend this model structure to all simplicial sets.

Our two new proofs of the left and right properness of the model structure are presented in Section 5. Here, it should be noted that, in contrast with the classical setting, left properness is not immediate since not every object is cofibrant. For right properness, one proof uses  $\mathrm{Ex}^\infty$  functor, while the other uses the Frobenius property. For left properness, both proofs use an argument dual to that for right properness using  $\mathrm{Ex}^\infty$ .

### 1.2. Remarks on constructivity

To fix ideas, we shall work in constructive Zermelo–Fraenkel set theory (CZF), a set theory based on constructive logic [1]. See [2] for more information on CZF. Readers who are unfamiliar with constructive set theory may think of our category of sets as being an arbitrary Grothendieck topos. By a finite set, we always mean a set with a bijection to a set of the form  $\{1, \dots, n\}$  for some  $n \in \mathbb{N}$ . Equality on such sets is always decidable, which is why they are sometimes referred to also as finite decidable sets. To simplify our presentation, we adopt an abuse of language and say ‘for all ... there exists ...’ to mean that we have a function giving witnesses for existential quantifiers. In particular, when we speak of a map having a right (or left) lifting property with respect to a given class of maps and in particular in the notion of a weak factorization system and a model structure, we mean that

**Table 1.** Notation for different classes of morphisms.

Acyclic map (weak equivalence)	$\xrightarrow{\sim}$		
Fibration	$\longrightarrow \twoheadrightarrow$	Cofibration	$\rightrightarrows$
Trivial fibration	$\longrightarrow \dashv \twoheadrightarrow$	Trivial cofibration	$\rightrightarrows \dashv$
Acyclic fibration	$\xrightarrow{\sim} \twoheadrightarrow$	Acyclic cofibration	$\rightrightarrows \xrightarrow{\sim}$
Monomorphism / Face operator	$\hookrightarrow$	Epimorphism / Degeneracy operator	$\twoheadrightarrow$

the map is equipped with a function providing diagonal fillers for the appropriate class of diagrams. Here, by a class of maps in a category we mean a class together with a forgetful function to the class of all maps of the category. However, after we have established the model structure, one can also derive a variation with subclasses, where for example the fibrations are the subclass of maps for which there exists a function providing diagonal fillers.

### 1.3. Notation

We will use distinct notations for various types of morphisms, which are summarized in Table 1 for the convenience of the readers.

## 2. Preliminaries

### 2.1. Decidable inclusions

We begin by verifying some basic properties of the category of sets in our constructive setting. In particular, we will show that it admits a weak factorization system consisting of decidable inclusions and split surjections. This will be useful to construct weak factorization systems on simplicial sets in Subsection 2.2. Recall that a map of sets  $i: A \rightarrow B$  is a *decidable inclusion* if there is a map  $j: C \rightarrow B$  such that  $i$  and  $j$  exhibit  $B$  as a coproduct of  $A$  and  $C$ . The map  $j$  (or just the set  $C$  by abuse of language) is called the *complement* of  $i$  (or  $A$ ). A *split surjection* is a function that admits a section. Note that, assuming EM, every injective function is a decidable inclusion and, assuming AC, every surjection is split.

We recall the notion of a van Kampen colimit (sometimes also called descent for colimits) in a category  $\mathcal{C}$  with pullbacks. A colimit in  $\mathcal{C}$  is said to be *van Kampen* if it is preserved by the bifunctor  $\mathcal{C} \downarrow -$  from  $\mathcal{C}^{\text{op}}$  to categories whose functorial action is given by reindexing. If  $\mathcal{C}$  is cocomplete, this concretely means the following for a colimit  $\text{colim} A$ : given a natural transformation  $u: X \rightarrow A$  that is Cartesian, that is, whose naturality squares are pullbacks, a cocone under  $X$  with summit  $\bar{X}$  and a map  $\bar{X} \rightarrow \text{colim} A$  cohering with  $X \rightarrow A$ , the cocone with summit  $\bar{X}$  is colimiting if and only if the square from  $X_s \rightarrow A_s$  to  $\bar{X} \rightarrow \text{colim} A$  is a pullback for each index  $s$ . If pullback functors preserve colimits, such as in **Set**, then the reverse direction of this equivalence holds.

Certain classes of van Kampen colimits have special names. We call  $\mathcal{C}$ :

- *extensive* if coproducts exist and are van Kampen;
- *adhesive* if pushouts along monomorphisms exist and are van Kampen;

- $\omega$ -exhaustive [34] if sequences of monomorphisms (forming  $\omega$ -indexed diagrams in  $\mathcal{C}$ ) have colimits that are van Kampen.

The next lemma is well-known classically and essentially straightforward also constructively.

LEMMA 2.1.1 *The category of sets is*

- (i) *extensive*;
- (ii) *adhesive*;
- (iii)  $\omega$ -*exhaustive*.

*Proof.* For part (i), the claim is that, given functions  $X_i \rightarrow A_i$  for  $i \in I$ , each square

$$\begin{array}{ccc} X_i & \longrightarrow & \coprod_i X_i \\ \downarrow & & \downarrow \\ A_i & \longrightarrow & \coprod_i A_i \end{array}$$

is a pullback. This is immediate.

The remaining parts are consequences of effectivity of quotients of equivalence relations. For part (ii), by [15, Theorem A], it suffices to show that pushouts along monomorphisms are pullbacks. Consider a pushout square

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ f \downarrow & & \downarrow \\ B & \longrightarrow & P \end{array}$$

with  $f: A \hookrightarrow B$  a monomorphism. The pushout  $P$  is the quotient of the equivalence relation on  $B \sqcup C$  generated by identifying the images of  $f(a)$  and  $g(a)$  for  $a \in A$ . Given  $b \in B$  and  $c \in C$  mapping to the same element of  $P$ , effectivity of the quotient implies that  $b$  and  $c$  are related in  $B \sqcup C$  by the equivalence relation. A direct calculation using the fact that  $f$  is a monomorphism shows that there exists a unique  $a \in A$  such that  $f(a) = b$  and  $g(a) = c$ . This makes the square a pullback.

For part (iii), consider a sequence of monomorphisms  $(A_i \hookrightarrow A_{i+1})_{i \in \omega}$ . The colimit  $A_\omega$  is the quotient of the equivalence relation on  $\coprod_i A_i$  that identifies  $a_j \in A_j$  and  $a_k \in A_k$  if they coincide in  $A_{\max(j,k)}$ . Effectivity of the quotient implies that the coprojections  $A_i \rightarrow A_\omega$  are monomorphisms. Consider now another sequence  $(X_i \rightarrow X_{i+1})_{i \in \omega}$  with a Cartesian natural transformation  $f: X \rightarrow A$ . In particular, each map  $X_i \rightarrow X_{i+1}$  is a monomorphism, and so we may construct the colimit  $X_\omega$  as above. Given  $i \in \omega$ , the claim is that the induced square

$$\begin{array}{ccc} X_i & \hookrightarrow & X_\omega \\ f_i \downarrow & & \downarrow f_\omega \\ A_i & \hookrightarrow & A_\omega \end{array}$$

is a pullback. Given  $x_\omega \in X_\omega$  such that  $f_\omega(x_\omega)$  lifts to  $A_i$ , we have to show that  $x_\omega$  lifts to  $X_i$ . By construction of  $X_\omega$  as a quotient of  $\coprod_j X_j$ , there exists  $j \geq i$  such that  $x_\omega$  lifts to  $x_j \in X_j$ . Since the naturality square of  $f$  at  $i \rightarrow j$  is Cartesian,  $x_j$  lifts to  $X_i$ .  $\square$

**REMARK 2.1.2** Categorically, parts (ii) and (iii) of Lemma 2.1.1 say that a Barr-exact category is adhesive if it is finitely extensive and  $\omega$ -exhaustive if it is countably extensive. Although this seems folklore, we were unable to locate precise references. For adhesivity, see [21, Lemma A.2.4.3] for a related argument in the context of topos theory (using a subobject classifier). For exhaustivity, see [34] for a criterion (mirroring that of [15, Theorem A] for adhesivity) that we have essentially followed in our proof.

**REMARK 2.1.3** Although we liberally use colimits of sets in this paper, all appearing colimits can be built from countable coproducts. In particular, pushouts along monomorphisms and sequential colimits of monomorphisms will be considered only for *decidable inclusions*. This makes the development independent from the availability of quotients.

The results in the rest of this subsection depend only on the fact that the category of sets is extensive. The other van Kampen properties established by Lemma 2.1.1 will be used in Corollary 4.2.7.

**LEMMA 2.1.4** *In the category of sets:*

- (i) *decidable inclusions are monomorphisms;*
- (ii) *decidable inclusions are closed under pullback along arbitrary functions;*
- (iii) *decidable inclusions are closed under retracts.*

*Proof.* All these are consequences of extensivity. Parts (i) and (ii) follow from [8, Proposition 2.6]. Moreover, if  $i$  is a retract of a decidable inclusion  $j$ , then it is also a pullback of  $j$ , and thus part (iii) is a consequence of part (ii).  $\square$

**LEMMA 2.1.5** *In the category of sets, pullbacks commute with coproducts.*

*Proof.* Let  $A_i \rightarrow B_i \leftarrow C_i$  be a family of cospans indexed by  $i \in I$ . In the cube

$$\begin{array}{ccccc}
 A_i \times_{B_i} C_i & \longrightarrow & \coprod_i A_i \times_{\coprod_i B_i} \coprod_i C_i & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & C_i & \longrightarrow & \coprod_i C_i \\
 & & \downarrow & & \downarrow \\
 A_i & \longrightarrow & \coprod_i A_i & & \\
 \searrow & & \downarrow & \searrow & \\
 & & B_i & \longrightarrow & \coprod_i B_i
 \end{array}$$

the left and right faces are pullbacks by construction. The front face is a pullback by extensivity and thus so is the back one. Using the extensivity once more, we conclude that  $\coprod_i A_i \times \coprod_{i \in B_i} \coprod_i C_i$  coincides with  $\coprod_i (A_i \times_{B_i} C_i)$ .  $\square$

**PROPOSITION 2.1.6** *Decidable inclusions and split surjections form a weak factorization system. The weak factorization system is cofibrantly generated by  $\{\emptyset \rightarrow 1\}$ .*

*Proof.* Every map  $f: S \rightarrow T$  factors in an evident way as  $S \rightarrow S \sqcup T \rightarrow T$ , where the first map is a decidable inclusion by construction and the second one has a section given by the coproduct inclusion  $T \rightarrow S \sqcup T$ . The lifting properties are immediate.

For the claim on cofibrant generation, first note that  $\emptyset \rightarrow 1$  is a decidable inclusion. Also, a map  $X \rightarrow Y$  with the right lifting property with respect to  $\emptyset \rightarrow 1$  is a split surjection. Indeed, it has the right lifting property also with respect to  $\emptyset \rightarrow Y$  since that map is the coproduct  $\coprod_{y \in Y} \emptyset \rightarrow \coprod_{y \in Y} 1$ .  $\square$

Since decidable inclusions  $A \rightarrow B$  are monomorphisms by part (i) of Lemma 2.1.4, they can be seen as subobjects of  $B$ . Subobjects corresponding to decidable inclusions will be called *decidable subsets*. We now establish some closure properties of decidable subsets.

**LEMMA 2.1.7**

- (i) *Decidable subsets are closed under finite unions.*
- (ii) *Decidable inclusions are closed under finite limits, that is, if  $X \rightarrow Y$  is a natural transformation between finite diagrams of sets that is a levelwise decidable inclusion, then so is the induced function  $\lim X \rightarrow \lim Y$ .*

*Proof.* For part (i), let  $B$  be an arbitrary set. The nullary case is clear since  $\emptyset$  is a decidable subset of  $B$ . For the binary case, consider decidable subsets  $A_0$  and  $A_1$  with complements  $C_0$  and  $C_1$ . The intersection and the union of  $A_0$  and  $A_1$  are given by the pullback square on the left and the pushout square on the right:

$$\begin{array}{ccc} A_\emptyset & \longrightarrow & A_0 \\ \downarrow & \lrcorner & \downarrow \\ A_1 & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} A_\emptyset & \longrightarrow & A_0 \\ \downarrow & \lrcorner & \downarrow \\ A_1 & \longrightarrow & A_{01} \\ & \searrow & \nearrow \\ & & B \end{array}$$

By the extensivity of the category of sets, we have an isomorphism  $B \cong (A_0 \times_B A_1) \sqcup (A_0 \times_B C_1) \sqcup (C_0 \times_B A_1) \sqcup (C_0 \times_B C_1)$ , where the first summand is  $A_\emptyset$ . Call the other three  $C'_0$ ,  $C'_1$  and  $C'_{01}$ . Then the map  $A_{01} \rightarrow B$  is isomorphic to

$$A_\emptyset \sqcup C'_0 \sqcup C'_1 \rightarrow A_\emptyset \sqcup C'_0 \sqcup C'_1 \sqcup C'_{01},$$

which is a decidable inclusion. The case of general finite unions follows by induction.

For part (ii), the conclusion holds trivially for terminal objects and so it will be enough to verify it for pullbacks. (See [3, Proposition 5.21].) Consider  $X$  and  $Y$  as cospans indexed over  $0 \rightarrow 01 \leftarrow 1$  and assume that  $X_0 \rightarrow Y_0$ ,  $X_1 \rightarrow Y_1$  and  $X_{01} \rightarrow Y_{01}$  are decidable inclusions. First, we treat the case when both  $X_0 \rightarrow Y_0$  and  $X_1 \rightarrow Y_1$  are isomorphisms. In this case, the rows of the diagram

$$\begin{array}{ccccc}
 X_0 & \longrightarrow & Y_0 & \longleftarrow & \emptyset \\
 \downarrow & & \downarrow & & \downarrow \\
 X_{01} & \longrightarrow & Y_{01} & \longleftarrow & C_{01} \\
 \uparrow & & \uparrow & & \uparrow \\
 X_1 & \longrightarrow & Y_1 & \longleftarrow & \emptyset
 \end{array}$$

are coproduct diagrams. Since pullbacks commute with coproducts by Lemma 2.1.5, the induced diagram  $X_0 \times_{X_{01}} X_1 \rightarrow Y_0 \times_{Y_{01}} Y_1 \leftarrow \emptyset \times_{C_{01}} \emptyset$  is also a coproduct. The map on the left is therefore a decidable inclusion. Next, assume only that  $X_0 \rightarrow Y_0$  is an isomorphism. We pull back the coproduct diagram  $X_1 \rightarrow Y_1 \leftarrow C_1$  along  $Y_0 \rightarrow Y_{01}$ , which yields the coproduct diagram  $Y_0 \times_{Y_{01}} X_1 \rightarrow Y_0 \times_{Y_{01}} Y_1 \leftarrow Y_0 \times_{Y_{01}} C_1$ . By the preceding case,  $X_0 \times_{X_{01}} X_1 \rightarrow Y_0 \times_{Y_{01}} X_1$  is a decidable inclusion and thus the composite  $X_0 \times_{X_{01}} X_1 \rightarrow Y_0 \times_{Y_{01}} Y_1$  is a decidable inclusion. The general statement is reduced to this case in the same way.  $\square$

Another approach to proving part (ii) above reduces limits in the arrow category to limits in slice categories. Let  $I$  denote the indexing category; we only require that  $I$  has a finite set of objects. In the slice over  $\lim Y$ , the object  $\lim X$  is the limit of  $\lim Y \times_{Y_i} X_i$  over  $i \in I$ . The maps  $\lim Y \times_{Y_i} X_i \rightarrow \lim Y$  are decidable inclusions by part (ii) of Lemma 2.1.4, in particular monomorphisms by part (i) of Lemma 2.1.4. As subobjects of  $\lim Y$ , their limit is isomorphic to their intersection. But decidable subsets are closed under finite intersection by the dual version of part (i).

## 2.2. Simplicial sets and the weak factorization systems

We now move on to consider the category of simplicial sets and define the two weak factorization systems that will be part of the constructive Kan–Quillen model structure. For this, let us fix some notation and terminology. We write  $\Delta$  for the *category of simplices*, that is, of non-empty finite ordinals, written  $[n]$ , for  $n \in \mathbb{N}$ , and order-preserving maps, to which we refer as *simplicial operators*. This category is a Reedy category [19, Chapter 5]. Morphisms of its direct part  $\Delta_{\#}$  are called *face operators*, with generators denoted  $\delta_i: [n-1] \rightarrow [n]$  (omitting  $i$ ); morphisms of its inverse part  $\Delta_b$  are called *degeneracy operators*, with generators denoted  $\sigma_i: [n+1] \rightarrow [n]$  (identifying  $i$  and  $i+1$ ). For a simplicial operator  $\varphi$ , we write  $\varphi = \varphi^{\#} \varphi^b$  for its unique decomposition into a degeneracy operator followed by a face operator.

The category of simplicial sets  $\mathbf{sSet}$  is the category of presheaves over  $\Delta$ . We write  $\Delta[m]$  for the representable simplicial set represented by  $[m]$ . Our convention is that simplicial operators act on the right, that is, if  $X \in \mathbf{sSet}$ ,  $x \in X_n$  and  $\varphi: [m] \rightarrow [n]$ , then the image of  $x$  under the action of  $\varphi$  is denoted by  $x\varphi$ .



Being a presheaf category,  $\mathbf{sSet}$  admits all (small) limits and colimits and is locally Cartesian closed. For  $M \in \mathbf{sSet}$ , we write  $\mathbf{sSet} \downarrow M$  for the slice category over  $M$ . With a slight abuse of notation, we sometimes refer to an object  $p: X \rightarrow M$  of  $\mathbf{sSet} \downarrow M$  simply by its domain and call  $p$  its *structure map*. For a map  $f: M \rightarrow N$ , we write

$$\mathbf{sSet} \downarrow N \xrightarrow{f^*} \mathbf{sSet} \downarrow M \quad \mathbf{sSet} \downarrow M \xrightarrow{\Pi_f} \mathbf{sSet} \downarrow N$$

for the induced pullback functor and its right adjoint, to which we refer as the *dependent product* along  $f$ .

For the following statement, recall the discussion of van Kampen colimits in Subsection 2.1.

LEMMA 2.2.1 *The category of simplicial sets is extensive, adhesive and  $\omega$ -exhaustive.*

*Proof.* The category of sets satisfies these properties by Lemma 2.1.1. Thus so does the category of simplicial sets since monomorphisms, pullbacks and colimits of presheaves are determined pointwise.  $\square$

Denote the inclusions  $\{0\} \hookrightarrow \Delta[1]$  and  $\{1\} \hookrightarrow \Delta[1]$  by  $\iota_0$  and  $\iota_1$ , respectively. The simplex  $\Delta[1]$  will serve as an interval object, but we will occasionally use other intervals (such as  $J$  in the proof of Lemma 2.4.15) with endpoint inclusions also denoted by  $\iota_0$  and  $\iota_1$ . The *cylinder* on a simplicial set  $X$  is the product  $X \times \Delta[1]$ . Let  $f$  and  $g$  simplicial maps  $X \rightarrow Y$ . A *homotopy* from  $f$  to  $g$  is a simplicial map  $H: X \times \Delta[1] \rightarrow Y$  such that  $f = H(X \times \iota_0)$  and  $g = H(X \times \iota_1)$  (we will usually abbreviate these to  $H_{\iota_0}$  and  $H_{\iota_1}$ ). Simplicial maps  $f, g: X \rightarrow Y$  are *homotopic*, written  $f \sim g$ , if they can be connected by a zigzag of homotopies. The *constant* homotopy on  $f: X \rightarrow Y$  is the composite  $f\pi$  where  $\pi: X \times \Delta[1] \rightarrow X$  is the projection. If  $X$  and  $Y$  are simplicial sets over  $M$ , then a homotopy  $H$  as above is *fibrewise* (over  $M$ ) if it becomes constant when composed with the structure map  $Y \rightarrow M$ .

For  $m \geq 0$ , the *boundary inclusion*  $\partial\Delta[m] \hookrightarrow \Delta[m]$  corresponds to the sieve on  $[m]$  of all maps that lift through  $\delta_j: [m-1] \rightarrow [m]$  for some  $j$ . We write  $I$  for the set of boundary inclusions. We say that a map is a *trivial fibration* if it has the right lifting property with respect to  $I$  and that a map is a *cofibration* if it has the left lifting property with respect to trivial fibrations. A simplicial set  $X$  is *cofibrant* if the map  $\emptyset \rightarrow X$  is a cofibration.

For  $m > 0$  and  $0 \leq i \leq m$ , the *horn inclusion*  $\Lambda^i[m] \hookrightarrow \Delta[m]$  corresponds to the sieve on  $[m]$  of all maps that lift through  $\delta_j: [m-1] \rightarrow [m]$  for some  $j \neq i$ . We write  $J$  for the set of horn inclusions. We say that a map is a *Kan fibration* if it has the right lifting property with respect to  $J$  and that a map is a *trivial cofibration* if it has the left lifting property with respect to Kan fibrations. By definition, a simplicial set is a *Kan complex* if the map  $X \rightarrow \Delta[0]$  is a Kan fibration.

Given a set of maps  $K$ , a  $K$ -cell complex of height  $\alpha \leq \omega$  is an  $\alpha$ -composition of pushouts of coproducts of maps in  $K$ .

LEMMA 2.2.2 *Every trivial cofibration is a cofibration and every trivial fibration is a fibration.*

*Proof.* It suffices to verify that every horn inclusion is a cofibration. Indeed, a horn inclusion is an  $I$ -cell complex of height 2.  $\square$

In order to define the weak factorization systems, we adopt a slight variation of the well-known small object argument. For this, we need a few results about colimits of diagrams of (trivial) fibrations that will be needed also later on. In our constructive setting, these statements are somewhat more delicate than usual to prove. Recall that a fibration is a map together with choice of lifts (from the right) against horn inclusions. A *structure morphism* of fibrations from  $X \twoheadrightarrow Y$  to  $X' \twoheadrightarrow Y'$  is a commuting square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

such that for any lifting problem of a horn inclusion  $\Lambda^i[m] \hookrightarrow \Delta[m]$  against  $X \twoheadrightarrow Y$ , the following diagram of chosen lifts commutes:

$$\begin{array}{ccccc} \Lambda^i[m] & \longrightarrow & X & \longrightarrow & X' \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \Delta[m] & \longrightarrow & Y & \longrightarrow & Y' \end{array}$$

The notion of structure morphism of trivial fibrations is defined similarly. For the benefit of the readers, we remark that the use of these notions is confined to this subsection and to the derivation of the fibration extension property in Corollary 4.2.7.

#### LEMMA 2.2.3

- (i) Let  $p$  be a sequential diagram of (trivial) fibrations  $p_k: X_k \twoheadrightarrow Y_k$  such that the naturality squares of  $p$  are structure morphisms. Then  $\operatorname{colim} X \twoheadrightarrow \operatorname{colim} Y$  is a (trivial) fibration.
- (ii) Let  $p$  be a diagram of (trivial) fibrations  $p_k: X_k \twoheadrightarrow Y_k$  such that the naturality squares of  $p$  are structure morphisms and are pullbacks. Assume that the induced square from  $X_k \rightarrow Y_k$  to  $\operatorname{colim} X \rightarrow \operatorname{colim} Y$  is a pullback. Then  $\operatorname{colim} X \twoheadrightarrow \operatorname{colim} Y$  is a (trivial) fibration.

*Proof.* Part (i) is a formal consequence of the fact that  $\Lambda^i[m]$  and  $\Delta[m]$  are finite colimits of representables, hence finitely presented, that is, mapping out of them preserves filtered colimits, and that pullbacks commute with filtered colimits; we provide the details of the case of fibrations for the convenience of the readers. We wish to construct a section of the map

$$\mathbf{sSet}(\Delta[m], \operatorname{colim} X) \rightarrow \mathbf{sSet}(\Delta[m], \operatorname{colim} Y) \times_{\mathbf{sSet}(\Lambda^i[m], \operatorname{colim} Y)} \mathbf{sSet}(\Lambda^i[m], \operatorname{colim} X).$$

Since  $\Lambda^i[m]$  and  $\Delta[m]$  are finitely presented, this is

$$\operatorname{colim}_k \mathbf{sSet}(\Delta[m], X_k) \rightarrow (\operatorname{colim}_k \mathbf{sSet}(\Delta[m], Y_k)) \times_{\operatorname{colim}_k \mathbf{sSet}(\Lambda^i[m], Y_k)} (\operatorname{colim}_k \mathbf{sSet}(\Lambda^i[m], X_k)).$$

Since pullbacks commute with sequential colimits, this is

$$\operatorname{colim}_k \mathbf{sSet}(\Delta[m], X_k) \rightarrow \operatorname{colim}_k \mathbf{sSet}(\Delta[m], Y_k) \times_{\mathbf{sSet}(\Lambda^i[m], Y_k)} \mathbf{sSet}(\Lambda^i[m], X_k).$$

By functoriality of colimits, it thus suffices to have sections of

$$\mathbf{sSet}(\Delta[m], X_k) \rightarrow \mathbf{sSet}(\Delta[m], Y_k) \times_{\mathbf{sSet}(\Lambda^i[m], Y_k)} \mathbf{sSet}(\Lambda^i[m], X_k),$$

naturally in  $k$ . We have such a section for each  $k$  since  $X_k \rightarrow Y_k$  is a fibration and they are natural since the naturality squares of  $p$  are structure morphisms of fibrations.

Part (ii) is a formal consequence of the fact that mapping out of  $\Delta[m]$  preserves arbitrary colimits. Again, we do the case of fibrations in detail. We wish to construct a section of the map

$$\mathbf{sSet}(\Delta[m], \operatorname{colim} X) \rightarrow \mathbf{sSet}(\Delta[m], \operatorname{colim} Y) \times_{\mathbf{sSet}(\Lambda^i[m], \operatorname{colim} Y)} \mathbf{sSet}(\Lambda^i[m], \operatorname{colim} X).$$

Since mapping out of the representable  $\Delta[m]$  preserves colimits, this is

$$\operatorname{colim}_k \mathbf{sSet}(\Delta[m], X_k) \rightarrow (\operatorname{colim}_k \mathbf{sSet}(\Delta[m], Y_k)) \times_{\mathbf{sSet}(\Lambda^i[m], \operatorname{colim} Y)} \mathbf{sSet}(\Lambda^i[m], \operatorname{colim} X).$$

Since pullbacks preserve colimits, this is

$$\operatorname{colim}_k \mathbf{sSet}(\Delta[m], X_k) \rightarrow \operatorname{colim}_k \mathbf{sSet}(\Delta[m], Y_k) \times_{\mathbf{sSet}(\Lambda^i[m], \operatorname{colim} Y)} \mathbf{sSet}(\Lambda^i[m], \operatorname{colim} X).$$

By assumption,  $\operatorname{colim} X$  pulls back along  $Y_k \rightarrow \operatorname{colim} Y$  to  $X_k$ , so this is

$$\operatorname{colim}_k \mathbf{sSet}(\Delta[m], X_k) \rightarrow \operatorname{colim}_k \mathbf{sSet}(\Delta[m], Y_k) \times_{\mathbf{sSet}(\Lambda^i[m], Y_k)} \mathbf{sSet}(\Lambda^i[m], X_k).$$

From here, we conclude as in part (i). □

LEMMA 2.2.4 *In any commuting square*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array},$$

with  $X \rightarrow Y$  and  $X' \rightarrow Y'$  (trivial) fibrations and  $X \rightarrow X'$  and  $Y \rightarrow Y'$  levelwise decidable inclusions, there is a replacement for the choice of lifts of  $X' \rightarrow Y'$  such that the square forms a structure morphism.

*Proof.* We only discuss the case of fibrations. Since  $\Lambda^i[m]$  is a finitely presented, the inclusion  $\mathbf{sSet}(\Lambda^i[m], X) \rightarrow \mathbf{sSet}(\Lambda^i[m], X')$  is decidable by part (ii) of Lemma 2.1.7 and so is  $\mathbf{sSet}(\Delta[m], Y) \rightarrow \mathbf{sSet}(\Delta[m], Y')$ . If we now consider the set of lifting problems of horn inclusions against  $X' \rightarrow Y'$ , its subset consisting of those lifting problems that factor via the given square is a

decidable subset. Thus, given a lifting problem of a horn inclusion against  $X' \rightarrow Y'$ , we may perform a case distinction and pick the lift given by either the fibration  $X \rightarrow Y$  or the fibration  $X' \rightarrow Y'$ .  $\square$

**PROPOSITION 2.2.5** *Let  $p$  be an sequential diagram of (trivial) fibrations  $p_k: X_k \twoheadrightarrow Y_k$  where  $X_k \rightarrow X_{k+1}$  and  $Y_k \rightarrow Y_{k+1}$  are levelwise decidable inclusions for  $k \geq 0$ . Then  $\text{colim} X \rightarrow \text{colim} Y$  is a (trivial) fibration.*

*Proof.* Recursively in  $k$ , we apply Lemma 2.2.4 to the naturality square of  $p$  at  $k \rightarrow k+1$  to put a new choice of lifts on  $X_{k+1} \rightarrow Y_{k+1}$  such that this square becomes a structure morphism. Then the conclusion follows from part (i) of Lemma 2.2.3.  $\square$

**LEMMA 2.2.6** *Let  $p$  be a sequential diagram of objects  $p_k: X_k \rightarrow Y$  over  $Y$  where  $X_k \rightarrow X_{k+1}$  is levelwise decidable inclusion for  $k \geq 0$ . Assume that for each horn inclusion (boundary inclusion)  $A \rightarrow B$ , we have lifts for any lifting problem against  $X_{k+1} \rightarrow Y$  that factors through  $X_k \rightarrow X_{k+1}$ . Then  $\text{colim} X \rightarrow Y$  is a (trivial) fibration.*

*Proof.* We have a sequential diagram of triangles

$$\begin{array}{ccc} X_k & \xrightarrow{\quad} & X_{k+1} \\ & \searrow & \swarrow \\ & Y & \end{array} \quad (1)$$

each of which has *relative right lifts* against  $A \rightarrow B$ , meaning that for any lifting problem of  $A \rightarrow B$  against the left map, the induced lifting problem against the right map has a lift. Since  $A$  is finitely presented, the maps  $\mathbf{sSet}(A, X_k) \rightarrow \mathbf{sSet}(A, X_{k+1})$  are decidable inclusion. Thus, as in Lemma 2.2.4, we can choose relative right lifts at stage  $k+1$  that cohere with those at stage  $k$  whenever the relative lifting problem factors. Recursively in  $k$ , as in Proposition 2.2.5, we obtain a choice of relative right lifts for (1) that is natural in  $k$ . Taking the sequential colimit as in part (i) of Lemma 2.2.3, we obtain relative right lifts for

$$\begin{array}{ccc} \text{colim} X & \xrightarrow{\quad} & \text{colim} X, \\ & \searrow & \swarrow \\ & Y & \end{array}$$

that is, lifts of  $A \rightarrow B$  against  $\text{colim} X \rightarrow Y$ .  $\square$

For the next statement, recall the sets  $I$  of boundary inclusions and  $J$  of horn inclusions.

**PROPOSITION 2.2.7** (Weak factorization systems).

- (i) *Every map  $X \rightarrow Y$  factors functorially as a cofibration followed by a trivial fibration. Every cofibration is a codomain retract of a relative  $I$ -cell complex of height  $\omega$ .*

- (ii) Every map  $X \rightarrow Y$  factors functorially as a trivial cofibration followed by a fibration. Every trivial cofibration is a codomain retract of a relative  $J$ -cell complex of height  $\omega$ .

*Proof.* We prove this by a constructive version of the small object argument. We take  $X_0 = X$  and  $X_k \rightarrow X_{k+1}$  as the pushout of a coproduct of horn inclusions (boundary inclusions) indexed by lifting problems against  $X_k \rightarrow Y$ . Importantly, since horn inclusions (boundary inclusions) are levelwise decidable inclusions and these are closed under coproducts and pushout, so is  $X_k \rightarrow X_{k+1}$ . That means we can use Lemma 2.2.6 to conclude that  $X_\omega \rightarrow Y$  is a (trivial) fibration. The relative cell complex  $X \rightarrow X_\omega$  is a trivial cofibration (cofibration) by standard saturation properties of the left lifting closure. The claim about trivial cofibrations (cofibrations) follows from the retract argument.  $\square$

For any simplicial set  $M$ , the weak factorization systems of Proposition 2.2.7 induce weak factorization systems in the slice  $\mathbf{sSet} \downarrow M$ . A map over  $M$  will be called a (trivial) (co)fibration if it is a (trivial) (co)fibration in  $\mathbf{sSet}$ .

**REMARK 2.2.8** A map is *acyclic* if it is a weak equivalence in a context-dependent sense of *fibrewise homotopy equivalence* of Subsection 2.6 or *weak homotopy equivalence* of Subsections 3.1 and 4.3. In particular, an *acyclic (co)fibration* is a (co)fibration that is also a weak equivalence. We will be careful about the distinction between the notions of acyclic (co)fibrations and trivial (co)fibrations until they are proved equivalent. The notation introduced in Table 1 is intended to help readers keep track of the different notions.

We conclude this subsection with two short but critical lemmas.

**LEMMA 2.2.9** Both  $\Delta[m]$  and  $\partial\Delta[m]$  are cofibrant for all  $m$ . In fact,  $\partial\Delta[m]$  is a cell complex with respect to  $\{\partial\Delta[k] \rightarrow \Delta[k] \mid k < m\}$ .

*Proof.* For  $\partial\Delta[m]$  consider a filtration  $X^{(-1)} \hookrightarrow X^{(0)} \hookrightarrow \dots \hookrightarrow X^{(m-1)}$  where

$$X_i^{(k)} = \{\varphi: [i] \rightarrow [m] \mid \varphi \text{ factors through } [k]\}.$$

Then we have  $X^{(-1)} = \emptyset$ ,  $X^{(m-1)} = \partial\Delta[m]$  and for each  $k \in \{0, \dots, m-1\}$  there is a pushout square

$$\begin{array}{ccc} \Delta_{\#}([k], [m]) \times \partial\Delta[k] & \longrightarrow & X^{(k-1)} \\ \downarrow & & \downarrow \\ \Delta_{\#}([k], [m]) \times \Delta[k] & \longrightarrow & X^{(k)} \end{array}$$

so  $\partial\Delta[m]$  is indeed a cell complex with respect to  $\{\partial\Delta[k] \rightarrow \Delta[k] \mid k < m\}$ . It follows that  $\Delta[m]$  is cofibrant as well.  $\square$

As a consequence we derive a cancellation property for trivial fibrations, which is crucially used in later arguments to extend certain results about cofibrant simplicial sets to all simplicial sets.

LEMMA 2.2.10 *If  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are simplicial maps such that  $f$  and  $gf$  are trivial fibrations, then so is  $g$ .*

*Proof.* Consider a lifting problem

$$\begin{array}{ccc} \partial \Delta[m] & \xrightarrow{u} & Y \\ i \downarrow & & \downarrow g \\ \Delta[m] & \xrightarrow{v} & Z. \end{array}$$

Since  $\partial \Delta[m]$  is cofibrant by Lemma 2.2.9,  $u$  lifts along  $f$ :

$$\begin{array}{ccc} & & X \\ & \nearrow \tilde{u} & \downarrow f \\ \partial \Delta[m] & \xrightarrow{u} & Y. \end{array}$$

This leads to a lifting problem

$$\begin{array}{ccc} \partial \Delta[m] & \xrightarrow{\tilde{u}} & X \\ i \downarrow & & \downarrow gf \\ \Delta[m] & \xrightarrow{v} & Z. \end{array}$$

Call its solution  $\tilde{w}: \Delta[m] \rightarrow X$ , then  $w = f\tilde{w}$  is a solution to the original problem. Indeed,

$$gw = gf\tilde{w} = v \text{ and } wi = f\tilde{w}i = f\tilde{u} = u.$$

□

### 2.3. Pushout product properties

We now establish that the weak factorization systems of cofibrations and trivial fibrations and of trivial cofibrations and fibrations satisfy various forms of the pushout product property. In order to do this, we need to introduce some notation. Given  $M \in \mathbf{sSet}$  and two objects  $X \rightarrow M$  and  $Y \rightarrow M$  in the slice category  $\mathbf{sSet} \downarrow M$ , we write  $\exp_M(X, Y) \rightarrow M$  for their exponential in  $\mathbf{sSet} \downarrow M$ . The category  $\mathbf{sSet} \downarrow M$  is also a  $\mathbf{sSet}$ -enriched category in a canonical way and we write  $\mathrm{hom}_M(X, Y)$  for the simplicial hom-object. When considering slices over the terminal object  $\Delta[0]$  of  $\mathbf{sSet}$ , we often drop the subscript and write  $Y^X$  for the common value of  $\exp_{\Delta[0]}(X, Y)$  and  $\mathrm{hom}_{\Delta[0]}(X, Y)$ . As a  $\mathbf{sSet}$ -enriched category,  $\mathbf{sSet} \downarrow M$  admits tensors and cotensors. For  $A \in \mathbf{sSet}$  and  $X \in \mathbf{sSet} \downarrow M$ , the tensor is written  $A \odot X \in \mathbf{sSet} \downarrow M$  and is given by the Cartesian product  $A \times X$  (the structure map is the composite of the product projection and the structure map of  $X$ ). The cotensor is written

$A \pitchfork X \in \mathbf{sSet} \downarrow M$  and is given by the pullback of  $X^A \rightarrow M^A$  along the map  $M \rightarrow M^A$  (the adjoint transpose of the product projection). For maps  $i: A \rightarrow B$  and  $p: X \rightarrow Y$ , we write

$$(A \times Y) \sqcup_{A \times X} (B \times X) \xrightarrow{i \widehat{\times} p} B \times Y \qquad X^B \xrightarrow{\widehat{\exp}(f, g)} X^A \times_{Y^A} Y^B$$

for their pushout product and their pullback exponential, respectively. Analogous notation is used when we consider the tensor and cotensor functors instead of the product and exponential functors.

The following proposition is well-known and easy to prove in classical logic. However, it is less trivial constructively due to the delicate nature of cofibrations and thus we provide a complete proof.

**PROPOSITION 2.3.1** *Let  $M \in \mathbf{sSet}$ .*

- (i) *If  $i: A \rightarrow B$  and  $j: C \rightarrow D$  are cofibrations in  $\mathbf{sSet} \downarrow M$ , then so is their pushout product  $i \widehat{\times}_M j$ .*
- (ii) *If  $i: A \rightarrow B$  is a cofibration and  $p: X \rightarrow Y$  is a trivial fibration in  $\mathbf{sSet} \downarrow M$ , then their pullback exponential  $\widehat{\exp}_M(i, p)$  is also a trivial fibration. In particular, if  $A$  is cofibrant and  $p: X \rightarrow Y$  is a trivial fibration, then  $\exp_M(A, p)$  is a trivial fibration.*

*Proof.* For part (i), it suffices to verify that for any pair of simplices  $\Delta[m] \rightarrow M$  and  $\Delta[n] \rightarrow M$  the pushout product of  $\partial\Delta[m] \rightarrow \Delta[m]$  and  $\partial\Delta[n] \rightarrow \Delta[n]$  is a cofibration. Indeed, there is a filtration

$$\partial\Delta[m] \times_M \Delta[n] \cup \Delta[m] \times_M \partial\Delta[n] = X^{(-1)} \hookrightarrow X^{(0)} \hookrightarrow \dots \hookrightarrow X^{(m+n)} = \Delta[m] \times_M \Delta[n]$$

where  $X^{(k)}$  is formed iteratively by taking pushouts

$$\begin{array}{ccc} S_k \times \partial\Delta[k] & \longrightarrow & X^{(k-1)} \\ \downarrow & \lrcorner & \downarrow \\ S_k \times \Delta[k] & \longrightarrow & X^{(k)} \end{array}$$

with

$$S_k = \{(\varphi, \psi) \in \Delta[m] \times_M \Delta[n] \mid \text{at least one of } \varphi: [k] \rightarrow [m] \text{ or } \psi: [k] \rightarrow [n] \text{ is surjective} \\ \text{and } (\varphi, \psi): [k] \rightarrow [m] \times [n] \text{ is injective.}\}$$

Part (ii) follows from part (i) by adjointness.  $\square$

**COROLLARY 2.3.2** *If  $A \rightarrow B$  is a simplicial map with  $A$  cofibrant, then the induced pullback functor  $\mathbf{sSet} \downarrow B \rightarrow \mathbf{sSet} \downarrow A$  preserves cofibrations.*

*Proof.* If  $X \rightarrow Y$  is a cofibration over  $B$ , then the induced map  $A \times_B X \rightarrow A \times_B Y$  coincides with the pushout product (in  $\mathbf{sSet} \downarrow B$ ) of  $\emptyset \rightarrow A$  and  $X \rightarrow Y$ . The conclusion follows from part (i) of Proposition 2.3.1.  $\square$

## PROPOSITION 2.3.3

- (i) If  $i: A \rightarrowtail B$  is a trivial cofibration and  $j: C \rightarrowtail D$  is a cofibration then their pushout product  $i \hat{\times} j$  is a trivial cofibration.
- (ii) If  $i: A \rightarrowtail B$  is a cofibration and  $p: X \rightarrowtail Y$  is a Kan fibration, then their pullback exponential  $\widehat{\text{hom}}(i, p)$  is a Kan fibration. In particular, if  $i: C \rightarrowtail D$  is a cofibration and  $K$  a Kan complex, then  $\widehat{\text{hom}}(i, K)$  is a Kan fibration. In particular, if  $A$  is cofibrant and  $K$  is a Kan complex, then  $K^A$  is a Kan complex.
- (iii) If  $i: A \rightarrowtail B$  is a trivial cofibration and  $p: X \rightarrowtail Y$  is a Kan fibration, then their pullback exponential  $\widehat{\text{hom}}(i, p)$  is a trivial fibration. In particular, if  $i: A \rightarrowtail B$  is a trivial cofibration and  $K$  a Kan complex, then  $\widehat{\text{hom}}(i, K)$  is a trivial fibration.

*Proof.* Part (i) is proved in [11, Proposition IV.2.2] with a constructive argument. Parts (ii) and (iii) follow by adjointness.  $\square$

## COROLLARY 2.3.4

- (i) If  $j: A \rightarrowtail B$  is a cofibration and  $i: C \rightarrowtail D$  is a trivial cofibration over  $M$ , then their pushout tensor  $j \hat{\otimes} i$  is a trivial cofibration.
- (ii) If  $i: A \rightarrowtail B$  is a trivial cofibration and  $j: C \rightarrowtail D$  is a cofibration over  $M$ , then their pushout tensor  $i \hat{\otimes} j$  is a trivial cofibration.
- (iii) If  $i: A \rightarrowtail B$  is a cofibration and  $p: X \rightarrowtail Y$  is a Kan fibration over  $M$ , then their pullback cotensor  $i \hat{\pitchfork} p$  is a Kan fibration.
- (iv) If  $j: A \rightarrowtail B$  is a trivial cofibration and  $p: X \rightarrowtail Y$  is a Kan fibration over  $M$ , then their pullback cotensor  $j \hat{\pitchfork} p$  is a trivial fibration.

*Proof.* Parts (i) and (ii) follow from Part (i) of Proposition 2.3.3 since the underlying map of the pushout tensor is the ordinary pushout product. Parts (iii) and (iv) follow from these by adjointness.  $\square$

## 2.4. Cofibrations as Reedy decidable inclusions

The aim of this subsection is to exhibit the weak factorization system of cofibrations and trivial fibrations on simplicial sets of Proposition 2.2.7 as the Reedy weak factorization system induced by the weak factorization system of decidable inclusions and split surjections on sets and use this fact to give streamlined proofs of several closure properties of cofibrations and cofibrant objects. For background on Reedy weak factorization systems, and in particular the definition of latching and matching objects, we refer to [19, Chapter 5] and [32].

A simplicial map  $A \rightarrow B$  is a *Reedy decidable inclusion* if for all  $m$  the relative latching map  $A_m \sqcup_{L_m A} L_m B \rightarrow B_m$  is a decidable inclusion. Dually, a simplicial map  $X \rightarrow Y$  is a *Reedy split surjection* if all its relative matching maps  $X_m \rightarrow Y_m \times_{M_m Y} M_m X$  are split surjections. We define Reedy decidable inclusions of cosimplicial sets similarly.

**LEMMA 2.4.1** *Reedy decidable inclusions coincide with cofibrations, and Reedy split surjections coincide with trivial fibrations.*



*Proof.* This follows from Proposition 2.1.6 and the fact that the associated Reedy weak factorization system of a cofibrantly generated weak factorization system is also cofibrantly generated with generators given by pushout products of the original generators and boundary inclusions of representable functors. See [32, Lemmas 7.3 and 7.4 and Corollary 6.7] for details.  $\square$

LEMMA 2.4.2 *Let  $X$  be a simplicial set.*

- (i) *Every degeneracy operator  $[m] \twoheadrightarrow [n]$  acts by a monomorphism  $X_n \rightarrow X_m$ .*
- (ii) *The latching map  $L_m X \rightarrow X_m$  is a monomorphism. Moreover, as a subset of  $X_m$ ,  $L_m X$  is the union of the subsets  $X_n$  indexed over non-identity degeneracy operators  $[m] \twoheadrightarrow [n]$ .*

*Let  $A \rightarrow B$  be a monomorphism of simplicial sets.*

- (iii) *For every degeneracy operator  $[m] \twoheadrightarrow [n]$ , the set  $A_n$  is the intersection of the subobjects  $A_m$  and  $B_n$  of  $B_m$ .*
- (iv)  *$L_m A$  is the intersection of the subobjects  $A_m$  and  $L_m B$  of  $B_m$ .*

*Proof.* Part (i) holds since  $[m] \twoheadrightarrow [n]$  has a section.

For part (ii), we use the fact that every span of degeneracy operators in  $\Delta$  has an absolute pushout<sup>1</sup> (that is, one that is preserved by all functors) which is proven in [24, Theorem 1.2.1]. It follows that each latching diagram  $(\partial([m] \downarrow \Delta_b))^{\text{op}} \rightarrow \mathbf{Set}$  induced by  $X$  sends pushouts in  $\partial([m] \downarrow \Delta_b)$  to pullbacks, and hence the family of subsets  $X_n \rightarrow X_m$  (as  $[m] \twoheadrightarrow [n]$  varies over non-identity degeneracy operators with source  $[m]$ ) is closed under intersection. Thus, its union coincides with its colimit  $L_m X$  in the slice over  $X_m$ . In particular,  $L_m X \rightarrow X_m$  is a monomorphism.

For part (iii), we need to show that the square

$$\begin{array}{ccc} A_n & \longrightarrow & B_n \\ \downarrow & & \downarrow \\ A_m & \longrightarrow & B_m \end{array}$$

is a pullback. Indeed, let  $X$  be a set and  $u: X \rightarrow A_m$  and  $v: X \rightarrow B_n$  be maps such that  $v\sigma = i_m u$ . Let  $\delta: [n] \rightarrow [m]$  be a section of  $\sigma$ . Then, by a direct calculation,  $w = u\delta: X \rightarrow A_n$  is the unique map such that  $w\sigma = u$  and  $i_n w = v$ .

Part (iv) follows from parts (ii) and (iii) using the fact that intersecting with  $A_m$  preserves unions.  $\square$

Lemma 2.4.2 implies that Lemma 2.4.1 is a special case of [17, Proposition 5.1.4], there proved without explicit reference to Reedy weak factorization systems.

Given a simplicial set  $A$  and a cosimplicial set  $W$ , the set  $A \times_{\Delta} W$  is defined as the coend  $\int^k A_k \times W_k$ . This set is also known as the *weighted colimit* of  $A$  with weight  $W$ . While the following two statements do not depend on the theory of weighted colimits and their pushout constructions, readers familiar with it may find it useful to think of them in these terms (cf. [32]).

<sup>1</sup>This means that  $\Delta$  is an *elegant Reedy category* in the sense of [5].

**LEMMA 2.4.3** *The bifunctor  $A, W \mapsto A \times_{\Delta} W$  satisfies the pushout product property with respect to Reedy decidable inclusions, that is, if  $A \rightarrow B$  and  $V \rightarrow W$  are Reedy decidable inclusions of simplicial and cosimplicial sets, respectively, then  $A \times_{\Delta} W \sqcup_{A \times_{\Delta} V} B \times_{\Delta} V \rightarrow B \times_{\Delta} W$  is a decidable inclusion.*

*Proof.* It suffices to verify this property on generators. ([32, Corollary 6.7] implies that cosimplicial Reedy decidable inclusions are generated by  $\{L_m \Delta([m], -) \rightarrow \Delta([m], -) \mid m \in \mathbb{N}\}$ .) Applying the construction to  $\partial \Delta[n] \rightarrow \Delta[n]$  and  $L_m \Delta([m], -) \rightarrow \Delta([m], -)$  yields the inclusion of  $\{\varphi \in \Delta([m], [n]) \mid \varphi \neq \text{id}\}$  into  $\Delta([m], [n])$ , which is decidable.  $\square$

We will now prove a useful characterization of cofibrations.

**PROPOSITION 2.4.4** *The following are equivalent for a simplicial map  $A \rightarrow B$ :*

- (i)  $A \rightarrow B$  is a cofibration,
- (ii)  $A_0 \rightarrow B_0$  is a decidable inclusion and, for each generating degeneracy operator  $[n+1] \rightarrow [n]$ , the induced map  $A_{n+1} \sqcup_{A_n} B_n \rightarrow B_{n+1}$  is a decidable inclusion.
- (iii)  $A \rightarrow B$  is a levelwise decidable inclusion and, for each degeneracy operator  $[m] \rightarrow [n]$ , the induced map  $A_m \sqcup_{A_n} B_n \rightarrow B_m$  is a decidable inclusion.

*Proof.* We go from (i) to (ii). The Reedy condition in dimension 0 means that  $A_0 \rightarrow B_0$  is a decidable inclusion. Next, take a degeneracy operator  $\sigma: [m] \rightarrow [n]$ . We will check that the induced map of cosimplicial sets  $\Delta([n], -) \rightarrow \Delta([m], -)$  is a Reedy decidable inclusion. Indeed, for each  $k$ , the map  $\Delta([n], [k]) \rightarrow \Delta([m], [k])$  is (isomorphic to) the inclusion of the subset of those simplicial operators  $[m] \rightarrow [k]$  that factor through  $\sigma$ , which is decidable. For each  $a$ , note that  $L_k \Delta([a], -) \rightarrow \Delta([a], [k])$  is the (decidable) subset of non-surjective operators. Thus,  $L_k \Delta([n], -)$  is the intersection over  $\Delta([m], [k])$  of  $L_k \Delta([m], -)$  and  $\Delta([n], [k])$ . It follows by part (i) of Lemma 2.1.7 that  $\Delta([n], [k]) \sqcup_{L_k \Delta([n], -)} L_k \Delta([m], -) \rightarrow \Delta([m], [k])$  is a decidable inclusion. The map  $A_m \sqcup_{A_n} B_n \rightarrow B_m$  coincides with  $A \times_{\Delta} \Delta([m], -) \sqcup_{A \times_{\Delta} \Delta([n], -)} B \times_{\Delta} \Delta([n], -) \rightarrow B \times_{\Delta} \Delta([m], -)$  induced by  $A \rightarrow B$  and  $\Delta([n], -) \rightarrow \Delta([m], -)$ , so it is a decidable inclusion by Lemma 2.4.3.

We go from (ii) to (iii). Each degeneracy operator  $[m] \rightarrow [n]$  is a finite composition of generators. Thus,  $A_m \sqcup_{A_n} B_n \rightarrow B_m$  is a finite composition of pushouts of maps  $A_{k+1} \sqcup_{A_k} B_k \rightarrow B_{k+1}$  for generators  $\sigma_i: [k+1] \rightarrow [k]$  and hence is a decidable inclusion. For each  $m$ , the map  $A_m \rightarrow B_m$  factors as  $A_m \rightarrow A_m \sqcup_{A_0} B_0 \rightarrow B_m$  (where the first map is a pushout of  $A_0 \rightarrow B_0$ ) using the degeneracy operator  $[m] \rightarrow [0]$  and hence is a decidable inclusion.

We go from (iii) to (i). Given  $m$  and working over  $B_m$ , we have to show that  $A_m \sqcup_{L_m A} L_m B$  is a decidable subobject. Recall from part (ii) of Lemma 2.4.2 that  $L_m B$  is the finite union of  $B_n$  indexed over non-identity degeneracy operators  $[m] \rightarrow [n]$ . By part (iii) of Lemma 2.4.2,  $A_m \sqcup_{A_n} B_n$  is the union of the subobjects  $A_m$  and  $B_n$ . Similarly, by part (iv) of Lemma 2.4.2,  $A_m \sqcup_{L_m A} L_m B$  is the union of the subobjects  $A_m$  and  $L_m B$ . Distributing unions, it follows that  $A_m \sqcup_{L_m A} L_m B$  is the union of the decidable subobject  $A_m$  with the finite union of decidable subobjects  $A_m \sqcup_{A_n} B_n$  and hence is a decidable subobject by part (i) of Lemma 2.1.4.  $\square$

**COROLLARY 2.4.5** *A simplicial set is cofibrant if and only if all (generating) degeneracy operators act on it by decidable inclusions.*

*Proof.* This is Proposition 2.4.4 for a map with initial domain.  $\square$

## COROLLARY 2.4.6

- (i) *Every fibration is a levelwise decidable inclusion.*
- (ii) *A levelwise decidable inclusion with cofibrant codomain is a cofibration.*

*In particular, a map between cofibrant objects is a cofibration if and only if it is a levelwise decidable inclusion.*

*Proof.* Part (i) was shown in Proposition 2.4.4. For part (ii), let  $A \rightarrow B$  be a levelwise decidable inclusion and let  $B$  be cofibrant. Then, for any degeneracy operator  $[m] \rightarrow [n]$ , the map  $B_n \rightarrow B_m$  is a decidable inclusion by Corollary 2.4.5. Thus part (i) of Lemma 2.1.7 implies that  $A_m \sqcup_{A_n} B_n$  is a decidable subobject of  $B_m$  as the union of decidable subobjects  $B_n$  and  $A_m$ . The conclusion follows from Corollary 2.4.5.  $\square$

Next, we obtain some closure properties of cofibrations and cofibrant objects.

LEMMA 2.4.7 *Cofibrant objects are closed under finite limits.*

*Proof.* This follows from part (ii) of Lemma 2.1.7 and Corollary 2.4.5.  $\square$

LEMMA 2.4.8 *If  $A \hookrightarrow B$  is a monomorphism and  $B$  is cofibrant, then so is  $A$ .*

*Proof.* Let  $[m] \rightarrow [n]$  be a degeneracy operator. The map  $B_n \rightarrow B_m$  is a decidable inclusion by Corollary 2.4.5. The map  $A_n \rightarrow A_m$  is a pullback of  $B_n \rightarrow B_m$  by part (iii) of Lemma 2.4.2 and hence is a decidable inclusion by part (ii) of Lemma 2.1.4. Using Corollary 2.4.5 again, we conclude that  $A$  is cofibrant.  $\square$

The above statement is a special case of the closure of cofibrations under pullback along monomorphisms. However, we will not need that more general fact.

REMARK 2.4.9 The notion of cofibration and the associated statements in this subsection up to Lemma 2.4.8 generalize, with the same arguments, to presheaves over an elegant Reedy category (of countable height) with decidable identities (as defined below), finite slices of face operators and finite coslices of degeneracy operators. In Subsection 3.3, we will need the case of bisimplicial sets, where a cofibration is defined as a Reedy decidable inclusion of presheaves over the elegant Reedy category  $\Delta \times \Delta$ . There, we will use bisimplicial versions of the current simplicial statements. In particular, a bisimplicial set is cofibrant if and only if all degeneracy operators act on it by decidable inclusions. (For this, it suffices to separately examine the two cases of a degeneracy operator that is an identity in one of the two directions.)

COROLLARY 2.4.10 *Every horn  $\Lambda^i[m]$  is cofibrant.*

*Proof.* The horn inclusion  $\Lambda^i[m] \rightarrow \Delta[m]$  is a cofibration by Lemma 2.2.2 and thus the horn  $\Lambda^i[m]$  is cofibrant by Lemma 2.4.8.  $\square$

As a consequence of the characterization of cofibrant objects of Corollary 2.4.5 we can distinguish a class of categories with cofibrant nerves, which will be useful in Section 3. Let us say that a category

$J$  has *decidable identities* if the function  $\text{ob } J \rightarrow \text{mor } J$  that sends each object to its identity morphism is a decidable inclusion. A functor  $I \rightarrow J$  is a *decidable inclusion* if both functions  $\text{ob } I \rightarrow \text{ob } J$  and  $\text{mor } I \rightarrow \text{mor } J$  are decidable inclusions. For example, the category  $\Delta$  and its wide subcategories  $\Delta_b$  and  $\Delta_\#$  have decidable identities.

LEMMA 2.4.11

- (i) *If  $J$  is a category with decidable identities, then  $\text{NJ}$  is cofibrant.*
- (ii) *If  $I \rightarrow J$  is a decidable inclusion, then  $\text{NI} \rightarrow \text{NJ}$  is a levelwise decidable inclusion. In particular, if  $I$  and  $J$  have moreover decidable identities, then  $\text{NI} \rightarrow \text{NJ}$  is a cofibration.*

*Proof.* For part (i), for each  $m$  the set  $(\text{NJ})_m$  is the iterated  $m$ -fold pullback  $\text{mor } J \times_{\text{ob } J} \dots \times_{\text{ob } J} \text{mor } J$ . By Corollary 2.4.5, it will be enough to check that each generating degeneracy operator  $\sigma_i : [m+1] \rightarrow [m]$  acts by a decidable inclusion. The set  $(\text{NJ})_{m+1}$  is the iterated  $(m+1)$ -fold pullback  $\text{mor } J \times_{\text{ob } J} \dots \times_{\text{ob } J} \text{mor } J$ . The set  $(\text{NJ})_m$  can be seen as an analogous  $(m+1)$ -fold pullback with the  $i$ -th occurrence of  $\text{mor } J$  replaced by  $\text{ob } J$ . Then the action of  $\sigma_i$  is induced by a transformation of these limit diagrams that consists of identities and the map  $\text{ob } J \rightarrow \text{mor } J$ . The conclusion follows by part (ii) of Lemma 2.1.7.

Part (ii) is a direct consequence of part (ii) of Lemma 2.1.7, using the pullback presentation of  $\text{NI}$  and  $\text{NJ}$  as in the proof of part (i).  $\square$

Given a presheaf  $F$  on a category  $\mathcal{C}$ , we write  $\mathcal{C} \downarrow F$  for the category of elements of  $F$ .

LEMMA 2.4.12 *Let  $F$  be a presheaf on a category  $\mathcal{C}$ . If  $\mathcal{C}$  has decidable identities, then so does  $\mathcal{C} \downarrow F$ .*

*Proof.* We form the below diagram, in which the right square is a pullback:

$$\begin{array}{ccccc}
 \text{ob}(\mathcal{C} \downarrow F) & \xrightarrow{\text{id}} & \text{mor}(\mathcal{C} \downarrow F) & \xrightarrow{\text{cod}} & \text{ob}(\mathcal{C} \downarrow F) \\
 \downarrow & & \downarrow & \lrcorner & \downarrow \\
 \text{ob } \mathcal{C} & \xrightarrow{\text{id}} & \text{mor } \mathcal{C} & \xrightarrow{\text{cod}} & \text{ob } \mathcal{C}
 \end{array}$$

By pullback pasting, the left square is also a pullback. Since the identity structure map of  $\mathcal{C} \downarrow F$  is a pullback of the identity structure map of  $\mathcal{C}$ , it is a decidable inclusion by part (ii) of Lemma 2.1.4.  $\square$

For the reader familiar with discrete Grothendieck fibrations, we note that the above lemma has a natural phrasing in terms of a discrete Grothendieck fibration  $\mathcal{E} \rightarrow \mathcal{C}$ .

In order to prove closure of cofibrant objects under certain exponentials, we need some preliminary lemmas. A simplicial map  $A \rightarrow B$  is called a *generalised degeneracy* if for every cofibrant simplicial set  $X$  the induced map  $\mathbf{sSet}(B, X) \rightarrow \mathbf{sSet}(A, X)$  is a decidable inclusion.

LEMMA 2.4.13 *The class of generalized degeneracies contains the degeneracy operators and is closed under composition, finite colimits, pushouts along arbitrary maps and retracts.*

*Proof.* Degeneracy operators are generalized degeneracies by Corollary 2.4.5. The closure properties hold since the class of decidable inclusions satisfies the dual closure properties by Lemmas 2.1.4 and 2.1.7.  $\square$

LEMMA 2.4.14 *If a simplicial set  $A$  is a finite colimit of representables, then the product projection  $A \times \Delta[1] \rightarrow A$  is a generalized degeneracy.*

*Proof.* Since  $A$  is a finite colimit of representables, the claim reduces to the case  $A = \Delta[m]$  by Lemma 2.4.13. The projection  $\Delta[m] \times \Delta[1] \rightarrow \Delta[m]$  is the row-wise colimit of the diagram

$$\begin{array}{ccccccc} \Delta[m+1] & \xleftarrow{\delta_1} & \Delta[m] & \xrightarrow{\delta_1} & \Delta[m+1] & \xleftarrow{\dots} & \Delta[m+1] \xleftarrow{\delta_m} \Delta[m] \xrightarrow{\delta_m} \Delta[m+1] \\ \sigma_0 \downarrow & & \downarrow & & \sigma_1 \downarrow & & \downarrow \sigma_{m-1} \quad \downarrow \quad \downarrow \sigma_m \\ \Delta[m] & \xleftarrow{\quad} & \Delta[m] & \xrightarrow{\quad} & \Delta[m] & \xleftarrow{\quad} & \Delta[m] \xrightarrow{\quad} \Delta[m] \end{array}$$

(where all unlabelled morphisms are identities), that is, it is a finite colimit of degeneracy operators and identity maps. Hence the conclusion follows from Lemma 2.4.13.  $\square$

A *deformation section* of a map  $f: A \rightarrow B$  is a map  $s: B \rightarrow A$  such that  $fs = \text{id}_B$  and there is a zigzag of fibrewise homotopies from  $sf$  to  $\text{id}_A$  over  $A$ . A map that admits a deformation section is called *shrinkable*. Note that shrinkability is an entirely fibrewise notion. It follows that shrinkable maps are closed under pullback.

LEMMA 2.4.15 *A shrinkable map between simplicial sets that are finite colimits of representables is a generalized degeneracy.*

*Proof.* Let  $f: A \rightarrow B$  be a shrinkable map with a deformation section  $s$ . A zigzag of homotopies connecting  $sf$  to  $\text{id}_A$  can be represented as a map  $H: A \times J \rightarrow A$  where  $J$  is its indexing zigzag, that is, a colimit of a finite number of copies of  $\Delta[1]$  attached to each other at their endpoints in the directions matching the directions of the homotopies. The product projection  $A \times J \rightarrow A$  is a finite colimit of projections  $A \times \Delta[1] \rightarrow A$  (and identity maps) and thus it is a generalized degeneracy by Lemmas 2.4.13 and 2.4.14. In the diagram

$$\begin{array}{ccccc} A & \longrightarrow & A \times J & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & B \sqcup_A (A \times J) & \longrightarrow & B, \end{array}$$

the left square is a pushout by construction and so is the outer rectangle since the horizontal composites are identities. Thus the right square is also a pushout. Moreover,  $H$  exhibits  $f$  a retract of  $B \sqcup_A (A \times J) \rightarrow B$  and thus Lemma 2.4.13 completes the proof.  $\square$

LEMMA 2.4.16 *Any degeneracy operator  $\sigma: \Delta[m] \rightarrow \Delta[n]$  is shrinkable.*

*Proof.* Let  $\delta: \Delta[n] \rightarrow \Delta[m]$  be the minimal section of  $\sigma$ . Then we have  $\delta\sigma(i) \leq i$  for each  $i \in [m]$ , which induces a fibrewise homotopy from  $\delta\sigma$  to  $\text{id}_{\Delta[m]}$ .  $\square$

**COROLLARY 2.4.17** *Let  $A \rightarrow B$  be a pullback of a degeneracy operator  $\Delta[m] \rightarrow \Delta[n]$  with  $B$  a finite colimit of representables. Then  $A \rightarrow B$  is a generalized degeneracy.*

*Proof.* Simplicial sets that are finite colimits of representables are closed under finite limits, in particular pullback. It follows that also  $A$  is a finite colimit of representables. Since  $\Delta[m] \rightarrow \Delta[n]$  is shrinkable by Lemma 2.4.16, so is its pullback  $A \rightarrow B$ . Then  $A \rightarrow B$  is a generalized degeneracy by Lemma 2.4.15.  $\square$

**COROLLARY 2.4.18** *Let  $A$  be a finite colimit of simplices. Exponentiation with  $A$  preserves cofibrant simplicial sets.*

*Proof.* Let  $X$  be a cofibrant simplicial set. To show that  $X^A$  is cofibrant, it suffices to show for every degeneracy operator  $\Delta[m] \rightarrow \Delta[n]$  that  $A \times \Delta[m] \rightarrow A \times \Delta[n]$  is a generalized degeneracy. Since simplicial sets that are finite colimits of representables are closed under finite limits, in particular products,  $A \times \Delta[n]$  is a finite colimit of representables. Then the claim is given by Corollary 2.4.17.  $\square$

We can alternatively derive Corollary 2.4.18 from the following, more general statement.

**PROPOSITION 2.4.19** *Let  $f: X \rightarrow Y$  be a map such that  $Y$  is cofibrant and the pullback of  $X$  along every map  $\Delta[n] \rightarrow Y$  is a finite colimit of representables. Then the dependent product  $\Pi_f: \mathbf{sSet} \downarrow X \rightarrow \mathbf{sSet} \downarrow Y$  preserves cofibrant objects.*

*Proof.* Consider  $A \rightarrow X$  with  $A$  cofibrant. Take a degeneracy operator  $\sigma: [m] \rightarrow [n]$  and a simplex  $y \in Y_n$ . By assumption,  $f^* \Delta[n]$  is a finite colimit of representables, so  $f^* \Delta[m] \rightarrow f^* \Delta[n]$  is a generalized degeneracy by Corollary 2.4.17. Thus,  $\mathbf{sSet}(f^* \Delta[n], A) \rightarrow \mathbf{sSet}(f^* \Delta[m], A)$  and  $\mathbf{sSet}(f^* \Delta[n], X) \rightarrow \mathbf{sSet}(f^* \Delta[m], X)$  are decidable inclusions. Since  $(\mathbf{sSet} \downarrow X)(f^* \Delta[n], A)$  is a fibre of  $\mathbf{sSet}(f^* \Delta[n], A) \rightarrow \mathbf{sSet}(f^* \Delta[n], X)$  and similarly for  $m$ , the case for pullbacks of part (ii) of Lemma 2.1.7 implies that

$$(\mathbf{sSet} \downarrow X)(f^* \Delta[n], A) \rightarrow (\mathbf{sSet} \downarrow X)(f^* \Delta[m], A)$$

is a decidable inclusion. Finally, we have

$$(\Pi_f A)_n = \coprod_{y \in Y_n} (\mathbf{sSet} \downarrow X)(f^* \Delta[n], A)$$

and similarly for  $m$ , hence  $(\Pi_f A)_n \rightarrow (\Pi_f A)_m$  decomposes into a coproduct of decidable inclusions followed by a decidable inclusion induced by  $Y_n \rightarrow Y_m$  and thus is a decidable inclusion itself.  $\square$

The following statement is also proved in [12, Lemma 5.1] by different methods.

**COROLLARY 2.4.20** *Let  $i: X \rightarrow Y$  be a cofibration between cofibrant objects. Then the dependent product  $\Pi_i: \mathbf{sSet} \downarrow X \rightarrow \mathbf{sSet} \downarrow Y$  preserves cofibrant objects.*

*Proof.* This reduces to Proposition 2.4.19 once we verify for every map  $\Delta[n] \rightarrow Y$  that  $i^* \Delta[n]$  is a finite colimit of representables. By Corollary 2.3.2, the map  $i^* \Delta[n] \rightarrow \Delta[n]$  is a cofibration. Finally, a levelwise decidable subobject of a finite colimit of representables is again a finite colimit of representables.  $\square$

**REMARK 2.4.21** We limited ourselves to proving only the facts on cofibrations needed for the constructive Kan–Quillen model structure, but more can be said. For example, a map over a cofibrant simplicial set  $X$  is a cofibration if and only if its pullback along every map  $\Delta[m] \rightarrow X$  is a cofibration.

## 2.5. Homotopies and homotopy equivalences

The issue of defining the weak equivalences of the Kan–Quillen model structure is fairly delicate in the constructive framework. In fact, our two proofs use different definitions that are only concluded to agree once both arguments are complete. However, in both cases the definitions ultimately go back to the notion of a homotopy equivalence between cofibrant Kan complexes. In this section we establish some basic properties of homotopy equivalences common to both approaches.

**LEMMA 2.5.1** *Let  $f, g$  and  $h$  be maps  $X \rightarrow Y$  and let  $G$  and  $H$  be homotopies from  $f$  to  $g$  and from  $f$  to  $h$  respectively. If  $X$  is cofibrant and there is a Kan fibration  $p: Y \rightarrow M$  such that  $pG = pH$  (in particular,  $pg = ph$ ), then there is a fibrewise homotopy from  $g$  to  $h$  over  $M$ .*

*Proof.* There is a commutative square

$$\begin{array}{ccc} X \times \Lambda^0[2] & \xrightarrow{[G,H]} & Y \\ \downarrow & & \downarrow p \\ X \times \Delta[2] & \xrightarrow{pG\sigma_1} & M, \end{array}$$

which has a diagonal filler  $J$  by part (i) of Proposition 2.3.3. Then  $J\delta_0$  is a homotopy from  $g$  to  $h$  over  $M$ .  $\square$

Using Lemma 2.5.1, note that maps  $f, g: X \rightarrow Y$  from a cofibrant object  $X$  to a fibrant object  $Y$  are homotopic if and only if there is a homotopy from  $f$  to  $g$  or vice versa, that is, the homotopy relation is witnessed by single-step homotopies. A similar remark applies to fibrewise homotopies. We will frequently use this without further reference.

A simplicial map  $f: X \rightarrow Y$  is a *homotopy equivalence* if there is a map  $g: Y \rightarrow X$  such that  $gf$  is homotopic to  $\text{id}_X$  and  $fg$  is homotopic to  $\text{id}_Y$ . If  $X$  and  $Y$  are both cofibrant and fibrant, this means there is a homotopy from  $gf$  to  $\text{id}_X$  and from  $fg$  to  $\text{id}_Y$ .

LEMMA 2.5.2 *Homotopy equivalences satisfy:*

- (i) *2-out-of-6 and*
- (ii) *closure under retracts.*

*Proof.* Recall that the homotopy relation forms a congruence. A map is a homotopy equivalence if and only if it is an isomorphism under this congruence. Since isomorphisms satisfy 2-out-of-6 and closure under retracts, so do homotopy equivalences.  $\square$

LEMMA 2.5.3 *If a simplicial map  $f: X \rightarrow Y$  is a homotopy equivalence, then for all simplicial sets  $Z$  so is the induced map  $Z^Y \rightarrow Z^X$ .*

*Proof.* The functor  $Z^{(-)}$  is simplicial and thus preserves homotopies.  $\square$

Dually to the definition of a shrinkable map of the preceding section, a *strong deformation retraction* of a map  $f: X \rightarrow Y$  is a map  $r: Y \rightarrow X$  such that  $rf = \text{id}_X$  and there is a homotopy  $H$  from  $fr$  to  $\text{id}_Y$  under  $X$ , that is,  $Hf$  is constant.

LEMMA 2.5.4

- (i) *A trivial fibration between cofibrant simplicial sets is shrinkable. In particular, it is a fibrewise homotopy equivalence over its codomain.*
- (ii) *A trivial cofibration between fibrant simplicial admits a strong deformation retraction. In particular, it is a homotopy equivalence.*

*Proof.* We only do part (i) as part (ii) is dual. Let  $p: X \rightarrow Y$  be a trivial fibration. Consider the square

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow p \\ Y & \xrightarrow{\text{id}_Y} & Y, \end{array}$$

which has a lift  $s: Y \rightarrow X$  since  $Y$  is cofibrant. Then there is a square

$$\begin{array}{ccc} X \times \partial \Delta[1] & \xrightarrow{[\text{id}_X, sp]} & X \\ \downarrow & & \downarrow p \\ X \times \Delta[1] & \xrightarrow{p\pi} & Y, \end{array}$$

which also admits a lift  $H: X \times \Delta[1] \rightarrow X$  by part (i) of Proposition 2.3.3 since  $X$  is cofibrant. Therefore,  $s$  is a deformation section of  $p$ .  $\square$



## 2.6. The fibration category of Kan fibrations over a base in cofibrant simplicial sets

In this final subsection of Section 2, we establish that if  $M$  is a cofibrant simplicial set, then the full subcategory of cofibrant simplicial sets over  $M$  spanned by fibrant objects admits a fibration category structure (Theorem 2.6.5). This result completes the preliminaries needed to carry out our proofs of the existence of the constructive Kan–Quillen model structure. Let us begin by recalling that a *fibration category* is a category  $\mathcal{C}$  equipped with a subcategory of *weak equivalences* (denoted by  $\xrightarrow{\sim}$ ) and a subcategory of *fibrations* (denoted by  $\twoheadrightarrow$ ) subject to the axioms (F1–4) listed below; note that a fibration is called *acyclic* if it is a weak equivalence.

- (F1)  $\mathcal{C}$  has a terminal object 1 and all objects are fibrant.
- (F2) Pullbacks along fibrations exist in  $\mathcal{C}$  and (acyclic) fibrations are stable under pullback.
- (F3) Every morphism factors as a weak equivalence followed by a fibration.
- (F4) Weak equivalences satisfy the 2-out-of-6 property.

Let  $M$  be a simplicial set and  $X$  and  $Y$  simplicial sets over  $M$ . A simplicial map  $f: X \rightarrow Y$  over  $M$  is a *fibrewise homotopy equivalence* if there is a map  $g: Y \rightarrow X$  over  $M$  such that  $gf$  is fibrewise homotopic to  $\text{id}_X$  and  $fg$  is fibrewise homotopic to  $\text{id}_Y$ . In this subsection, we call a map in  $\mathbf{sSet}_{\text{cof}} \downarrow M$  *acyclic* if it is a fibrewise homotopy equivalence.

LEMMA 2.6.1 *If a simplicial map  $f: X \rightarrow Y$  is a homotopy equivalence, then for all simplicial sets  $Z$  over  $M$  the map  $f \pitchfork Z: Y \pitchfork Z \rightarrow X \pitchfork Z$  is a fibrewise homotopy equivalence.*

*Proof.* The functor  $-\pitchfork Z$  is simplicial and thus carries homotopies to fibrewise homotopies. □

Let  $\mathbf{sSet}_{\text{cof}}$  be the full subcategory of  $\mathbf{sSet}$  spanned by the cofibrant simplicial sets and let  $\mathbf{sSet}_{\text{cof}} \downarrow M$  be the full subcategory of  $\mathbf{sSet}_{\text{cof}} \downarrow M$  spanned by Kan fibrations over  $M$ . Recall the notion of a shrinkable map from Subsection 2.4.

LEMMA 2.6.2 *An acyclic Kan fibration in  $\mathbf{sSet}_{\text{cof}} \downarrow M$  is shrinkable.*

*Proof.* Let  $K$  and  $L$  be objects of  $\mathbf{sSet}_{\text{cof}} \downarrow M$  and let  $p: K \rightarrow L$  be an acyclic Kan fibration over  $M$ . Since  $p$  is a homotopy equivalence, there exist a map  $\tilde{s}: L \rightarrow K$  and homotopies  $\tilde{H}$  from  $p\tilde{s}$  to  $\text{id}_L$  and  $\tilde{G}$  from  $\tilde{s}p$  to  $\text{id}_K$ . The commutative square

$$\begin{array}{ccc} L \times \{0\} & \xrightarrow{\tilde{s}} & K \\ \downarrow & & \downarrow p \\ L \times \Delta[1] & \xrightarrow{\tilde{H}} & L \end{array}$$

admits a lift  $H: L \times \Delta[1] \rightarrow K$  by part (i) of Proposition 2.3.3. Set  $s = H\iota_1$  so that  $H$  is a homotopy from  $\tilde{s}$  to  $s$ . Then we have  $ps = pH\iota_1 = \tilde{H}\iota_1 = \text{id}_L$ . Moreover,  $sp \sim \tilde{s}p$  so we can pick a direct homotopy  $\hat{G}$  from  $sp$  to  $\text{id}_K$  by Lemma 2.5.1. Now,  $sp\hat{G}$  is a homotopy from  $spsp = sp$  to  $sp$  and we also have  $psp\hat{G} = p\hat{G}$ . Thus by Lemma 2.5.1 there is a homotopy  $G$  from  $ps$  to  $\text{id}_K$  over  $L$ . □

Note that the proof above only used the assumption that  $p$  is a homotopy equivalence, which is in general weaker than a fibrewise homotopy equivalence. This argument implies that these two notions coincide for fibrations, but in the current section we put emphasis on fibrewise homotopy equivalence.

**LEMMA 2.6.3** *A shrinkable Kan fibration is trivial.*

*Proof.* Assume that  $p: K \rightarrow L$  is a shrinkable fibration with a deformation section  $s: L \rightarrow K$  with  $ps = \text{id}_L$  and a zigzag of homotopies from  $sp$  to  $\text{id}_K$  over  $L$ . Let this zigzag be represented by  $H: K \times J \rightarrow K$  with  $J$  as its indexing zigzag. Consider a cofibration  $i: A \rightarrowtail B$  and the first of the following two squares.

$$\begin{array}{ccc} A & \xrightarrow{u} & K \\ i \downarrow & & \downarrow p \\ B & \xrightarrow{v} & L \end{array} \qquad \begin{array}{ccc} A \times J \cup B & \xrightarrow{[Hu, \tilde{w}]} & K \\ \downarrow & & \downarrow p \\ B \times J & \xrightarrow{v\pi} & L \end{array}$$

Set  $\tilde{w} = sv$ . Then  $\tilde{w}i = svi = spu$ , so  $Hu$  is a zigzag of homotopies from  $\tilde{w}i$  to  $u$  over  $L$ . Moreover,  $p\tilde{w} = psv = v$  and so the right square above commutes and admits a lift  $G: B \times J \rightarrow K$  by part (i) of Proposition 2.3.3 (note that the inclusion of an endpoint into  $J$  is a trivial cofibration). Set  $w = G_{\iota_1}$ . Then  $pw = pG_{\iota_1} = v\pi_{\iota_1} = v$  and  $wi = G_{\iota_1}i = Gi_{\iota_1} = Hu_{\iota_1} = u$  so that  $w$  is a lift in the original square and hence  $p$  is a trivial fibration.  $\square$

Alternatively, Lemma 2.6.3 can be concluded as a special case of Lemma 4.1.1, at least for shrinkable maps with homotopies indexed over  $\Delta[1]$ , which is sufficient for our purposes (see Remark 4.1.2 for the general case). Indeed, such a shrinkable Kan fibration is a strong homotopy equivalence and hence a retract of its pullback exponential with  $\{0\} \rightarrow \Delta[1]$ , which is a trivial fibration by Proposition 2.3.3. However, we gave a self-contained proof above for the sake of clarity.

**COROLLARY 2.6.4** *A morphism in  $\mathbf{sSet}_{\text{cof}} \downarrow M$  is an acyclic Kan fibration if and only if it is a trivial fibration.*

*Proof.* An acyclic Kan fibration is trivial by Lemmas 2.6.2 and 2.6.3. Conversely, a trivial fibration is a Kan fibration by Lemma 2.2.2. Moreover, it is shrinkable by part (i) of Lemma 2.5.4. Hence it is a fibrewise homotopy equivalence over its codomain and thus also over  $M$ .  $\square$

**THEOREM 2.6.5** *If  $M$  is cofibrant, the category  $\mathbf{sSet}_{\text{cof}} \downarrow M$  with fibrewise homotopy equivalences and (underlying) Kan fibrations is a fibration category. The acyclic fibrations are the trivial fibrations.*

*Proof.* Axiom (F1) holds since  $\text{id}_M$  is the terminal object and all objects are fibrant by definition. For axiom (F2), Kan fibrations are stable under pullback since they are defined by a right lifting property (and cofibrancy is preserved by Lemma 2.4.7). The same argument applies to acyclic fibrations, which are the trivial fibrations by Corollary 2.6.4. To verify axiom (F3) it suffices

(by [7, p. 421, Factorization lemma]) to factor the diagonal map  $K \rightarrow K \times_M K$ . In the factorization  $K \rightarrow \Delta[1] \lrcorner K \rightarrow \partial\Delta[1] \lrcorner K \cong K \times_M K$ , the first map is a fibrewise homotopy equivalence by Lemma 2.6.1 (since  $\Delta[1] \rightarrow \Delta[0]$  is a homotopy equivalence) and the second one is a Kan fibration by part (iii) of Corollary 2.3.4. Note that  $\Delta[1] \lrcorner K$  is cofibrant by Corollary 2.4.18 and Lemma 2.4.7. Fibrewise homotopy equivalences satisfy the 2-out-of-6 property (axiom F4) by the same argument as in the proof of part (i) of Lemma 2.5.2.  $\square$

### 3. The model structure via the $\text{Ex}^\infty$ functor

In this section we present the first proof of our main theorem. Our approach follows closely classical simplicial homotopy theory, and readers familiar with that area will recognize many standard concepts and ideas such as Kan’s  $\text{Ex}$  functor (following the treatment of Latch–Thomason–Wilson [26]), diagonals of bisimplicial sets or Quillen’s Theorem A. Constructively, however, much of that theory is valid only for cofibrant simplicial sets and requires more delicate arguments, which occupy most of this section. It is only in the final subsection where we are able to go beyond cofibrant objects and establish enough of their properties to construct the full Kan–Quillen model structure.

#### 3.1. The weak homotopy equivalences

Recall that in Section 2 we defined the fibrations and cofibrations of the Kan–Quillen model structure. We now move on to introduce its weak equivalences, called *weak homotopy equivalences*. However, we do not give a direct general definition. Instead, we split it into four cases, each building on the previous one.

Some of these definitions will depend on the notion of a *strong cofibrant replacement* of a simplicial set  $X$ , which is a cofibrant simplicial set  $\tilde{X}$  equipped with a trivial fibration  $\tilde{X} \rightarrow X$ . A strong cofibrant replacement of a simplicial map  $f: X \rightarrow Y$  is a simplicial map  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  equipped with a square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{+} & X \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{Y} & \xrightarrow{+} & Y \end{array}$$

where  $\tilde{X}$  and  $\tilde{Y}$  are cofibrant and both horizontal maps are trivial fibrations. A strong cofibrant replacement always exists by Proposition 2.2.7.

Let  $f: X \rightarrow Y$  be a simplicial map.

- (W1) If  $X$  and  $Y$  are cofibrant Kan complexes, then  $f$  is a weak homotopy equivalence if it is a homotopy equivalence.
- (W2) If  $X$  and  $Y$  are Kan complexes, then  $f$  is a weak homotopy equivalence if it has a strong cofibrant replacement that is a weak homotopy equivalence in the sense of (W1).
- (W3) If  $X$  and  $Y$  are cofibrant, then  $f$  is a weak homotopy equivalence if for every Kan complex  $K$  the induced map  $f^*: K^Y \rightarrow K^X$  is a weak homotopy equivalence in the sense of (W2). Note that this is a valid definition since  $K^X$  and  $K^Y$  are Kan complexes by part (ii) of Proposition 2.3.3.

(W4) If  $X$  and  $Y$  are arbitrary, then  $f$  is a weak homotopy equivalence if it has a strong cofibrant replacement that is a weak homotopy equivalence in the sense of (W3).

We collect some basic properties of definition (W1).

LEMMA 3.1.1 *In the category of cofibrant Kan complexes:*

- (i) *weak homotopy equivalences in the sense of (W2) satisfy 2-out-of-6;*
- (ii) *weak homotopy equivalences in the sense of (W2) are closed under retracts;*
- (iii) *every trivial fibration satisfies (W2).*

*Proof.* Parts (i) and (ii) were verified in Lemma 2.5.2. Part (iii) was verified in part (i) of Lemma 2.5.4.  $\square$

We proceed to establish the analogous properties of the remaining definitions. Having done that, we will be able to show, in Proposition 3.1.8 below, that these apparently different notions are mutually consistent.

LEMMA 3.1.2 *Definition (W2) does not depend on the choice of strong cofibrant replacements.*

*Proof.* Let  $f: X \rightarrow Y$  be a map between Kan complexes and let  $\tilde{f}$  and  $\tilde{f}'$  be strong cofibrant replacements of it. Form a diagram

$$\begin{array}{ccccc}
 \tilde{X}'' & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \tilde{X}' \\
 \downarrow & & \downarrow & \searrow & \downarrow \tilde{f}' \\
 & & & \tilde{X} & \xrightarrow{\quad} & X \\
 & & \tilde{f} & & \downarrow & \\
 \tilde{Y}'' & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \tilde{Y}' \\
 & & \downarrow & \searrow & \downarrow \\
 & & \tilde{Y} & \xrightarrow{\quad} & Y \\
 & & & & \downarrow f
 \end{array}$$

where the top and bottom squares are pullbacks and the square on the left is another strong cofibrant replacement. Then the maps  $\tilde{X}'' \rightarrow \tilde{X}$  and  $\tilde{X}'' \rightarrow \tilde{X}'$  (as well as their counterparts for  $Y$ ) are trivial fibrations between cofibrant Kan complexes and hence satisfy (W1) by part (iii) of Lemma 3.1.1. Thus, by part (i) of Lemma 3.1.1,  $\tilde{f}$  satisfies Lemma (W1) if and only if  $\tilde{f}'$  does.  $\square$

LEMMA 3.1.3 *In the category of all Kan complexes:*

- (i) *weak homotopy equivalences in the sense of (W2) satisfy 2-out-of-6;*
- (ii) *weak homotopy equivalences in the sense of (W2) are closed under retracts;*
- (iii) *every trivial fibration satisfies (W2);*
- (iv) *every homotopy equivalence satisfies (W2).*

*Proof.* For parts (i) and (ii), it follows from functorial cofibrant replacement (as given by Proposition 2.2.7) and Lemma 3.1.2 that weak homotopy equivalences in the sense of (W2) are created by a functor from weak homotopy equivalences in the sense of (W1). The latter satisfy 2-out-of-6 and closure under retracts by parts (i) and (ii) of Lemma 3.1.1 and hence so do the former.

For part (iii), if  $f: X \rightarrow Y$  is a trivial fibration between Kan complexes, pick a strong cofibrant replacement  $\tilde{X} \twoheadrightarrow X$ . The identity on  $\tilde{X}$  is a strong cofibrant replacement of  $f$ , which therefore satisfies (W2).

For part (iv), first note that the endpoint projections  $X^{\Delta[1]} \rightarrow X$  of path objects are trivial fibrations by part (iii) of Proposition 2.3.3 and hence weak homotopy equivalences by part (iii). Next, every map homotopic to a weak homotopy equivalence is a weak homotopy equivalence. For this, we use 2-out-of-3 to see that in every diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow & \searrow & \\ Y & \xleftarrow{\sim} & Y^{\Delta[1]} & \xrightarrow{\sim} & Y \end{array}$$

if one of the maps  $X \rightarrow Y$  is a weak homotopy equivalence, then so is the other. Finally, let  $f: X \rightarrow Y$  be a map with a homotopy inverse  $g$ . Since  $gf$  and  $fg$  are homotopic to identities, they are weak homotopy equivalences. Then  $f$  is a weak homotopy equivalence by 2-out-of-6.  $\square$

**REMARK 3.1.4** One may avoid using functoriality of cofibrant replacements in the proof of Lemma 3.1.3 by using Reedy cofibrant replacements. For this, we note that the indexing categories of the respective diagrams ([3] in part (i) and the walking retract in part (ii)) are Reedy categories.

**LEMMA 3.1.5** *In the category of cofibrant simplicial sets:*

- (i) *weak homotopy equivalences in the sense of (W3) satisfy 2-out-of-6;*
- (ii) *weak homotopy equivalences in the sense of (W3) are closed under retracts;*
- (iii) *every trivial fibration satisfies (W3);*
- (iv) *every homotopy equivalence satisfies (W3).*

*Proof.* Parts (i) and (ii) reduce to the respective parts of Lemma 3.1.3. We prove part (iv) before dealing with part (iii). If  $f: X \rightarrow Y$  is a homotopy equivalence with  $X$  and  $Y$  are cofibrant and  $K$  is a Kan complex, then  $f^*: K^Y \rightarrow K^X$  is a homotopy equivalence by Lemma 2.5.3. Thus, it satisfies (W2) by part (iv) of Lemma 3.1.3, and so  $f$  satisfies (W3). For part (iii), a trivial fibration between cofibrant simplicial sets is a homotopy equivalence by part (iii) of Lemma 3.1.1 and thus it satisfies (W3) by part (iv).  $\square$

**LEMMA 3.1.6** *Definition (W4) does not depend on the choice of strong cofibrant replacements.*

*Proof.* This follows by the same argument as Lemma 3.1.2, using Lemma 3.1.5 instead of Lemma 3.1.1.  $\square$

**LEMMA 3.1.7** *In the category of all simplicial sets:*

- (i) *weak homotopy equivalences in the sense of (W4) satisfy 2-out-of-6;*
- (ii) *weak homotopy equivalences in the sense of (W4) are closed under retracts;*
- (iii) *every trivial fibration satisfies (W4);*
- (iv) *every homotopy equivalence satisfies (W4).*

*Proof.* Parts (i), (ii) and (iii) follow by the arguments for the respective parts of Lemma 3.1.3, with Lemmas 3.1.5 and 3.1.6 used in the place of Lemmas 3.1.1 and 3.1.2. For part (iv), pick a strong cofibrant replacement  $\tilde{X} \rightarrow X$ . Then the square

$$\begin{array}{ccc} \tilde{X}^{\Delta[1]} & \xrightarrow{+} & X^{\Delta[1]} \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{+} & X \end{array}$$

is a strong cofibrant replacement of the projection  $X^{\Delta[1]} \rightarrow X$  (for either endpoint) by Corollary 2.4.18 and part (ii) of Proposition 2.3.1. The projection  $\tilde{X}^{\Delta[1]} \rightarrow \tilde{X}$  satisfies (W3) by part (iv) of Lemma 3.1.5 since it is a homotopy equivalence by Lemma 2.5.3, so  $X^{\Delta[1]} \rightarrow X$  satisfies (W4) (and similarly for  $Y$ ). Therefore,  $f$  also satisfies (W4) by part (i) and the argument of part (iv) of Lemma 3.1.3.  $\square$

**PROPOSITION 3.1.8** *Let  $f: X \rightarrow Y$  be a simplicial map.*

- (i) *If  $X$  and  $Y$  are cofibrant Kan complexes, then  $f$  satisfies (W1) if and only if it satisfies (W2).*
- (ii) *If  $X$  and  $Y$  are cofibrant Kan complexes, then  $f$  satisfies (W1) if and only if it satisfies (W3).*
- (iii) *If  $X$  and  $Y$  are Kan complexes, then  $f$  satisfies (W2) if and only if it satisfies (W4).*
- (iv) *If  $X$  and  $Y$  are cofibrant simplicial sets, then  $f$  satisfies (W3) if and only if it satisfies (W4).*

*Proof.* Parts (i) and (iv) follow from Lemmas 3.1.2 and 3.1.6, respectively. Part (iii) is a consequence of part (i).

It remains to check part (ii). If  $f$  satisfies (W1), then it satisfies (W3) by Lemma 2.5.3 and part (iv) of Lemma 3.1.5. Conversely, assume that  $f$  satisfies (W3) and pick a strong cofibrant replacement of  $f^*: X^Y \rightarrow X^X$ :

$$\begin{array}{ccc} E_Y & \xrightarrow{u_Y} & X^Y \\ \tilde{f} \downarrow & & \downarrow f^* \\ E_X & \xrightarrow{u_X} & X^X \end{array}$$

so that  $\tilde{f}: E_Y \rightarrow E_X$  is a homotopy equivalence. Let  $\varphi: E_X \rightarrow E_Y$  be its homotopy inverse. Since  $u_X$  is a trivial fibration, we can pick  $i \in E_X$  such that  $u_X i = \text{id}_X$ . If we set  $g = u_Y \varphi i: Y \rightarrow X$ , then we have

$$gf = f^* g = f^* u_Y \varphi i = u_X \tilde{f} \varphi i \sim u_X i = \text{id}_X.$$

Next, choose a strong cofibrant replacement of  $f^*: Y^Y \rightarrow Y^X$ :

$$\begin{array}{ccc}
E'_Y & \xrightarrow{u'_Y} & Y^Y \\
\tilde{f}' \downarrow & & \downarrow f^* \\
E'_X & \xrightarrow{u'_X} & Y^X
\end{array}$$

with  $\varphi' : E'_X \rightarrow E'_Y$  a homotopy inverse of  $\tilde{f}'$ . Since  $u'_Y$  is a trivial fibration, we can pick  $i', j' \in E'_Y$  such that  $u'_Y i' = \text{id}_Y$  and  $u'_Y j' = fg$ . Then we have  $u'_X \tilde{f}' i' = f^* u'_Y i' = f^* \text{id}_Y = f$  and  $u'_X \tilde{f}' j' = f^* u'_Y j' = f^*(fg) = fgf$ . However, we already know that  $gf \sim \text{id}_X$ , and thus  $fgf \sim f$ . Since  $u'_X$  is a trivial fibration, the latter homotopy lifts to  $\tilde{f}' j' \sim \tilde{f}' i'$ , and hence

$$fg = u'_Y j' \sim u'_Y \varphi' \tilde{f}' j' \sim u'_Y \varphi' \tilde{f}' i' \sim u'_Y i' = \text{id}_Y.$$

Therefore,  $g$  is a homotopy inverse of  $f$ , that is,  $f$  satisfies (W1).  $\square$

Having established that our definitions of weak homotopy equivalences are mutually compatible, we will use them interchangeably, often without comment. In particular, ‘acyclic (co)fibration’ will refer to a (co)fibration that is also a weak homotopy equivalence. See Table 1 for the notation used to denote these maps.

We conclude the subsection with an observation that will be useful later. The following statements hold also without the cofibrancy assumption since strong cofibrant replacements are closed under product (as cofibrant objects and trivial fibrations are). However, we will not need that stronger statement.

**COROLLARY 3.1.9** *If  $f : X \rightarrow Y$  is a weak homotopy equivalence between cofibrant simplicial sets and  $A$  is a cofibrant simplicial set, then  $f \times A$  is a weak homotopy equivalence.*

*Proof.* For any Kan complex  $K$ , we have a commutative square

$$\begin{array}{ccc}
K^{Y \times A} & \xrightarrow{(f \times A)^*} & K^{X \times A} \\
\cong \downarrow & & \downarrow \cong \\
(K^A)^Y & \xrightarrow{f^*} & (K^A)^X.
\end{array}$$

By part (ii) of Proposition 2.3.3,  $K^A$  is a Kan complex, and therefore  $f^*$  is a weak homotopy equivalence (in the sense of (W2)). Hence so is  $(f \times A)^*$ , and thus  $f \times A$  is a weak homotopy equivalence (in the sense of (W3)).  $\square$

### 3.2. The (co)fibration category of (co)fibrant simplicial sets

In this subsection, we establish the fibration category of fibrant simplicial sets (that is, Kan complexes) and the cofibration category of cofibrant simplicial sets. This will give us sufficient

understanding of constructive simplicial homotopy theory to prove a number of intermediate results in the following subsections. We will then use these results to derive the full model structure. The case of the fibration category reduces directly to Theorem 2.6.5 (specialized to  $M = \Delta[0]$ ).

**PROPOSITION 3.2.1** *A simplicial map between Kan complexes is an acyclic Kan fibration if and only if it is a trivial fibration.*

*Proof.* Let  $f: X \rightarrow Y$  be an acyclic Kan fibration between Kan complexes. Choose a strong cofibrant replacement  $\tilde{Y} \twoheadrightarrow Y$ , take a pullback and another strong cofibrant replacement:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{+} \twoheadrightarrow & \bullet & \xrightarrow{+} \twoheadrightarrow & X \\ & & \downarrow & & \downarrow f \\ & & \tilde{Y} & \xrightarrow{+} \twoheadrightarrow & Y \end{array}$$

Then both  $\tilde{X} \rightarrow \bullet$  and  $\tilde{Y} \rightarrow Y$  are trivial fibrations, and so is the composite  $\tilde{X} \rightarrow \tilde{Y}$  by Corollary 2.6.4. It follows that  $f$  is also a trivial fibration by Lemma 2.2.10.

Conversely, a trivial fibration is a Kan fibration by Lemma 2.2.2. Moreover, it is acyclic by part (iii) of Lemma 3.1.3.  $\square$

The fibration category of Kan complexes established in the next theorem satisfies certain additional axioms beyond (F1–4) given in Subsection 2.6, listed below. They assert that certain infinite limits exist and are well-behaved with respect to fibrations and acyclic fibrations. Such a fibration category is called *complete*. The only reason why the fibration category of Theorem 2.6.5 is not complete is that cofibrant objects are not generally closed under infinite limits.

(F5)  $\mathcal{C}$  has products and (acyclic) fibrations are stable under products.

(F6)  $\mathcal{C}$  has limits of countable towers of fibrations and (acyclic) fibrations are stable under such limits.

**THEOREM 3.2.2** *The category of Kan complexes (that is, the full subcategory of the category of simplicial sets spanned by the Kan complexes) with weak homotopy equivalences (in the sense of (W2)) and Kan fibrations is a complete fibration category.*

*Proof.* Axiom (F1) holds since  $\Delta[0]$  is the terminal Kan complex and all Kan complexes are fibrant by definition. For axiom (F2), Kan fibrations are stable under pullback since they are defined by a right lifting property. The same argument applies to acyclic fibrations by Proposition 3.2.1. For axiom (F3), it suffices (by [7, p. 421, Factorization lemma]) to factor the diagonal map  $K \rightarrow K \times K$ . In the factorization  $K \rightarrow K^{\Delta[1]} \rightarrow K \times K$ , the first map is a homotopy equivalence by Lemma 2.5.3 and hence a weak homotopy equivalence by part (iv) of Lemma 3.1.3 and the second one is a Kan fibration by part (ii) of Proposition 2.3.3. For axiom (F4), weak homotopy equivalences satisfy the 2-out-of-6 property by part (i) of Lemma 3.1.3. Finally, axioms (F5) and (F6) follow by the same argument as axiom (F2).  $\square$

We go on to show that the category of cofibrant simplicial sets carries a structure of a *cocomplete cofibration category*, that is, it satisfies axioms (C1–6) dual to the axioms (F1–4) of Subsection 2.6



and (F5–6) above. As usual, the critical difficulty lies in showing that acyclic cofibrations are closed under pushout. In contrast to the arguments of Theorems 2.6.5 and 3.2.2, we do not establish any lifting property of acyclic cofibrations. Instead, the theorem below is proved by reduction to Theorem 3.2.2 via the exponential functors  $K^{(-)}$  for all Kan complexes  $K$  (which justifies the choice of (W3) as the definition of weak homotopy equivalences between cofibrant objects). Later, in Proposition 3.6.4, we will show that acyclic cofibrations coincide with trivial cofibrations and thus are characterized by a lifting property.

**THEOREM 3.2.3** *The category of cofibrant simplicial sets with weak homotopy equivalences (in the sense of (W3)) and cofibrations is a cocomplete cofibration category.*

*Proof.* Axiom (C1) holds since  $\emptyset$  is the initial cofibrant simplicial sets and all objects are cofibrant by definition. For axiom (C2), cofibrations are stable under pushout since they are defined by a left lifting property. To verify that acyclic cofibrations are stable under pushout, consider a pushout square

$$\begin{array}{ccc} C & \xrightarrow{u} & X \\ i \downarrow \sim & & \downarrow j \\ D & \xrightarrow{v} & Y \end{array}$$

where  $i$  is an acyclic cofibration (and all objects are cofibrant). Then for any Kan complex  $K$  there is an induced pullback square of Kan complexes (by part (ii) of Proposition 2.3.3)

$$\begin{array}{ccc} K^Y & \xrightarrow{v^*} & K^D \\ j^* \downarrow \sim & & \downarrow i^* \\ K^X & \xrightarrow{u^*} & K^C \end{array}$$

where  $i^*$  is a Kan fibration by part (ii) of Proposition 2.3.3 and a weak homotopy equivalence by definition. Thus  $j^*$  is a weak homotopy equivalence by Theorem 3.2.2, and hence  $j$  is a weak homotopy equivalence. To verify axiom (C3), it suffices (by [7, p. 421, Factorization lemma]) to factor the codiagonal map  $X \sqcup X \rightarrow X$ . In the factorization  $X \sqcup X \rightarrow X \times \Delta[1] \rightarrow X$ , the first map is a cofibration by part (i) of Proposition 2.3.1 and the second one is a weak homotopy equivalence by part (iv) of Lemma 3.1.5. For axiom (C4), weak homotopy equivalences satisfy the 2-out-of-6 property by part (i) of Lemma 3.1.5. Axioms (C5) and (C6) follow by the same argument as (C2).  $\square$

### 3.3. Diagonals of bisimplicial sets

In this subsection we prove Proposition 3.3.5, a constructive version of the classical result saying that the diagonal of a bisimplicial map that is a levelwise weak homotopy equivalence is itself a weak

homotopy equivalence. We can establish this only under a suitable cofibrancy assumption, but our argument follows a standard approach (that is, as in [16, Proposition 1.9]), which relies only on the cocomplete cofibration category of cofibrant simplicial sets that we constructed in Theorem 3.2.3.

For purposes of the present subsection, we consider bisimplicial sets as simplicial objects in the category of simplicial sets. In particular, we will use the fact that a cofibration of bisimplicial sets (that is, a Reedy decidable inclusion over  $\Delta \times \Delta$ ) is the same thing as a Reedy cofibration over  $\Delta$  with respect to cofibrations of simplicial sets (see [32, Example 4.6]). If  $A$  and  $B$  are simplicial sets, then their *external product* is the bisimplicial set  $A \times B$  given as  $(A \times B)_{m,n} = A_m \times B_n$ .

The  $k$ -th *skeleton* of a bisimplicial set  $X$  is the coend  $\mathrm{Sk}^k X = \int^{[m] \in \Delta} X_m \times (\mathrm{Sk}^k \Delta[m])$  where  $\mathrm{Sk}^k \Delta[m]$  stands for the simplicial subset of  $\Delta[m]$  spanned by its  $k$ -simplices. For completeness, we will establish a few basic facts about skeleta, a more abstract discussion can be found in [32, Section 6].

**LEMMA 3.3.1** *The skeleta of a bisimplicial set  $X$  come with canonical morphisms  $\mathrm{Sk}^k X \rightarrow \mathrm{Sk}^l X$  for all  $l \geq k \geq -1$ . These morphisms exhibit  $X$  as the colimit of the resulting sequence*

$$\mathrm{Sk}^{-1} X \rightarrow \mathrm{Sk}^0 X \rightarrow \mathrm{Sk}^1 X \rightarrow \dots$$

*Proof.* The functors in the definition of the skeleta come with maps

$$\mathrm{Sk}^{-1} \Delta[-] \rightarrow \mathrm{Sk}^0 \Delta[-] \rightarrow \mathrm{Sk}^1 \Delta[-] \rightarrow \dots$$

and the colimit of this sequence is  $\Delta[-]$ . Thus the colimit of

$$\mathrm{Sk}^{-1} X \rightarrow \mathrm{Sk}^0 X \rightarrow \mathrm{Sk}^1 X \rightarrow \dots$$

is  $\int^{[m]} X_m \times \Delta[m] \cong X$ . □

**LEMMA 3.3.2** *For all natural numbers  $k, m$  and  $n$ , the square*

$$\begin{array}{ccc} \partial \Delta[k]_m \times \Delta([k], [n]) \cup \Delta[k]_m \times \partial \Delta([k], -)_n & \longrightarrow & \mathrm{Sk}^{k-1} \Delta[n]_m \\ \downarrow & & \downarrow \\ \Delta[k]_m \times \Delta([k], [n]) & \longrightarrow & \mathrm{Sk}^k \Delta[n]_m \end{array}$$

*is a pushout.*

*Proof.* The set  $\partial \Delta([k], -)$  is the boundary of the cosimplicial set  $\Delta([k], -)$ , that is,  $\partial \Delta([k], -)_n$  consists of simplicial operators  $\alpha: [k] \rightarrow [n]$  such that  $\alpha^b \neq \mathrm{id}_{[k]}$ . Similarly,  $\partial \Delta[k]_m$  consists of operators  $\beta: [m] \rightarrow [k]$  such that  $\beta^\# \neq \mathrm{id}_{[k]}$ . Thus the union in the upper left corner is the set of pairs  $(\beta, \alpha)$  such that the composite  $\alpha\beta$  factors through  $[k-1]$ . The conclusion follows since  $\mathrm{Sk}^k \Delta[n]_m$  is the set of operators  $[m] \rightarrow [n]$  that factor through  $[k]$  and  $\mathrm{Sk}^{k-1} \Delta[n]_m$  is the set of those that factor through  $[k-1]$ . □

COROLLARY 3.3.3 *For any bisimplicial set  $X$  the square*

$$\begin{array}{ccc} \partial \Delta[k] \times X_k \cup \Delta[k] \times L_k X & \longrightarrow & \mathrm{Sk}^{k-1} X \\ \downarrow & & \downarrow \\ \Delta[k] \times X_k & \longrightarrow & \mathrm{Sk}^k X \end{array}$$

*is a pushout.*

*Proof.* When  $m$  and  $n$  vary in the square of the preceding lemma, we obtain a pushout square of functors  $\Delta \rightarrow \mathbf{sSet}$ . Applying the coend  $\int^{[m]} X_m \times (-)_m$  yields the required pushout square.  $\square$

In the remainder of this section, we will freely use the cocomplete cofibration category of cofibrant simplicial sets established in Theorem 3.2.3 in order to invoke some standard results from [31].

LEMMA 3.3.4 *If  $X \rightarrow Y$  is a map between cofibrant bisimplicial sets such that  $X_k \rightarrow Y_k$  is a weak homotopy equivalence for all  $k$ , then the induced map  $L_k X \rightarrow L_k Y$  is also a weak homotopy equivalence for all  $k$ .*

*Proof.* The latching object  $L_k X$  can be written as a colimit over  $\partial([k] \downarrow \Delta_b)^{\mathrm{op}}$  (the opposite of the latching category). That category is direct and the diagram (sending  $[k] \rightarrow [l]$  to  $X_l$ ) is Reedy cofibrant since  $X$  is (as follows from the fact that the latching categories of  $\partial([k] \downarrow \Delta_b)$  are isomorphic to the latching categories of  $\Delta_b$ ). Thus the conclusion follows from [31, Theorem 9.3.5 (1c)].  $\square$

PROPOSITION 3.3.5 *If  $X \rightarrow Y$  is a map between cofibrant bisimplicial sets such that  $X_k \rightarrow Y_k$  is a weak homotopy equivalence for all  $k$ , then the induced map  $\mathrm{diag} X \rightarrow \mathrm{diag} Y$  is also a weak homotopy equivalence.*

*Proof.* First, we will prove by induction with respect to  $k$  that the induced map  $\mathrm{diag} \mathrm{Sk}^k X \rightarrow \mathrm{diag} \mathrm{Sk}^k Y$  is a weak homotopy equivalence.

For  $k = -1$ , both  $\mathrm{diag} \mathrm{Sk}^k X$  and  $\mathrm{diag} \mathrm{Sk}^k Y$  are empty, so the statement holds. For  $k \geq 0$ , consider the cube

$$\begin{array}{ccccc} L_k X \times \Delta[k] \cup X_k \times \partial \Delta[k] & \longrightarrow & \mathrm{diag} \mathrm{Sk}^{k-1} X & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & X_k \times \Delta[k] & \longrightarrow & \mathrm{diag} \mathrm{Sk}^k X & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ L_k Y \times \Delta[k] \cup Y_k \times \partial \Delta[k] & \longrightarrow & \mathrm{diag} \mathrm{Sk}^{k-1} Y & & \\ & \downarrow & \downarrow & \searrow & \\ & Y_k \times \Delta[k] & \longrightarrow & \mathrm{diag} \mathrm{Sk}^k Y & \end{array}$$

of which the top and bottom squares arise by applying  $\text{diag}$  to the squares of Corollary 3.3.3 since  $\text{diag}$  carries external products to products. These squares are pushouts since  $\text{diag}$  preserves pushouts.

The map  $L_k X \rightarrow L_k Y$  is a weak homotopy equivalence by Lemma 3.3.4 and therefore so are  $L_k X \times \Delta[k] \rightarrow L_k Y \times \Delta[k]$  and  $L_k X \times \partial\Delta[k] \rightarrow L_k Y \times \partial\Delta[k]$  as well as  $X_k \times \partial\Delta[k] \rightarrow Y_k \times \partial\Delta[k]$  by Corollary 3.1.9. Thus the left vertical map in the back of the cube is a weak homotopy equivalence by the Gluing Lemma [31, Lemma 1.4.1 (1b)] (using the fact that  $X$  and  $Y$  are cofibrant as well as part (i) of Proposition 2.3.3). The map  $X_k \times \Delta[k] \rightarrow Y_k \times \Delta[k]$  is a weak homotopy equivalence by Corollary 3.1.9 and so is  $\text{diagSk}^{k-1} X \rightarrow \text{diagSk}^{k-1} Y$  by the inductive hypothesis. Moreover, the diagonal maps on the left of the cube are cofibrations by cofibrancy of  $X$  and  $Y$  and part (i) of Proposition 2.3.1. Thus the Gluing Lemma [31, Lemma 1.4.1 (1b)] implies that  $\text{diagSk}^k X \rightarrow \text{diagSk}^k Y$  is a weak homotopy equivalence.

It also follows that the diagonal maps on the right are cofibrations. Hence, Lemma 3.3.1 and [31, Theorem 9.3.5 (1c)] imply that  $\text{diag} X \rightarrow \text{diag} Y$  is a weak homotopy equivalence.  $\square$

### 3.4. The $\text{Ex}^\infty$ functor

We turn to the constructive treatment of the  $\text{Ex}^\infty$  functor. Classically, it is a fibrant replacement functor with some convenient properties, most notably preservation of finite limits and Kan fibrations. In the constructive setting, we are only able to show that it is a fibrant replacement functor in the subcategory of cofibrant simplicial sets. Some of the material below is treated also in [18, Section 3]. However, we do not need to establish some of the more intricate results on trivial cofibrations proved therein.

If  $P$  is a poset, then let  $\text{sd}P$  denote the poset of finite, non-empty, totally ordered subsets of  $P$  ordered by inclusion. (This defines a functor  $\text{sd}: \mathbf{Pos} \rightarrow \mathbf{Pos}$ .) Let  $\max_P: \text{sd}P \rightarrow P$  denote the (natural, order-preserving) map sending a finite, non-empty, totally ordered subset of  $P$  to its maximal element. Let  $\text{Sd}: \mathbf{sSet} \rightarrow \mathbf{sSet}$  be the essentially unique colimit-preserving functor such that  $\text{Sd}\Delta[m] = N\text{sd}[m]$  (as functors  $\Delta \rightarrow \mathbf{sSet}$ ) and  $\mu: \text{Sd} \rightarrow \text{id}_{\mathbf{sSet}}$  be the natural transformation such that  $\mu_{\Delta[m]} = N\max_{[m]}$  (up to isomorphism, there is a unique such transformation since  $\text{Sd}$  is the left Kan extension of the functor  $N\text{sd}: \Delta \rightarrow \mathbf{sSet}$  along the Yoneda embedding). For a poset  $P$ , we have a natural isomorphism  $\text{Sd}NP \cong N\text{sd}P$ . Moreover, if  $K$  is a simplicial subset of  $NP$ , then  $\text{Sd}K$  is a simplicial subset of  $N\text{sd}P$ ; it consists of those simplices whose vertices lie in  $K$  (when seen as non-degenerate simplices of  $NP$ ).

Before discussing homotopy theoretic properties of the subdivision, we establish two preliminary lemmas needed for the verification of cofibrancy of various simplicial and bisimplicial sets constructed later in this section.

**LEMMA 3.4.1** *If  $\sigma: [m] \rightarrow [n]$  is a degeneracy operator, then the induced map  $\text{Sd}\Delta[m] \rightarrow \text{Sd}\Delta[n]$  is a generalized degeneracy.*

*Proof.* Let  $\delta: [n] \rightarrow [m]$  be a section of  $\sigma$ . Then  $\text{Sd}\delta$  is a section of  $\text{Sd}\sigma$  and we will show that it is a deformation section. Let  $f: \text{sd}[m] \rightarrow \text{sd}[n]$  be given by  $A \mapsto A \cup \delta\sigma A$  for every finite non-empty subset  $A \subseteq [m]$ . Then we have  $A \subseteq fA \supseteq (\text{sd}\delta)(\text{sd}\sigma)A$ , which induces a zigzag of fibrewise homotopies  $\text{id}_{\text{Sd}\Delta[m]} \rightarrow Nf \leftarrow (\text{Sd}\delta)(\text{Sd}\sigma)$ . Thus,  $\text{Sd}\sigma$  is shrinkable and hence a generalized degeneracy by Lemma 2.4.15.  $\square$

LEMMA 3.4.2 *For every  $m$ , the map  $\mu_{\Delta[m]}: \text{Sd } \Delta[m] \rightarrow \Delta[m]$  is shrinkable. In particular, it is a homotopy equivalence and a generalized degeneracy.*

*Proof.* Define an order-preserving map  $\iota_{[m]}: [m] \rightarrow \text{sd}[m]$  by  $\iota_{[m]}(i) = \{0, \dots, i\}$ . Then  $\max_{[m]} \iota_{[m]} = \text{id}_{[m]}$  and  $\iota_{[m]} \max_{[m]} \supseteq \text{id}_{\text{sd}[m]}$  so that  $N_{\iota_{[m]}}$  is a deformation section of  $\mu_{\Delta[m]}$ . Thus  $\mu_{\Delta[m]}$  is shrinkable and a homotopy equivalence. It is a generalized degeneracy by Lemma 2.4.15.  $\square$

The subdivision functor  $\text{Sd}: \mathbf{sSet} \rightarrow \mathbf{sSet}$  has a right adjoint denoted by  $\text{Ex}$ , which can be constructed as  $(\text{Ex } X)_m = \mathbf{sSet}(\text{Sd } \Delta[m], X)$ . Under this adjunction,  $\mu: \text{Sd} \rightarrow \text{id}_{\mathbf{sSet}}$  corresponds to a natural transformation  $\text{id}_{\mathbf{sSet}} \rightarrow \text{Ex}$ , which will be denoted by  $\nu$ .

PROPOSITION 3.4.3 *The functor  $\text{Ex}$  satisfies the following conditions.*

- (i) *It preserves limits.*
- (ii) *It preserves Kan fibrations.*
- (iii) *It preserves trivial fibrations.*
- (iv) *If  $X$  is cofibrant, then  $\nu_X$  is a cofibration. In particular,  $\text{Ex } X$  is cofibrant.*

*Proof.* Part (i) holds since  $\text{Ex}$  is a right adjoint.

For part (ii), it suffices to show that  $\text{Sd}$  carries horn inclusions to trivial cofibrations. Note that any permutation of  $[m]$  induces a simplicial automorphism of  $\text{Sd } \Delta[m]$  that carries horns to horns. Thus, it is enough to check that  $\text{Sd } \Lambda^0[m] \hookrightarrow \text{Sd } \Delta[m]$  is a trivial cofibration.  $\text{Sd } \Lambda^0[m]$  is the nerve of the subposet of  $\text{sd}[m]$  spanned by all elements except  $[m]$  and  $[m] \setminus \{0\}$ . Thus, the inclusion in question factors as  $\text{Sd } \Lambda^0[m] \hookrightarrow \text{Sd } \Lambda^0[m] \cup X \hookrightarrow \text{Sd } \Delta[m]$  where  $X$  is the nerve of the subposet of  $\text{sd}[m]$  spanned by all elements  $S \subseteq [m]$  such that  $0 \in S$ . Moreover, let  $Y$  be the nerve of the subposet of  $\text{sd}[m]$  spanned by all elements except  $\{0\}$ . Then there are pushout squares

$$\begin{array}{ccc} \text{Sd } \Lambda^0[m] \cap X & \hookrightarrow & \text{Sd } \Lambda^0[m] \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \text{Sd } \Lambda^0[m] \cup X \end{array} \quad \begin{array}{ccc} (\text{Sd } \Lambda^0[m] \cup X) \cap Y & \hookrightarrow & \text{Sd } \Lambda^0[m] \cup X \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & \text{Sd } \Delta[m]. \end{array}$$

In the left one,  $\text{Sd } \Lambda^0[m] \cap X$  is the nerve of the subposet of  $\text{sd}[m]$  spanned by all elements  $S \subseteq [m]$  such that  $0 \in S$  except  $[m]$ . Thus  $\text{Sd } \Lambda^0[m] \cap X \hookrightarrow X$  can be identified with the  $m$ -fold pushout product of  $\Lambda^0[1] \rightarrow \Delta[1]$  and so it is a trivial cofibration by part (i) of Proposition 2.3.3. Hence  $\text{Sd } \Lambda^0[m] \hookrightarrow \text{Sd } \Lambda^0[m] \cup X$  is also a trivial cofibration. Similarly, in the right square,  $(\text{Sd } \Lambda^0[m] \cup X) \cap Y \hookrightarrow Y$  can be identified with the pushout product of  $\Lambda^1[1] \hookrightarrow \Delta[1]$  and  $\text{Sd } \partial \Delta[m-1] \hookrightarrow \text{Sd } \Delta[m-1]$  and so it is a trivial cofibration by part (i) of Proposition 2.3.3. (Under this identification,  $\Delta[m-1]$  corresponds to the face  $\delta_0: \Delta[m-1] \hookrightarrow \Delta[m]$ .) Hence,  $\text{Sd } \Lambda^0[m] \cup X \hookrightarrow \text{Sd } \Delta[m]$  is also a trivial cofibration and thus so is  $\text{Sd } \Lambda^0[m] \hookrightarrow \text{Sd } \Delta[m]$ .

For part (iii), we need to verify for all  $m$  that the map  $\text{Sd } \partial \Delta^i[m] \rightarrow \text{Sd } \Delta[m]$  is a cofibration. This follows by Lemma 2.4.11 since it is the nerve of a decidable inclusion between categories with decidable identities.

For part (iv), we first check that  $\text{Ex } X$  is cofibrant. Indeed, by Corollary 2.4.5 it is enough to check that each degeneracy operator  $[m] \rightarrow [n]$  induces a decidable inclusion  $(\text{Ex } X)_n \rightarrow (\text{Ex } X)_m$ .

This inclusion is induced by  $\mathrm{Sd} \Delta[m] \rightarrow \mathrm{Sd} \Delta[n]$ , so the conclusion follows from Lemma 3.4.1. Next, by Corollary 2.4.6 it suffices to verify that  $\nu_X: X \rightarrow \mathrm{Ex} X$  is a levelwise decidable inclusion. This is a consequence of Lemma 3.4.2 since  $X_m \rightarrow (\mathrm{Ex} X)_m$  is induced by  $\mu_{\Delta[m]}: \mathrm{Sd} \Delta[m] \rightarrow \Delta[m]$ .  $\square$

For a simplicial set  $X$ , we define  $\mathrm{Ex}^\infty X$  to be the colimit of the sequence

$$X \xrightarrow{\nu_X} \mathrm{Ex} X \xrightarrow{\nu_{\mathrm{Ex} X}} \mathrm{Ex}^2 X \longrightarrow \dots$$

We write  $\nu_X^\infty: X \rightarrow \mathrm{Ex}^\infty X$  for the resulting natural map.

Proposition 3.4.4 below extends some properties of the  $\mathrm{Ex}$  functor to the  $\mathrm{Ex}^\infty$  functor.

**PROPOSITION 3.4.4** *The functor  $\mathrm{Ex}^\infty$  satisfies the following conditions:*

- (i) *It preserves finite limits.*
- (ii) *It preserves Kan fibrations between cofibrant objects.*
- (iii) *It preserves trivial fibrations between cofibrant objects.*
- (iv) *If  $X$  is cofibrant, then  $\nu_X^\infty$  is a cofibration. In particular  $\mathrm{Ex}^\infty X$  is cofibrant. Moreover,  $\mathrm{Ex}^\infty X$  is a Kan complex.*

*Proof.* For part (i), observe that  $\mathrm{Ex}^\infty$  is a filtered colimit of functors and each of these functors preserve limits by part (i) of Proposition 3.4.3. Hence it preserves finite limits itself.

Parts (ii) and (iii) follow by Proposition 2.2.5 from the corresponding parts (ii) and (iii) of Proposition 3.4.3 using part (iv) of Proposition 3.4.3 to satisfy the requirement that the step maps in the colimit are cofibrations.

For part (iv),  $\nu_X^\infty$  is a cofibration by part (iv) of Proposition 3.4.3 and thus  $\mathrm{Ex}^\infty X$  is cofibrant. To show that it is a Kan complex, we appeal to Lemma 2.2.6. The step maps  $\mathrm{Ex}^k X \rightarrow \mathrm{Ex}^{k+1} X$  are cofibrations by part (iv) and hence levelwise decidable. It remains to construct the indicated lift in any lifting problem

$$\begin{array}{ccccc} \Lambda^i[m] & \xrightarrow{x} & \mathrm{Ex}^k X & \xrightarrow{\nu_{\mathrm{Ex}^k X}} & \mathrm{Ex}^{k+1} X. \\ \downarrow & & & \nearrow & \\ \Delta[m] & & & & \end{array}$$

It suffices to have this only for  $k \geq 1$ . Then by adjointness this problem rewrites as

$$\begin{array}{ccccc} \mathrm{Sd}^2 \Lambda^i[m] & \xrightarrow{\mathrm{Sd} \mu_{\Lambda^i[m]}} & \mathrm{Sd} \Lambda^i[m] & \xrightarrow{\tilde{x}} & \mathrm{Ex}^{k-1} X. \\ \downarrow & \nearrow \varphi & & \nearrow & \\ \mathrm{Sd}^2 \Delta[m] & & & & \end{array}$$

For this, it will suffice to construct the dashed map  $\varphi$ . We do so by first defining a map  $\varphi: \mathrm{Sd}^2 \Delta[m] \rightarrow \mathrm{Sd} \Delta[m]$ , showing that its image is in  $\mathrm{Sd} \Lambda^i[m]$  and finally checking the required

commutativity. All simplicial sets in the left triangle are nerves of finite posets, so it will be enough to define  $\varphi$  on the underlying posets. First, for a finite non-empty subset  $A \subseteq [m]$ , define

$$\mu'A = \begin{cases} i & \text{if } A \in \{[m], [m] \setminus \{i\}\}, \\ \max A & \text{otherwise.} \end{cases}$$

We then let

$$\varphi(A_0 \subset \dots \subset A_p) = \{\mu'A_0, \dots, \mu'A_p\}, \quad (2)$$

so that  $\varphi$  is an order-preserving map. We verify that the image of  $\varphi$  lies in  $\text{Sd } \Lambda^i[m]$ . Since  $\text{Sd } \Lambda^i[m]$  is the nerve of the poset of the faces of  $\Lambda^i[m]$  and  $\Lambda^i[m] \hookrightarrow \Delta[m]$  is a levelwise decidable inclusion by Lemma 2.2.2 and Corollary 2.4.6, we can do so by ruling out two cases. The first is that  $\varphi(A_0 \subset \dots \subset A_p) = [m] \setminus \{i\}$ . In this case, we have  $i \notin \varphi(A_0 \subset \dots \subset A_p)$  so that  $\varphi(A_0 \subset \dots \subset A_p) = \{\max A_0, \dots, \max A_p\}$  and  $[m] \setminus \{i\} \subseteq A_p$ . This implies that  $i \in \varphi(A_0 \subset \dots \subset A_p)$  and that is not possible. The second case is that  $\varphi(A_0 \subset \dots \subset A_p) = [m]$ . But then we would have  $p = m$  and  $\varphi(A_0 \subset \dots \subset A_{m-1}) = [m] \setminus \{i\}$ , which is also not possible, as above. Finally, by the very definition in (2),  $\varphi$  restricts to  $\text{Sd } \mu_{\Lambda^i[m]}$ , as required.  $\square$

**PROPOSITION 3.4.5** *For a cofibrant simplicial set  $X$ , the map  $\nu_X: X \rightarrow \text{Ex } X$  is a weak homotopy equivalence.*

*Proof.* We use the argument of [26, Theorem 4.1] enhanced with cofibrancy checks necessary to make it constructive. We begin by noticing that  $\text{Ex}$  preserves homotopies. Indeed, a homotopy  $X \times \Delta[1] \rightarrow Y$  gives a map  $\text{Ex } X \times \Delta[1] \rightarrow \text{Ex } X \times \text{Ex } \Delta[1] \rightarrow \text{Ex } Y$ . Thus,  $\text{Ex}$  also preserves homotopy equivalences.

Consider the commutative square

$$\begin{array}{ccc} \text{sSet}(\Delta[m] \times \Delta[0], X) & \longrightarrow & \text{sSet}(\Delta[m] \times \Delta[n], X) \\ \downarrow & & \downarrow \\ \text{sSet}(\text{Sd } \Delta[m] \times \Delta[0], X) & \longrightarrow & \text{sSet}(\text{Sd } \Delta[m] \times \Delta[n], X) \end{array}$$

in the category of sets, which becomes a square of bisimplicial sets when  $m$  and  $n$  vary. We will verify that all these are cofibrant. By Remark 2.4.9, it is enough to verify that all degeneracy operators act on them via decidable inclusions. For the left objects, this amounts to cofibrancy of  $X$  and  $\text{Ex } X$  (part (iv) of Proposition 3.4.3). For the right objects, this reduces to cofibrancy of  $X^{\Delta[m]}$ ,  $X^{\text{Sd } \Delta[m]}$ ,  $X^{\Delta[n]}$  and  $\text{Ex}(X^{\Delta[n]})$ , which follows from Corollary 2.4.18 and part (iv) of Proposition 3.4.3. Since bisimplicial Reedy cofibrancy coincides with iterated simplicial Reedy cofibrancy, all these bisimplicial sets are Reedy cofibrant simplicial objects in  $\text{sSet}$  in both directions.

By fixing either  $m$  or  $n$  we obtain two squares of simplicial sets

$$\begin{array}{ccc}
\bullet & \longrightarrow & X^{\Delta[m]} \\
\downarrow & & \downarrow \sim \\
\bullet & \longrightarrow & X^{\text{Sd}\Delta[m]}
\end{array}
\qquad
\begin{array}{ccc}
X^{\Delta[0]} & \xrightarrow{\sim} & X^{\Delta[n]} \\
\downarrow & & \downarrow \\
\text{Ex}(X^{\Delta[0]}) & \xrightarrow{\sim} & \text{Ex}(X^{\Delta[n]})
\end{array}$$

where the right map in the left square is a homotopy equivalence as the image of the homotopy equivalence  $\text{Sd}\Delta[m] \rightarrow \Delta[m]$  of Lemma 3.4.2 under  $X^{(-)}$ , the top map in the right square is a homotopy equivalence as the image of the homotopy equivalence  $\Delta[n] \rightarrow \Delta[0]$  under  $X^{(-)}$  and then the bottom map in the right square is a homotopy equivalence since  $\text{Ex}$  preserves homotopy equivalences as noted above. In the first two cases we use Lemma 2.5.3 to show that  $X^{(-)}$  preserves homotopy equivalences.

Homotopy equivalences are weak homotopy equivalences by part (iv) of Lemma 3.1.5 and, consequently, taking the diagonal simplicial sets in the original square (that is, setting  $m = n$ ) yields

$$\begin{array}{ccc}
X & \xrightarrow{\sim} & \bullet \\
\downarrow & & \downarrow \sim \\
\text{Ex } X & \xrightarrow{\sim} & \bullet
\end{array}$$

in which the horizontal and right vertical maps are weak homotopy equivalences by Proposition 3.3.5. Thus,  $X \rightarrow \text{Ex } X$  is also a weak homotopy equivalence by the 2-out-of-3 property.  $\square$

**PROPOSITION 3.4.6** *For a cofibrant simplicial set  $X$  the map  $\nu_X^\infty : X \rightarrow \text{Ex}^\infty X$  is a weak homotopy equivalence.*

*Proof.* This follows from part (iv) of Proposition 3.4.3, Proposition 3.4.5 and [31, Theorem 9.3.5 (1c)].  $\square$

### 3.5. An explicit cofibrant replacement functor

Up to this point, we have developed a fair amount of homotopy theory of cofibrant simplicial sets. To move beyond cofibrant objects, we need a sufficiently well-behaved cofibrant replacement functor. (Specifically, we need it to preserve pushouts and cofibrations.) There are a few functors that are suitable. We use a functor  $T$  where  $TX$  is defined as the nerve of the category of simplices of  $X$ . However, even to prove all necessary facts about  $T$ , we implicitly use another cofibrant replacement functor which is the variation of  $T$  using the subcategory of face operators of the category of simplices. Yet another cofibrant replacement functor, denoted  $LU$  and obtained from an adjunction between simplicial and semisimplicial sets, will be discussed in Section 5.

To deal with homotopy theory of nerves of categories, we employ the classical Theorem A of Quillen. As usual, the standard proof technique [30, p. 93] is applicable but only for cofibrant objects.

A simplicial set is *contractible* if the map  $X \rightarrow \Delta[0]$  is a homotopy equivalence. It is *weakly contractible* if  $X \rightarrow \Delta[0]$  is a weak homotopy equivalence.



**THEOREM 3.5.1** (Quillen’s Theorem A) *Let  $f: I \rightarrow J$  be a functor between categories with decidable identities. If for every  $y \in J$  the nerve  $N(f \downarrow y)$  is weakly contractible, then the induced map  $NI \rightarrow NJ$  is a weak homotopy equivalence.*

*Proof.* Let  $Sf$  be a bisimplicial set whose  $(m, n)$ -bisimplices are triples  $(x, y, \varphi)$  where  $x: [m] \rightarrow I$ ,  $y: [n] \rightarrow J$  and  $\varphi: fx_m \rightarrow y_0$ . It comes with two bisimplicial maps

$$NI \times \Delta[0] \longleftarrow Sf \longrightarrow \Delta[0] \times NJ.$$

All the bisimplicial sets in this diagram are cofibrant. For the codomains, recall that  $NI$  and  $NJ$  are cofibrant by part (i) of Lemma 2.4.11. Part (ii) of Lemma 2.1.7, Corollary 2.4.5 and Remark 2.4.9 imply that the external product (defined in Subsection 3.3) of cofibrant simplicial sets is cofibrant. For  $Sf$ , the set of  $(m, n)$ -bisimplices can be written as the pullback  $(NI)_m \times_{\text{ob } J} \text{mor } J \times_{\text{ob } J} (NI)_n$ , and so the cofibrancy of  $Sf$  follows from cofibrancy of  $NI$  and  $NJ$  and part (ii) of Lemma 2.1.7.

For a fixed  $m$ , the left map becomes

$$\coprod_{[m] \rightarrow I} \Delta[0] \longleftarrow \coprod_{x: [m] \rightarrow I} N(fx_m \downarrow J),$$

which is a weak homotopy equivalence since each  $N(fx_m \downarrow J)$  is contractible (as  $fx_m \downarrow J$  has an initial object). For a fixed  $n$ , the left map becomes

$$\coprod_{y: [n] \rightarrow J} N(f \downarrow y_0) \longrightarrow \coprod_{[n] \rightarrow J} \Delta[0],$$

which is a weak homotopy equivalence since each  $N(f \downarrow y_0)$  is contractible by assumption. (Here, we use the fact that weak homotopy equivalences between cofibrant objects are closed under coproducts, see Theorem 3.2.3.)

Thus, by taking diagonals we obtain the diagram

$$\begin{array}{ccccc} NI & \xleftarrow{\sim} & \text{diag } Sf & \xrightarrow{\sim} & NJ \\ \downarrow & & \downarrow & & \downarrow \\ NJ & \xleftarrow{\sim} & \text{diag } S \text{ id}_J & \xrightarrow{\sim} & NJ \end{array}$$

where all horizontal maps are weak homotopy equivalences by Proposition 3.3.5 (using the fact that  $\text{id}_{NJ}$  satisfies the hypotheses of the theorem). The left map is  $Nf$  and the right one is  $\text{id}_{NJ}$ . It follows by 2-out-of-3 that  $Nf$  is a weak homotopy equivalence.  $\square$

For a simplicial set  $X$ , set  $TX = N(\Delta \downarrow X)$ . There is a natural map  $\tau_X: TX \rightarrow X$  given as follows. An  $m$ -simplex of  $TX$  is a functor  $[m] \rightarrow \Delta \downarrow X$ , that is, a sequence of simplices  $\Delta[k_0] \rightarrow \Delta[k_1] \rightarrow \dots \Delta[k_m] \rightarrow X$ . Write  $\varphi_i$  for the map  $[k_i] \rightarrow [k_m]$  of this sequence and let  $\bar{\varphi}: [m] \rightarrow [k_m]$  be given by  $\bar{\varphi}i = \varphi_i k_i$ . Then  $\tau_X$  sends the simplex above to  $\Delta[m] \rightarrow \Delta[k_m] \rightarrow X$  induced by  $\bar{\varphi}$ .

LEMMA 3.5.2 *The functor  $T$  carries homotopy equivalences to weak homotopy equivalences.*

*Proof.* For a simplicial set  $X$ , consider the projection  $p: X \times \Delta[1] \rightarrow X$  and the induced functor  $p_*: \Delta \downarrow (X \times \Delta[1]) \rightarrow \Delta \downarrow X$ . We will begin by verifying that it satisfies the hypothesis of Theorem 3.5.1. Note that all categories involved have decidable identities by Lemma 2.4.12. Moreover,  $p$  is a pullback of  $\Delta[1] \rightarrow \Delta[0]$  and thus a slice of  $p_*$  over  $x: \Delta[m] \rightarrow X$  coincides with the slice of the induced functor  $\Delta \downarrow [1] \rightarrow \Delta$  over  $[m]$ . The latter slice is isomorphic to  $\Delta \downarrow ([m] \times [1])$  so it suffices to show that the nerve of this category is contractible. Let  $s: \Delta \downarrow ([m] \times [1]) \rightarrow \Delta \downarrow ([m] \times [1])$  be a functor that sends  $(\varphi, \psi): [k] \rightarrow [m] \times [1]$  to  $(\varphi', \psi'): [k+1] \rightarrow [m] \times [1]$  where  $\varphi' i = \varphi i$  and  $\psi' i = \psi i$  for  $i \in [k]$  while  $\varphi'(k+1) = m$  and  $\psi'(k+1) = 1$ . This functor admits natural transformations  $c_{m,1} \rightarrow s \leftarrow \text{id}_{\Delta \downarrow ([m] \times [1])}$  where  $c_{m,1}$  is the constant functor at  $(m, 1): [0] \rightarrow [m] \times [1]$  which proves the claim.

Theorem 3.5.1 implies that  $Np_* = Tp$  is a weak homotopy equivalence. Consequently,  $T$  also carries the cylinder inclusions  $X \rightarrow X \times \Delta[1]$  to weak homotopy equivalences, and thus it carries all homotopy equivalences to weak homotopy equivalences.  $\square$

LEMMA 3.5.3 *For every simplicial set  $X$ , the inclusion functor  $j_X: \Delta_{\#} \downarrow X \rightarrow \Delta \downarrow X$  induces a weak homotopy equivalence on the nerves.*

*Proof.* As a preliminary step, we verify that the nerve of  $j \downarrow [m]$  is contractible where  $j$  is the inclusion  $\Delta_{\#} \rightarrow \Delta$ . Indeed, let  $s$  be an endofunctor of that category given by  $s\varphi = \varphi'$  where  $\varphi'0 = 0$  and  $\varphi' i = \varphi(i+1)$  (assuming  $\varphi: [k] \rightarrow [m]$  so that  $\varphi': [1+k] \rightarrow [m]$ ). Then the diagram

$$\begin{array}{ccccc} [0] & \xrightarrow{0} & [1+k] & \xleftarrow{\delta_0} & [k] \\ & \searrow & \downarrow & \swarrow & \\ & 0 & \varphi' & \varphi & \\ & & [m] & & \end{array}$$

exhibits natural transformations  $c_0 \rightarrow s \leftarrow \text{id}_{j \downarrow [m]}$  where  $c_0$  is the constant functor at  $0: [0] \rightarrow [m]$ . Thus,  $N(j \downarrow [m])$  is contractible as claimed.

Finally, note that for each  $x \in X_m$ , seen as an object of  $\Delta \downarrow X$ , the slice category  $j_X \downarrow x$  is isomorphic to  $j \downarrow [m]$  (since a morphism  $j_X y \rightarrow x$  is uniquely determined by a simplicial operator  $[n] \rightarrow [m]$  for any  $y \in X_n$ ). Hence the conclusion follows from Theorem 3.5.1 (note that  $\Delta_{\#} \downarrow X$  has decidable identities by Lemma 2.4.12 since the projection to  $\Delta_{\#}$  is a discrete Grothendieck fibration).  $\square$

LEMMA 3.5.4 *The functor  $T$  carries trivial fibrations to weak homotopy equivalences.*

*Proof.* Let  $p: X \rightarrow Y$  be a trivial fibration. By Lemma 3.5.3, it will be enough to show that  $p_*: \Delta_{\#} \downarrow X \rightarrow \Delta_{\#} \downarrow Y$  induces a weak homotopy equivalence on the nerves.

First, we construct a section  $s$  of  $p_*$  as follows. Given  $y: \Delta[m] \rightarrow Y$ , assume inductively that  $s$  has been already defined on all object of degree less than  $m$  (and morphisms between them). We define  $s y$  as the solution to the lifting problem

$$\begin{array}{ccc}
 \partial \Delta[m] & \xrightarrow{[s(y\delta_i) \mid i \in [m]]} & X \\
 \downarrow & \searrow sy & \downarrow p \\
 \Delta[m] & \xrightarrow{y} & Y
 \end{array}$$

where functoriality is guaranteed by the commutativity of the upper triangle.

Similarly, we define a ‘homotopy’  $H: \Delta_{\#} \downarrow X \rightarrow \Delta_{\#} \downarrow (X^{\Delta[1]})$  from  $sp_*$  to  $\text{id}_{\Delta_{\#} \downarrow X}$  that is fibrewise in the sense that it becomes the ‘constant homotopy’ at  $p_*$  when composed with the functor  $\Delta_{\#} \downarrow (X^{\Delta[1]}) \rightarrow \Delta_{\#} \downarrow (Y^{\Delta[1]})$  induced by  $p$ . Assuming that  $H$  was defined at objects of degree less than  $m$  and given  $x: [m] \rightarrow X$ , we set  $Hx$  to be the solution to the lifting problem

$$\begin{array}{ccc}
 \partial \Delta[m] \times \Delta[1] \cup \Delta[m] \times \partial \Delta[1] & \xrightarrow{[[H(x\delta_i) \mid i \in [m]], spx, x]} & X \\
 \downarrow & \searrow Hx & \downarrow p \\
 \Delta[m] \times \Delta[1] & \xrightarrow{px\pi} & Y
 \end{array}$$

(which exists by part (i) of Proposition 2.3.1).

To conclude the proof, we note that Lemmas 3.5.2 and 3.5.3 imply that both projections  $\Delta_{\#} \downarrow (X^{\Delta[1]}) \rightarrow \Delta_{\#} \downarrow X$  induce weak homotopy equivalences on the nerves. Therefore, by an argument analogous to part (iv) of Lemma 3.1.3,  $Np_*$  is also a weak homotopy equivalence.  $\square$

All properties of the functor  $T$  that will be needed later in the paper are summarized in the following proposition. (Although we will not use it, the functor  $N(\Delta_{\#} \downarrow -)$  appearing above satisfies the same properties.)

**PROPOSITION 3.5.5** *The functor  $T$  satisfies the following conditions.*

- (i) *It preserves colimits.*
- (ii) *It takes values in cofibrant simplicial sets.*
- (iii) *It preserves cofibrations.*
- (iv) *For every simplicial set  $X$ ,  $\tau_X: TX \rightarrow X$  is a weak homotopy equivalence.*

*Proof.* Part (i) follows from the fact that we can write

$$(TX)_m = \coprod_{[k_0] \rightarrow \dots \rightarrow [k_m]} X_{k_m}.$$

Part (ii) is a consequence of part (i) of Lemma 2.4.11 and Lemma 2.4.12.

For Part (iii), it is enough to check that  $T\partial\Delta[m] \rightarrow T\Delta[m]$  is a cofibration. By the previous part and Part (ii) of Lemma 2.4.11, it is sufficient to verify that  $\Delta \downarrow \partial\Delta[m] \rightarrow \Delta \downarrow \Delta[m]$  is a decidable inclusion. That in turn follows from the fact that  $\partial\Delta[m] \rightarrow \Delta[m]$  is a levelwise decidable inclusion.

We now prove part (iv). First, observe that  $T\Delta[m]$  is the nerve of a category with a terminal object and hence is contractible. Therefore,  $\tau_{\Delta[m]}$  is a weak homotopy equivalence. Next, we show that

$\tau_{\Delta^I[m]}$  is a weak homotopy equivalence by induction. By Lemma 2.2.9,  $\partial\Delta[m]$  is a cell complex with respect to  $\{\partial\Delta[k] \rightarrow \Delta[k] \mid k < m\}$ . By the inductive hypothesis and the preceding observation,  $\tau$  is a weak homotopy equivalence on the domains and codomains of maps in this set. Since  $T$  preserves colimits and cofibrations, Theorem 3.2.3 and the Gluing Lemma [31, Lemma 1.4.1 (1b)] imply that  $\tau_{\Delta^I[m]}$  is a weak homotopy equivalence as well. The same argument shows that  $\tau_X$  is a weak homotopy equivalence for all  $I$ -cell complexes  $X$  (where  $I = \{\partial\Delta[m] \hookrightarrow \Delta[m] \mid m \geq 0\}$ ). Since weak homotopy equivalences are closed under retracts by part (ii) of Lemma 3.1.5, the same holds for all cofibrant  $X$ . Finally, the conclusion follows for arbitrary  $X$  by Lemma 3.5.4.  $\square$

### 3.6. Conclusion of the first proof

We are now ready to establish the Kan–Quillen model structure. We need to show that acyclic fibrations coincide with trivial fibrations. In Proposition 3.2.1, we have already verified this for maps between Kan complexes. We follow the argument in [28, Section 17.6], attributed by the authors to Bousfield, to extend this result to maps between arbitrary simplicial sets. This uses the  $\text{Ex}^\infty$  functor, but our situation is more subtle due to the fact that it is a fibrant replacement on cofibrant objects only.

We also need to verify that acyclic cofibrations coincide with trivial cofibrations, which will follow by a general retract argument as soon as we know that trivial cofibrations are weak homotopy equivalences. This is, however, non-trivial for maps between non-cofibrant objects and relies on good properties of the cofibrant replacement functor  $T$ .

**LEMMA 3.6.1** *If  $X$  and  $Y$  are cofibrant and  $p: X \twoheadrightarrow Y$  is an acyclic Kan fibration, then all fibres of  $p$  are contractible.*

*Proof.* For each  $y \in Y_0$ , form the diagram

$$\begin{array}{ccccc}
 F_y & \xrightarrow{\quad} & X & & \\
 \downarrow & \searrow \sim & \downarrow p & \searrow \sim & \\
 & \text{Ex}^\infty F_y & \xrightarrow{\quad} & \text{Ex}^\infty X & \\
 \downarrow & \downarrow & \downarrow & \downarrow \text{Ex}^\infty p & \\
 \Delta[0] & \xrightarrow{\quad} & Y & & \\
 \downarrow & \searrow \sim & \downarrow & \searrow \sim & \\
 & \text{Ex}^\infty \Delta[0] & \xrightarrow{\quad} & \text{Ex}^\infty Y &
 \end{array}$$

where the back square is a pullback that exhibits  $F_y$  as the fibre of  $p$  over  $y$ . The front square is also a pullback by part (i) of Proposition 3.4.4 and all back-to-front maps are weak homotopy equivalences by Proposition 3.4.6. Since  $p$  is acyclic,  $\text{Ex}^\infty p$  is an acyclic Kan fibration by part (ii) of Proposition 3.4.4 and part (i) of Lemma 3.1.5. Moreover, all objects in the front face are Kan complexes by part (iv) of Proposition 3.4.4, and hence  $\text{Ex}^\infty F_y \rightarrow \text{Ex}^\infty \Delta[0]$  is a weak homotopy equivalence by Theorem 3.2.2. Thus, so is  $F_y \rightarrow \Delta[0]$ , that is,  $F_y$  is contractible.  $\square$

LEMMA 3.6.2 *A Kan fibration with contractible fibres is trivial.*

*Proof.* Let  $p: X \rightarrow Y$  be a Kan fibration with contractible fibres. Consider a lifting problem

$$\begin{array}{ccc} \partial \Delta[m] & \xrightarrow{u} & X \\ \downarrow & & \downarrow p \\ \Delta[m] & \xrightarrow{v} & Y. \end{array}$$

Let  $H: \Delta[m] \times \Delta[1] \rightarrow \Delta[m]$  be a homotopy from the constant map at 0 to the identity map. It yields another lifting problem

$$\begin{array}{ccc} \partial \Delta[m] \times \{1\} & \xrightarrow{u} & X \\ \downarrow & & \downarrow p \\ \partial \Delta[m] \times \Delta[1] & \xrightarrow{vH|_{\partial \Delta[m] \times \Delta[1]}} & Y, \end{array}$$

which has a solution  $\tilde{G}: \partial \Delta[m] \times \Delta[1] \rightarrow X$  by part (i) of Proposition 2.3.3. Such  $\tilde{G}$  is a homotopy from  $\tilde{u}$  to  $u$  and the former factors through the fibre  $F_{v_0}$  of  $p$  over  $v_0$  to yield a diagram

$$\begin{array}{ccccc} \partial \Delta[m] & \longrightarrow & F_{v_0} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow p \\ \Delta[m] & \longrightarrow & \Delta[0] & \xrightarrow{v_0} & Y. \end{array}$$

The left square of this diagram has a lift since  $F_{v_0}$  is contractible. (Since  $\Delta[0]$  and  $F_{v_0}$  are Kan complexes, the acyclic fibration  $F_{v_0} \rightarrow \Delta[0]$  is trivial by Proposition 3.2.1.) Let  $\tilde{w}: \Delta[m] \rightarrow X$  denote the resulting composite, which leads to yet another lifting problem

$$\begin{array}{ccc} \Delta[m] \times \{0\} \cup \partial \Delta[m] \times \Delta[1] & \xrightarrow{[\tilde{w}, \tilde{G}]} & X \\ \downarrow & & \downarrow p \\ \Delta[m] \times \Delta[1] & \xrightarrow{vH} & Y. \end{array}$$

Again, the left map is a trivial cofibration by part (i) of Proposition 2.3.3, so there is a solution  $G: \Delta[m] \times \Delta[1] \rightarrow X$ , which is a homotopy from  $\tilde{w}$  to  $w$ . That  $w$  is a solution to the original lifting problem. Indeed,  $pw = pG\iota_1 = vH\iota_1 = v$  and  $w|_{\partial \Delta[m]} = G\iota_1|_{\partial \Delta[m]} = \tilde{G}\iota_1 = u$ .  $\square$

**PROPOSITION 3.6.3** *A simplicial map is an acyclic Kan fibration if and only if it is a trivial fibration.*

*Proof.* This follows by the same argument as Proposition 3.2.1 except it uses Lemmas 3.6.1 and 3.6.2 in the place of Corollary 2.6.4 and Lemma 3.1.7 in the place of Lemma 3.1.3.  $\square$

**PROPOSITION 3.6.4** *A simplicial map is an acyclic cofibration if and only if it is a trivial cofibration.*

*Proof.* An acyclic cofibration is a cofibration by Lemma 2.2.2.

First, we check the acyclicity in the subcategory of cofibrant objects. Let  $f: X \twoheadrightarrow Y$  be a trivial cofibration between cofibrant simplicial sets. For any Kan complex  $K$ , the map  $f^*: K^Y \rightarrow K^X$  is a trivial fibration between Kan complexes by parts (ii) and (ii) of Proposition 2.3.3. Thus,  $f^*$  is a weak homotopy equivalence in the sense of (W2) by part (iii) of Lemma 3.1.3, and it follows that  $f$  is a weak homotopy equivalence in the sense of (W3).

In the general case, note that the class of cofibrations  $i$  such that  $Ti$  is a weak homotopy equivalence is closed under coproducts, pushouts, sequential colimits and retracts by Theorem 3.2.3 and part (ii) of Lemma 3.1.7 and since  $T$  preserves colimits and cofibrations and takes values in cofibrant objects (Proposition 3.5.5). By the preceding argument, this class also contains the horn inclusions and hence all trivial cofibrations by part (ii) of Proposition 2.2.7.

Conversely, let  $f$  be an acyclic cofibration and pick a factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow i & \nearrow p \\ & \widehat{Y} & \end{array}$$

where  $i$  is a trivial cofibration and  $p$  is a Kan fibration. Then  $i$  is a weak homotopy equivalence by the argument above and thus so is  $p$  by 2-out-of-3. Hence,  $p$  is a trivial fibration by Proposition 3.6.3. This implies that the lifting problem

$$\begin{array}{ccc} X & \xrightarrow{i} & \widehat{Y} \\ f \downarrow & & \downarrow p \\ Y & \xrightarrow{\text{id}_Y} & Y \end{array}$$

has a solution, which exhibits  $f$  as a retract of  $i$ . The former is thus a trivial cofibration.  $\square$

**THEOREM 3.6.5** *The cofibrations, Kan fibrations and weak homotopy equivalences form a model structure on simplicial sets.*

*Proof.* Weak homotopy equivalences satisfy 2-out-of-6 by part (i) of Lemma 3.1.7. By Proposition 2.2.7, we have weak factorization systems of cofibrations and trivial fibrations and of trivial cofibrations and Kan fibrations. By Proposition 3.6.3, acyclic fibrations coincide with trivial fibrations. By Proposition 3.6.4, trivial cofibrations coincide with acyclic cofibrations.  $\square$

**REMARK 3.6.6** Let us briefly highlight the differences between our first proof and Simon Henry’s proof in [18]. First of all, the definitions of the weak equivalences are different. Indeed, Henry first introduces weak equivalences only between objects that are either fibrant or cofibrant using the homotopy category, as in his work on weak model categories [17], and then extends this definition to all objects using the adjunction to the category of semisimplicial sets and a weak model structure on it. Secondly, the two proofs are organized in very different ways. In particular, Henry’s proof first establishes the existence of a model structure with the required weak equivalences and cofibrations [18, Theorem 2.2.8] and then exploits  $\text{Ex}^\infty$  to show that the fibrations of the model structure are the Kan fibrations. This second step is particularly complex since it relies on an intricate auxiliary notion of a P-structure introduced in [27]. Moreover, it depends on the theory of degeneracy quotients (called collapses by Joyal [23]), which we replace by the closely related, but less technical, theory of generalized degeneracies. (Every degeneracy quotient between finite colimits of simplices is a generalized degeneracy.)

#### 4. The model structure via the equivalence extension property

In this section, we present our second proof of the existence of the constructive Kan–Quillen model structure. It is based on the development in [14, 33]. The strategy of the approach is as follows. We first build the restriction of the model structure to cofibrant objects. For this, we follow the approach of [33] in the setting of cofibrant simplicial sets. We then obtain the extension to simplicial sets using formal reasoning.

The development of the model structure in cofibrant simplicial sets occupies most of this section. The cornerstones are the Frobenius property (Proposition 4.1.6) in Subsection 4.1 and the trivial fibration extension (Corollary 4.2.5) and fibration extension (Corollary 4.2.7) properties in Subsection 4.2. The latter two statements are consequences of the equivalence extension property (Proposition 4.2.3), which is the namesake of Subsection 4.2. (Although we refer to all these statements as properties, we recall that constructively we ought to think of them as operations.) In Subsection 4.3, we define notions of weak homotopy equivalences and verify the model structure properties, first in cofibrant objects and then in all of simplicial sets. This subsection is entirely formal and uses from the earlier two subsections only the three statements referred to above.

##### 4.1. The Frobenius property

The first step is to prove a restricted version of the Frobenius property for the weak factorization system of trivial cofibrations and fibrations, asserting that pullback along a fibration with cofibrant domain preserves trivial cofibrations. For this, we follow [14], but make explicit the cofibrancy assumption on the domain of the fibration (cf the proof of Lemma 4.1.5).

We begin by recalling from [14] the notion of a strong homotopy equivalence in simplicial sets. Let  $f: A \rightarrow B$  be a homotopy equivalence with homotopy inverse  $g: B \rightarrow A$ . We say that  $f$  is *0-oriented* if  $gf \sim \text{id}_A$  and  $fg \sim \text{id}_B$  are witnessed by a homotopy  $u$  from  $gf$  to  $\text{id}_A$  and a homotopy  $v$  from  $fg$  to  $\text{id}_B$ , respectively. If the homotopies go in the reverse direction, we say that  $f$  is *1-oriented*. In either case, we say that  $f$  is a *strong homotopy equivalence* if  $fu = vf$ . Note that if  $u$  is the constant homotopy,  $f$  specializes to a map with a strong deformation retraction as in Subsection 2.5. Dually, if  $v$  is the constant homotopy,  $f$  specializes to a shrinkable map in the sense of Subsection 2.4 (cf Remark 4.1.2).

A key property of the notion of strong homotopy equivalence is that it admits a characterisation in terms of retracts. For this, we consider the following square in simplicial sets:

$$\begin{array}{ccc} \emptyset & \xrightarrow{!} & \{0\} \\ ! \downarrow & & \downarrow \iota_0 \\ \{1\} & \xrightarrow{\iota_1} & \Delta[1]. \end{array}$$

In the arrow category  $\mathbf{sSet}^{[1]}$ , this square induces a map  $\theta_0: ! \rightarrow \iota_0$  when read horizontally and a map  $\theta_1: ! \rightarrow \iota_1$  when read vertically. Note that  $!$  is the unit for the pushout product. We then have the following characterization.

LEMMA 4.1.1 ([14, Lemma 4.3]) *The following are equivalent for a map  $f$  and  $k \in \{0, 1\}$ :*

- (i)  $f$  is a  $k$ -oriented strong homotopy equivalence,
- (ii)  $f \widehat{\times} \theta_k: f \rightarrow f \widehat{\times} \iota_k$  has a retraction and
- (iii)  $\widehat{\text{hom}}(\theta_k, f): \widehat{\text{hom}}(\iota_k, f) \rightarrow f$  has a section.

*Proof.* The equivalence of (ii) and (iii) follows by adjointness. For the equivalence of (i) and (ii), we only consider the case  $k = 0$  (the case  $k = 1$  is dual). The data of a retraction of  $f \widehat{\times} \theta_0: f \rightarrow f \widehat{\times} \iota_0$  are shown in the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & A \times \Delta[1] \sqcup_{A \times \{0\}} B \times \{0\} & \xrightarrow{[g, u]} & A \\ f \downarrow & & f \widehat{\times} \iota_0 \downarrow & & \downarrow f \\ B & \xrightarrow{B \times \iota_1} & B \times \Delta[1] & \xrightarrow{v} & B \end{array}$$

(we have omitted the identity arrows for  $A$  and  $B$ ; the top left map is induced by  $A \times \iota_1$ ). Commutativity of the diagram precisely amounts to  $g, u, v$  making  $f$  into a 0-oriented strong homotopy equivalence.  $\square$

REMARK 4.1.2 The general setting for the theory of strong homotopy equivalences is that of an (adjoint) functorial cylinder. One may then take the 0-oriented notion as default and obtain the 1-oriented notion from the cylinder with mirrored endpoints. Indeed, this viewpoint allows us to support zigzags of homotopies (as they occur in our notions of homotopy equivalence and shrinkable map). We merely have to consider the cylinder functor induced by a walking zigzag  $J$ , seen as an interval object.

LEMMA 4.1.3 *The horn inclusion  $\Lambda^k[n] \rightarrow \Delta[n]$  is a 0-oriented strong homotopy equivalence for  $k < n$  and a 1-oriented strong homotopy equivalence for  $k > 0$ . In particular, every horn inclusion is a strong homotopy equivalence.*



This is [11, IV.2.1.3] (describing strong homotopy equivalences via condition (ii) of Lemma 4.1.1). For self-containedness, we include a proof.

*Proof.* We let  $k < n$  and show that  $h: \Lambda^k[n] \rightarrow \Delta[n]$  is a 0-oriented strong homotopy equivalence (the other case is dual). The initial segment  $s: [k] \rightarrow [n]$  has a unique retraction  $r: [n] \rightarrow [k]$ . Because  $k < n$ , the map  $s$  induces a monomorphism  $i: \Delta[k] \rightarrow \Lambda^k[n]$ . We define the homotopy inverse of  $h$  as  $ir: \Delta[n] \rightarrow \Lambda^k[n]$ . The homotopy from  $h(ir)$  to  $\text{id}_{\Delta[n]}$  is given by the nerve of the map  $[1] \times [n] \rightarrow [n]$  corresponding to the unique map  $sr \rightarrow \text{id}_{[n]}$ . This restricts on  $h$  to a homotopy from  $\text{id}_{\Lambda^k[n]}$  to  $(ir)h$ .  $\square$

**COROLLARY 4.1.4** *The weak factorization system of trivial cofibrations and fibrations is cofibrantly generated by cofibrations between cofibrant objects that are strong homotopy equivalences.*

*Proof.* Every horn inclusion is a cofibration between cofibrant objects by Lemma 2.2.2 and Corollary 2.4.10 and a strong homotopy equivalence by Lemma 4.1.3. In the other direction, every cofibration  $i$  that is a  $k$ -oriented strong homotopy equivalence is a retract of  $\iota_k \hat{\times} f$  by Lemma 4.1.1, which is a trivial cofibration by part (i) of Proposition 2.3.3.  $\square$

**LEMMA 4.1.5** (cf. [14, Lemma 4.7]) *In any pullback square*

$$\begin{array}{ccc} A & \longrightarrow & B \\ f \downarrow & \lrcorner & \downarrow g \\ X & \longrightarrow\!\!\!\twoheadrightarrow & Y \end{array}$$

*with  $X$  cofibrant, if  $g$  is a  $k$ -oriented strong homotopy equivalence, then so is  $f$ .*

*Proof.* We only consider the case  $k = 0$  (the case  $k = 1$  is dual). We work with 0-oriented strong homotopy equivalences as characterized by condition (ii) of Lemma 4.1.1. To make  $f$  into one, we will construct the dotted map in the following (vertical) map of retracts in the arrow category:

$$\begin{array}{ccccc} f & \xrightarrow{f \hat{\times} \theta_0} & f \hat{\times} \iota_0 & \dashrightarrow & f \\ \downarrow & & \downarrow & & \downarrow \\ g & \xrightarrow{g \hat{\times} \theta_0} & g \hat{\times} \iota_0 & \longrightarrow & g. \end{array}$$

Since  $f \rightarrow g$  is a pullback square, it suffices to produce this map on codomains:

$$\begin{array}{ccccc}
X & \xrightarrow{X \times \iota_1} & X \times \Delta[1] & \dashrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
Y & \xrightarrow{Y \times \iota_1} & Y \times \Delta[1] & \longrightarrow & Y.
\end{array}$$

This is a lifting problem

$$\begin{array}{ccc}
X & \xrightarrow{\text{id}_X} & X \\
\downarrow X \times \iota_1 & \nearrow \text{dashed} & \downarrow \\
X \times \Delta[1] & \longrightarrow & Y.
\end{array}$$

Here, the left map is the pushout product of  $\emptyset \rightarrow X$  with  $\{1\} \rightarrow \Delta[1]$  and hence is a trivial cofibration by part (i) of Proposition 2.3.3 since  $X$  is cofibrant.  $\square$

**PROPOSITION 4.1.6** (restricted Frobenius property, cf. [14, Theorem 3.8]) *Let  $X \rightarrowtail Y$  be a fibration with  $X$  cofibrant. Pullback along  $X \rightarrowtail Y$  preserves trivial cofibrations.*

*Proof.* Since pullback along  $X \rightarrowtail Y$  is a left adjoint, it suffices to show that it preserves the generators of Corollary 4.1.4, that is, cofibrations between cofibrant objects that are strong homotopy equivalences. For this, we note the following.

- Pullback along  $X \rightarrowtail Y$  preserves cofibrations (and hence cofibrant objects) by Corollary 2.3.2.
- Let  $A \rightarrow B$  be a map over  $Y$  with  $B$  cofibrant such that  $A \rightarrow B$  is a strong homotopy equivalence. Then its pullback  $X \times_Y A \rightarrow X \times_Y B$  is a strong homotopy equivalence. This follows from Lemma 4.1.5 since  $X \times_Y B \rightarrow B$  is a fibration (as a pullback of  $X \rightarrowtail Y$ ) with cofibrant domain.

$\square$

**REMARK 4.1.7** Let us explain the name of Proposition 4.1.6. The *Frobenius property* [4] of a weak factorization system in a category with finite limits refers to the condition that left maps are closed under pullback along right maps. In our setting, we do not quite obtain the Frobenius property of the weak factorization system of trivial cofibrations and fibrations, but only a restricted version where the source of the map we pull back along is cofibrant. This is because cofibrations are not generally closed under pullback.

Proposition 4.1.6 in particular encompassed the Frobenius property in cofibrant simplicial sets. This is what we will use to verify the model structure properties in cofibrant simplicial sets. However, Proposition 4.1.6 is more general: the target of the fibration does not need to be cofibrant. This will be used to extend the model structure to the entirety of simplicial sets.

#### 4.2. The equivalence extension property

In this subsection, we prove the equivalence extension property and derive its corollaries, the (trivial) fibration extension properties. All of this happens entirely in the cofibrant fragment of simplicial sets. Before we delve into the proof, we record some basic facts about mapping path space factorizations.

The context of the following definition and lemmas is the slice of cofibrant simplicial sets over an object  $M$ . The *mapping path space factorization* of a map  $X \rightarrow Y$  is

$$X \longrightarrow X \times_Y \Delta[1] \rightharpoonup Y. \quad (3)$$

We adopt the convention that the position of the pullback symbol with respect to  $\Delta[1] \rightharpoonup Y$  indicates whether the pullback is taken with respect to the left or right endpoint projection. The first factor is induced by the constant map  $Y \rightarrow \Delta[1] \rightharpoonup Y$  and the second factor is induced by the right endpoint projection. The middle object is cofibrant by Lemma 2.4.7 and Corollary 2.4.18.

The following lemma already features implicitly in the proof of Theorem 2.6.5.

**LEMMA 4.2.1** *If  $X$  and  $Y$  are fibrant in (3), then the second factor in (3) is a fibration.*

*Proof.* The second factor decomposes as a pullback of the pullback cotensor of  $Y \rightrightarrows 1$  with  $\partial\Delta[1] \rightrightarrows \Delta[1]$  followed by a pullback of  $X \rightrightarrows 1$ . So it is a fibration by part (ii) of Proposition 2.3.3.  $\square$

**LEMMA 4.2.2** *Assume that  $Y$  is fibrant in (3). Then the following are equivalent:*

- (i)  $X$  is fibrant and  $X \rightarrow Y$  is a fibrewise homotopy equivalence and
- (ii) the second factor in (3) is a trivial fibration.

*Proof.* First note that the first factor in (3) admits a retraction (given by the first projection). Thus,  $X$  is a retract of the middle object. If the second factor is a fibration, then the middle object is fibrant, and hence so is  $X$ . This shows that condition (ii) makes  $X$  fibrant.

Now assume that  $X$  is fibrant. Using the fibration category of Theorem 2.6.5, it remains to show that  $X \rightarrow Y$  is a fibrewise homotopy equivalence if and only if the second factor in (3) is a fibrewise homotopy equivalence. This holds by 2-out-of-3 after we check that the first factor in (3) is a fibrewise homotopy equivalence. Indeed, that map is a retraction of a pullback of the first endpoint projection  $\Delta[1] \rightharpoonup Y \rightarrow Y$ , a trivial fibration by part (iv) of Corollary 2.3.4.  $\square$

We are now ready to prove the equivalence extension property. While in [25, 12] this property is proved in context of a model structure to establish univalence of a classifying fibration, it was observed in [33] that it can reversely be used to establish that very model structure. Indeed, the statement of the equivalence extension property does not refer to the weak equivalences of a model structure, but the more elementary notion of fibrewise homotopy equivalence. Thus, the natural setting for its direct proof is the fibration category of Kan fibrations over a base in cofibrant simplicial sets established in Theorem 2.6.5. The idea of a proof using elementary means, not making use of an ambient model structure, goes back to [10].

**PROPOSITION 4.2.3** (Equivalence extension, cf. [33, Proposition 5.1]). *In cofibrant simplicial sets, consider the solid part of the diagram*

$$\begin{array}{ccccc}
 X_0 & \overset{\sim}{\dashrightarrow} & Y_0 & & \\
 \downarrow & \searrow \scriptstyle \sim & \downarrow & \searrow \scriptstyle \sim & \\
 & X_1 & \xrightarrow{\quad} & Y_1 & \\
 \downarrow & \swarrow & \downarrow & \swarrow & \\
 A & \xrightarrow{\quad i \quad} & B & & 
 \end{array}
 \tag{4}$$

where  $A \rightarrow B$  is a cofibration,  $X_0 \rightarrow A$  and  $Y_1 \rightarrow B$  are fibrations, the lower square is a pullback, and the map  $X_0 \rightarrow X_1$  is a fibrewise homotopy equivalence over  $A$ . Then there is  $Y_0$  fitting into the diagram as indicated such that the back square is a pullback, the map  $Y_0 \rightarrow B$  is a fibration and the map  $Y_0 \rightarrow Y_1$  is a fibrewise homotopy equivalence over  $B$ .

*Proof.* In the following, we will make use of the adjunction  $i^* \dashv \Pi_i$ . We identify  $X_1$  with  $i^*Y_1$  over  $A$ . Since  $i$  is a monomorphism, the unit of the adjunction  $\Sigma_i \dashv i^*$  is invertible. By adjointness, the counit of the adjunction  $i^* \dashv \Pi_i$  is invertible. That is,  $\Pi_i$  is a reflective embedding with reflector  $i^*$ . Note that  $\Sigma_i i^*$  is isomorphic to the functor of product with  $A$  over  $B$ . By adjointness, this means  $\Pi_i i^* \simeq \exp_B(A, -)$ . By Corollary 2.3.2, the functor  $i^*$  preserves cofibrations. By adjointness, the functor  $\Pi_i$  preserves trivial fibrations. In the following, we use freely that  $\Pi_i$  preserves cofibrant objects by Corollary 2.4.20 and that cofibrant objects are closed under finite limits (Lemma 2.4.7). With this, all our constructions remain within cofibrant objects.

In the slice over  $B$ , we define  $Y_0$  and its map to  $Y_1$  via the pullback

$$\begin{array}{ccc}
 Y_0 & \xrightarrow{\quad} & \Pi_i X_0 \\
 \downarrow & \lrcorner & \downarrow \\
 Y_1 & \xrightarrow{\quad} & \Pi_i i^* Y_1
 \end{array}
 \tag{5}$$

where the right map is the image of  $X_0 \rightarrow X_1$  under  $\Pi_i$  (identifying  $X_1$  with  $i^*Y_1$  over  $A$ ) and the bottom map is a unit map of  $i^* \dashv \Pi_i$ . Since  $\Pi_i$  is a reflective embedding, the bottom map in (5) becomes invertible after pulling back along  $i: A \rightarrow B$ :

$$\begin{array}{ccc}
 i^* Y_0 & \xrightarrow{\sim} & i^* \Pi_i X_0 \\
 \downarrow & \lrcorner & \downarrow \\
 i^* Y_1 & \xrightarrow{\sim} & i^* \Pi_i i^* Y_1.
 \end{array}
 \tag{6}$$

Furthermore, the right map above is isomorphic to  $X_0 \rightarrow i^*Y_1$ , that is,  $X_0 \rightarrow X_1$ . This shows that  $Y_0$  over  $Y_1$  pulls back along  $i$  to  $X_0$  over  $X_1$  in (4) as required.

For the rest of the proof, we work in the slice over  $B$  unless stated otherwise. It remains to show that  $Y_0$  is fibrant and  $Y_0 \rightarrow Y_1$  is a fibrewise homotopy equivalence. For this, we consider the mapping path space factorization of  $Y_0 \rightarrow Y_1$ :

$$Y_0 \longrightarrow Y_0 \times_{Y_1} \Delta[1] \pitchfork Y_1 \longrightarrow Y_1. \quad (7)$$

Using the reverse direction of Lemma 4.2.2, it remains to show that its second factor is a trivial fibration.

By pullback pasting with (5), the middle object in (7) is isomorphic to  $\Pi_i X_0 \times_{\Pi_i i^* Y_1} \Delta[1] \pitchfork Y_1$ . Under this isomorphism, the right map in (7) corresponds to the top map in the below diagram, induced by the right endpoint projection:

$$\begin{array}{ccc} \Pi_i X_0 \times_{\Pi_i i^* Y_1} \Delta[1] \pitchfork Y_1 & \xrightarrow{\quad} & Y_1 \\ & \searrow & \nearrow \\ \Pi_i X_0 \times_{\Pi_i i^* Y_1} \Pi_i i^*(\Delta[1] \pitchfork Y_1) \times_{\Pi_i i^* Y_1} Y_1 & & \end{array} \quad (8)$$

To show that it is a trivial fibration, we produce a factorization into two trivial fibrations as indicated above.

Associating the pullbacks in the middle object of (8) to the right, the first factor in (8) is the pullback along  $\Pi_i X_0 \rightarrow \Pi_i i^* Y_1$  of the map

$$\Delta[1] \pitchfork Y_1 \longrightarrow \Pi_i i^*(\Delta[1] \pitchfork Y_1) \times_{\Pi_i i^* Y_1} Y_1.$$

It suffices to show this is a trivial fibration. Rewriting using  $\Pi_i i^* \simeq \exp_B(A, -)$ , this is the pullback exponential (over  $B$ ) of the right endpoint projection  $\Delta[1] \pitchfork Y_1 \rightarrow Y_1$  with  $A \rightarrow B$ . Since  $Y_1$  is fibrant, the right endpoint projection  $\Delta[1] \pitchfork Y_1 \rightarrow Y_1$  is a trivial fibration by part (iii) of Proposition 2.3.3. Since  $A \rightarrow B$  is a cofibration, the pullback exponential is a trivial fibration by part (ii) of Proposition 2.3.1.

Associating the pullbacks in the middle object of (8) to the left, the second factor in (8) is the pullback along  $Y_1 \rightarrow \Pi_i i^* Y_1$  of the following map, again induced by the right endpoint projection:

$$\Pi_i X_0 \times_{\Pi_i i^* Y_1} \Pi_i i^*(\Delta[1] \pitchfork Y_1) \longrightarrow \Pi_i i^* Y_1.$$

Since the right adjoint  $\Pi_i$  preserves pullbacks, this is the image of the map  $X_0 \times_{i^* Y_1} i^*(\Delta[1] \pitchfork Y_1) \rightarrow i^* Y_1$  under  $\Pi_i$ . And since  $\Pi_i$  preserves trivial fibrations, it suffices to show this is a trivial fibration. Since  $i^*$  preserves cotensors and using the identification  $X_1 \simeq i^* Y_1$ , this is  $X_0 \times_{X_1} \Delta[1] \pitchfork X_1 \rightarrow X_1$ , the second factor in the mapping path space factorization of  $X_0 \rightarrow X_1$ . It is a trivial fibration using the forward direction of Lemma 4.2.2.  $\square$

An important example of a fibrewise homotopy equivalence arises in cofibrant simplicial sets from a fibration with target  $A \times \Delta[1]$ . Note that  $A \times \Delta[1]$  is cofibrant if  $A$  is cofibrant by Lemmas 2.2.9 and 2.4.7. For the below statement, recall that  $(A \times \iota_k)^*$  denotes pullback along the inclusion  $\iota_k: A \rightarrow A \times \Delta[1]$  for  $k = 0, 1$ .

**LEMMA 4.2.4** *Let  $p: X \rightarrow A \times \Delta[1]$  be a fibration with  $A$  and  $X$  cofibrant. There is a fibrewise homotopy equivalence over  $A$  (in either direction) between  $(A \times \iota_0)^* X$  and  $(A \times \iota_1)^* X$ .*

*Proof.* We take the pullback

$$\begin{array}{ccc} P & \longrightarrow & \exp(\Delta[1], X) \\ \downarrow & \lrcorner & \downarrow \exp(\Delta[1], p) \\ A & \xrightarrow{\eta_A} & \exp(\Delta[1], A \times \Delta[1]) \end{array}$$

where the bottom map is a unit map of the evident adjunction. We record that  $P$  is cofibrant by Lemma 2.4.7 and Corollary 2.4.18, and that  $P \twoheadrightarrow A$  is a fibration by Subsection 2.3.3.

We will argue that there are trivial fibrations from  $P$  to  $(A \times \iota_0)^*X$  and  $(A \times \iota_1)^*X$  over  $A$ . These trivial fibrations are fibrewise homotopy equivalences over  $A$  by Corollary 2.6.4. Reversing and composing them in the fibration category of Theorem 2.6.5 as needed gives the claim.

We only construct the trivial fibration from  $P$  to  $(A \times \iota_0)^*X$  (the other case is dual). By pullback pasting, we can see  $(A \times \iota_0)^*X$  as being obtained by two successive pullbacks:

$$\begin{array}{ccccc} (A \times \iota_0)^*X & \longrightarrow & X \times_{A \times \Delta[1]} \exp(\Delta[1], A \times \Delta[1]) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow p \\ A & \xrightarrow{\eta_A} & \exp(\Delta[1], A \times \Delta[1]) & \xrightarrow{\exp(\iota_0, A \times \Delta[1])} & A \times \Delta[1]. \end{array}$$

Taking the pullback exponential of  $p$  with  $\iota_0: \{0\} \rightarrow \Delta[1]$ , we obtain

$$\exp(\Delta[1], X) \rightarrow X \times_{A \times \Delta[1]} \exp(\Delta[1], A \times \Delta[1])$$

over  $\exp(\Delta[1], X)$ . This is a trivial fibration by Proposition 2.3.3. Pulling back along  $\eta_A$  yields the desired trivial fibration.  $\square$

In cofibrant simplicial sets, we say that a (trivial) fibration  $X \rightarrow A$  *extends* along a cofibration  $A \rightarrow B$  if we can find a (trivial) fibration  $Y \rightarrow B$  that restricts to  $X \rightarrow A$  along the given cofibration:

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ A & \longrightarrow & B. \end{array} \quad (9)$$

Our ultimate goal in this subsection is to show that fibrations extend along trivial cofibrations.

**COROLLARY 4.2.5** (Trivial fibration extension) *In cofibrant simplicial sets, trivial fibrations extend along cofibrations.*

*Proof.* This is the special case of Proposition 4.2.3 where the maps  $X_1 \rightarrow A$  and  $Y_1 \rightarrow B$  are identities. To see this, we recall that a map  $X \rightarrow Y$  is a trivial fibration exactly if it is a fibration and fibrewise homotopy equivalence over  $Y$ . This is recorded in Theorem 2.6.5 (with  $M$  instantiated to  $Y$ ).  $\square$

LEMMA 4.2.6 *In cofibrant simplicial sets, fibrations extend along cofibrations with a retraction up to homotopy. In particular, they extend along cofibrations that are strong homotopy equivalences.*

*Proof.* Let  $i: A \rightarrow B$  be a cofibration with  $r: B \rightarrow A$  and a homotopy relating  $ri$  and  $\text{id}_A$  (in either direction). We will solve an extension problem

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow{i} & B. \end{array} \quad (10)$$

We have the solid part of the diagram

$$\begin{array}{ccccc} X & \dashrightarrow & Y & & \\ & \searrow \simeq & & \swarrow \simeq & \\ & (ri)^*X & \xrightarrow{\quad} & r^*X & \\ & \swarrow & & \searrow & \\ A & \xrightarrow{i} & B & & \end{array}$$

where the squares going from left to right are pullbacks. For the fibrewise homotopy equivalence on the left, we apply Lemma 4.2.4 to the pullback of  $X$  along the homotopy  $\Delta[1] \times A \rightarrow A$  relating  $ri$  and  $\text{id}_A$ . We then complete the diagram using equivalence extension, Proposition 4.2.3.  $\square$

COROLLARY 4.2.7 (Fibration extension). *In cofibrant simplicial sets, fibrations extend along trivial cofibrations.*

*Proof.* We work exclusively in cofibrant simplicial sets. Consider the class of cofibrations that fibrations extend along. By Lemma 4.2.6, this includes the horn inclusions (they are cofibrations between cofibrant objects by Lemma 2.2.2 and Corollary 2.4.10 and strong homotopy equivalences by Lemma 4.1.3). Using the presentation of trivial cofibrations given by part (ii) of Proposition 2.2.7, it remains to show this class is closed under coproducts, pushouts, sequential colimits and codomain retracts. In the case of a codomain retract  $A \rightarrow B'$  of  $A \rightarrow B$ , we simply extend along  $A \rightarrow B$  and then pull back along  $B' \rightarrow B$  (cofibrancy is assured by Lemma 2.4.7); by pullback pasting, this gives the required extension along  $A \rightarrow B'$ . In the other cases, all involved colimits in simplicial sets are van Kampen by Lemma 2.2.1 (since cofibrations are monomorphisms).

For a coproduct of cofibrations  $A_s \rightarrow B_s$ , consider a fibration  $X \rightarrow \coprod_s A_s$ . We pull it back to each  $A_s$ , extend separately for each  $s$  along  $A_s \rightarrow B_s$ , and then take the coproduct of the resulting fibrations to get a map  $Y \rightarrow \coprod_s B_s$ . Since  $\coprod_s B_s$  is van Kampen, this map pulls back to the individual fibrations over  $B_s$  for each  $s$ . So it is a fibration by part (ii) of Lemma 2.2.3. Using that  $\coprod_s A_s$  is van Kampen, the map also pulls back to  $X \rightarrow \coprod_s A_s$ .

In the remaining two cases, it is convenient to improve fibration extension to *structured fibration extension*. Recall that a fibration is a map together with a choice of lifts against horn inclusions; recall also the notion of structure morphism of fibrations from Subsection 2.2. In any fibration extension square (9), also the upper horizontal map is a cofibration by Corollary 2.3.2, hence the horizontal maps are levelwise decidable. Using Lemma 2.2.4, we can choose lifts for the right map such that the square becomes a structure morphism of fibrations. We note finally that the colimits in question are all van Kampen (since cofibrations are monomorphisms).

For a pushout  $A' \rightarrowtail B'$  of a cofibration  $A \rightarrowtail B$ , consider a fibration  $X' \twoheadrightarrow A'$ . We first pull it back to a fibration  $X \twoheadrightarrow A$  (using Lemma 2.4.7 for cofibrancy of  $X$ ). Note that the canonically induced choice of lifts for  $X \twoheadrightarrow A$  makes this pullback square into a structure morphism of fibrations. We then use structured fibration extension along  $A \rightarrowtail B$  to produce a fibration  $Y \twoheadrightarrow B$ . Finally, we take the pushout of all three fibrations. Since the given pushout is van Kampen, this map pulls back to the individual fibrations. So it is a fibration by part (ii) of Lemma 2.2.3.

For a sequential colimit of cofibrations  $A_0 \rightarrowtail A_1 \rightarrowtail \dots$ , consider a fibration with target  $A_0$ . We recursively use structured fibration extension to produce a fibration with target  $A_k$  for each  $k$  and then take the sequential colimit of the resulting fibrations. Since the given sequential colimit is van Kampen, this map pulls back to the fibration over  $A_k$  for each  $k$ . So it is a fibration by part (ii) of Lemma 2.2.3.

Note that the domain of the extended fibrations in these three cases is cofibrant by construction. This uses that the top map in any fibration extension square (9) is a cofibration.  $\square$

#### 4.3. Conclusion of the second proof

This subsection is entirely formal. It is divided into two parts. In the first part, ending with Remark 4.3.4, we develop the properties of the model structure on *cofibrant simplicial sets*. We define its weak equivalences from homotopy equivalences via fibrant replacement. The fact that trivial (co)fibrations and acyclic (co)fibrations coincide is deduced from the Frobenius and (trivial) fibration extension properties, all in cofibrant simplicial sets.

In the second part, we extend to the desired model structure on *simplicial sets* (Theorem 4.3.8). We define its weak equivalences from the above ones via cofibrant replacement. The fact that trivial (co)fibrations and acyclic (co)fibrations coincide reduces to the cofibrant case using the restricted Frobenius property and a cancellation property of trivial fibrations.

We define the notion of *strong fibrant replacement* dual to the notion of strong cofibrant replacement of Subsection 3.1. A strong fibrant replacement of a simplicial set  $X$  is a Kan complex  $\bar{X}$  equipped with a trivial cofibration  $X \rightarrow \bar{X}$ . Note that  $\bar{X}$  is cofibrant if  $X$  is cofibrant. A strong fibrant replacement of a map  $f: X \rightarrow Y$  is a map  $\bar{f}: \bar{X} \rightarrow \bar{Y}$  equipped with a square

$$\begin{array}{ccc} X & \xrightarrow{+} & \bar{X} \\ f \downarrow & & \downarrow \bar{f} \\ Y & \xrightarrow{+} & \bar{Y} \end{array}$$

where  $\bar{X}$  and  $\bar{Y}$  are Kan complexes and both horizontal maps are trivial cofibrations. Strong fibrant replacements can be constructed using Proposition 2.2.7.



Using strong fibrant replacement, we may lift the notion of homotopy equivalence for cofibrant Kan complexes to a notion of weak equivalence in cofibrant simplicial sets. We say that a map in cofibrant simplicial sets is a *weak homotopy equivalence* if it has a strong fibrant replacement that is a homotopy equivalence.

LEMMA 4.3.1 *In cofibrant simplicial sets:*

- (i) *the notion of weak homotopy equivalence does not depend on the choice of strong fibrant replacements,*
- (ii) *a map between Kan complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence and*
- (iii) *weak homotopy equivalences satisfy 2-out-of-6.*

*Proof.* Part (i) holds by the dual of the argument of Lemma 3.1.2 (considering strong fibrant replacement instead of strong cofibrant replacement). This uses the fact that trivial cofibrations between cofibrant Kan complexes are homotopy equivalences by part (ii) of Lemma 4.3.2. With this, we prove the remaining parts. For part (ii), we take the map itself as its strong cofibrant replacement. For part (iii), functorial fibrant replacement (via Proposition 2.2.7) creates weak homotopy equivalences from homotopy equivalences between cofibrant Kan complexes. The latter satisfy 2-out-of-6 by part (i) of Lemma 2.5.2 and hence so do the former.  $\square$

LEMMA 4.3.2 *Let  $f: X \rightarrow Y$  be a fibration in cofibrant simplicial sets. Then  $f$  is a trivial fibration if and only if it is a weak homotopy equivalence.*

*Proof.* Assume first that  $f: X \rightarrow Y$  is a trivial fibration. We take a strong fibrant replacement  $Y \rightarrow \bar{Y}$  and extend the given trivial fibration along it using Corollary 4.2.5, obtaining a pullback square

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \bar{X} \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{\quad} & \bar{Y}. \end{array}$$

By the Frobenius property in cofibrant simplicial sets (Proposition 4.1.6),  $X \rightarrow \bar{X}$  is a trivial cofibration, so  $X \rightarrow \bar{X}$  is a strong fibrant replacement. As a trivial fibration between cofibrant objects,  $\bar{X} \rightarrow \bar{Y}$  is a homotopy equivalence by part (i) of Lemma 2.5.4. Thus,  $X \rightarrow Y$  is a weak homotopy equivalence.

Conversely, assume that  $f: X \rightarrow Y$  is a weak homotopy equivalence. We extend  $f$  along  $Y \rightarrow \bar{Y}$  using Corollary 4.2.7, obtaining a pullback square

$$\begin{array}{ccc} X & \dashrightarrow & \bar{X} \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{\quad} & \bar{Y}. \end{array}$$

By Proposition 4.1.6,  $X \rightarrow \bar{X}$  is a trivial cofibration. This makes the above square a strong fibrant replacement. So  $\bar{X} \rightarrow \bar{Y}$  is a homotopy equivalence by part (i) of Lemma 4.3.1. Since it is also a fibration between cofibrant Kan complexes, it is a trivial fibration by Corollary 2.6.4. As its pullback, so is  $f$ .  $\square$

**LEMMA 4.3.3** *Let  $f: A \rightarrow B$  be a cofibration in cofibrant simplicial sets. Then  $f$  is a trivial cofibration if and only if it is a weak homotopy equivalence.*

*Proof.* Assume that  $f$  is a trivial cofibration. Take a strong fibrant replacement  $B \rightarrow \bar{B}$ . The identity on  $\bar{B}$  is a strong fibrant replacement of  $f$ , which is thus a weak homotopy equivalence.

The converse direction is a formal consequence of the forward direction and Lemma 4.3.2. In detail, factor  $f$  as a trivial cofibration  $j$  followed by a fibration  $p$ . Assuming  $f$  is a weak homotopy equivalence, so is  $p$  by 2-out-of-3. By Lemma 4.3.2,  $p$  is a trivial fibration. Since  $f$  lifts against its second factor  $p$ , it is a retract of its first factor  $j$ , a trivial cofibration.  $\square$

The above statements make weak homotopy equivalences and the two weak factorization systems under consideration a model structure on cofibrant simplicial sets.

**REMARK 4.3.4** The approach of [33] (in the setting of cofibrant simplicial sets) uses a different definition of weak homotopy equivalences: those maps factoring as a trivial cofibration followed by a trivial fibration. Then acyclic (co)fibrations coincide with trivial (co)fibrations for formal reasons (proved here as Lemmas 4.3.3 and 4.3.2) and the goal shifts to showing that weak homotopy equivalences satisfy 2-out-of-3 (holding here essentially by construction). However, this is only a superficial difference in setup: the overall work ends up being the same. We have chosen our current definitions to achieve greater similarity with the below extension to arbitrary simplicial sets (via strong cofibrant replacement) and the setup in Subsection 3.1.

We now extend our notion of weak homotopy equivalence to arbitrary simplicial sets, not necessarily cofibrant. Recall the notion of *strong cofibrant replacement* of Subsection 3.1, dual to that of strong fibrant replacement. We say that a map in simplicial sets is a *weak homotopy equivalence* if it has a strong cofibrant replacement that is a weak homotopy equivalence in cofibrant simplicial sets.

**LEMMA 4.3.5** *In simplicial sets:*

- (i) *the notion of weak homotopy equivalence does not depend on the choice of strong cofibrant replacements,*
- (ii) *a map between cofibrant simplicial sets is a weak homotopy equivalence (in the above sense) if and only if it is a weak homotopy equivalence in cofibrant simplicial sets and*
- (iii) *weak homotopy equivalences satisfy 2-out-of-6.*

*Proof.* This is dual to Lemma 4.3.1. Part (i) holds by the argument of Lemma 3.1.2 (replace Kan complexes by arbitrary simplicial sets, (W1) by weak homotopy equivalence in cofibrant simplicial sets and (W2) by weak homotopy equivalences). This uses the fact that trivial fibrations in cofibrant simplicial sets are weak homotopy equivalences by Lemma 4.3.2. With this, parts (ii) and (iii) reduce to the respective parts of Lemma 4.3.1.  $\square$

In light of part (ii) of Lemma 4.3.5, the meanings of the term ‘weak homotopy equivalence’ in cofibrant simplicial sets and simplicial sets are compatible. We may thus use the term without qualification.

**LEMMA 4.3.6** *Let  $f: X \rightarrow Y$  be a fibration in simplicial sets. Then  $f$  is a trivial fibration if and only if it is a weak homotopy equivalence.*

*Proof.* Assume that  $f$  is a trivial fibration. Take a strong cofibrant replacement  $\tilde{X} \twoheadrightarrow X$ . The identity on  $\tilde{X}$  is a strong cofibrant replacement of  $f$ , which is thus a weak homotopy equivalence.

Conversely, let  $f$  be an acyclic fibration. Take a strong cofibrant replacement  $\tilde{Y} \twoheadrightarrow Y$  followed by a strong cofibrant replacement of the pullback of  $X$  along this map:

$$\begin{array}{ccccc} \tilde{X} & \twoheadrightarrow & X' & \twoheadrightarrow & X \\ & & \downarrow & \lrcorner & \downarrow \\ & & \tilde{Y} & \twoheadrightarrow & Y. \end{array}$$

Since  $X \rightarrow Y$  is a weak homotopy equivalence, its strong cofibrant replacement  $\tilde{X} \rightarrow \tilde{Y}$  is a weak homotopy equivalence between cofibrant objects. By Lemma 4.3.2, it is a trivial fibration. Applying Lemma 2.2.10 in the above diagram, we obtain that  $X \rightarrow Y$  is a trivial fibration.  $\square$

**LEMMA 4.3.7** *Let  $f: A \rightarrow B$  be a cofibration in simplicial sets. Then  $f$  is a trivial cofibration if and only if it is a weak homotopy equivalence.*

*Proof.* Let  $A \twoheadrightarrow B$  be a trivial cofibration. Take a strong cofibrant replacement  $\tilde{B} \twoheadrightarrow B$ . Consider the pullback square

$$\begin{array}{ccc} \tilde{A} & \twoheadrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ \tilde{B} & \twoheadrightarrow & B. \end{array}$$

Note that  $\tilde{A}$  is cofibrant by Corollary 2.3.2. In particular  $\tilde{A} \rightarrow \tilde{B}$  is a strong cofibrant replacement of  $A \rightarrow B$ . Since  $\tilde{B}$  is cofibrant, the restricted Frobenius property (Proposition 4.1.6) makes  $\tilde{A} \rightarrow \tilde{B}$  a trivial cofibration. By Lemma 4.3.3, it is a weak homotopy equivalence in cofibrant simplicial sets. This makes  $A \rightarrow B$  a weak homotopy equivalence.

The converse direction follows from the forward direction together with Lemma 4.3.6 using 2-out-of-3 by the retract argument as in Lemma 4.3.3.  $\square$

**THEOREM 4.3.8** *The cofibrations, Kan fibrations and weak homotopy equivalences form a model structure on simplicial sets.*

*Proof.* Weak homotopy equivalences satisfy 2-out-of-6 by part (iii) of Lemma 4.3.5. We have established the two weak factorization systems in Proposition 2.2.7. By Lemma 4.3.6, acyclic fibrations coincide with trivial fibrations. By Lemma 4.3.7, trivial cofibrations coincide with acyclic cofibrations.  $\square$

REMARK 4.3.9 Any model structure is determined by its classes of cofibrations and fibrations. So the model structure of Theorem 4.3.8 coincides with the one of Theorem 3.6.5. It follows, after the fact, that all considered notions of weak homotopy equivalence are equivalent in their respective subcategories of cofibrant and/or fibrant simplicial sets.

REMARK 4.3.10 Following up on Remark 3.6.6, our second construction of the constructive Kan-Quillen model structure differs from [18] in that it avoids the  $\text{Ex}^\infty$  functor and semisimplicial homotopy theory. Instead, following [33], it is based on a restricted version of the Frobenius property [14] and the equivalence extension property in cofibrant simplicial sets.

## 5. Right and left properness

PROPOSITION 5.1 *The Kan-Quillen model structure on simplicial sets is right proper.*

*Proof.* ( $\text{Ex}^\infty$  argument) Consider a fibration  $p: X \twoheadrightarrow Y$  and a weak homotopy equivalence  $f: B \rightarrow Y$ . Form the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{\quad} & X & & \\
 \nwarrow \sim & & \nwarrow \sim & & \\
 \tilde{A} & \xrightarrow{\quad} & \tilde{X} & & \\
 \searrow \sim & & \searrow \sim & & \\
 & \text{Ex}^\infty \tilde{A} & \xrightarrow{\quad} & \text{Ex}^\infty \tilde{X} & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 B & \xrightarrow{\quad} & Y & & \\
 \nwarrow \sim & & \nwarrow \sim & & \\
 \tilde{B} & \xrightarrow{\quad} & \tilde{Y} & & \\
 \searrow \sim & & \searrow \sim & & \\
 & \text{Ex}^\infty \tilde{B} & \xrightarrow{\quad} & \text{Ex}^\infty \tilde{Y} &
 \end{array}$$

as follows.

- (i) The back square is a pullback.
- (ii)  $\tilde{Y} \twoheadrightarrow Y$  is a strong cofibrant replacement.

- (iii)  $\tilde{B} \rightarrowtail B$  and  $\tilde{X} \rightarrowtail X$  are strong cofibrant replacements, the latter chosen so that  $\tilde{X} \rightarrow X \times_Y \tilde{Y}$  is a trivial fibration.
- (iv) The middle square is a pullback.
- (v) The front square is obtained by applying  $\text{Ex}^\infty$  to the middle one and the transformation between them is  $\nu^\infty$  of Subsection 3.4.

Denote the maps  $\tilde{X} \rightarrow \tilde{Y}$  and  $\tilde{B} \rightarrow \tilde{Y}$  by  $\tilde{p}$  and  $\tilde{f}$ , respectively. Note that  $\tilde{A} \rightarrow A$  is a trivial fibration as the composite of pullbacks of  $\tilde{B} \rightarrow B$  and  $\tilde{X} \rightarrow X \times_Y \tilde{Y}$ . Thus all middle to back maps are weak homotopy equivalences by part (iii) of Lemma 3.1.7. Moreover,  $\tilde{A}$  is cofibrant by Lemma 2.4.7 so Proposition 3.4.6 implies that all middle to front maps are also weak homotopy equivalences. It follows that  $\text{Ex}^\infty \tilde{f}$  is a weak homotopy equivalence since  $f$  is. Moreover, the front square is a pullback by part (i) of Proposition 3.4.4,  $\text{Ex}^\infty \tilde{p}$  is a Kan fibration by part (ii) of Proposition 3.4.4 and all objects in the front face are Kan complexes by part (iv) of Proposition 3.4.4. It follows that  $\text{Ex}^\infty A \rightarrow \text{Ex}^\infty X$  is a weak homotopy equivalence by Theorem 3.2.2 and [31, Lemma 1.4.2 (2b)] and thus so is  $A \rightarrow X$ .  $\square$

*Proof (Frobenius property argument).* It suffices to show that the pullback of a trivial cofibration  $B \rightarrow Y$  along a fibration  $X \rightarrow Y$  is a weak equivalence. Consider a further pullback along a cofibrant replacement  $\tilde{X} \rightarrow X$ :

$$\begin{array}{ccccc} A' & \xrightarrow{\sim} & A & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \sim \\ \tilde{X} & \xrightarrow{\sim} & X & \longrightarrow & Y. \end{array}$$

Since  $\tilde{X}$  is cofibrant,  $A' \rightarrow \tilde{X}$  is a trivial cofibration by the restricted Frobenius property (Proposition 4.1.6). Then  $A \rightarrow X$  is a weak equivalence by 2-out-of-3.  $\square$

Semisimplicial sets are presheaves over  $\Delta_\sharp$ , the subcategory of face operators in  $\Delta$ . Let  $U$  be the forgetful functor from simplicial to semisimplicial sets and let  $L$  be its left adjoint, with unit  $\eta$  and counit  $\varepsilon$ .

Let  $\Delta_\sharp[m]$  denote the semisimplicial set represented by  $[m]$ ; its boundary  $\partial\Delta_\sharp[m]$  is obtained by removing the top simplex  $\text{id}_{[m]}$ . The category of semisimplicial sets carries a weak factorization system generated by the boundary inclusions  $\partial\Delta_\sharp[m] \hookrightarrow \Delta_\sharp[m]$ . A *cofibration* is a morphism of the left class of this weak factorization system.

**LEMMA 5.2** *A semisimplicial map is a cofibration if and only if it is a levelwise decidable inclusion.*

*Proof.* Cofibrations coincide with Reedy decidable inclusions by the same argument as in Lemma 2.4.1. Since  $\Delta_\sharp$  is a direct category, the latter are the same as levelwise decidable inclusions.  $\square$

**COROLLARY 5.3** *The functor  $L$  preserves cofibrations.*

*Proof.* It suffices to check that  $L$  preserves boundary inclusions. Indeed, we have  $\partial\Delta_{\#}[m] = \operatorname{colim}_{[k] \xrightarrow{\partial} [m]} \Delta_{\#}[k]$  and thus  $L\partial\Delta_{\#}[m] = \operatorname{colim} L\Delta_{\#}[k] = \operatorname{colim} \Delta[k] = \partial\Delta[m]$ .  $\square$

**COROLLARY 5.4** *The functor  $U$  sends levelwise decidable inclusions to cofibrations. In particular, it preserves cofibrations.*

*Proof.* The first claim follows from Lemma 5.2. With this, the second claim follows from Corollary 2.4.6.  $\square$

**PROPOSITION 5.5** *The functor  $LU$  satisfies the following conditions:*

- (i) *It preserves colimits.*
- (ii) *It takes values in cofibrant simplicial sets.*
- (iii) *It preserves cofibrations.*
- (iv) *For every simplicial set  $X$ , the counit  $\varepsilon_X: LUX \rightarrow X$  is a weak homotopy equivalence.*

*Proof.* Part (i) holds since both  $L$  and  $U$  are left adjoints. For part (ii), note that every semisimplicial set is cofibrant by Lemma 5.2 while  $L$  preserves cofibrant objects by Corollary 5.3. Part (iii) follows directly from Corollaries 5.3 and 5.4. We postpone the proof of part (iv) until we show an auxiliary lemma.  $\square$

**REMARK 5.6** In Subsection 3.5, the verification of properties of the cofibrant replacement functor  $T$  makes use of another cofibrant replacement functor, sending a simplicial set  $X$  to  $N(\Delta_{\#} \downarrow UX)$ , the nerve of the category of elements of the semisimplicial set underlying  $X$ . This functor factors as  $LU$  followed by subdivision.

Semisimplicial sets carry a closed symmetric monoidal product called the *geometric product*. It is uniquely determined by setting the geometric product of  $\Delta_{\#}[m]$  and  $\Delta_{\#}[n]$  to the semisimplicial set consisting of the non-degenerate simplices of  $\Delta[m] \times \Delta[n]$ . More precisely, if we denote that semisimplicial set by  $P_{m,n}$ , then the geometric product of semisimplicial sets  $X$  and  $Y$  is the coend  $\int^{m,n} X_m \times Y_n \times P_{m,n}$ . It follows that the geometric product preserves colimits in each variable. Since  $L$  also preserves colimits and satisfies  $LP_{m,n} = \Delta[m] \times \Delta[n]$ , it follows that it is monoidal, that is, it carries the geometric product to the Cartesian product.

**LEMMA 5.7** *The functor  $LU$  sends trivial fibrations to weak equivalences.*

*Proof.* Given a trivial fibration  $p: X \rightarrow Y$ , we will show that  $LU p: LUX \rightarrow LUY$  is a homotopy equivalence. It is then a weak equivalence for example by part (iv) of Lemma 3.1.7.

Note that  $LUX$  and  $LUY$  are cofibrant by part (ii) of Proposition 5.5. We take a lift

$$\begin{array}{ccc}
\emptyset & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow p \\
LUY & \xrightarrow{\varepsilon_Y} & Y.
\end{array}$$

Transposing the square using the adjunction  $L \dashv U$  and applying  $L$  gives a section  $s: LUY \rightarrow LUX$  of  $LU p$ .

Using part (i) of Proposition 2.3.3, we take a lift

$$\begin{array}{ccccc}
LUX \times \partial\Delta[1] & \xrightarrow{[s(LUp), \text{id}]} & LUX & \xrightarrow{\varepsilon_X} & X \\
\downarrow & \nearrow u' & & & \downarrow p \\
LUX \times \Delta[1] & \longrightarrow & LUY \times \Delta[1] & \longrightarrow & LUY \xrightarrow{\varepsilon_Y} Y.
\end{array}$$

Note that the left arrow lies in the essential image of  $L$  (it is isomorphic to the image of the geometric product of  $\partial\Delta[1] \rightarrow \Delta[1]$  with  $UX$ ) and so does the top left arrow (by construction of  $s$ ). Using terminality of  $(UX, \varepsilon_X)$  in  $L \downarrow X$  and applying  $L$ , we obtain the dashed arrow in the commuting triangle

$$\begin{array}{ccc}
LUX \times \partial\Delta[1] & \xrightarrow{[s(LUp), \text{id}]} & LUX. \\
\downarrow & \nearrow u & \\
LUX \times \Delta[1] & & 
\end{array}$$

We thus have  $s(LUp) \sim \text{id}_{LUX}$  and  $(LUp)s = \text{id}_{LUY}$  as desired.  $\square$

**REMARK 5.8** The proof of Lemma 5.7 is the combination and unfolding of the following elementary observations regarding semisimplicial sets, which carry notions of homotopy and homotopy equivalence based on the geometric product:

- $U$  preserves trivial fibrations (defined also in semisimplicial sets as lifting against boundary inclusions),
- for a simplicial set  $X$ , the identity of  $UX$  admits an endohomotopy,
- in semisimplicial sets, every trivial fibration  $Y \rightarrow X$  with an endohomotopy on the identity on  $X$  extends to a homotopy equivalence,
- $L$  preserves homotopy equivalences.

*Proof of part (iv) of Proposition 5.5.* First, note that  $\varepsilon_{\Delta[m]}: LU\Delta[m] \rightarrow \Delta[m]$  is a weak homotopy equivalence for all  $m$ . Indeed,  $LU\Delta[m]$  is the nerve of a category  $[m]'$  which is obtained from  $[m]$  by adjoining an idempotent endomorphism to every object that acts trivially on morphisms of  $[m]$ . This category admits a natural transformation from the endofunctor constant at 0 to the identity endofunctor. Thus, its nerve is contractible and the conclusion follows. Since we have already verified

that  $LU$  preserves colimits and cofibrations, the argument of part (iv) of Proposition 3.5.5 shows that  $\varepsilon_X$  is a weak homotopy equivalence for all cofibrant  $X$ . (That argument is an instance of the following general fact for a natural transformation  $u$  between endofunctors of  $\mathbf{sSet}$  that preserve colimits and cofibrations: if  $u$  is a weak homotopy equivalence on simplices, then  $u$  is a weak homotopy equivalence on all cofibrant objects.) The same follows for arbitrary  $X$  by Lemma 5.7.  $\square$

**PROPOSITION 5.9** *The Kan–Quillen model structure on simplicial sets is left proper.*

*Proof.* The following argument uses the functor  $T$  of Subsection 3.5 and its properties listed in Proposition 3.5.5. An alternative argument is obtained by substituting  $LU$  for  $T$  since the parallel properties have been verified in Proposition 5.5.

Consider a cofibration  $A \rightarrowtail B$  and a weak homotopy equivalence  $A \rightarrow X$ . Form the diagram

$$\begin{array}{ccccc}
 TA & \xrightarrow{\quad} & TX & & \\
 \downarrow & \searrow \sim & \downarrow & \searrow \sim & \\
 & A & \xrightarrow{\quad} & X & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 TB & \xrightarrow{\quad} & TY & & \\
 \downarrow & \searrow \sim & \downarrow & \searrow \sim & \\
 & B & \xrightarrow{\quad} & Y & 
 \end{array}$$

where the front square is a pushout and the remainder of the cube is given by the copointed functor  $T$ . Part (iv) of Proposition 3.5.5 implies that all back to front maps are weak homotopy equivalences. It follows that  $TA \rightarrow TX$  is a weak homotopy equivalence since  $A \rightarrow X$  is. Moreover, the back square is a pushout by part (i) of Proposition 3.5.5,  $TA \rightarrow TB$  is a cofibration by part (iii) of Proposition 3.5.5 and all objects in the back face are cofibrant by part (ii) of Proposition 3.5.5. It follows that  $TB \rightarrow TY$  is a weak homotopy equivalence by Theorem 3.2.3 and [31, Lemma 1.4.2 (1b)], and thus so is  $B \rightarrow Y$ .  $\square$

Using the cofibrant replacement functor  $LU$ , we obtain a slightly stronger statement.

**PROPOSITION 5.10** *In the category of simplicial sets, pushout along a levelwise decidable inclusion preserves weak homotopy equivalences. In particular, every pushout along a levelwise decidable inclusion is a homotopy pushout.*

*Proof.* By Corollaries 5.4 and 5.3, the functor  $LU$  carries levelwise decidable inclusions to cofibrations. With this, the conclusion follows exactly as in the proof of Proposition 5.9. (Here, we used the fact that levelwise decidable inclusions are stable under pushout by Proposition 2.1.6 to reduce to pushout squares as in that proof.)  $\square$



REMARK 5.11 We conclude by describing the relationship between our proofs of properness and the one given in [18]. For left properness, the proof in [18] relies on the existence of a weak model structure (in the sense of [17]) on the category of semisimplicial sets and uses its interaction with the adjunction  $L \dashv U$  with simplicial sets.

Here, in one version of the argument, we use the comonad  $LU$  of the adjunction  $L \dashv U$  to model cofibrant replacement, but using only the notion of cofibration in semisimplicial sets and not any further semisimplicial homotopy theory. In particular, we do not need to know that  $U$  preserves cylinder objects. The other version also circumvents semisimplicial homotopy theory and uses a completely different cofibrant replacement functor  $T$  in place of  $LU$ , although there is a relation as explained in Remark 5.6.

For right properness, our first proof is very similar to the one in [18], but we included it for completeness, also because it follows naturally from our discussion of the  $\text{Ex}^\infty$  functor in Subsection 3.4. Our second proof is entirely different and goes via the restricted Frobenius property proved directly in Subsection 4.1.

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