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Article

Investment decisions with two-factor uncertainty

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Abstract: This paper considers investment problems in real options with non-homogeneous two-factor uncertainty. We derive some analytical properties of the resulting optimal stopping problem and present a finite difference algorithm to approximate the firm's value function and optimal exercise boundary. An important message of our paper is that the frequently applied quasi-analytical approach underestimates the impact of uncertainty. This is caused by the fact that the quasi-analytical solution does not satisfy the partial differential equation that governs the value function. As a result, the quasi-analytical approach may wrongly advise to invest in a substantial part of the state space.

Keywords: Investment analysis; Optimal stopping time problem; Two-factor uncertainty.

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1. Introduction

Since the seminal works of Dixit and Pindyck (1994) and Trigeorgis (1996), it has become clear that real investments should be valued using a real options approach when decision makers are exposed to a significant amount of uncertainty. In these cases, application of the standard net present value decision rule can lead to investment decisions that are significantly sub-optimal, as is extensively demonstrated in the above books. Since firm investment decisions lie at the basis of economic growth, it is crucial to take these decisions in the right way. From this perspective it is clear that it is of main importance to work on the development of the theory of real options.

In the basic analysis the real options model consists of a single firm having the opportunity to invest in a project of given size, with revenue that is subject to uncertainty that is governed by a single stochastic process. Several authors have extended this framework in different directions. Smets (1991) is the first to consider a scenario where two firms can invest in the same market. The revenue in this market is still governed by one stochastic process, also after both firms have already invested and thus are active in this market. The assumption "project of given size" is relaxed in Bar-Ilan and Strange (1999) and Dangl (1999), in which the firm not only needs to decide about the time, but also about the size of the investment. Huisman and Kort (2015) combine the two extensions by considering a duopoly market where both firms also have to determine the investment size.

Most of the real options literature uses a single (one-dimensional) stochastic process to model the evolution of random shocks affecting the investment's value. This can be a major shortcoming, especially when analysing problems with, e.g., multiple firms or products. Such investment problems are especially common in the field of energy and environmental economics (Agaton and Collera 2022; Deeney et al. 2021; Li and Cao

2022; Zhang et al. 2021). The transition to a circular and low-carbon, bio-based economy requires firms to shift to the use of renewable resources, to cooperate with firms in other markets, and to valorize their waste streams (The European Commission 2019 2021). These decisions expose firms to different types of risks, creating the need for models that account for multiple sources of uncertainty. Therefore, in this paper we consider a real options problem with multiple uncertain factors, which is an important extension to the basic real options analysis, especially from a practical perspective.

The first real options model with two-factor uncertainty occurs in McDonald and Siegel (1986). Their value function is homogenous of degree one, and the two stochastic processes are the output price and the investment cost. In such cases, the investment threshold level can be determined for the price-to-cost ratio. This allows to reformulate the problem in terms of the relative price, and to reduce the number of stochastic variables to one. In this way a standard one-factor real options model is obtained for which a closed-form solution exists. The result of this analysis is, however, not a threshold point but a threshold boundary at which it is optimal to invest (Nunes and Pimentel 2017). Hu and Øksendal (1998) generalize this solution to the n-dimensional case. Armada et al. (2013) consider a problem where the output price and quantity are stochastic. Here the dimension of the state space can be reduced to a one-dimensional space, because the only relevant payoff variable is revenue (price times quantity). The problem then reduces to finding an optimal revenue threshold that makes investment optimal.

Several authors have tried to use this dimension-reduction approach to cases characterized by multiple stochastic processes and a constant sunk cost. Huisman et al. (2013) and Compernolle et al. (2017) consider price and cost uncertainty and determine the investment threshold level for the price-to-cost ratio. However, there are some problems with this approach. In the presence of a constant sunk cost investment, homogeneity does not hold. For this reason, the state space cannot be reduced to a one-dimensional one. This is also revealed in these papers, because two processes (price/cost and cost) remain present in the equations.

For problems of this kind, Adkins and Paxson (2011b) propose a quasi-analytical approach that results in a set of equations to determine the optimal investment boundary. They solve this set of equations simultaneously while keeping one of the stochastic threshold variables fixed. The present paper shows that this methodology can lead to sub-optimal solutions. In fact, the results of the quasi-analytical approach will generically-speaking be incorrect and there is no guarantee that it converges in any meaningful sense to the correct solution. To put it succinctly, the main problem is that Adkins and Paxson (2011b) use a "local" approach to solve a "global" problem, which can lead to misleading results. Consequently, while the method is intuitively appealing and relatively easy to implement, we argue that care is required in checking that the results conform to economic intuition.

In this paper we provide numerical examples for which the quasi-analytical approach violates certain properties of the optimal boundary that can be analytically established. The point is, in a nutshell, that when solving the partial differential equation that governs the value function, two power parameters turn out to be a function of the state variables, where the quasi-analytical approach starts with the assumption that these parameters are constant.

Our contribution to the literature is three-fold. Firstly, we alert the research community to potential pitfalls in a regularly-used numerical method. In the literature, we find several papers concerning investment problems where the uncertainty is driven by multi-dimensional stochastic processes, and where no analytical solution can be derived. In such cases, the authors propose ways to circumvent the problem and come out with an approximation of the solution. For instance, we refer to Kauppinen et al. (2018), where the model proposed by Adkins and Paxson (2011b) is extended, by adding time to build to the investment problem. In the context of replacement options, Adkins and Paxson (2013a) examine premature and postponed replacement in the presence of technological progress, where revenue and operating costs are treated as geometric Brownian motions. Adkins and Paxson (2017b) use a general replacement model to investigate when it is optimal to replace an asset whose operating cost and salvage value deteriorate stochastically.

The need to take into account multi-sources of uncertainty is also present in problems related with investments in the energy sector. For example, Adkins and Paxson (2011a) solve a switching model for two alternative energy inputs with fixed switching costs. Boomsma and Linnerud (2015) examine investment in a renewable energy project under both market and policy uncertainty. Fleten et al. (2016) study investment decisions in the renewable energy sector, where the revenue comes from selling electricity and from receiving subsidies, both stochastic. Adkins and Paxson (2016) consider the optimal investment policy for an energy facility with price and quantity uncertainty under different subsidy schemes. Støre et al. (2018) determine the optimal timing to switch from oil to gas production in the tail production phase, with the price of oil and gas following (correlated) Geometric Brownian motions. Finally, we refer to Heydari et al. (2012), who extend the quasi-analytical approach proposed in Adkins and Paxson (2011b) to a three-factor model, which is employed to value the choice between two emissions-reduction technologies assuming that the value of each option depends on fuel, electricity and CO_2 prices, all following (correlated) Geometric Brownian motions.

Secondly, while it could, a priori, be the case that the approximation obtained by the quasi-analytical approach is close enough to the true solution to be of practical value, we show that for the models under consideration in much of the literature this is not necessarily the case. For example, in the model with two uncertain revenue flows we find that in some situations the investment boundary is *de*creasing in uncertainty environment. This violates one of the major qualitative result from real options theory: "an increase in uncertainty leads to an increase in project value". We formally prove that this feature also holds for the model under consideration.

Thirdly, we develop a numerical algorithm which is based on a finite difference scheme and we apply this algorithm to a model with two stochastic revenue streams. We determine the optimal timing of investment in the presence of a constant sunk investment cost. Note that this model is different from the one analysed in Adkins and Paxson (2011b), who analyse a stochastic revenue and a (possibly correlated) stochastic cost. Importantly, our finite difference scheme *does* exhibit the expected behavior in relation to an increase in uncertainty.

In the literature most finite-difference schemes have been developed to solve models with a one-dimensional stochastic process and a finite time horizon. This method typically employs a backward induction argument in the time dimension to approximate the optimal exercise boundary and value function in a step-by-step fashion; see, e.g., (Dixit and Pindyck 1994, Appendix 10.A). For a two-dimensional problem this approach does not work, because both processes can move up or down in any time step. Therefore, we suggest a finite difference scheme that starts with a hypothesized boundary, after which the value function is approximated at all points in the two-dimensional finite grid at once. A discretized smooth pasting condition (in two dimensions) can then be used to judge the quality of the hypothesized boundary. This procedure is repeated until an acceptable approximation to the optimal boundary is found.

Note that to solve multidimensional optimal stopping problems, also other numerical approaches could be applied. Among the relevant contributions is Lange et al. (2020). In this paper, the authors consider that the decision to stop can only be taken at specific times, generated by an exogenous Poisson process with intensity rate λ. This means that the set of admissible stopping times for the optimization problem is the set of events of a Poisson process, independent of the filtration generated by the state variables. In this setting, the optimization problem may be written as a fixed-point problem, for which the authors propose a numerical scheme, providing proof and rate of convergence.

In our paper, we prove some analytical properties of the optimal boundary, notably we prove that is convex, non-increasing and continuous in $\mathbb{R}^+$. In Dammann and Ferrari (2021) we may find similar results, but using different arguments, that rely on a probabilistic representation of the Value Function. Moreover, the authors show that the boundary is the unique solution of an integral condition. By use of this integral equation, they prove monotonicty of the value function with respect to drift and volatility of the involved parameters. Finally, they propose a numerical approach to find the boundary, based on the integral equation using Monte-Carlo simulation.

The remainder of this paper is organized as follows. Section 2 introduces a non-homogeneous investment problem characterized by two uncertain revenue flows. Section 3 applies the methodological approach in Adkins and Paxson (2011b) to solve the model presented in Section 2, and highlights the mathematical problems with the solution. Section 4 proposes an alternative numerical approach to solve the model. Section 5 concludes. Proofs of propositions are presented in Appendix A.

2. Investment decision given two uncertain revenue flows

Consider a profit-maximizing, risk-neutral firm that has the opportunity to invest in a production plant by paying a constant investment cost, I. The plant can produce two different products, the prices of which are stochastic and follow correlated geometric Brownian motions X and Y, i.e.,

$$dX_t = \alpha_1 X_t dt + \sigma_1 X_t dW_{X,t}, \quad dY_t = \alpha_2 Y_t dt + \sigma_2 Y_t dW_{Y,t} \tag{1}$$

with

$$X_0 = x, Y_0 = y \tag{2}$$

being the initial values of the processes X and Y, respectively. We note that in Equation (1), α_1 (α_2) denotes the drift of the process X (Y), whereas σ_1 (σ_2) is the volatility of X (Y). Following the usual notation, we let $\{W_t, t \geq 0\}$ denote a standard two-dimensional Brownian motion, which we index by X and Y, respectively. We allow these processes to be correlated, so that $\mathbb{E}[dW_{X,t} dW_{Y,t}] = \rho dt$ for some $\rho \in (-1, 1)$, where $\rho > 0$ ($\rho < 0$) means that W_X and W_Y are positively (negatively) correlated.

At any instant, if the prices of the two products are x and y, respectively, then the instantaneous profit of the firm is given by:

$$\pi(x, y) = Q_1 x + Q_2 y, \tag{3}$$

where Q_1 and Q_2 denote the quantities of the products produced. Moreover, at that instant the firm's value is equal to the perpetual revenue flow from selling two products:[1]

$$F(x, y) = \mathbb{E}_{(x,y)}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(X_t, Y_t)dt - I\right] = \frac{Q_1 x}{\delta_1} + \frac{Q_2 y}{\delta_2} - I, \tag{4}$$

with $\delta_i := r - \alpha_i, i \in \{1, 2\}$, and r being the discount rate. To ensure finite integrals, we assume that $r > \max\{0, \alpha_1, \alpha_2\}$. Equation (4) gives the expected value of the discounted stream of profits that result from operating the production process forever, given current prices x and y. That is, $\mathbb{E}_{(x,y)}$ denotes the expectation operator, conditional on the initial state being $(X_0, Y_0) = (x, y)$.

The firm needs to determine the optimal time to undertake the investment, and, thus, solves the following optimal stopping problem

$$V(X_t, Y_t) = \sup_{\tau \geq t} \mathbb{E}_{(X_t, Y_t)}\left[\mathrm{e}^{-r(\tau - t)} F(X_\tau, Y_\tau)\right] \tag{5}$$

[1] $\mathbb{E}_{(x,y)}$ denotes the expectation conditional on $(X_0, Y_0) = (x, y)$.

where the supremum is taken over all stopping times τ with respect to the filtration generated by the joint process (W_X, W_Y). That is, we are looking for the optimal time to invest in the production plant, given the current values for the price of each one of the two types of product, such that we maximize the expected value of the overall profit. By maximizing over stopping times we recognize that the optimal time to invest may depend on the stochastic evolution of the product prices.

Using standard calculations from optimal stopping theory (see, e.g., Øksendal and Sulem (2007)), we derive the following Hamilton-Jacobi-Bellman (HJB) equation for this problem:

$$\min\{rV(x,y) - \mathcal{L}V(x,y), V(x,y) - F(x,y)\} = 0, \quad \forall\,(x,y) \in \Re_+ \times \Re_+. \tag{6}$$

Here $\mathcal{L}$ denotes the infinitesimal generator of the process (X,Y):

$$\mathcal{L}V(x,y) = \lim_{h \downarrow 0} \frac{E[V(x+h, y+h)] - V(x,y)}{h}$$

which is given by (Øksendal and Sulem (2007)):

$$\begin{aligned}\mathcal{L}V(x,y) &= \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 V(x,y)}{\partial x^2} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 V(x,y)}{\partial y^2} + \rho\sigma_1\sigma_2 xy \frac{\partial^2 V(x,y)}{\partial x \partial y} \\ &+ \alpha_1 x \frac{\partial V(x,y)}{\partial x} + \alpha_2 y \frac{\partial V(x,y)}{\partial y}.\end{aligned} \tag{7}$$

This equation should be understood as follows: before the investment takes place, and assuming that the current prices of the products are x and y, the value of the firm, $V(x,y)$, is such that $V(x,y) > F(x,y)$ (and thus investment is not yet optimal) and that optimality of the function V requires that $rV(x,y) - \mathcal{L}V(x,y) = 0$. The latter equation essentially states that the investment's value today is equal to the discounted expected value of the investment a short amount of time later. Then, as soon as investment is optimal, it holds that $V(x,y) = F(x,y)$ and that the value of immediate investment exceeds the discounted expected value of the investment a short amount of time later, i.e. $rV(x,y) - \mathcal{L}V(x,y) > 0$.

Moreover, we let the set $D := \{(x,y) \in \Re_+^2 | V(x,y) > F(x,y)\}$ denote the *continuation region*, and $S := \Re_+^2 \setminus D = \{(x,y) \in \Re_+^2 | V(x,y) = F(x,y)\}$ denote the *stopping region*. Following the general theory of optimal stopping, it then follows that τ^*, the time at which the investment should be undertaken, is given by the first exit time of the continuation region, i.e.,

$$\tau^* = \inf\{t \geq 0; (X_t, Y_t) \notin D\}. \tag{8}$$

Therefore, τ^* is the first time that the value function is equal to the expected value from immediately investing in the production plant.

In view of the equation (6), it follows that

$$rV(x,y) - \mathcal{L}V(x,y) \geq 0 \;\wedge\; V(x,y) \geq F(x,y),\; \forall\,(x,y) \in \Re_+^2.$$

Moreover,

$$rV(x,y) - \mathcal{L}V(x,y) = 0 \;\wedge\; V(x,y) \geq F(x,y),\; \forall\,(x,y) \in D, \tag{9}$$

whereas

$$rF(x,y) - \mathcal{L}F(x,y) \geq 0 \;\wedge\; V(x,y) = F(x,y),\; \forall\,(x,y) \in S. \tag{10}$$

The solution of the HJB equation, V, must satisfy the following boundary condition:

$$V(0+, 0+) = 0, \tag{11}$$

which reflects the fact that the value of the firm will be zero if the prices are zero. Also the following value-matching and smooth-fit conditions should hold (see Pham (1997), Tankov (2003) and Larbi and Kyprianou (2005)):

$$V(x,y) = F(x,y) \quad \text{and} \quad \nabla V(x,y) = \nabla F(x,y), \quad \text{for } (x,y) \in \partial D. \tag{12}$$

Here ∂D denotes the boundary of D, which we call *critical boundary*, and ∇ is the gradient operator. Therefore, the solution of the problem is continuous at the critical boundary, not only for itself but also for its derivatives. The resulting threshold is a curve separating the two regions (the continuation and the stopping regions).

Note that $x = 0$ and $y = 0$ are absorbing barriers. Consequently, at these boundaries, the firm only receives revenues from one product and only one stochastic process is in use. Therefore, the threshold at these points corresponds to the standard solution for a one-dimensional problem. It follows that the investment triggers at the y and x axis are

$$x^* = \frac{\beta_1}{\beta_1 - 1}\delta_1 I, \text{ if } y = 0, \quad \text{and} \quad y^* = \frac{\eta_1}{\eta_1 - 1}\delta_2 I, \text{ if } x = 0, \tag{13}$$

respectively. We refer, for instance, to Dixit and Pindyck (1994) for derivation of these values.

These thresholds can be interpreted as follows. If $y = 0$, then the firm should still invest in this plant as soon as the price of the other product reaches the value x^*. The intuition is analogous for $x = 0$ and y^*. The parameters $\beta_1 > 1$ and $\eta_1 > 1$ are the positive roots of the quadratic equations

$$\frac{1}{2}\sigma_1\beta(\beta - 1) + \alpha_1\beta - r = 0, \quad \text{and} \quad \frac{1}{2}\sigma_2\eta(\eta - 1) + \alpha_2\eta - r = 0,$$

respectively.

Solving problem (5) means, in particular, that we need to derive the set of values for x and y where stopping is optimal, i.e., where investment should take place. In particular, we want to derive the boundary between D and S, as crossing this boundary means that investment should be undertaken right away. We call it the *threshold boundary*. As we have two state variables, we may define this threshold boundary as a surface in $\mathbb{R}^2$, as follows: given that the price of one product is $x \in \mathbb{R}^+$, the firm should undertake the investment if the price of the other product is larger or equal to $b(x)$. If it is smaller, than the firm should wait before investment. The next theorem derives some qualitative features of the threshold boundary for the problem defined in (5).

Theorem 1. *The boundary between D and S can be described by a mapping $x \mapsto b(x)$, where:*

1. $b(x) = \sup\{\, y \in \Re_+ \mid V(x,y) > F(x,y) \,\}$ *for all* $x \in (0, x^*)$;
2. *b is non-increasing on* $(0, x^*)$;
3. *b is convex on* $(0, x^*)$;
4. *b is continuous;*
5. $b(x) < y^*$ *on* $(0, x^*)$, *and* $b(x) = 0$ *on* $[x^*, \infty)$.

In addition, the stopping set S is:

1. *closed;*
2. *convex.*

Finally, the value function V satisfies:

1. $V > 0$ *on* $\Re^2_{++}$;
2. *V is convex;*
3. *V is continuous;*
4. *V is increasing in x and y.*

Remark 1. *Theorem 1 leads to the following observations.*

1. *We can write*

$$D = \left\{ (x,y) \in \Re^2_+ \,\middle|\, y < b(x) \right\}, \quad \text{and} \quad S = \left\{ (x,y) \in \Re^2_+ \,\middle|\, y \geq b(x) \right\}.$$

2. *The optimal stopping boundary can never lie below the Net Present Value boundary $\bar{b}$, i.e.*

$$b(x) > \bar{b}(x) := \delta_2(I - x/\delta_1), \quad \text{all } x \in (0, \delta_1 I).$$

Thus in order to solve (5) we need to find V and, at the same time, $b(x)$ for $x \leq x^*$, such that the properties enumerated in Theorem 1 hold. In particular for V, conditions usually known in the literature as *fit conditions* are checked: the value matching condition (for the continuity of the value function) and smooth-pasting (for the smoothness of the value function).

3. The quasi-analytical approach

Following the approach in Adkins and Paxson (2011b), we start by postulating a solution to equation (6) of the following form:

$$v(x,y) = Ax^\beta y^\eta, \tag{14}$$

where A, β, and η are constants. Simple calculations lead to the conclusion that for (14) to be a solution to (7) it must hold that β and η are the roots of the characteristic root equation:

$$\mathcal{Q}(\beta,\eta) = \frac{1}{2}\sigma_2^2\eta(\eta-1) + \frac{1}{2}\sigma_1^2\beta(\beta-1) + \rho\sigma_1\sigma_2\beta\eta + \alpha_1\beta + \alpha_2\eta - r = 0. \tag{15}$$

The set of solutions to (15) defines an ellipse that intersects all quadrants of $\Re^2$, with β (η) on the horizontal (vertical) axis.

Adkins and Paxson (2011b) hypothesize that the boundary between the continuation and stopping regions is of the form $x \mapsto b(x)$. As Theorem 1 shows, this is correct. In order to find this boundary, Adkins and Paxson (2011b) try to extend the standard value-matching and smooth-pasting conditions to a two-dimensional setting. The way this is done is as follows: on the boundary it should hold for every $\hat{x} \in (0, x^*)$, with x^* as given in Equation (13):

$$v(\hat{x}, b(\hat{x})) = \frac{Q_1\hat{x}}{r-\alpha_1} + \frac{Q_2 b(\hat{x})}{r-\alpha_2} - I \quad \text{(value matching)} \tag{16}$$

$$\frac{\partial v(x,y)}{\partial x}_{|x=\hat{x},y=b(\hat{x})} = \frac{Q_1}{r-\alpha_1} \quad \text{(smooth pasting in } x\text{-direction)} \tag{17}$$

$$\frac{\partial v(x,y)}{\partial y}_{|x=\hat{x},y=b(\hat{x})} = \frac{Q_2}{r-\alpha_2} \quad \text{(smooth pasting in } y\text{-direction).} \tag{18}$$

Now, *if* the value function is of the form[2]

$$v(x,y) = Ax^\beta y^\eta,$$

[2] Note that Adkins and Paxson (2011b) assume this is the case.

then it should hold that $\beta, \eta > 0$ since the boundary conditions $\lim_{x\downarrow 0} v(x,y) = \lim_{y\downarrow 0} v(x,y) = 0$ should be satisfied. Therefore, for every $\hat{x} \in (0, x^*)$ we can solve the system of non-linear equations

$$A\hat{x}^{\beta} b(\hat{x})^{\eta} = \frac{Q_1 \hat{x}}{r-\alpha_1} + \frac{Q_2 b(\hat{x})}{r-\alpha_2} - I \tag{19}$$

$$\beta A\hat{x}^{\beta-1} b(\hat{x})^{\eta} = \frac{Q_1}{r-\alpha_1} \tag{20}$$

$$\eta A\hat{x}^{\beta} b(\hat{x})^{\eta-1} = \frac{Q_2}{r-\alpha_2} \tag{21}$$

$$\mathcal{Q}(\beta,\eta) = 0, \tag{22}$$

in b, A, β, and η, under the condition that $\beta, \eta > 0$.

Using the approach presented in Støre et al. (2018) to solve this system, we find the explicit solution for the boundary[3]:

$$\hat{b}(x) = x\frac{\eta(x)(r-\alpha_2)Q_1}{\beta(x)(r-\alpha_1)Q_2}, \tag{23}$$

where

$$\beta(x) = \frac{\sigma_1^2 - 2\alpha_1 + C^*(x)(2\alpha_2+\sigma_2^2)}{2(\sigma_1^2+\sigma_2^2 C^*(x))} + \sqrt{\left(\frac{\sigma_1^2 - 2\alpha_1 + C^*(x)(2\alpha_2+\sigma_2^2)}{2(\sigma_1^2+\sigma_2^2 C^*(x))}\right)^2 + 2\frac{r-\alpha_2}{\sigma_1^2+\sigma_2^2 C^*(x)}}, \tag{24}$$

$$\eta(x) = 1 - \beta(x)C^*(x), \tag{25}$$

$$C^*(x) = 1 - \frac{(r-\alpha_1)I}{xQ_1}. \tag{26}$$

In the previous equations we use $\beta(x)$ and $\eta(x)$ instead of β and η to emphasize the dependency on the state variable x.

Therefore, solving (20)-(22) leads to values of β and η that *do* depend on the value of x and, thus, cannot be treated as fixed parameters. This is also the case for the problem in Adkins and Paxson (2011b), as illustrated by their Figure 3.[4] The same holds for A and b.

Let $\mathbf{u} = [\,\beta(x) \quad \eta(x) \quad A(x) \quad \hat{b}(x)\,]^T$ denote the vector of solutions resulting from (20)-(22). Then at $\hat{b}(x)$, the value of the firm can be written as

$$v(x,\hat{b}(x)) = A(x)x^{\beta(x)}\hat{b}^{\eta(x)}(x). \tag{27}$$

Note that from (9) the partial differential equation $rV(x,y) - \mathcal{L}V(x,y) = 0$ must also hold along the threshold boundary, implying that

$$\frac{1}{2}\sigma_1^2 x^2\left(\frac{\partial^2 v(x,\hat{b})}{\partial x^2} + \frac{\partial}{\partial \mathbf{u}}\left(\frac{\partial v(x,\hat{b})}{\partial x}\right)\frac{\partial \mathbf{u}}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial v(x,\hat{b})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial x}\right)\right) + \frac{1}{2}\sigma_2^2\hat{b}^2\frac{\partial^2 v(x,\hat{b})}{\partial \hat{b}^2}$$
$$+\alpha_1 x\left(\frac{\partial v(x,\hat{b})}{\partial x} + \frac{\partial v(x,\hat{b})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial x}\right) + \alpha_2\hat{b}\frac{\partial v(x,\hat{b})}{\partial \hat{b}} - rv(x,\hat{b}) = 0. \tag{28}$$

[3] For simplicity, henceforth we assume that $\rho = 0$.

[4] The same holds for (Adkins and Paxson 2011a), see Table 2; Adkins and Paxson (2017a), see Table of Figure 1; Heydari et al. (2012), see equation (19); Adkins and Paxson (2013a), see equation (9); Adkins and Paxson (2013b), see Figure 2; Fleten et al. (2016), see equation (17); Støre et al. (2018), see equation (18); and Adkins and Paxson (2017b), see Table 3.

Using (27) we can rewrite (28) as

$$\frac{1}{2}\sigma_1^2 x^2 \left(\frac{\partial}{\partial \mathbf{u}} \left(\frac{\partial v(x,\hat{b})}{\partial x} \right) \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial v(x,\hat{b})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x} \right) \right) + \alpha_1 x \left(\frac{\partial v(x,\hat{b})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x} \right)$$
$$+ A x^{\beta} \hat{b}^{\eta} \left(\frac{1}{2}\sigma_2^2 \eta(\eta - 1) + \frac{1}{2}\sigma_1^2 \beta(\beta - 1) + \alpha_1 \beta + \alpha_2 \eta - r \right) = 0, \quad (29)$$

The first two terms in (29) represent the contributions of the partial derivatives of $\hat{b}$, A, β and η with respect to x, whereas the last term is equal to $Ax^{\beta}\hat{b}^{\eta}\mathcal{Q}(\beta,\eta)$. If the solution proposed in (14) is correct, then the latter should be equal to zero, and we can still use the system (19)-(22) to determine the threshold boundary. In what follows we verify whether the contribution of the partial derivatives is negligible for the numerical example in Table 1.

$\hat{x}$	Contribution of partial derivatives
10	-10841.14
20	-54 856.60
30	-9040.40

Table 1: The value of the first two terms of equation (29) for the following set of the parameter values: $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, and $I = 2000$.

For $\hat{x} = 10$, and the set of parameter values in Table 1, $F(\hat{x}, \hat{b}(\hat{x})) = 7281.23$. Therefore, we conclude that the contribution of the partial derivatives cannot be neglected. As a result, substitution of the solution (27) in (7) leads to the conclusion that the condition for η and β is no longer (15). In fact, (15) needs to be modified to incorporate terms involving $\beta'(x)$, $\eta'(x)$, $\beta''(x)$, $\eta''(x)$, $A'(x)$, $A''(x)$, $\hat{b}'(x)$ and $\hat{b}''(x)$. The implication is that solving the system (19)-(22) for different values of $\hat{x}$ does not result in a correct threshold boundary.

3.1. Results of the quasi-analytical approach

After having shown that the boundary $\hat{b}$, as determined by the quasi-analytical approach, is not the true boundary b, it could still be the case that $\hat{b}$ is a good approximation of b. This section, however, provides an argument that this is not the case, at least for the problem in Section 2.

We start out by presenting the following proposition.

Proposition 1. *The value function, V, is monotonically increasing in both σ_1 and σ_2.*

Proof. The result follows from Proposition 3 in Olsen and Stensland (1992) using the fact that the optimal value function is convex, as stated in our Theorem 1. □

In the following, we study the behaviour of the investment boundary as a function of the volatilities of the involved processes, σ_1 and σ_2. Thus we let $b(\sigma_1, \sigma_2; x)$ denote the boundary, given that the current price of the first product is x, and that the volatilities are σ_1 (for X) and σ_2 (for Y).

Corollary 1. *Let $b(\sigma_1, \sigma_2; x)$ denote the optimal investment threshold boundary for a given level of x. Then it holds that $b(\sigma_1, \sigma_2; x)$ is increasing in both σ_1 and σ_2.*

Proof. We prove the result by contradiction. Without loss of generality we only consider a change in σ_1. Consider two different values of σ_1, such that $\hat{\sigma}_1 > \bar{\sigma}_1$, and $b(\hat{\sigma}_1, \sigma_2; x) <$

$b(\bar{\sigma}_1, \sigma_2; x)$ for some x. Let $V(\sigma_1, \sigma_2; x, y)$ denote the optimal value function for a given level of x. Then,

$$V(\hat{\sigma}_1, \sigma_2; x, y) = \begin{cases} > F(x, y) & \text{for } y < b(\hat{\sigma}_1, \sigma_2; x), \\ = F(x, y) & \text{for } y \geq b(\hat{\sigma}_1, \sigma_2; x), \end{cases} \tag{30}$$

$$V(\bar{\sigma}_1, \sigma_2; x, y) = \begin{cases} > F(x, y) & \text{for } y < b(\bar{\sigma}_1, \sigma_2; x), \\ = F(x, y) & \text{for } y \geq b(\bar{\sigma}_1, \sigma_2; x). \end{cases} \tag{31}$$

If $b(\hat{\sigma}_1, \sigma_2; x) < b(\bar{\sigma}_1, \sigma_2; x)$, then for $y \in (b(\hat{\sigma}_1, \sigma_2; x), b(\bar{\sigma}_1, \sigma_2; x))$, it holds that $V(\bar{\sigma}_1, \sigma_2; x, y) > F(x, y) = V(\hat{\sigma}_1, \sigma_2; x, y)$, which contradicts Proposition 1. □

Figure 1 illustrates the quasi-analytical threshold boundaries for different values of σ_2.

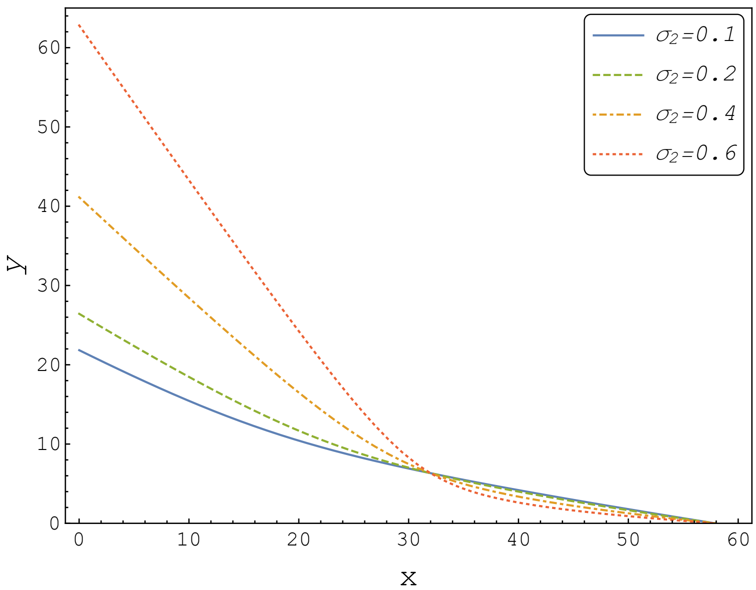

Figure 1. The threshold boundaries, $\hat{b}$, for the following set of parameter values: $\sigma_1 = 0.2$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$, and different values of σ_2.

Evidently, the numerical example violates Corollary 1, since the threshold boundaries intersect. Moreover, this result does not correspond to what we would expect from real options theory, i.e. that the firm invests for a larger threshold level in a more uncertain environment. In fact, for $x > 32$, the quasi-analytical approach suggests that the firm should invest for a lower threshold level when σ_2 is larger. This clearly leads to a sub-optimal decision, so the quasi-analytical solution falls short in being a useful approximation to the optimal solution in this case.

4. Numerical Solution

This section develops a finite difference algorithm to solve the optimal stopping problem in (5). The results of the numerical approach are different from the results obtained by the analytical approach *and* in line with Theorem 1 and Proposition 1.

We start by generating a discrete grid over the domain of the partial differential equation in (7). Thus we assume that the intervals $(0, x^*]$ and $(0, x^*]$ are divided in $N_x + 1$ and $N_y + 1$ equally spaced subintervals, respectively, and we let

$$x_i = ih \qquad i = 0, 1, 2, .., N_x, \qquad h = \frac{x^*}{N_x}, \tag{32}$$

$$y_j = jg \qquad i = 0, 1, 2, .., N_y, \qquad g = \frac{y^*}{N_y}, \tag{33}$$

where x^* and y^* are the optimal investment triggers in case the other state variable is zero and, thus, are the natural end points of the grid. Moreover, we consider the following notation: $V_{i,j}$ denotes the value of the firm at the grid points (x_i, y_j):

$$V_{(i,j)} = V(x_i, Y_i)$$

293 with V defined in Equation (5). Finally we let $\mathbf{v}$ be vector of unknown grid points, which
294 can be ordered in the following way

$$\mathbf{v} = \begin{bmatrix} V_{0,0} \\ V_{0,1} \\ \vdots \\ V_{0,N_j} \\ \vdots \\ V_{N_i,N_j} \end{bmatrix} \tag{34}$$

Then we are able to derive a linear system of equations that allows to solve for the discrete grid points simultaneously, as follows. We discretize the partial differential equation using a weighted sum of the function values at the neighboring point approximations to the partial derivatives. This yields

$$\frac{1}{2}\sigma_1^2 p_1^2 \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h^2} + \frac{1}{2}\sigma_2^2 p_2^2 \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{g^2} + \alpha_1 x \frac{V_{i+1,j} - V_{i,j}}{h} + \alpha_2 y \frac{V_{i,j+1} - V_{i,j}}{g}$$
$$+\rho\sigma_1\sigma_2 xy \frac{V_{i+1,j+1} - V_{i+1,j-1} - V_{i-1,j+1} + V_{i-1,j-1}}{4hg} - rV_{i,j} = 0. \tag{35}$$

Rearranging the terms, gives

$$V_{i,j}\left(-\sigma_1^2 i^2 - \alpha_1 ih - \sigma_2^2 \frac{j^2h^2}{g^2} - \alpha_2 \frac{jh^2}{g} - rh^2\right) + V_{i,j+1}\left(\frac{1}{2}\sigma_2^2 \frac{j^2h^2}{g^2} + \alpha_2 \frac{jh^2}{g}\right) +$$
$$V_{i+1,j}\left(\frac{1}{2}\sigma_1^2 i^2 + \alpha_1 ih\right) + V_{i,j-1}\left(\frac{1}{2}\sigma_2^2 \frac{j^2h^2}{g^2}\right) + V_{i-1,j}\left(\frac{1}{2}\sigma_1^2 i^2\right) +$$
$$+\rho\sigma_1\sigma_2 \frac{ijh}{4g}(V_{i+1,j+1} - V_{i+1,j-1} - V_{i-1,j+1} + V_{i-1,j-1}) = 0. \tag{36}$$

295 Then the partial differential equation (35) can be represented as a system of linear
296 equations

$$B\mathbf{v} = 0, \tag{37}$$

297 where B is the matrix of coefficients resulting from (36).
298 This system can be solved by applying appropriate boundary conditions. We use
299 the fact that the value at $(x^*, 0)$ and $(0, y^*)$ must equal the value of the immediate
300 investment. In addition, if either x_i or y_j is equal to zero the problem is reduced to
301 one-dimension, and the grid points together with the threshold boundary can be found

analytically. Given a candidate threshold function, the system (37) in combination with the boundary conditions in zero and final nodes, yield a solution for the unknown grid points. To determine the optimal threshold we implement the following procedure. First, we propose a shape of the exercise boundary. For example, the results that we present in Figure 2 are based on the quadratic function, i.e. $y = a + bx + cx^2$. The unknown parameters, a and b can be determined using the analytical threshold boundaries when either x_i or y_j is zero. In order to find c, we compute the derivative of the option value at the candidate threshold boundary at each node, and compare it with the derivatives resulting from the smooth pasting conditions. Next, we compute the sum squared error of the differences and minimize it with respect to unknown parameter c, which allows to determine the optimal threshold in such a way that the smooth pasting condition is satisfied.

We now replicate Figure 1 using our finite difference scheme and depict it in Figure 2.

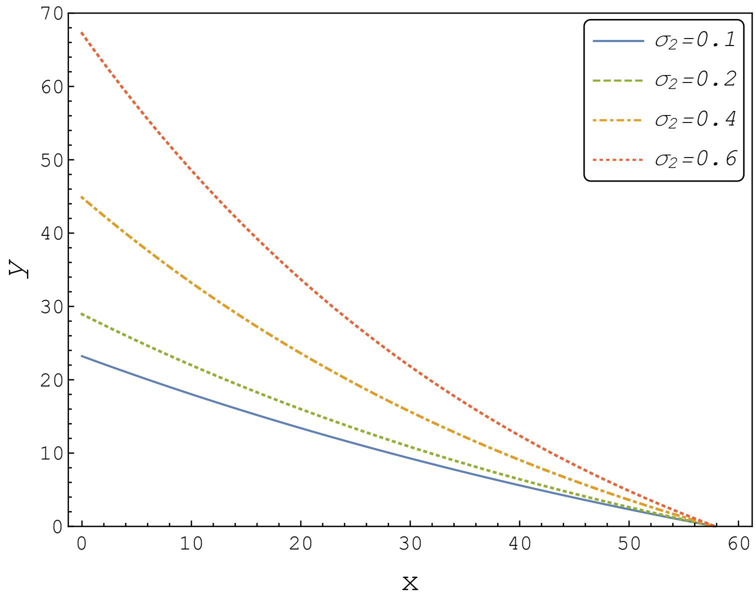

Figure 2. The numerical threshold boundary for the following set of parameter values: $\sigma_1 = 0.2$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$, and different values of σ_2.

This numerical example results in a more intuitive shape of thresholds boundaries and represent a standard result from the real options theory. Namely, an increase in volatility leads to an increase of the optimal investment threshold.

In addition, finite difference also allows for the calculation of an approximation to the value function that is implied by the quasi-analytical boundary $\hat{b}$. This can be done by solving (37) for the boundary in (23). Figure 3 illustrates the comparison between the implied value function and the numerical solution represented by quadratic boundary for a fixed level of x and different values of y.

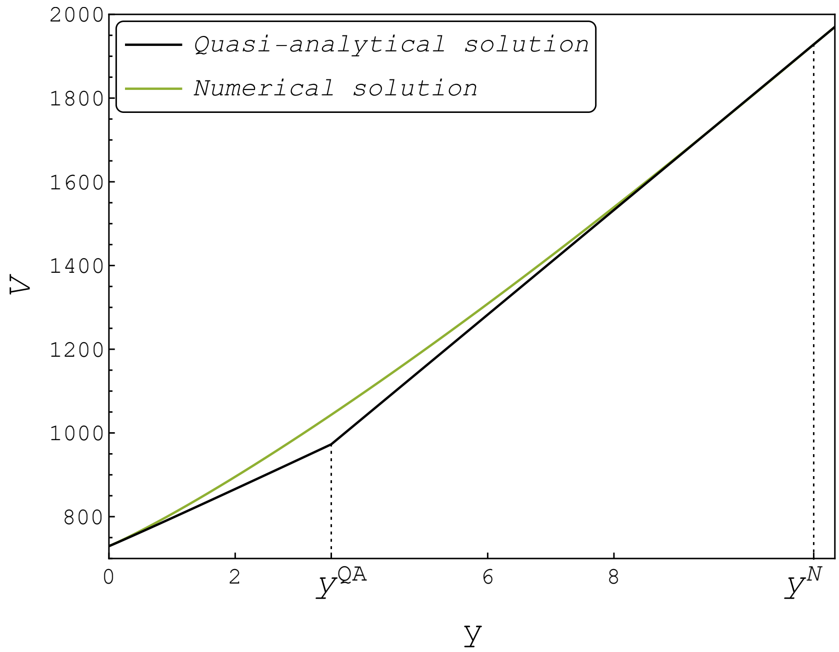

Figure 3. The numerical threshold boundary for the following set of parameter values: $x = 40.52$, $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$, and different values of y.

From Figure 3 it is evident that the value function implied by the quasi-analytical solution has a kink at the boundary point $y^{QA} = \hat{b}(40.52) = 2.49$, violating the smooth-pasting condition. Consequently, the quasi-analytical approach underestimates the true value function, which leads to a sub-optimal investment decision rule for large values of x. Note that the quasi-analytical approach suggests a much lower trigger than our finite difference scheme. For $x = 40.52$, the numerical procedure based on the finite-difference algorithm gives the boundary point $y^{QA} = 10.26$, such that the smooth-pasting condition holds. Figure 4 illustrates the value function for different values of x and y, as well as the threshold boundary.

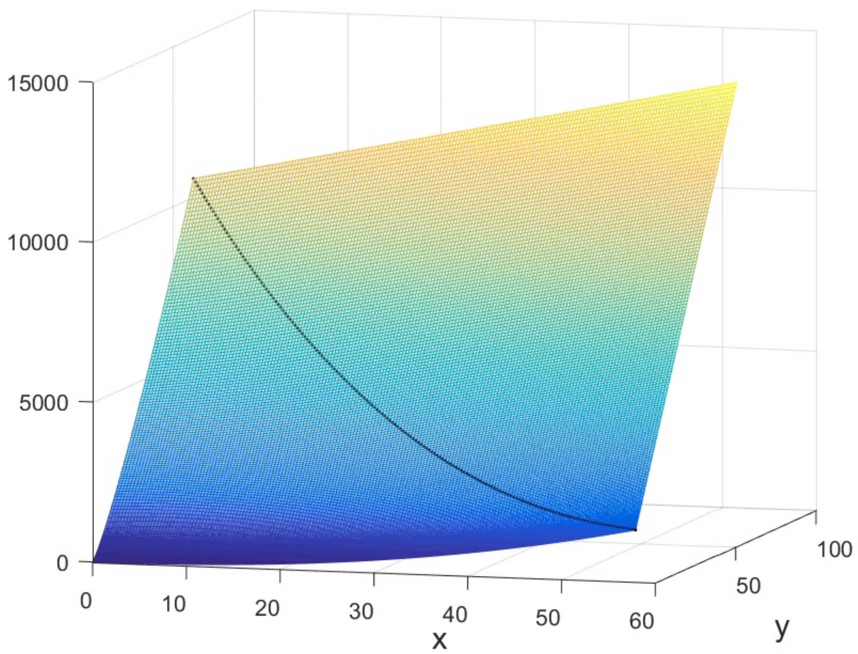

Figure 4. The numerical value function and threshold boundary (solid black curve) for the following set of parameter values: $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$, and different values of x and y.

As can be seen, the value appears to be smooth for different values of x and y in the grid. The average squared error resulting from the numerical procedure is equal to 0.44, which corresponds to 0.17% of the true value of total derivative of the value function. Therefore, we conclude that the proposed numerical method is a good approximation for the true value function and optimal threshold.

Lastly, in order to give an indication how often a firm would make a poorly timed investment decision, we simulate the passage time for the processes X_t and Y_t to reach the quasi-analytical boundary. We then run the procedure 5000 times for a specific set of starting values (x_0, y_0), and calculate the percentage of cases of the threshold being reached within the next 5 years. We perform a similar procedure, to determine the investment probabilities for our numerical solution. The results for the different starting points are presented in Table 2.

(x_0, y_0)	5	10	15
10	10.06%	23.97%	39.03%
15	21.69%	42.87%	61.69%
20	40.32%	68.34%	90.21%

(a) Quasi-analytical boundary

(x_0, y_0)	5	10	15
10	5.40%	5.56%	5.51%
15	5.57%	5.41%	5.57%
20	5.43%	5.24%	5.59%

(b) Numerical boundary

Table 2: Percentage of cases when a firm undertakes an investment within the next 5 years for the set of parameter values: $\sigma_1 = 0.2$, $\sigma_2 = 0.6$, $\alpha_1 = 0.02$, $\alpha_2 = 0.02$, $r = 0.1$, $\rho = 0$, $Q_1 = 5$, $Q_2 = 10$, $I = 2000$.

Table 2b shows that, for example, for the starting values (15,10) the firm should invest in 5.41% of the cases. According to the quasi-analytical approach however, the firm invests in 42.87% of the cases, implying that many times the firm invests while it is in fact not optimal to do so.

5. Conclusion

This paper develops an easy-to-implement finite difference algorithm to solve real options models with two-factor uncertainty. The proposed framework is, thus, highly relevant for the evaluation of business opportunities involving multiple end-products, a switch in feed-stock or end-product, a cooperation between firms that are operative in different markets, and investments in new technologies incentivized by market-based policy instruments.

We apply it to a particular investment problem, where after investment the firm is able to produce two different products. The output prices of these products follow two geometric Brownian motion processes, possibly correlated. The investment cost is constant and sunk. We contrast our solution approach to the quasi-analytical approach developed by Adkins and Paxson (2011b) to address such problems. The latter has already been adopted by several other authors, as the overview in Section 1 shows. This paper argues, however, that this quasi-analytical method does not always result in the correct investment decision rule.

From the analysis of this two-factor real options problem we obtain that the quasi-analytical investment decision rule in some cases also fails to be a reasonable approximation to the optimal decision. In particular, we find that the quasi-analytical solution does not comply with the (analytical) result that the investment threshold boundary must be monotonically increasing in the volatility parameters of both stochastic processes.

The ultimate conclusion is that non-homogenous real options problems with two-factor uncertainty should be solved using a different numerical procedure. Or at the very least, the quality of the quasi-analytical approximation should be discussed. Note, however, that if our two-factor uncertainty problem is homogenous, then a standard (cf. McDonald and Siegel (1986)) reduction in dimensionality can be obtained, leading to an analytical solution.

Appendix A

Proof of Theorem 1.
Throughout the proof, we will denote the unique solution to (1) for given starting point $(X_0, Y_0) \in \Re^2_+ \setminus \{0\}$ by (X^x, Y^y). Note that $(X^x, Y^y) = (xX^1, yY^1)$.

1. ($V > 0$ on $\Re^2_{++}$) On S the result is trivial. Let $(x, y) \in D \cap \Re^2_{++}$. Consider the stopping time

$$\tau = \inf\{t \geq 0 | F(X_\tau, Y_\tau) > 0\}.$$

Since $e^{-r\tau}F(X_\tau, Y_\tau) = 0$ on $\{\tau = \infty\}$ (since $r > \max\{\alpha_1, \alpha_2\}$) and $\mathsf{P}(\tau < \infty) > 0$, it holds that

$$V(x, y) \geq \mathbb{E}\big[e^{-r\tau}F(X_\tau, Y_\tau)\big] > 0.$$

2. (Convexity of V) On S the result is trivial. Take $(x', y'), (x'', y'') \in D$ and $\lambda \in (0, 1)$. Define $(x, y) := \lambda(x', y') + (1-\lambda)(x'', y'')$. It then holds that

$$\begin{aligned}
V(x, y) =& \sup_\tau \mathbb{E}\big[e^{-r\tau}F(x', y')\big] \\
=& \sup_\tau \mathbb{E}\left[e^{-r\tau}\left(\frac{xX^1_\tau}{\delta_1} + \frac{yY^1_\tau}{\delta_2} - I\right)\right] \\
=& \sup_\tau \mathbb{E}\left[e^{-r\tau}\left(\frac{(\lambda x' + (1-\lambda)x'')X^1_\tau}{\delta_1} + \frac{(\lambda y' + (1-\lambda)y'')Y^1_\tau}{\delta_2} - I\right)\right] \\
=& \sup_\tau \mathbb{E}\left[\lambda e^{-r\tau}\left(\frac{x'X^1_\tau}{\delta_1} + \frac{y'Y^1_\tau}{\delta_2} - I\right) + (1-\lambda)e^{-r\tau}\left(\frac{x''X^1_\tau}{\delta_1} + \frac{y''Y^1_\tau}{\delta_2} - I\right)\right] \\
\leq& \lambda \sup_\tau \mathbb{E}\big[e^{-r\tau}F(x', y')\big] + (1-\lambda)\sup_\tau \mathbb{E}\big[e^{-r\tau}F(x'', y'')\big] \\
=& \lambda V(x', y') + (1-\lambda)V(x'', y'').
\end{aligned}$$

3. (Continuity of V) This property follows from the general theory of stochastic processes, see, e.g., (Krylov 1980, Theorem 3.1.5).

4. (Monotonicity of V) We prove that V is (strictly) increasing in x. Again, the result is trivial on S. Take $(x,y) \in D$ and let $\varepsilon > 0$ be such that $(x+\varepsilon, y) \in D$ (such ε exists since D is open; see below). Take any stopping time τ. It then holds that

$$\mathbb{E}\left[e^{-r\tau}\left(\frac{(x+\varepsilon)X_\tau^1}{\delta_1} + \frac{yY_\tau^1}{\delta_2} - I\right)\right] \geq \mathbb{E}\left[e^{-r\tau}\left(\frac{xX_\tau^1}{\delta_1} + \frac{yY_\tau^1}{\delta_2} - I\right)\right],$$

with equality only when $\{\tau = \infty\}$ a.s.. Note that τ with $\{\tau = \infty\}$ a.s. is never optimal. Take $\tau^* = \inf\{t \geq 0 | Y_t \geq \delta_1 I + 1\}$. Then $\mathsf{P}(\tau^* < \infty) > 0$ and, thus, we have that $\mathbb{E}\left[e^{-r\tau^*}F(X_{\tau^*}^x, Y_{\tau^*}^y)\right] > 0$.] Therefore, $V(x+\varepsilon, y) > V(x,y)$.

5. (Closedness of D) Take a sequence $(x^{(n)}, y^{(n)})_{n\in\aleph}$ in S with limit (x,y). Then $V(x^{(n)}, y^{(n)}) = F(x^{(n)}, y^{(n)})$ for all $n \in \aleph$. Since $\lim_{n\to\infty} F(x^{(n)}, y^{(n)}) = F(x,y)$ and V is continuous, it holds that $V(x,y) = F(x,y)$. This implies that $(x,y) \in S$.

6. (Convexity of D) Suppose there exists $(x', y'), (x'', y'') \in S$ and $\lambda \in (0,1)$ such that $(x,y) := \lambda(x',y') + (1-\lambda)(x'', y'') \in D$. It then holds that

$$V(x,y) > F(x,y) = \lambda F(x',y') + (1-\lambda)F(x'',y'') = \lambda V(x',y') + (1-\lambda)V(x'',y'').$$

This contradicts convexity of V.

7. ($b(x)$ can be written as a sup) Take $(x,y) \in D$. The there exists a stopping time τ^* such that $(X_{\tau^*}, Y_{\tau^*}) \in D$, a.s.. Hence,

$$V(x,y) = \sup_\tau \mathbb{E}\left[e^{-r\tau}F(X_\tau, Y_\tau)\right] \geq \mathbb{E}\left[e^{-r\tau^*}F(X_{\tau^*}, Y_{\tau^*})\right] > F(x,y).$$

Now take $\varepsilon \in (0,y)$. Then

$$\begin{aligned}
V(x, y-\varepsilon) \geq & \mathbb{E}\left[e^{-r\tau^*}F(X_{\tau^*}, Y_{\tau^*})\right] \\
= & \mathbb{E}\left[e^{-r\tau^*}\left(\frac{xX_{\tau^*}^1}{\delta_1} + \frac{(y-\varepsilon)Y_{\tau^*}^1}{\delta_2} - I\right)\right] \\
= & \mathbb{E}\left[e^{-r\tau^*}\left(\frac{xX_{\tau^*}^1}{\delta_1} + \frac{yY_{\tau^*}^1}{\delta_2} - I\right)\right] - \mathbb{E}\left[e^{-r\tau^*}\frac{\varepsilon Y_{\tau^*}^1}{\delta_2}\right] \\
\overset{(*)}{\geq} & \mathbb{E}\left[e^{-r\tau^*}\left(\frac{xX_{\tau^*}^1}{\delta_1} + \frac{yY_{\tau^*}^1}{\delta_2} - I\right)\right] - \frac{\varepsilon}{\delta_2} \\
> & \mathbb{E}\left[e^{-r\tau^*}\left(\frac{xX_{\tau^*}^1}{\delta_1} + \frac{yY_{\tau^*}^1}{\delta_2} - I\right)\right] > F(x,y) > F(x, y-\varepsilon),
\end{aligned}$$

where $(*)$ follows from the fact that $e^{-rt}Y_t$ is a supermartingale. Therefore, $(x, y-\varepsilon) \in D$.

8. (b is non-increasing) This follows from the fact that for all $(x,y) \in D$ and all $\varepsilon \in (0,x)$ it holds that $(x-\varepsilon, y) \in D$. This can be proved using a similar argument as above.

9. (b is convex) Convexity of b follows from the fact that its epigraph is the convex set S.

10. (b is continuous) Continuity of b on $(0,\infty)$ is immediate, because it is a convex function on an open convex set (see, for example, Berge 1963, Theorem 8.5.7). Continuity at $x = 0$ follows from the fact that the stopping set is closed.

11. (boundedness of b) The boundedness properties follow from continuity and x^* and y^* being the solutions of the optimal stopping problem on $\Re_+ \times \{0\}$ and $\{0\} \times \Re_+$, respectively. $\square$

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