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# Investment decisions with two-factor uncertainty

Tine Compernelle<sup>1,2</sup>, Kuno J.M. Huisman<sup>3,4</sup>, Peter M. Kort<sup>3</sup>, Maria Lavrutich<sup>5\*</sup>, Cláudia Nunes<sup>6</sup> and Jacco J.J. Thijssen<sup>7</sup>

- <sup>1</sup> University of Antwerp, Antwerp, Belgium  
<sup>2</sup> Geological Survey of Belgium, Royal Belgian Institute of Natural Sciences, Brussels, Belgium  
<sup>3</sup> Tilburg University, Tilburg, The Netherlands  
<sup>4</sup> ASML Netherlands B.V., Veldhoven, The Netherlands  
<sup>5</sup> Norwegian University of Science and Technology, Trondheim, Norway  
<sup>6</sup> Instituto Superior Técnico, Lisbon, Portugal  
<sup>7</sup> University of York, York, United Kingdom  
\* Correspondence: maria.lavrutich@ntnu.no

**Abstract:** This paper considers investment problems in real options with non-homogeneous two-factor uncertainty. We derive some analytical properties of the resulting optimal stopping problem and present a finite difference algorithm to approximate the firm's value function and optimal exercise boundary. An important message of our paper is that the frequently applied quasi-analytical approach underestimates the impact of uncertainty. This is caused by the fact that the quasi-analytical solution does not satisfy the partial differential equation that governs the value function. As a result, the quasi-analytical approach may wrongly advise to invest in a substantial part of the state space.

**Keywords:** Investment analysis; Optimal stopping time problem; Two-factor uncertainty.

## 1. Introduction

Since the seminal works of Dixit and Pindyck (1994) and Trigeorgis (1996), it has become clear that real investments should be valued using a real options approach when decision makers are exposed to a significant amount of uncertainty. In these cases, application of the standard net present value decision rule can lead to investment decisions that are significantly sub-optimal, as is extensively demonstrated in the above books. Since firm investment decisions lie at the basis of economic growth, it is crucial to take these decisions in the right way. From this perspective it is clear that it is of main importance to work on the development of the theory of real options.

In the basic analysis the real options model consists of a single firm having the opportunity to invest in a project of given size, with revenue that is subject to uncertainty that is governed by a single stochastic process. Several authors have extended this framework in different directions. Smets (1991) is the first to consider a scenario where two firms can invest in the same market. The revenue in this market is still governed by one stochastic process, also after both firms have already invested and thus are active in this market. The assumption "project of given size" is relaxed in Bar-Ilan and Strange (1999) and Dangi (1999), in which the firm not only needs to decide about the time, but also about the size of the investment. Huisman and Kort (2015) combine the two extensions by considering a duopoly market where both firms also have to determine the investment size.

Most of the real options literature uses a single (one-dimensional) stochastic process to model the evolution of random shocks affecting the investment's value. This can be a major shortcoming, especially when analysing problems with, e.g., multiple firms or products. Such investment problems are especially common in the field of energy and environmental economics (Agaton and Collera 2022; Deeney et al. 2021; Li and Cao

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2022; Zhang et al. 2021). The transition to a circular and low-carbon, bio-based economy requires firms to shift to the use of renewable resources, to cooperate with firms in other markets, and to valorize their waste streams (The European Commission 2019 2021). These decisions expose firms to different types of risks, creating the need for models that account for multiple sources of uncertainty. Therefore, in this paper we consider a real options problem with multiple uncertain factors, which is an important extension to the basic real options analysis, especially from a practical perspective.

The first real options model with two-factor uncertainty occurs in McDonald and Siegel (1986). Their value function is homogenous of degree one, and the two stochastic processes are the output price and the investment cost. In such cases, the investment threshold level can be determined for the price-to-cost ratio. This allows to reformulate the problem in terms of the relative price, and to reduce the number of stochastic variables to one. In this way a standard one-factor real options model is obtained for which a closed-form solution exists. The result of this analysis is, however, not a threshold point but a threshold boundary at which it is optimal to invest (Nunes and Pimentel 2017). Hu and Øksendal (1998) generalize this solution to the  $n$ -dimensional case. Armada et al. (2013) consider a problem where the output price and quantity are stochastic. Here the dimension of the state space can be reduced to a one-dimensional space, because the only relevant payoff variable is revenue (price times quantity). The problem then reduces to finding an optimal revenue threshold that makes investment optimal.

Several authors have tried to use this dimension-reduction approach to cases characterized by multiple stochastic processes and a constant sunk cost. Huisman et al. (2013) and Compennolle et al. (2017) consider price and cost uncertainty and determine the investment threshold level for the price-to-cost ratio. However, there are some problems with this approach. In the presence of a constant sunk cost investment, homogeneity does not hold. For this reason, the state space cannot be reduced to a one-dimensional one. This is also revealed in these papers, because two processes (price/cost and cost) remain present in the equations.

For problems of this kind, Adkins and Paxson (2011b) propose a quasi-analytical approach that results in a set of equations to determine the optimal investment boundary. They solve this set of equations simultaneously while keeping one of the stochastic threshold variables fixed. The present paper shows that this methodology can lead to sub-optimal solutions. In fact, the results of the quasi-analytical approach will generically-speaking be incorrect and there is no guarantee that it converges in any meaningful sense to the correct solution. To put it succinctly, the main problem is that Adkins and Paxson (2011b) use a “local” approach to solve a “global” problem, which can lead to misleading results. Consequently, while the method is intuitively appealing and relatively easy to implement, we argue that care is required in checking that the results conform to economic intuition.

In this paper we provide numerical examples for which the quasi-analytical approach violates certain properties of the optimal boundary that can be analytically established. The point is, in a nutshell, that when solving the partial differential equation that governs the value function, two power parameters turn out to be a function of the state variables, where the quasi-analytical approach starts with the assumption that these parameters are constant.

Our contribution to the literature is three-fold. Firstly, we alert the research community to potential pitfalls in a regularly-used numerical method. In the literature, we find several papers concerning investment problems where the uncertainty is driven by multi-dimensional stochastic processes, and where no analytical solution can be derived. In such cases, the authors propose ways to circumvent the problem and come out with an approximation of the solution. For instance, we refer to Kauppinen et al. (2018), where the model proposed by Adkins and Paxson (2011b) is extended, by adding time to build to the investment problem. In the context of replacement options, Adkins

89 and Paxson (2013a) examine premature and postponed replacement in the presence  
90 of technological progress, where revenue and operating costs are treated as geometric  
91 Brownian motions. Adkins and Paxson (2017b) use a general replacement model to  
92 investigate when it is optimal to replace an asset whose operating cost and salvage value  
93 deteriorate stochastically.

94 The need to take into account multi-sources of uncertainty is also present in prob-  
95 lems related with investments in the energy sector. For example, Adkins and Paxson (  
96 2011a) solve a switching model for two alternative energy inputs with fixed switching  
97 costs. Boomsma and Linnerud (2015) examine investment in a renewable energy project  
98 under both market and policy uncertainty. Fleten et al. (2016) study investment deci-  
99 sions in the renewable energy sector, where the revenue comes from selling electricity  
100 and from receiving subsidies, both stochastic. Adkins and Paxson (2016) consider the  
101 optimal investment policy for an energy facility with price and quantity uncertainty  
102 under different subsidy schemes. Støre et al. (2018) determine the optimal timing to  
103 switch from oil to gas production in the tail production phase, with the price of oil and  
104 gas following (correlated) Geometric Brownian motions. Finally, we refer to Heydari  
105 et al. (2012), who extend the quasi-analytical approach proposed in Adkins and Paxson  
106 (2011b) to a three-factor model, which is employed to value the choice between two  
107 emissions-reduction technologies assuming that the value of each option depends on  
108 fuel, electricity and CO<sub>2</sub> prices, all following (correlated) Geometric Brownian motions.

109 Secondly, while it could, a priori, be the case that the approximation obtained by  
110 the quasi-analytical approach is close enough to the true solution to be of practical  
111 value, we show that for the models under consideration in much of the literature this is  
112 not necessarily the case. For example, in the model with two uncertain revenue flows  
113 we find that in some situations the investment boundary is *decreasing* in uncertainty  
114 environment. This violates one of the major qualitative result from real options theory:  
115 “an increase in uncertainty leads to an increase in project value”. We formally prove that  
116 this feature also holds for the model under consideration.

117 Thirdly, we develop a numerical algorithm which is based on a finite difference  
118 scheme and we apply this algorithm to a model with two stochastic revenue streams.  
119 We determine the optimal timing of investment in the presence of a constant sunk  
120 investment cost. Note that this model is different from the one analysed in Adkins and  
121 Paxson (2011b), who analyse a stochastic revenue and a (possibly correlated) stochastic  
122 cost. Importantly, our finite difference scheme *does* exhibit the expected behavior in  
123 relation to an increase in uncertainty.

124 In the literature most finite-difference schemes have been developed to solve models  
125 with a one-dimensional stochastic process and a finite time horizon. This method  
126 typically employs a backward induction argument in the time dimension to approximate  
127 the optimal exercise boundary and value function in a step-by-step fashion; see, e.g., (  
128 Dixit and Pindyck 1994, Appendix 10.A). For a two-dimensional problem this approach  
129 does not work, because both processes can move up or down in any time step. Therefore,  
130 we suggest a finite difference scheme that starts with a hypothesized boundary, after  
131 which the value function is approximated at all points in the two-dimensional finite grid  
132 at once. A discretized smooth pasting condition (in two dimensions) can then be used  
133 to judge the quality of the hypothesized boundary. This procedure is repeated until an  
134 acceptable approximation to the optimal boundary is found.

135 Note that to solve multidimensional optimal stopping problems, also other numeri-  
136 cal approaches could be applied. Among the relevant contributions is Lange et al. (2020).  
137 In this paper, the authors consider that the decision to stop can only be taken at specific  
138 times, generated by an exogenous Poisson process with intensity rate  $\lambda$ . This means that  
139 the set of admissible stopping times for the optimization problem is the set of events of a  
140 Poisson process, independent of the filtration generated by the state variables. In this  
141 setting, the optimization problem may be written as a fixed-point problem, for which  
142 the authors propose a numerical scheme, providing proof and rate of convergence.

143 In our paper, we prove some analytical properties of the optimal boundary, notably  
 144 we prove that is convex, non-increasing and continuous in  $\mathbb{R}^+$ . In [Dammann and](#)  
 145 [Ferrari \(2021\)](#) we may find similar results, but using different arguments, that rely on a  
 146 probabilistic representation of the Value Function. Moreover, the authors show that the  
 147 boundary is the unique solution of an integral condition. By use of this integral equation,  
 148 they prove monotonicity of the value function with respect to drift and volatility of the  
 149 involved parameters. Finally, they propose a numerical approach to find the boundary,  
 150 based on the integral equation using Monte-Carlo simulation.

151 The remainder of this paper is organized as follows. Section 2 introduces a non-  
 152 homogeneous investment problem characterized by two uncertain revenue flows. Sec-  
 153 tion 3 applies the methodological approach in [Adkins and Paxson \(2011b\)](#) to solve the  
 154 model presented in Section 2, and highlights the mathematical problems with the solu-  
 155 tion. Section 4 proposes an alternative numerical approach to solve the model. Section 5  
 156 concludes. Proofs of propositions are presented in Appendix A.

## 157 2. Investment decision given two uncertain revenue flows

Consider a profit-maximizing, risk-neutral firm that has the opportunity to invest in  
 a production plant by paying a constant investment cost,  $I$ . The plant can produce two  
 different products, the prices of which are stochastic and follow correlated geometric  
 Brownian motions  $X$  and  $Y$ , i.e.,

$$dX_t = \alpha_1 X_t dt + \sigma_1 X_t dW_{X,t}, \quad dY_t = \alpha_2 Y_t dt + \sigma_2 Y_t dW_{Y,t} \quad (1)$$

with

$$X_0 = x, Y_0 = y \quad (2)$$

158 being the initial values of the processes  $X$  and  $Y$ , respectively. We note that in Equation  
 159 (1),  $\alpha_1$  ( $\alpha_2$ ) denotes the drift of the process  $X$  ( $Y$ ), whereas  $\sigma_1$  ( $\sigma_2$ ) is the volatility of  $X$   
 160 ( $Y$ ). Following the usual notation, we let  $\{W_t, t \geq 0\}$  denote a standard two-dimensional  
 161 Brownian motion, which we index by  $X$  and  $Y$ , respectively. We allow these processes  
 162 to be correlated, so that  $\mathbb{E}[dW_{X,t}dW_{Y,t}] = \rho dt$  for some  $\rho \in (-1, 1)$ , where  $\rho > 0$  ( $\rho < 0$ )  
 163 means that  $W_X$  and  $W_Y$  are positively (negatively) correlated.

At any instant, if the prices of the two products are  $x$  and  $y$ , respectively, then the  
 instantaneous profit of the firm is given by:

$$\pi(x, y) = Q_1 x + Q_2 y, \quad (3)$$

where  $Q_1$  and  $Q_2$  denote the quantities of the products produced. Moreover, at that  
 instant the firm's value is equal to the perpetual revenue flow from selling two products:<sup>1</sup>

$$F(x, y) = \mathbb{E}_{(x,y)} \left[ \int_0^\infty e^{-rt} \pi(X_t, Y_t) dt - I \right] = \frac{Q_1 x}{\delta_1} + \frac{Q_2 y}{\delta_2} - I, \quad (4)$$

164 with  $\delta_i := r - \alpha_i, i \in \{1, 2\}$ , and  $r$  being the discount rate. To ensure finite integrals, we  
 165 assume that  $r > \max\{0, \alpha_1, \alpha_2\}$ . Equation (4) gives the expected value of the discounted  
 166 stream of profits that result from operating the production process forever, given current  
 167 prices  $x$  and  $y$ . That is,  $\mathbb{E}_{(x,y)}$  denotes the expectation operator, conditional on the initial  
 168 state being  $(X_0, Y_0) = (x, y)$ .

The firm needs to determine the optimal time to undertake the investment, and,  
 thus, solves the following optimal stopping problem

$$V(X_t, Y_t) = \sup_{\tau \geq t} \mathbb{E}_{(X_t, Y_t)} \left[ e^{-r(\tau-t)} F(X_\tau, Y_\tau) \right] \quad (5)$$

<sup>1</sup>  $\mathbb{E}_{(x,y)}$  denotes the expectation conditional on  $(X_0, Y_0) = (x, y)$ .

169 where the supremum is taken over all stopping times  $\tau$  with respect to the filtration  
 170 generated by the joint process  $(W_X, W_Y)$ . That is, we are looking for the optimal time to  
 171 invest in the production plant, given the current values for the price of each one of the  
 172 two types of product, such that we maximize the expected value of the overall profit.  
 173 By maximizing over stopping times we recognize that the optimal time to invest may  
 174 depend on the stochastic evolution of the product prices.

Using standard calculations from optimal stopping theory (see, e.g., [Øksendal and Sulem \(2007\)](#)), we derive the following Hamilton-Jacobi-Bellman (HJB) equation for this problem:

$$\min\{rV(x, y) - \mathcal{L}V(x, y), V(x, y) - F(x, y)\} = 0, \quad \forall (x, y) \in \mathfrak{R}_+ \times \mathfrak{R}_+. \quad (6)$$

Here  $\mathcal{L}$  denotes the infinitesimal generator of the process  $(X, Y)$ :

$$\mathcal{L}V(x, y) = \lim_{h \downarrow 0} \frac{E[V(x+h, y+h)] - V(x, y)}{h}$$

175 which is given by ([Øksendal and Sulem \(2007\)](#)):

$$\begin{aligned} \mathcal{L}V(x, y) &= \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 V(x, y)}{\partial x^2} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 V(x, y)}{\partial y^2} + \rho\sigma_1\sigma_2 xy \frac{\partial^2 V(x, y)}{\partial x \partial y} \\ &+ \alpha_1 x \frac{\partial V(x, y)}{\partial x} + \alpha_2 y \frac{\partial V(x, y)}{\partial y}. \end{aligned} \quad (7)$$

176 This equation should be understood as follows: before the investment takes place, and  
 177 assuming that the current prices of the products are  $x$  and  $y$ , the value of the firm,  
 178  $V(x, y)$ , is such that  $V(x, y) > F(x, y)$  (and thus investment is not yet optimal) and that  
 179 optimality of the function  $V$  requires that  $rV(x, y) - \mathcal{L}V(x, y) = 0$ . The latter equation  
 180 essentially states that the investment's value today is equal to the discounted expected  
 181 value of the investment a short amount of time later. Then, as soon as investment is  
 182 optimal, it holds that  $V(x, y) = F(x, y)$  and that the value of immediate investment  
 183 exceeds the discounted expected value of the investment a short amount of time later,  
 184 i.e.  $rV(x, y) - \mathcal{L}V(x, y) > 0$ .

185 Moreover, we let the set  $D := \{(x, y) \in \mathfrak{R}_+^2 \mid V(x, y) > F(x, y)\}$  denote the *contin-*  
 186 *uation region*, and  $S := \mathfrak{R}_+^2 \setminus D = \{(x, y) \in \mathfrak{R}_+^2 \mid V(x, y) = F(x, y)\}$  denote the *stopping*  
 187 *region*. Following the general theory of optimal stopping, it then follows that  $\tau^*$ , the  
 188 time at which the investment should be undertaken, is given by the first exit time of the  
 189 continuation region, i.e.,

$$\tau^* = \inf\{t \geq 0; (X_t, Y_t) \notin D\}. \quad (8)$$

190 Therefore,  $\tau^*$  is the first time that the value function is equal to the expected value from  
 191 immediately investing in the production plant.

In view of the equation (6), it follows that

$$rV(x, y) - \mathcal{L}V(x, y) \geq 0 \wedge V(x, y) \geq F(x, y), \quad \forall (x, y) \in \mathfrak{R}_+^2.$$

Moreover,

$$rV(x, y) - \mathcal{L}V(x, y) = 0 \wedge V(x, y) \geq F(x, y), \quad \forall (x, y) \in D, \quad (9)$$

whereas

$$rF(x, y) - \mathcal{L}F(x, y) \geq 0 \wedge V(x, y) = F(x, y), \quad \forall (x, y) \in S. \quad (10)$$

The solution of the HJB equation,  $V$ , must satisfy the following boundary condition:

$$V(0+, 0+) = 0, \quad (11)$$

which reflects the fact that the value of the firm will be zero if the prices are zero. Also the following value-matching and smooth-fit conditions should hold (see Pham (1997), Tankov (2003) and Larbi and Kyprianou (2005)):

$$V(x, y) = F(x, y) \quad \text{and} \quad \nabla V(x, y) = \nabla F(x, y), \quad \text{for } (x, y) \in \partial D. \quad (12)$$

192 Here  $\partial D$  denotes the boundary of  $D$ , which we call *critical boundary*, and  $\nabla$  is the gradient  
193 operator. Therefore, the solution of the problem is continuous at the critical boundary,  
194 not only for itself but also for its derivatives. The resulting threshold is a curve separating  
195 the two regions (the continuation and the stopping regions).

Note that  $x = 0$  and  $y = 0$  are absorbing barriers. Consequently, at these boundaries, the firm only receives revenues from one product and only one stochastic process is in use. Therefore, the threshold at these points corresponds to the standard solution for a one-dimensional problem. It follows that the investment triggers at the  $y$  and  $x$  axis are

$$x^* = \frac{\beta_1}{\beta_1 - 1} \delta_1 I, \quad \text{if } y = 0, \quad \text{and} \quad y^* = \frac{\eta_1}{\eta_1 - 1} \delta_2 I, \quad \text{if } x = 0, \quad (13)$$

196 respectively. We refer, for instance, to Dixit and Pindyck (1994) for derivation of these  
197 values.

These thresholds can be interpreted as follows. If  $y = 0$ , then the firm should still invest in this plant as soon as the price of the other product reaches the value  $x^*$ . The intuition is analogous for  $x = 0$  and  $y^*$ . The parameters  $\beta_1 > 1$  and  $\eta_1 > 1$  are the positive roots of the quadratic equations

$$\frac{1}{2} \sigma_1 \beta (\beta - 1) + \alpha_1 \beta - r = 0, \quad \text{and} \quad \frac{1}{2} \sigma_2 \eta (\eta - 1) + \alpha_2 \eta - r = 0,$$

198 respectively.

199 Solving problem (5) means, in particular, that we need to derive the set of values  
200 for  $x$  and  $y$  where stopping is optimal, i.e., where investment should take place. In  
201 particular, we want to derive the boundary between  $D$  and  $S$ , as crossing this boundary  
202 means that investment should be undertaken right away. We call it the *threshold boundary*.  
203 As we have two state variables, we may define this threshold boundary as a surface in  
204  $\mathbb{R}^2$ , as follows: given that the price of one product is  $x \in \mathbb{R}^+$ , the firm should undertake  
205 the investment if the price of the other product is larger or equal to  $b(x)$ . If it is smaller,  
206 than the firm should wait before investment. The next theorem derives some qualitative  
207 features of the threshold boundary for the problem defined in (5).

208 **Theorem 1.** *The boundary between  $D$  and  $S$  can be described by a mapping  $x \mapsto b(x)$ , where:*

- 209 1.  $b(x) = \sup \{ y \in \mathfrak{R}_+ \mid V(x, y) > F(x, y) \}$  for all  $x \in (0, x^*)$ ;
- 210 2.  $b$  is non-increasing on  $(0, x^*)$ ;
- 211 3.  $b$  is convex on  $(0, x^*)$ ;
- 212 4.  $b$  is continuous;
- 213 5.  $b(x) < y^*$  on  $(0, x^*)$ , and  $b(x) = 0$  on  $[x^*, \infty)$ .

214 *In addition, the stopping set  $S$  is:*

- 215 1. closed;
- 216 2. convex.

217 *Finally, the value function  $V$  satisfies:*

- 218 1.  $V > 0$  on  $\mathfrak{R}_{++}^2$ ;
- 219 2.  $V$  is convex;
- 220 3.  $V$  is continuous;
- 221 4.  $V$  is increasing in  $x$  and  $y$ .

222 **Remark 1.** *Theorem 1 leads to the following observations.*

1. We can write

$$D = \left\{ (x, y) \in \mathbb{R}_+^2 \mid y < b(x) \right\}, \quad \text{and} \quad S = \left\{ (x, y) \in \mathbb{R}_+^2 \mid y \geq b(x) \right\}.$$

2. The optimal stopping boundary can never lie below the Net Present Value boundary  $\bar{b}$ , i.e.

$$b(x) > \bar{b}(x) := \delta_2(I - x/\delta_1), \quad \text{all } x \in (0, \delta_1 I).$$

223 Thus in order to solve (5) we need to find  $V$  and, at the same time,  $b(x)$  for  $x \leq x^*$ ,  
 224 such that the properties enumerated in Theorem 1 hold. In particular for  $V$ , conditions  
 225 usually known in the literature as *fit conditions* are checked: the value matching condition  
 226 (for the continuity of the value function) and smooth-pasting (for the smoothness of the  
 227 value function).

### 228 3. The quasi-analytical approach

Following the approach in [Adkins and Paxson \(2011b\)](#), we start by postulating a solution to equation (6) of the following form:

$$v(x, y) = Ax^\beta y^\eta, \quad (14)$$

where  $A$ ,  $\beta$ , and  $\eta$  are constants. Simple calculations lead to the conclusion that for (14) to be a solution to (7) it must hold that  $\beta$  and  $\eta$  are the roots of the characteristic root equation:

$$\mathcal{Q}(\beta, \eta) = \frac{1}{2}\sigma_2^2\eta(\eta - 1) + \frac{1}{2}\sigma_1^2\beta(\beta - 1) + \rho\sigma_1\sigma_2\beta\eta + \alpha_1\beta + \alpha_2\eta - r = 0. \quad (15)$$

229 The set of solutions to (15) defines an ellipse that intersects all quadrants of  $\mathbb{R}^2$ , with  
 230  $\beta$  ( $\eta$ ) on the horizontal (vertical) axis.

[Adkins and Paxson \(2011b\)](#) hypothesize that the boundary between the continuation and stopping regions is of the form  $x \mapsto b(x)$ . As Theorem 1 shows, this is correct. In order to find this boundary, [Adkins and Paxson \(2011b\)](#) try to extend the standard value-matching and smooth-pasting conditions to a two-dimensional setting. The way this is done is as follows: on the boundary it should hold for every  $\hat{x} \in (0, x^*)$ , with  $x^*$  as given in Equation (13):

$$v(\hat{x}, b(\hat{x})) = \frac{Q_1 \hat{x}}{r - \alpha_1} + \frac{Q_2 b(\hat{x})}{r - \alpha_2} - I \quad (\text{value matching}) \quad (16)$$

$$\frac{\partial v(x, y)}{\partial x} \Big|_{x=\hat{x}, y=b(\hat{x})} = \frac{Q_1}{r - \alpha_1} \quad (\text{smooth pasting in } x\text{-direction}) \quad (17)$$

$$\frac{\partial v(x, y)}{\partial y} \Big|_{x=\hat{x}, y=b(\hat{x})} = \frac{Q_2}{r - \alpha_2} \quad (\text{smooth pasting in } y\text{-direction}). \quad (18)$$

Now, if the value function is of the form<sup>2</sup>

$$v(x, y) = Ax^\beta y^\eta,$$

<sup>2</sup> Note that [Adkins and Paxson \(2011b\)](#) assume this is the case.



then it should hold that  $\beta, \eta > 0$  since the boundary conditions  $\lim_{x \downarrow 0} v(x, y) = \lim_{y \downarrow 0} v(x, y) = 0$  should be satisfied. Therefore, for every  $\hat{x} \in (0, x^*)$  we can solve the system of non-linear equations

$$A\hat{x}^\beta b(\hat{x})^\eta = \frac{Q_1\hat{x}}{r - \alpha_1} + \frac{Q_2b(\hat{x})}{r - \alpha_2} - I \quad (19)$$

$$\beta A\hat{x}^{\beta-1} b(\hat{x})^\eta = \frac{Q_1}{r - \alpha_1} \quad (20)$$

$$\eta A\hat{x}^\beta b(\hat{x})^{\eta-1} = \frac{Q_2}{r - \alpha_2} \quad (21)$$

$$\mathcal{Q}(\beta, \eta) = 0, \quad (22)$$

231 in  $b$ ,  $A$ ,  $\beta$ , and  $\eta$ , under the condition that  $\beta, \eta > 0$ .

232 Using the approach presented in [Støre et al. \(2018\)](#) to solve this system, we find the  
233 explicit solution for the boundary<sup>3</sup>:

$$\hat{b}(x) = x \frac{\eta(x)(r - \alpha_2)Q_1}{\beta(x)(r - \alpha_1)Q_2}, \quad (23)$$

234 where

$$\beta(x) = \frac{\sigma_1^2 - 2\alpha_1 + C^*(x)(2\alpha_2 + \sigma_2^2)}{2(\sigma_1^2 + \sigma_2^2 C^*(x))} + \sqrt{\left(\frac{\sigma_1^2 - 2\alpha_1 + C^*(x)(2\alpha_2 + \sigma_2^2)}{2(\sigma_1^2 + \sigma_2^2 C^*(x))}\right)^2 + 2\frac{r - \alpha_2}{\sigma_1^2 + \sigma_2^2 C^*(x)}}, \quad (24)$$

$$\eta(x) = 1 - \beta(x)C^*(x), \quad (25)$$

$$C^*(x) = 1 - \frac{(r - \alpha_1)I}{xQ_1}. \quad (26)$$

235 In the previous equations we use  $\beta(x)$  and  $\eta(x)$  instead of  $\beta$  and  $\eta$  to emphasize the  
236 dependency on the state variable  $x$ .

237 Therefore, solving (20)-(22) leads to values of  $\beta$  and  $\eta$  that *do* depend on the value of  
238  $x$  and, thus, cannot be treated as fixed parameters. This is also the case for the problem  
239 in [Adkins and Paxson \(2011b\)](#), as illustrated by their Figure 3.<sup>4</sup> The same holds for  $A$   
240 and  $b$ .

241 Let  $\mathbf{u} = [\beta(x) \quad \eta(x) \quad A(x) \quad \hat{b}(x)]^T$  denote the vector of solutions resulting from  
242 (20)-(22). Then at  $\hat{b}(x)$ , the value of the firm can be written as

$$v(x, \hat{b}(x)) = A(x)x^{\beta(x)}\hat{b}^{\eta(x)}(x). \quad (27)$$

243 Note that from (9) the partial differential equation  $rV(x, y) - \mathcal{L}V(x, y) = 0$  must  
244 also hold along the threshold boundary, implying that

$$\begin{aligned} & \frac{1}{2}\sigma_1^2 x^2 \left( \frac{\partial^2 v(x, \hat{b})}{\partial x^2} + \frac{\partial}{\partial \mathbf{u}} \left( \frac{\partial v(x, \hat{b})}{\partial x} \right) \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial v(x, \hat{b})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x} \right) \right) + \frac{1}{2}\sigma_2^2 \hat{b}^2 \frac{\partial^2 v(x, \hat{b})}{\partial \hat{b}^2} \\ & + \alpha_1 x \left( \frac{\partial v(x, \hat{b})}{\partial x} + \frac{\partial v(x, \hat{b})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x} \right) + \alpha_2 \hat{b} \frac{\partial v(x, \hat{b})}{\partial \hat{b}} - rv(x, \hat{b}) = 0. \quad (28) \end{aligned}$$

<sup>3</sup> For simplicity, henceforth we assume that  $\rho = 0$ .

<sup>4</sup> The same holds for ([Adkins and Paxson 2011a](#)), see Table 2; [Adkins and Paxson \(2017a\)](#), see Table of Figure 1; [Heydari et al. \(2012\)](#), see equation (19); [Adkins and Paxson \(2013a\)](#), see equation (9); [Adkins and Paxson \(2013b\)](#), see Figure 2; [Fleten et al. \(2016\)](#), see equation (17); [Støre et al. \(2018\)](#), see equation (18); and [Adkins and Paxson \(2017b\)](#), see Table 3.

245 Using (27) we can rewrite (28) as

$$\begin{aligned} & \frac{1}{2}\sigma_1^2 x^2 \left( \frac{\partial}{\partial \mathbf{u}} \left( \frac{\partial v(x, \hat{b})}{\partial x} \right) \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial v(x, \hat{b})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x} \right) \right) + \alpha_1 x \left( \frac{\partial v(x, \hat{b})}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x} \right) \\ & + Ax^\beta \hat{b}^\eta \left( \frac{1}{2}\sigma_2^2 \eta(\eta - 1) + \frac{1}{2}\sigma_1^2 \beta(\beta - 1) + \alpha_1 \beta + \alpha_2 \eta - r \right) = 0, \end{aligned} \quad (29)$$

246 The first two terms in (29) represent the contributions of the partial derivatives of  
247  $\hat{b}$ ,  $A$ ,  $\beta$  and  $\eta$  with respect to  $x$ , whereas the last term is equal to  $Ax^\beta \hat{b}^\eta Q(\beta, \eta)$ . If the  
248 solution proposed in (14) is correct, then the latter should be equal to zero, and we can  
249 still use the system (19)-(22) to determine the threshold boundary. In what follows we  
250 verify whether the contribution of the partial derivatives is negligible for the numerical  
251 example in Table 1.

$\hat{x}$	Contribution of partial derivatives
10	-10841.14
20	-54 856.60
30	-9040.40

Table 1: The value of the first two terms of equation (29) for the following set of the parameter values:  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.6$ ,  $\alpha_1 = 0.02$ ,  $\alpha_2 = 0.02$ ,  $r = 0.1$ ,  $\rho = 0$ ,  $Q_1 = 5$ ,  $Q_2 = 10$ , and  $I = 2000$ .

252 For  $\hat{x} = 10$ , and the set of parameter values in Table 1,  $F(\hat{x}, \hat{b}(\hat{x})) = 7281.23$ . There-  
253 fore, we conclude that the contribution of the partial derivatives cannot be neglected. As  
254 a result, substitution of the solution (27) in (7) leads to the conclusion that the condition  
255 for  $\eta$  and  $\beta$  is no longer (15). In fact, (15) needs to be modified to incorporate terms  
256 involving  $\beta'(x)$ ,  $\eta'(x)$ ,  $\beta''(x)$ ,  $\eta''(x)$ ,  $A'(x)$ ,  $A''(x)$ ,  $\hat{b}'(x)$  and  $\hat{b}''(x)$ . The implication is  
257 that solving the system (19)-(22) for different values of  $\hat{x}$  does not result in a correct  
258 threshold boundary.

### 259 3.1. Results of the quasi-analytical approach

260 After having shown that the boundary  $\hat{b}$ , as determined by the quasi-analytical  
261 approach, is not the true boundary  $b$ , it could still be the case that  $\hat{b}$  is a good approxima-  
262 tion of  $b$ . This section, however, provides an argument that this is not the case, at least  
263 for the problem in Section 2.

264 We start out by presenting the following proposition.

265 **Proposition 1.** *The value function,  $V$ , is monotonically increasing in both  $\sigma_1$  and  $\sigma_2$ .*

266 **Proof.** The result follows from Proposition 3 in Olsen and Stensland (1992) using the  
267 fact that the optimal value function is convex, as stated in our Theorem 1.  $\square$

268 In the following, we study the behaviour of the investment boundary as a function  
269 of the volatilities of the involved processes,  $\sigma_1$  and  $\sigma_2$ . Thus we let  $b(\sigma_1, \sigma_2; x)$  denote the  
270 boundary, given that the current price of the first product is  $x$ , and that the volatilities  
271 are  $\sigma_1$  (for  $X$ ) and  $\sigma_2$  (for  $Y$ ).

272 **Corollary 1.** *Let  $b(\sigma_1, \sigma_2; x)$  denote the optimal investment threshold boundary for a given level  
273 of  $x$ . Then it holds that  $b(\sigma_1, \sigma_2; x)$  is increasing in both  $\sigma_1$  and  $\sigma_2$ .*

274 **Proof.** We prove the result by contradiction. Without loss of generality we only consider  
275 a change in  $\sigma_1$ . Consider two different values of  $\sigma_1$ , such that  $\hat{\sigma}_1 > \bar{\sigma}_1$ , and  $b(\hat{\sigma}_1, \sigma_2; x) <$

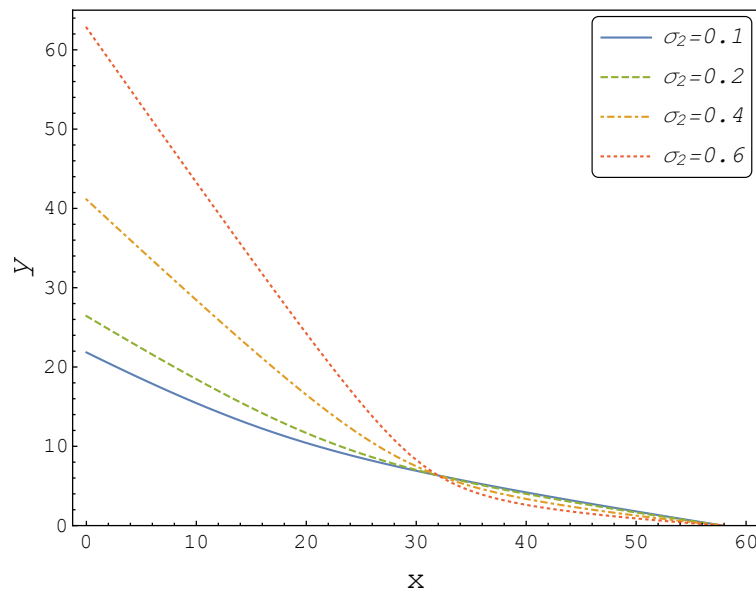
276  $b(\hat{\sigma}_1, \sigma_2; x)$  for some  $x$ . Let  $V(\sigma_1, \sigma_2; x, y)$  denote the optimal value function for a given  
 277 level of  $x$ . Then,

$$V(\hat{\sigma}_1, \sigma_2; x, y) = \begin{cases} > F(x, y) & \text{for } y < b(\hat{\sigma}_1, \sigma_2; x), \\ = F(x, y) & \text{for } y \geq b(\hat{\sigma}_1, \sigma_2; x), \end{cases} \quad (30)$$

$$V(\bar{\sigma}_1, \sigma_2; x, y) = \begin{cases} > F(x, y) & \text{for } y < b(\bar{\sigma}_1, \sigma_2; x), \\ = F(x, y) & \text{for } y \geq b(\bar{\sigma}_1, \sigma_2; x). \end{cases} \quad (31)$$

278 If  $b(\hat{\sigma}_1, \sigma_2; x) < b(\bar{\sigma}_1, \sigma_2; x)$ , then for  $y \in (b(\hat{\sigma}_1, \sigma_2; x), b(\bar{\sigma}_1, \sigma_2; x))$ , it holds that  
 279  $V(\bar{\sigma}_1, \sigma_2; x, y) > F(x, y) = V(\hat{\sigma}_1, \sigma_2; x, y)$ , which contradicts Proposition 1.  $\square$

280 Figure 1 illustrates the quasi-analytical threshold boundaries for different values of  
 281  $\sigma_2$ .



**Figure 1.** The threshold boundaries,  $\hat{b}$ , for the following set of parameter values:  $\sigma_1 = 0.2$ ,  $\alpha_1 = 0.02$ ,  $\alpha_2 = 0.02$ ,  $r = 0.1$ ,  $\rho = 0$ ,  $Q_1 = 5$ ,  $Q_2 = 10$ ,  $I = 2000$ , and different values of  $\sigma_2$ .

282 Evidently, the numerical example violates Corollary 1, since the threshold bound-  
 283 aries intersect. Moreover, this result does not correspond to what we would expect  
 284 from real options theory, i.e. that the firm invests for a larger threshold level in a more  
 285 uncertain environment. In fact, for  $x > 32$ , the quasi-analytical approach suggests that  
 286 the firm should invest for a lower threshold level when  $\sigma_2$  is larger. This clearly leads  
 287 to a sub-optimal decision, so the quasi-analytical solution falls short in being a useful  
 288 approximation to the optimal solution in this case.

#### 289 4. Numerical Solution

290 This section develops a finite difference algorithm to solve the optimal stopping  
 291 problem in (5). The results of the numerical approach are different from the results  
 292 obtained by the analytical approach *and* in line with Theorem 1 and Proposition 1.

We start by generating a discrete grid over the domain of the partial differential equation in (7). Thus we assume that the intervals  $(0, x^*]$  and  $(0, y^*]$  are divided in  $N_x + 1$  and  $N_y + 1$  equally spaced subintervals, respectively, and we let

$$x_i = ih \quad i = 0, 1, 2, \dots, N_x, \quad h = \frac{x^*}{N_x}, \quad (32)$$

$$y_j = jg \quad i = 0, 1, 2, \dots, N_y, \quad g = \frac{y^*}{N_y}, \quad (33)$$

where  $x^*$  and  $y^*$  are the optimal investment triggers in case the other state variable is zero and, thus, are the natural end points of the grid. Moreover, we consider the following notation:  $V_{i,j}$  denotes the value of the firm at the grid points  $(x_i, y_j)$ :

$$V_{(i,j)} = V(x_i, Y_i)$$

293 with  $V$  defined in Equation (5). Finally we let  $\mathbf{v}$  be vector of unknown grid points, which  
294 can be ordered in the following way

$$\mathbf{v} = \begin{bmatrix} V_{0,0} \\ V_{0,1} \\ \vdots \\ V_{0,N_y} \\ \vdots \\ V_{N_x,N_y} \end{bmatrix} \quad (34)$$

Then we are able to derive a linear system of equations that allows to solve for the discrete grid points simultaneously, as follows. We discretize the partial differential equation using a weighted sum of the function values at the neighboring point approximations to the partial derivatives. This yields

$$\begin{aligned} \frac{1}{2} \sigma_1^2 p_1^2 \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h^2} + \frac{1}{2} \sigma_2^2 p_2^2 \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{g^2} + \alpha_1 x \frac{V_{i+1,j} - V_{i,j}}{h} + \alpha_2 y \frac{V_{i,j+1} - V_{i,j}}{g} \\ + \rho \sigma_1 \sigma_2 xy \frac{V_{i+1,j+1} - V_{i+1,j-1} - V_{i-1,j+1} + V_{i-1,j-1}}{4hg} - rV_{i,j} = 0. \end{aligned} \quad (35)$$

Rearranging the terms, gives

$$\begin{aligned} V_{i,j} \left( -\sigma_1^2 i^2 - \alpha_1 ih - \sigma_2^2 \frac{j^2 h^2}{g^2} - \alpha_2 \frac{jh^2}{g} - rh^2 \right) + V_{i,j+1} \left( \frac{1}{2} \sigma_2^2 \frac{j^2 h^2}{g^2} + \alpha_2 \frac{jh^2}{g} \right) + \\ V_{i+1,j} \left( \frac{1}{2} \sigma_1^2 i^2 + \alpha_1 ih \right) + V_{i,j-1} \left( \frac{1}{2} \sigma_2^2 \frac{j^2 h^2}{g^2} \right) + V_{i-1,j} \left( \frac{1}{2} \sigma_1^2 i^2 \right) + \\ + \rho \sigma_1 \sigma_2 \frac{ijh}{4g} (V_{i+1,j+1} - V_{i+1,j-1} - V_{i-1,j+1} + V_{i-1,j-1}) = 0. \end{aligned} \quad (36)$$

295 Then the partial differential equation (35) can be represented as a system of linear  
296 equations

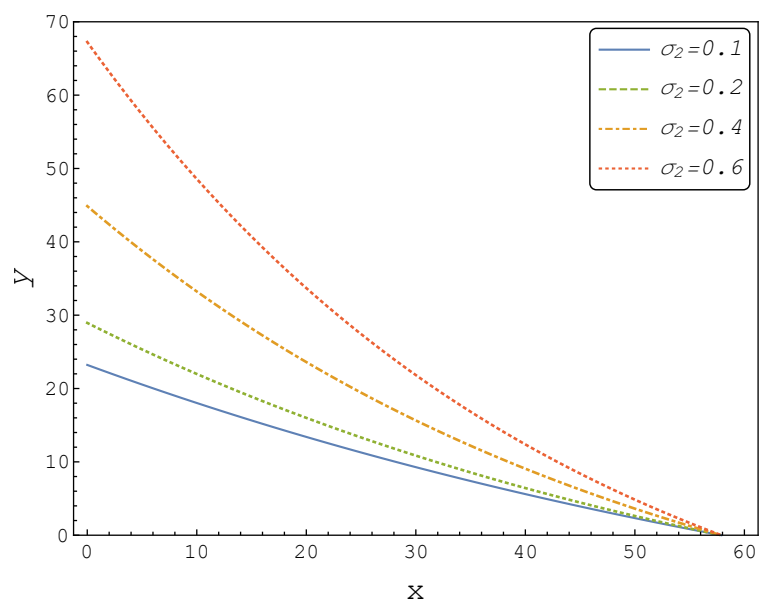
$$B\mathbf{v} = 0, \quad (37)$$

297 where  $B$  is the matrix of coefficients resulting from (36).

298 This system can be solved by applying appropriate boundary conditions. We use  
299 the fact that the value at  $(x^*, 0)$  and  $(0, y^*)$  must equal the value of the immediate  
300 investment. In addition, if either  $x_i$  or  $y_j$  is equal to zero the problem is reduced to  
301 one-dimension, and the grid points together with the threshold boundary can be found

302 analytically. Given a candidate threshold function, the system (37) in combination with  
 303 the boundary conditions in zero and final nodes, yield a solution for the unknown grid  
 304 points. To determine the optimal threshold we implement the following procedure. First,  
 305 we propose a shape of the exercise boundary. For example, the results that we present  
 306 in Figure 2 are based on the quadratic function, i.e.  $y = a + bx + cx^2$ . The unknown  
 307 parameters,  $a$  and  $b$  can be determined using the analytical threshold boundaries when  
 308 either  $x_i$  or  $y_j$  is zero. In order to find  $c$ , we compute the derivative of the option value  
 309 at the candidate threshold boundary at each node, and compare it with the derivatives  
 310 resulting from the smooth pasting conditions. Next, we compute the sum squared error  
 311 of the differences and minimize it with respect to unknown parameter  $c$ , which allows  
 312 to determine the optimal threshold in such a way that the smooth pasting condition is  
 313 satisfied.

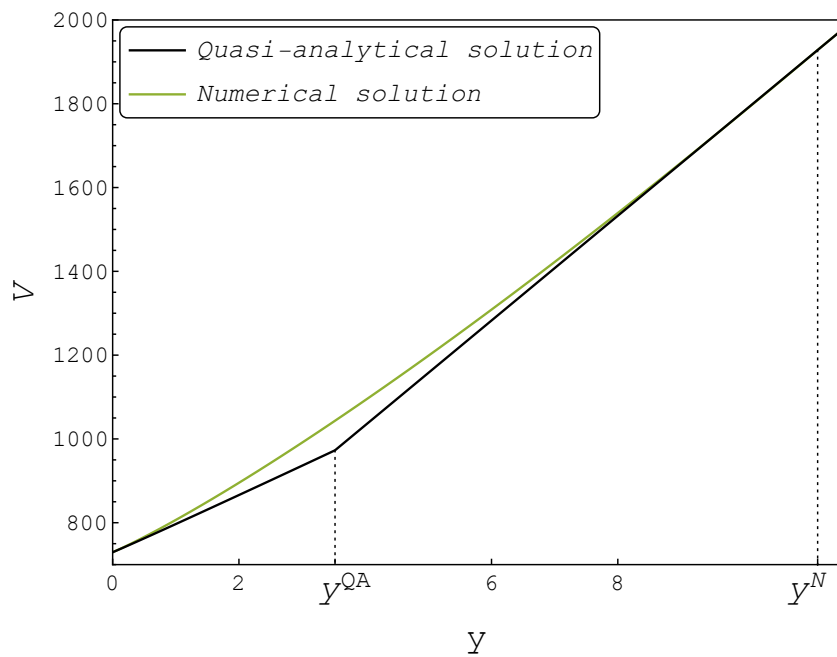
314 We now replicate Figure 1 using our finite difference scheme and depict it in Figure  
 315 2.



**Figure 2.** The numerical threshold boundary for the following set of parameter values:  $\sigma_1 = 0.2$ ,  $\alpha_1 = 0.02$ ,  $\alpha_2 = 0.02$ ,  $r = 0.1$ ,  $\rho = 0$ ,  $Q_1 = 5$ ,  $Q_2 = 10$ ,  $I = 2000$ , and different values of  $\sigma_2$ .

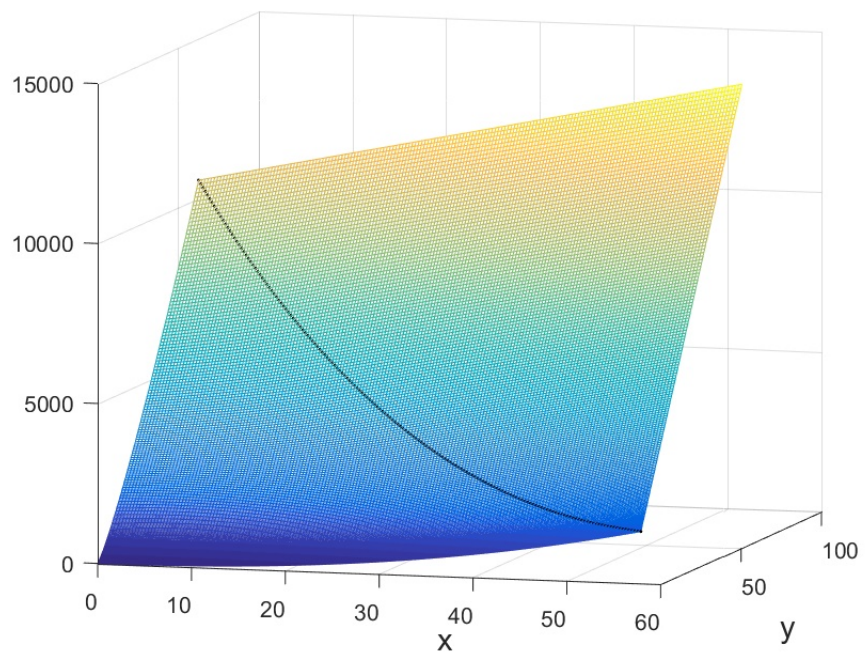
316 This numerical example results in a more intuitive shape of thresholds boundaries  
 317 and represent a standard result from the real options theory. Namely, an increase in  
 318 volatility leads to an increase of the optimal investment threshold.

319 In addition, finite difference also allows for the calculation of an approximation to  
 320 the value function that is implied by the quasi-analytical boundary  $\hat{b}$ . This can be done  
 321 by solving (37) for the boundary in (23). Figure 3 illustrates the comparison between the  
 322 implied value function and the numerical solution represented by quadratic boundary  
 323 for a fixed level of  $x$  and different values of  $y$ .



**Figure 3.** The numerical threshold boundary for the following set of parameter values:  $x = 40.52$ ,  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.6$ ,  $\alpha_1 = 0.02$ ,  $\alpha_2 = 0.02$ ,  $r = 0.1$ ,  $\rho = 0$ ,  $Q_1 = 5$ ,  $Q_2 = 10$ ,  $I = 2000$ , and different values of  $y$ .

324 From Figure 3 it is evident that the value function implied by the quasi-analytical  
 325 solution has a kink at the boundary point  $y^{QA} = \hat{b}(40.52) = 2.49$ , violating the smooth-  
 326 pasting condition. Consequently, the quasi-analytical approach underestimates the true  
 327 value function, which leads to a sub-optimal investment decision rule for large values of  
 328  $x$ . Note that the quasi-analytical approach suggests a much lower trigger than our finite  
 329 difference scheme. For  $x = 40.52$ , the numerical procedure based on the finite-difference  
 330 algorithm gives the boundary point  $y^{QA} = 10.26$ , such that the smooth-pasting condition  
 331 holds. Figure 4 illustrates the value function for different values of  $x$  and  $y$ , as well as  
 332 the threshold boundary.



**Figure 4.** The numerical value function and threshold boundary (solid black curve) for the following set of parameter values:  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.6$ ,  $\alpha_1 = 0.02$ ,  $\alpha_2 = 0.02$ ,  $r = 0.1$ ,  $\rho = 0$ ,  $Q_1 = 5$ ,  $Q_2 = 10$ ,  $I = 2000$ , and different values of  $x$  and  $y$ .

333 As can be seen, the value appears to be smooth for different values of  $x$  and  $y$  in the  
 334 grid. The average squared error resulting from the numerical procedure is equal to 0.44,  
 335 which corresponds to 0.17% of the true value of total derivative of the value function.  
 336 Therefore, we conclude that the proposed numerical method is a good approximation  
 337 for the true value function and optimal threshold.

338 Lastly, in order to give an indication how often a firm would make a poorly timed  
 339 investment decision, we simulate the passage time for the processes  $X_t$  and  $Y_t$  to reach  
 340 the quasi-analytical boundary. We then run the procedure 5000 times for a specific set  
 341 of starting values  $(x_0, y_0)$ , and calculate the percentage of cases of the threshold being  
 342 reached within the next 5 years. We perform a similar procedure, to determine the  
 343 investment probabilities for our numerical solution. The results for the different starting  
 344 points are presented in Table 2.

$(x_0, y_0)$	5	10	15	$(x_0, y_0)$	5	10	15
10	10.06%	23.97%	39.03%	10	5.40%	5.56%	5.51%
15	21.69%	42.87%	61.69%	15	5.57%	5.41%	5.57%
20	40.32%	68.34%	90.21%	20	5.43%	5.24%	5.59%

(a) Quasi-analytical boundary

(b) Numerical boundary

Table 2: Percentage of cases when a firm undertakes an investment within the next 5 years for the set of parameter values:  $\sigma_1 = 0.2$ ,  $\sigma_2 = 0.6$ ,  $\alpha_1 = 0.02$ ,  $\alpha_2 = 0.02$ ,  $r = 0.1$ ,  $\rho = 0$ ,  $Q_1 = 5$ ,  $Q_2 = 10$ ,  $I = 2000$ .

345 Table 2b shows that, for example, for the starting values (15,10) the firm should  
 346 invest in 5.41% of the cases. According to the quasi-analytical approach however, the  
 347 firm invests in 42.87% of the cases, implying that many times the firm invests while it is  
 348 in fact not optimal to do so.

## 349 5. Conclusion

350 This paper develops an easy-to-implement finite difference algorithm to solve real  
351 options models with two-factor uncertainty. The proposed framework is, thus, highly  
352 relevant for the evaluation of business opportunities involving multiple end-products, a  
353 switch in feed-stock or end-product, a cooperation between firms that are operative in  
354 different markets, and investments in new technologies incentivized by market-based  
355 policy instruments.

356 We apply it to a particular investment problem, where after investment the firm  
357 is able to produce two different products. The output prices of these products follow  
358 two geometric Brownian motion processes, possibly correlated. The investment cost is  
359 constant and sunk. We contrast our solution approach to the quasi-analytical approach  
360 developed by [Adkins and Paxson \(2011b\)](#) to address such problems. The latter has  
361 already been adopted by several other authors, as the overview in Section 1 shows. This  
362 paper argues, however, that this quasi-analytical method does not always result in the  
363 correct investment decision rule.

364 From the analysis of this two-factor real options problem we obtain that the quasi-  
365 analytical investment decision rule in some cases also fails to be a reasonable approxima-  
366 tion to the optimal decision. In particular, we find that the quasi-analytical solution does  
367 not comply with the (analytical) result that the investment threshold boundary must be  
368 monotonically increasing in the volatility parameters of both stochastic processes.

369 The ultimate conclusion is that non-homogenous real options problems with two-  
370 factor uncertainty should be solved using a different numerical procedure. Or at the  
371 very least, the quality of the quasi-analytical approximation should be discussed. Note,  
372 however, that if our two-factor uncertainty problem is homogenous, then a standard (cf.  
373 [McDonald and Siegel \(1986\)](#)) reduction in dimensionality can be obtained, leading to an  
374 analytical solution.

## 375 Appendix A

### Proof of Theorem 1.

Throughout the proof, we will denote the unique solution to (1) for given starting point  
( $X_0, Y_0$ )  $\in \mathbb{R}_+^2 \setminus \{0\}$  by  $(X^x, Y^y)$ . Note that  $(X^x, Y^y) = (xX^1, yY^1)$ .

1. ( $V > 0$  on  $\mathbb{R}_{++}^2$ ) On  $S$  the result is trivial. Let  $(x, y) \in D \cap \mathbb{R}_{++}^2$ . Consider the  
stopping time

$$\tau = \inf\{t \geq 0 \mid F(X_t, Y_t) > 0\}.$$

Since  $e^{-r\tau}F(X_\tau, Y_\tau) = 0$  on  $\{\tau = \infty\}$  (since  $r > \max\{\alpha_1, \alpha_2\}$ ) and  $P(\tau < \infty) > 0$ , it  
holds that

$$V(x, y) \geq \mathbb{E}[e^{-r\tau}F(X_\tau, Y_\tau)] > 0.$$

2. (Convexity of  $V$ ) On  $S$  the result is trivial. Take  $(x', y'), (x'', y'') \in D$  and  $\lambda \in (0, 1)$ .  
Define  $(x, y) := \lambda(x', y') + (1 - \lambda)(x'', y'')$ . It then holds that

$$\begin{aligned} V(x, y) &= \sup_{\tau} \mathbb{E}[e^{-r\tau}F(x', y')] \\ &= \sup_{\tau} \mathbb{E}\left[e^{-r\tau}\left(\frac{xX_\tau^1}{\delta_1} + \frac{yY_\tau^1}{\delta_2} - I\right)\right] \\ &= \sup_{\tau} \mathbb{E}\left[e^{-r\tau}\left(\frac{(\lambda x' + (1 - \lambda)x'')X_\tau^1}{\delta_1} + \frac{(\lambda y' + (1 - \lambda)y'')Y_\tau^1}{\delta_2} - I\right)\right] \\ &= \sup_{\tau} \mathbb{E}\left[\lambda e^{-r\tau}\left(\frac{x'X_\tau^1}{\delta_1} + \frac{y'Y_\tau^1}{\delta_2} - I\right) + (1 - \lambda)e^{-r\tau}\left(\frac{x''X_\tau^1}{\delta_1} + \frac{y''Y_\tau^1}{\delta_2} - I\right)\right] \\ &\leq \lambda \sup_{\tau} \mathbb{E}[e^{-r\tau}F(x', y')] + (1 - \lambda) \sup_{\tau} \mathbb{E}[e^{-r\tau}F(x'', y'')] \\ &= \lambda V(x', y') + (1 - \lambda)V(x'', y''). \end{aligned}$$



3. (Continuity of  $V$ ) This property follows from the general theory of stochastic processes, see, e.g., (Krylov 1980, Theorem 3.1.5).

4. (Monotonicity of  $V$ ) We prove that  $V$  is (strictly) increasing in  $x$ . Again, the result is trivial on  $S$ . Take  $(x, y) \in D$  and let  $\varepsilon > 0$  be such that  $(x + \varepsilon, y) \in D$  (such  $\varepsilon$  exists since  $D$  is open; see below). Take any stopping time  $\tau$ . It then holds that

$$\mathbb{E} \left[ e^{-r\tau} \left( \frac{(x + \varepsilon)X_\tau^1}{\delta_1} + \frac{yY_\tau^1}{\delta_2} - I \right) \right] \geq \mathbb{E} \left[ e^{-r\tau} \left( \frac{xX_\tau^1}{\delta_1} + \frac{yY_\tau^1}{\delta_2} - I \right) \right],$$

with equality only when  $\{\tau = \infty\}$  a.s.. Note that  $\tau$  with  $\{\tau = \infty\}$  a.s. is never optimal. Take  $\tau^* = \inf\{t \geq 0 | Y_t \geq \delta_1 I + 1\}$ . Then  $P(\tau^* < \infty) > 0$  and, thus, we have that  $\mathbb{E} \left[ e^{-r\tau^*} F(X_{\tau^*}^x, Y_{\tau^*}^y) \right] > 0$ .] Therefore,  $V(x + \varepsilon, y) > V(x, y)$ .

5. (Closedness of  $D$ ) Take a sequence  $(x^{(n)}, y^{(n)})_{n \in \mathbb{N}}$  in  $S$  with limit  $(x, y)$ . Then  $V(x^{(n)}, y^{(n)}) = F(x^{(n)}, y^{(n)})$  for all  $n \in \mathbb{N}$ . Since  $\lim_{n \rightarrow \infty} F(x^{(n)}, y^{(n)}) = F(x, y)$  and  $V$  is continuous, it holds that  $V(x, y) = F(x, y)$ . This implies that  $(x, y) \in S$ .

6. (Convexity of  $D$ ) Suppose there exists  $(x', y'), (x'', y'') \in S$  and  $\lambda \in (0, 1)$  such that  $(x, y) := \lambda(x', y') + (1 - \lambda)(x'', y'') \in D$ . It then holds that

$$V(x, y) > F(x, y) = \lambda F(x', y') + (1 - \lambda)F(x'', y'') = \lambda V(x', y') + (1 - \lambda)V(x'', y'').$$

This contradicts convexity of  $V$ .

7. ( $b(x)$  can be written as a sup) Take  $(x, y) \in D$ . Then there exists a stopping time  $\tau^*$  such that  $(X_{\tau^*}, Y_{\tau^*}) \in D$ , a.s.. Hence,

$$V(x, y) = \sup_{\tau} \mathbb{E} \left[ e^{-r\tau} F(X_\tau, Y_\tau) \right] \geq \mathbb{E} \left[ e^{-r\tau^*} F(X_{\tau^*}, Y_{\tau^*}) \right] > F(x, y).$$

Now take  $\varepsilon \in (0, y)$ . Then

$$\begin{aligned} V(x, y - \varepsilon) &\geq \mathbb{E} \left[ e^{-r\tau^*} F(X_{\tau^*}, Y_{\tau^*}) \right] \\ &= \mathbb{E} \left[ e^{-r\tau^*} \left( \frac{xX_{\tau^*}^1}{\delta_1} + \frac{(y - \varepsilon)Y_{\tau^*}^1}{\delta_2} - I \right) \right] \\ &= \mathbb{E} \left[ e^{-r\tau^*} \left( \frac{xX_{\tau^*}^1}{\delta_1} + \frac{yY_{\tau^*}^1}{\delta_2} - I \right) \right] - \mathbb{E} \left[ e^{-r\tau^*} \frac{\varepsilon Y_{\tau^*}^1}{\delta_2} \right] \\ &\stackrel{(*)}{\geq} \mathbb{E} \left[ e^{-r\tau^*} \left( \frac{xX_{\tau^*}^1}{\delta_1} + \frac{yY_{\tau^*}^1}{\delta_2} - I \right) \right] - \frac{\varepsilon}{\delta_2} \\ &> \mathbb{E} \left[ e^{-r\tau^*} \left( \frac{xX_{\tau^*}^1}{\delta_1} + \frac{yY_{\tau^*}^1}{\delta_2} - I \right) \right] > F(x, y) > F(x, y - \varepsilon), \end{aligned}$$

376 where  $(*)$  follows from the fact that  $e^{-rt}Y_t$  is a supermartingale. Therefore,  $(x, y - \varepsilon) \in D$ .

377

378 8. ( $b$  is non-increasing) This follows from the fact that for all  $(x, y) \in D$  and all  $\varepsilon \in (0, x)$   
379 it holds that  $(x - \varepsilon, y) \in D$ . This can be proved using a similar argument as above.

380

381 9. ( $b$  is convex) Convexity of  $b$  follows from the fact that its epigraph is the convex  
382 set  $S$ .

383

384 10. ( $b$  is continuous) Continuity of  $b$  on  $(0, \infty)$  is immediate, because it is a convex  
385 function on an open convex set (see, for example, Berge 1963, Theorem 8.5.7). Continuity

386 at  $x = 0$  follows from the fact that the stopping set is closed.

387

388 **11.** (boundedness of  $b$ ) The boundedness properties follow from continuity and  $x^*$   
 389 and  $y^*$  being the solutions of the optimal stopping problem on  $\mathbb{R}_+ \times \{0\}$  and  $\{0\} \times \mathbb{R}_+$ ,  
 390 respectively.  $\square$

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