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Efficient Kidney Exchange with Dichotomous Preferences[1]

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Abstract: This paper studies a general kidney exchange model with compatible patient-donor pairs, incompatible patient-donor pairs, single donors, and patients on the waiting list. We derive an explicit formula of the maximal number of feasible kidney transplants under several sizes of cycles and chains of exchange, analyze the effect of different ways of exchange on efficiency, and provide substantial simulation results based on the USA data. Our results further show that kidney exchange can be decentralized for relatively large populations, and that allowing compatible pairs and single donors to exchange with incompatible pairs can significantly increase the number of feasible kidney transplants. A more general model of two-category type-compatible exchanges is also established.

Keywords: Kidney Exchange, Kidney Transplant, Efficiency, Simulation.
JEL classification: C78, D47, I190.

1 Introduction

Every year in the world hundreds and thousands patients of severe kidney disease need kidney transplants. The difficulty of having suitable kidney transplants arises in three major aspects. Firstly, there is a significant shortage of kidneys from deceased donors. For instance, in the United States in 2017 more than 92,000 patients were waiting for kidney transplants and only about 14,077 received transplants from deceased donors and 5,536 received transplants from living donors. While in waiting, over 4,000 patients passed away and about 4,414 were getting too sick to have a transplant and were therefore removed from the waiting list (see the USA OPTN/SRTR annual report by Hart et al. 2017). Secondly, a patient may receive a kidney from a living donor who can be a family member, a relative, or a friend. In this case the patient and the donor are called a patient-donor

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pair, and the patient is a paired patient and the donor a paired donor. But the patient may not be compatible with the donor and therefore is unable to use the kidney directly because of blood or tissue incompatibility. Kidney transplants can be conducted in two ways: The first is called *a cycle of exchange* which involves typically several incompatible patient-donor pairs and refers to a process in which, say two pairs, the 1st paired patient receives a compatible kidney from the 2nd paired donor, and the 2nd paired patient receives a compatible kidney from the 1st paired donor. The other is *a chain of exchange* which involves an altruistic donor, patient-donor pairs, and a patient on the waiting list and is a process in which the 1st paired patient if any receives a compatible kidney from an altruistic donor, the 2nd paired patient receives a compatible kidney from the 1st paired donor, etc, and finally the patient on the waiting list receives a compatible kidney from the last paired donor. *A k-way exchange* means a cycle or a chain of exchange involving k patients and k donors. Thirdly, although most people have one more kidney than they need, it is almost universally illegal to buy or sell a kidney.

The operation of a suitable kidney transplant must satisfy three basic constraints. The first are two medical constraints: the patient must be both blood-compatible and tissue-compatible with the donor. The second is the incentive constraint. The incentive issue arises when it involves patient-donor pairs. If a paired patient is incompatible with his/her paired donor, the patient need to exchange one kidney for another. Then the order of implementing kidney transplant becomes crucial to incentive-compatible exchange. To avoid the moral hazard caused by the ordering, exchanges between the two pairs must be carried out simultaneously. Because transplants are performed simultaneously, four simultaneous surgical treatments are required for two pair exchanges. The third is the capacity constraint. In practice there is a limit to the number of possible kidney transplants in each hospital. It is desirable to have short chains or cycles of exchange.

Kidney exchange has been previously studied by medical researchers (see Rapaport 1986, Ross et al. 1997, Ross and Woodle 2000, Zenios et al. 2001, etc). Roth et al. (2004) initialized economic analysis of kidney exchange, stimulating considerable interest in the subject. They proposed an efficient and incentive compatible mechanism extending the top trading cycle procedure from Shapley and Scarf (1974); see also Abdulkadiroğlu and Sönmez (1999) for a related mechanism. In this case cycles and chains could be long. Roth et al. (2007a) considered a simpler but more practical model where patients are indifferent between compatible kidneys and prefer compatible kidneys to incompatible ones, i.e., patients have dichotomous preferences. Their model consists of multiple incompatible patient-donor pairs. They proved that using two-way and three-way cycles could significantly increase the number of possible exchanges, and at most four-way cycles suffice to capture all potential gains of exchange. Roth et al. (2007a) and Saidman et al. (2006)

provided simulation results using the USA OPTN/SRTR data from 1993 to 2002.

In this paper we generalize the model of Roth et al. (2007a) by allowing all possible exchanges among incompatible, compatible patient-donor pairs, (altruistic) single donors (decreased or living), and patients on the waiting list and making the model closer to the reality. Our aim is to explore ways of maximizing the number of feasible kidney transplants. Because the model of Roth et al. (2007a) consists of only incompatible patient-donor pairs, only cycles of exchange will be needed. In our model, both cycles and chains of exchange will be needed. We derive for each case of k-way exchange, $k = 2, 3, 4$, an explicit formula of the maximal number of feasible kidney transplants and propose a procedure to find efficient allocations of kidney transplants. We show that two-way cycles and chains of exchange can gain most of efficient exchange, three-ways can still make a visible effect and at most four-ways will be sufficient to capture all potential gains of exchange. This demonstrates that the key insights of Roth et al. (2007a) still hold for the current more general model, although the analysis inevitably becomes substantially more involved and complicated.

We further prove that in every k-way ($k = 2, 3, 4$) of exchange, each cycle contains at most two blood-type compatible pairs and each chain comprises at most one blood-type compatible pair. We provide substantial simulation results based on two data sets from the USA OPTN/SRTR from 1993 to 2002 (see Table 3) and from 1995 to 2016 (see Table 4). The second data set covers a longer period of time with more and better information than those collected in the first set. We highlight two important findings: Firstly, we find that under at most three-way exchanges allowing compatible pairs to exchange with incompatible pairs can have at least 10% net increase of feasible kidney transplants and that allowing both compatible pairs and single donors to exchange with incompatible pairs can have at least 30% net increase of feasible kidney transplants (see Table 8). This net increase of possible kidney transplants is very significant, meaning that many more patients can be saved when both compatible pairs and single donors are allowed to exchange with incompatible pairs. Secondly, our simulations clearly indicate that as the number of incompatible patient-donor pairs in the population reaches 100, the slope of matching rates (in percentage) of incompatible paired patients getting transplants becomes almost flat, albeit slightly upwards. This has some novel policy implication: Kidney exchange can be decentralized in the sense that in a country with a relatively large population, separate kidney exchange programs/centers can be established in several major regions. This means that patients and donors can have kidney transplant operations in their own regional or nearby center and save their traveling cost and time and that medical resources can be evenly distributed across the country and need not be concentrated in one place.

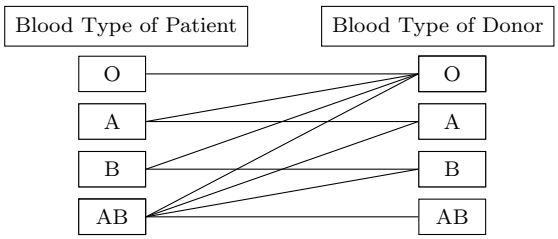
Furthermore, we propose a general and abstract model of two-category type-compatible exchanges and prove that it is sufficient to fully achieve efficiency by using only n-way cycles

and chains of exchange where n is the number of the primary types and is independent of the number of secondary types. In the context of kidney exchange, primary types in the first category correspond to blood types and have only four types and secondary types in the second category correspond to the classification of tissue types and can vary from one country to another. This abstract model has extended the general model of Roth et al. (2007a) and could have applications beyond kidney exchange.

We conclude this introduction by briefly reviewing several other related studies. Ross and Woodle (2000) suggested the idea of allowing compatible pairs to exchange with incompatible pairs. Roth et al. (2005a) examined a model with both compatible and incompatible pairs under two-way exchanges and dichotomous preferences and developed constrained-efficient and strategy-proof procedures. Sönmez and Ünver (2014) studied a similar model under two-way exchange by exploring the structure of Pareto-efficient kidney transplants via the Gallai-Edmonds theorem from graph theory and also conducting comparative static analysis. Roth et al. (2005b) and Gentry et al. (2007) discussed the issue of kidney paired donation with compatible pairs and provided some simulation results; see Roth et al. (2007b) for some historical aspect. Ünver (2010) studied efficient kidney exchanges in a dynamic environment in which agents arrive according to a stochastic Poisson process. Yilmaz (2011) characterized an efficient and egalitarian two-way exchange mechanism involving paired donations and list exchanges. Nicoló and Rodríguez-Álvarez (2012) presented a model with match quality and patients' preferences. They proposed an efficient and truthful two-way exchange mechanism. Sönmez and Ünver (2013) offered a survey on the subject.

Ausubel and Morrill (2014) observed that incentive compatibility for kidney exchange requires only kidney donation to occur no later than the associated kidney receipt. They showed that sequential exchanges can increase the number of beneficial exchanges. Bilgel and Galle (2015) examined financial incentives for kidney donation by using a synthetic control method; see also Roth et al. (2007a) for competitive compensation. Nicoló and Rodríguez-Álvarez (2017) studied a model in which patients prefer kidneys from compatible younger donors to kidneys from older ones. They proposed sequential two-way exchange rules being individually rational, strategy-proof and non-bossy. Biró et al. (2019) and Sönmez et al. (2019) explored ways of incentivizing compatible pairs to participate in exchanges with incompatible pairs. Agarwal et al. (2019) reported market failure in kidney exchange and offered some remedy. Andersson and Kratz (2020) introduced a model of kidney exchange that tries to relax the traditional blood type compatibility. In their model, every patient prefers a fully acceptable donor to any donor who is not fully acceptable, and yet prefers an acceptable donor to any unacceptable donor. They showed that more efficiency can be made with also supporting simulation results. See Chun et al. (2015) and

Figure 1: Blood-type compatibility between patients and donors.



Sönmez et al. (2018) for independent related studies.

This paper is organized as follows. The model and basic concepts are introduced in Section 2. Two-, three-, and four-way exchanges are examined with focus on two-way exchanges in Section 3. A general and abstract multi-way exchange model is discussed in Section 4. Simulations are presented in Section 5 and conclusion is given in Section 6.

2 The Model

Kidney exchanges involve patients and donors. A kidney can be transplanted from a willing donor to a patient if the donor's kidney is compatible to the patient both in blood type and tissue type. There are four blood types, A, B, AB, and O. A patient of O type can receive a kidney only from a donor of O type, a patient of A type can receive a kidney from a donor of A or O type, a patient of B type can receive a kidney from a donor of B or O type, while a patient of AB type can receive a kidney from a donor of any blood type. Blood-compatibility is shown in Figure 1. Another medical test concerns tissue. Tissue-compatibility is determined by six HLA (human leukocyte antigen) proteins (three from the father and another three from the mother). If the potential recipient shows antibodies against HLA in the donor kidney called a positive crossmatch, then the donor kidney cannot be transplanted to the patient.

Formally our kidney exchange model consists of a set D^S of single donors, a set P^W of patients on the waiting list (on TWL in short) and a set PD of patient-donor pairs. Single donors are all altruistic and can be cadavers or living people. Patients on TWL are also called single patients. A patient-donor pair describes a designated patient and a living donor who is willing to give a kidney to the patient or to exchange a kidney with another kidney for the designated patient. A patient (donor) in a patient-donor pair will be called a paired patient (donor). Patients are indifferent between compatible kidneys, indifferent between incompatible kidneys, and prefer compatible kidneys to incompatible ones. In reality there is always a large pool of patients on the waiting list so that such

patients can be found to match compatibly with any given kidney.

Our primary objective is to enable as many patients as possible to receive compatible kidneys, i.e., to achieve a maximal number of feasible kidney transplants between patients and donors. More precisely, we follow and extend the approach of Roth et al. (2007a) by deriving and calculating the exact number of maximum possible feasible kidney transplants and proposing a procedure for realizing this efficiency.

In our paper, the symbol (X, Y) indicates a pair of a patient with blood type X and a donor with blood type Y, and $(X, Y)^i$ ($(X, Y)^c$) means a pair of patient and donor who are tissue-incompatible (tissue-compatible). Furthermore, we use $\#X^d$ to denote the number of single donors with blood-type X, $\#Y^p$ the number of patients on the waiting list with blood-type Y, and $\#(X, Y)$ the number of patient-donor pairs with blood-type X for patients and blood-type Y for donors. For any real number k, $\lfloor k \rfloor$ stands for the largest integer no bigger than k.

An outcome of the kidney exchange problem is a *matching* of kidneys (i.e., donors)/the waiting list option to patients such that each paired patient is either assigned a compatible kidney (i.e., donor) or stays with his paired donor, each patient on the waiting list is either assigned a compatible kidney (i.e., donor) or stays put, and no kidney (i.e., donor) is assigned to more than one patient. A matching μ is *efficient or maximal* if there exists no other matching ν such that $|\nu| > |\mu|$ where $|\mu|$ is the number of possible kidney transplants for the matching μ.

A matching can be made through several ways of exchange between patients and donors. A *one-way cycle of exchange* comprises only one patient-donor pair in which the patient is compatible with the donor and is directly transplanted a kidney from the donor. A *two-way cycle of exchange* involves two patient-donor pairs in which each patient is compatible with the other patient's donor. For instance, we have two patient-donor pairs (A, B) and (B, A) and use $(A, B) - (B, A)$ to indicate a two-way cycle exchange in which blood-type A patient in first pair receives the kidney from blood-type A paired donor in second pair and blood-type B patient in second pair can receive the kidney from blood-type B paired donor in first pair. A *three-way cycle of exchange* involves three patient-donor pairs in which the patient in the first pair is compatible with the donor in the second pair, the patient in the second pair is compatible with the donor in the third pair, and the patient in the third pair is compatible with the donor in the first pair. An example consists of three pairs (X, Z), (Z, Y), and (Y, X), and the three-way cycle of exchange is given by $(X, Z) - (Z, Y) - (Y, X)$ in which each patient receives a compatible kidney. Similarly we can define a four-way cycle of exchange.

We also need to use chain exchanges. A *one-way chain of exchange* involves a single donor, denoted by X^d, and a compatible patient, denoted by Y^p, on the waiting list. We

write this exchange as $X^d - Y^p$. A *two-way chain of exchange* is a chain $X^d - (X, Y) - Y^p$ in which the patient of blood-type X in the pair receives the kidney from the single donor X^d, and the patient Y^p on the waiting list receives the kidney from the donor in the pair. A *three-way chain of exchange* is a chain $X^d - (X, Y) - (Y, Z) - Z^p$ in which the single donor X^d gives her compatible kidney to the patient X in the first pair, the donor Y in the first pair gives hers to the patient Y in the second pair, and the donor Z in the second pair gives hers to the patient Z^p in waiting. Four-way chains of exchange can be defined analogously. It should be noted that chains discussed here are markedly different from those in Roth et al. (2004). They defined a chain to be a sequence of exchanges in which a patient-donor pair (p, d) donates a kidney to either another paired patient or a person on the waiting list, in return for the paired patient p getting a high priority on the waiting list. So in their definition chains involve only patient-donor pairs and the waiting option.

For a given positive integer k, we say that a matching μ is *k-efficient* if there exists no other matching ν such that $|\nu| > |\mu|$ when the maximum size of kidney exchanges is no more than k-way cycles or chains of exchange. In the following *when we say a k-way exchange, it can be an l-way cycle or chain of exchange for any* $1 \leq l \leq k$.

It is natural to bring compatible patient-donor pairs and single donors into exchange with incompatible pairs as more patients can be benefited from their involvement. In practice single donors play a significant role. For instance, the number of total kidney donors in USA is 19,613 including 14,077 deceased donors and 5536 living donors in 2017 according to Hart et al. (2017).

Note that cycles involve only living people who are paired patients or paired donors, while every chain involves only one single donor and only one patient on the waiting list and possibly patient-donor pairs. The single donor is altruistic, decreased or living. In practice, kidneys from deceased donors cannot be preserved too long. Melcher et al. (2016), OPTN/UNOSKT (2017), Molmenti et al. (2018), and Cornelio et. al. (2019) reported and discussed practical implementation of deceased donor-initiated chains.

To derive an analytical expression for the maximum number of feasible transplants among the whole kidney exchange pool, we impose the following three basic assumptions.

Assumption 1 (Upper Bound Assumption): *Every patient on the waiting list is tissue-compatible with every blood-type compatible donor and every paired patient is tissue-compatible with a blood-type compatible single donor or paired donor of any other paired patient.*

This assumption is a slight generalization of Assumption 1 of Roth et al. (2007a, p. 831) and seems extremely difficult to be relaxed as like Roth et al. (2007a, p. 831) we want to obtain the exact number of the maximal feasible kidney transplants. Note that advances in medical technology and practice can mitigate the tissue-incompatibility issue

considerably. Unlike blood-compatibility, tissue-compatibility does not require exact HLA match between a patient and a donor. The percentage of tissue-incompatibility is also very low; see Zenios et al. (2001). With evolving clinical practice, the significance of HLA matching has diminished (Su et al. 2004). To decide whether a person can donate a kidney or not, the level of HLA level does not play a central role. This is consistent with the practical evidence from OPTN/SRTR Annual Data Report in Hart et al. (2017) that most of transplanted patients have HLA mistakes with donors (see page 45 of the report). We will have more discussion on the issue in Section 5.

Assumption 2 $\#(A,B) > \#(B,A)$.

Terasaki et al. (1998) and OPTN & SRTR annual data report in 2017 have provided statistical evidence for this assumption that the number of pairs (A,B) is greater than the number of pairs (B,A). This assumption is used as Assumption 3 in Roth et al. (2007a, p. 834).

Assumption 3 *Let* (X,Y) *denote a blood-compatible type from* (A,A), (B,B), (AB,AB), (O,O), (A,O), (B,O), (AB,O), (AB,A) *and* (AB,B). *There exists either no pair of type* (X,Y) *or at least one tissue-compatible pair of type* (X,Y).

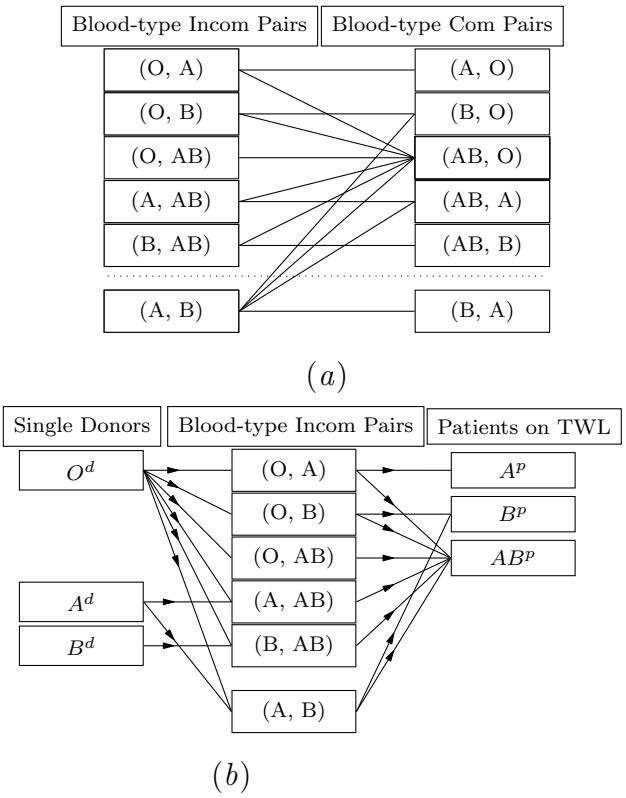
This assumption can be easily satisfied for a relatively large population and is similar to Assumption 4 of Roth et al. (2007a, p. 834).

For a relatively large population, due to blood-compatibility constraints, there will be likely higher demand for kidneys of type O than type A or B, and higher demand for kidneys of type A or B than type AB. As a result, pairs of type (O,A), (O,B), (O,AB), (A,AB), or (B,AB) are on the long side of the exchange and will have to wait longer for a feasible exchange than pairs of other types. Their opposite blood-type compatible but tissue-type incompatible pairs are on the short side. This is used as their Assumption 2 of Roth et al. (2007a, p. 832). Our model will dispense with this assumption and can handle cases that violate or satisfy this assumption.

3 Efficient Kidney Exchange

In this section we derive a maximum number of feasible kidney transplants, when one-way, two-way, three-way, or four-way cycles or chains of exchange are used. In Section 4 we will show that at most four-way cycles and chains suffice to extract all the potential gains of kidney exchange. This result follows from our basic Theorem 1 for a general and abstract matching model of any finite ways of exchanges with two-category-type compatibility constraints like blood types and tissue types. As the current model is arguably very general,

Figure 2: Two-way cycles (a) and chains (b) of exchange.



our analysis becomes inevitably much more involved due to a large number of combinatorial cases caused by the presence of compatible or incompatible patient-donor pairs, single donors and patients on the waiting list. For this reason we will focus on two-way exchanges and give an intuitive but less formal treatment of the essential cases of three-way and four-way exchanges. For the formal and rigorous analysis of three-way and four-way exchanges, see Cheng and Yang (2017a, b) of more than 240-page discussion paper.

3.1 Two-Way Exchange

Recall that to distinguish blood-type compatible but tissue-type incompatible pairs and compatible pairs, we use $(X,Y)^i$ to denote the first group and $(X,Y)^c$ to denote the second group. Obviously $\#(X,Y) = \#(X,Y)^i + \#(X,Y)^c$. In the following, the notation $(A,B) - (C,D)/(X,Y)$ means that $(A,B) - (C,D)$ and/or $(A,B) - (X,Y)$, and $(A,B)/(C,D) - (X,Y)$ means that $(A,B) - (X,Y)$ and/or $(C,D) - (X,Y)$.

Figure 2 shows several basic two-way cycles and chains of exchange but do not include pairs (X,X). In Figure 2(a) the right column above the dot line represents blood-type compatible pairs while the left column above the dot line stands for the blood-type incom-

patible pairs. By Assumption 3 all tissue incompatible pairs of type $(X,Y)^i$ on the right side can be matched by two-way cycle $(X,Y)^i-(X,Y)^i$ or two-way cycle $(X,Y)^i-(X,Y)^c$. The problem becomes how to take full advantage of blood-type compatible pairs and single donors to match a maximum number of blood-type incompatible pairs because blood-type incompatible pairs cannot match with each other in two-way cycles.

A cell in the left column linking a cell in the right column means a two-way cycle, for instance, $(O,A)-(A,O)$ and $(O,A)-(AB,O)$. In Figure 2(b) a cell in the left column linking a cell in the middle column linking a cell in the right column implies a two-way chain, for instance, $O^d-(O,A)-A^p$, $O^d-(O,B)-B^p$ and $O^d-(O,B)-AB^p$. Using this idea we propose a sequential matching procedure to find a maximal number of (feasible) transplants when at most two-way cycles or chains of exchange will be used. We call it a *sequential 2-way matching procedure*, which is delegated to the appendix.

We use the following example to show how each matching procedure assigns compatible kidneys to patients and how efficiency will be improved as more ways of exchange are permitted.

Example 1 *There are 32 incompatible patient-donor pairs consisting of three incompatible pairs of type* $(AB,AB)^i$*, five pairs of type* (O,A)*, one pairs of type* (O,B)*, one pair of type* (O,AB)*, two pairs of type* (A,AB)*, seven pairs of type* (B,AB)*, seven pairs of type* (A,B)*, one incompatible pair of each type of* $(A,O)^i$*,* $(B,O)^i$*,* $(AB,O)^i$*,* $(AB,A)^i$*,* $(AB,B)^i$ *and* (B,A)*; three compatible patient-donor pairs consisting of one compatible pair of each type* $(AB,AB)^c$*,* $(AB,O)^c$ *and* $(A,O)^c$*; and five single donors consisting of three single donors of type* A^d*, one single donor of type* B^d *and one single donor of type* AB^d*, and a large number of single patients.*

Observe that in the example there are in total 35 patient-donor pairs including 32 incompatible pairs and three compatible ones and many single patients. Table 1 shows that if two-way exchange is implemented, 24 paired patients and 5 single patients can receive kidney transplants and all three compatible pairs are involved in kidney exchange with incompatible pairs. Four pairs of type (B,AB), three pairs of type (O,A) and four pairs of type $(A,B)/(A,AB)$ stay put. In Table 1, Step 1 has two cycles, i.e., $(AB,AB)^i-(AB,AB)^i$ and $(AB,AB)^i-(AB,AB)^c$. Note that we can randomly pick kidney exchanges from cycles $(AB,O)^c-(O,A)/(O,B)/(B,AB)$ and chains $O^d-(O,A)/(O,B)/(B,AB)-Y^p$ in Step 4.

We have the following observation immediately from Figure 2 or by definition. What is interesting or nontrivial is that the same conclusion holds true also for three or four or higher-way exchanges.

Table 1: The illustration of the two-way matching procedure.

Steps	Number of Cycles or Chains	Cycles or Chains	Number of Remaining Pairs and Donors
Step 1	2	$(AB,AB)^i-(AB,AB)^i$ $(AB,AB)^i-(AB,AB)^c$	
Step 2	1	$(O,A)-(A,O)^i$	4 (O,A)
	1	$(O,B)-(B,O)^i$	
	1	$(O,AB)-(AB,O)^i$	
	1	$(A,AB)-(AB,A)^i$	(A,AB)
	1	$(B,AB)-(AB,B)^i$	6 (B,AB)
	1	$(A,B)-(B,A)$	6 (A,B)
Step 3	1	$(O,A)-(A,O)^c$	3 (O,A)
	1	$A^d-(A,AB)-AB^p$	2 A^d
	1	$B^d-(B,AB)-AB^p$	5 (B,AB)
	2	$A^d-(A,B)-B^p/AB^p$	4 (A,B)
Step 4	1	$(AB,O)^c-(B,AB)$	4 (B,AB)
Step 5 (End)	1	AB^d-AB^p	

Lemma 1 *Assume that the kidney exchange model satisfies the Assumptions 1 and 3. Let μ be a 2-efficient matching. Then in μ every cycle contains at most two blood-type compatible pairs and every chain contains at most one blood-type compatible pair.*

Our next result shows that we can actually obtain an explicit formula of the maximal number of feasible kidney transplants under two-way exchanges and the realization of this efficient exchange can be achieved by the sequential 2-way matching procedure. As the formula is too long, it will be given in the appendix together with the proof of the proposition.

Proposition 1 *Assume that the kidney exchange model obeys the Assumptions 1, 2, and 3. Then the matching μ obtained from the above mechanism is 2-efficient and the maximum number of transplants through two-way exchanges can be given explicitly as in the appendix.*

Now we compare the lower bound of the number in Proposition 1 with the case in which incompatible patient-donor pairs, compatible patient-donor pairs, and patients in waiting and single donors are treated separately under two-way exchange. We consider the most common situation that the number of blood-type incompatible pairs of each type: $\#(O,A)$, $\#(O,B)$, $\#(O,AB)$, $\#(A,AB)$, and $\#(B,AB)$, is at least as large as the number of its opposite blood-type compatible but tissue-type incompatible pairs: $\#(A,O)^i$, $\#(B,O)^i$, $\#(AB,O)^i$, $\#(AB,A)^i$, and $\#(AB,B)^i$, respectively. We can do similar comparison for other situations. Hence, the maximum number of feasible transplants for the group of

incompatible patient-donor pairs under two-way cycles is

$$\begin{aligned}&2(\#(A,O)^i+\#(B,O)^i+\#(AB,O)^i+\#(AB,A)^i+\#(AB,B)^i)\\&+2\#(B,A)+2(\lfloor\tfrac{\#(A,A)^i}{2}\rfloor+\lfloor\tfrac{\#(B,B)^i}{2}\rfloor+\lfloor\tfrac{\#(AB,AB)^i}{2}\rfloor+\lfloor\tfrac{\#(O,O)^i}{2}\rfloor)\end{aligned}$$

The maximum number of transplants for patients on the waiting list under one/two-way chains equals $(\#A^d+\#B^d+\#AB^d+\#O^d)$ because the number of patients on the waiting list exceeds the number of single donors so that a single donor can always find a compatible patient on the waiting list to donate. The maximum number of transplants for the group of compatible patient-donor pairs equals $\#(A,O)^c+\#(B,O)^c+\#(AB,O)^c+\#(AB,A)^c+\#(AB,B)^c+\#(A,A)^c+\#(B,B)^c+\#(O,O)^c+\#(AB,AB)^c$ because every patient in a compatible pair can receive the kidney from its own paired donor.

Since for any blood-type compatible pair of type (X,Y), we have $\#(X,Y)=\#(X,Y)^i+\#(X,Y)^c$, the maximum number of transplants in the whole pool becomes

$$\begin{aligned}&\#(A,O)+\#(B,O)+\#(AB,O)+\#(AB,A)+\#(AB,B)\\&+\#(A,O)^i+\#(B,O)^i+\#(AB,O)^i+\#(AB,A)^i+\#(AB,B)^i\\&+2\#(B,A)+2(\lfloor\tfrac{\#(A,A)^i}{2}\rfloor+\lfloor\tfrac{\#(B,B)^i}{2}\rfloor+\lfloor\tfrac{\#(AB,AB)^i}{2}\rfloor+\lfloor\tfrac{\#(O,O)^i}{2}\rfloor)\\&+\#(A,A)^c+\#(B,B)^c+\#(AB,AB)^c+\#(O,O)^c\\&+\#A^d+\#B^d+\#AB^d+\#O^d\end{aligned}$$

We compare the above number with the lower bound of the number in Proposition 1 and obtain

$$\begin{aligned}&\min\{N_1,N_2,N_3,N_4,N_5,N_6,N_7,N_8,N_9,N_{10},N_{11},N_{12},N_{13},N_{14},N_{15},N_{16},N_{17}\}\\&-(\#(A,O)^i+\#(B,O)^i+\#(AB,O)^i+\#(AB,A)^i+\#(AB,B)^i+\#(B,A))\\&+\#(A,A)+\#(B,B)+\#(AB,AB)+\#(O,O)\\&-(\#(A,A)^c+\#(B,B)^c+\#(AB,AB)^c+\#(O,O)^c)\\&-2(\lfloor\tfrac{\#(A,A)^i}{2}\rfloor+\lfloor\tfrac{\#(B,B)^i}{2}\rfloor+\lfloor\tfrac{\#(AB,AB)^i}{2}\rfloor+\lfloor\tfrac{\#(O,O)^i}{2}\rfloor)\\&\geq 0\end{aligned}$$

where the numbers N_1, N_2, $\cdots$, N_{17} are parts of the formula in Proposition 1 and are given in the appendix. This shows the benefits of allowing compatible patient-donor pairs to join incompatible pairs for exchange and adding two-way chain exchange.

3.2 Three-Way Exchange

To improve the potential gains of exchange, three-way cycles and three-way chains of exchange can be used.

Figures 3 and 4 show all possible three-way cycles and chains under Assumptions 1, 2 and 3. Note that these figures do not include two-way exchanges which are discussed

in the previous subsection. Recall that blood-compatible pairs can always be matched by Assumption 3. To have more transplants we can make the best use of every blood-compatible pair to match with a blood-incompatible pair. As a result, three-way cycles can be formed.

Here we will discuss several essential cases of three-way exchanges. Firstly, consider some beneficial three-way cycles or chains with two blood-incompatible pairs. Under three-way exchanges, blood-compatible pair (AB, O) (the right column) can involve not one but two blood-type incompatible pairs through 4 three-way cycles $(AB, O) - (O, A) - (A, AB)$, $(AB, O) - (O, A) - (A, B)$, $(AB, O) - (A, B) - (B, AB)$ and $(AB, O) - (O, B) - (B, AB)$. For blood-compatible pair (B, O), we have just one three-way cycle $(B, O) - (O, A) - (A, B)$. For blood-compatible pair (AB, A), we have also just one three-way cycle $(AB, A) - (A, B) - (B, AB)$. Similarly, we can use single donors to match with two blood-incompatible pairs and patients on the waiting list. Consequently, three-way chains can be generated. With one-way and two-way chains, each single donor can trade with at most one blood-incompatible pair. If three-way chains are allowed, single donor O^d can trade with two blood-incompatible pairs through three-way chains $O^d - (O, A) - (A, AB) - AB^p$, $O^d - (O, A) - (A, B) - B^p$, $O^d - (A, B) - (B, AB) - AB^p$ and $O^d - (O, B) - (B, AB) - AB^p$. Moreover, if there is any (A, B) left, type (A, B) can bring an extra blood-incompatible pair into chains through three-way chains $A^d - (A, B) - (B, AB) - AB^p$.

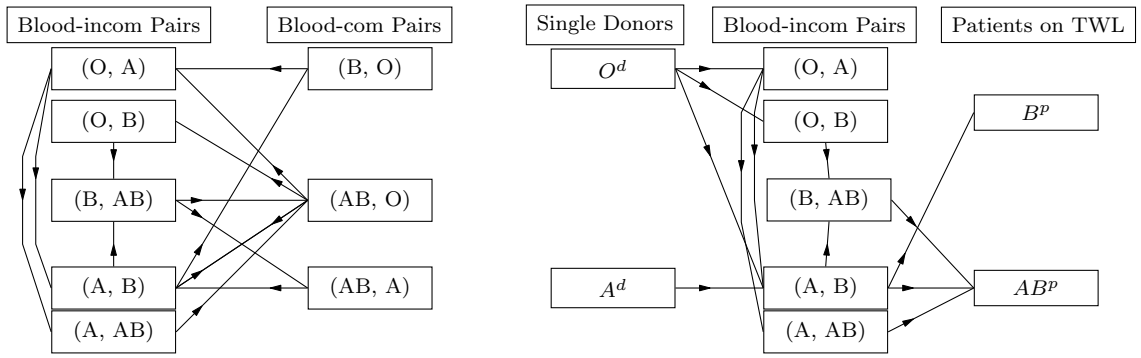
Secondly, consider some beneficial three-way cycles or chains with one pair (B, A) or with one blood-incompatible pair. Observe that (B, A) pairs are on the short side by Assumption 2. These pairs can be very beneficial in the following situations: Firstly, there are pairs or singles, (A, O), (O, B), $A^d/(AB, A)$, and (B, AB). In this case, we cannot match blood-incompatible pairs (O, B) and (B, AB) in a two-way cycle. But if we break two-way cycle $(A, B) - (B, A)$, we can make three-way cycles $(A, O) - (O, B) - (B, A)$ and $(AB, A) - (A, B) - (B, AB)$ and thus increase the number of transplants. Also three-way cycles $(A, O) - (O, B) - (B, A)$ and chains $A^d - (A, B) - (B, AB) - AB^p$ can yield more transplants. Secondly, there are pairs or singles, (A, AB), (B, O), $B^d/(AB, B)$, and (O, A). In this case, we cannot match blood-incompatible pairs (O, A) and (A, AB) in a two-way cycle, but we can make three-way cycles $(B, O) - (O, A) - (A, B)$ and $(AB, B) - (B, A) - (A, AB)$ and increase the number of transplants. Also three-way cycles $(B, O) - (O, A) - (A, B)$ and three-way chains $B^d - (B, A) - (A, AB) - AB^p$ can bring more transplants.

Furthermore, it is easy to see that $(AB, A) - (A, O)$, or $(AB, B) - (B, O)$ can make a three-way cycle of exchange with any pair (X, Y), and that $A^d - (A, O)$ or $B^d - (B, O)$ can yield a three-way chain with any pair (X, Y). In particular, when there are pairs (B, AB), (O, B) and (O, AB), it is impossible to use them in two-way exchange but it is easy to combine them with $(AB, A) - (A, O)$ to yield three-way exchange $(AB, A) -$

$(A, O) - (B, AB)/(O, B)/(O, AB)$. Similarly, we can make three-way exchanges $A^d - (A, O) - (B, AB)/(O, B)/(O, AB) - Y^p$, $(AB, B) - (B, O) - (A, AB)/(O, A)/(O, AB)$ and $B^d - (B, O) - (A, AB)/(O, A)/(O, AB) - Y^p$.

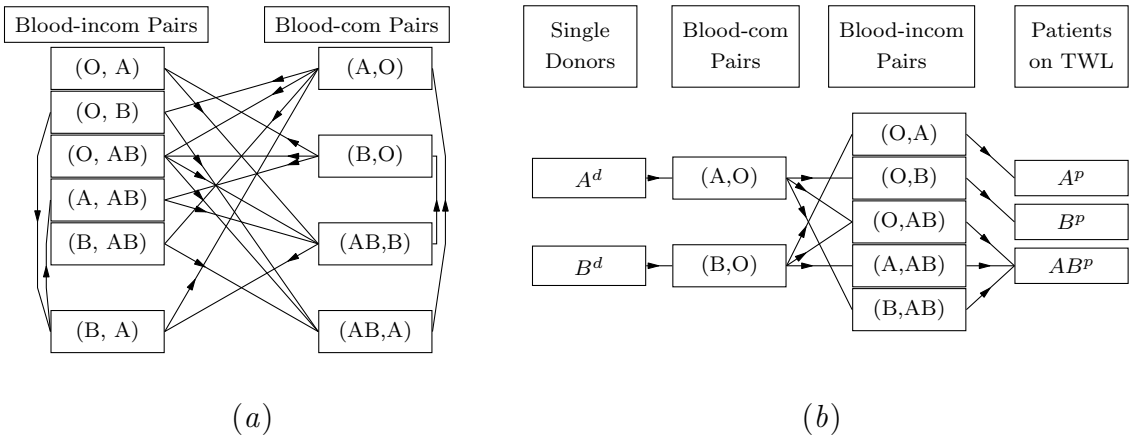
We have developed an efficient three-way exchange procedure and obtained results similar to Proposition 1 which are omitted here but given in Cheng and Yang (2017a, b) in detail. We use Example 1 to demonstrate how much efficiency can be gained by three-way exchange compared with two-way exchange. In the case of three-way exchange, 31 paired patients and five single patients will receive kidney transplants and four pairs of type (B, AB), (A, B), (O, B) and (O, AB) stay put. Compared with the previous two-way exchange, three-way exchange increases the maximum number of kidney transplants by seven.

Figure 3: Three-way cycles (a) and chains (b) of exchange with two blood-incompatible pairs.



(a) (b)

Figure 4: Three-way cycles (a) and chains (b) of exchange with (B, A) and three-way cycles of exchange with one blood-incompatible pair.



(a) (b)

The following proposition shows that in three-way exchanges every cycle contains at most two blood-type compatible pairs and every chain comprises at most one blood-type compatible pair, extending the insight of Lemma 1.

Proposition 2 *Assume that the kidney exchange model satisfies the Assumptions 1 and 3. Then every* 3*-efficient matching* μ *can be transformed to another* 3*-efficient matching in which every cycle contains at most two blood-type compatible pairs and every chain contains at most one blood-type compatible pair.*

Proof. Consider any given 3-efficient matching μ as stated in the lemma. If μ consists only of cycles with no more than two blood-type compatible pairs and chains with no more than one blood-type compatible pair, we are done. Suppose to the contrary that μ contains a cycle with more than two blood-type compatible pairs or a chain with more than one blood-type compatible pair. We only need to consider the case of three-way cycles or chains. We will show that a three-way cycle with three blood-type compatible pairs can be decomposed into three single blood-compatible pairs and a three-way chain with two blood-compatible pairs can be decomposed into two single blood-compatible pairs and a one-way chain in which the single donor donates its kidney to a patient on the waiting list.

Because a blood-type compatible and tissue-type compatible pair can directly do transplant, all blood-type compatible and tissue-type compatible pairs can do transplants separately. Let $\mathcal{D}$ be the set of all blood-type compatible but tissue-type incompatible pairs in a three-way cycle or chain under consideration. Let $(X, Y)^i$ present the type of a blood-type compatible but tissue-type incompatible pair. If there exists two or more pairs of type $(X, Y)^i$, we can have a two-way cycle among them $(X, Y)^i - (X, Y)^i$. Therefore, at most

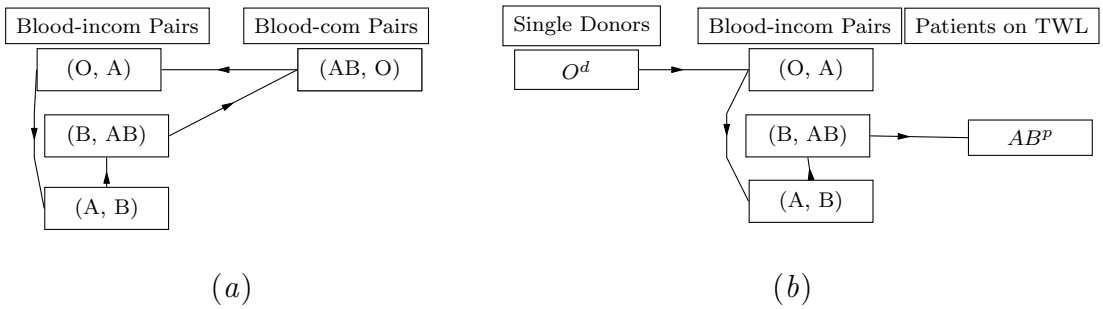
one pair of type $(X,Y)^i$ left after the process. By Assumption 3, there exists at least one blood-type and tissue-type compatible pair of type $(X,Y)^c$. If the compatible pair $(X,Y)^c$ does not involve in any cycle or chain, then we can match the remaining pair $(X,Y)^i$ with pair $(X,Y)^c$. Otherwise, the compatible pair $(X,Y)^c$ involves in a cycle consisting of no more than two blood-type compatible pairs or a chain consisting of no more than one blood-type compatible pair. Then we can use pair $(X,Y)^i$ instead of $(X,Y)^c$ based on Assumption 1 and pair $(X,Y)^c$ do transplant directly. Therefore, all remaining pairs of type $(X,Y)^i$ can be matched. □

3.3 Four-Way Exchange

If four-way cycles and chains of exchange are used, more kidney transplants will be made possible. Figures 5 and 6 show all four-way cycles and chains of exchange but do not include two- or three-way exchange, which are already discussed in the previous two subsections.

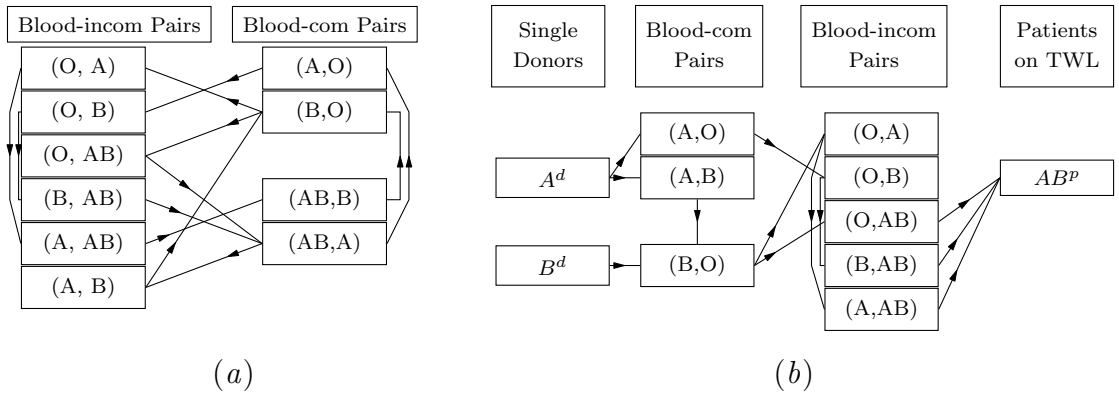
In this case we have a four-way cycle with three blood-incompatible pairs $(AB,O) - (O,A) - (A,B) - (B,AB)$, a four-way chain with three blood-incompatible pairs $O^d - (O,A) - (A,B) - (B,AB) - AB^p$, two four-way cycles with two blood-compatible pairs $(AB,A)-(A,O)-(O,B)-(B,AB)$ and $(AB,B)-(B,O)-(O,A)-(A,AB)$, two four-way chains with one blood-compatible pair $A^d-(A,O)-(O,B)-(B,AB)$ and $B^d-(B,O)-(O,A)-(A,AB)$, one four-way cycle $(AB,A)-(A,B)-(B,O)-(X,Y)$ and one four-way chain $A^d-(A,B)-(B,O)-(X,Y)-Z^p$, where (X,Y) is any pair and Z^p is any single patient.

Figure 5: Four-way cycles (a) and chains (b) of exchange with three blood-incompatible pairs.



(a) (b)

Figure 6: Four-way cycles (a) and chains (b) of exchange with two blood-incompatible pairs.



(a) (b)

An efficient four-way exchange procedure has been proposed and also results similar to Proposition 1 are obtained in Cheng and Yang (2017a, b). If we apply four-way exchange to Example 1, then 32 paired patients and five single patients will receive kidney transplants and three pairs of type (A, AB), (O, B) and (O, AB) will be left. In comparison with the previous three-way exchange, the four-way exchange increases the maximum number of kidney transplants by only one. This indicates that four-way exchanges can make only minor improvement of efficiency.

The next result demonstrates that the same insight of Proposition 2 holds also for four-way exchanges. Namely, as in three-way exchanges, in four-way exchanges, for every four-way cycle we need at most two compatible pairs and for every four-way chain we need at most one compatible pair. It proof can be found in Cheng and Yang (2017a, b).

Proposition 3 *Assume that the kidney exchange model satisfies the Assumptions 1 and 3. Then every* 4*-efficient matching* μ *can be transformed to another* 4*-efficient matching in which every cycle contains at most two blood-type compatible pairs and every chain contains at most one blood-type compatible pair.*

In the next section we will show in Corollary 1 that at most four-way exchanges are sufficient to achieve all the potential gains of possible and feasible kidney transplants under Assumptions 1 and 3 and this number of four is nothing but the number of four blood types. This will easily follow from a basic theorem (i.e., Theorem 1) for a general and abstract matching model of multi-way exchanges.

4 Multi-Way Cycles and Chains of Exchange

In the previous sections we have focused on two-way, three-way, and four-way cycles and chains of exchange and derived the upper bounds of the possible number of kidney trans-

plants under those given assumptions. In the current section, we consider a more general model of kidney exchange and show that under similar conditions, five or higher-way cycles and chains of exchange even if available will not further increase the number of feasible kidney transplants. In other words, four or less-way exchanges are sufficient to capture all the potential gains of kidney exchange.

Our general model consists of pairs, single donors and patients on the waiting list. We also call a patient on the waiting list *a single patient.* Each pair i has a patient P_i^p and a donor D_i^p. Each single patient is denoted by P_i^s and each single (decreased or living) donor is denoted as D_i^s.

Let $\mathcal{B}$ be the family of primary types such as blood shared by patients and donors with $|\mathcal{B}| = n > 2$. In other words, all patients and donors have their types X in $\in \mathcal{B}$. For any given two primary types $X, Y \in \mathcal{B}$, $X \succeq Y$ means that agent of type X is primary type compatible with agent of type Y. In the context of kidney exchange, a patient of type Y is blood-type compatible with a donor of type X. Following Roth et al. (2007a), we assume that the compatibility relation $\succeq$ for primary types satisfies reflexivity, asymmetry and transitivity properties:

1. (Reflexivity) $X \succeq X$ for any $X \in \mathcal{B}$,
2. (Asymmetry) $X \succeq Y$ and $X \neq Y \Rightarrow Y \not\succeq X$ for any $X, Y \in \mathcal{B}$, and
3. (Transitivity) $X \succeq Y$ and $Y \succeq Z \Rightarrow X \succeq Z$ for any $X, Y \in \mathcal{B}$.

Blood-type compatibility possesses the properties of reflexivity, asymmetry and transitivity.

Different from and improving Roth et al. (2007a), we also introduce secondary types to reflect and grasp important properties like tissue shared by patients and donors and also to accommodate other economic models going beyond blood-type and tissue-type compatibility. Although we know from the medical practice that the requirement on tissue-type compatibility is not as stringent as on blood-type compatibility, some degree of tissue-type compatibility is still required. As will be discussed in the following section, tissue types can be roughly divided into several groups according to their percentage of incompatibility. In this way, it is desirable and reasonable to imbed secondary types into the model rather than to simply assume away or ignore them.

Let $\mathcal{C}$ be the family of secondary types with $|\mathcal{C}| = m \geq 2$. For any given two secondary types $Z, W \in \mathcal{C}$, $Z \sim W$ means that agent of type Z is secondary type compatible with agent of type W. In the context of kidney exchange, a patient of type Z is tissue-type compatible with a donor of type W. We assume that the compatibility relation $\sim$ for secondary types satisfies symmetry and intransitivity properties:

I. (Symmetry) $Z \sim W \Rightarrow W \sim Z$ for any $Z, W \in \mathcal{C}$, and

II. (Intransitivity) $Z \sim W$ and $W \sim L \nRightarrow Z \sim L$ for any $Z, W, L \in \mathcal{C}$.

Tissue-type compatibility possesses the properties of symmetry and intransitivity.

An agent of primary type $X \in \mathcal{B}$ and secondary type $Z \in \mathcal{C}$ is compatible with an agent of primary type $Y \in \mathcal{B}$ and secondary type $W \in \mathcal{C}$ if and only if $X \succeq Y$ and $Z \sim W$. In the context of kidney exchange, a patient of type $Y \in \mathcal{B}$ and $W \in \mathcal{C}$ can accept a kidney from a donor of type $X \in \mathcal{B}$ and $Z \in \mathcal{C}$.

Because the compatibility of secondary types is symmetric and intransitive, we use symbol i to stand for $\nsim$ and symbol c to stand for $\sim$. Let $(X, Y)^t$ describe a pair which has a patient of primary type $X \in \mathcal{B}$ and a donor of primary type $Y \in \mathcal{B}$ and the compatibility relation of secondary types between the patient and the doctor is $t \in \{i, c\}$. Therefore, we can divide all pairs into four groups:

1. $(X, Y)^i$ for any $X, Y \in \mathcal{B}$, and $Y \not\succeq X$,
2. $(X, Y)^c$ for any $X, Y \in \mathcal{B}$, and $Y \not\succeq X$,
3. $(X, Y)^i$ for any $X, Y \in \mathcal{B}$, and $Y \succeq X$,
4. $(X, Y)^c$ for any $X, Y \in \mathcal{B}$, and $Y \succeq X$.

In this model, group 4 demonstrates compatible pairs and the other three groups cover incompatible pairs. To simplify the notation, we write incompatible pairs from groups 1 and 2 as (X, Y) for which donors are primary type incompatible with patients, i.e., $Y \not\succeq X$.

We can describe a three-way cycle as

$$E = ((P_1^p, D_1^p), (P_2^p, D_2^p), (P_3^p, D_3^p)),$$

which means that the paired donor D_1^p is matched with the paired patient P_2^p, the paired donor D_2^p is matched with the paired patient P_3^p, and the paired donor D_3^p is matched with the paired patient P_1^p. Any size cycle can be defined similarly. A cycle E is *feasible* if the type of each donor in E is compatible with the type of patient who is matched with the donor. Also, we can describe a three-way chain as

$$C = (D_1^s, (P_1^p, D_1^p), (P_2^p, D_2^p), P_1^s),$$

in which the single donor D_1^s is matched with the paired patient P_1^p, the paired donor D_1^p is matched with the paired patient P_2^p, and the paired donor D_2^p is matched with the single patient P_1^s. Any size chain can be defined in a similar way. A chain C is *feasible* if the type of every donor in C is compatible with the type of patient who is matched with the donor.

We can recast and generalize the Assumptions 1 and 3 into the present more general model, respectively.

Assumption 4 *Every single agent of primary type $X \in \mathcal{B}$ and secondary type $Z \in \mathcal{C}$ is $Z \sim W$ with every agent of type $Y \in \mathcal{B}$ and $W \in \mathcal{C}$ who is $Y \succeq X$. Every agent in a pair of type $X \in \mathcal{B}$ and $Z \in \mathcal{C}$ is $Z \sim W$ with every agent other than agents in the pair of type $Y \in \mathcal{B}$ and $W \in \mathcal{C}$ who is $Y \succeq X$.*

Assumption 5 *Let $X, Y \in \mathcal{B}$ be such that $Y \succeq X$. There exists either no pair of type (X, Y) or at least one pair of type $(X, Y)^c$.*

We will show that when the compatibility relation of primary type satisfies reflexivity, asymmetry and transitivity, and the compatibility relation of secondary type satisfies symmetry and intransitivity, a maximal size of exchange in the model can be achieved through no more than n-way cycles and n-way chains. The next two results generalize those of Roth et al. (2007a) to the setting which comprises all kinds of pairs, single donors and patients on the waiting list and uses both cycles and chains of exchange. It is worth pointing out that our model studies two-category type-compatible exchanges with the first category $\mathcal{B}$ of primary types and the second category $\mathcal{C}$ of secondary types and the number of types in each category can be different, while the model of Roth et al. (2007a) examines one-category type-compatible exchanges. Observe that in the following theorem the maximal n-way exchange depends only on the number n of types in the first category $\mathcal{B}$ and is independent of the number m of types in the second category $\mathcal{C}$, although the matching requires two-category type-compatibility. This result somewhat bears a resemblance to the famous Modigliani-Miller Theorem (1958) in finance that a firm's capital structure is irrelevant to the firm's value; see also Stiglitz (1969).

Theorem 1 *Assume that there are $n \geq 2$ primary types in $\mathcal{B}$ and $m \geq 2$ secondary types in $\mathcal{C}$ and the Assumptions 4 and 5 hold. Let μ be any maximal matching in the sense that any size of kidney exchanges is permitted in the matching. Then there exists a maximal matching ν which contains at most n-way cycles and chains of exchange but has the same set of patients matched with compatible donors as in the matching μ.*

A basic intuition behind this general theorem comes from a key observation that we can always reduce any large $k(> n)$-efficient matching ν into a lower $(k-1)$-efficient matching ν', which contains the same set of compatible matched pairs as ν does. We can do this exercise recursively until we reach the number of n-the total number of primary types, which is the number of building blocks of the general matching model. The proof of this theorem is given in the appendix.

The following is an immediate consequence of the theorem, because in the context of kidney exchange we have only four blood types, corresponding to $n = 4$, the total number of primary types in the theorem. As a result, at most four-way exchanges will be needed to capture all the potential gain of feasible kidney transplants.

Corollary 1 *Consider a kidney exchange model under the Assumptions 1 and 3. Let μ be any maximal matching without any restriction on the size of exchange. Then there exists a maximal matching ν which contains at most four-way exchanges but has the same set of patients who can benefit from exchanges as in the matching μ.*

5 Simulations Based on the USA Data

In this section, we use two data sets from the U.S. Organ Procurement and Transplantation Network (OPTN) and the Scientific Registry of Transplant Recipients (SRTR) from 1993 to 2002 and from 1995 to 2016, respectively,[4] to generate simulated data reflecting the characteristics of the population involved and to test how well our theoretical results can predict. Although the simulated population which is almost identical or very close to the real life situation may not fully meet the simplifying assumptions (in particular Assumption 1) made for the model, we find that the predicted maximum number of transplants given by our derived formulas is surprisingly close to the number of transplants that can be actually realized.

Dickerson et al. (2019) have reported a fairly high percentage of last minute failures in planned matches recommended by kidney exchange algorithms/procedures. This is a type of failure before the transplant surgery takes place. One of the main reasons concerns the tissue-type incompatibility because it is not possible to obtain the full tissue-type compatibility data prior to any actual kidney transplants as this information is very specific to the donor and the intended recipient. Fortunately, there are now several ways of resolving the tissue-type incompatibility problem. Dickerson et al. (2019) suggested to take failure-awareness and some uncertainty into exchange design and showed a significant gain of successful transplants. Another effective and practical approach is to make use of the historic data of actual crossmatch tests between patients and donors; see Sönmez and Ünver (2013) for detailed and informed discussions. Also the desensitization in HLA or ABO incompatible living donors to their kidney transplantation to end-stage kidney disease patients has been successfully carried out in experienced hospitals; see Becker et al. (2013) and Thukral et al. (2019).

[4]They are retrieved from `http://optn.transplant.hrsa.gov/data/view-data-reports/national-data`.

5.1 Data Construction

Data is collected for two time slots. The first time slot data is from 1993 to 2002 and is shown in Table 3, and the second time slot data is from 1995 to 2016 and is shown in Table 4. These data sets illustrate the national characteristics of the USA population involved in kidney exchanges. The first period data from 1993 to 2002 is largely similar to those used by Roth et al. (2007a), and Saidman et al. (2006), except that in our new data set we include more relevant information like the distribution of compatible patient-donor pairs and single donors, which are not used in Roth et al. (2007a), and Saidman et al. (2006). The second data set from 1995 to 2016 stretches over a long period of time and reveals more and better information on the national characteristics of the USA population.

5.1.1 Patient-Donor Pairs and Single Donors Construction

Following Roth et al. (2007a), to avoid the complications of possible impact of genetics on immunological incompatibilities we exclude all blood-related incompatible patient-donor pairs in all our samples.

In the first time slot from 1993 to 2002, we use the same characteristics of incompatible pairs as that of Roth et al. (2007a) but add the blood-type characteristics for compatible patients, compatible donors and single donors; see Table 3. The second time slot data from 1995 to 2016 covers a long period of time and contains more detailed information about characteristics of the population. Compared to three levels of PRA (Percent Reactive Antibody) of patients from the data of the first time slot, five levels of PRA called CPRA (Calculated Percent Reactive Antibody) are provided in the data of the second time slot. The second time slot data contains also the information of compatible paired patient gender, compatible paired patient CPRA types and the blood-type information of incompatible paired donor; see Table 4. It should be noted that the percentages 39.83 and 36.02 of single donors on Tables 3 and 4, respectively, include only those (deceased or living) single donors whose kidneys have been transplanted.

It is important to point out that in the OPTN/SRTR annual report there is no clear information about the number of incompatible patient-donor pairs. Following Roth et al. (2007a) we use newly-added patients on the waiting list every year as approximately incompatible paired patients and the blood-type distribution of donors whose kidneys have been transplanted as the blood-type distribution of incompatible paired donors.

Because there exist a large number of patients on the waiting list, we can always find a patient who is compatible with any given kidney. Hence, we do not need to simulate any data for patients on the waiting list.

5.1.2 Tissue-type Incompatibility

Tissue-type compatibility is the second condition for kidney transplants. In our simulations of the first time slot from 1993 to 2002, we adopt the same method as used by Roth et al. (2007a) such that patients are divided into three groups based on the difficult level of tissue-type compatible with a random donor. In the first group called Low PRA group, patients are tissue-type incompatible with less than 10 percent of the population. The second group called Medium PRA contains patients who are tissue-type incompatible with 10-80 percent of the population. And, the third one called High PRA has patients who have a tissue-type incompatibility problem with more than 80 percent of the population. We use the following categories as used by Roth et al. (2007a):

1. In Low PRA group, each patient is tissue-type incompatible with 5 percent of the population,

2. In Medium PRA group, each patient is tissue-type incompatible with 45 percent of the population, and

3. In High PRA group, each patient is tissue-type incompatible with 90 percent of the population.

In our simulations for the second time slot from 1995 to 2016, CPRA index is used to check whether a patient is sensitive or not according to OPTN/SRTR database. Five levels are calculated in CPRA index, which are 0, 1-19, 20-79, 80-97, and 98-100. If a patient CPRA equals 0, it means the patient has no PRA problem with potential donors; 1-19 means the patient has 1 percent to 19 percent to have problem with potential donors and so on. In this simulation, we divide patients into five groups based on the difficult levels of tissue-type compatibility with a random donor. Based on the CPRA data, we use the following five groups:

1. In 0 CPRA group, each paired patient is tissue-type incompatible with 0 percent of the population;

2. In 1-19 CPRA group, each paired patient is tissue-type incompatible with 9.5 percent of the population;

3. In 20-79 CPRA group, each paired patient is tissue-type incompatible with 50 percent of the population;

4. In 80-97 CPRA group, each paired patient is tissue-type incompatible with 88 percent of the population;

5. In 98-100 CPRA group, each paired patient is tissue-type incompatible with 99 percent of the population;

Because the data from 1995 to 2016 contains more detailed information on the tissue-type compatibility of patients and donors, it provides more accurate information than the first time slot data does. This has important implications: it will yield better results as

shown in the subsequent section.

According to Zenios et al. (2001), a female patient is more likely to have a positive crossmatch with her husband. For instance, when positive crossmatch probability is 11.1 percent between random pairs, it becomes 33.3 percent between female patients and their donor husbands. Hence, when a patient is female and her potential donor is her husband, we adjust the probability of tissue-type incompatibility between them by using the formulas

$$PRA^* = 100 - 0.75(100 - PRA) \quad and \quad CPRA^* = 100 - 0.75(100 - CPRA).$$

5.2 Simulations

We generate a Monte-Carlo simulation size of 5,000 random population constructions for five population sizes n of 25, 50, 100, 150 and 200 incompatible patient-donor pairs together with the corresponding population sizes of compatible patient-donor pairs and single donors according to the population distributions given by Table 3 based on the 1993-2002 data set and by Table 4 based on the 1995-2016 data set, respectively. In addition we do a Monte-Carlo simulation size of 500 random population constructions for two big population sizes of 300 and 400 incompatible patient-donor pairs. Note that for these big population sizes we only generate 500 instead of 5,000 random population constructions in order to save time as it involves a relatively large and computationally difficult integer programming problem. By comparison, Roth et al. (2007a) have done Monte-Carlo simulations of 500 random population constructions for 25, 50, and 100 incompatible patient-donor pairs and Saidman et al. (2006) have tested the case of 25 and 100 incompatible patient-donor pairs both papers based on the 1993-2002 data set.

For each sample of the population, we try to compute the maximal number of incompatible paired patients who can receive a compatible kidney when

(1) exchanges are allowed only among incompatible patient-donor pairs. This is called *the exclusive exchange,* which is the case studied by Roth et al. (2007a);

(2) compatible patient-donor pairs are allowed to exchange with incompatible patient-donor pairs. This is called *the first degree inclusive exchange;*

(3) compatible patient-donor pairs, single donors, and patients on the waiting list are allowed to exchange with incompatible patient-donor pairs. It is called *the second degree inclusive exchange;*

(4) only two-way exchanges are allowed; or two-way and three-way exchanges are allowed; or two-way, three-way and four-way exchanges are allowed.

This maximal number will be simply called *simulation*. We compare these numbers with those predicted by the formula given by Proposition 1 in Section 3 and Propositions 3.5 and 3.7 in Cheng and Yang (2017a) to see how close or far the actual maximal number of kidney transplants can be from the predicted number based on the formula. For two-way exchanges we use Edmonds' algorithm-an improvement of the Hungarian method (Edmonds, 1965). For three-way and four-way exchanges we use the software package called CPLEX. It is well-known in mathematics that two-way exchanges can be solved in polynomial time, but three-way or higher-way exchanges are NP-complete in the sense that it can be extremely time-consuming. In our simulation we did not observe any visible net increase of efficiency through four-way exchanges for population sizes $n = 25$, $n = 50$ and $n = 100$. Because it was extremely time-consuming and also very costly, we did not pursue four-way exchanges for higher sizes.

Following Roth et al. (2007a), we make use of two types of upper bound:

Upper Bound (UB) 1. This is the number given by the formula in Proposition 1 in Section 3 and Propositions 3.5 and 3.7 in Cheng and Yang (2017a) for the simulated population sample of 25, 50, 100, 150, 200, 300, and 400 incompatible patient-donor pairs.

Upper Bound (UB) 2. For each simulated population sample, there may exist some patients who cannot find a compatible donor in the simulated population. We exclude those hopeless patients from the sample and compute the number given by the formula in Proposition 1 in Section 3 and Propositions 3.5 and 3.7 in Cheng and Yang (2017a) for the remaining population. This number is called the Upper Bound 2 and clearly gives a more accurate upper bound for the number of feasible transplants that can be realized.

For each population size of 25, 50, 100, 150, and 200 incompatible patient-donor pairs, we generate 5000 random samples and calculate the average of all 5000 simulations, upperbound 1's and upperbound 2's. For each population size of 300 and 400 incompatible patient-donor pairs, we generate 500 random samples and calculate the average of all 500 simulations, upperbound 1's and upperbound 2's. All results are collected in Tables 5 and 6 for the period of 1993-2002 and the period of 1995-2016, respectively.

5.3 Discussion of the Simulation Results

Before highlighting the major simulation results, we introduce two performance measures. The first is the deviation of each simulation from the upper bound 1 and upper bound 2 by

$$\frac{\text{upper bound i} - \text{simulation}}{\text{upper bound i}}, \quad \text{i=1, 2}$$

All deviations are given in Table 7. It is clear that as the size of the population increases, the deviation becomes smaller.

The second is the matching rate for each case of feasible transplants for incompatible paired patients over the number of incompatible patient-donor pairs under each exchange by

$$\frac{\text{the number of feasible transplants for incompatible paired patients}}{\text{the number of all incompatible paired patients}}$$

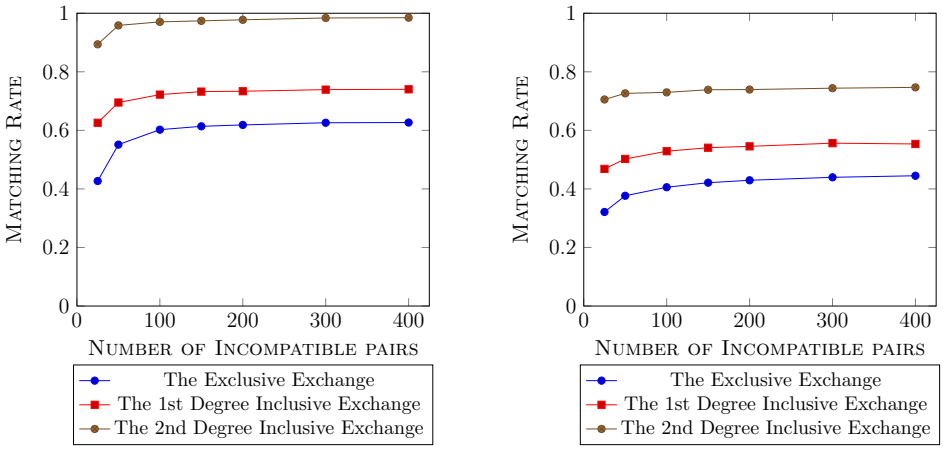
All matching rates are collected in Table 8 and shown in Figure 7. It is also obvious that as the size of the population increases, the matching rate increases.

The major findings are summarized as follows:

A. Figure 7 demonstrates that overall the slope of matching rate is upward and when the number of incompatible patient-donor pairs is below 100-a kind of threshold, the slope is relatively steep, but after 100, the slope becomes almost flat albeit upward, i.e., efficiency of exchange is nearly a constant or efficiency of exchange becomes asymptotically constant. This has important and novel policy implications: Kidney exchanges could be *decentralized* in the sense that any country with a large population like USA can have several separate kidney exchange programs/centers spread across the country where each program/center covers a sufficient number of patients and donors, say, no less than 100 of incompatible patient-donor pairs. Patients and donors can have kidney transplant operations in their own regional or nearby center. This can be very important and useful in practice because saving traveling cost can be extremely helpful for patients and the life of kidneys. This also means that medical resources can be evenly distributed across the country and need not be concentrated in one place. Recall that in order to avoid the moral hazard problem, kidney exchanges involving patient-donor pairs have to be operated simultaneously in one hospital or very close hospitals.

B. When both compatible patient-donor pairs and single donors participate in kidney exchanges with incompatible pairs, efficiency of exchange increases significantly. More precisely, allowing compatible pairs to exchange with incompatible pairs (the first degree inclusive exchange) can have at least 10% net increase of feasible kidney transplants and allowing compatible pairs and single donors to exchange with incompatible pairs (the second degree inclusive exchange) can have at least 30% net increase of feasible kidney transplants. To see this, let us look at Table 8 for the 1993-2002 data set. By comparison with the exclusive exchange, for the size $n = 25$ the net increase of efficiency by using the 1st deg. inclusive exchange is equal to $0.62592 - 0.472 \approx 0.15$, i.e., 15%. For all seven sizes, we take the smallest difference among all as the net increase of efficiency which is given by $\min\{0.15, 0.14, 0.12, 0.12, 0.12, 0.11, 0.11\}$, i.e., 11%. Similarly, by comparison with the exclusive exchange, the 2nd deg. inclusive exchange brings the net increase of efficiency by $\min\{0.42, 0.41, 0.37, 0.36, 0.36, 0.36, 0.36\}$, i.e., 36%. Now look also at Table 8 for the 1995-2016 data set. By comparing with the exclusive exchange, the 1st deg. inclusive exchange yields the net increase of efficiency by $\min\{0.15, 0.13, 0.12, 0.12, 0.12, 0.12,$

Figure 7: Matching rates of incompatible paired patients through at most three-way exchanges based on the 1993-2002 data (a) and based on the 1995-2016 data (b).



0.10}, i.e., 10%, and the 2nd deg. inclusive exchange raises min{0.38, 0.35, 0.32, 0.32, 0.31, 0.30, 0.30}, i.e., 30%.

C. The simulation results are very close to the theoretical bounds predicted by the formula in Proposition 1. Note that all our simulated population samples contain tissue-type incompatibilities, whereas our model basically assumes away the issue of tissue-type incompatibility. For the two data sets, Table 7 shows that the 2nd degree inclusive exchange performs better than the 1st degree inclusive exchange which outperforms the exclusive exchange. First let us look at the case of the 1993-2002 data set. The deviations for Upper Bound 1 under the exclusive exchange are 17.7%, 7.9%, 2.3%, 1%, 0.6%, 0.3%, 0.2%, under the 1st deg. inclusive exchange are 8%, 3%, 0.8%, 0.4%, 0.3%, 0.1%, 0.1%, and under the 2nd deg. inclusive exchange are 3.1%, 0.5%, 0.0%, 0.0%, 0.0%, 0.0%, 0.0%. Now look at the case of the 1995-2016 data set. The deviations for Upper Bound 1 under the exclusive exchange are 17.6%, 12.5%, 8.5%, 6.2%,
5.2%, 3.7%, 3%, under the 1st deg. inclusive exchange are 9.7%, 6.6%, 4.5%, 3.6%, 2.7%, 1.9%, 1.3%, and under the 2nd deg. inclusive exchange are 5.4%, 3.7%, 2.6%, 1.7%, 1.4%, 0.8%, 0.5%.

D: Simulation in Table 8 shows for the seven sizes of the population two increasing series of matching rates 0.89384, 0.95848, 0.97054, 0.973973, 0.97763, 0.983867, 0.98478 for the data set of 1993-2002 and 0.70544, 0.72636, 0.72972, 0.738667, 0.73936, 0.74404, 0.746815 for the data set of 1995-2016. In other words, the matching rate is an increasing function of the population size. This is also consistent with those found by Akbarpour et al. (2017) for general matching markets.

E: Two-way exchanges can reap most benefits of exchange and will gain more benefits of exchange as the size of the population increases. Three-way exchanges can make some visible gain of exchange. As stated earlier in our simulation for the population sizes 25, 50 and 100 the effect of four-way exchanges on efficiency is negligible and we expect it remains so for larger sizes of population (see Point A above) because the matching rate increases rapidly before $n = 100$ but the rate becomes almost flat after $n = 100$. Let us look at Table 5 of the 1993-2002 data set. For instance, when $n = 25$ through the 2nd deg. inclusive exchange (simulation), three-way exchanges yields a net increase of feasible transplants by $22.346 - 19.5904 = 2.7556$, i.e., $\frac{2.7556}{22.346}\% = 12\%$, and two-way exchanges produce 19.5904 feasible transplants, i.e., $\frac{19.5904}{22.346}\% = 88\%$. For the seven sizes of population, two-way exchanges contribute roughly $87\%, 89\%, 92\%, 94\%, 94\%, 95\%, 95\%$ to the number of feasible transplants and the remaining is due to three-way exchanges. Now turn to Table 6 for the 1995-2016 data set. For the seven sizes of population, similarly two-way exchanges contribute roughly $91\%, 93\%, 95\%, 95\%, 96\%, 96\%,$
96% to the number of feasible transplants and the remaining is due to three-way exchanges.

F: More accurate information can improve the quality of transplants and at the same time reduce the matching rate. This will be explained in the following subsection.

5.3.1 An Explanation of the Matching Rate on the Second Dataset

In this subsection we explain why the matching rate in the 1995-2016 data set (the second time slot) is lower than in the 1993-2002 dataset (the first time slot). In our simulations, we first draw a population of n incompatible pairs from the pool. Each incompatible pair is either blood-type incompatible or tissue-type compatible or both. When a compatible pair is drawn, we put the compatible pair back to the pool and keep drawing pairs from the pool until the population of n incompatible pairs is generated.

From the information given in Tables 3 and 4, we can calculate the percentage of incompatible pairs in the pool. The percentage of blood-type incompatible pairs for the first time slot and the second time slot are 0.3163 [5] and 0.30767 [6], respectively.

We give an example of the calculation by using the first group of each time slot. 89.24 percent of patients have no tissue type problem (CPRA=0) in the second time slot while 70.19 percent of patients have a low PRA value of 5 percent in the first time slot. Therefore, the percentages of drawing incompatible pairs from this group in the first and second time slots are given as follows, respectively:

$$\text{(Low PRA): } 5\% + 95\% * 0.3163 = 0.05 + 0.300485 = 0.350485$$

[5] $48.14\% * (1 - 48.14\%) + (33.73\% * 14.28\%) * 2 + 33.73\% * 3.85\% + 3.85\% * 14.28\% = 0.3163$.

[6] $48.46\% * (1 - 55.3\%) + 33.22\% * 9.9\% + 14.48\% * 32.46\% + 33.22\% * 2.34\% + 2.34\% * 14.48\% = 0.30767$.

(0): $0\% + 100\% * 0.30767 = 0.30767$.

When an incompatible paired patient is tissue-type compatible with a paired donor, the patient is blood-type incompatible with the donor. We have seven types of blood-type incompatible pairs (O, A), (O, B), (O, AB), (A, B), (B, A), (A, AB) and (B, AB). From the theoretical part, we can see that the blood-type incompatible pairs are difficult to find compatible pairs because they cannot match with each other except $(A, B) - (B, A)$, especially among incompatible pairs.

Table 2: The percentage of incompatible pairs in the pool

Groups from 1992-2003	The rate of tissue type incompatible pairs (%)	The rate of blood-type incompatible but tissue type compatible pairs (%)	The rate of incompatible pairs (%)
Low PRA	5	30.0485	35.0485
Medium PRA	45	17.3965	62.3965
High PRA	9	3.163	93.163
Average	21.3385	24.88	46.2185
Groups from 1995-2016	The rate of tissue type incompatible pairs (%)	The rate of blood-type incompatible but tissue type compatible pairs (%)	The rate of incompatible pairs (%)
0	0	30.767	30.767
1-19	9.5	27.844	37.344
20-79	50	15.3835	65.3835
80-97	88	3.692	91.692
98-100	99	0.30767	99.30767
Average	5.658	29.026	34.68445

We can see that blood-type incompatible pairs account for 53.89 (0.213385*0.3163+ 0.2488/0.462185) percent of the total incompatible pairs in the first time slot. While blood-type incompatible pairs account for 83.69 (0.05658*0.30767 + 0.29026/0.3468) percent of the total incompatible pairs in the second time slot, which is 29.8 percent higher than that of the first time slot. This means that the number of blood-type incompatible pairs from the second time slot is larger than those from the first time slot.

On the other hand, the number of blood-type compatible but tissue type incompatible pairs $(0.213385 * (1 - 0.3163) = 0.146)$ in the first time slot is larger than that in the second one $(0.05658 * (1 - 0.30767) = 0.039)$. Since blood-type incompatible pairs cannot be matched except $(A, B) - (B, A)$ with each other, it will be more difficult for incompatible paired patients to be matched in the second time slot than in the first time slot. This shows why the matching rate in the second time slot is lower than that in the second time slot.

6 Conclusion

The current study was motivated by practical and theoretical issues concerning kidney exchange. The first one is very practical and concerns kidney exchanges in a real life environment. In this environment as in our current model, there are many compatible patient-donor pairs, incompatible patient-donor pairs, patients on the waiting list, and single donors who are altruistic living or cadaver donors, and kidney exchanges can be done mostly by two-way, occasionally by three-way, and rarely by four-way. We have examined what will be the maximal number of possible feasible transplants in the environment. The second one is theoretical and concerns the derivation of the maximum number of feasible kidney transplants from two-way, three-way, and four-way exchanges, respectively.

Following and extending Roth et al. (2007a) on exchanges among incompatible patient-donor pairs, our current study has focused on deriving and calculating the exact number of maximum possible feasible kidney transplants and providing ways of realizing these efficient kidney transplants. The number of possible transplants can be known before any practical implementation or simulation. This approach is different from most other studies, which have discussed and proposed ways and procedures of achieving efficient kidney exchanges. In such cases, the number of possible kidney transplants will be known only after the proposed procedure is implemented by computer or in hospital.

Besides, our other major contributions include: Firstly, we have derived a precise maximum number of feasible kidneys transplants under two-way, three-way, and four-way exchanges respectively. We have shown that even for this general model at most four-way cycles or chains will be sufficient to accomplish all potential gains of kidney exchange, and that two-way exchanges can achieve most benefits of exchange. We have also proposed a general but abstract model of two-category type-compatible exchanges and proved that it suffices to fully achieve efficiency by using only n-way cycles and chains of exchanges where n is the number of the primary types in the first category and is independent of the number of secondary types in the second category. The number of primary types can be different from the number of secondary types. We found that in every efficient exchange, each cycle contains at most two blood-type compatible pairs and each chain contains at most one blood-type compatible pair. Secondly, we have provided substantial simulation results based on the USA data for the periods of 1993-2002 and of 1995-2016. We found that even just under two-way exchanges allowing compatible pairs to exchange with incompatible pairs can increase the number of feasible transplants considerably (net increase at least 10%), and that allowing both compatible pairs and single donors to participate in exchange with incompatible pairs can significantly increase the number of feasible transplants (net increase at least 30%).

Thirdly, our results demonstrate that our theory can predict very well in the sense that

the actual maximal number of feasible kidney transplants is very close to the predicated number given by our derived formula. As the size of the population increases, the predictive power of our theory becomes stronger; two-way exchange can accomplish most of the potential gains of exchange. If the population is large enough, it is sufficient to use two-way exchange to clear all incompatible pairs. Finally, our results have a novel and meaningful policy implication: kidney exchange can be decentralized in the sense that in a country with a large population, several separate kidney exchange programs/centers can be established across the country, not just one centralized program/center for the entire country.

The current paper leaves some important open questions for further investigation: The first one is how to estimate and ultimately conduct the maximum number of possible feasible kidney transplants when patients or/and donors appear at random or follow some reasonable stochastic process; see Ünver (2010). The second is to conduct a study on a setting extending our current model by adopting the approach of Andersson and Kratz (2020) that relaxes the standard blood type compatibility constraint.

We hope this study will be useful in helping design practical kidney exchange program and stimulate further research.

The Appendix

In the following matching procedure, whenever cycles or chains of exchange are going to be made, priority is given to incompatible pairs.

A Sequential Two-Way Matching Procedure

Step 1: Make a maximum number of two-way cycles of exchange $(A,A)^i-(A,A)^i$. Then make a maximum number of two-way cycles of exchange $(A,A)^i-(A,A)^c$ if any. Carry out transplants for the remaining pairs $(A,A)^c$. Repeat the same process for each type (B,B), (O,O), (AB,AB), respectively.

Step 2: Make a maximum number of two-way cycles of exchange $(O,A)-(A,O)^i$, $(O,B)-(B,O)^i$, $(O,AB)-(AB,O)^i$, $(A,AB)-(AB,A)^i$, $(B,AB)-(AB,B)^i$, and $(A,B)-(B,A)$, respectively.

Step 3: Make a maximum number of two-way cycles or chains of exchange $(O,A)-(A,O)^c$, $(O,B)-(B,O)^c$, $(A,AB)-(AB,A)^c$, $(B,AB)-(AB,B)^c$, $A^d-(A,B)-AB^p/B^p$, $A^d-(A,AB)-AB^p$, and $B^d-(B,AB)-AB^p$, respectively. Match a maximum number of two-way cycles $(B,O)^c-(A,B)$, $(AB,A)^c-(A,B)$, $(B,O)^i-(A,B)$, $(AB,A)^i-(A,B)$ and two-way chain $A^d-(A,B)-Y^p$.

Step 4: Make a maximal number of two-way cycles of exchange

$$(AB,O)^c/(AB,O)^i - (O,A)/(O,B)/(O,AB)/(A,AB)/(B,AB)/(A,B),$$

respectively. And then match a maximum number of single donors O^d with the remaining pairs $(O,A)/(O,B)/(O,AB)/(A,AB)/(B,AB)/(A,B)$, respectively.

Step 5: Match a maximum number of the remaining single donors O^d, A^d, B^d, AB^d with any remaining single patients O^p, A^p, B^p, AB^p. Match a maximum number of two-way cycles of exchange $(A,O)^i - (A,O)^i$. Then make a maximum number of two-way cycles of exchange $(A,O)^i - (A,O)^c$ if any. Repeat the same process for each type $(B,O)^i$, $(AB,O)^i$, $(AB,A)^i$, $(AB,B)^i$. Match any remaining paired patients from compatible patient-donor pairs with their own paired donors.

This is the end of the description of the procedure.
The maximum number of transplants through two-way exchanges in Proposition 1 is given by the following formula:

$$\begin{aligned}
&\#(A,O)+\#(B,O)+\#(AB,O)+\#(AB,A)+\#(AB,B)+\#(B,A)\\
&+\#(A,A)+\#(B,B)+\#(O,O)+\#(AB,AB)\\
&+\#A^d+\#B^d+\#AB^d+\#O^d\\
&+\min\{N_1,N_2,N_3,N_4,N_5,N_6,N_7,N_8,N_9,N_{10},N_{11},N_{12},N_{13},N_{14},N_{15},N_{16},N_{17}\}
\end{aligned}$$

where

$$\begin{aligned}
N_1 &= \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_2 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#(B,AB)+\#(A,B)\\
N_3 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_4 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#B^d+\#(AB,B)+\#(A,B)\\
N_5 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#B^d+\#(AB,B)\\
N_6 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#(B,AB)\ +\#(A,B)\\
N_7 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_8 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#B^d+\#(AB,B)+\#(A,B)
\end{aligned}$$

$$\begin{aligned}
N_9 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#B^d+\#(AB,B)\\
N_{10} &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#(B,AB)+\#(B,A)\\
N_{11} &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_{12} &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#B^d+\#(AB,B)+\#(B,A)\\
N_{13} &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#B^d+\#(AB,B)\\
N_{14} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#(B,AB)+\#(B,A)\\
N_{15} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_{16} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#B^d+\#(AB,B)+\#(B,A)\\
N_{17} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#B^d+\#(AB,B)
\end{aligned}$$

This is the end of the description of the formula.

Proof of Proposition 1: Under Assumptions 1 to 3, all blood-type compatible but tissue-type incompatible pairs and pairs of type (B, A) can be matched through two-way cycles. All compatible pairs can be matched because even if paired patients from compatible pairs are not involved into two-way cycles, they can receive their own donors. All pairs of types (A, A), (B, B), (O, O), (AB, AB) can be also matched in two-way cycles. As long as a kidney can be allocated to a patient in waiting, we can always find a compatible patient in waiting because of the large population of patients in waiting. Hence, the maximal number of transplantations for patients in waiting, paired patients from blood-type compatible pairs and paired patients from pairs of type (B, A) is

$$\begin{aligned}&\#(A,O)+\#(B,O)+\#(AB,O)+\#(AB,A)+\#(AB,B)\\&+\#(B,A)+\#(A,A)+\#(B,B)+\#(AB,AB)+\#(O,O)\\&+\#A^d+\#B^d+\#AB^d+\#O^d\end{aligned}$$

Next, let N be the maximum number of transplants for blood-type incompatible paired patients of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B). The number of two-way cycles $(A, B) - (B, A)$ is bounded by $\#(B, A)$ by Assumption 2. The number of two-way cycles $(O, A) - (A, O)$ is bounded by $\min\{\#(O, A), \#(A, O)\}$. Similarly, the number of two-way cycles $(O, B) - (B, O)$ is bounded by $\min\{\#(O, B), \#(B, O)\}$; the number of two-way cycles and chains $(AB, A) - (A, AB)$, $A^d - (A, AB) - Y^p$ is bounded by $\min\{\#A^d + \#(AB, A), \#(A, AB)\}$; the number of two-way cycles and chains $(AB, A) - (A, B)$, $A^d - (A, B) - Y^p$, $(B, O) - (A, B)$ is bounded by $\min\{\#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\} + \#(B, O) - \min\{\#(O, B), \#(B, O)\}, \#(A, B) - \#(B, A)\}$; the number of two-way cycles and chains $(AB, B) - (B, AB)$, $B^d - (B, AB) - AB^p$ is bounded by $\min\{\#B^d + \#(AB, B), \#(B, AB)\}$; and the number of two-way cycles and chains

$$(AB, O) - (O, A)/(O, B)/(O, AB)/(A, B)/(A, AB)/(B, AB), \text{ and,}$$

$$O^d - (O, A)/(O, B)/(O, AB)/(A, B)/(A, AB)/(B, AB) - Y^w$$

is bounded either by $\#O^d + \#(AB, O)$ or all blood-type incompatible paired patients are matched. Therefore, we have either

$$\begin{aligned}N \leq\ & \#(B,A)+\min\{\#(O,A),\#(A,O)\}+\min\{\#(O,B),\#(B,O)\}+\\&\min\{\#A^d+\#(AB,A),\#(A,AB)\}+\\&\min\{\#A^d+\#(AB,A)-\min\{\#A^d+\#(AB,A),\#(A,AB)\}\\&+\#(B,O)-\min\{\#(O,B),\#(B,O)\},\#(A,B)-\#(B,A)\}+\\&\min\{\#B^d+\#(AB,B),\#(B,AB)\}+\\&\min\{\#B^d+\#(AB,B),\#(B,AB)\}+\#O^d+\#(AB,O)\end{aligned}$$

or

$$N \leq \#(O,A) + \#(O,B) + \#(O,AB) + \#(A,AB) + \#(A,B) + \#(B,AB).$$

The expressions can be rewritten as follows

$N \leq \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}$ and hence the maximum number of transplants can be reached is:

$N = \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}$.

We now prove that the sequential matching procedure achieves the maximum number of kidney transplants.

Since for every one/two-way chains, we can always find a compatible single patient, the number of transplantations for single patients equals $\#A^d + \#B^d + \#AB^d + \#O^d$.

By Assumption 3, all pairs of type $(A,A)^i$, $(B,B)^i$, $(O,O)^i$, $(AB,AB)^i$ can be matched through two-way cycles in Step 1. By Assumption 3, all remaining blood-type compatible but tissue-type incompatible pairs $(A,O)^i$, $(B,O)^i$, $(AB,O)^i$, $(AB,A)^i$, $(AB,B)^i$ can be matched through two-way cycles in Step 5. All compatible pairs $(A,O)^c$, $(B,O)^c$, $(AB,O)^c$, $(AB,A)^c$, $(AB,B)^c$, $(A,A)^c$, $(B,B)^c$, $(O,O)^c$, $(AB,AB)^c$ can be matched either through two-way cycles or doing transplantations with their own donors. Moreover, by Assumption 2, all pairs of type (B,A) can be matched through two-way cycle $(A,B)-(B,A)$ in Step 2 so that the remaining number of pairs of type (A,B) is $\#(A,B)-\#(B,A)$. Hence, the number of transplants for compatible pairs, blood-type compatible pairs and pairs of type (B,A) in the procedure is

$$\begin{aligned}&\#(A,O) + \#(B,O) + \#(AB,O) + \#(AB,A) + \#(AB,B)\\ &+\#(B,A) + \#(A,A) + \#(B,B) + \#(AB,AB) + \#(O,O)\end{aligned}$$

Next, we prove that the maximum number of transplants for blood-type incompatible pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) can be achieved in the procedure.

Denote X_1 as the number of blood-type incompatible paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 2 so that

$$X_1 = \#(B,A) + e_1 + e_2 + e_3 + e_4 + e_5$$

where

$$\begin{aligned}e_1 &= \min\{\#(O,A), \#(A,O)^i\}\\ e_2 &= \min\{\#(O,B), \#(B,O)^i\}\\ e_3 &= \min\{\#(O,AB), \#(AB,O)^i\}\\ e_4 &= \min\{\#(A,AB), \#(AB,A)^i\}\\ e_5 &= \min\{\#(B,AB), \#(AB,B)^i\}\end{aligned}$$

Denote X_2 as the number of blood-type incompatible paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 3 so that

$$X_2 = \ a_1 + a_2 + b_1 + b_2 + b_3$$

where

$$\begin{aligned}
a_1 = &\ \min\{\#(O,A) - e_1, \#(A,O)^c\} \\
a_2 = &\ \min\{\#(O,B) - e_2, \#(B,O)^c\} \\
b_1 = &\ \min\{\#A^d + \#(AB,A)^c, \#(A,AB) - e_4\} \\
b_2 = &\ \min\{\#B^d + \#(AB,B)^c, \#(B,AB) - e_5\} \\
b_3 = &\ \min\{\#A^d + \#(AB,A)^c + \#(AB,A)^i - e_4 - b_1 + \#(B,O)^c \\
&\ +\#(B,O)^i - e_2 - b_2, \#(A,B) - \#(B,A)\}
\end{aligned}$$

Denote X_3 as the number of blood-type incompatible paired patients from pairs of types (O,A),(O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 4 so that

$$\begin{aligned}
X_3 = &\ \min\{\#O^d + \#(AB,O)^c + \#(AB,O)^i - e_3, \#(O,A) - e_1 - a_1 \\
&\ +\#(O,B) - e_2 - a_2 + \#(O,AB) - e_3 + \#(A,AB) - e_4 - b_1 \\
&\ +\#(B,AB) - e_5 - b_2 + \#(A,B) - \#(B,A) - b_3\}
\end{aligned}$$

Therefore, the total number of transplants for paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) in the procedure is $X = X_1 + X_2 + X_3$; one may refer to Tables from A1 to A15 in Supplement A of Cheng and Yang (2017b) for detail. Then the equation can be rewritten as follows:

$X = \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}$.

Therefore, the total number of transplants can be achieved in the mechanism is that

$$\begin{aligned}
&\#(A,O) + \#(B,O) + \#(AB,O) + \#(AB,A) + \#(AB,B) \\
&+\#(B,A) + \#(A,A) + \#(B,B) + \#(AB,AB) + \#(O,O) \\
&+\#A^d + \#B^d + \#AB^d + \#O^d \\
&+ \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}
\end{aligned}$$

We proved that every matching produced by the procedure achieves the maximum number of transplants in the pool and hence the procedure is 2-efficient. □

Proof of Theorem 1: Consider an efficient matching μ for a population stated in the theorem. If the maximal matching μ only consists of n-way cycles and chains, or smaller cycles and chains, we are done. Otherwise, we will prove that there exists a matching ν which consists of at most n-way cycles or chains can match the same set of receiving agents as matching μ.

We will prove the theorem for the case in which the largest exchanges (cycles or chains) in matching μ is (n+1)-way. The same proof can be applied to show that for any maximal matching in which the largest exchanges is k-way where $k > (n+1)$, there exists another matching which matches the same set of receiving agent through $(k-1)$-way or smaller exchanges. Then, repeating the same argument yields the desired result. It is worth noting that this argument holds only for $k(> n+1)$-way exchanges. When the maximal matching μ only consists of at most n-way cycles and chains, the argument may not hold. To see this point, consider the blood-type as the primary type. Suppose that there are four patient-donor pairs $(AB-O)$,$(O-A)$, $(A-B)$, $(B-AB)$ in the pool, and patients have no tissue-type problem with donors. The maximum number of feasible transplants can be reached through a 4-way exchange $(AB,O)-(O,A)-(A,B)-(B,AB)$ which is unique and cannot be reduced to 3 or 2-way exchanges.

Three cases may occur in the matching μ which has $(n+1)$-way cycles and chains, or only $(n+1)$-way cycles or $(n+1)$-way chains. We will prove the most complicated case that μ consists of both $(n+1)$-way cycles and $(n+1)$-way chains. Then, the other two case follow automatically.

Let

$$E^0 = ((P_1^p, D_1^p), (P_2^p, D_2^p), (P_3^p, D_3^p), ..., (P_n^p, D_n^p), (P_{n+1}^p, D_{n+1}^p))$$
$$C^0 = (D_1^s, (P_1^p, D_1^p), (P_2^p, D_2^p), ..., (P_n^p, D_n^p), P_1^s)$$

be any $(n+1)$-way cycle and chain respectively in matching μ. We will prove that all receiving agents in these two ways of exchange can be matched via smaller ways of exchange without changing the set of pairs that are matched.

Because we have only n types, there are at least two receiving agents in cycle E^0 who have the same type. Pick any two such receiving agents. We have two cases to consider.

Case 1. The two receiving agents are not matched together.

Suppose these receiving agents are P_1^p and P_n^p in cycle E^0. The receiving agent P_1^p is matched with donating agent D_{n+1}^p and the receiving agent P_n^p is matched with donating agent D_{n-1}^p. Since agents P_1^p and P_n^p have the same type, donating agents D_{n-1}^p and D_{n+1}^p are compatible with the two receiving agents P_1^p and P_n^p. Hence, the $(n+1)$-way cycle can be divided into two smaller cycles as follows.

$$E_1^1 = ((P_1^p, D_1^p), (P_2^p, D_2^p), (P_3^p, D_3^p), ..., (P_{n-1}^p, D_{n-1}^p)),\ E_2^1 = ((P_n^p, D_n^p), (P_{n+1}^p, D_{n+1}^p))$$

Suppose these receiving agents are P_1^p and P_1^s in chain C^0. The receiving agent P_1^p is matched with donating agent D_1^s and the receiving agent P_1^s is matched with donating agent D_n^p. Since agents P_1^p and P_1^s have the same type, donating agents D_1^s and D_n^p are compatible with both the two agents. Hence, the $(n+1)$-way chain C^0 can be divided into

one cycle and one chain as follows.

$$C_2^1 = (D_1^s, P_1^s),\ E_2^1 = ((P_1^p, D_1^p), (P_2^p, D_2^p), ..., (P_n^p, D_n^p))$$

Case 2. The two receiving agents are matched together. Suppose agents P_1^p and P_2^p have the same type.

Under cycle E^0, since agent P_1^p is matched with donating agent D_{n+1}, donating agent D_{n+1} is compatible with the receiving agent P_2^p. Hence, the following n-way exchange is feasible.

$$E_1^2 = ((P_2^p, D_2^p), (P_3^p, D_3^p), ..., (P_{n-1}^p, D_{n-1}^p))$$

Under chain C^0, since agent P_1^p is matched with donating agent D_1^s, donating agent D_1^s is compatible with receiving agent P_2^p. Hence, the following n-way chain is feasible.

$$C_2^2 = (D_1^s, (P_2^p, D_2^p), ..., (P_{n-1}^p, D_{n-1}^p), (P_n^p, D_n^p), P_1^s)$$

Now, we will prove that the remaining pair (P_1^p, D_1^p) can be matched in an exchange without affecting pairs that are matched under μ. Because of the Assumption 4, we can directly use "type" to present the primary type. Let pair (P_1^p, D_1^p) be of type $(X, Y)^t$ where $t \in \{i, c\}$, and hence receiving agent P_2^p is type X. Since donating agent D_1^p of type Y is compatible with receiving agent P_2^p, we have $Y \succeq X$. Therefore, pair of type (X, Y) is primary type compatible pair.

Let $\mathcal{A}$ be the set of $n+1$-way cycles and $n+1$-way chains in Case 2. From the above proof, every cycle can be separated into an n-way cycle and one remaining primary type compatible pair and every chain can be separated into an n-way chain and one remaining primary type compatible pair. Let $\mathcal{D}$ be the set of remaining primary type compatible pairs in $\mathcal{A}$. Then, we have $Y \succeq X$. If remaining pairs are compatible, we can do transplants directly. Otherwise, let $(X, Y)^i$ present the type of a primary type compatible but secondary type incompatible pair. If there exists two or more pairs of type $(X, Y)^i$, we can match them by two-way cycles $(X, Y)^i - (X, Y)^i$. Therefore, at most one pair of type $(X, Y)^i$ will be left. By Assumption 5, there exists at least one pair of type $(X, Y)^c$. If the pair $(X, Y)^c$ does not involve in a cycle or a chain, we can match the remaining pair $(X, Y)^i$ with pair $(X, Y)^c$. Otherwise, pair $(X, Y)^c$ involves in a cycle or chain no larger than n-way, then the remaining pair $(X, Y)^i$ can replace the position of pair $(X, Y)^c$ and pair $(X, Y)^c$ can do the transplant straightforwardly. □

Table 3: Patient-donor pair and single donor distributions used in simulations based on OPTN/SRTR database from 1993 to 2002, retrieved from https://optn.transplant.hrsa.gov/data/view-data-reports/national-data.

Incompatible paired patient blood type	**Percent**
O	48.14
A	33.73
B	14.28
AB	3.85
Patient gender	**Percent**
Female	40.9
Male	59.1
Relationship of patient-donor pair	**Percent**
Spouse	48.97
Other	51.03
PRA types	**Percent**
Low PRA	70.19
Medium PRA	20.00
High PRA	9.81
Compatible paired patient blood type	**Percent**
O	45.12
A	38.54
B	12.64
AB	3.7
Compatible paired donor blood type	**Percent**
O	63.74
A	27.12
B	8.08
AB	1.06
Single donor blood type	**Percent**
O	47.31
A	38.14
B	11.16
AB	3.39
Transplant ratio by donor types	**Percent**
Single Donors	39.83
Paired Donors	22.77

Table 4: Patient-donor pair and single donor distributions used in simulations based on OPTN/SRTR database from 1995 to 2016, retrieved from https://optn.transplant.hrsa.gov/data/view-data-reports/national-data.

Incompatible paired patient blood type	**Percent**	**S.D.**
O	48.46	0.0032
A	33.22	0.0047
B	14.48	0.0028
AB	3.84	0.0011
Incompatible paired patient gender	**Percent**	**S.D.**
Female	40.1	0.0117
Male	59.9	0.0117
Incompatible paired patient CPRA type	**Percent**	**S.D.**
0	89.24	0.0145
1-19	2.79	0.0071
20-79	4.64	0.005
80-97	2.03	0.001
98-100	1.3	0.002
Compatible paired patient blood type	**Percent**	**S.D.**
O	44.71	0.0092
A	38.47	0.0075
B	12.99	0.0044
AB	3.83	0.0029
Compatible paired patient gender	**Percent**	**S.D.**
Female	39.95	0.0204
Male	60.05	0.0204
Compatible paired patient CPRA type	**Percent**	**S.D.**
0	73.11	0.0241
1-19	9.43	0.0154
20-79	12.82	0.0084
80-97	3.38	0.0041
98-100	1.26	0.0025
Relationship of patient-donor pair	**Percent**	**S.D.**
Spouse	35.8	0.1201
Other	64.2	0.1201
Incompatible paired donor blood type	**Percent**	**S.D.**
O	55.3	0.0122
A	32.46	0.0081
B	9.9	0.0041
AB	2.34	0.0022
Compatible paired donor blood type	**Percent**	**S.D.**
O	64.66	0.011
A	26.45	0.0074
B	7.91	0.0044
AB	0.98	0.0021
Single donor blood type	**Percent**	**S.D.**
O	47.59	0.0068
A	37.41	0.0084
B	11.57	0.0055
AB	3.43	0.0026
Transplant ratio by donor type	**Percent**	**S.D.**
Single Donors	36.02	0.0398
Paired Donors	19.9	0.039

Table 5: Simulation results about average maximal number of incompatible paired patients receiving transplants and average predicted number by the formula based on the 1993-2002 data.

Size of Incompat. Pairs	Method	Number of incompatible paired patients getting transplants					
		At most two-way exchanges			At most three-way exchanges		
		The Exclusive Exchange	The 1st Deg. Inclusive Exchange	The 2nd Deg. Inclusive Exchange	The Exclusive Exchange	The 1st Deg. Inclusive Exchange	The 2nd Deg. Inclusive Exchange
n=25	Simulation	8.9992 (3.3465)	12.8388 (3.36736)	19.5904 (3.1966)	11.8 (4.1432)	15.6480 (3.347)	22.3460 (2.7910)
	UB 1	12.4444 (3.62319)	15.8782 (3.55402)	21.919 (3.0039)	14.3480 (3.7968)	17.1380 (3.2815)	23.062 (2.5806)
	UB 2	9.7012 (3.69614)	14.0782 (3.59381)	20.964 (3.02684)	12.424 (4.2280)	16.242 (3.3636)	22.6540 (2.626)
n=50	Simulation	21.7872 (5.04759)	29.599 (5.17304)	42.8134 (4.77275)	27.566 (5.6835)	34.754 (5.1818)	47.9240 (3.0537)
	UB 1	27.0408 (5.16082)	33.5676 (5.31818)	45.413 (4.45821)	29.96 (5.2792)	35.896 (5.1418)	48.178 (2.9448)
	UB 2	23.7656 (5.47378)	31.9192 (5.4182)	44.8486 (4.41678)	28.468 (5.676)	35.412 (5.1788)	48.096 (2.9448)
n=100	Simulation	49.8772 (7.36965)	64.2164 (7.4473)	89.8862 (6.9542)	60.232 (7.434)	72.214 (7.1707)	97.054 (4.4116)
	UB 1	56.7104 (7.36069)	68.614 (7.58903)	92.2014 (6.59551)	61.696 (7.2799)	72.814 (7.1843)	97.15 (4.3733)
	UB 2	53.4844 (7.70327)	67.4584 (7.6945)	92.0746 (6.57535)	60.934 (7.3839)	72.69 (7.2076)	97.142 (4.369)
n=150	Simulation	78.9256 (9.29992)	100.014 (9.42842)	137.567 (8.63815)	92.072 (8.9213)	109.852 (9.1063)	146.096 (5.3979)
	UB 1	86.692 (9.1035)	104.442 (9.48898)	139.417 (8.33299)	93.038 (8.7667)	110.33 (9.0828)	146.1080 (5.3869)
	UB 2	83.6704 (9.54597)	103.647 (9.58259)	139.383 (8.31955)	92.686 (8.9142)	110.29 (9.0811)	146.108 (5.3869)
n=200	Simulation	108.716 (10.7764)	135.571 (10.9588)	184.819 (10.3357)	123.714 (9.8055)	146.72 (10.6069)	195.5260 (5.9048)
	UB 1	116.799 (10.5688)	139.742 (11.0306)	186.254 (10.1569)	124.474 (9.8062)	147.174 (10.6329)	195.5420 (5.8993)
	UB 2	114.232 (10.9591)	139.168 (11.0965)	186.245 (10.1546)	124.258 (9.8394)	147.148 (10.6529)	195.542 (5.8993)
n=300	Simulation	170.54 (13.8317)	208.974 (13.8698)	280.91 (13.4347)	187.8 (12.8155)	221.74 (13.5413)	295.16 (7.4243)
	UB 1	178.668 (13.6163)	212.676 (14.0197)	281.688 (13.4062)	188.442 (12.8354)	222.136 (13.5083)	295.162 (7.4233)
	UB 2	176.948 (13.9028)	212.404 (14.0379)	281.688 (13.4062)	188.386 (12.8539)	222.136 (13.5083)	295.162 (7.4233)
n=400	Simulation	231.628 (15.1099)	281.492 (15.1398)	375.198 (15.2474)	250.7081 (14.9844)	296.082 (15.0974)	393.912 (9.0612)
	UB 1	239.524 (14.592)	284.636 (15.0674)	375.65 (15.2176)	251.2801 (15.0512)	296.464 (15.1327)	393.912 (9.0612)
	UB 2	238.36 (14.8267)	284.466 (15.1155)	375.65 (15.2176)	251.2411 (15.0499)	296.464 (15.1327)	393.912 (9.0612)

Table 6: Simulation results about average maximal number of incompatible paired patients receiving transplants and average predicted number by the formula based on the 1995-2016 data.

Size of Incompat. Pairs	Method	Number of incompatible paired patients getting transplants					
		At most two-way exchanges			At most three-way exchanges		
		The Exclusive Exchange	The 1st Deg. Inclusive Exchange	The 2nd Deg. Inclusive Exchange	The Exclusive Exchange	The 1st Deg. Inclusive Exchange	The 2nd Deg. Inclusive Exchange
n=25	Simulation	6.6844 (3.02308)	9.6722 (3.16884)	16.1756 (3.39085)	8.032 (3.4985)	11.71 (3.4753)	17.636 (3.3892)
	UB 1	8.3772 (3.29944)	11.3094 (3.4135)	17.6964 (3.55437)	9.754 (3.5387)	12.968 (3.4454)	18.654 (3.4546)
	UB 2	6.832 (3.12092)	10.0494 (3.30825)	16.7298 (3.50851)	8.208 (3.5976)	11.898 (3.5203)	17.824 (3.4212)
n=50	Simulation	15.008 (4.5394)	21.5734 (4.71549)	33.8482 (4.9819)	18.83 (5.4835)	25.118 (4.6548)	36.318 (5.1957)
	UB 1	18.5984 (4.79534)	24.1956 (4.9522)	36.1456 (5.1907)	21.5260 (5.2418)	26.894 (4.751)	37.734 (5.2416)
	UB 2	16.0188 (4.75009)	22.3364 (4.88135)	34.7956 (5.1391)	19.228 (5.4714)	25.37 (4.6661)	36.46 (5.2164)
n=100	Simulation	34.496 (6.8107)	46.3272 (7.05924)	69.7068 (7.42242)	40.57 (7.078)	52.874 (6.3209)	72.972 (6.8957)
	UB 1	39.6832 (6.96165)	50.2572 (7.27533)	73.0118 (7.62722)	44.368 (6.7903)	55.378 (6.4366)	74.936 (6.9059)
	UB 2	35.8428 (7.01817)	47.6532 (7.22253)	71.1594 (7.55584)	41.196 (6.9782)	53.302 (6.4019)	73.376 (6.9255)
n=150	Simulation	54.2632 (8.65407)	71.5348 (8.9778)	105.994 (9.24828)	63.19 (8.573)	81.106 (7.6125)	110.8 (8.3508)
	UB 1	60.934 (8.82225)	76.3784 (9.16827)	110.046 (9.4449)	67.438 (8.5153)	84.154 (7.6895)	112.816 (8.3451)
	UB 2	56.2608 (8.89426)	73.313 (9.17326)	107.991 (9.44535)	63.986 (8.5644)	81.78 (7.6938)	111.364 (8.4599)
n=200	Simulation	74.134 (10.0771)	96.6472 (10.4245)	142.411 (10.6297)	85.9 (10.4493)	109.078 (9.7596)	147.926 (10.5832)
	UB 1	81.8596 (10.1768)	102.143 (10.691)	146.966 (10.8525)	90.666 (10.2872)	112.206 (9.9683)	150.066 (10.6476)
	UB 2	76.5832 (10.2132)	98.7708 (10.659)	144.941 (10.8549)	87.078 (10.3492)	110.008 (9.8715)	148.63 (10.6966)
n=300	Simulation	114.904 (11.8696)	147.89 (12.4127)	215.976 (13.0876)	131.898 (12.8419)	166.8680 (11.4042)	223.212 (12.5156)
	UB 1	124.292 (11.9257)	154.37 (12.6063)	221.19 (13.282)	137.078 (12.7128)	170.142 (11.4801)	225.194 (12.5067)
	UB 2	118.272 (12.0396)	150.724 (112.6819)	219.358 (13.3162)	133.566 (2.7571)	168.064 (11.4839)	224.002 (12.5993)
n=400	Simulation	155.572 (13.846)	198.49 (14.1654)	288.54 (14.4048)	177.972 (13.8102)	221.428 (12.7008)	298.7260 (14.6946)
	UB 1	166.024 (13.8123)	205.776 (14.4168)	294.304 (14.7397)	183.614 (13.4576)	224.45 (12.7107)	300.412 (14.7551)
	UB 2	159.384 (13.8554)	202.008 (14.3976)	292.802 (14.7343)	179.95 (13.5565)	222.762 (12.7487)	299.6260 (14.7936)

Table 7: Deviation from upper bounds 1 and 2 in simulation through at most three-way exchanges based on the 1993-2002 data and 1995-2016 data.

Data from 1993-2002				
Population Size of Incompatible Pairs	Method	Deviation Value		
		The Exclusive Exchange	The First Degree Inclusive Exchange	The Second Degree Inclusive Exchange
n=25	Upper Bound 1	0.177586	0.08694	0.03105
	Upper Bound 2	0.050225	0.03657	0.013596
n=50	Upper Bound 1	0.07991	0.031814	0.005272
	Upper Bound 2	0.031685	0.01858	0.003576
n=100	Upper Bound 1	0.023729	0.00824	0.00099
	Upper Bound 2	0.01152	0.006548	0.00091
n=150	Upper Bound 1	0.01038	0.004332	0.000082
	Upper Bound 2	0.006624	0.00397	0.000082
n=200	Upper Bound 1	0.00611	0.003085	0.000081
	Upper Bound 2	0.004378	0.002909	0.000081
n=300	Upper Bound 1	0.003407	0.001783	0.000007
	Upper Bound 2	0.003111	0.001783	0.000007
n=400	Upper Bound 1	0.002276	0.001288	0
	Upper Bound 2	0.002122	0.001288	0
Data from 1995-2016				
Population Size of Incompatible Pairs	Method	Deviation Value		
		The Exclusive Exchange	The First Degree Inclusive Exchange	The Second Degree Inclusive Exchange
n=25	Upper Bound 1	0.176543	0.09701	0.05457
	Upper Bound 2	0.021443	0.0158	0.01055
n=50	Upper Bound 1	0.12524	0.06604	0.037526
	Upper Bound 2	0.0207	0.009933	0.003895
n=100	Upper Bound 1	0.0856	0.04522	0.02621
	Upper Bound 2	0.015196	0.00803	0.005506
n=150	Upper Bound 1	0.06299	0.03622	0.01787
	Upper Bound 2	0.01244	0.00824	0.005065
n=200	Upper Bound 1	0.052567	0.027877	0.01426
	Upper Bound 2	0.01353	0.008454	0.004737
n=300	Upper Bound 1	0.037789	0.019243	0.0088
	Upper Bound 2	0.012488	0.007116	0.003268
n=400	Upper Bound 1	0.030728	0.013464	0.005612
	Upper Bound 2	0.01099	0.005988	0.003004

Table 8: Matching rates of incompatible paired patients in simulation through at most three-way exchanges based on the 1993-2002 data and 1995-2016 data.

Data from 1993-2002				
Population Size of Incompatible Pairs	Method	Matching Rate		
		The Exclusive Exchange	The First Degree Inclusive Exchange	The Second Degree Inclusive Exchange
n=25	Simulation	0.472	0.62592	0.89384
	Upper Bound 1	0.57392	0.68552	0.92248
	Upper Bound 2	0.49696	0.64968	0.90616
n=50	Simulation	0.55132	0.69508	0.95848
	Upper Bound 1	0.5992	0.71792	0.96356
	Upper Bound 2	0.56936	0.70824	0.96192
n=100	Simulation	0.60232	0.72214	0.97054
	Upper Bound 1	0.61696	0.72814	0.9715
	Upper Bound 2	0.60934	0.7269	0.97142
n=150	Simulation	0.613813	0.732347	0.973973
	Upper Bound 1	0.620253	0.73553	0.974053
	Upper Bound 2	0.617907	0.735267	0.974053
n=200	Simulation	0.61857	0.7336	0.97763
	Upper Bound 1	0.62237	0.73587	0.97771
	Upper Bound 2	0.62129	0.73574	0.97771
n=300	Simulation	0.626	0.739133	0.983867
	Upper Bound 1	0.62814	0.74045	0.983873
	Upper Bound 2	0.62795	0.74045	0.983873
n=400	Simulation	0.62677	0.740205	0.98478
	Upper Bound 1	0.6282	0.74116	0.98478
	Upper Bound 2	0.6281	0.74116	0.98478
Data from 1995-2016				
Population Size of Incompatible Pairs	Method	Matching Rate		
		The Exclusive Exchange	The First Degree Inclusive Exchange	The Second Degree Inclusive Exchange
n=25	Simulation	0.32128	0.4684	0.70544
	Upper Bound 1	0.39016	0.51872	0.74616
	Upper Bound 2	0.32832	0.47592	0.71296
n=50	Simulation	0.3766	0.50236	0.72636
	Upper Bound 1	0.43052	0.53788	0.75468
	Upper Bound 2	0.38456	0.5074	0.7292
n=100	Simulation	0.4057	0.52874	0.72972
	Upper Bound 1	0.44368	0.55378	0.74936
	Upper Bound 2	0.41196	0.53302	0.73376
n=150	Simulation	0.421267	0.540707	0.738667
	Upper Bound 1	0.449587	0.561027	0.752107
	Upper Bound 2	0.426573	0.5452	0.742427
n=200	Simulation	0.4295	0.54539	0.73936
	Upper Bound 1	0.45333	0.56103	0.75033
	Upper Bound 2	0.43539	0.55004	0.74315
n=300	Simulation	0.43966	0.556227	0.74404
	Upper Bound 1	0.456927	0.56714	0.750647
	Upper Bound 2	0.44522	0.560213	0.746673
n=400	Simulation	0.44493	0.55357	0.746815
	Upper Bound 1	0.459035	0.561125	0.75103
	Upper Bound 2	0.448975	0.556905	0.749065

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