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# REES ALGEBRAS OF SPARSE DETERMINANTAL IDEALS

ELA CELIKBAS, EMILIE DUFRESNE, LOUIZA FOULI, ELISA GORLA, KUEI-NUAN LIN,  
CLAUDIA POLINI, AND IRENA SWANSON

**ABSTRACT.** We determine the defining equations of the Rees algebra and of the special fiber ring of the ideal of maximal minors of a  $2 \times n$  sparse matrix. We prove that their initial algebras are ladder determinantal rings. This allows us to show that the Rees algebra and the special fiber ring are Cohen-Macaulay domains, they are Koszul, they have rational singularities in characteristic zero and are F-rational in positive characteristic.

## 1. INTRODUCTION

Given an ideal  $I$  in a Noetherian ring  $R$ , one can associate an algebra to  $I$  known as the Rees algebra  $\mathcal{R}(I)$  of  $I$ . This algebra  $\mathcal{R}(I) = \bigoplus_{i \geq 0} I^i t^i$  is a subalgebra of  $R[t]$ , where  $t$  is an indeterminate. It was introduced by Rees in 1956 in order to prove what is now known as the Artin-Rees Lemma [34]. Geometrically, the Rees algebra corresponds to the blowup of  $\text{Spec}(R)$  along  $V(I)$ . If  $R$  is local with maximal ideal  $\mathfrak{m}$  or graded with homogeneous maximal ideal  $\mathfrak{m}$ , the special fiber ring of  $I$  is the algebra  $\mathcal{F}(I) = \mathcal{R}(I) \otimes R/\mathfrak{m}$ . This algebra corresponds to the special fiber of the blowup of  $\text{Spec}(R)$  along  $V(I)$ . Besides its connections to resolution of singularities, the study of Rees algebras plays an important role in many other active areas of research including multiplicity theory, equisingularity theory, asymptotic properties of ideals, and integral dependence.

Although blowing up is a fundamental operation in the study of birational varieties, an explicit understanding of this process remains an open problem. In particular, a key objective in this area is to express the Rees algebra and the special fiber ring as quotients of a polynomial ring, henceforth to determine their defining ideals.

This question is wide open even for the simplest classes of ideals, including ideals generated by forms of the same degree in a polynomial ring. These are precisely the ideals generated by forms parametrizing a variety in projective space. The implicit equations of these varieties can be obtained from the defining ideal of the Rees ring. Indeed, the bihomogeneous coordinate ring of the graph of the morphism defined by the forms is the Rees algebra of the ideal  $I$ . The homogeneous coordinate ring of the variety parametrized by the forms is the special fiber ring.

As the graph of a map carries more information than its image, even a partial understanding of the Rees ring such as the bigraded degrees of its defining equations, the Betti

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*Key words and phrases.* Rees algebra, special fiber ring, determinantal ideal, sparse matrix, toric ring, Koszul algebra, ladder determinantal ring, SAGBI basis, Gröbner basis, Plücker relations.

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numbers, or the regularity of the defining ideal can be instrumental to the study of the variety. Determining the defining equations of the Rees algebra is a difficult problem in elimination theory, studied by commutative algebraists, algebraic geometers, and applied mathematicians in geometric modeling, see e.g. [7, 8, 9, 14, 37]. Answers to these questions also have applications to the study of chemical reaction networks [15].

The goal of this paper is to determine the defining equations of the Rees algebra and of the special fiber ring of the ideals generated by the maximal minors of sparse  $2 \times n$  matrices. Sparse matrices are matrices whose entries are either zeroes or distinct variables. Their degeneracy loci were first studied by Giusti and Merle in the 80's. In [23] they compute the codimension of their defining ideals and characterize when these ideals are prime or Cohen-Macaulay. Boocher in 2012 proved in [2] that a minimal free resolution of the ideals of maximal minors of sparse matrices can be obtained from the Eagon-Northcott complex via a pruning method. In the same paper, he shows that the natural generators form a universal Gröbner basis.

In the case of a generic matrix, that is a matrix whose entries are distinct variables, the special fiber ring of the ideal of maximal minors is the coordinate ring of a Grassmannian variety. The fact that the Plücker relations define the Grassmannian variety is a classical theorem, see e.g. [6]. The Rees algebra and the special fiber ring of ideals of maximal minors of generic matrices are Algebras with Straightening Laws (ASLs) in the sense of [16], see [18, 19]. Since the straightening relations come from the Plücker relations and the defining equations of the symmetric algebra, it follows that  $\mathcal{R}(I)$  is of fiber type, see [16, 18], [5, Lemma 2.2.1], [6, Proposition 4.2].

In addition, as the posets defining  $\mathcal{R}(I)$  and  $\mathcal{F}(I)$  are wonderful in the sense of [19], it follows that both algebras are Cohen-Macaulay, see [19, Proposition 2.6]. The normality of  $\mathcal{F}(I)$  follows immediately from the Cohen-Macaulay property since the Grassmannian variety is smooth. Trung in [40] proved that the powers and symbolic powers of  $I$  coincide and therefore  $\mathcal{R}(I)$  is normal. From the deformation theorem developed in [12], see for instance [4, Proposition 3.6], one can see that the Rees algebra of the ideal of maximal minors is defined by a Gröbner basis of quadrics. The same statement holds for  $\mathcal{F}(I)$ . Therefore, in the generic case both  $\mathcal{R}(I)$  and  $\mathcal{F}(I)$  are Koszul algebras and according to [3] the ideal  $I$  and all its powers have a linear resolution.

Our main result shows that the above properties of the blowup algebras of ideals of maximal minors of a generic matrix still hold in the case of sparse  $2 \times n$  matrices.

Now let  $I$  be the ideal of maximal minors of a  $2 \times n$  sparse matrix. Inspired by the pioneering work of Conca, Herzog, and Valla [12], we study the initial algebra of the Rees algebra of  $I$ . Our main technique is SAGBI bases [30, 35], an analogue for algebras of Gröbner bases for ideals.

This approach was successfully used to study the Rees algebras of other families of ideals [1, 12, 32].

First we prove that the initial algebra of the Rees algebra of  $I$  is the Rees algebra of the initial ideal of  $I$  with respect to a suitable order (see Theorem 3.3). Using deformation theory, we transfer properties from the Rees algebra of the initial ideal to the Rees algebra of the ideal itself. One advantage of this approach is that it allows us to reduce to the study

of the Rees algebra of the initial ideal, which is not just a monomial algebra, but also the Rees algebra of the edge ideal of a graph and a ladder determinantal ring. These objects have been studied extensively and one can draw a plethora of information that allows us to describe these algebras in full detail, see among others [10, 13, 36, 41, 42, 43].

A key step in our proof that  $\mathcal{R}(\text{in}_\tau(I))$  is the initial algebra of  $\mathcal{R}(I)$  is Lemma 3.2, where we prove that taking the initial ideal commutes with powers. The main idea behind the proof is a comparison of the Hilbert functions of  $I^2$  and  $(\text{in}_\tau(I))^2$ , an approach which was first used in [25]. We then use a lifting technique to obtain the defining equations of the Rees algebra and of the special fiber ring. Interestingly, they turn out to be the specialization of the defining equations of the Rees algebra and of the special fiber ring in the generic case.

The general question of understanding the Rees algebra and the special fiber ring of the ideal  $I$  of maximal minors of a sparse  $m \times n$  matrix is still open. In Remark 4.11 we propose a different approach, which applies to sparse matrices whose zero region has a special shape. These sparse matrices are exactly those that have the property that a maximal minor is non-zero if and only if the product of the elements on its diagonal is non-zero. This yields a nice combinatorial description of the initial ideal of  $I$ . Our arguments allow us to compute the equations of the special fiber ring and the Rees algebra and to establish algebraic properties such as normality, Cohen-Macaulayness, and Koszulness. We conjecture that these properties hold in general.

Our main results are summarized in the following.

**Theorem 1.1.** *Let  $X$  be a sparse  $2 \times n$  matrix and  $I = I_2(X)$ .*

- (a) *The defining ideal of  $\mathcal{F}(I)$  is generated by the Plücker relations on the  $2 \times 2$ -minors of  $X$  and these form a Gröbner basis of the ideal they generate.*
- (b)  *$\mathcal{R}(I)$  is of fiber type, that is, its defining ideal is generated by the relations of the symmetric algebra of  $I$  and by the Plücker relations on the  $2 \times 2$ -minors of  $X$ . Moreover, these equations form a Gröbner basis of the ideal they generate.*
- (c)  *$\mathcal{R}(I)$  and  $\mathcal{F}(I)$  have rational singularities in characteristic zero and they are  $F$ -rational in positive characteristic. In particular, they are Cohen-Macaulay normal domains.*
- (d)  *$\mathcal{R}(I)$  and  $\mathcal{F}(I)$  are Koszul algebras. In particular, the powers of  $I$  have a linear resolution.*
- (e) *The natural algebra generators of  $\mathcal{R}(I)$  and  $\mathcal{F}(I)$  are SAGBI bases of the algebras they generate.*

## 2. NOTATION

This section is devoted to the setup and notation we will use throughout the paper. Let  $K$  be a field,  $n \geq 3$  a positive integer, and  $X = (x_{ij})$  a  $2 \times n$  sparse matrix of indeterminates over  $K$ , i.e., a matrix whose entries  $x_{ij}$  are either distinct indeterminates or zero. We use the notation  $x_{i,j}$  when it is not clear what the two subscripts are. Let  $R = K[X]$  denote the polynomial ring over  $K$  in the variables appearing in  $X$  and let  $I$  be the ideal generated by the  $2 \times 2$  minors of  $X$ .

Up to permuting the rows and the columns of  $X$ , we may assume that there exist  $r$  and  $s$  with  $0 \leq n - r - s \leq r < n$  such that

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,r} & x_{1,r+1} & \cdots & x_{1,r+s} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & x_{2,r+1} & \cdots & x_{2,r+s} & x_{2,r+s+1} & \cdots & x_{2,n} \end{pmatrix}.$$

If  $r = 0$ , then  $n = s$  and  $X$  is a generic matrix. Since the results obtained in this paper are known in this case, we may assume without loss of generality that  $r \geq 1$ . Let  $f_{ij}$  denote the  $2 \times 2$  minor of  $X$  that involves columns  $i$  and  $j$ . Then

$$I = (f_{ij} \mid 1 \leq i < j \leq n) \subseteq R.$$

Notice that  $f_{ij} = -f_{ji}$  and that  $f_{ij}$  is a monomial or zero, unless  $r + 1 \leq i, j \leq r + s$ .

By the form of  $X$ , if a minor is non-zero, then the product of the entries on its diagonal is also non-zero. This means that we can choose a diagonal term order on  $K[X]$ , that is, a term order with the property that the leading term of each minor of  $X$  is the product of the elements on its diagonal. An example of a diagonal term order is the lexicographic order with  $x_{1,1} > \cdots > x_{1,r+s} > x_{2,r+1} > \cdots > x_{2,n}$ . The maximal minors of  $X$  form a diagonal Gröbner basis for  $I$  by [2, Proposition 5.4].

Throughout the paper, we fix a diagonal term order  $\tau$ . The minimal generating set for  $\text{in}_\tau(I)$  with respect to  $\tau$  is described next.

**Proposition 2.1** ([2], Proposition 5.4). *Let  $X$  be a sparse  $2 \times n$  matrix,  $I$  the ideal of  $2 \times 2$  minors of  $X$ , and  $\tau$  a diagonal term order. The initial ideal of  $I$  is*

$$\text{in}_\tau(I) = (x_{1,i}x_{2,j} \mid 1 \leq i \leq r + s, \max\{r, i\} < j \leq n).$$

It turns out that  $\text{in}_\tau(I)$  corresponds to a *Ferrers diagram* as in Figure 1.

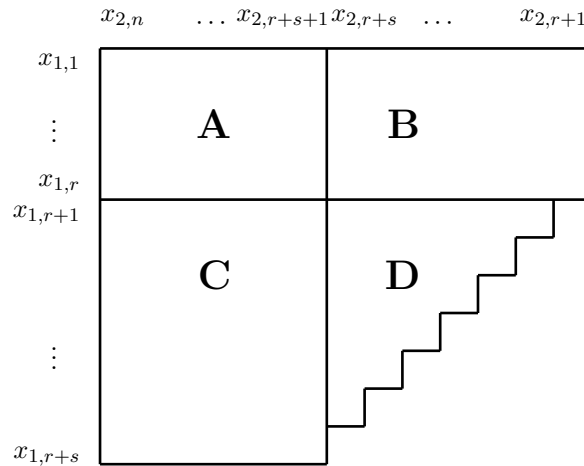


FIGURE 1. Ferrers diagram

**Corollary 2.2.** *The ideal  $\text{in}_\tau(I)$  is the edge ideal of a Ferrers bipartite graph whose vertex sets are*

$$V = \{x_{1,1}, \dots, x_{1,r+s}\}, \quad W = \{x_{2,n}, \dots, x_{2,r+1}\},$$

and whose partition is

$$\lambda = (\underbrace{n-r, \dots, n-r}_{r \text{ times}}, n-r-1, n-r-2, \dots, n-r-s).$$

In other words, the first  $r$  elements of  $V$  have edges connecting them to all  $n-r$  elements of  $W$ . Moreover, for  $i > r$ , the  $i$ th element of  $V$  is connected by an edge to the first  $n-i$  elements of  $W$ .

We say that  $\text{in}_\tau(I)$  is the *Ferrers ideal*  $I_\lambda$ . See [13] for more on Ferrers graphs, diagrams, and ideals.

**Example 2.3.** For the matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & 0 & 0 \\ 0 & 0 & 0 & x_{24} & x_{25} & x_{26} & x_{27} & x_{28} & x_{29} \end{pmatrix}$$

the ideal  $\text{in}_\tau(I)$  is the Ferrers ideal  $I_\lambda$  for the partition  $\lambda = (6, 6, 6, 5, 4, 3, 2)$ . Its Ferrers diagram is depicted in Figure 2.

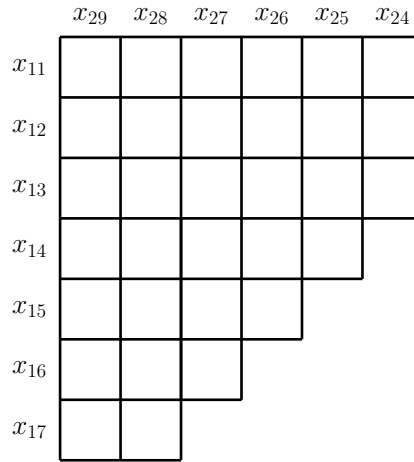


FIGURE 2. Ferrers diagram for  $\lambda = (6, 6, 6, 5, 4, 3, 2)$

**Definition 2.4** ([13]). The *one-sided ladder associated to a Ferrers ideal*  $I_\lambda$  is the ladder  $L_\lambda$  with the same shape as the Ferrers diagram for  $\lambda$  and for which the entry in row  $x_{1i}$  and column  $x_{2j}$  is  $y_{ij}$ .

The *ladder associated to*  $\text{in}_\tau(I) = I_\lambda$  is the two-sided ladder  $L'_\lambda$  obtained from  $L_\lambda$  by adding a row of  $n-r$  boxes at the top and a column of  $r+s$  boxes on the left. The new boxes are filled with the variables from  $W$  and  $V$ , respectively, in their listed order.

Illustrations of  $L_\lambda$  and  $L'_\lambda$  for  $\lambda = (6, 6, 6, 5, 4, 3, 2)$  are in Figure 3.

[illegible]

FIGURE 3. Ladders  $L_\lambda$  and  $L'_\lambda$  for  $\lambda = (6, 6, 6, 5, 4, 3, 2)$

Notice that the  $x_{ij}$  are only row and column markers in  $L_\lambda$ , whereas in  $L'_\lambda$  they are entries of the ladder. The entries in  $L_\lambda$  and in  $L'_\lambda$  are distinct variables. Therefore, results for ladder determinantal ideals apply, see for instance [10, 33].

### 3. REES ALGEBRAS OF $I$ AND OF ITS INITIAL IDEAL

The Rees algebra of an ideal can be realized as a quotient of a polynomial ring. When  $I$  is an ideal generated by  $n$  elements, say  $f_1, \dots, f_n$ , we let  $y_1, \dots, y_n$  be variables over  $R$  and consider the map from  $R[y_1, \dots, y_n]$  to  $\mathcal{R}(I)$  that maps  $y_i$  to  $f_i t$ . Hence  $\mathcal{R}(I) \cong R[y_1, \dots, y_n]/J$ , where  $J$  is the *defining ideal* of the algebra, and the *defining equations* are a system of generators of  $J$ . The defining equations of the Rees algebra are in general difficult to compute or determine theoretically. The largest  $y$ -degree of a minimal generator of the defining ideal  $J$  is the *relation type* of the ideal and plays an important role in the study of blowup algebras. Finally, we say that the Rees algebra of  $I$  is of *fiber type* if the defining ideal of the Rees algebra is generated by the defining equations of the symmetric algebra and the defining equations of the special fiber ring.

Given a term order  $\tau$  on a polynomial ring  $R$  over a field  $K$ , one can extend it to a term order  $\tau'$  on  $R[t]$  as follows. Let  $a, b \in R$  be monomials and let  $i, j$  be non-negative integers. Define

$$(3.1) \quad at^i <_{\tau'} bt^j \Leftrightarrow i < j \quad \text{or} \quad i = j \text{ and } a <_{\tau} b.$$

For a  $K$ -subalgebra  $A$  of  $R[t]$ , one defines  $\text{in}_{\tau'}(A)$  as the  $K$ -algebra generated by all initial monomials of elements in  $A$ . When  $A$  is homogeneous in the variable  $t$ , then  $\text{in}_{\tau'}(A) = \bigoplus_{i \geq 0} \text{in}_{\tau'}(A_i)$ , where  $A_i$  is the set of elements of  $A$  that are homogeneous of degree  $i$  in  $t$ . In particular, in our setting

$$\mathrm{in}_{\tau'}(\mathcal{R}(I)) = \oplus_{i>0} \mathrm{in}_{\tau'}(\mathcal{R}(I)_i) = \oplus_{i>0} \mathrm{in}_{\tau}(I^i)t^i.$$

Since  $\mathcal{R}(\text{in}_\tau(I)) = \bigoplus_{i \geq 1} (\text{in}_\tau(I))^i t^i$ , in order to prove that  $\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$ , it suffices to show that  $(\text{in}_\tau(I))^i = \text{in}_\tau(I^i)$  for all  $i \geq 1$ . A result by Conca, Herzog, and Valla states that it suffices to check the equality up to the relation type of  $\text{in}_\tau(I)$ .

**Theorem 3.1.** [12, Corollary 2.8] *Let  $R$  be a polynomial ring,  $I$  a homogeneous ideal in  $R$ , and  $\tau$  a term order on  $R$ . Suppose that  $(\text{in}_\tau(I))^i = \text{in}_\tau(I^i)$  for  $1 \leq i \leq \text{reltype}(\text{in}_\tau(I))$ . Then*

$$(\text{in}_\tau(I))^i = \text{in}_\tau(I^i) \text{ for all } i \geq 1 \quad \text{and} \quad \text{reltype}(I) \leq \text{reltype}(\text{in}_\tau(I)).$$

The equality  $\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$  was established by Conca in [11, Theorem 2.1] for the ideal  $I$  of maximal minors of a generic  $m \times n$  matrix. The following lemma is the key to establishing  $\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$  in our case.

**Lemma 3.2.** *Let  $X$  be a sparse  $2 \times n$  matrix,  $I$  the ideal of  $2 \times 2$  minors of  $X$ , and  $\tau$  a diagonal term order on  $R = K[X]$ . Then  $\text{in}_\tau(I^2) = (\text{in}_\tau(I))^2$ .*

*Proof.* We proceed by induction on  $n \geq 2$ . For  $n = 2$ , the ideal  $I$  is principal and the result holds automatically. To proceed we choose a decomposition of  $I$  as follows.

Let  $I_1$  be the ideal generated by the variables  $x_{2j}$  for  $r+1 \leq j \leq n$  and let  $I_2$  be the ideal generated by the  $2 \times 2$  minors of the submatrix of  $X$  obtained by deleting the first column of  $X$ . Since  $r \geq 1$ , then  $I = x_{11}I_1 + I_2$ . By Proposition 2.1,  $\text{in}_\tau(I) = x_{11}I_1 + \text{in}_\tau(I_2)$ . Notice here that for  $I_2$  we may have that its corresponding  $r$  is 0. In that case  $\text{in}_\tau(I_2^2) = (\text{in}_\tau(I_2))^2$  by [11, Theorem 2.1]. Otherwise, the equality holds by induction.

Certainly  $(\text{in}_\tau(I))^2 \subseteq \text{in}_\tau(I^2)$ . To prove equality it is enough to show that the Hilbert functions of  $(\text{in}_\tau(I))^2$  and  $I^2$  are the same, since the Hilbert function of  $I^2$  is the same as the Hilbert function of  $\text{in}_\tau(I^2)$ .

By induction on  $n$ , the Hilbert functions of  $I_2^2$  and  $(\text{in}_\tau(I_2))^2$  are the same. We have

$$I^2 = (x_{11}^2 I_1^2 + x_{11} I_1 I_2) + I_2^2 \quad \text{and} \quad (\text{in}_\tau(I))^2 = (x_{11}^2 I_1^2 + x_{11} I_1 \text{in}_\tau(I_2)) + (\text{in}_\tau(I_2))^2.$$

Both of these ideals are of the form  $J^2 = (x_{11}^2 I_1^2 + x_{11} I_1 J_2) + J_2^2$ , with  $J_2 = I_2$  and  $J = I$  in the former case and with  $J_2 = \text{in}_\tau(I_2)$  and  $J = \text{in}_\tau(I)$  in the latter case. Each case gives rise to the short exact sequence:

$$(3.2) \quad 0 \longrightarrow (x_{11}^2 I_1^2 + x_{11} I_1 J_2) \cap J_2^2 \longrightarrow (x_{11}^2 I_1^2 + x_{11} I_1 J_2) \oplus J_2^2 \longrightarrow J^2 \longrightarrow 0.$$

Since the variable  $x_{11}$  is a non-zerodivisor on  $J_2^2$ , and since  $J_2 \subseteq I_1$ , the intersection on the left-hand side of the sequence is:

$$\begin{aligned} (x_{11}^2 I_1^2 + x_{11} I_1 J_2) \cap J_2^2 &= (x_{11}^2 I_1^2 + x_{11} I_1 J_2) \cap (x_{11}) \cap J_2^2 \\ &= (x_{11}^2 I_1^2 + x_{11} I_1 J_2) \cap x_{11} J_2^2 \\ &= x_{11} J_2^2. \end{aligned}$$

This means that the short exact sequence simplifies to

$$(3.3) \quad 0 \longrightarrow x_{11} J_2^2 \longrightarrow (x_{11}^2 I_1^2 + x_{11} I_1 J_2) \oplus J_2^2 \longrightarrow J^2 \longrightarrow 0.$$

By the induction hypothesis, the two incarnations of  $J_2^2$  and hence of  $x_{11} J_2^2$  have the same Hilbert function, so that to prove that the two incarnations  $I^2$  and  $(\text{in}_\tau(I))^2$  of  $J^2$  have the



same Hilbert function it suffices to prove that the two incarnations of  $x_{11}^2 I_1^2 + x_{11} I_1 J_2$  have the same Hilbert function. As before we get a short exact sequence:

$$(3.4) \quad 0 \longrightarrow x_{11} I_1^2 \cap I_1 J_2 \longrightarrow x_{11} I_1^2 \oplus I_1 J_2 \longrightarrow x_{11} I_1^2 + I_1 J_2 \longrightarrow 0,$$

and since  $x_{11}$  is a non-zerodivisor on  $I_1 J_2$  and  $J_2 \subseteq I_1$ , the short exact sequence simplifies to

$$(3.5) \quad 0 \longrightarrow x_{11} I_1 J_2 \longrightarrow x_{11} I_1^2 \oplus I_1 J_2 \longrightarrow x_{11} I_1^2 + I_1 J_2 \longrightarrow 0.$$

The conclusion will follow once we show that in the two incarnations of  $J_2$  the Hilbert function of  $I_1 J_2$  is the same, that is  $\text{in}_\tau(I_1 I_2) = I_1 \text{in}_\tau(I_2)$ . Clearly,  $I_1 \text{in}_\tau(I_2) \subseteq \text{in}_\tau(I_1 I_2)$ . Define a grading on  $R$  by setting  $\deg(x_{1i}) = 0$  and  $\deg(x_{2j}) = 1$  for all  $i, j$ . With this grading,  $I_1$  is generated by elements of degree 1,  $I_2$  is generated by elements of degree 1, and  $I_1 I_2$  is generated by elements of degree 2. It suffices to show that for every homogeneous  $f \in I_1 I_2$ ,  $\text{in}_\tau(f) \in I_1 \text{in}_\tau(I_2)$ . Clearly  $\text{in}_\tau(f)$  has degree at least 2 and is in  $\text{in}_\tau(I_2)$ . By Proposition 2.1,  $\text{in}_\tau(I_2)$  is generated by elements of degree 1. Thus by degree reasons, to write  $\text{in}_\tau(f)$  as an element of  $\text{in}_\tau(I_2)$ , the coefficient must have degree at least 1, i.e., the coefficient must be in  $(x_{2i} \mid i = r+1, \dots, n) = I_1$ . It follows that  $\text{in}_\tau(f) \in I_1 \text{in}_\tau(I_2)$ .  $\square$

We are now ready to prove the main theorem of this section. Let  $A$  be a subalgebra of a polynomial ring over a field  $K$ . We recall that a subset  $C$  of  $A$  is a *SAGBI basis* for  $A$  with respect to a term order  $\tau$  if  $\text{in}_\tau(A)$  is generated as a  $K$ -algebra by the initial monomials of the elements in  $C$  with respect to  $\tau$ . In general, SAGBI bases are difficult to compute and may not even be finite. When  $I$  is the ideal of maximal minors of a generic  $m \times n$  matrix  $X$ , Conca showed that  $\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$  with respect to a diagonal term order  $\tau$  [11, Theorem 2.1]. Moreover, Narasimhan [33, Corollary 3.4] showed that the maximal minors of  $X$  form a Gröbner basis for  $I$ . It then follows that the natural algebra generators of  $\mathcal{R}(I)$  are a SAGBI basis of it. The following theorem extends these results to the case of the ideal maximal minors of a sparse  $2 \times n$  matrix.

**Theorem 3.3.** *Let  $X$  be a sparse  $2 \times n$  matrix and let  $I$  be the ideal of  $2 \times 2$  minors of  $X$ . Let  $\tau$  be a diagonal term order on  $R = K[X]$  and let  $\tau'$  be its extension to  $R[t]$  as in (3.1). Then*

$$\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I)) \quad \text{and} \quad \text{reltype}(I) \leq 2.$$

*Moreover, the set  $\{x_{1i}, x_{2j} \mid 1 \leq i \leq r+s, r+1 \leq j \leq n\} \cup \{f_{ij}t \mid 1 \leq i < j \leq n\}$  is a SAGBI basis of  $\mathcal{R}(I)$  with respect to  $\tau'$ .*

*Proof.* According to [13, Proposition 5.1], the defining ideal of the special fiber ring of  $I_\lambda = \text{in}_\tau(I)$  is the ideal generated by the  $2 \times 2$  minors of  $L_\lambda$ . By [41, Theorem 3.1] we know that  $\mathcal{R}(\text{in}_\tau(I))$  is of fiber type and hence the relation type of  $\text{in}_\tau(I)$  is at most 2.

By Lemma 3.2 we have that  $\text{in}_\tau(I^2) = (\text{in}_\tau(I))^2$ , and thus by Theorem 3.1, the relation type of  $I$  is at most 2 and  $\text{in}_\tau(I^i) = (\text{in}_\tau(I))^i$  for all  $i$ . Therefore,

$$\text{in}_{\tau'}(\mathcal{R}(I)) = \bigoplus_{i \geq 0} \text{in}_\tau(I^i) = \bigoplus_{i \geq 0} (\text{in}_\tau(I))^i = \mathcal{R}(\text{in}_\tau(I)).$$

The last part of the claim now follows, since the  $2 \times 2$  minors of  $X$  are a Gröbner basis of  $I$  by [2, Proposition 5.4]. Therefore the set  $\{x_{1i}, x_{2j} \mid 1 \leq i \leq r+s, r+1 \leq j \leq n\} \cup \{f_{ij}t \mid 1 \leq i < j \leq n\}$  is a SAGBI basis of  $\mathcal{R}(I)$ , since the leading terms of these elements generate  $\mathcal{R}(\text{in}_\tau(I))$  as  $K$ -algebra and  $\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$ .  $\square$

An immediate consequence of the equality  $\text{in}_\tau(I^i) = (\text{in}_\tau(I))^i$  is the following corollary.

**Corollary 3.4.** *Let  $X$  be a sparse  $2 \times n$  matrix and let  $I$  the ideal of  $2 \times 2$  minors of  $X$ . Let  $\tau$  be a diagonal term order on  $R = K[X]$ . The  $i$ -fold products of maximal minors of  $X$  are a  $\tau$ -Gröbner basis of  $I^i$  for every  $i$ .*

Using the theory of SAGBI bases, one can now deduce properties of the Rees algebra of  $I$  from those of its initial algebra.

**Corollary 3.5.** *Let  $X$  be a sparse  $2 \times n$  matrix and let  $I$  the ideal of  $2 \times 2$  minors of  $X$ . Then  $\mathcal{R}(I)$  has rational singularities if the field  $K$  has characteristic 0 and it is  $F$ -rational if  $K$  has positive characteristic. In particular,  $\mathcal{R}(I)$  is a Cohen-Macaulay normal domain.*

*Proof.* Let  $\tau$  be a diagonal term order on  $R = K[X]$  and let  $\tau'$  be its extension to  $R[t]$  as in (3.1). By [36, Corollary 5.3 and Theorem 5.9],  $\mathcal{R}(\text{in}_\tau(I))$  is a Cohen-Macaulay normal domain. Thus  $\text{in}_{\tau'}(\mathcal{R}(I))$  is a Cohen-Macaulay normal domain by Theorem 3.3. The conclusion then follows from [12, Corollary 2.3].  $\square$

#### 4. THE DEFINING EQUATIONS OF THE REES ALGEBRA

In this section we obtain the defining equations of the Rees algebra of the ideal of  $2 \times 2$  minors of a sparse  $2 \times n$  matrix  $X$ . In addition to the setup and notation from Section 2, we will also adopt the following.

**Setting and Notation 4.1.** *Let  $S = K[L'_\lambda]$  and  $T = K[L_\lambda]$  be the polynomial rings over  $K$  in the variables that appear in  $L'_\lambda$  and  $L_\lambda$ , respectively. To simplify our notation, we make the identifications:*

- $x_{ui} = 0$  if  $u = 1$  and  $i > r + s$  or if  $u = 2$  and  $i \leq r$ ,
- $y_{ij} = 0$  if  $i, j \in \{1, \dots, r\}$  or  $i, j \in \{r + s + 1, \dots, n\}$ ,

to give meaning to all other  $x_{ui}, y_{ij}$  with  $u \in \{1, 2\}$ ,  $1 \leq i < j \leq n$ . Define the following standard presentations of the symmetric algebra  $\mathcal{S}(I)$ , the Rees algebra  $\mathcal{R}(I)$  of  $I$ , and the special fiber ring  $\mathcal{F}(I)$ :

$$\begin{aligned}\sigma : S &\longrightarrow \mathcal{S}(I), \\ \rho : S &\longrightarrow \mathcal{R}(I), \\ \varphi : T &\longrightarrow \mathcal{F}(I),\end{aligned}$$

where for all  $i, j$ ,  $\sigma(x_{ij}) = \rho(x_{ij}) = x_{ij}$ ,  $\sigma(y_{ij}) = \varphi(y_{ij}) = f_{ij}$ , and  $\rho(y_{ij}) = f_{ij}t$ . We let  $\mathcal{L} = \ker(\sigma)$ ,  $\mathcal{J} = \ker(\rho)$ , and  $\mathcal{K} = \ker(\varphi)$ . The ideals  $\mathcal{L}, \mathcal{J}, \mathcal{K}$  are called the defining ideals of the symmetric algebra, the Rees algebra, and the special fiber ring of  $I$ , respectively.

We similarly define the presentations of the symmetric algebra, of the Rees algebra, and of the special fiber ring of  $\text{in}_\tau(I)$ :

$$\begin{aligned}\sigma' : S &\longrightarrow \mathcal{S}(\text{in}_\tau(I)), \\ \rho' : S &\longrightarrow \mathcal{R}(\text{in}_\tau(I)) = \text{in}_{\tau'}(\mathcal{R}(I)), \\ \varphi' : T &\longrightarrow \mathcal{F}(\text{in}_\tau(I)),\end{aligned}$$

where for all  $i, j$ ,  $\sigma(x_{ij}) = \rho(x_{ij}) = x_{ij}$ ,  $\sigma(y_{ij}) = \varphi(y_{ij}) = \text{in}_\tau(f_{ij})$ , and  $\rho(y_{ij}) = \text{in}_\tau(f_{ij})t$ . Moreover, let  $\mathcal{L}' = \ker(\sigma')$ ,  $\mathcal{J}' = \ker(\rho')$ , and  $\mathcal{K}' = \ker(\varphi')$  denote the defining ideals of the symmetric algebra, the Rees algebra, and the special fiber of  $\text{in}_\tau(I)$ , respectively.

**Remark 4.2.** Let  $\ell_{uijk} = x_{ui}y_{jk} - x_{uj}y_{ik} + x_{uk}y_{ij}$  and  $p_{ijkl} = y_{ij}y_{kl} - y_{ik}y_{jl} + y_{il}y_{jk}$  for  $u \in \{1, 2\}$  and  $1 \leq i < j < k < l \leq n$ . With the identifications from the Setting and Notation 4.1,  $\ell_{uijk}, p_{ijkl}$  are in  $S$  for all  $u \in \{1, 2\}$ ,  $1 \leq i < j < k < l \leq n$ . We call the  $\ell_{uijk}$ 's the linear relations of  $I$  and the  $p_{ijkl}$ 's the Plücker relations of  $I$ . Clearly  $\ell_{uijk} \in \mathcal{L}$  and  $\ell_{uijk}, p_{ijkl} \in \mathcal{J}$  for all  $u \in \{1, 2\}$  and  $1 \leq i < j < k < l \leq n$ .

In the next proposition we describe the defining equations of the symmetric algebra, the special fiber, and the Rees algebra of  $\text{in}_\tau(I)$ . In addition, we show that the  $\ell_{uijk}$ 's are indeed the defining equations of the symmetric algebra of  $I$ .

**Proposition 4.3.** *Adopt Setting and Notation 4.1. Then*

- (a)  $\mathcal{L} = (\ell_{uijk} \mid u = 1, 2; 1 \leq i < j < k \leq n)$ .
- (b)  $\mathcal{L}'$  is generated by the  $2 \times 2$  minors of  $L'_\lambda$  that involve either the first column or the first row.
- (c)  $\mathcal{K}' = I_2(L_\lambda)$ .
- (d)  $\mathcal{J}' = I_2(L'_\lambda) = \mathcal{L}' + I_2(L_\lambda)S$ .

Moreover, the natural generators of  $I_2(L_\lambda)$  and  $I_2(L'_\lambda)$  are Gröbner bases of  $\mathcal{K}'$  and  $\mathcal{J}'$ , respectively, with respect to a diagonal term order.

*Proof.* (a) By [2, Theorem 4.1 and its proof] the presentation matrix of  $I$  is obtained from the Eagon-Northcott resolution by “pruning”. In particular, the relations on the generators  $f_{ij}$  of  $I$  arise from taking the  $3 \times 3$  minors of the  $3 \times n$  matrix obtained from  $X$  by doubling one of the two rows. These yield precisely the relations on the symmetric algebra of the given form.

Item (b) follows from [26, Theorem 5.1] since Ferrers ideals satisfy the  $\ell$ -exchange property, see e.g. [31, Lemma 6.3].

(c) This follows from [13, Proposition 5.1].

(d) Since  $\text{in}_\tau(I)$  is a Ferrers ideal, it is the edge ideal of a bipartite graph. Therefore the Rees algebra of  $\text{in}_\tau(I)$  is of fiber type by [41, Theorem 3.1].

The last part follows from the fact that  $I_2(L_\lambda)$  and  $I_2(L'_\lambda)$  are ladder determinantal ideals. Therefore, by [33, Corollary 3.4], the  $2 \times 2$  minors of  $L_\lambda$  and  $L'_\lambda$  are Gröbner bases of  $I_2(L_\lambda)$  and  $I_2(L'_\lambda)$ , respectively, with respect to a diagonal term order.  $\square$

The description of the defining equations of the Rees algebra of  $\text{in}_\tau(I)$ , in combination with our result from the previous section that shows that  $\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$ , allows us to deduce that the Rees algebra of  $I$  is a Koszul algebra.

**Corollary 4.4.** *Let  $X$  be a sparse  $2 \times n$  matrix,  $I$  the ideal of  $2 \times 2$  minors of  $X$ . Then  $\mathcal{R}(I)$  is a Koszul algebra and  $I$  has linear powers. In particular,  $\text{reg}(I^k) = 2k$  for all  $k \in \mathbb{N}$ .*

*Proof.* The defining ideal of  $\mathcal{R}(\text{in}_\tau(I))$  has a Gröbner basis of quadrics by Proposition 4.3, so  $\mathcal{R}(\text{in}_\tau(I))$  is a Koszul algebra, see [21] and [20, Theorem 6.7]. By Theorem 3.3 we have  $\text{in}_\tau(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$  and since  $\mathcal{R}(\text{in}_\tau(I))$  is a Koszul algebra, then  $\mathcal{R}(I)$  is a Koszul algebra, by [12, Corollary 2.6]. According to Blum [3, Corollary 3.6], if the Rees algebra of an ideal is Koszul, then the ideal has linear powers, i.e., the powers of the ideal have a linear resolution.  $\square$

We now are ready to prove the main result of this article.

**Theorem 4.5.** *Let  $X$  be a sparse  $2 \times n$  matrix,  $I$  the ideal of  $2 \times 2$  minors of  $X$ . Adopt Setting and Notation 4.1.*

- (a) *The defining ideal  $\mathcal{J}$  of  $\mathcal{R}(I)$  is generated by the linear relations  $\ell_{uijk}$  for  $u \in \{1, 2\}$  and  $1 \leq i < j < k \leq n$ , and the Plücker relations  $p_{ijkl}$  for  $1 \leq i < j < k < l \leq n$ . Moreover, these generators form a Gröbner basis of  $\mathcal{J}$  with respect to a suitable weight.*
- (b) *The Rees algebra  $\mathcal{R}(I)$  is of fiber type, i.e.,*

$$\mathcal{J} = \mathcal{L} + \mathcal{K}S.$$

- (c) *The Plücker relations  $p_{ijkl}$  for  $1 \leq i < j < k < l \leq n$  are the defining equations of  $\mathcal{F}(I)$ .*

*Proof.* (a) We let  $G$  be the set of all  $\ell_{uijk}$  and  $p_{ijkl}$  as in Remark 4.2. We claim that  $\mathcal{J} = (G)$ . Define a weight  $\omega$  on  $R[t]$  and a weight  $\pi$  on  $S$  as follows:  $\omega(x_{1j}) = 1$ ,  $\omega(x_{2j}) = j$ ,  $\omega(t) = 1$ ,  $\pi(x_{1j}) = 1$ ,  $\pi(x_{2j}) = j$ ,  $\pi(y_{ij}) = \omega(\text{in}_\tau(f_{ij})t) = j + 2$ .

By Proposition 4.3,  $I_2(L'_\lambda)$  is the defining ideal of  $\mathcal{R}(\text{in}_\tau(I)) = \text{in}_{\tau'}(\mathcal{R}(I))$ . The weight  $\omega$  represents the term order  $\tau$ , that is,  $\text{in}_\omega(f_{ij}) = \text{in}_\tau(f_{ij})$ . Therefore, by [38, Proposition in Lecture 3.1] (see also [39, Theorem 11.4]) we conclude that  $I_2(L'_\lambda) = \text{in}_\pi(\mathcal{J})$ .

Since  $G \subseteq \mathcal{J}$ , to prove that  $G$  is a Gröbner basis of  $\mathcal{J}$  with respect to  $\pi$ , it suffices to prove that each  $2 \times 2$  minor of  $L'_\lambda$  is the leading form with respect to  $\pi$  of some element in  $G$ . This also implies that  $\mathcal{J}$  is generated by  $G$ .

We first analyze the  $2 \times 2$  minors of  $L'_\lambda$  that involve the first column. These are of the form  $E_{1ijk} = x_{1i}y_{jk} - x_{1j}y_{ik}$  with  $i < j < k$  and they are homogeneous of weight  $k + 3$ . Since  $x_{1k}y_{ij}$  has weight  $j + 3$ ,  $E_{1ijk}$  is the leading form of  $\ell_{1ijk} \in G$ .

We next analyze the  $2 \times 2$  minors of  $L'_\lambda$  that involve the first row. These are of the form  $E_{2ijk} = x_{2k}y_{ij} - x_{2j}y_{ik}$  with  $i < j < k$  and they are homogeneous of weight  $j + k + 2$ . Since  $x_{2i}y_{jk}$  has weight  $i + k + 2$ ,  $E_{2ijk}$  is the leading form of  $\ell_{2ijk} \in G$ .

It remains to prove that each  $2 \times 2$  minor of  $L_\lambda$  is a unit multiple of a leading form of an element in  $G$ . Such a minor is of the form  $F_{ijkl} = y_{il}y_{jk} - y_{ik}y_{jl}$  for some  $i < j < k < l$  and it is homogeneous of weight  $k + l + 4$ . Since  $y_{ij}y_{kl}$  has weight  $j + l + 4$ ,  $F_{ijkl}$  is the leading form of  $p_{ijkl} \in G$ .

Items (b) and (c) follow immediately from (a).  $\square$

The following corollary is a direct consequence of Theorem 4.5.

**Corollary 4.6.** *Let  $X$  be a sparse  $2 \times n$  matrix,  $I$  the ideal of  $2 \times 2$  minors of  $X$ , and  $\tau$  a diagonal term order on  $R = K[X]$ . Then  $\text{in}_\tau(\mathcal{F}(I)) = \mathcal{F}(\text{in}_\tau(I))$ . Moreover, the  $2 \times 2$*

minors of  $X$  are a SAGBI basis of the  $K$ -algebra  $\mathcal{F}(I)$ , the Plücker relations form a Gröbner basis for  $\mathcal{K}$ , and  $\mathcal{F}(I)$  is a Koszul algebra.

*Proof.* By Proposition 4.3, the defining equations of  $\mathcal{F}(\text{in}_\tau(I))$  are the  $2 \times 2$  minors of  $L_\lambda$ , i.e.,  $\mathcal{F}(\text{in}_\tau(I)) = K[\text{in}_\tau(f_{ij})] \cong T/I_2(L_\lambda)$ . For any  $i < j < k < l$ , let  $F_{ijkl} = y_{il}y_{jk} - y_{ik}y_{jl}$  be a generator of  $I_2(L_\lambda)$ . In the proof of Theorem 4.5 we showed that  $p_{ijkl} = F_{ijkl} + y_{ij}y_{kl}$ . Hence  $\varphi(F_{ijkl}) = -\varphi(y_{ij}y_{kl})$  and  $\text{in}_\tau \varphi(F_{ijkl}) = \text{in}_\tau \varphi(y_{ij}y_{kl})$ . Then by [12, Proposition 1.1] the generators of  $I$  are a SAGBI basis of  $\mathcal{F}(I)$ , in particular  $\text{in}_\tau(\mathcal{F}(I)) = \mathcal{F}(\text{in}_\tau(I))$ . Moreover, the Plücker relations form a Gröbner basis of  $\mathcal{K}$  by [39, Corollary 11.6], hence  $\mathcal{F}(I)$  is a Koszul algebra.  $\square$

Finally, we obtain the analogous result for  $\mathcal{F}(I)$  as in Corollary 3.5.

**Corollary 4.7.** *Let  $X$  be a sparse  $2 \times n$  matrix and  $I$  the ideal of  $2 \times 2$  minors of  $X$ . Then  $\mathcal{F}(I)$  has rational singularities if the field  $K$  has characteristic 0 and is  $F$ -rational if  $K$  has positive characteristic. In particular,  $\mathcal{F}(I)$  is a Cohen-Macaulay normal domain.*

*Proof.* By [36, Theorem 7.1] and Corollary 3.5,  $\mathcal{F}(\text{in}_\tau(I))$  is normal. Thus, by [28, Theorem 1]  $\mathcal{F}(\text{in}_\tau(I))$  is Cohen-Macaulay as well. By Corollary 4.6, we have  $\text{in}_\tau(\mathcal{F}(I)) = \mathcal{F}(\text{in}_\tau(I))$ . The result now follows from [12, Corollary 2.3].  $\square$

**Corollary 4.8.** *Let  $X$  be a sparse  $2 \times n$  matrix and  $I$  the ideal of  $2 \times 2$  minors of  $X$ . Then*

- (a)  $\mathcal{F}(I)$  has dimension  $\min\{n + s - 1, 2n - r - 2\}$ .
- (b)  $\mathcal{F}(I)$  is Gorenstein if  $r \leq 2$ .
- (c)  $\text{reg}(\mathcal{F}(I)) = \min\{n - 3, n - r - 1, r + s - 1\}$ .
- (d) The  $a$ -invariant of  $\mathcal{F}(I)$  is  $a(\mathcal{F}(I)) = \min\{-r - s, r - n, -s - 2\}$ , unless  $n = r + s$  in which case  $a(\mathcal{F}(I)) = -n$  if  $r = 1$  or  $a(\mathcal{F}(I)) = -n + 1$  otherwise.

*Proof.* (a) The formula follows from [12, Corollary 2.6] in combination with [36] or [10, Section 4].

(b) By [10, Proposition 2.5]  $\mathcal{F}(\text{in}_\tau(I))$  is Gorenstein if and only if  $r \leq 2$ . The statement now follows from [12, Corollary 2.3].

(c) From [13, Proposition 5.7] we obtain the regularity of  $\mathcal{F}(\text{in}_\tau(I))$ , which simplifies to  $\text{reg}(\mathcal{F}(\text{in}_\tau(I))) = \min\{n - 3, n - r - 1, r + s - 1\}$ . By [12, Corollary 2.5] we have  $a(\mathcal{F}(\text{in}_\tau(I))) = a(\mathcal{F}(I))$  and since  $\mathcal{F}(\text{in}_\tau(I))$  is Cohen-Macaulay, then  $\text{reg}(\mathcal{F}(I)) = \text{reg}(\mathcal{F}(\text{in}_\tau(I)))$ .

(d) The result follows from the fact that  $a(\mathcal{F}(I)) = -\dim \mathcal{F}(I) + \text{reg}(\mathcal{F}(I))$ . Alternatively, the formula is a special case of [22, Corollary 9]. In the special case  $n = r + s$ , we have  $\dim \mathcal{F}(I) = 2n - r - 2$  and  $\text{reg}(\mathcal{F}(I)) = \min\{n - 3, n - r - 1\}$ .  $\square$

**Corollary 4.9.** *Let  $X$  be a sparse  $2 \times n$  matrix and  $I$  the ideal of  $2 \times 2$  minors of  $X$ . Let  $\lambda$  be the partition for the Ferrers ideal  $\text{in}_\tau(I_\lambda)$ . Then the normalized Hilbert series of  $\mathcal{F}(I)$  is  $p_\lambda(z) = 1 + h_1(\lambda)z + \dots + h_{r+s-1}(\lambda)z^{r+s-1}$ , where  $h_k(\lambda) = \binom{r+s-1}{k} \binom{n-r-1}{k} - \binom{s+1}{k+1} \binom{n-3}{k-1}$  with the convention that  $\binom{j}{i} = 0$  if  $j < i$ . Moreover, the multiplicity of  $\mathcal{F}(I)$  is  $e(\mathcal{F}(I)) = \binom{n+s-2}{n-r-1} - \binom{n+s-2}{n-1}$ .*

*Proof.* First notice that the  $h$ -vectors for  $\mathcal{F}(I)$  and  $\mathcal{F}(\text{in}_\tau(I))$  coincide and in particular,  $e(\mathcal{F}(I)) = e(\mathcal{F}(\text{in}_\tau(I)))$  by [12, Corollary 2.5]. Since  $\mathcal{F}(\text{in}_\tau(I))$  is a ladder determinantal ring the formula for the Hilbert series is obtained in [42, Theorem 4.7]. One can then deduce the formula for the multiplicity immediately. The formula for the multiplicity is also worked out explicitly in [43, Corollary 4.2].  $\square$

**Corollary 4.10.** *Let  $X$  be a sparse  $2 \times n$  matrix and  $I$  the ideal of  $2 \times 2$  minors of  $X$ . Then*

- (a) *The regularity of  $\mathcal{R}(I)$  is  $\text{reg}(\mathcal{R}(I)) = \min\{n-1, n-r, r+s\}$ .*
- (b) *The  $a$ -invariant of  $\mathcal{R}(I)$  is  $a(\mathcal{R}(I)) = \min\{-s-2, -s-r-1, r-n-1\}$ .*
- (c) *The normalized Hilbert series of  $\mathcal{R}(I)$  is  $p_\lambda(z) = 1 + h_1(\lambda)z + \dots + h_{r+s}(\lambda)z^{r+s}$ , where  $h_1(\lambda) = (r+s)(n-r) - \binom{s+1}{2} - 1$  and  $h_k(\lambda) = \binom{r+s}{k} \binom{n-r}{k} - \binom{s+1}{k+1} \binom{n-1}{k-1}$  for  $k \neq 1$ , with the convention that  $\binom{j}{i} = 0$  if  $j < i$ .*
- (d) *The multiplicity of  $\mathcal{R}(I)$  is  $e(\mathcal{R}(I)) = \binom{n+s}{n-r} - \binom{n+s}{n+1} - 1$ .*

*Proof.* Let

$$\lambda = (\underbrace{n-r, \dots, n-r}_{r \text{ times}}, n-r-1, n-r-2, \dots, n-r-s),$$

$$\mu = (\underbrace{n-r+1, \dots, n-r+1}_{r+1 \text{ times}}, n-r, n-r-1, \dots, n-r-s+1),$$

and let  $I_2(L'_\lambda)$  be the defining ideal of  $\mathcal{R}(\text{in}_\tau(I))$ . By [24, Theorem 2.1], the ideal  $I_2(L_\mu)$  is obtained from the ideal  $I_1(L_\lambda)$  generated by the entries of  $L_\lambda$  by an ascending G-biliaison of height 1 on  $I_2(L'_\lambda)$ . Specifically, one has

$$(4.1) \quad yI_2(L_\mu) + I_2(L'_\lambda) = fI_1(L_\lambda) + I_2(L'_\lambda),$$

where  $y$  is the variable appearing in position  $(2, 2)$  of  $L_\mu$  and  $f$  is the  $2 \times 2$ -minor of the first two rows and columns of  $L_\mu$ .

(a) By [12, Corollary 2.5] we have  $a(\mathcal{R}(\text{in}_\tau(I))) = a(\mathcal{R}(I))$  and since  $\mathcal{R}(\text{in}_\tau(I))$  and  $\mathcal{R}(I)$  are Cohen-Macaulay, then  $\text{reg}(\mathcal{R}(I)) = \text{reg}(\mathcal{R}(\text{in}_\tau(I)))$ . Since  $\text{reg}(K[L_\mu]/I_1(L_\lambda)) = 0$ , then

$$\text{reg}(\mathcal{R}(I)) = \text{reg}(K[L_\mu]/I_2(L_\mu)) = \min\{n-1, n-r, r+s\},$$

where the first equality follows from [17, Theorem 3.1] and the second from Corollary 4.8.

(b) The formula for the  $a$ -invariant of  $\mathcal{R}(I)$  now follows from the fact that  $a(\mathcal{R}(I)) = -\dim \mathcal{R}(I) + \text{reg}(\mathcal{R}(I)) = \min\{-s-2, -s-r-1, r-n-1\}$ .

We now prove (c) and (d) together. The normalized Hilbert series of  $\mathcal{R}(I)$  and  $\mathcal{R}(\text{in}_\tau(I))$  coincide and in particular  $e(\mathcal{R}(I)) = e(\mathcal{R}(\text{in}_\tau(I)))$  by [12, Corollary 2.5]. Using the short exact sequences

$$0 \longrightarrow I_2(L'_\lambda)(-1) \longrightarrow I_2(L_\mu)(-1) \oplus I_2(L'_\lambda) \longrightarrow yI_2(L_\mu) + I_2(L'_\lambda) \longrightarrow 0$$

and

$$0 \longrightarrow I_2(L'_\lambda)(-2) \longrightarrow I_1(L_\lambda)(-2) \oplus I_2(L'_\lambda) \longrightarrow fI_1(L_\lambda) + I_2(L'_\lambda) \longrightarrow 0$$

together with (4.1), one obtains

$$HS_{K[L_\mu]/I_2(L_\mu)}(z) = zHS_{K[L_\mu]/I_1(L_\lambda)}(z) + (1-z)HS_{K[L_\mu]/I_2(L'_\lambda)}(z),$$

where  $HS_A(z)$  denotes the Hilbert series of the algebra  $A$ . Therefore, by Corollary 4.9

$$h_1(\lambda) = h_1(\mu) - 1 = (r+s)(n-r) - \binom{s+1}{2} - 1$$

and

$$h_k(\lambda) = h_k(\mu) = \binom{r+s}{k} \binom{n-r}{k} - \binom{s+1}{k+1} \binom{n-1}{k-1} \quad \text{for } k \neq 1,$$

where  $1 + h_1(\mu)z + \dots + h_{r+s}(\mu)z^{r+s}$  is the normalized Hilbert series of  $K[L_\mu]/I_2(L_\mu)$ . In particular, the multiplicity of  $\mathcal{R}(\text{in}_\tau(I))$  is

$$e(\mathcal{R}(\text{in}_\tau(I))) = e(K[L_\mu]/I_2(L_\mu)) - e(K[L_\mu]/J) = \binom{n+s}{n-r} - \binom{n+s}{n+1} - 1. \quad \square$$

We close with the following remark that gives an alternative path for a proof of our results for special types of  $m \times n$  sparse matrices.

**Remark 4.11.** Let  $X$  be an  $m \times n$  sparse matrix,  $m \leq n$ . We assume that, after row and column permutations, the variables that appear in  $X$  form a two-sided ladder as in Figure 4.

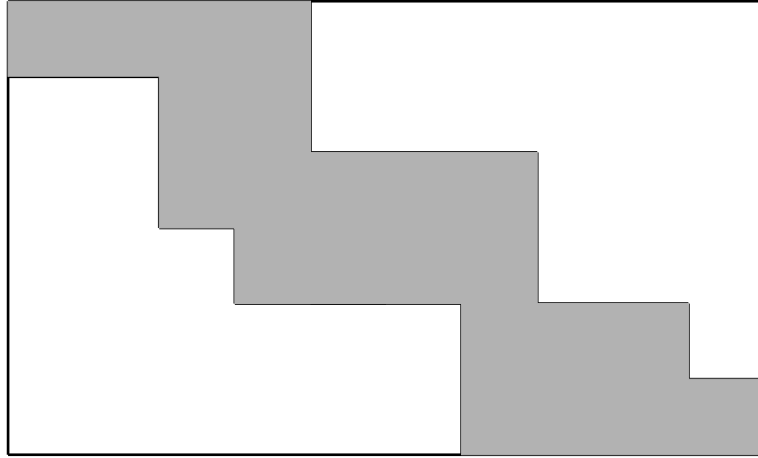


FIGURE 4. A matrix with a shaded two-sided ladder.

The case of a sparse  $2 \times n$  matrix  $X$  is a special case of the above type of matrix with  $m = 2$ , where the ladder has one lower inside corner in position  $(1, r+1)$  and one upper inside corner in position  $(2, r+s)$ . Let  $I = I_m(X)$  and let  $\tau$  be a diagonal term order. Notice that an  $m \times m$ -minor of  $X$  is non-zero if and only if all the entries on its diagonal are non-zero. Moreover, the  $m \times m$ -minors of  $X$  form a universal Gröbner basis of  $I_m(X)$  by [2, Proposition 5.4]. It follows that  $\text{in}_\tau(I)$  is generated by the products of the elements on the diagonals of the  $m \times m$  non-zero minors of  $X$ .

One can show that  $\text{in}_\tau(I)$  is a sortable ideal with respect to the lexicographic order induced by the following order of the variables:  $x_{i,j} > x_{k,l}$  if either  $i < k$  or  $i = k$  and  $j < l$ . Therefore, the defining ideal of  $\mathcal{F}(\text{in}_\tau(I))$  is the toric ideal generated by the binomial relations obtained by the sorting [20, Theorem 6.16]. Moreover, these generators are a quadratic Gröbner basis of the defining ideal of  $\mathcal{F}(\text{in}_\tau(I))$ . In particular, the sorting we use allows us to conclude

that  $\text{in}_\tau(I)$  is a generalized Hibi ideal and  $\mathcal{F}(\text{in}_\tau(I))$  is a generalized Hibi ring, see [27] and [20, Section 6.3]. Therefore,  $\mathcal{F}(\text{in}_\tau(I))$  is a normal, Cohen-Macaulay, Koszul algebra.

Furthermore,  $\text{in}_\tau(I)$  is a weakly polymatroidal ideal with respect to the same order on the variables as above. For the definition of a weakly polymatroidal ideal, see [29] or [20, Definition 6.25]. Hence, by [20, Proposition 6.26]  $\text{in}_\tau(I)$  satisfies the  $\ell$ -exchange property with respect to the sorting order. Therefore,  $\mathcal{R}(\text{in}_\tau(I))$  is of fiber type and the defining equations of  $\mathcal{R}(\text{in}_\tau(I))$  are precisely the defining equations of the special fiber ring  $\mathcal{F}(\text{in}_\tau(I))$  and the linear relations by [20, Theorem 6.24].

Because of the special shape of our matrix, one can use the criterion in [35] (see also [12, Proposition 1.1]) and proceed in a similar manner as in [20, Theorem 6.46] to show that the Plücker relations are the defining equations of  $\mathcal{F}(I)$  and the maximal minors of  $X$  are a SAGBI basis of  $\mathcal{F}(I)$ . One can also show that Plücker relations along with the linear relations of  $\mathcal{S}(I)$  are the defining equations of  $\mathcal{R}(I)$  and the maximal minors of  $X$  along with the variables of  $R$  are a SAGBI basis of  $\mathcal{R}(I)$ . In particular  $\text{in}_{\tau'}(\mathcal{R}(I)) = \mathcal{R}(\text{in}_\tau(I))$  and  $\text{in}_\tau(\mathcal{F}(I)) = \mathcal{F}(\text{in}_\tau(I))$ , where  $\tau'$  is a suitable term order. One then obtains similar results for  $\mathcal{R}(I)$  and  $\mathcal{F}(I)$  as in Corollaries 3.5, 4.4, 4.6, and 4.7.

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ELA CELIKBAS, DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, W.V. 26506

*Email address:* [ela.celikbas@math.wvu.edu](mailto:ela.celikbas@math.wvu.edu)

*URL:* <https://math.wvu.edu/~ec0029>

EMILIE DUFRESNE, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF YORK, YORK, UK

*Email address:* [emilie.dufresne@york.ac.uk](mailto:emilie.dufresne@york.ac.uk)

*URL:* <https://www.york.ac.uk/maths/staff/emilie-dufresne/>

LOUIZA FOULI, DEPARTMENT OF MATHEMATICAL SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, NM 88003

*Email address:* [lfouli@nmsu.edu](mailto:lfouli@nmsu.edu)

*URL:* <http://www.web.nmsu.edu/~lfouli>

ELISA GORLA, INSTITUT DE MATHÉMATIQUES, UNIVERSITÉ DE NEUCHÂTEL, RUE EMILE-ARGAND 11, CH-2000 NEUCHÂTEL, SWITZERLAND

*Email address:* [elisa.gorla@unine.ch](mailto:elisa.gorla@unine.ch)

*URL:* <http://members.unine.ch/elisa.gorla/>

KUEI-NUAN LIN, DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, GREATER ALLEGHENY CAMPUS, MCKEESPORT, PA 15132

*Email address:* [kul20@psu.edu](mailto:kul20@psu.edu)

*URL:* <https://sites.psu.edu/kul20>

CLAUDIA POLINI, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, 255 HURLEY, NOTRE DAME, IN 46556

*Email address:* [cpolini@nd.edu](mailto:cpolini@nd.edu)

*URL:* <https://www3.nd.edu/~cpolini>

IRENA SWANSON, DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY STREET, WEST LAFAYETTE, IN 47907

*Email address:* [irena@purdue.edu](mailto:irena@purdue.edu)

*URL:* <https://www.math.purdue.edu/~iswanso>