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CORRIGENDUM TO 'MUTATION OF FROZEN JACOBIAN ALGEBRAS'

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Abstract. Proposition 5.16 of the author's paper 'Mutations of frozen Jacobian algebras' [11] is false. We give several possible remedies with different applications.

In [11, Prop. 5.16] it is stated that if T is a cluster-tilting object in a Hom-finite Frobenius cluster category $\mathcal{E}$, then the quiver of $A = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ has no loops or 2-cycles. The part of the statement concerning 2-cycles is, however, false, and the error in the claimed proof is the use of [11, Lem. 5.12], which applies only when A is isomorphic to a frozen Jacobian algebra. Examples in which 2-cycles appear may be constructed in the following way. Given any finite-dimensional algebra A of global dimension at most 3, the category $\operatorname{proj} A$ is a Hom-finite Frobenius cluster category (albeit a highly degenerate one that does not categorify an interesting cluster algebra, since its stable category is zero). Up to additive equivalence, the unique cluster-tilting object of $\operatorname{proj} A$ is A itself, with endomorphism algebra $\operatorname{End}_A(A)^{\mathrm{op}} \xrightarrow{\sim} A$, and it is not hard to arrange that A is not a frozen Jacobian algebra, or that its quiver has 2-cycles. Indeed, an explicit example is given by

$$A = k\left(1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \right)/\langle\alpha\beta\rangle$$

The easiest way to correct the statement in a way that still permits its application in the proof of [11, Prop. 5.17] is to add a further assumption, as in the following version, which is essentially due to Buan, Iyama, Reiten and Scott [2].

Proposition 1. *Let $\mathcal{E}$ be a Hom-finite Frobenius cluster category equivalent as an exact category to a full subcategory, closed under subobjects, of an abelian category. Then if $T \in \mathcal{E}$ is a cluster-tilting object, the quiver of $A = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ has no loops or 2-cycles.*

Proof. Hom-finiteness of $\mathcal{E}$ implies that A is finite dimensional, and it is part of the definition of a Frobenius cluster category that A has finite global dimension. Thus we may obtain the result exactly as in the proof of [2, Prop. II.1.11(b)]. □

In the proof of [11, Prop. 5.17], the relevant Frobenius cluster category is the category $\operatorname{Sub} Q_k$ of submodules of direct sums of copies of an injective module Q_k for a preprojective algebra Π, which is a subobject-closed subcategory of the abelian category $\operatorname{mod}\Pi$, and so Proposition 1 may be applied in place of [11, Prop. 5.16]. We note that another proof of [11, Prop. 5.17], using essentially the same strategy, is given in [6, Prop. 4.2].

Proposition 1 also applies to some of the more general Frobenius cluster categories equivalent to the category $\operatorname{GP}(B)$ of Gorenstein projective modules over an Iwanaga–Gorenstein algebra. By [7, Thm. 2.7], any Hom-finite Frobenius cluster category with finitely many isoclasses of indecomposable projectives is equivalent to such a category, and the Gorenstein dimension of B is bounded above by 3 (see also [10, Cor. 3.10] for the result in this language). In practice, the Gorenstein dimension of B is often strictly smaller than 3—if it is either 0 or 1 then $\operatorname{GP}(B)$ is a subobject-closed subcategory of $\operatorname{mod} B$, and so Proposition 1 once again applies.

Under the original assumptions of [11, Prop. 5.16], we can at least rule out the existence of loops, and of 2-cycles in the quiver of the stable endomorphism algebra of a cluster-tilting object.

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Proposition 2. *Let $\mathcal{E}$ be a Hom-finite Frobenius cluster category, and $T \in \mathcal{E}$ a cluster-tilting object. Then the quiver of $A = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ has no loops, and the quiver of the stable endomorphism algebra $\underline{\operatorname{End}}_{\mathcal{E}}(T)^{\mathrm{op}}$ has no loops or 2-cycles.*

Proof. As in the setting of Proposition 1, A is a finite dimensional algebra of finite global dimension, so it has no loops by the no loops theorem [5, 9]. In [4, Prop. 3.11] (which is also used in the proof of [2, Prop. II.1.11(b)] cited above), Geiß, Leclerc and Schröer show that the quiver of A has no 2-cycles provided $\operatorname{Ext}^2_A(S, S) = 0$ for any simple A-module S. In fact, their proof is 'local', and shows that there is no 2-cycle between vertices i and j provided $\operatorname{Ext}^2_A(S_i, S_i) = 0 = \operatorname{Ext}^2_A(S_j, S_j)$, where S_i and S_j are the simple A-modules supported on these vertices.

If i is a vertex corresponding to a non-projective summand of T, then we have

$$\operatorname{Ext}^2_A(S_i, S_i) = \mathrm{D}\operatorname{Ext}^1_A(S_i, S_i) = 0,$$

the first equality by [8, §5.4] (see also [10, Thm. 3.4]), and the second since there is no loop at i. Hence by [4, Prop. 3.11] any 2-cycle in the quiver of A must pass through a vertex corresponding to a projective summand of T. Since the quiver of $\underline{\operatorname{End}}_{\mathcal{E}}(T)^{\mathrm{op}}$ is obtained from that of A by deleting these vertices and their incident arrows, it has no 2-cycles. □

When making connections to cluster theory, we give the quiver of A the structure of an ice quiver by declaring those vertices corresponding to projective summands of T to be frozen. Since arrows between frozen vertices play no role in constructing a cluster algebra from this ice quiver, for applications to cluster algebras we need only rule out 2-cycles in the quiver passing through at least one mutable vertex (cf. [2, §II.1]). We do not currently know whether the assumptions of Proposition 2 and [11, Prop. 5.16] are sufficient for this purpose, and indeed our counterexamples to the original statement from [11] only exhibit 2-cycles between frozen vertices.

Another purpose of [11, Prop. 5.16] was to provide sufficient conditions under which [11, Thm. 5.15] could be applied, this theorem explaining when mutations of cluster-tilting objects in a Frobenius cluster category are compatible with mutations of frozen Jacobian algebras, and with (extended) Fomin–Zelevinsky mutations of quivers. In this context, we are given a Frobenius cluster category $\mathcal{E}$ containing a cluster-tilting object T for which $\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ is isomorphic to a frozen Jacobian algebra. To be in the setting of [11, Thm. 5.15], we need to rule out 2-cycles in the quivers of $\operatorname{End}_{\mathcal{E}}(\hat{T})^{\mathrm{op}}$ for cluster-tilting objects $\hat{T}$ mutation equivalent to T. The next proposition shows that this follows from the other assumptions of the theorem when $\mathcal{E}$ is Hom-finite.

Proposition 3. *Let $\mathcal{E}$ be a Hom-finite Frobenius cluster category, and let $T \in \mathcal{E}$ be a cluster-tilting object such that $\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}} \cong \mathcal{J}(Q, F, W)$ for a reduced ice quiver with potential (Q, F, W). If Q has no loops or 2-cycles, then the quiver of $\operatorname{End}_{\mathcal{E}}(\hat{T})^{\mathrm{op}}$ has no loops or 2-cycles for any $\hat{T}$ mutation equivalent to T.*

Proof. The quiver of $\operatorname{End}_{\mathcal{E}}(\hat{T})^{\mathrm{op}}$ has no loops by Proposition 2, the assumptions of which are weaker than those of the present statement, and so it is sufficient for us to rule out 2-cycles.

The quiver Q, which is the quiver of $\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ since (Q, F, W) is reduced, has no 2-cycles by assumption. Let T' be a cluster-tilting object obtained from T by a single mutation, say at the summand corresponding to vertex $k \in Q_0$. Writing $(Q', F', W') = \mu_k(Q, F, W)$ for the mutation of (Q, F, W) at this vertex, it follows from [11, Thm. 5.14] that there is an isomorphism

$$A' = \operatorname{End}_{\mathcal{E}}(T')^{\mathrm{op}} \cong \mathcal{J}(Q', F', W'),$$

and so in particular A' is isomorphic to a frozen Jacobian algebra. As a result, the argument given in [11, Prop. 5.16] does in fact apply in this case, and we summarise it here. Since A' is a frozen Jacobian algebra, there is an exact sequence

$$\bigoplus_{\substack{\beta \in (Q')^{\mathrm{m}}_1 \\ h\beta = i}} P_{t\beta} \to \bigoplus_{\substack{\alpha \in Q'_1 \\ t\alpha = i}} P_{h\alpha} \to P_i \to S_i \to 0 \tag{1}$$

for any $i \in Q'_0$, where S_i is the simple module at i and $P_j = A'e_j$ is the indecomposable projective module with top S_j. We recall also the notation $(Q')^{\mathrm{m}}_1$ for the set of unfrozen arrows of Q'. Since

(Q', F', W') is reduced by the definition of mutation, Q' is the quiver of A', and thus has no loops by Proposition 2. It follows that $t\beta \neq i$ whenever $h\beta = i$, and so

$$\mathrm{Hom}_{A'}\left(\bigoplus_{\substack{\beta\in(Q')_1^{\mathrm{m}}\\ h\beta=i}} P_{t\beta}, S_i\right) = 0.$$

Since $\mathrm{Ext}^2_{A'}(S_i, S_i)$ is a subquotient of this space, (1) being the start of a projective resolution of S_i, we also have $\mathrm{Ext}^2_{A'}(S_i, S_i) = 0$. It follows that the quiver Q' of A' has no 2-cycles by [4, Prop. 3.11]. Applying the argument inductively, we extend this conclusion to the entire mutation class of T. □

Remark 4. We take this opportunity to make an additional, more minor, correction to [11], namely that the ideal generated by commutators appearing in [11, Defs. 2.8 and 2.10] (and in the text between these definitions) should be replaced by the closure of the vector subspace spanned by commutators, as in [11, Def. 4.8].

Secondly, Chang and Zhang point out [3, Rem. 2.9] that the ice quivers with potential associated to (k, n)-Postnikov diagrams [1] are not rigid in the sense of [11, Def. 4.8]. Since this family should be particularly well-behaved (and these ice quivers with potential are non-degenerate by [11, Prop. 5.17] and [6, Prop. 4.2]), our interpretation of their remark is that the definition of rigidity given in [11] is too strong, and should be replaced by a better notion. However, at this time we do not have an alternative suggestion.

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