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CONGRUENCES OF SAITO-KUROKAWA LIFTS AND DENOMINATORS OF CENTRAL SPINOR L-VALUES

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Abstract. Following Ryan and Tornaría, we prove that moduli of congruences of Hecke eigenvalues, between Saito-Kurokawa lifts and non-lifts (certain Siegel modular forms of genus 2), occur (squared) in denominators of central spinor L-values (divided by twists) for the non-lifts. This is conditional on Böcherer's conjecture and its analogues, and is viewed in the context of recent work of Furusawa, Morimoto and others. It requires a congruence of Fourier coefficients, which follows from a uniqueness assumption, or can be proved in examples. We explain these factors in denominators via a close examination of the Bloch-Kato conjecture.

1. Introduction

Let $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ be an elliptic curve defined over $\mathbb{Q}$, with its rational point at infinity the neutral element for an abelian group structure. Associated with E is an L-function $L(E, s) = \prod_p L_p(E, s)$, where for any prime p of good reduction (all but finitely many), $L_p(E, s) = (1-a_pp^{-s}+p^{1-2s})^{-1}$, with $1 + p - a_p$ the number of points mod p on E. This converges for $\Re(s) > \frac{3}{2}$, by the Hasse bound $|a_p| < 2p^{1/2}$, but has an analytic continuation to the whole complex plane, thanks to the modularity of E: the function $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ on the upper half plane is a modular form of weight 2 for the congruence subgroup $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \; N \mid c \right\}$, holomorphic at cusps and satisfying $f\left(\frac{az+b}{cz+d}\right) = (cz + d)^2 f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Here N, the conductor of E, is an integer divisible by certain powers of the primes of bad reduction.

The Birch and Swinnerton-Dyer conjecture equates the rank of the finitely-generated group $E(\mathbb{Q})$ of rational points with the order of vanishing of $L(E, s)$ at $s = 1$, which is the central point with respect to a functional equation relating values at s and $2 - s$. It also gives a formula for the leading term in the Taylor expansion about $s = 1$. When the order of vanishing is 0, this is just a formula for the value:

$$L(E, 1) = \frac{\prod_p c_p \,\#\text{Ш}\,\Omega}{(\#\, E(\mathbb{Q}))^2},$$

where the c_p are certain integer factors coming from primes p of bad reduction, Ш is the Shafarevich-Tate group, and Ω is the integral of a Néron differential over the real locus.

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Supposing that $\ell \mid \# E(\mathbb{Q})$ for some prime ℓ, one might hope to be able somehow to detect the factor ℓ^2 in the denominator of the rational number $\frac{L(E,1)}{\Omega}$. The existence of a rational point of order ℓ is equivalent to a congruence

$$a_p \equiv 1 + p \pmod{\ell}, \tag{1}$$

for all primes p of good reduction. In this paper, we consider an analogous situation, replacing $L(E,1) = L(f,1)$ by $L(k-1, F, \mathrm{spin})$, the central value of the spinor L-function attached to a suitable Siegel modular form of genus 2 (i.e. involving a 4×4 symplectic group) and weight k. The congruence (1) is replaced by a congruence of Hecke eigenvalues

$$\mu_p(F) \equiv a_p(f) + p^{k-1} + p^{k-2} \pmod{\lambda}$$

between F and another Siegel modular form, the Saito-Kurokawa lift, which comes from a genus 1 form f of weight $2k-2$. In these circumstances the Bloch-Kato conjecture, a wide generalisation of the Birch and Swinnerton-Dyer conjecture, puts a factor λ^2 in the denominator of a formula for $L(k-1, F, \mathrm{spin})$.

To prove that this factor actually occurs, in the denominator of an algebraic number obtained by dividing $L(k-1, F, \mathrm{spin})$ by a suitable twisted value $L(k-1, F, \mathrm{spin}, \chi_d)$, we need something on central twisted spinor L-values for F. Our main results are conditional on conjectures of Böcherer type, relating these to linear combinations of Fourier coefficients of F. In the remainder of this introduction we go into considerably more detail, at least in the case of level 1, then briefly summarise the contents of the paper, which cover also odd, square-free $\Gamma_0^{(2)}(M)$ level, and paramodular level.

Let $F(Z) = \sum_S a(F,S) e^{2\pi i \mathrm{tr}(SZ)}$ be a Siegel cusp form of genus 2 and weight k for $\mathrm{Sp}_2(\mathbb{Z}) := \left\{ g \in M_4(\mathbb{Z}) : \, {}^t g \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \right\}$. In particular,

$$F((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k F(Z) \quad \forall \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_2(\mathbb{Z}).$$

Here, $Z \in \mathfrak{H}_2 := \{Z \in M_2(\mathbb{C}) : \, {}^t Z = Z, \mathrm{Im}(Z) > 0\}$, S runs over matrices of the form $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, with $a, b, c \in \mathbb{Z}$, $a > 0$, $\mathrm{disc}(S) := b^2 - 4ac < 0$ and $\mathrm{cont}(S) := \gcd\{a, b, c\}$. Let $\langle F, F \rangle$ be the Petersson norm of F (normalised as in [16]). We have $a(F, {}^t U S U) = a(F, S)$ for any $U \in \mathrm{SL}_2(\mathbb{Z})$. Let $-D < 0$ be a fundamental discriminant, and $K = \mathbb{Q}(\sqrt{-D})$, with associated quadratic character χ_{-D}, $w(K)$ roots of unity and ideal class group Cl_K. There is a natural bijection between elements of Cl_K and $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of the S (of fixed discriminant $-D$), with $a(F,S)$ depending only on the class of S, so we have a well-defined $a(F,c)$ for each $c \in \mathrm{Cl}_K$. Let $\Lambda : \mathrm{Cl}_K \to \mathbb{C}^\times$ be any character, and define

$$R(F, K, \Lambda) := \sum_{c \in \mathrm{Cl}_K} a(F,c) \Lambda^{-1}(c).$$

Let $\mathcal{AI}(\Lambda^{-1})$ be the automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ automorphically induced from Λ^{-1} (viewed as a character of $\mathrm{GL}_1(\mathbb{A}_K)$), where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. This is associated with the theta series

$$\theta_{\Lambda^{-1}}(z) = \sum_{\mathfrak{a} \subseteq O_K} \Lambda^{-1}(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z}$$

(sum over integral ideals), which is of weight 1, character χ_{-D}. If F is a Hecke eigenform, let π_F be the associated cuspidal, irreducible, automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$, $L(s,\pi_F)$ the degree 4 spinor L-function, $L(s,\pi_F\times\mathcal{AI}(\Lambda^{-1}))$ the degree 8 tensor product L-function, and $L(s,\pi_F,\mathrm{ad})$ the degree 10 adjoint L-function.

The following is a generalisation of a refinement of a well-known conjecture of Böcherer [8], and is Conjecture 1.3 in [16].

Conjecture 1.1. *Suppose that F is a cuspidal Hecke eigenform of weight $k\geq 2$ for $\mathrm{Sp}_2(\mathbb{Z})$, not a Saito-Kurokawa lift, and $K=\mathbb{Q}(\sqrt{-D})$ any imaginary quadratic field as above. Then*

$$\frac{|R(F,K,\Lambda)|^2}{\langle F,F\rangle}=\frac{2^{4k-4}\pi^{2k+1}}{(2k-2)!}w(K)^2D^{k-1}\frac{L(1/2,\pi_F\times\mathcal{AI}(\Lambda^{-1}))}{L(1,\pi_F,\mathrm{ad})}.$$

Note that here we are normalising L-functions in such a way that $s=1/2$ is the central point of the functional equation. The spinor L-function of F is $L(s,F,\mathrm{spin})=L(s-(k-3/2),\pi_F)$. Dickson, Pitale, Saha and Schmidt [16] have shown that this conjecture is implied by Y. Liu's refined global Gan-Gross-Prasad conjecture [31], using their calculations of local integrals, and the identification of $R(F,K,\Lambda)$ with a Bessel period [41, 1-26], [20, (4.3.4)], [38, Proposition 4.3]. In the case $\Lambda=1$, which has been proved by Furusawa and Morimoto [21], the tensor product L-function decomposes as $L(s,\pi_F)L(s,\pi_F\times\chi_{-D})$, and it becomes a refined version of Böcherer's conjecture, [16, Theorem 1.16].

Suppose now that k is even, and let f be a cuspidal Hecke eigenform of weight $k':=2k-2$ for $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathbb{Q}(f)$ be the number field generated by the Hecke eigenvalues of f. Associated with f is its Saito-Kurokawa lift $\hat{f}$, a cuspidal Hecke eigenform of weight k for $\mathrm{Sp}_2(\mathbb{Z})$, such that

$$L(s,\hat{f},\mathrm{spin})=L(s,f)\zeta(s-(k-1))\zeta(s-(k-2)),\text{ and}$$
$$L(s,\hat{f},\mathrm{st})=\zeta(s)L(s+(k-1),f)L(s+(k-2),f).$$

Let $\mathbb{T}$ be the ring generated over $\mathbb{Z}$ by (for all primes p) the Hecke operators denoted $T(p)$ and $T_1(p^2)$ in [25, §4]. (Note that $T_2(p^2)$ acts on the space of weight k cuspforms $S_k(\mathrm{Sp}_2(\mathbb{Z}))$ as multiplication by p^{2k-6}.) If F is a cuspidal Hecke eigenform of weight $k\geq 3$ for $\mathrm{Sp}_2(\mathbb{Z})$, and $T\in\mathbb{T}$, let $\mu_F(T)$ denote the eigenvalue for T acting on F. As noted in [25, §4], this is always an algebraic integer. Let $\mathbb{Q}(F)$ be the number field generated by all the $\mu_F(T)$. Let $\{F_1,F_2,\dots,F_t\}$ be a basis of Hecke eigenforms for $S_k(\mathrm{Sp}_2(\mathbb{Z}))$, and let E be the field generated by the Hecke eigenvalues of all the F_i, i.e. the compositum of the $\mathbb{Q}(F_i)$. The following is essentially a theorem proved independently by Brown and Katsurada [10, 25].

Theorem 1.2. *Let f be as above, and $\ell>2k-4$ a prime number. Let $\lambda'\mid\ell$ be a prime of $\mathbb{Q}(f)$, and suppose that there exists a fundamental discriminant $-D<0$, and even m with $2<m<k-2$, such that*

$$\mathrm{ord}_{\lambda'}\left(\frac{\zeta(m)L(m+k-2,f)L(m+k-1,f)D^{k-(3/2)}L(f,k-1,\chi_{-D})}{\langle f,f\rangle\pi^{3m+2k-4}L(f,k)}\right)<0.$$

Then there exists a cuspidal Hecke eigenform F of weight k for $\mathrm{Sp}_2(\mathbb{Z})$, orthogonal to $\hat{f}$, and a prime $\lambda\mid\lambda'$ in E, such that for all $T\in\mathbb{T}$,

$$\mu_F(T)\equiv\mu_{\hat{f}}(T)\pmod{\lambda}.$$

We prove this at the beginning of the next section.

Remark 1.3. There exist periods $\Omega^+ \in \mathbb{R}$, $\Omega^- \in i\mathbb{R}$ such that each $L(f,t)$, for $1 \leq t \leq 2k-3$, is of the form $L_{\mathrm{alg}}(f,t)(2\pi i)^t \Omega^{(-1)^t}$, and

$$L(f,k-1,\chi_{-D}) = L_{\mathrm{alg}}(f,k-1,\chi_{-D})(2\pi i)^{k-1} i\sqrt{D}\Omega^+,$$

with $L_{\mathrm{alg}}(f,t), L_{\mathrm{alg}}(f,k-1,\chi_{-D}) \in \mathbb{Q}(f)$. These periods are defined only up to $\mathbb{Q}(f)^\times$-multiples, but there is a natural choice (up to divisors of small primes) such that if $\mathrm{ord}_{\lambda'}\left(\frac{\langle f,f\rangle}{i\Omega^+\Omega^-}\right) > 0$ then there is a congruence of Hecke eigenvalues between f and some other cuspidal Hecke eigenform of weight $2k-2$ for $\mathrm{SL}_2(\mathbb{Z})$ (not a multiple of f), i.e λ' is a congruence prime for f in $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$. This is a theorem of Hida, essentially [24, Theorem 7.1]. Adopting this choice, the condition of Theorem 1.2 is satisfied if the following hold.

(1) $\mathrm{ord}_{\lambda'}(\zeta(1-m)L_{\mathrm{alg}}(m+k-1)L_{\mathrm{alg}}(m+k-2)) = 0$.
(2) There exists a fundamental discriminant $-D < 0$ such that

$$\mathrm{ord}_{\lambda'}(D^{k-1}L_{\mathrm{alg}}(k-1,f,\chi_{-D})) = 0.$$

(3) λ' is not a congruence prime for f in $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$.
(4) $\mathrm{ord}_{\lambda'}(L_{\mathrm{alg}}(k,f)) > 0$.

We may think then of $L_{\mathrm{alg}}(k,f)$ (the critical value immediately to the right of the centre) as the origin of the modulus of the congruence. Condition (3) implies that F is not a Saito-Kurokawa lift, and conditions (1) and (2) are very weak.

Remark 1.4. Theorem 1.5 below is then telling us that the divisor λ, which starts in the numerator of $L_{\mathrm{alg}}(k,f)$, also occurs in the denominator of $L_{\mathrm{alg}}(k-1,F,\mathrm{spin})$. This is analogous to how 691, which comes from the numerator of $\frac{\zeta(12)}{\pi^{12}}$, also occurs in the denominator of $L_{\mathrm{alg}}(11,\Delta)$, with the congruence between the Saito-Kurokawa lift and non-lift playing the rôle of Ramanujan's congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$. Here, $\Delta(z) = \sum_{n=1}^{\infty}\tau(n)q^n$ is the normalised cusp form of level 1 and weight 12 for $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 1.5. *Let f and λ be as in Theorem 1.2, and suppose the following.*

(1) *The F in Theorem 1.2 is unique, up to scaling.*
(2) *F is not a Saito-Kurokawa lift.*
(3) *There exists a fundamental discriminant $-D < 0$ such that* $\mathrm{ord}_{\lambda'}(c(D))$ *(as in the proof of Theorem 1.2) is minimal* and *the class number h_K of $K = \mathbb{Q}(\sqrt{-D})$ is not divisible by ℓ.*

Then Conjecture 1.1 implies that $L(k-1,F,\mathrm{spin}) \neq 0$, and that for any fundamental discriminant $d > 0$, coprime to ℓD, assuming also that $L(k-1,F,\mathrm{spin},\chi_d) \neq 0$,

$$\mathrm{ord}_\lambda\left(\frac{L(k-1,F,\mathrm{spin})}{L(k-1,F,\mathrm{spin},\chi_d)}\right) \leq -2.$$

Remark 1.6. Remark 10 in [21] suggests that Furusawa and Morimoto may soon prove Y. Liu's conjecture for general Λ. This would, by [16, Theorem 1.13], prove Conjecture 1.1, making Theorem 1.5 unconditional.

Theorem 1.5 is one of the main results of the paper, and its proof occupies §2. By examining the proof of Katsurada's Theorem 5.2 [25], which led to our Theorem 1.2, we prove in Proposition 2.1, under a uniqueness condition on F, a congruence of Fourier coefficients between $\hat{f}$ and F. The explicit formula for the Fourier coefficients of $\hat{f}$, in particular the fact that $a(\hat{f},S)$ depends only on the discriminant of S, leads to $R(\hat{f}, K, \Lambda) = 0$ for non-trivial Λ. The congruence of Fourier coefficients between $\hat{f}$ and F then gives $R(F, K, \Lambda) \equiv 0 \pmod{\lambda}$. Applying Conjecture 1.1, for trivial and non-trivial Λ (with different K), carefully chosen as in [36, Proposition 5.1], then leads to Theorem 1.5. So far we have largely followed Ryan and Tornaría, the only novelty being to link their work, via our Proposition 2.1, with that of Katsurada and Brown on divisors of L-values as moduli of congruences.

In §3 we introduce a new ingredient, taking a careful look at the Bloch-Kato conjecture on special values of motivic L-functions, applied to the central twisted spinor L-values we are concerned with here, and, in Proposition 3.1, connect it with Theorem 1.5. We view a global torsion term as contributing λ^2 to the denominator of $L(k-1, F, \mathrm{spin})$. The purpose of dividing by $L(k-1, F, \mathrm{spin}, \chi_d)$ is then to cancel an unwanted Deligne period.

So far we have only considered forms of level 1. In §4 we generalise the results of §2 and §3 to the case of $\Gamma_0^{(2)}(M)$-level, for odd, squarefree M. Agarwal and Brown have proved the Hecke eigenvalue congruences analogous to those of Katsurada and Brown in Theorem 1.2, namely Theorem 4.2. While the uniqueness condition (leading to a congruence of Fourier coefficients) may be less practical here, we are motivated by the work of Dickson, Pitale, Saha and Schmidt, showing that in this case the direct analogue of Conjecture 1.1 (i.e. Conjecture 4.1) would follow from Y. Liu's refined Gan-Gross-Prasad conjecture, which has been proved for trivial Λ by Furusawa and Mizumoto, and might be extended by them to non-trivial Λ.

In §5 we consider the case of paramodular level $\Gamma^{\mathrm{para}}(M)$, for odd squarefree M. Conjecture 5.1 (which implies Ryan and Tornaría's Conjecture C) is a weak analogue of Conjecture 1.1, good enough for our purposes, and in Proposition 5.3 we prove that it would follow from Y. Liu's refined Gan-Gross-Prasad conjecture. Theorem 5.4 shows how congruences of Fourier coefficients between Gritsenko lifts and non-lifts F would then imply $\mathrm{ord}_\lambda\left(\frac{L(k-1,F,\mathrm{spin})}{L(k-1,F,\mathrm{spin},\chi_d)}\right) \leq -2$, for suitable d. We finish by applying this to known numerical examples in weights $k = 2$ and 3. Weight 2, paramodular level, was the main focus of Ryan and Tornaría, and Poor and Yuen. V. Golyshev and A. Mellit's interest in a weight 3 example of Poor and Yuen, including the appearance of the congruence modulus in Hecke and spinor L-values, was what led me to look at all this.

2. Proof of Theorem 1.5

First we must come back to a proof of Theorem 1.2.

Proof. As in equation (5) in [19, §6],

(2)
$$\frac{\zeta(m)L(m+k-2,f)L(m+k-1,f)D^{k-(3/2)}L(f,k-1,\chi_{-D})}{\langle f,f\rangle\pi^{3m+2k-4}L(f,k)} = \frac{L(m,\hat{f},\mathrm{st})\,c(D)^2}{\pi^{3m+2k-3}\langle\hat{f},\hat{f}\rangle},$$

where $\tilde{f}(z) = \sum c(n)q^n \in S_{k-(1/2)}(\Gamma_0(4))^+$ is a half-integral weight modular form mapping to f under Kohnen's correspondence, via which one constructs $\hat{f}$. To obtain this formula (following [10] or [25]) one uses the factorization of $L(s, \hat{f}, \mathrm{st})$, a formula of Kohnen and Skoruppa for $\frac{\langle\hat{f},\hat{f}\rangle}{\langle\tilde{f},\tilde{f}\rangle}$ [27], and a formula of Kohnen and Zagier for $\frac{\langle\tilde{f},\tilde{f}\rangle}{\langle f,f\rangle}$ [28]. The scaling of $\tilde{f}$ determines that of $\hat{f}$, in fact if $\mathrm{disc}(S) = -D$, a fundamental discriminant, then $a(\hat{f},S) = c(D)$. It follows that if $\mathcal{I}_{\hat{f}}$ is the fractional ideal of $\mathbb{Q}(\hat{f}) = \mathbb{Q}(f)$ generated by the Fourier coefficients of $\hat{f}$ then, under our assumption,

$$\mathrm{ord}_\lambda\left(\frac{L(m,\hat{f},\mathrm{st})\,\mathcal{I}_{\hat{f}}^2}{\pi^{3m+2k-3}\langle\hat{f},\hat{f}\rangle}\right) < 0.$$

The congruence of Hecke eigenvalues is now a direct application of a theorem of Katsurada [25, Theorem 5.2]. □

Proposition 2.1. *Let f, m and λ be as in Theorem 1.2, and suppose that the F there is unique, up to scaling. Then for an appropriate choice of scalings, such that the $a(F,S)$ and the $a(\hat{f},S)$ are all integral at λ but neither the $a(F,S)$ nor the $a(\hat{f},S)$ are all divisible by λ, there is a congruence of Fourier coefficients*

$$a(F,S) \equiv a(\hat{f},S) \pmod{\lambda} \qquad \forall S.$$

Proof. Let $\{F_1,F_2,\ldots,F_t\}$ be a basis of Hecke eigenforms for $S_k(\mathrm{Sp}_2(\mathbb{Z}))$, with $F_1 = \hat{f}$ and $F_2 = F$, and recall that E is the field generated by the Hecke eigenvalues of all the F_i. Now $S_k(\mathrm{Sp}_2(\mathbb{Z}))$ has a basis with rational Fourier coefficients [4], spanning over $\mathbb{Q}$ the set of all forms with rational Fourier coefficients. With respect to this basis each Hecke operator $T(p)$ or $T_1(p^2)$, preserving rationality of Fourier coefficients, acts by a rational matrix. It follows that we may assume that each F_i has Fourier coefficients in the field $\mathbb{Q}(F_i)$ generated by its Hecke eigenvalues, hence in E. (For the generalisations beyond level 1 in later sections, we can use [39] in place of [4].)

Katsurada applies a certain differential operator to the Siegel-Eisenstein series of genus 4, then, fixing any S_0, considers a partial Fourier coefficient $\mathcal{F}_{m+2,k;S_0}(Z)$ in its restriction to $\mathfrak{H}_2\times\mathfrak{H}_2$. By [25, Theorem 4.4],

$$\mathcal{F}_{m+2,k;S_0}(Z) = C_{k,m}\sum_{i=1}^{t}\frac{L(m,F_i,\mathrm{st})}{\pi^{3m+2k-3}\langle F_i,F_i\rangle}a(F_i,S_0)F_i(Z),$$

where $C_{k,m}$ is some rational constant such that $\mathrm{ord}_\ell(C_{k,m}) = 0$ (because $\ell > 2k-4$, cf. formula preceding [25, Theorem 4.3]), while $\mathcal{F}_{m+2,k;S_0}(Z)$ has rational Fourier coefficients, integral at ℓ (denominators divisible at worst by primes less than or equal to $2m+3 \leq 2k-5$). Note that E is totally real, so, looking at Katsurada's formula, $\overline{F_i(-\overline{Z})} = F_i(Z)$.

Thanks to (2), in Theorem 1.2 we may choose $-D$, among negative fundamental discriminants, in such a way that $\mathrm{ord}_{\lambda'}(c(D))$ is minimal. If we then scale $\tilde{f}$ so that $\mathrm{ord}_{\lambda'}(c(D)) = 0$ then all the coefficients of $\tilde{f}$ are integral at λ', since for any fundamental discriminant $-D' < 0$, the non-fundamental coefficients $c(D'n^2)$ are determined by the formal equality of Dirichlet series

$$L(s-(k-2),\chi_{-D'})\sum_{n=1}^{\infty}c(D'n^2)n^{-s} = c(D')\sum_{n=1}^{\infty}a_n(f)n^{-s}.$$

Moreover, all the Fourier coefficients of $\hat{f}$ are integral at λ', because of the formula

$$a(\hat{f},S) = \sum_{b\mid\mathrm{cont}(S)} c\left(\frac{|\mathrm{disc}(S)|}{b^2}\right),$$

which follows from Theorem 1 and Proposition 3 of [29].

If S_0 has discriminant $-D$, chosen as above, then $\mathrm{ord}_{\lambda'}(a(\hat{f}, S_0)) = 0$, because $a(\hat{f}, S_0) = c(D)$ (with the scaling of $\hat{f}$ determined by that of $\tilde{f}$). We now know that $\hat{f}$ is scaled as in the statement of the proposition.

Given that, by (2),

$$\mathrm{ord}_{\lambda'}\left(\frac{L(m, \hat{f}, \mathrm{st})\, c(D)^2}{\pi^{3m+2k-3}\langle \hat{f}, \hat{f}\rangle}\right) < 0,$$

and $\mathrm{ord}_{\lambda'}(c(D)) = \mathrm{ord}_{\lambda'}(a(\hat{f}, S_0)) \geq 0$, we see that if

$$\mathcal{F}_{m+2,k;S_0}(Z) = \sum_{i=1}^{t} c_i F_i(Z)$$

then $\mathrm{ord}_{\lambda'}(c_1) < 0$.

By uniqueness of F, for each $i > 2$ there exists $T_i \in \mathbb{T}$ such that

$$\mathrm{ord}_{\lambda}(\mu_{\hat{f}}(T_i) - \mu_{F_i}(T_i)) = 0.$$

We can apply the operator $\prod_{i=3}^{t}(T_i - \mu_{F_i}(T_i))$ to both sides of

$$\mathcal{F}_{m+2,k;S_0}(Z) = \sum_{i=1}^{t} c_i F_i(Z)$$

to kill all the F_i for $i > 2$, obtaining an equation of the form

$$\mathcal{F}(Z) = b_1 \hat{f} + b_2 F,$$

where $\mathcal{F}$ has Fourier coefficients integral at λ, and $\mathrm{ord}_{\lambda}(b_1) = \mathrm{ord}_{\lambda}(c_1) < 0$, because

$$b_1 = c_1 \prod_{i=3}^{t}(\mu_{\hat{f}}(T_i) - \mu_{F_i}(T_i)).$$

Dividing both sides by b_1, and replacing F by $(-b_2/b_1)F$, we find a congruence mod λ of Fourier coefficients between $\hat{f}$ and F (which forces the scaling of F to be as in the statement of the proposition). $\square$

Lemma 2.2. *Let f be a cuspidal Hecke eigenform of weight $2k-2$ for $\mathrm{SL}_2(\mathbb{Z})$, with Saito-Kurokawa lift $\hat{f}$. Let $-D < 0$ be a fundamental discriminant, $K = \mathbb{Q}(\sqrt{-D})$ and $\Lambda : \mathrm{Cl}_K \to \mathbb{C}^\times$ a* non-trivial *character. Then*

$$R(\hat{f}, K, \Lambda) = 0.$$

Proof.

$$R(\hat{f}, K, \Lambda) = \sum_{c \in \mathrm{Cl}_K} a(\hat{f}, c)\Lambda^{-1}(c) = c(D) \sum_{c \in \mathrm{Cl}_K} \Lambda^{-1}(c) = 0.$$

$\square$

Let $-D$ be a fundamental discriminant, and $d > 0$ a fundamental discriminant coprime to D. Let $K = \mathbb{Q}(\sqrt{-Dd})$ and $L = \mathbb{Q}(\sqrt{-Dd}, \sqrt{-D})$. Then L is an unramified quadratic extension of K. Let Λ be the quadratic character of $\mathrm{Cl}_K \simeq \mathrm{Gal}(H/K)$ whose kernel is $\mathrm{Gal}(H/L)$, where H is the Hilbert class field of K. Using Frobenius reciprocity,

$$\mathrm{Ind}_K^{\mathbb{Q}}(\Lambda^{-1}) \simeq \chi_{-D} \oplus \chi_d,$$

hence

$$L(1/2, \pi_F \times \mathcal{AI}(\Lambda^{-1})) = L(1/2, \pi_F \times \chi_{-D})L(1/2, \pi_F \times \chi_d).$$

Similarly, using $K' = \mathbb{Q}(\sqrt{-D})$ and the trivial character id of $\mathrm{Cl}_{K'}$,

$$L(1/2, \pi_F \times \mathcal{AI}(\mathrm{id})) = L(1/2, \pi_F \times \chi_{-D})L(1/2, \pi_F).$$

It follows that if Conjecture 1.1 is true then (as long as $\ell \nmid d$)

$$\mathrm{ord}_\lambda \left(\frac{L(k-1, F, \mathrm{spin})}{L(k-1, F, \mathrm{spin}, \chi_d)} \right) = \mathrm{ord}_\lambda \left(\frac{|R(F, K', \mathrm{id})|^2}{|R(F, K, \Lambda)|^2} \right).$$

To complete the proof of Theorem 1.5, it suffices to show that $\mathrm{ord}_\lambda(R(F, K', \mathrm{id})) = 0$ and $\mathrm{ord}_\lambda(R(F, K, \Lambda)) > 0$, with D chosen as in condition (3) of Theorem 1.5, and F scaled as in the proof of Proposition 2.1. But, using Proposition 2.1,

$$R(F, K', \mathrm{id}) \equiv R(\hat{f}, K', \mathrm{id}) = c(D)h_{K'} \not\equiv 0 \pmod{\lambda}$$

(which implies that $L(k-1, F, \mathrm{spin}) \neq 0$) and, by Lemma 2,

$$R(F, K, \Lambda) \equiv R(\hat{f}, K, \Lambda) = 0 \pmod{\lambda}.$$

Remark 2.3. The choice of Λ above was inspired by [36, §2.2]. The proof of Lemma 2 is taken from [36, Proposition 4.2]. The proof of Theorem 1.5 is essentially that of [36, Proposition 5.1], combined with our Proposition 2.1.

Remark 2.4. Because k is even and f has level 1, the sign in the functional equation of $L(s, f)$ is $(-1)^{k-1} = -1$, hence $L(k-1, f) = 0$, but since $L(s, \hat{f}, \mathrm{spin}) = L(s, f)\zeta(s - (k-1))\zeta(s - (k-2))$, then $L(k-1, \hat{f}, \mathrm{spin}) = 0$. So the fact that $R(\hat{f}, K', \mathrm{id}) \neq 0$ implies that Conjecture 1.1 (which excludes Saito-Kurokawa lifts) cannot be extended to $\hat{f}$ for the trivial character. But since $\chi_d(-1) = 1$ we find that also $L(k-1, \hat{f}, \mathrm{spin}, \chi_d) = 0$, so the fact that $R(\hat{f}, K, \Lambda) = 0$ can be viewed as proving an extension, to $\hat{f}$ and the non-trivial character Λ, of Conjecture 1.1, cf. [36, §4]. However, note also that in the case of Saito-Kurokawa lifts, using Waldspurger's theorem, Böcherer proved his original conjecture, where $L(k-1, \hat{f}, \mathrm{spin}, \chi_{-D})$ is still present, but other factors such as $L(k-1, \hat{f}, \mathrm{spin})$ are replaced by an undetermined, but *non-zero* constant.

Example 2.5. (From [26, §4].) When $k = 22$, so $2k - 2 = 42$, $S_{42}(\mathrm{SL}_2(\mathbb{Z}))$ is spanned by Galois conjugate Hecke eigenforms $f = f_1, f_2, f_3$ with Hecke eigenvalues in a field E generated by the roots of a polynomial $g(x) := x^3 + 7181x^2 - 2766919456x - 4705905729536$. There is a prime ideal $\lambda \mid \ell$, with $\ell = 1423$, such that $\mathrm{ord}_\lambda(L_{\mathrm{alg}}(f, k)) > 0$, and 22 is the smallest value of k for which this happens with an $\ell > 2k - 4$. There is an eigenform F with rational Fourier coefficients (Υ_{22} in Katsurada's notation) such that $\{\hat{f}_1, \hat{f}_2, \hat{f}_3, F\}$ is a basis for $S_{22}(\mathrm{Sp}_2(\mathbb{Z}))$ and there is a congruence mod λ of Hecke eigenvalues between F and $\hat{f}$. Using the computations in [26, §4], ℓ does not divide the discriminant of $g(x)$, or the index of the order in $\langle 1, \theta, \theta^2 \rangle_{\mathbb{Z}}$ generated by the Fourier coefficients of f, where θ is a root of $g(x)$. It follows that there is no mod λ congruence of Hecke eigenvalues between f and either of its Galois conjugates, hence F has the desired uniqueness property. Moreover, with $\hat{f}$ scaled as in [26], $\mathcal{I}_{\hat{f}}$ is an algebraic integer, not divisible by λ, while $\mathrm{ord}_\lambda(a(\hat{f}, S)) = 0$, where S is Katsurada's $A_0 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, of discriminant $D = -3$, and $h_{\mathbb{Q}(\sqrt{-3})} = 1$. The remaining conditions of Theorem 1.5 are easily checked.

3. Interpretation via the Bloch-Kato conjecture

Let F and λ be as in Theorem 1.5. By a theorem of Weissauer [42], there exists a 4-dimensional E_λ-vector space V_λ, with a continuous action $\rho_{F,\lambda}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, such that for each prime $p \neq \ell$, the Euler factor at p in $L(s, F, \mathrm{spin})$ is

$$L_p(s, F, \mathrm{spin})^{-1} = \det(I - \rho_{F,\lambda}(\mathrm{Frob}_p^{-1})p^{-s}|V_\lambda^{I_p}),$$

where I_p is an inertia subgroup at p, and Frob_p is a Frobenius element mapping to a generator $x \mapsto x^p$ of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

For us, since F is level 1, ρ_λ is unramified at all $p \neq \ell$, so $V_\lambda^{I_p} = V_\lambda$. But when we look at $V_\lambda \otimes \chi_d$, where the Galois action is twisted by a character corresponding to the Dirichlet character χ_d, there will be ramification at primes dividing d, and the local Euler factors at such primes will in fact be trivial. For any integer j, let $V_\lambda(j)$ be the j^{th} Tate twist, so we have tensored with the j^{th} power of the ℓ-adic cyclotomic character, multiplying the action of each Frob_p by p^j. Let T_λ be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathcal{O}_\lambda$-lattice in V_λ, and $A_\lambda := V_\lambda/T_\lambda$.

Poincare duality gives an isomorphism of $E_\lambda[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules

$$V_\lambda \simeq V_\lambda^*(3-2k),$$

where $V^* := \mathrm{Hom}_{E_\lambda}(V, E_\lambda)$ with the natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. We may now define another $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathcal{O}_\lambda$-lattice T'_λ in V_λ by

$$T'_\lambda := T_\lambda^*(3-2k),$$

where $T_\lambda^* = \mathrm{Hom}_{\mathcal{O}_\lambda}(T_\lambda, \mathcal{O}_\lambda)$, and then

$$A'_\lambda := V'_\lambda/T'_\lambda.$$

Were the residual representation $\overline{\rho}_{F,\lambda}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $T_\lambda/\lambda T_\lambda$ irreducible, T'_λ would necessarily be the same as T_λ (up to scaling). However, for us $\overline{\rho}_{F,\lambda}$ will be reducible (therefore actually dependent on the choice of T_λ), and we need to avoid assuming that T'_λ is the same as T_λ.

Conjecturally, V_λ is the λ-adic realisation $M_{F,\lambda}$ of M_F, a Grothendieck motive over $\mathbb{Q}$ with coefficients in E, associated to F. We assume at least the existence of a premotivic structure (collection of realisations and comparison isomorphisms) in the sense of [15, 1.1.1]. The Hodge type of the de Rham realisation $M_{F,\mathrm{dR}}$ is $\{(0, 2k-3), (k-2, k-1), (k-1, k-2), (2k-3, 0)\}$, and the central point $s = k-1$ gives the unique critical value of the L-function $L(s, F, \mathrm{spin})$ of M_F.

The following formulation of the λ-part of the Bloch-Kato conjecture [7], applied to the central critical value of $L(s, F, \mathrm{spin})$, is based on [15, (59)] (where, however, there is a non-empty set Σ of bad primes), using the exact sequence in their Lemma 2.1.

$$\mathrm{ord}_\lambda\left(\frac{L(k-1, F, \mathrm{spin})}{\Omega}\right) = \mathrm{ord}_\lambda\left(\frac{(\prod_p c_p)\,\#H^1_f(\mathbb{Q}, A'_\lambda(k-1))}{\#H^0(\mathbb{Q}, A'_\lambda(k-1))\,\#H^0(\mathbb{Q}, A_\lambda(k-1))}\right),$$

where "$\#B$" denotes the Fitting ideal of B. Here $H^1_f(\mathbb{Q}, A'_\lambda(k-1))$ is a Bloch-Kato Selmer group, defined by certain local conditions, Ω is a Deligne period, and the c_p are local Tamagawa factors. The Ω and the c_p depend (up to units in the localisation $\mathcal{O}_{E,(\lambda)}$) on choices of $\mathcal{O}_{E,(\lambda)}$-lattices T_B and T_{dR} in the Betti and de Rham realisations of M_F (which are E-vector spaces). The choice of T_B determines

$T_\lambda = T_B \otimes \mathcal{O}_\lambda$, which is required to be $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable. Note that, since (assuming Conjecture 1.1) $L(k-1, F, \mathrm{spin}) \neq 0$, conjecturally then $H^1_f(\mathbb{Q}, A'_\lambda(k-1))$ *is* finite.

The congruence of Hecke eigenvalues in Theorem 1.2 may be read as an equality of traces of Frob_p^{-1} between $\overline{\rho}_{F,\lambda}$ and $\mathbb{F}_\lambda(1-k) \oplus \overline{\rho}_{f,\lambda} \oplus \mathbb{F}_\lambda(2-k)$, thus identifying the composition factors of $\overline{\rho}_{F,\lambda}$. (Typically $\overline{\rho}_{f,\lambda}$, the residual representation attached to f, will be irreducible, but we do not have to assume this.) We now make a choice of T_λ in such a way that $\mathbb{F}_\lambda(1-k)$ is a submodule of $\overline{\rho}_{F,\lambda}$. There can be no non-trivial extension of $\mathbb{F}_\lambda(2-k)$ by $\mathbb{F}_\lambda(1-k)$ inside $\overline{\rho}_{F,\lambda}$, since this would force $\mathrm{ord}_\ell(B_2) > 0$, using Herbrand's Theorem as in [10, §8]. It follows that we can also arrange for $\mathbb{F}_\lambda(2-k)$ to be a quotient of $\overline{\rho}_{F,\lambda}$.

Then the trivial representation $\mathbb{F}_\lambda$ is a submodule of $\overline{\rho}_{F,\lambda}(k-1)$, contributing a factor of λ to $\#H^0(\mathbb{Q}, A_\lambda(k-1))$. Since there is a perfect Galois-equivariant pairing $T'_\lambda/\lambda T'_\lambda \times T_\lambda/\lambda T_\lambda \to \mathbb{F}_\lambda(3-2k)$, the quotient $\mathbb{F}_\lambda(2-k)$ must be paired with a submodule $\mathbb{F}_\lambda(1-k)$, and $T'_\lambda/\lambda T'_\lambda$ must have its composition factors in the same order as $T_\lambda/\lambda T_\lambda$, in particular also with a submodule $\mathbb{F}_\lambda(1-k)$. This contributes a factor λ also to $\#H^0(\mathbb{Q}, A'_\lambda(k-1))$. We may view these contributions as the origin of the -2 in Theorem 1.5, but we need to pay some attention to the various other factors.

We assume that there is "no particular reason" for there to be a non-trivial element of the Selmer group $H^1_f(\mathbb{Q}, A'_\lambda(k-1))$. Having chosen T_λ as above, we then choose T_{dR} in such a way that $\mathbb{V}(T_{\mathrm{dR}} \otimes \mathcal{O}_\lambda) = T_\lambda$, where $\mathbb{V}$ is the version of the Fontaine-Lafaille functor used in [15]. Consequently, if $\ell > 2k-2$ then $\mathrm{ord}_\lambda(c_\ell) = 0$, by [7, Theorem 4.1(iii)]. For $p \neq \ell$, $\mathrm{ord}_\lambda(c_p)$ is also 0 for any p at which $\rho_{F,\lambda}$ is unramified, which is all $p \neq \ell$ for us, since F has level 1.

We must also consider $M_F \otimes M_{\chi_d}$, whose L-function is $L(s, F, \mathrm{spin}, \chi_d)$, where M_{χ_d} is the premotivic structure associated to the even Dirichlet character χ_d. For natural choices of bases for the Betti and de Rham realisations of M_{χ_d}, the Deligne period is $1/\sqrt{d}$ [13, §6]. It follows that the Deligne period for $M_F \otimes M_{\chi_d}$ (with the implied choices of lattices) is $\Omega/(\sqrt{d})^2 = \Omega/d$. The Bloch-Kato conjecture says that

$$\begin{aligned}\mathrm{ord}_\lambda&\left(\frac{L(k-1, F, \mathrm{spin}, \chi_d)}{\Omega/d}\right)\\ &= \mathrm{ord}_\lambda\left(\frac{\left(\prod_p c_p(\chi_d)\right) \#H^1_f(\mathbb{Q}, A'_\lambda(k-1, \chi_d))}{\#H^0(\mathbb{Q}, A'_\lambda(k-1, \chi_d))\#H^0(\mathbb{Q}, A_\lambda(k-1, \chi_d))}\right).\end{aligned}$$

We shall assume that $\ell \nmid d$. If $p \mid d$ then I_p, which acted trivially on V_λ, now acts non-trivially through a quotient of order 2, via χ_d. Hence $H^0(\mathbb{Q}_p, A_\lambda(k-1, \chi_d))$ is trivial, from which it follows that $\mathrm{ord}_\lambda(c_p(\chi_d))$=0 and that $H^0(\mathbb{Q}, A_\lambda(k-1, \chi_d))$ is trivial. Similarly $H^0(\mathbb{Q}, A'_\lambda(k-1, \chi_d))$ is trivial. As before, we have $\mathrm{ord}_\lambda(c_p(\chi_d)) = 0$ for $p \nmid \ell d$, and (if $\ell > 2k-2$), $\mathrm{ord}_\lambda(c_\ell(\chi_d)) = 0$. (We have tensored together the choices of lattices we already made in realisations of M_F and M_{χ_d}.) We have arrived at the following.

Proposition 3.1. *Let F and λ be as in Theorem 1.5, with $\ell > 2k-2$, and $d > 0$ a fundamental discriminant such that $\ell \nmid d$. Let T_λ be chosen as above (with $\mathbb{F}_\lambda(1-k)$ a submodule of $T_\lambda/\lambda T_\lambda$). The Bloch-Kato conjecture predicts that if*

$H^1_f(\mathbb{Q}, A'_\lambda(k-1))$ *is trivial (and if* $L(k-1, F, \mathrm{spin}, \chi_d) \neq 0$*) then*

$$\mathrm{ord}_\lambda\left(\frac{L(k-1, F, \mathrm{spin})}{L(k-1, F, \mathrm{spin}, \chi_d)}\right) \leq -2.$$

This seems to fit well with Theorem 1.5, but we should examine the notion that there is "no particular reason" for $H^1_f(\mathbb{Q}, A'_\lambda(k-1))$ to be non-trivial, by looking at some closely related situations where there *is* a reason to believe that a Selmer group is non-trivial. Suppose that we choose T_λ a different way, with $\mathbb{F}_\lambda(1-k)$ not a submodule of $T'_\lambda/\lambda T'_\lambda$ (but then necessarily $\overline{\rho}_{f,\lambda}$ or $\mathbb{F}_\lambda(2-k)$ is a submodule instead). Then we can no longer rely on the contribution of $H^0(\mathbb{Q}, A'_\lambda(k-1))$ to help explain Theorem 1.5, and (unless $\lambda^2 \mid \#H^0(\mathbb{Q}, A_\lambda(k-1))$) require instead non-triviality of $H^1_f(\mathbb{Q}, A'_\lambda(k-1, \chi_d))$. We sketch how this arises, in the two cases.

Note that this non-triviality of $H^1_f(\mathbb{Q}, A'_\lambda(k-1, \chi_d))$ (or $\lambda^2 \mid \#H^0(\mathbb{Q}, A_\lambda(k-1))$) would be forced by isogeny-invariance of the Bloch-Kato conjecture, but *only* if we assume triviality of $H^1_f(\mathbb{Q}, A'_\lambda(k-1))$ for the original choice of T_λ. So the independent constructions below lend credence to such an assumption.

(1) Let the 2-dimensional Galois representation $\rho_{f,\lambda}$ associated to f be on a space $V_{f,\lambda}$. Since χ_d is an even character, the sign in the functional equation of $L(s, f, \chi_d)$ is -1, like that of $L(s, f)$. By a parity result of Nekovář [32, Theorem B] (assuming irreducibility of $\overline{\rho}_{f,\lambda}$), the dimension of the λ-adic Selmer group $H^1_f(\mathbb{Q}, V_{f,\lambda}(k-1, \chi_d))$ is odd, so in particular $H^1_f(\mathbb{Q}, V_{f,\lambda}(k-1, \chi_d))$ is non-trivial. Suppose we choose T_λ in such a way that $\overline{\rho}_{f,\lambda}$ is a submodule of $T'_\lambda/\lambda T'_\lambda$. Then using a non-zero element of $H^1_f(\mathbb{Q}, V_{f,\lambda}(k-1, \chi_d))$, and the injection from the λ-torsion $A_{f,\lambda}[\lambda](k-1, \chi_d)$ of $A_{f,\lambda}(k-1, \chi_d)$ to that of $A'_\lambda(k-1, \chi_d)$, we can construct a non-zero element of $H^1(\mathbb{Q}, A'_\lambda(k-1, \chi_d))$, and using the fact that it came from something in $H^1_f(\mathbb{Q}, V_{f,\lambda}(k-1, \chi_d))$, satisfying the Bloch-Kato local conditions, it is not difficult to show that it too satisfies those conditions, so lies in $H^1_f(\mathbb{Q}, A'_\lambda(k-1, \chi_d))$.

    If one tried to construct an element in $H^1_f(\mathbb{Q}, A'_\lambda(k-1))$ (which we do not really want) in this manner, $H^0(\mathbb{Q}, A'_\lambda[\lambda](k-1)/A'_{f,\lambda}[\lambda](k-1)$ might be non-trivial, because of the composition factor $\mathbb{F}_\lambda$ of $A'_\lambda[\lambda](k-1)/A'_{f,\lambda}[\lambda](k-1)$, so the constructed element may be 0.

(2) If we choose T_λ in such a way that $\mathbb{F}_\lambda(2-k)$ is a submodule of $T'_\lambda/\lambda T'_\lambda$, we may similarly construct a non-zero element of $H^1_f(\mathbb{Q}, A'_\lambda(k-1, \chi_d))$, starting from the non-triviality of $H^1_f(\mathbb{Q}, \mathbb{Q}_\ell(1, \chi_d))$, which follows from the infinitude of the group of units of $\mathbb{Q}(\sqrt{d})$.

Remark 3.2. If we abandon the hypothesis in Theorem 1.5 concerning uniqueness of F, and let G be another such eigenform in $S_k(\mathrm{Sp}_2(\mathbb{Z}))$, satisfying the Hecke eigenvalue congruence with $\hat{f}$, then $\overline{\rho}_{G,\lambda}$, like $\overline{\rho}_{F,\lambda}$, has composition factors $\mathbb{F}_\lambda(1-k)$, $\overline{\rho}_{f,\lambda}$ and $\mathbb{F}_\lambda(2-k)$. Even if we ensure (by choice of $\mathcal{O}_\lambda$-lattices before reducing) that $\mathbb{F}_\lambda(1-k)$ is a submodule for both $\overline{\rho}_{F,\lambda}$ and $\overline{\rho}_{G,\lambda}$, because of the reducibility there is no guarantee that $\overline{\rho}_{F,\lambda}$ and $\overline{\rho}_{G,\lambda}$ are isomorphic. Proposition 5.1 of [6] gives a condition that would ensure that they are. *If* they are then, letting r be minimal such that $\rho_{F,\lambda}$ and $\rho_{G,\lambda}$ (on the $\mathcal{O}_\lambda$-lattices) are different modulo λ^{r+1},

$$\rho_{G,\lambda}(\sigma) \equiv \rho_{F,\lambda}(\sigma)(I + \lambda^r\,\theta(\sigma)) \pmod{\lambda^{r+1}}$$

defines a cocycle θ representing a non-zero cohomology class in $H^1(\mathbb{Q}, \mathrm{ad}(\overline{\rho}_{F,\lambda}))$. Composing with projection to the quotient module $\mathbb{F}_\lambda(2-k)$ of $\overline{\rho}_{F,\lambda}$ would land us in $H^1(\mathbb{Q}, A'_\lambda[\lambda](k-1))$, so could conceivably lead to a non-zero element of the Selmer group $H^1_f(\mathbb{Q}, A'_\lambda(k-1))$. If so, the Bloch-Kato conjecture could be compatible with failure of the conclusion of Theorem 1.5. There is an analogous remark in [18, §8], in a different situation.

4. Squarefree level

Let $M \geq 1$ be an integer, and $\Gamma_0^{(2)}(M) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}) : \ C \in M\, M_2(\mathbb{Z}) \right\}$. As explained in [16, §1.4, §3.2], there is a notion of old and new subspaces of $S_k(\Gamma_0^{(2)}(M))$, and a newform is, in the new subspace, an eigenform F of $T(p)$ and $T_1(p^2)$ for all primes $p \nmid M$ and of operators $U(p)$ for all $p \mid M$. It generates an irreducible cuspidal automorphic representation π_F of $\mathrm{GSp}_2(\mathbb{A})$. Let $\mathbb{T}$ be the Hecke algebra generated by the $T(p)$ and $T_1(p^2)$ for all primes $p \nmid M$.

Let $f = \sum a_f(n)q^n \in S_{2k-2}(\Gamma_0(M))$ be a normalised newform (of genus 1), with $k \geq 2$ even and M odd and squarefree. As explained in [2], there is a Saito-Kurokawa lift $\hat{f} \in S_k(\Gamma_0^{(2)}(M))$. It is a Hecke eigenform for $\mathbb{T}$, with $\mu_{\hat{f}}(T(p)) = a_f(p) + p^{k-2} + p^{k-1}$ for all primes $p \nmid M$.

Theorem 1.13 of [16] is that the following is a consequence of Yifeng Liu's refined Gan-Gross-Prasad conjecture.

Conjecture 4.1. *Suppose that $F \in S_k(\Gamma_0^{(2)}(M))$, with M odd and squarefree, $k \geq 3$, is a newform, not a Saito-Kurokawa lift or a Yoshida lift. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field, with $\chi_{-D}(p) = -1$ for all primes $p \mid M$, Λ a character of Cl_K, and $R(F, K, \Lambda)$ defined as before. Then*

$$\frac{|R(F,K,\Lambda)|^2}{\langle F, F\rangle} = \frac{2^{4k-4}\pi^{2k+1}}{(2k-2)!} w(K)^2 D^{k-1} \frac{L(1/2, \pi_F \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi_F, \mathrm{ad})} \prod_{p \mid M} J_p,$$

where

$$J_p = \begin{cases} (1+p^{-2})(1+p^{-1}) & \text{if } \pi_{F,p} \text{ is of type IIIa [37, §2.2];} \\ 2(1+p^{-2})(1+p^{-1}) & \text{if } \pi_{F,p} \text{ is of type VIb;} \\ 0 & \text{otherwise.} \end{cases}$$

The following may serve the role of Theorem 1.2. It is taken from results of Agarwal and Brown [1, Theorem 6.5, Theorem 7.4, Corollary 7.5].

Theorem 4.2. *Let $k \geq 6$ be even, M odd and squarefree, $f \in S_{2k-2}(\Gamma_0(M))$ a newform, $\hat{f} \in S_k(\Gamma_0^{(2)}(M))$ a Saito-Kurokawa lift. For primes $p \mid M$, let $w_p(f)$ be the Atkin-Lehner eigenvalues. Suppose that there exists an even character χ of conductor $N > 1$, with $M \mid N$, a fundamental discriminant $-D < 0$ coprime to M, with $\chi_{-D}(p) = w_p(f)$ for all primes $p \mid M$, and a prime divisor $\lambda \mid \ell > 2k-2$ (with $\ell \nmid ND$) in a sufficiently large number field E, such that*

$$\mathrm{ord}_\lambda \left(\frac{L^N(3-k,\chi)L(2k-4,f,\chi)L(2k-3,f,\chi)D^{k-(3/2)}L(f,k-1,\chi_{-D})}{\langle f, f\rangle \pi^{4k-7} L(f,k)} \right) < 0.$$

(Here the superscript N indicates omission of Euler factors at $p \mid N$.) Suppose that there is no mod λ congruence of Hecke eigenvalues between f and any newform of

level dividing M. Then there exists a Hecke eigenform (for $\mathbb{T}$) $F \in S_k(\Gamma_0^{(2)}(M))$, not a Saito-Kurokawa lift, such that for all $T \in \mathbb{T}$,

$$\mu_F(T) \equiv \mu_{\hat{f}}(T) \pmod{\lambda}.$$

If additionally $\ell \nmid (p^2-1)$ for all primes $p \mid M$, then F is not a Yoshida lift.

In Theorem 1.2, the existence of the auxiliary even m with $2 < m < k-2$ requires $k > 6$, which was automatically satisfied at level 1 because $2k-2 \geq 12$. But for $M > 1$ in general it is a real condition. In fact, Agarwal and Brown only require $k \geq 6$, because the twist by χ allows them to use $m = k-2$, which also avoids the application of a differential operator to the Eisenstein series. This condition also ensures convergence of the genus 4 Siegel-Eisenstein series whose pullback formula is being applied, and integrality at ℓ of its Fourier coefficients. They also strengthen our condition "< 0" in Theorems 1.2 and 4.2 to "$< -\mathrm{ord}_\lambda\left(\frac{\langle f,f\rangle}{i\Omega^+\Omega^-}\right)$", thus ensuring that F may be taken not to be a Saito-Kurokawa lift, even if λ is a congruence prime for f.

As in the case $M = 1$, there is a half-integral weight form $\tilde{f} = \sum c(n)q^n \in S^+_{k-(1/2)}(\Gamma_0(4M))$ in the background. Note that, for a fundamental discriminant $-D < 0$, $c(D) = 0$ unless $\chi_{-D}(p) = w_p(f)$ for all primes $p \mid M$, by [30, Corollary 1, Remark]. It is easy to prove the following analogue of Theorem 1.5. For the formulas $c(D) = a(\hat{f}, S_0)$ and

$$a(\hat{f}, S) = \sum_{\substack{b \mid \mathrm{cont}(S) \\ b \nmid M}} c\left(\frac{|\mathrm{disc}(S)|}{b^2}\right),$$

used in the proof of Theorem 1.5 in the special case $M = 1$, we can in general use [2, §3].

Theorem 4.3. *Let f and λ be as in Theorem 4.2, and suppose the following.*

1. *$w_p(f) = -1$ for all primes $p \mid M$.*
2. *The F in Theorem 4.2 is unique, up to scaling.*
3. *There exists a fundamental discriminant $-D < 0$, with $\chi_{-D}(p) = -1$ for all primes $p \mid M$, such that $\mathrm{ord}_\lambda(c(D))$ is minimal* and *the class number h_K of $K = \mathbb{Q}(\sqrt{-D})$ is not divisible by ℓ.*

Then Conjecture 4.1 implies that $L(k-1, F, \mathrm{spin}) \neq 0$, and that for any fundamental discriminant $d > 0$, coprime to ℓD, such that $\chi_d(p) = 1$ for all primes $p \mid M$, assuming also that $L(k-1, F, \mathrm{spin}, \chi_d) \neq 0$,

$$\mathrm{ord}_\lambda\left(\frac{L(k-1, F, \mathrm{spin})}{L(k-1, F, \mathrm{spin}, \chi_d)}\right) \leq -2.$$

The condition on $\chi_d(p)$ guarantees that $\chi_{-dD}(p) = -1$ for all primes $p \mid M$, and happily coincides with that appearing in Proposition 4.4 below. The congruence ensures that for no $p \mid M$ can we be in one of the cases where $J_p = 0$ in Conjecture 4.1, since this would imply $R(F, K, \mathrm{id}) = 0$. This would contradict

$$R(F, K, \mathrm{id}) \equiv R(\hat{f}, K, \mathrm{id}) = h(K)c(D) \not\equiv 0 \pmod{\lambda}.$$

Note that J_p is independent of K and Λ, so it cancels when we take the ratio. Its value does not matter to us, as long as it is non-zero. The uniqueness of F forces it to be a newform. Remark 1.6 applies also here, i.e. Conjecture 4.1 may soon be proved by Furusawa and Morimoto.

It is easy to prove the following generalisation of Proposition 3.1.

Proposition 4.4. *Let $k \geq 2$ be even, M odd and squarefree, $f \in S_{2k-2}(\Gamma_0(M))$ a newform, $\hat{f} \in S_k(\Gamma_0^{(2)}(M))$ a Saito-Kurokawa lift. Let $\mathbb{T}$ be the Hecke algebra generated by the $T(p)$ and $T_1(p^2)$ for all primes $p \nmid M$. Let $F \in S_k(\Gamma_0^{(2)}(M))$ be a newform, not a Saito-Kurokawa lift, such that for all $T \in \mathbb{T}$,*

$$\mu_F(T) \equiv \mu_{\hat{f}}(T) \pmod{\lambda}.$$

Here λ is a prime divisor in a suitable number field E, dividing a rational prime $\ell > 2k-2$. Let $d > 0$ a fundamental discriminant such that $\ell \nmid d$ and $\chi_d(p) = 1$ for all primes $p \mid M$. Let T_λ (a lattice in the λ-adic Galois representation attached to F) be chosen with $\mathbb{F}_\lambda(1-k)$ a submodule of $T_\lambda/\lambda T_\lambda$. The Bloch-Kato conjecture predicts that if $H^1_f(\mathbb{Q}, A'_\lambda(k-1))$ is trivial (and if $L(k-1, F, \mathrm{spin}), L(k-1, F, \mathrm{spin}, \chi_d) \neq 0$) then

$$\mathrm{ord}_\lambda\left(\frac{L(k-1, F, \mathrm{spin})}{L(k-1, F, \mathrm{spin}, \chi_d)}\right) \leq -2.$$

The condition $\chi_d(p) = 1$ for $p \mid M$ (so that the Galois character χ_d is trivial when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$) ensures that c_p and $c_p(\chi_d)$, though not necessarily trivial when $p \mid M$, are at least the same, and so cancel in the ratio.

5. Paramodular level

Let M be an odd squarefree integer, and let $f \in S_{2k-2}(\Gamma_0(M))$ be a newform with sign -1 in the functional equation of its L-function. We require $k \geq 2$, but no longer that k is even. As explained in [11, §6], there is a paramodular Saito-Kurokawa lift $\hat{f} \in S_k(\Gamma^{\mathrm{para}}(M))$, a.k.a. the Gritsenko lift $\mathrm{Grit}(\phi)$ of an associated Jacobi form ϕ of weight k, level 1 and index M. The paramodular group of level M is defined by

$$\Gamma^{\mathrm{para}}(M) = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \frac{1}{M}\mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & M\mathbb{Z} \\ M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix} \cap \mathrm{Sp}_2(\mathbb{Q})$$

$$= \gamma M_4(\mathbb{Z})\gamma^{-1} \cap \mathrm{Sp}_2(\mathbb{Q}), \quad \gamma = \mathrm{diag}(1, 1, M, 1).$$

To get from f to ϕ, this time we do not go via a modular form of half-integral weight, rather we go in one step by the inverse of the isomorphism in Theorem 5 of [40].

A new feature of the paramodular case is that in the Fourier expansion $F = \sum_S a(F, S)e^{2\pi i \mathrm{tr}(SZ)}$, we now have $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ with a restriction $M \mid a$, and $a(F, U^t SU) = a(F, S)$ for $U \in \Gamma_0(M)$. Let $\mathcal{Q}_{M,-D}$ be the set of all such S, with fixed fundamental discriminant $b^2 - 4ac = -D$. Note that the condition $M \mid a$ arises from the fact that $\begin{pmatrix} I_2 & \begin{pmatrix} \frac{1}{M} & 0 \\ 0 & 0 \end{pmatrix} \\ 0_2 & I_2 \end{pmatrix} \in \Gamma^{\mathrm{para}}(M)$, so F is invariant under the translation $\tau \mapsto \tau + \frac{1}{M}$, where $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$.

Conjecture 5.1. *For M odd and squarefree, $k \geq 2$, suppose that $F \in S_k(\Gamma^{\mathrm{para}}(M))$ is a newform (cf. [37]), not a Saito-Kurokawa lift. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field, with $\chi_{-D}(p) = 1$ for all primes $p \mid M$, and Λ a character of Cl_K such that $\Lambda([\mathfrak{p}]) = 1$ for all prime ideals $\mathfrak{p} \mid M$. Then*

$$|R(F, K, \Lambda)|^2 = c_F w(K)^2 D^{k-1} L(1/2, \pi_F \times \mathcal{AI}(\Lambda^{-1})),$$

where c_F is some constant independent of K and Λ, and now

$$R(F, K, \Lambda) := \sum_{[S] \in \mathcal{Q}_{M,-D}/\Gamma_0(M)} a(F, S)\Lambda^{-1}([S]).$$

Here, to evaluate Λ^{-1} on $[S]$, one simply projects from $\mathcal{Q}_{M,-D}$ to $\mathcal{Q}_{1,-D}$, replacing the $\Gamma_0(M)$-equivalence class of S by its $\mathrm{SL}_2(\mathbb{Z})$-equivalence class, which is then identified with an element of Cl_K in the usual way.

Remark 5.2. For the choice of Λ occurring in the proof of Theorem 1.5, with $K = \mathbb{Q}(\sqrt{-Dd})$ and $\chi_{-D}(p) = \chi_d(p) = 1$ for all primes $p \mid M$, this conjecture becomes a conjecture of Ryan and Tornaría [36, Conjecture C], though this is not immediately obvious. Equation (4) in [22, §I.1] gives a surjective map $\Phi : \mathcal{Q}_{M,-Dd}/\Gamma_0(M) \to \mathcal{Q}_{1,-Dd}/\mathrm{SL}_2(\mathbb{Z})$. If we replaced S by $\Phi(S)$ in Conjecture 5.1 then we would get Ryan and Tornaría's Conjecture C. Now if $S = \begin{pmatrix} \alpha M & b/2 \\ b/2 & c \end{pmatrix}$ then $\Phi(S) = \begin{pmatrix} \alpha M_1 & b/2 \\ b/2 & cM_2 \end{pmatrix}$, for a certain factorisation $M = M_1 M_2$. Under the usual bijection between $\mathcal{Q}_{1,-Dd}/\mathrm{SL}_2(\mathbb{Z})$ and Cl_K, $[S]$ maps to (the class of) $\tilde{S} := \langle 2\alpha M, b + \sqrt{-Dd} \rangle_{\mathbb{Z}}$. Although this is the $\mathbb{Z}$-span, it is easy to check that it is in fact an ideal in $\mathcal{O}_K$. Likewise, $\Phi(S)$ maps to $\widetilde{\Phi(S)} := \langle 2\alpha M_1, b + \sqrt{-Dd} \rangle_{\mathbb{Z}}$. Since $\tilde{S} \subseteq \widetilde{\Phi(S)}$, there is an ideal $\mathfrak{a}$ of $\mathcal{O}_K$ such that $\tilde{S} = \mathfrak{a}\widetilde{\Phi(S)}$. If $p \mid M$ then p splits in $L = \mathbb{Q}(\sqrt{-dD}, \sqrt{-D})$, hence $\Lambda([\mathfrak{p}]) = 1$ for any $\mathfrak{p}$ of norm p. This shows that Λ satisfies the hypothesis of Conjecture 5.1. But since $\mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{a}) = M_2 \mid M$, it also shows that $\Lambda([\mathfrak{a}]) = 1$, so $\Lambda([S]) = \Lambda([\Phi(S)])$, as required.

Proposition 5.3. *Conjecture 5.1 follows from Y. Liu's refined Gan-Gross-Prasad conjecture.*

Proof. Looking at Y. Liu's conjecture as stated in [16, Conjecture 1.12], first note that $\pi_{F,v}$ is generic for almost all places v, given that F is not a Gritsenko lift, as in [16, §1.4, (ii) non-CAP]. The formula [16, (9)] is

$$\frac{|B(\phi, \Lambda)|^2}{\langle \phi, \phi \rangle} = \frac{C_T}{S_{\pi_F}} \frac{\zeta(2)\zeta(4)L(1/2, \pi_F \times \mathcal{AI}(\Lambda^{-1}))}{L(1, \pi_F, \mathrm{ad})L(1, \chi_{-D})} \prod_v J_v(\phi_v).$$

Here S_{π_F} is either 4 or 2, depending on whether or not F is a Yoshida lift. We proceed as in [16, §3.3]. In their (75) we replace S by some fixed S in $\mathcal{Q}_{M,-D}$, then in their K_0, at $p \mid M$ we replace their $T_S(\mathbb{Q}_p) \cap \mathrm{GL}_2(\mathbb{Z}_p)$ by $T_S(\mathbb{Q}_p) \cap I_p$, where $I_p := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) : p \mid \gamma \right\}$. It is still the case that $T_S(\mathbb{Q}) \cap K_0 \simeq \mathcal{O}_K$, since $T_S(\mathbb{Q}) = A^\times$, with $A := \{x + y\xi_S : x, y \in \mathbb{Q}\}$, and $\xi_S = \begin{pmatrix} b/2 & c \\ -a & -b/2 \end{pmatrix}$ such that $\xi_S^2 = -D/4$, and $M \mid a$. Thus we arrive at their Proposition 3.5 relating the

Bessel period $B(\phi,\Lambda)$ with $R(F,K,\Lambda)$, and Proposition 3.6 that

$$\frac{|R(F,K,\Lambda)|^2}{\langle F,F\rangle} = e^{4\pi \mathrm{tr}(S)}\left(\frac{D^{1/2}w(K)^2}{2^3 S_{\pi_F}}\right)\frac{L(1/2,\pi_F\times\mathcal{AI}(\Lambda^{-1}))}{L(1,\pi_F,\mathrm{ad})}\frac{J_\infty}{\mathrm{vol}(\mathbb{R}^\times\backslash T_S(\mathbb{R}))}\prod_{p\mid M}J(\phi_p).$$

As in [16, (105)], $\frac{J_\infty}{\mathrm{vol}(\mathbb{R}^\times\backslash T_S(\mathbb{R}))}=2^{2k}D^{k-(3/2)}e^{-4\pi\mathrm{tr}(S)}$, so we now have the correct power of D, and it remains to show that the $J(\phi_p)$, for primes $p\mid M$, are independent of K and Λ.

Viewing Λ as a character of $\mathbb{A}_K^\times/K^\times$, the conditions $\chi_{-D}(p)=1$ and $\Lambda([\mathfrak{p}])=1$ ensure that locally at $p\mid M$, $(K\otimes\mathbb{Q}_p)^\times\simeq\mathbb{Q}_p^\times\times\mathbb{Q}_p^\times$, and the restriction of Λ is trivial. However, on the face of it, $J(\phi_p)$ may depend on the way that $\mathbb{Q}_p^\times\times\mathbb{Q}_p^\times$ is embedded in $\mathrm{GL}_2(\mathbb{Q}_p)$ as $T_S(\mathbb{Q}_p)$, depending on S and therefore on K. Looking at the definition of $J(\phi_p)$ preceding [16, Conjecture 1.12], and bearing in mind that Λ_p is trivial, we must show that

$$\int_{\mathbb{Q}_p^\times\backslash T_S(\mathbb{Q}_p)}\int_{N(\mathbb{Q}_p)}\frac{\langle\pi_{F,p}(t_pn_p)\phi_p,\phi_p\rangle}{\langle\phi_p,\phi_p\rangle}\,\theta_S^{-1}(n_p)\,dn_p\,dt_p$$

is independent of K. Here $\pi_{F,p}$ is the local component at p of the automorphic representation of $\mathrm{GSp}_2(\mathbb{A})$ associated with F, $\langle\cdot,\cdot\rangle$ is an invariant inner product on its space, and $T_S(\mathbb{Q}_p)$ is embedded in $\mathrm{GSp}_2(\mathbb{Q}_p)$ via $g\mapsto\begin{pmatrix}g&0\\0&(\det(g))\,{}^tg^{-1}\end{pmatrix}$. Also,

$$N(\mathbb{Q}_p)=\{n(X):X\in M_2(\mathbb{Q}_p),\ {}^tX=X\},\text{with } n(X):=\begin{pmatrix}I&X\\0&I\end{pmatrix},$$

and $\theta_S(n(X)):=\psi_p(\mathrm{tr}(SX))$, with $\psi_p:\mathbb{Q}_p\to\mathbb{C}$ a standard additive character.

We seek $\gamma\in I_p$ such that ${}^t\gamma S\gamma=\begin{pmatrix}0&1/2\\1/2&0\end{pmatrix}$. Then let $\tilde\gamma:=\begin{pmatrix}\gamma&0\\0&(\det(\gamma))\,{}^t\gamma^{-1}\end{pmatrix}\in K_p$, the local component at p of an open compact subgroup of $\mathrm{GSp}_2(\mathbb{A}_f)$ whose intersection with $\mathrm{GSp}_2(\mathbb{Q})$ is $\Gamma_0^{\mathrm{para}}(M)$. By invariance of the inner product, and the fact that ϕ_p is K_p-fixed,

$$\begin{aligned}\frac{\langle\pi_{F,p}(t_pn_p)\phi_p,\phi_p\rangle}{\langle\phi_p,\phi_p\rangle}&=\frac{\langle\pi_{F,p}(\tilde\gamma^{-1}t_pn_p\tilde\gamma)\pi_{F,p}(\tilde\gamma^{-1})\phi_p,\pi_{F,p}(\tilde\gamma^{-1})\phi_p\rangle}{\langle\phi_p,\phi_p\rangle}\\&=\frac{\langle\pi_{F,p}(\tilde\gamma^{-1}t_pn_p\tilde\gamma)\phi_p,\phi_p\rangle}{\langle\phi_p,\phi_p\rangle}.\end{aligned}$$

Now $\mathrm{tr}(SX)=\mathrm{tr}\left(\frac{{}^t\gamma S\gamma}{\det\gamma}\cdot(\det(\gamma))\gamma^{-1}X\,{}^t\gamma^{-1}\right)$ and

$$n((\det(\gamma)\gamma^{-1}X\,{}^t\gamma^{-1})=\tilde\gamma^{-1}n(X)\tilde\gamma.$$

Using $\xi_S=\iota S$ with $\iota:=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, and the fact that $\gamma^{-1}\iota\,{}^t\gamma^{-1}=(\det(\gamma))^{-1}\iota$ (for *any* invertible 2-by-2 matrix γ), we find that $\gamma^{-1}\xi_S\gamma=\xi_{S'}$, where $S'=\frac{{}^t\gamma S\gamma}{\det(\gamma)}=\begin{pmatrix}\frac{1}{2\det(\gamma)}&0\\0&-\frac{1}{2\det(\gamma)}\end{pmatrix}$, which gives a standard $T_{S'}=\begin{pmatrix}\mathbb{Q}_p^\times&0\\0&\mathbb{Q}_p^\times\end{pmatrix}$. Conjugation by $\tilde\gamma^{-1}$ maps T_S to $T_{S'}$ (inside $\mathrm{GSp}_2(\mathbb{Q}_p)$) and $N(\mathbb{Q}_p)$ to $N(\mathbb{Q}_p)$, preserving the measures and converting the integral for $J(\phi_p)$ to a standard form independent of K and Λ (subject to the conditions of the proposition).

It remains to show that γ exists. If $c = 0$ then $ax^2 + bxy + cy^2 = x(ax + by) = XY$, where $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, so $\gamma^{-1} = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \in I_p$ since $p \mid a$. If $c \neq 0$ then, applying first a γ of the form $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, we may suppose that $p \nmid c$. By the assumption $\chi_{-d}(p) = 1$, and since p is odd, there exists $\beta \in \mathbb{Z}_p$ with $\beta^2 = -d$. Since $\beta^2 = b^2 - 4ac \equiv b^2 \pmod{p}$, we may choose $\beta \equiv b \pmod{p}$. Then $ax^2 + bxy + cy^2 = XY$, where $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} (b+\beta)/2c & 1 \\ (b-\beta)/2 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, so $\gamma^{-1} = \begin{pmatrix} (b+\beta)/2c & 1 \\ (b-\beta)/2 & c \end{pmatrix} \in I_p$ since $p \mid (b - \beta)$ and $p \nmid c$. $\square$

The analogue of Proposition 4.4 is almost identical, including the condition $\ell > 2k-2$, but in the following there is no condition on the prime number ℓ such that $\lambda \mid \ell$.

Theorem 5.4. *For M odd and squarefree, $k \geq 2$, suppose that $F \in S_k(\Gamma^{\mathrm{para}}(M))$ is a newform (cf. [37]), not a Saito-Kurokawa lift, with Fourier coefficients integral at λ, not all divisible by λ. Suppose that ϕ is a Jacobi form of weight k, level 1 and index M, and that there is a congruence of Fourier coefficients $F \equiv \mathrm{Grit}(\phi) \pmod{\lambda}$. Suppose that there exists a fundamental discriminant $-D < 0$ such that $\chi_{-D}(p) = 1$ for all primes $p \mid M$, and*

$$R(F, \mathbb{Q}(\sqrt{-D}), \mathrm{id}) \not\equiv 0 \pmod{\lambda},$$

equivalently

$$R(\mathrm{Grit}(\phi), \mathbb{Q}(\sqrt{-D}), \mathrm{id}) \not\equiv 0 \pmod{\lambda}.$$

Then Conjecture 5.1 implies that $L(k-1, F, \mathrm{spin}) \neq 0$, and that for any fundamental discriminant $d > 0$, coprime to ℓD, such that $\chi_d(p) = 1$ for all primes $p \mid M$, assuming also that $L(k-1, F, \mathrm{spin}, \chi_d) \neq 0$,

$$\mathrm{ord}_\lambda\left(\frac{L(k-1, F, \mathrm{spin})}{L(k-1, F, \mathrm{spin}, \chi_d)}\right) \leq -2.$$

We do not assume that ϕ is a Hecke eigenform, though the case where $\mathrm{Grit}(\phi) = \hat{f}$, for a newform $f \in S_{2k-2}(\Gamma_0(M))$ with sign -1 in the functional equation of $L(s, f)$, is of particular interest.

Proof. For $S = \begin{pmatrix} \alpha M & b/2 \\ b/2 & c \end{pmatrix}$ of discriminant $-dD$, $a(\mathrm{Grit}(\phi), S)$ depends only on the discriminant and on $b \pmod{2M}$, cf. [36, around Proposition 4.2]. (Note that this is unlike the $\Gamma_0^{(2)}(M)$ case, where a Fourier coefficient of a Saito-Kurokawa lift depended only on the discriminant.) For each $\rho \pmod{2M}$ with $\rho^2 \equiv -dD \pmod{4M}$, we may define a subset $\mathcal{Q}_{M,-dD,\rho}$ of $\mathcal{Q}_{M,-dD}$, comprising those S for which $b \equiv \rho \pmod{2M}$. Gross, Kohnen and Zagier show [22, §I.1] that the map $\Phi : \mathcal{Q}_{M,-Dd}/\Gamma_0(M) \to \mathcal{Q}_{1,-Dd}/\mathrm{SL}_2(\mathbb{Z})$ remains surjective even when restricted to $\mathcal{Q}_{M,-dD,\rho}/\Gamma_0(M)$. The size of the pre-image in $\mathcal{Q}_{M,-dD,\rho}$ of any $c \in \mathcal{Q}_{1,-Dd}/\mathrm{SL}_2(\mathbb{Z})$ is a power of 2 independent of c (but possibly depending on ρ).

Letting $\Lambda : \mathrm{Cl}_{\mathbb{Q}(\sqrt{-dD})} \to \mathbb{C}^\times$ be as in Remark 5.2, the unramified quadratic character associated to $\mathbb{Q}(\sqrt{-D},\sqrt{d})/\mathbb{Q}(\sqrt{-dD})$, we see that for each ρ the subsum

$$\sum_{[S]\in\mathcal{Q}_{M,-dD,\rho}/\Gamma_0(M)} a(\mathrm{Grit}(\phi),S)\Lambda^{-1}(\Phi([S]))=0.$$

By Remark 5.2, $\Lambda(\Phi([S]))=\Lambda([S])$, so, summing over ρ,

$$R(\mathrm{Grit}(\phi),\mathbb{Q}(\sqrt{-dD}),\Lambda)=0.$$

The congruence implies that

$$R(F,\mathbb{Q}(\sqrt{-dD}),\Lambda)\equiv 0 \pmod{\lambda}.$$

We may now proceed as in the proof of Theorem 1.5, noting that we must be in a case where $J_p\neq 0$ for all primes $p\mid M$, since $R(F,\mathbb{Q}(\sqrt{-D}),\mathrm{id})\neq 0$. $\square$

Results of Brown and Li [11, Theorem 6.9, Corollary 6.14] provide a substitute for Theorems 1.2 and 4.2 in the paramodular case, and we could use them to prove congruences of Fourier coefficients between paramodular Saito-Kurokawa lifts and non-lifts. Then we could apply Theorem 5.4 to these congruences to get some analogue of Theorems 1.5 and 4.3. But perhaps it is more interesting to apply Theorem 5.4 to some numerical examples with $k=2$ or 3.

Example 5.5. By [34, Theorem 7.3], $S_2(\Gamma^{\mathrm{para}}(277))$ is 11-dimensional, with a 10-dimensional subspace of Gritsenko lifts, and a non-lift $\mathbb{T}$-eigenform F, with rational integer Fourier coefficients (g.c.d. 1), satisfying a congruence

$$a(F,S)\equiv a(\mathrm{Grit}(\phi),S) \pmod{15}\ \forall S.$$

Here ϕ is the first Fourier-Jacobi coefficient of F and $\mathrm{Grit}(\phi)$ is its Gritsenko lift. Ryan and Tornaria [36, Table 4] computed $R(F,\mathbb{Q}(\sqrt{-7}),\mathrm{id})=2$. Note that $\chi_{-7}(277)=1$. Let $K=\mathbb{Q}(\sqrt{-7d})$, with $d>0$ a fundamental discriminant such that $\chi_d(277)=1$, and let Λ be the quadratic character of Cl_K in the proof of Theorem 1.5. The fact that $R(F,K,\Lambda)\equiv 0 \pmod{15}$ is illustrated numerically for several values of d in [36, Table 4]. This is essentially [36, Proposition 5.1]. Anyway, Theorem 5.4 says that Conjecture 5.1 implies

$$\mathrm{ord}_\ell\left(\frac{L(1,F,\mathrm{spin})}{L(1,F,\mathrm{spin},\chi_d)}\right)\le -2,$$

for $\ell=3$ and $\ell=5$ (assuming as always that the denominator does not vanish). Recall that Proposition 5.3 shows that Conjecture 5.1 follows from Y. Liu's refined Gan-Gross-Prasad conjecture, so this inequality will become unconditional if Furusawa and Morimoto extend their proof to non-trivial Λ, as will those in the examples below. There is also experimental support for several values of d in [36, Table 4]. They approximated L-values numerically, obtaining coefficients in the Dirichlet series by counting points mod p on the hyperelliptic curve below.

It has been proved recently by Brumer et. al. [12] that $L(s,F,\mathrm{spin})$ is the L-function of the jacobian J of the genus 2 curve $y^2+y=x^5+5x^4+8x^3+6x^2+2x$, which has a rational point of order 15. The Birch and Swinnerton-Dyer conjecture then predicts a contribution of 15^2 to the denominator of $L(1,F,\mathrm{spin})/\Omega$, where Ω is a determinant of periods of Néron differentials. This factor of 15^2 is exposed by considering the ratio $\frac{L(1,F,\mathrm{spin})}{L(1,F,\mathrm{spin},\chi_d)}$. The trivial composition factors generated by rational torsion points in the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules $J[3]$ and $J[5]$ can also be connected with congruences of Hecke eigenvalues between Hecke eigenforms, $F \equiv \hat{f}_1 \pmod 3$ and $F \equiv \hat{f}_2 \pmod \lambda$, where f_1 has rational Hecke eigenvalues and $\lambda \mid 5$ in a number field of degree 9. These congruences of Hecke eigenvalues extend to congruences of Fourier coefficients, as noted following the proof of Theorem 7.3 in [34].

Example 5.6. By [33, Theorem A.1], $S_2(\Gamma^{\mathrm{para}}(731))$ (where $731 = 17 \times 43$) is 19-dimensional, with an 18-dimensional subspace of Gritsenko lifts, and a non-lift $\mathbb{T}$-eigenform F, with rational integer Fourier coefficients (g.c.d. 1), satisfying a congruence

$$a(F,S) \equiv a(\hat{f},S) \pmod 5 \ \forall S.$$

Here, as in [6, §9.1], $f \in S_2(\Gamma_0(731))$ is a normalised newform with rational coefficients, associated with the elliptic curve (Cremona label $731a1$) $y^2+xy+y = x^3-539x+4765$. We have $\chi_{-8}(17) = \chi_{-8}(43) = 1$. Also $R(F,\mathbb{Q}(\sqrt{-2}),\mathrm{id}) =$

$$a\left(F,\begin{pmatrix}731 & 462/2\\ 462/2 & 73\end{pmatrix}\right)+a\left(F,\begin{pmatrix}731 & 54/2\\ 54/2 & 1\end{pmatrix}\right) = -1+(-1) = -2 \not\equiv 0 \pmod 5.$$

Letting $d>0$ be a fundamental discriminant such that $2 \nmid d$ and $\chi_d(17) = \chi_d(43) = 1$, Theorem 5.4 says that Conjecture 5.1 implies

$$\mathrm{ord}_5\left(\frac{L(1,F,\mathrm{spin})}{L(1,F,\mathrm{spin},\chi_d)}\right) \le -2.$$

It has been proved recently by Berger and Klosin [6] that $L(s,F,\mathrm{spin})$ is the L-function of the jacobian of the genus 2 curve $y^2+(x^3+x^2)y = x^5+2x^4-x-3$, which has a rational point of order 5. The Birch and Swinnerton-Dyer conjecture then predicts a contribution of 5^2 to the denominator of $L(1,F,\mathrm{spin})/\Omega$, where Ω is a determinant of periods of Néron differentials, which is exposed by considering the ratio $\frac{L(1,F,\mathrm{spin})}{L(1,F,\mathrm{spin},\chi_d)}$.

Example 5.7. Poor and Yuen [34, §8, Example 1] showed that $S_3(\Gamma^{\mathrm{para}}(61))$ is 7-dimensional, with a 6-dimensional subspace of Gritsenko lifts, and a non-lift $\mathbb{T}$-eigenform F, with rational integer Fourier coefficients (g.c.d. 1), satisfying a congruence

$$a(F,S) \equiv a(\mathrm{Grit}(\phi),S) \pmod{43} \ \forall S.$$

Here ϕ is a certain Jacobi cusp form of weight 3 and index 61, and $\mathrm{Grit}(\phi)$ is its Gritsenko lift. We have $\chi_{-3}(61) = 1$. Also

$$R(F,\mathbb{Q}(\sqrt{-3}),\mathrm{id}) = a\left(F,\begin{pmatrix}61 & -27/2\\ -27/2 & 3\end{pmatrix}\right) = 3 \not\equiv 0 \pmod{43}.$$

Let $d>0$ be a fundamental discriminant such that $3 \nmid d$, $\chi_d(61) = 1$. Then Theorem 5.4 says that Conjecture 5.1 implies

$$\mathrm{ord}_{43}\left(\frac{L(2,F,\mathrm{spin})}{L(2,F,\mathrm{spin},\chi_d)}\right) \le -2.$$

To link this with the analysis in §3, we can easily derive, from the congruence between F and $\mathrm{Grit}(\phi)$, a congruence mod λ, of Hecke eigenvalues or even of Fourier coefficients, between F and $\hat{f}$, where $f \in S_4(\Gamma_0(61))$ is a newform with coefficients in a number field E of degree 6, and $\lambda \mid 43$ is a prime divisor in E. (Expressing $\mathrm{Grit}(\phi)$ as a linear combination of $\hat{f}_1,\ldots,\hat{f}_6$, where $\{f = f_1, f_2, \ldots, f_6\}$ is a basis of Hecke eigenforms for $S_4(\Gamma_0(61))$, in the relation $F \equiv \mathrm{Grit}(\phi) \pmod \lambda$, one applies an element of $\mathbb{T}$ to kill the $\hat{f}_2, \dots, \hat{f}_6$ components, in the manner of the proof of Theorem 1.2.)

As in [11, §7], this congruence of Hecke eigenvalues, interpreted as mod λ reducibility of a Galois representation, leads to an element in a Selmer group which, via the Bloch-Kato conjecture, predicts that $\lambda \mid L_{\mathrm{alg}}(3, f)$. (We might have remarked earlier that in general, the Bloch-Kato conjecture links the congruence to the appearance of λ in both the numerator of $L_{\mathrm{alg}}(k, f)$ and the denominator of $L_{\mathrm{alg}}(k-1, F, \mathrm{spin})$.) Using the command LRatio in the Magma computer package [9], one readily checks that, as expected, $43 \mid \mathrm{Nm}_{E/\mathbb{Q}}(L_{\mathrm{alg}}(3, f))$. Corollary 6.14 in [11], which proves a congruence of Hecke eigenvalues from divisibility of $L_{\mathrm{alg}}(k, f)$, does not apply here, since the condition $k \geq 6$ is far from being satisfied. It would be very nice to be able to prove this connection between L-values and congruences for such a low value as $k = 3$. Regarding Examples 5.5 and 5.6, $k = 2$ is too low, since then $k > 2k - 3$, so $L(k, f)$ is not even a critical value.

Using their methods, Poor and Yuen were able to produce the Euler factors at $2, 3$ and 5 of $L(s, F, \mathrm{spin})$. Some time around the end of 2010, in collaboration with V. Golyshev, A. Mellit computed the first 1000 coefficients of a Dirichlet series, using the requirement that it should satisfy experimentally a functional equation of the type expected of $L(s, F, \mathrm{spin})$, cf. [17, §6]. Not only were these compatible with the above Hecke eigenvalue congruence with $\hat{f}$, they were also sufficient to check numerically that 43^2 appears to divide the ratio of the central L-value to a twisted central L-value, using the algorithm of T. Dokchitser implemented in Magma [17] to approximate the L-values.

The data also supports a congruence

$$\mu_F(T(p)) \equiv 1 + p^3 + p a_p(g) \pmod{\lambda'},$$

for all primes $p \neq 61$, where $g = \sum a_n(g) q^n \in S_2(\Gamma_0(61))$ is a normalised newform with coefficients in a cubic field E', and $\lambda' \mid 19$ is a divisor in E'. (Golyshev conjectured the existence of a second congruence for F, beyond the one involving 43, then K. Buzzard found it, having realised the possibility of it involving weight 2 rather than weight 4.) For over ten years this remained unproved, but recent computations of Rama and Tornaría using quinary forms have led to a proof, which will be published in due course.

The left hand side of the congruence was obtained experimentally from the coefficients of the Dirichlet series. For yet another way to produce the first 100 coefficients of this Dirichlet series, using algebraic modular forms for the orthogonal group of a quinary quadratic form, see [23, Appendix B] (computations that have been extended to the first 3000 coefficients by Rama and Tornaría [35]). It is expected that $L(s, F, \mathrm{spin})$ is the L-function attached to the 3^{rd} cohomology of some Calabi-Yau 3-fold defined over $\mathbb{Q}$, with Hodge numbers $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$. In support of this, Golyshev has recently matched it experimentally with a fibre in one of the 4^{th} order motivic variations in the AESZ database [3].

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