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Bayesian persuasion in unlinked games

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Abstract
Originating from Kamenica and Gentzkow (Am Econ Rev 101(6):2590–2615, 2011), we analyze multi-receiver Bayesian persuasion games with heterogeneous beliefs without strategic interactions among receivers, which we call *unlinked*. We show that given the receivers' best-responses, the sender's rationalizable strategies are obtained from a single linear programming problem.

Keywords Bayesian persuasion · Multiple receivers · Heterogeneous beliefs · Rationalizability

JEL Classification C72 · D83

1 Introduction

In this paper, we extend the analysis of Kamenica and Gentzkow (2011) to a broader class of games. In their seminal study, Kamenica and Gentzkow (2011) added a new dimension to the literature on strategic information transmission by introducing a novel type of message space. In the pioneering work of Crawford and Sobel (1982), the sender chooses different messages depending on the states the sender privately observes. The departure in Kamenica and Gentzkow (2011) is that, while the messages still depend on the states, the sender commits herself to a (verifiable) distribution over

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the messages conditional on the *unknown state*. This implies that both the sender's belief and the receiver's belief regarding the states play important roles. Examples of such a communication method include examinations and experiments. Kamenica and Gentzkow (2011) analyzed games with this type of communication methods: *Bayesian persuasion games*.

In Kamenica and Gentzkow (2011), there is one sender and one receiver who share a common (initial) belief regarding the states. After observing the sender's strategy and the message, the receiver updates her belief about the states and chooses a best-response. By identifying the receiver's best-response for each strategy and message, the sender chooses a strategy which maximizes her expected payoff. *The sender influences the receiver's behavior through the receiver's belief update.* Kamenica and Gentzkow (2011) provide a condition under which the sender benefits from persuasion and analyze optimal strategy. Their technical contribution is that for the sender's optimization problem, she can only focus on the interim (updated) belief which needs to satisfy a consistency condition called *Bayes plausibility*.[1]

The purpose of our study is to generalize their framework. Namely, we analyze *multi-receiver* Bayesian persuasion games with *heterogeneous beliefs* where there is no strategic interaction among receivers. We call such games *unlinked*.[2] Kamenica and Gentzkow (2011, Section VI) suggested that their approach could be extended to Bayesian persuasion games with multiple receivers where (i) the sender's preferences are separable with respect to the receivers' actions, *and* (ii) each receiver only cares about her action. While (ii) is assumed in our study, (i) is not, implying that the approach in Kamenica and Gentzkow (2011) cannot be readily applied. Given that there are cases naturally arising with multiple audiences and heterogeneous beliefs, the development of the framework to analyze such scenarios is of great importance.

Our main result (Sect. 4) shows that the identification of optimal strategies for the sender in any unlinked game can be seen as *a single linear programming problem*.[3] In addition to the computational tractability, this immediately implies that the sender has the unique rationalizable strategy *generically*.

Since our focus is on unlinked games, each receiver's optimization is rather a single-person decision problem at the second stage. In addition, we first analyze the second stage (receivers), and then the first stage (sender). Thus, the approach resembles back-

[1] This is also known as the concavification approach. See Aumann and Maschler (1995).

[2] Regarding heterogeneous beliefs, one can solve the sender's optimization problem as if they share the common prior since in the sender's and the receivers' payoff functions, π—sender's strategy—is multiplied by their priors. There is no need for such a detour in our approach. See Alonso and Câmara (2016a) for heterogenous priors with one receiver. Alonso and Câmara (2016b), Yun (2013), Chan et al. (2019) analyze voting games with multiple receivers. For the literature on cheap talk (a closely related field), see Farrell and Gibbons (1989) and Goltsman and Pavlov (2011) for multiple receivers and Miura and Yamashita (2020) for the notion of maximal miscommunication.

[3] Studies with similar approaches include the following. By assuming a linear-quadratic specification, Tamura (2018) uses the semidefinite programming approach. Kolotilin (2018) assumes (i) both the sender and the receiver have their own types with supports that are compact (and the sender's type is realized only after her message is sent), (ii) the receiver's action space is binary, (iii) one action leads to a payoff of zero (normalization) for both the sender and the receiver, and (iv) the single-crossing assumption for the receiver's preferences from the other action; there exists a threshold of her private information with which her expected payoff is zero. The sender only needs to make sure that the receiver's expected payoff from the other action is zero.

ward induction (with incomplete information). This allows us to deviate in two aspects. First, given its close relationship to backward induction (e.g., Battigalli (1997)), our linear programming approach directly corresponds to the notion of Δ-*rationalizability*, introduced and developed by Battigalli and Siniscalchi (2003).[4] In particular, Δ-rationzalibility explicitly allows heterogeneous beliefs. Second, instead of dealing with the receivers' interim beliefs as the variables for the sender, as in Kamenica and Gentzkow (2011), we directly look at the sender's strategies—distributions over the set of messages conditional on the states—which implies that there is no need to check whether the solution satisfies the Bayes plausibility condition.

Given the relationship between Nash equilibrium and linear programming for zero-sum games, for example, one may suspect that our main result is rather a straightforward exercise.[5] Given the structure of unlinked games, however, we need to establish several results to reach our main result. Our approach takes the following steps. First, given an action profile for the receivers, we identify the set of strategies for the sender which induce the receivers to choose the corresponding action profile. If the set is nonempty, the sender can identify an optimal strategy within this set. This set can be seen as a "constraint" for the sender, and the identification of the sender's optimal strategy under the "constraint" can be seen as a linear programming problem. Second, we show that the sender can indeed identify her rationalizable strategy via a *single* linear programming problem with the "largest constraint", implying that there is no need to solve the sender's optimization problem for each possible constraint.

There are two main hurdles. First, the sets of strategies for the sender which induce the receivers to choose certain action profiles (which we called "constraints" above) are *not* closed, implying the possibility that rationalizable strategies for the sender do not exist. Although the sender's optimization problem becomes well-defined by taking the *closure* of each "constraint", this still does not guarantee the existence of solutions. We show that taking the closure of each constraint is indeed sufficient to identify the sender's rationalizable strategies (Sect. 4.1). Second, the argument suggests that we need to consider *multiple* optimization problems (for all the non-empty constraints). We show the existence of the largest constraint under which a single optimization problem leads to the identification of rationalizable strategies for the sender (Sect. 4.2). Our result also shows how the largest constraint can be constructed.

As shown in Kamenica and Gentzkow (2011), the receivers may be indifferent among multiple actions after observing certain messages. The receivers nevertheless choose the actions which the sender intends for them in our solutions. Kamenica and Gentzkow (2011) utilized the notion of sender-preferred equilibrium, which ensures that given the receiver's (sender-preferred) best-response, the sender's expected payoff is upper-semicontinuous in the interim belief, implying the existence of the sender's optimal strategy. We do not impose such restrictions. This is because if the receivers are expected to choose other (unintended) actions, there is simply no corresponding

4 Given a rationalizable strategy for the sender in an unlinked game, the corresponding strategy profile can also be supported as a Bayesian Nash equilibrium. As a by-product, the employment of rationalizability provides a solid epistemic foundation for our solutions; see, for example, Dekel and Siniscalchi (2015). The notion of *forward induction* implied by rationalizability can give a sharper prediction when there are strategic interactions among the receivers. See Shimoji (2016).

5 See for example Adler (2013).

rationalizable strategy for the sender (Sect. 4.3).[6] We also investigate the properties of the constraints (Sect. 4.4)

There are several recent studies on Bayesian persuasion, some of which utilize similar strategy spaces for the sender. Brocas and Carrillo (2007) consider a model with two states where instead of choosing signal distributions conditional on the states, the sender chooses the number of times (binary) signals are revealed given the pre-fixed signal distribution conditional on the states. Rayo and Segal (2010) consider a model in which the set of actions is binary: "accept" or "reject". The sender chooses the distribution of signals conditional on the states (prospects). If the receiver accepts, players' payoffs depend on the state. If the receiver rejects, she receives the realization of her uniformly distributed reservation payoff (she knows the value when she chooses her action) while the sender's payoff is pre-determined. Ostrovsky and Schwarz (2010) consider job matchings between students and potential employers. It is the school that knows the students' types and controls the information revelation to the potential employers. Hörner and Skrzypacz (2016) analyze the model with multiple rounds of persuasion stages. Kolotilin et al. (2017) study persuasion when the receiver privately observes her type (different from the state). Bergemann and Morris (2016) consider correlated equilibrium for games with incomplete information under a common prior and show that their approach can analyze Bayesian persuasion games with multiple receivers. The decision rule (mediator) which recommends actions would correspond to the message in the current paper. Other recent applications include Hernández and Neeman (2021), Lipnowski and Mathevet (2018) and Taneva (2019). See Bergemann and Morris (2019) and Sobel (2013) for further discussions.

After providing the set-up in Sect. 2, we show several examples in Sect. 3 to demonstrate how our approach works. The examples also highlight some of the results in the paper. In Sect. 4, we establish the results of the linear programming approach for unlinked games.

2 Preliminaries

There is one sender. Let N be the finite set of receivers with $|N| = n \geq 1$. The finite set of states is Θ with θ being a typical element. Let p_S^0 be the sender's (commonly known) initial belief over Θ. Likewise, for each receiver $i \in N$, let p_i^0 be her (commonly known) initial belief over Θ. We assume that $p_S^0(\theta) > 0$ for each $\theta \in \Theta$ and $p_i^0(\theta) > 0$ for each $\theta \in \Theta$ and $i \in N$. We allow heterogeneous beliefs. It is important to specify the beliefs not only for the receivers but also for the sender. This is because (i) the sender chooses her strategy without knowing the actual state, and (ii) difference in beliefs could lead to different outcomes. The examples in Sect. 3 demonstrate the latter point.

At the first stage, the sender chooses her strategy while the receivers choose their actions at the second stage. Let A_i be the finite set of actions for receiver $i \in N$ with a_i being a typical element and $A = \times_{j \in N} A_j$. The sender's payoff function is

[6] A similar observation can be found in the ultimatum game with the continuous action space for the first mover, for example.

$u_S : A \times \Theta \to \mathbb{R}$ while for each receiver $i \in N$, we have $u_i : A_i \times \Theta \to \mathbb{R}$. Note that each receiver's payoff does not depend on the other receivers' actions. We call this class of games *unlinked*.

First stage A *message* for each receiver is simply a recommendation of which action to take.[7] We thus take A_i as the set of messages for each receiver $i \in N$ as well. The sender's strategy is a distribution over A conditional on Θ, denoted by π. Let Π be the set of the sender's strategies. Given $\pi \in \Pi$, for each $i \in N$, let

$$\mathcal{A}_i(\pi) = \{a_i \in A_i \mid \pi(a_i, a_{-i} \mid \theta) > 0 \text{ for some } a_{-i} \in A_{-i} \text{ and } \theta \in \Theta\}.$$

In other words, $\mathcal{A}_i(\pi)$ is the set of realizable messages for receiver $i \in N$ under π. Note that by choosing π, the sender is also choosing the set of messages, $\mathcal{A}(\pi) = \times_{j \in N} \mathcal{A}_j(\pi)$. Note also that given $\pi \in \Pi$, there may exist $a \in \mathcal{A}(\pi)$ which will not be realized, since (i) $\mathcal{A}(\pi)$ is a product set and (ii) π allows correlations.

Second stage We assume (i) that the sender's strategy π is observable to the receivers, and (ii) that the realized message, $a_i \in \mathcal{A}_i(\pi)$, is private information for each receiver $i \in N$. For each $i \in N$, let $M_i \subset \Pi \times A_i$ be such that for any $(\pi, a_i) \in M_i$, $a_i \in \mathcal{A}_i(\pi)$.[8] Given π, the marginal $\pi(a_i \mid \theta) = \sum_{a_{-i} \in A_{-i}} \pi((a_i, a_{-i}) \mid \theta)$ for each $a_i \in \mathcal{A}_i(\pi)$ and $\theta \in \Theta$ is computed for each receiver $i \in N$. After observing $(\pi, a_i) \in M_i$, each receiver $i \in N$ revises her belief regarding each $\theta \in \Theta$ via Bayes' rule:

$$p_i^\pi(\theta \mid a_i) = \frac{\pi(a_i \mid \theta) p_i^0(\theta)}{\sum_{\tilde{\theta} \in \Theta} \pi(a_i \mid \tilde{\theta}) p_i^0(\tilde{\theta})}.$$

Each receiver $i \in N$ then simultaneously chooses her action under her interim (updated) belief, $p_i^\pi(\cdot \mid a_i)$. Let (i) $s_i : M_i \to A_i$ be a pure strategy for receiver $i \in N$ with S_i being the set of strategies for receiver $i \in N$, and (ii) $s_i(\pi, a_i) \in A_i$ be receiver i's action after observing $(\pi, a_i) \in M_i$. Note that the message itself has no meaning for the receivers who only care about their interim beliefs. This means that any permutation of A_i as the set of messages works as well as A_i itself. To simplify our analysis, our focus is on the case where the receivers follow the messages. We will discuss this in Sect. 4.

The sender's strategy π is called *null for player* $i \in N$ if it induces an interim belief identical to the initial belief for each message.[9] If π is null for each player $i \in N$, we say that π is *null*, which is denoted by π^0. For each receiver $i \in N$, let $A_i^0 \subseteq A_i$ be the set of actions which are best-responses to her initial belief. For any $i \in N$, if $|A_i^0| = 1$, we simply let $A_i^0 = \{a_i^0\}$, i.e., a_i^0 is the unique best-response for receiver i's initial belief.

[7] Kamenica and Gentzkow (2011) call such strategies *straightforward*.

[8] Remember that $\pi \in \Pi$ uniquely determines $\mathcal{A}_i(\pi)$ and an element of $\mathcal{A}_i(\pi)$ is chosen by nature.

[9] This is the case if $\mathcal{A}_i(\pi)$ is singleton, or for each $a_i \in \mathcal{A}_i(\pi)$, $\pi(a_i \mid \theta') = \pi(a_i \mid \theta'')$ for each $\theta', \theta'' \in \Theta$. Note that since the sender chooses π over $A \neq \emptyset$, the receivers always observe a message.

Our approach follows the notion of Δ-rationalizability (Battigalli and Siniscalchi 2003). In this approach, "Δ" corresponds to players' initial (possibly heterogeneous) beliefs which are p_S^0 and p_i^0 for each $i \in N$ in the current study. In general, (i) the set of rationalizable strategies is obtained by iteratively eliminating strategies which are not best-responses for each player, and (ii) as a consequence, the application of rationalizability is considered to be challenging. For the current context (i.e., unlinked games), the identification of rationalizable strategies for each receiver is rather straightforward since each receiver's optimization problem is a single-person decision problem. The identification of rationalizable strategies for the sender thus takes into account each receiver's decision *à la* backward induction. This order of operation is indeed what we have for the linear programming approach, which is discussed in Remark 5.[10]

3 Examples

In this section, we consider three examples to demonstrate how our approach works. In particular, we show (i) that by utilizing one example from Kamenica and Gentzkow (2011), our approach chooses the same outcome as theirs, (ii) that heterogeneity in beliefs would lead to different outcomes, and (iii) that our approach can handle multiple receivers.

3.1 Examples: one receiver

In this subsection, we first use the example from Kamenica and Gentzkow (2011) to demonstrate that our approach also chooses the equilibrium outcome identified in their example. Second, in the modified example, we demonstrate that outcomes could be substantially different depending on players' heterogeneous beliefs. The two examples share the same discussion for the receiver's behavior, which we provide below. Differences in the two examples are (i) the sender's preferences and (ii) heterogeneity in beliefs.

Receiver 1 has two actions, $A_1 = \{a_1', a_1''\}$. There are two states, $\Theta = \{\theta', \theta''\}$. Let p_S^0 and p_1^0 be the probabilities assigned to θ' for the sender and receiver 1 respectively. The sender chooses $\pi \in \Pi$. We assume that $u_1(a_1', \theta') = u_1(a_1'', \theta'') = 1$ and $u_1(a_1', \theta'') = u_1(a_1'', \theta') = 0$.

Receiver's behavior This part is shared by the first two examples we discuss below. First, consider the case where $|\mathcal{A}_1(\pi)| = 1$ (i.e., only one message is realized) as the benchmark. Receiver 1 chooses

$$\begin{Bmatrix} a_1' \\ a_1'' \end{Bmatrix} \text{ only if } p_1^0 \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} \tfrac{1}{2}.$$

Second, consider the case where $\mathcal{A}_1(\pi) = A_1$. After observing π and $\tilde{a}_1 \in \mathcal{A}_1(\pi)$, receiver 1's expected payoffs are

[10] The application of Δ-rationalizability can also be seen as that of *conditional dominance* (Shimoji and Watson 1998) with incomplete information.

$$\left\{\begin{array}{l} p_1^\pi(\theta' \mid \tilde{a}_1)u_1(a_1', \theta') + p_1^\pi(\theta'' \mid \tilde{a}_1)u_1(a_1', \theta'') = p_1^\pi(\theta' \mid \tilde{a}_1) \\ p_1^\pi(\theta' \mid \tilde{a}_1)u_1(a_1'', \theta') + p_1^\pi(\theta'' \mid \tilde{a}_1)u_1(a_1'', \theta'') = p_1^\pi(\theta'' \mid \tilde{a}_1) \end{array}\right\}$$
$$\text{from choosing } \left\{\begin{array}{c} a_1' \\ a_1'' \end{array}\right\}.$$

Thus, receiver 1 chooses

$$\left\{\begin{array}{c} a_1' \\ a_1'' \end{array}\right\} \text{ only if } p_1^\pi(\theta' \mid \tilde{a}_1) \left\{\begin{array}{c} \geq \\ \leq \end{array}\right\} p_1^\pi(\theta'' \mid \tilde{a}_1)$$

or equivalently

$$\left\{\begin{array}{c} a_1' \\ a_1'' \end{array}\right\} \text{ only if } \pi(\tilde{a}_1 \mid \theta'') \left\{\begin{array}{c} \leq \\ \geq \end{array}\right\} \left[\frac{p_1^0}{1-p_1^0}\right] \pi(\tilde{a}_1 \mid \theta').$$

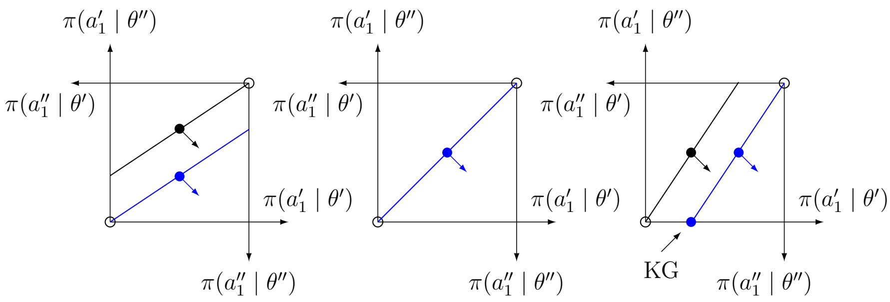
Receiver 1 follows the realized message, i.e., choosing a_1' after observing a_1' and a_1'' after observing a_1'' only if

$$\begin{aligned} \pi(a_1' \mid \theta'') &\leq \left[\frac{p_1^0}{1-p_1^0}\right] \pi(a_1' \mid \theta') \\ \pi(a_1'' \mid \theta'') &\geq \left[\frac{p_1^0}{1-p_1^0}\right] \pi(a_1'' \mid \theta'). \end{aligned} \tag{1}$$

Depending on the value of p_1^0, (1) is visualized in Fig. 1. Since there are two states and two actions, the sender chooses two parameters, corresponding to one message for each state; e.g., $\pi(a_1' \mid \theta')$ and $\pi(a_1' \mid \theta'')$. While we can then modify (1) by substituting $\pi(a_1'' \mid \theta') = 1 - \pi(a_1' \mid \theta')$ and $\pi(a_1'' \mid \theta'') = 1 - \pi(a_1' \mid \theta'')$ to have only two variables, we can also use the box diagram to illustrate the expressions in (1), as shown in Fig. 1. Each side is equal to one, and any point in the box diagram represents the sender's strategy π. Note that we exclude the origins since we have $\mathcal{A}_1(\pi) = A_1$. In other words, we cannot have $\pi(a_1' \mid \theta') = \pi(a_1' \mid \theta'') = 0$ and $\pi(a_1'' \mid \theta') = \pi(a_1'' \mid \theta'') = 0$. The first expression in (1) has the origin in the bottom-left corner while the second expression in (1) has the origin in the top-right corner. For each expression, the slope is equal to $\frac{p_1^0}{1-p_1^0}$ (and hence the two lines in each box diagram are parallel to each other).

Figure 1 includes three qualitatively different cases: (a) $p_1^0 < \frac{1}{2}$, (b) $p_1^0 = \frac{1}{2}$, and (c) $p_1^0 > \frac{1}{2}$. The line from each origin indicates the set of π's with which the corresponding expression holds with equality. The set of π's under which each expression holds with strict inequality is indicated by the corresponding arrow. The intersection of two inequalities is therefore the set of π's with which the sender can make the receiver choose the intended actions, depending on the realized messages.

Sender's behavior We provide two examples. The first example is from Kamenica and Gentzkow (2011), with which we demonstrate that our approach chooses the



(a) $p_1^0 < \frac{1}{2}$ **(b)** $p_1^0 = \frac{1}{2}$ **(c)** $p_1^0 > \frac{1}{2}$

Fig. 1 Visualization of (1)

same outcome. In the second example, we show that heterogenous beliefs could lead to different outcomes.

Example 1: Kamenica and Gentzkow (2011) This corresponds to the original example from Kamenica and Gentzkow (2011, p. 2591). The sender's payoffs are such that $u_S(a_1', \theta) = 0$ and $u_S(a_1'', \theta) = 1$ for each $\theta \in \Theta$; her payoffs are state-independent. Let $p_S^0 = p_1^0 = 0.7$, corresponding to Fig. 1c. Receiver 1 chooses a_1' without persuasion, which the sender avoids. The sender's expected payoff is higher if the chance of a_1'' is higher; i.e, towards the southwest. The sender therefore chooses $\pi(a_1'' \mid \theta') = \frac{1-0.7}{0.7} = \frac{3}{7}$ and $\pi(a_1'' \mid \theta'') = 1$, consistent with the solution in Kamenica and Gentzkow (2011).

Remark 1 If receiver 1 observes a_1', corresponding to the *worst action* from the sender's point of view, she knows that the state is θ' since $\pi(a_1' \mid \theta'') = 0$. This observation is reflected in Proposition 4 of Kamenica and Gentzkow (2011).[11] We will discuss this for the case of multiple receivers in Example 4.

Example 2: Heterogeneous beliefs The sender's payoffs are such that $u_S(a_1', \theta'') = u_S(a_1'', \theta') = 1$ and $u_S(a_1', \theta') = u_S(a_1'', \theta'') = 0$; two players have completely opposite preferences. Assume that $p_1^0 < \frac{1}{2}$, corresponding to Fig. 1a. Receiver 1 chooses a_1'' without persuasion, in which case the sender's expected payoff is p_S^0. Suppose instead that the sender employs two distinct messages and makes the receiver follow the realized message. The sender then chooses π to maximize

$$\pi(a_1' \mid \theta'')[1 - p_S^0] + \pi(a_1'' \mid \theta')p_S^0 \Leftrightarrow \pi(a_1' \mid \theta'')[1 - p_S^0] + [1 - \pi(a_1' \mid \theta')]p_S^0.$$

Note (i) that the higher $\pi(a_1' \mid \theta'')$ and $\pi(a_1'' \mid \theta')$ are, the higher the sender's expected payoff is (i.e., moving towards the northwest in the box diagram), and (ii) the slope of the sender's "indifference curve" is $\frac{d\pi(a_1'|\theta'')}{d\pi(a_1'|\theta')} = \frac{p_S^0}{1-p_S^0}$. Three possibilities depending on the sender's prior are visualized in Fig. 2:

[11] Strictly speaking, the receiver may not know the state if there are multiple states where the corresponding (worst) action is the unique best-response for the receiver.

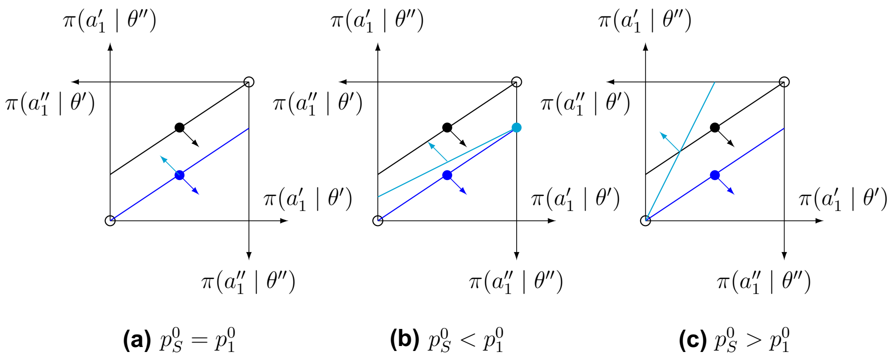


Fig. 2 Heterogeneous beliefs

(a) $p_S^0 = p_1^0$ or $\frac{p_S^0}{1-p_S^0} = \frac{p_1^0}{1-p_1^0}$: There is a continuum of solutions. Since $\pi(a_1' \mid \theta'') = \left[\frac{p_S^0}{1-p_S^0}\right]\pi(a_1' \mid \theta')$, the sender's expected payoff is p_S^0. The sender is indifferent between this strategy and π^0.

(b) $p_S^0 < p_1^0$ or $\frac{p_S^0}{1-p_S^0} < \frac{p_1^0}{1-p_1^0}$: Since receiver 1 overestimates the possibility of θ', it is costly for the sender to confuse receiver 1 when the state is θ'; $\pi(a_1'' \mid \theta') = 0$.[12] In return, the sender makes sure that $\pi(a_1' \mid \theta'')$ takes the highest possible value, $\pi(a_1' \mid \theta'') = \frac{p_1^0}{1-p_1^0}$. The expected payoff for the sender is $\frac{p_1^0[1-p_S^0]}{1-p_1^0}$. Since

$$\frac{p_1^0[1-p_S^0]}{1-p_1^0} - p_S^0 = \frac{p_1^0 - p_S^0}{1-p_1^0} > 0,$$

the sender strictly prefers this strategy to π^0. Note that the constraint for a_1' binds.

(c) $p_S^0 > p_1^0$ or $\frac{p_S^0}{1-p_S^0} > \frac{p_1^0}{1-p_1^0}$: No solution exists within the "constraint". It turns out that the sender's expected payoff with this strategy is strictly less than p_S^0. The sender's rationalizable strategy is π^0 (i.e., no persuasion) with which receiver 1 choose a_1''.

Remark 2 In the last scenario, while (c) of Fig. 2 suggests one of the origins is the solution, it does not belong to the constraint, as noted above. In addition, the sender's expected payoff from any π in the constraint is strictly lower than the expected payoff from no persuasion, and thus we can simply dismiss this constraint. We generalize this observation in Lemma 2.

[12] Since $\pi(a_1'' \mid \theta') = 0$, receiver 1 knows the state (θ'') when the message is a_1''.

$\theta = A$, player 2 in the columns, player 1 in the rows:

1 \ 2	a_2	b_2
a_1	1, 1, 1	0, 1, 0
b_1	0, 0, 1	2, 0, 0

$\theta = B$, player 2 in the columns, player 1 in the rows:

1 \ 2	a_2	b_2
a_1	0, 0, 0	0, 0, 1
b_1	0, 1, 0	1, 1, 1

(a) Payoffs

$\theta = A$	$\pi(a_2 \mid A)$	$\pi(b_2 \mid A)$
$\pi(a_1 \mid A)$	w_a	x_a
$\pi(b_1 \mid A)$	y_a	z_a

$\theta = B$	$\pi(a_2 \mid B)$	$\pi(b_2 \mid B)$
$\pi(a_1 \mid B)$	w_b	x_b
$\pi(b_1 \mid B)$	y_b	z_b

(b) Conditional Probabilities

Fig. 3 Multiple receivers: Example 3

3.2 Examples: two receivers

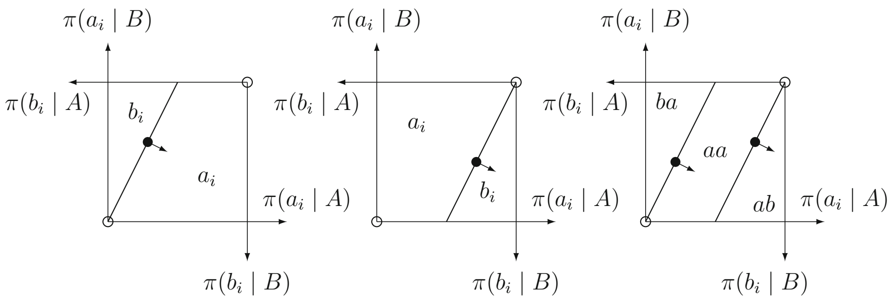
In the following examples, we have two receivers. The purpose of these examples is to show that our approach can handle the case of multiple receivers.

Example 3: Numerical example There are two possible states $\Theta = \{A, B\}$. Receivers 1 and 2 have two actions $A_i = \{a_i, b_i\}$ for each $i \in \{1, 2\}$. The payoffs are summarized in (a) of Fig. 3 where the first payoff in each cell is for the sender while the second and third payoffs are for receivers 1 and 2 respectively. Note that receiver $i \in \{1.2\}$ will choose (i) a_i only if the probability of A is at least $\frac{1}{2}$ and (ii) b_i only if the probability of B is at least $\frac{1}{2}$. We assume that $p_1^0(A) = \frac{8}{10}$ and $p_2^0(A) = \frac{6}{10}$ for receivers 1 and 2 respectively and $p_S^0(A) = \frac{1}{10}$ for the sender. Thus, without further information, each receiver $i \in \{1, 2\}$ will choose a_i, in which case the sender's expected payoff is $1 \cdot (\frac{1}{10}) + 0 \cdot (\frac{9}{10}) = \frac{1}{10}$.

Given that there are two actions for each receiver, we assume that the sender uses either one message or two messages from $\{a_i, b_i\}$ for each receiver i. In this example, we explicitly demonstrate how the iterative process for rationalizability works.

First round

Sender That no strategy has been eliminated for each receiver yet implies that for each $\pi \in \Pi$, we can construct a belief for the sender regarding the receivers' behavior



(a) a_i is realized **(b)** b_i is realized **(c)** Combining (a) and (b)

Fig. 4 Receiver's best-responses

to which π is a best-response; e.g., for any combination of $\pi \in \Pi$ and the realized message, each receiver i chooses b_i, which is the best scenario from the sender's point of view. That we identify the optimal π only after identifying the receivers' best-responses implies that the iterative process requires multiple rounds.

Receivers Since there is no strategic interaction between the receivers, we analyze a representative receiver's behavior. For the following argument, we consider all possible "information sets" for the representative receiver; i.e., any combination of $\pi \in \Pi$ and the realized message.

One message If only one message is employed, e.g., $\pi(b_i \mid \theta) = 1$ for each $\theta \in \{A, B\}$, receiver $i \in \{1, 2\}$ makes her decision with her initial belief. In this case, receiver $i \in \{1, 2\}$ will choose a_i, as noted above.

Two messages Suppose instead that both messages are employed, i.e., for each message $x \in \{a_i, b_i\}$, there exists $\theta \in \{A, B\}$ such that $\pi(x \mid \theta) > 0$.[13] Given the initial beliefs specified above, after observing the realized message $x \in \{a_i, b_i\}$, each receiver $i \in \{1, 2\}$ chooses

$$\begin{Bmatrix} a_i \\ b_i \end{Bmatrix} \text{ only if } p_i^{\pi}(A \mid x) = \frac{\pi(x|A)p_i^0(A)}{\pi(x|A)p_i^0(A)+\pi(x|B)p_i^0(B)} \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} \frac{1}{2}.$$

By noting $p_i^0(B) = 1 - p_i^0(A)$, the inequalities above can be rewritten as

$$\pi(x \mid B) \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} \left[\frac{p_i^0(A)}{1-p_i^0(A)}\right] \pi(x \mid A) \tag{2}$$

where $\frac{p_i^0(A)}{1-p_i^0(A)} > 1$ if $p_i^0(A) \in (\frac{1}{2}, 1)$, which is the case for each $i \in \{1, 2\}$.

Based on the inequalities in (2), we first show how the behavior of receiver $i \in \{1, 2\}$ depends on π for each possible realized message from $\{a_i, b_i\}$:

13 As noted earlier, this implies that we exclude the two origins in each box diagram in Fig. 4.

- a_i is observed ($x = a_i$). This case is shown in (a) of Fig. 4 where (i) the origin is the bottom-left corner and (ii) the line represents $\pi(a_i \mid B) = \left[\frac{p_i^0(A)}{1-p_i^0(A)}\right]\pi(a_i \mid A)$, i.e., $x = a_i$. After observing a_i, receiver i chooses (i) a_i if $(\pi(a_i \mid A), \pi(a_i \mid B))$ is below (south-east) the line, including any point on the line (except the origin), and (ii) b_i otherwise.[14]
- b_i is observed ($x = b_i$). This case is shown in (b) of Fig. 4 where (i) the origin is the top-right corner and (ii) the line represents $\pi(b_i \mid B) = \left[\frac{p_i^0(A)}{1-p_i^0(A)}\right]\pi(b_i \mid A)$, i.e., $x = b_i$. After observing b_i, receiver i chooses (i) b_i if $(\pi(b_i \mid A), \pi(b_i \mid B))$ is below (south-east) the line, including any point on the line (except the origin) and (ii) a_i otherwise.

The box diagram (c) of Fig. 4 is obtained by combining (a) and (b) of Fig. 4. The diagram shows how receiver i's behavior would depend on π and the realized message. There are three areas in (c) of Fig. 4:

ba : Receiver i chooses (i) b_i if the message a_i is realized and (ii) a_i if the message b_i is realized.
aa : Receiver i chooses a_i regardless of the realized message.
ab : Receiver i chooses (i) a_i if the message a_i is realized and (ii) b_i if the message b_i is realized.

Receiver i's behavior in the two areas *ba* and *ab* are qualitatively equivalent. While receiver i follows the realized message in the area *ab*, she chooses the action which is the opposite of the realized message in the area *ba*.[15]

The discussion above describes each receiver i's best response, which determines the sender's best-response in the next round.

Second round

Receivers No further elimination of strategies given the arguments in the first round.

Sender We partition Π into three different sets, depending on how the receivers behave after observing the realized messages.

Case 1 First, we consider π's with which the receivers choose the same action independent of the realized message. If the sender chooses π corresponding to the area *aa* in (c) of Fig. 4 for each receiver $i \in \{1, 2\}$ so that she chooses a_i regardless of the realized message, the sender's expected payoff is $\frac{1}{10}$, which is equal to that of the case where only one message is used (and thus each receiver $i \in \{1, 2\}$ chooses a_i). Since our focus is on the cases where the receivers follow the realized messages, we take $\pi(a_i \mid \theta) = 1$ for each $\theta \in \{A, B\}$ for each receiver $i \in \{1, 2\}$ for the current

[14] As discussed earlier, we assume that the receivers follow the realized messages if they are indifferent between two actions. We will discuss this point in Sect. 4.3.

[15] The diagram in (c) of Fig. 4 implies that it is not possible for the sender to persuade receiver i to choose b_i independent of the realized message. This is because $p_i^0(A) > \frac{1}{2}$ for each $i \in \{1, 2\}$.

case instead of (i) π's corresponding to the area aa in (c) of Fig. 4 for each receiver $i \in \{1, 2\}$ and (ii) $\pi(b_i \mid \theta) = 1$ for each $\theta \in \{A, B\}$ for each receiver $i \in \{1, 2\}$.

Case 2 Next, we consider π's with which receiver $i \in \{1, 2\}$ chooses a_i independent of the realized message while the action receiver $j \neq i$ chooses depends on the realized message. This implies that it is not possible to observe (b_1, b_2) for each state. Unless $\theta = A$ and the receivers choose (a_1, a_2) in which case the sender will receive a payoff of one, the sender's payoff is zero. Thus, the sender's expected payoff in this case does not exceed that of the case discussed above.

Case 3 Lastly, we consider π's with which each receiver i chooses a different action depending on the realized message. Since the area ba is qualitatively equivalent to the area ab as pointed out above, given that our focus is on the cases where the receivers follow the realized messages, we consider the area ab for the current case.[16] We consider the following parameters for the sender's strategy, which are also shown in (b) of Fig. 3:

$$\begin{array}{llll} \pi(a_1, a_2 \mid A) = w_a & \pi(a_1, b_2 \mid A) = x_a & \pi(b_1, a_2 \mid A) = y_a & \pi(b_1, b_2 \mid A) = z_a \\ \pi(a_1, a_2 \mid B) = w_b & \pi(a_1, b_2 \mid B) = x_b & \pi(b_1, a_2 \mid B) = y_b & \pi(b_1, b_2 \mid B) = z_b \end{array}$$

where $w_a + x_a + y_a + z_a = 1$ and $w_b + x_b + y_b + z_b = 1$, and thus

$$\begin{array}{llll} \pi(a_1 \mid A) = w_a + x_a & \pi(b_1 \mid A) = y_a + z_a & \pi(a_2 \mid A) = w_a + y_a & \pi(b_2 \mid A) = x_a + z_a \\ \pi(a_1 \mid B) = w_b + x_b & \pi(b_1 \mid B) = y_b + z_b & \pi(a_2 \mid B) = w_b + y_b & \pi(b_2 \mid B) = x_b + z_b. \end{array}$$

Given the inequalities in (2), receiver $i \in \{1, 2\}$ follows the realized message only if

$$\pi(a_1 \mid B) = (w_b + x_b) \leq \left[\frac{\frac{8}{10}}{1-\frac{8}{10}}\right] \pi(a_1 \mid A) = 4(w_a + x_a) \tag{3}$$

$$\pi(b_1 \mid B) = (y_b + z_b) \geq \left[\frac{\frac{8}{10}}{1-\frac{8}{10}}\right] \pi(b_1 \mid A) = 4(y_a + z_a) \tag{4}$$

for receiver 1 who has $p_1^0(A) = \frac{8}{10}$ and

$$\pi(a_2 \mid B) = (w_b + y_b) \leq \left[\frac{\frac{6}{10}}{1-\frac{6}{10}}\right] \pi(a_2 \mid A) = \frac{3}{2}(w_a + y_a) \tag{5}$$

$$\pi(b_2 \mid B) = (x_b + z_b) \geq \left[\frac{\frac{6}{10}}{1-\frac{6}{10}}\right] \pi(b_2 \mid A) = \frac{3}{2}(x_a + z_a) \tag{6}$$

for receiver 2 who has $p_2^0(A) = \frac{6}{10}$. The sender maximizes her expected payoff

$$\begin{aligned} &[\pi(a_1, a_2 \mid A) + 2\pi(b_1, b_2 \mid A)]\, p_S^0(A) + \pi(b_1, b_2 \mid B) p_S^0(B) \\ &\quad = [w_a + 2z_a]\left(\frac{1}{10}\right) + z_b\left(\frac{9}{10}\right) \end{aligned} \tag{7}$$

[16] While there are two inequalities to be satisfied for the area ab in (c) of Fig. 4, note that the inequality for the message b_i implies that of the message a_i. We discuss this observation in Sect. 4.4.

subject to the inequalities (3), (4), (5) and (6) as well as the assumptions on the parameters; i.e., they are non-negative and add up to one for each state. It is important to note that the expression (7) implies (i) that the sender's expected payoff is strictly increasing in w_a, z_a and z_b, and (ii) that the coefficient of z_a is strictly higher than that of w_a.

We now solve this constrained maximization problem. First, the inequalities (3) and (5) imply that the solution has $w_b = 0$. Given this observation, let $z_b = 1-(x_b+y_b)$. The inequalities above can then be modified as

$$x_b \leq 4(w_a + x_a) \;\; 1 - x_b \geq 4(y_a + z_a) \;\; y_b \leq \tfrac{3}{2}(w_a + y_a) \;\; 1 - y_b \geq \tfrac{3}{2}(x_a + z_a)$$

which imply that the solution has $x_b = y_b = 0$, and thus $z_b = 1$. The inequalities above can then be modified further as

$$0 \leq 4(w_a + x_a) \;\; 1 \geq 4(y_a + z_a) \;\; 0 \leq \tfrac{3}{2}(w_a + y_a) \;\; 1 \geq \tfrac{3}{2}(x_a + z_a).$$

The first and the third inequalities hold. The second and the fourth inequalities imply $x_a = y_a = 0$. We then have

$$0 \leq 4w_a \;\; 1 \geq 4z_a \;\; 0 \leq \tfrac{3}{2}w_a \;\; 1 \geq \tfrac{3}{2}z_a.$$

Given that the "marginal utility" from z_a is higher than that of w_a in (7), the second constraint binds, implying $z_a = \frac{1}{4}$ and thus $w_a = \frac{3}{4}$.[17] We have $(w_a, x_a, y_a, z_a) = (\frac{3}{4}, 0, 0, \frac{1}{4})$ and $(w_b, x_b, y_b, z_b) = (0, 0, 0, 1)$, and thus

$$\begin{array}{ll} \pi(a_1 \mid A) = \pi(a_2 \mid A) = \frac{3}{4} & \pi(b_1 \mid A) = \pi(b_2 \mid A) = \frac{1}{4} \\ \pi(a_1 \mid B) = \pi(a_2 \mid B) = 0 & \pi(b_1 \mid B) = \pi(b_2 \mid B) = 1 \end{array}$$

under which the receivers know that the state is A if they observe a_i.

The following observations confirm that the receivers follow the realized messages. Receiver 1's updated beliefs after observing $x \in \{a_1, b_1\}$ are:

$$p_1^{\pi}(A \mid a_1) = \frac{\pi(a_1|A)p_1^0(A)}{\pi(a_1|A)p_1^0(A)+\pi(a_1|B)p_1^0(B)} = \frac{\left(\frac{3}{4}\right)\left(\frac{8}{10}\right)}{\left(\frac{3}{4}\right)\left(\frac{8}{10}\right)+0\cdot\left(\frac{2}{10}\right)} = 1 \Rightarrow p_1^{\pi}(B \mid a_1) = 0$$

$$p_1^{\pi}(A \mid b_1) = \frac{\pi(b_1|A)p_1^0(A)}{\pi(b_1|A)p_1^0(A)+\pi(b_1|B)p_1^0(B)} = \frac{\left(\frac{1}{4}\right)\left(\frac{8}{10}\right)}{\left(\frac{1}{4}\right)\left(\frac{8}{10}\right)+1\cdot\left(\frac{2}{10}\right)} = \frac{1}{2} \Rightarrow p_1^{\pi}(B \mid b_1) = \frac{1}{2}.$$

Likewise, Receiver 2's updated beliefs after observing $x \in \{a_2, b_2\}$ are:

$$p_2^{\pi}(A \mid a_2) = \frac{\pi(a_2|A)p_2^0(A)}{\pi(a_2|A)p_2^0(A)+\pi(a_2|B)p_2^0(B)} = \frac{\left(\frac{3}{4}\right)\left(\frac{6}{10}\right)}{\left(\frac{3}{4}\right)\left(\frac{6}{10}\right)+0\cdot\left(\frac{4}{10}\right)} = 1 \Rightarrow p_2^{\pi}(B \mid a_2) = 0$$

[17] That the last constraint holds with strict inequality is due to the fact that we allow for heterogenous beliefs.

$\theta = A$ (1 = row player, 2 = column player)

	a_2	b_2
a_1	1, 1, 0	0, 1, 1
b_1	$-\varepsilon$, 0, 0	1, 0, 1

$\theta = B$ (1 = row player, 2 = column player)

	a_2	b_2
a_1	1, 0, 1	0, 0, 0
b_1	$-\varepsilon$, 1, 1	1, 1, 0

(a) Payoffs

$\theta = A$	$\pi(a_2 \mid A)$	$\pi(b_2 \mid A)$
$\pi(a_1 \mid A)$	w_a	x_a
$\pi(b_1 \mid A)$	y_a	z_a

$\theta = B$	$\pi(a_2 \mid B)$	$\pi(b_2 \mid B)$
$\pi(a_1 \mid B)$	w_b	x_b
$\pi(b_1 \mid B)$	y_b	z_b

(b) Conditional Probabilities

Fig. 5 Multiple receivers: Example 4

$$p_2^{\pi}(A \mid b_2) = \frac{\pi(b_2|A)p_2^0(A)}{\pi(b_2|A)p_2^0(A)+\pi(b_2|B)p_2^0(B)} = \frac{\left(\frac{1}{4}\right)\left(\frac{6}{10}\right)}{\left(\frac{1}{4}\right)\left(\frac{6}{10}\right)+1\cdot\left(\frac{4}{10}\right)} = \frac{3}{11} \Rightarrow p_2^{\pi}(B \mid b_2) = \frac{8}{11}.$$

These expressions imply that receiver $i \in \{1, 2\}$ chooses (i) a_i after observing the message a_i and (ii) b_i after observing the message b_i.

Since the sender's expected payoff is $[(\frac{3}{4}) + 2 \cdot (\frac{1}{4})](\frac{1}{10}) + (\frac{9}{10}) = \frac{41}{40} > \frac{1}{10}$, the solution to the constrained optimization problem corresponds to the sender's best-response, implying that the sender uses two messages instead of one.

Rationalizable strategies

Receivers Each receiver i's rationalizable strategy is such that while receiver i chooses a_i if one message is employed, if two messages are used, receiver i's behavior depends on both $\pi \in \Pi$ and the realized message and is summarized in (c) of Fig. 4.

Sender The distribution π with two messages described above, namely

$$\begin{array}{llll} \pi(a_1, a_2 \mid A) = \frac{3}{4} & \pi(a_1, b_2 \mid A) = 0 & \pi(b_1, a_2 \mid A) = 0 & \pi(b_1, b_2 \mid A) = \frac{1}{4} \\ \pi(a_1, a_2 \mid B) = 0 & \pi(a_1, b_2 \mid B) = 0 & \pi(b_1, a_2 \mid B) = 0 & \pi(b_1, b_2 \mid B) = 1 \end{array}$$

is the sender's rationalizable strategy.

Example 4: Multiple receivers There are two possible states $\Theta = \{A, B\}$. Receivers 1 and 2 have two actions $A_i = \{a_i, b_i\}$ for each $i \in \{1, 2\}$. The payoffs are summarized in (a) of Fig. 5 where the first payoff in each cell is for the sender while the second and third payoffs are for receivers 1 and 2 respectively. We assume that $\varepsilon > 0$ is an arbitrary small number. Regarding the receivers, note (i) that there is no strategic interaction between the receivers, and (ii) that each receiver has a strictly dominant action for each state. Regarding the sender, note (i) that the sender's payoffs are state-independent, and (ii) that (b_1, a_2) is the worst action profile from the sender's point of view.

Let $p_S^0(A) = p_S^0$ and $p_i^0(A) = p_i^0$ for each $i \in \{1, 2\}$. Receivers 1 and 2 will choose (a_1, a_2) if $p_1^0 > \frac{1}{2} > p_2^0$ or (b_1, b_2) if $p_1^0 < \frac{1}{2} < p_2^0$, in which case the sender simply chooses π^0. If $\min\{p_1^0, p_2^0\} > \frac{1}{2}$ or $\frac{1}{2} > \max\{p_1^0, p_2^0\}$, the sender attempts to change their beliefs so that there would be a chance that they will coordinate. We focus on the former, i.e., $\min\{p_1^0, p_2^0\} > \frac{1}{2}$. Without persuasion, receiver 1 chooses a_1 while receiver 2 chooses b_2, implying that the sender's payoff is zero.

The question is then whether the sender can achieve a higher expected payoff by persuading the receivers. The sender randomizes the messages, i.e., $\mathcal{A}_i(\pi) = A_i$ for each $i \in \{1, 2\}$. The set of non-negative parameters the sender chooses (i.e., probabilities conditional on the states) is shown in (b) of Fig. 5 where $w_a + x_a + y_a + z_a = 1$ and $w_b + x_b + y_b + z_b = 1$. Note that we allow correlations.

The condition under which each receiver follows the realized message is

$$\frac{\pi(a_1|A)p_1^0}{\pi(a_1|A)p_1^0+\pi(a_1|B)(1-p_1^0)} \geq \frac{\pi(a_1|B)(1-p_1^0)}{\pi(a_1|A)p_1^0+\pi(a_1|B)(1-p_1^0)}$$
$$\frac{\pi(b_1|A)p_1^0}{\pi(b_1|A)p_1^0+\pi(b_1|B)(1-p_1^0)} \leq \frac{\pi(b_1|B)(1-p_1^0)}{\pi(b_1|A)p_1^0+\pi(b_1|B)(1-p_1^0)}$$

or

$$\begin{aligned} \pi(a_1 \mid B) &\leq \left[\frac{p_1^0}{1-p_1^0}\right]\pi(a_1 \mid A) \\ \pi(b_1 \mid B) &\geq \left[\frac{p_1^0}{1-p_1^0}\right]\pi(b_1 \mid A) \end{aligned} \tag{8}$$

for receiver 1 and

$$\begin{aligned} \pi(a_2 \mid B) &\geq \left[\frac{p_2^0}{1-p_2^0}\right]\pi(a_2 \mid A) \\ \pi(b_2 \mid B) &\leq \left[\frac{p_2^0}{1-p_2^0}\right]\pi(b_2 \mid A) \end{aligned} \tag{9}$$

for receiver 2. The notations in (b) of Fig. 5 modify (8) and (9) as

$$\begin{aligned} (w_b + x_b) &\leq \frac{p_1^0}{1-p_1^0}(w_a + x_a) \\ \frac{p_1^0}{1-p_1^0}(y_a + z_a) &\leq (y_b + z_b) \end{aligned} \tag{10}$$

and

$$\begin{aligned} \frac{p_2^0}{1-p_2^0}(w_a + y_a) &\le (w_b + y_b) \\ (x_b + z_b) &\le \frac{p_2^0}{1-p_2^0}(x_a + z_a). \end{aligned} \tag{11}$$

respectively. The sender maximizes

$$(w_a + z_a - \varepsilon y_a)p_S + (w_b + z_b - \varepsilon y_b)(1 - p_S)$$

subject to (10) and (11).

For example, suppose $p_S^0 = 0.8$, $p_1^0 = 0.7$, $p_2^0 = 0.6$, and $\varepsilon = 0.01$. In this case, the solution is $(w_a, x_a, y_a, z_a) = (\frac{2}{3}, 0, 0, \frac{1}{3})$ and $(w_b, x_b, y_b, z_b) = (\frac{2}{9}, 0, \frac{7}{9}, 0)$. Hence, we have

$$\begin{aligned} &(\pi(a_1 \mid A), \pi(b_1 \mid A)) = \left(\tfrac{2}{3}, \tfrac{1}{3}\right) \ \ (\pi(a_1 \mid B), \pi(b_1 \mid B)) = \left(\tfrac{2}{9}, \tfrac{7}{9}\right) \\ &(\pi(a_2 \mid A), \pi(b_2 \mid A)) = \left(\tfrac{2}{3}, \tfrac{1}{3}\right) \ \ (\pi(a_2 \mid B), \pi(b_2 \mid B)) = (1, 0). \end{aligned}$$

The sender's expected payoff is $\frac{8}{10} + \frac{2}{10}[\frac{2}{9} + \frac{7}{9}(-\varepsilon)] = \frac{1}{45}(38 - 7\varepsilon) \approx 0.84$.[18] The receivers always coordinate when $\theta = A$ $(w_a + z_a = 1)$, while the chance that they will coordinate when $\theta = B$ is $\frac{2}{9}$ $\left(w_b + z_b = \frac{2}{9}\right)$.

Remark 3 While receiver 1 never knows the state, receiver 2 knows the state if she observes b_2 $(\theta = A)$. Since $y_b > 0$, it is possible that receiver 1 chooses b_1 and receiver 2 chooses a_2 when $\theta = B$, leading to the worst payoff for the sender, $-\varepsilon$. Remember that the original example from Kamenica and Gentzkow (2011) shows that if the receiver chooses the worst *action* (from the sender's point of view), she knows the state (Remark 1). In this example, when they choose the worst action profile (b_1, a_2), they do not know $\theta = B$.

4 Linear programming approach

We first describe our main result (Proposition 1). In the following subsections, we establish the existence of rationalizable strategies for the sender (Sect. 4.1) and show that a single linear programming problem leads to rationalizable strategies for the sender (Sect. 4.2). We further discuss the issue of multiple best-responses (Sect. 4.3) and the property of the constraints (Sect. 4.4).

[18] If the sender persuades only receiver 1, her expected payoff is $p_S^0\left[\frac{1-p_1^0}{p_1^0}\right] + (1 - p_S^0) = \frac{19}{35} \approx 0.54$ while it is $p_S^0\left[\frac{1-p_2^0}{p_2^0}\right] + (1 - p_S^0) = \frac{11}{15} \approx 0.73$ if the sender persuades only receiver 2.

In our approach, we identify the set of rationalizable strategies for the sender and the receivers *á la* backward induction.

Second stage The second stage only concerns the receivers' decision problems. Given $\pi \in \Pi$ and $a_i \in \mathcal{A}_i(\pi)$ for each $i \in N$, let $s_i^+(\pi, a_i) \in A_i$ be such that

$$\sum_{\theta\in\Theta} u_i(s_i^+(\pi, a_i), \theta) p_i^\pi(\theta \mid a_i) \geq \sum_{\theta\in\Theta} u_i(a_i', \theta) p_i^\pi(\theta \mid a_i)$$

or

$$\sum_{\theta\in\Theta} \left[u_i(s_i^+(\pi, a_i), \theta) - u_i(a_i', \theta)\right] \pi(a_i \mid \theta) p_i^0(\theta) \geq 0 \tag{12}$$

for each $a_i' \in A_i$. Let s_i^+ be the corresponding best-response strategy for receiver $i \in N$. The presence of multiple best-responses is possible, and we also use $s_i^+(\pi, a_i)$ as a best-response correspondence (abuse of notation). In our analysis, we pay our attention to the case where the receivers follow the sender's recommendation.

Definition 1 We say that π **implements** $\tilde{A} = \times_{j\in N} \tilde{A}_j$ where $\tilde{A}_j \subseteq A_j$ for each $j \in N$ if

1. $\mathcal{A}_i(\pi) = \tilde{A}_i$ for each $i \in N$, and
2. for each $i \in N$ and $a_i \in \mathcal{A}_i(\pi)$, $a_i \in s_i^+(\pi, a_i)$.

Let

$$\Pi(\tilde{A}) = \left\{\pi \in \Pi \mid \pi \text{ implements } \tilde{A}\right\}.$$

We say that $\tilde{A}$ is implementable if $\Pi(\tilde{A}) \neq \emptyset$.[19] For the following analysis, our focus is on the collection of implementable sets, meaning that the receivers follow the realized messages.[20] Note that some message profiles in $\tilde{A}$ may not be realized for certain states, since $\tilde{A}$ is a product set and π permits correlations. That $\pi^0 \in \Pi(A^0)$ means that there always exists an implementable set. Since our focus is on implementable sets, for each $i \in N$, if $a_i \in A_i^0$ is implemented, $\pi^0(a_i \mid \theta) = 1$ for each $\theta \in \Theta$.

Remark 4 The latter point in Definition 1 implies that the sender makes receiver i follow the realized message. If there are multiple best-responses, however, she may choose a different action $a_i' \in s_i^+(\pi, a_i)$. We will discuss this point in Sect. 4.3.

First stage For each implementable set $\tilde{A}$, the sender identifies $\pi \in \Pi(\tilde{A})$ which maximizes her expected payoff—the optimization problem under the constraint $\Pi(\tilde{A})$. Note that for each $a_i \in \tilde{A}_i$, $\Pi(\tilde{A})$ does not contain π such that $\pi(a_i \mid \theta) = 0$ for each $\theta \in \Theta$. This can be seen in the exclusion of the origins in the box diagrams in the previous examples. This implies (i) that $\Pi(\tilde{A})$ is not closed, and thus (ii) that there

[19] For example, $\{a_1\} \times \{b_2\}$, $A_1 \times \{b_2\}$, $\{a_1\} \times A_2$, and $A_1 \times A_2$ are implementable in Example 4.

[20] This means that we do not consider the permutations of $\tilde{A}_i$.

may not be a solution. For each implementable $\tilde{A}$, we instead look at the *closure* of $\Pi(\tilde{A})$, denoted by $\bar{\Pi}(\tilde{A})$.[21] For each implementable $\tilde{A}$, the sender identifies

$$\text{argmax}_{\pi \in \bar{\Pi}(\tilde{A})} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}(\pi)} u_S(a, \theta)\pi(a \mid \theta) p_S^0(\theta). \tag{13}$$

Since (13) and (12) are linear in π, this constrained maximization problem can be seen as a *linear programming problem*. Among the solutions for all implementable sets, the ones which reach the highest expected payoff for the sender are the rationalizable strategies for the sender.

Note (i) that the consideration of $\bar{\Pi}(\tilde{A})$ suggests a possibility that the set of rationalizable strategies for the sender can be empty, and (ii) that the procedure above leads to *a set of* linear programming problems (corresponding to different implementable sets). We show (i) that there always exists a rationalizable strategy for the sender (Sect. 4.1), and (ii) that a single linear programming problem leads to the identification of rationalizable strategies for the sender (Sect. 4.2). Regarding the latter, we show that there exists the *largest implementable set* $\tilde{A}^\sharp = \times_{j \in N} \tilde{A}_j^\sharp$ under which the sender's optimization problem is solved. For the identification of $\tilde{A}^\sharp$, we show that for each $i \in N$, $\tilde{A}_i^\sharp \subseteq A_i$ is the set of actions for receiver i, which are best-responses to *some* distributions over Θ.

A solution for the sender's optimization problem under $\bar{\Pi}(\tilde{A}^\sharp)$, denoted by π^*, may be that $\pi^* \in \bar{\Pi}(\tilde{A}^\sharp)\backslash\Pi(\tilde{A}^\sharp)$. In such a case, there exists an implementable set $\tilde{A}^* \subset \tilde{A}^\sharp$ such that π^* is the solution of the sender's optimization problem under $\Pi(\tilde{A}^*)$. We do not need to take the closure for $\Pi(\tilde{A}^*)$. We prove Proposition 1 below in the Sects. 4.1 and 4.2.

Proposition 1 *The sender's rationalizable strategy π^* is*

$$\pi^* \in \text{argmax}_{\pi \in \bar{\Pi}(A^\sharp)} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}(\pi)} u_S(a, \theta)\pi(a \mid \theta) p_S^0(\theta).$$

There exists an implementable set $\tilde{A}^* = \times_{j \in N} \tilde{A}_j^* \subseteq A^\sharp$ such that

$$\pi^* \in \text{argmax}_{\pi \in \Pi(\tilde{A}^*)} \sum_{\theta \in \Theta} \sum_{a \in \mathcal{A}(\pi)} u_S(a, \theta)\pi(a \mid \theta) p_S^0(\theta).$$

One immediate implication of Proposition 1 is that if $\pi^* \notin \Pi(A^0)$, π^* is generically unique.[22]

Remark 5 In our linear programming approach, we first identify each receiver's best-response (rationalizable strategy), and then the sender's rationalizable strategy. This is exactly the order of eliminations of strategies for rationalizability: In the first round, (i) no strategy for the sender is eliminated, and (ii) a best-response is identified for

[21] For each $\pi \in \bar{\Pi}(\tilde{A})\backslash\Pi(\tilde{A})$, there exist $i \in N$ and $a_i \in \tilde{A}_i$ such that $\pi(a_i \mid \theta) = 0$ for each $\theta \in \Theta$.

[22] For the uniqueness of the solution, see Mangasarian (1979).

each receiver. In the second round, given each receiver's best-response, the sender can identify her rationalizable strategy. Note that given that there is no strategic interaction at the second stage (unlinked), the argument above immediately implies a similarity between our approach and backward induction.

Remark 6 Our result on the largest implementable set holds since π is freely chosen by the sender. In other words, π is practically tailor-made for each receiver, implying that the constraints for different receivers do not interact. If there is a restriction on π and the constraints for different receivers interact, we may not be able to use the largest implementable set since this would imply the discontinuity of the sender's expected payoff in π. In such cases, we consider different constraints separately.

4.1 Existence

We first establish the existence of a rationalizable strategy for the sender. As discussed above, the issue is that for each $\tilde{A}$ with $|\tilde{A}_i| > 1$ for some $i \in N$, $\Pi(\tilde{A})$ is not closed, implying the possibility that no rationalizable strategy for the sender exists. In Example 2 (c), for the sender's optimization problem with $\Pi(A_1)$, $\pi(a_1' \mid \theta') = \pi(a_1' \mid \theta'') = 0$ is the "solution", which does not belong to $\Pi(A_1)$. This corresponds to the concern above. Note, however, that the sender's unique rationalizable strategy is π^0 and actually corresponds to the "solution" above. This suggests that given a constraint, once the "origin" is identified as a solution, we can find another constraint in which there exists a strategy for the sender identical to the "origin" in the original constraint. We show that this is always the case. This approach also allows us to use the closures of the constraints to include the origins.

In Example 2 (c), (i) we reduced the constraint from A_1 to $\{a_1''\}$, and (ii) while the "origin" for a_1' is contained in $\bar{\Pi}(A_1)$, the one for a_1'' is *not*. We generalize this observation for the reduction of constraints. Given $\tilde{A}$, for each $i \in N$, let

$$\tilde{A}_i^{\triangleleft} = \left\{a_i \in \tilde{A}_i \mid \text{ there exists } \pi \in \bar{\Pi}(\tilde{A}) \text{ such that } \pi(a_i \mid \theta) = 0 \text{ for each } \theta \in \Theta\right\}.$$

In other words, $\bar{\Pi}(\tilde{A})$ contains the "origin" for each $a_i \in \tilde{A}_i^{\triangleleft}$. We first establish the following result.

Lemma 1 *Given $\tilde{A}$, for any $i \in N$ with $|\tilde{A}_i| > 1$, take any $a_i' \in \tilde{A}_i^{\triangleleft}$ and let $\tilde{A}' = \tilde{A}_i \backslash \{a_i'\}$. Then, (i) $\tilde{A}_i' \times \tilde{A}_{-i}$ is implementable, and (ii) $\Pi(\tilde{A}_i' \times \tilde{A}_{-i}) \subset \bar{\Pi}(\tilde{A}) \backslash \Pi(\tilde{A})$.*

Proof Take any $\pi' \in \bar{\Pi}(\tilde{A}) \backslash \Pi(\tilde{A})$ such that $\pi'(a_i' \mid \theta) = 0$ for each $\theta \in \Theta$. Since this does not change (12) with respect to the comparison of any pair in $\tilde{A}_i'$, receiver i follows the observed message for each $a_i \in \tilde{A}_i'$. Let $\pi'(a_j \mid \theta) = \pi(a_j \mid \theta)$ for each $j \neq i$, $a_j \in \tilde{A}_j$, and $\theta \in \Theta$. This implies that $\tilde{A}_i' \times \tilde{A}_{-i}$ is implementable. The last part is immediately established by reversing the argument above. □

Lemma 1 implies the following result, which was mentioned in Remark 2. It says that given $\tilde{A}$, once the sender finds a solution in $\bar{\Pi}(\tilde{A}) \backslash \Pi(\tilde{A})$, she can find a smaller implementable set $\tilde{A}'$ (i.e., the reduction of the implementable set) and $\pi' \in \Pi(\tilde{A}')$ such that her expected payoff with π' is weakly higher than the ones with any $\pi \in \Pi(\tilde{A})$, as in Example 2 (c).[23] If it is strictly higher, $\tilde{A}$ can simply be dismissed.

Lemma 2 *Given $\tilde{A}$, if a solution for (13), π, is such that $\pi \in \bar{\Pi}(\tilde{A})\backslash\Pi(\tilde{A})$, there exists an implementable set $\tilde{A}' \subsetneq \tilde{A}$ with which there exists $\pi' \in \Pi(\tilde{A}')$ such that*

$$\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}(\pi')} u_S(a,\theta)\pi'(a \mid \theta)p_S^0(\theta) = \sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}(\pi)} u_S(a,\theta)\pi(a \mid \theta)p_S^0(\theta). \quad (14)$$

Proof Given π, for each $i \in N$, identify $\tilde{B}_i \subseteq \tilde{A}_i^{\triangleleft}$ such that for each $a_i' \in \tilde{B}_i$, $\pi(a_i' \mid \theta) = 0$ for each $\theta \in \Theta$. Let $\tilde{A}_i' = \tilde{A}_i\backslash\tilde{B}_i$ for each $i \in N$. Lemma 1 says that $\tilde{A}' = \times_{j\in N}\tilde{A}_j'$ is implementable. For each $i \in N$ and $a_i \in \tilde{A}_i'$, let $\pi'(a_i \mid \theta) = \pi(a_i \mid \theta)$ for each $\theta \in \Theta$. It is then immediate (i) that $\pi' \in \Pi(\tilde{A}')$ and (ii) that (14) holds. □

Note that for any implementable $\tilde{A}$ and $\pi \in \Pi(\tilde{A})$, there exist $i \in N$, $a_i \in \tilde{A}_i$, and $\theta \in \Theta$ such that $\pi(a_i \mid \theta) > 0$, meaning that any reduction of $\tilde{A}$ never leads to an empty set. Since A_i is finite for each $i \in N$, Lemma 2 implies the existence result.

Proposition 2 *There exist $\tilde{A}^*$ and $\pi^* \in \Pi(\tilde{A}^*)$ such that π^* is a rationalizable strategy for the sender.*

The results above seemingly suggest that to obtain π^*, it is necessary to solve (13) with respect to all possible implementable sets. The results in the next section show (i) that there exists the largest implementable set $\tilde{A}^\sharp$ under which a solution to (13) is π^*, and (ii) how we can construct $\tilde{A}^\sharp$. The former implies that a *single* linear programming problem leads to π^*.

4.2 Largest implementable set

In this subsection, we show the existence of the largest implementable set $\tilde{A}^\sharp$ under which we can solve (13) to show a rationalizable strategy for the sender, π^*.

Let (i) $\sigma \in \Delta(\Theta)$ be a distribution over Θ, and (ii) $\sigma(\theta) \in [0, 1]$ be a probability assigned to $\theta \in \Theta$ by σ. For each $i \in N$, let

$$A_i^\sharp = \left\{ a_i \in A_i \mid \begin{array}{c} \text{there exists } \sigma \in \Delta(\Theta) \text{ such that for each } a_i' \in A_i \\ \sum_{\theta\in\Theta} u_i(a_i,\theta)\sigma(\theta) \geq \sum_{\theta\in\Theta} u_i(a_i',\theta)\sigma(\theta) \end{array} \right\}.$$

and $A^\sharp = \times_{j\in N} A_j^\sharp$. In other words, $A_i^\sharp$ is the set of actions for receiver $i \in N$ which are best-responses to *some* distributions over Θ. For each $a_i \in A_i^\sharp$, let (i) $\sigma(a_i) \in \Delta(\Theta)$ be a distribution to which a_i is a best-response, and (ii) $\sigma(\theta, a_i) \in [0, 1]$ be a probability assigned to $\theta \in \Theta$ by $\sigma(a_i)$ (and hence $\sum_{\theta\in\Theta}\sigma(\theta, a_i) = 1$). For each $i \in N$, let

$$A_i^{\sharp,1} = \left\{ a_i \in A_i^\sharp \mid \text{ there exists } \theta \in \Theta \text{ such that } \sigma(\theta, a_i) = 1 \right\}.$$

[23] Note that Example 2 (a) is also consistent with this.

In other words, $A_i^{\sharp,1}$ is the set of actions for receiver $i \in N$ which are best-responses to some degenerate distributions over Θ.

Given the way $A^\sharp$ is constructed, if the largest implementable set $\tilde{A}^\sharp$ exists, it immediately follows that $\tilde{A}^\sharp \subseteq A^\sharp$. We have the consequent strong result, which immediately implies the existence of $\tilde{A}^\sharp$.

Lemma 3 $\tilde{A}^\sharp = A^\sharp$.

Proof We construct π such that $\pi \in \Pi(A^\sharp)$, which immediately implies $\tilde{A}^\sharp = A^\sharp$. First, consider $A_i^{\sharp,1}$. For each $a_i \in A_i^{\sharp,1}$, let

$$\Theta(a_i) = \{\theta \in \Theta \mid \text{there exists } \sigma(a_i) \text{ such that } \sigma(\theta, a_i) = 1\}.$$

Note (i) that for each $\theta \in \Theta$, there exists $a_i \in A_i^{\sharp,1}$ such that $\theta \in \Theta(a_i)$, and (ii) that for each $a_i \in A_i^{\sharp,1}$, $|\Theta(a_i)| \geq 1$. For each $i \in N$ and $a_i \in A_i^{\sharp,1}$, let

$$\pi(a_i \mid \theta) = \begin{Bmatrix} \alpha(a_i, \theta) \\ 0 \end{Bmatrix} \text{ for each } \theta \begin{Bmatrix} \in \\ \notin \end{Bmatrix} \Theta(a_i)$$

where $\alpha(a_i, \theta) \in (0, 1]$ is an arbitrary constant, which we will discuss later. Note that for each $\theta \in \Theta$, $p_i^\pi(\theta \mid a_i) > 0$ if and only if $\theta \in \Theta(a_i)$; i.e., for any $\alpha(a_i, \theta) \in (0, 1]$, a_i is a best-response after observing the message a_i.[24] □

Next, consider $A_i^\sharp \backslash A_i^{\sharp,1}$. It is possible that $A_1^\sharp \backslash A_1^{\sharp,1}$ is empty. For each $i \in N$ and $a_i \in A_i^\sharp \backslash A_i^{\sharp,1}$, let

$$\pi(a_i \mid \theta) = \beta(a_i)\sigma(\theta, a_i)$$

where $\beta(a_i) \in (0, 1]$ is again an arbitrary constant which will be discussed later. Since

$$p_i^\pi(\theta \mid a_i) = \frac{\beta(a_i)\sigma(\theta, a_i)}{\sum_{\tilde{\theta} \in \Theta} \beta(a_i)\sigma(\tilde{\theta}, a_i)} = \sigma(\theta, a_i),$$

for any $\beta(a_i) \in (0, 1]$, a_i is a best-response after observing the message a_i.

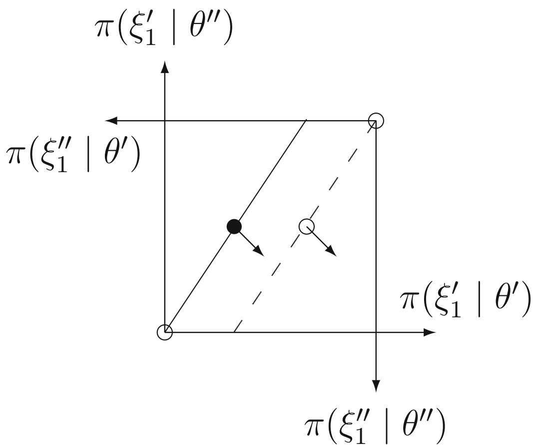
For each $\theta \in \Theta$, consider

$$\begin{aligned} \sum_{a_i \in A_i} \pi(a_i \mid \theta) &= \textstyle\sum_{a_i \in A_i^{*,1}} \pi(a_i \mid \theta) + \sum_{a_i \in A_i^* \backslash A_i^{*,1}} \pi(a_i \mid \theta) \\ &= \textstyle\sum_{a_i \in A_i^{\sharp,1}} \alpha(a_i, \theta) + \sum_{a_i \in A_i^\sharp \backslash A_i^{\sharp,1}} \beta(a_i)\sigma(\theta, a_i). \end{aligned}$$

Note that while the second term may be zero (i.e., $A_i^\sharp \backslash A_i^{\sharp,1} = \emptyset$), the first term is strictly positive since for each $\theta \in \Theta$, there exists $a_i \in A_i^{\sharp,1}$ such that $\theta \in \Theta(a_i)$. We need to have $\sum_{a_i \in A_i} \pi(a_i \mid \theta) = 1$. For each $a_i \in A_i^\sharp \backslash A_i^{\sharp,1}$, make $\beta(a_i) \in (0, 1]$ sufficiently small so that the second term is strictly less than one. For each $a_i \in A_i^{\sharp,1}$ and each

[24] If $|\Theta(a_i)| > 1$, receiver i does not know the actual state, and only knows the state is in $\Theta(a_i)$.

Fig. 6 Multiple best-responses: Example 1 revisited



$\theta \in \Theta(a_i)$, choose $\alpha(a_i, \theta) \in (0, 1]$ so that $\sum_{a_i \in A_i} \pi(a_i \mid \theta) = 1$. Remember that for each $a_i \in A_i^\sharp$, there exists $\theta \in \Theta$ such that $\pi(a_i \mid \theta) > 0$. This implies that $\pi \in \Pi(A^\sharp)$, completing the proof. $\square$

Lemmas 2 and 3 lead to Proposition 1.

4.3 Multiple best-responses

As discussed in Remark 4, if a receiver is indifferent about choosing between two actions, our focus is on the case where the receiver follows the realized message. However, the receiver may as well choose another best-response. The notion of sender-preferred equilibrium in Kamenica and Gentzkow (2011) avoids this. What would happen if such a restriction is not imposed? It turns out that the argument does not change. The question is simply whether such reversals of the receivers' behavior would create more rationalizable strategies for the sender. Remember that when such indifferences matter, it is because π^* lies on the boundaries of the constraints. Consider again Example 1 and suppose instead that receiver 1 chooses a_1' when she is indifferent between a_1' and a_1'' after observing the message a_1'', which is visualized in Fig. 6. If this is the case, there is simply no corresponding rationalizable strategy for the sender. Again, this is because the sender's expected payoff is not uppersemicontinuous at the points on the boundary of the constraint.[25] As this example suggests, the reversals of the receivers' behavior creates no additional rationalizable strategy for the sender.

4.4 Parallel constraints

In the box diagrams of the examples in Sect. 3.1, not only are the two constraints parallel with each other, but also one implies the other. We now show that this is true

[25] See, for example, Dufwenberg and Stegeman (2002).

by taking into account the probabilities assigned to two messages. Given $A^\sharp$, take $i \in N$ with $|A_i^\sharp| \geq 2$, and $a_i', a_i'' \in A_i^\sharp$ with $a_i' \neq a_i''$. For each $\pi \in \Pi(A^\sharp)$, we have

$$\sum_{\theta\in\Theta} \left[u_i(a_i', \theta) - u_i(a_i'', \theta)\right] \pi(a_i' \mid \theta) p_i^0(\theta) \geq 0 \tag{15}$$

$$\sum_{\theta\in\Theta} \left[u_i(a_i', \theta) - u_i(a_i'', \theta)\right] \pi(a_i'' \mid \theta) p_i^0(\theta) \leq 0. \tag{16}$$

Let $c(a_i', a_i'' \mid \theta) = \pi(a_i' \mid \theta) + \pi(a_i'' \mid \theta)$ for each $\theta \in \Theta$, and consider the following expression:

$$\sum_{\theta\in\Theta} \left[u_i(a_i', \theta) - u_i(a_i'', \theta)\right] c(a_i', a_i'' \mid \theta) p_i^0(\theta). \tag{17}$$

Note that (15) and (16) can be rewritten as

$$\sum_{\theta\in\Theta} \left[u_i(a_i', \theta) - u_i(a_i'', \theta)\right] c(a_i', a_i'' \mid \theta) p_i^0(\theta)$$
$$\geq \sum_{\theta\in\Theta} \left[u_i(a_i', \theta) - u_i(a_i'', \theta)\right] \pi(a_i'' \mid \theta) p_i^0(\theta) \tag{18}$$
$$\sum_{\theta\in\Theta} \left[u_i(a_i', \theta) - u_i(a_i'', \theta)\right] c(a_i', a_i'' \mid \theta) p_i^0(\theta)$$
$$\leq \sum_{\theta\in\Theta} \left[u_i(a_i', \theta) - u_i(a_i'', \theta)\right] \pi(a_i' \mid \theta) p_i^0(\theta). \tag{19}$$

We show that the relationship of (15) and (16) depends on the sign of (17), using the following definition.

Definition 2 (15) is said to imply (16) if every $\pi \in \Pi(A^\sharp)$ satisfying (15) satisfies (16). (16) is said to imply (15) if every $\pi \in \Pi(A^\sharp)$ satisfying (16) satisfies (15).

We first show the following result, which states that (17) is zero if and only if (15) and (16) hold simultaneously with equalities. Let

$$\Pi'(A^\sharp) = \left\{\pi \in \Pi(A^\sharp) \mid (15) \text{ holds with equality}\right\}$$
$$\Pi''(A^\sharp) = \left\{\pi \in \Pi(A^\sharp) \mid (16) \text{ holds with equality}\right\}.$$

Lemma 4 $\Pi'(A^\sharp) = \Pi''(A^\sharp)$ *if and only if (17) is zero.*

Proof *(only if)* Assume $\Pi'(A^\sharp) = \Pi''(A^\sharp)$ and take any $\pi \in \Pi'(A^\sharp) = \Pi''(A^\sharp)$ for which (15) and (16) and hence (18) and (19) hold with equalities. Suppose that (17) is positive. Then, the right-hand side of (19) is positive, contradicting (15) with equality. Likewise, suppose that (17) is negative. Then, the right-hand side of (18) is negative, contradicting (16) with equality. Hence, if $\Pi'(A^\sharp) = \Pi''(A^\sharp)$, (17) is zero.

[if] Assume that (17) is zero. Suppose $\Pi'(A^\sharp) \neq \Pi''(A^\sharp)$. First, suppose that there exists $\pi' \in \Pi'(A^\sharp)$ such that $\pi' \notin \Pi''(A^\sharp)$. For any such π', (18) holds with equality. That (17) is zero then implies $\pi' \in \Pi''(A^\sharp)$, thus producing a contradiction. Likewise, suppose instead that there exists $\pi'' \in \Pi''(A^\sharp)$ such that $\pi'' \notin \Pi'(A^\sharp)$. For any such π'', (19) holds with equality. That (17) is zero then implies $\pi'' \in \Pi'(A^\sharp)$, thus producing a contradiction. Hence, if (17) is zero, $\Pi'(A^\sharp) = \Pi''(A^\sharp)$. □

If one of the two actions a_i' or a_i'' is not a best-response to the prior, we have the following result.

Proposition 3 *Given $A^\sharp$, take receiver $i \in N$ with $|A_i^\sharp| \geq 2$ and $a_i', a_i'' \in A_i^\sharp$. Then,*

- (17) *is non-positive if and only if* (15) *implies* (16)*, and*
- (17) *is non-negative if and only if* (16) *implies* (15)*.*

Proof The following argument applies to the first part. The similar argument applies to the second part, which we omit. [only if] Suppose that (17) is non-positive. If (15) holds, so does (18). That (17) is non-negative means that (16) holds. [if] Suppose that (15) implies (16). That both (16) and (18) hold simultaneously means that (17) is non-positive. □

We have the following immediate result.

Corollary 1 *Suppose that there exists $i \in N$ such that $\tilde{A}_i^* = \{a_i', a_i''\}$ (i.e., $|\tilde{A}_i^*| = 2$) where $a_i' \in A_i^0$. Then, (16) implies (15).*

Proof Since $|\tilde{A}_i^*| = 2$, $c(a_i', a_i'' \mid \theta) = 1$ for each $\theta \in \Theta$. That $a_i^0 \in A_i^0$ implies that (17) is non-negative. Proposition 3 then implies the result. □

Given $\tilde{A}^*$ and $\pi^* \in \Pi(\tilde{A}^*)$, for each $i \in N$ and any pair of $a_i', a_i'' \in \tilde{A}_i^*$, (17) is generically non-zero. This is because while fixing the marginals for the other receivers, the sender's expected payoffs corresponding to a_i' and a_i'' are generically different. The sender simply shifts the π's from one action to the other to increase her expected payoff.

5 Conclusion

In this paper, we analyzed the Bayesian persuasion games from Kamenica and Gentzkow (2011). We showed that it is possible to analyze multiple-receiver Bayesian persuasion games with heterogeneous beliefs. Our departure from Kamenica and Gentzkow (2011) is that we directly analyze the sender's messages. With our approach, the sender's optimization problem turns to the examination of a single linear programming problem for unlinked Bayesian persuasion games.[26]

In unlinked games, there is no strategic interaction at the second stage. Once strategic interactions among the receivers are introduced, the framework would no longer

[26] If the stage game exhibits a *Bayesian potential*, the sender can utilize the potential function to analyze the receivers' behavior via Bayes Nash equilibrium. The definition of potential games is by Monderer and Shapley (1996). For Bayesian potential games, see van Heumen et al. (1996) and Ui (2009).

accept the linear programming approach discussed in this paper. Such frameworks would be at least as relevant as linked games. Further theoretical developments as well as applications are needed to improve our understanding.[27]

A Notation

- N: finite set of receivers with $|N| = n \geq 1$.
- Θ: finite set of states with θ being a typical element.
- p_S^0: sender's initial belief over Θ with $p_S^0(\theta) > 0$ for each $\theta \in \Theta$.
- p_i^0: initial belief for receiver $i \in N$ with $p_i^0(\theta) > 0$ for each $\theta \in \Theta$.
- A_i: finite set of (i) actions and (ii) messages (recommendation) for receiver $i \in N$ with $A = \times_{j \in N} A_j$.
- $u_S : A \times \Theta \to \mathbb{R}$: sender's payoff function.
- $u_i : A_i \times \Theta \to \mathbb{R}$: payoff function for receiver $i \in N$—it depends on A_i, not A.
- π: sender's strategy = a distribution over A conditional on Θ with Π being the set of sender's strategies.
- $\mathcal{A}_i(\pi)$: set of realizable messages for receiver $i \in N$ under π

  $$\mathcal{A}_i(\pi) = \{a_i \in A_i \mid \pi(a_i, a_{-i} \mid \theta) > 0 \text{ for some } a_{-i} \in A_{-i} \text{ and } \theta \in \Theta\}$$

  with $\mathcal{A}(\pi) = \times_{j \in N} \mathcal{A}_j(\pi)$.
- M_i: set of possible observations receiver $i \in N$ can have

  $$M_i \subset \Pi \times A_i \text{ such that for any } (\pi, a_i) \in M_i, a_i \in \mathcal{A}_i(\pi).$$

- $\pi(a_i \mid \theta)$: the marginal over a_i conditional on θ

  $$\pi(a_i \mid \theta) = \sum_{a_{-i} \in A_{-i}} \pi((a_i, a_{-i}) \mid \theta).$$

- $p_i^\pi(\theta \mid a_i)$: interim belief for receiver $i \in N$ after observing $(\pi, a_i) \in M_i$

  $$p_i^\pi(\theta \mid a_i) = \frac{\pi(a_i \mid \theta) p_i^0(\theta)}{\sum_{\tilde{\theta} \in \Theta} \pi(a_i \mid \tilde{\theta}) p_i^0(\tilde{\theta})}.$$

[27] In one example in Shimoji (2016), we showed a model in which "silence" has some meaning by utilizing the notion of forward induction from rationalizability.

- $s_i : M_i \to A_i$: pure strategy for receiver $i \in N$ with S_i being the set of strategies for i.
- $s_i(\pi, a_i) \in A_i$: receiver i's action after observing $(\pi, a_i) \in M_i$
- π^0: null message (interim belief = prior for each receiver)
- $A_i^0 \subseteq A_i$: set of actions which are best-responses to p_i^0. If A_i is singleton, a_i^0 represents the unique best-response to p_i^0.
- $s_i^+(\pi, a_i)$: (i) best-response after observing (π, a_i), and (ii) best-response correspondence if there are multiple best-responses (abuse of notation)
- $\tilde{A}$: implementable set—see Definition 1—with $\tilde{A} = \times_{j \in N} \tilde{A}_j$.
- $\Pi(\tilde{A})$: set of sender's strategies which implement $\tilde{A}$

$$\Pi(\tilde{A}) = \left\{ \pi \in \Pi \mid \pi \text{ implements } \tilde{A} \right\}$$

  with $\bar{\Pi}(\tilde{A})$ being the closure of $\Pi(\tilde{A})$.
- $\tilde{A}^\sharp$: largest implementable set
- π^*: sender's rationalizable strategy, maximizing her expected payoff under $\bar{\Pi}(\tilde{A}^\sharp)$
- $\tilde{A}_i^* \subseteq \tilde{A}_i^\sharp$: set of actions for receiver $j \in N$ which will be played with a positive probability under π^* with $\tilde{A}^* = \times_{j \in N} \tilde{A}_j^*$.
- $\tilde{A}_i^\triangleleft$: $\tilde{A}_i^\triangleleft \subseteq \tilde{A}_i$ where each element $a_i \in \tilde{A}_i^\triangleleft$ is such that the "origin" corresponding to a_i is included in $\bar{\Pi}(\tilde{A})$
- $\Delta(\Theta)$: the set of distributions over Θ with $\sigma(\theta)$ being a probability assigned to θ by σ.
- $A_i^\sharp$: set of actions which are best-responses to some $\sigma \in \Delta(\Theta)$
- $\sigma(a_i) \in \Delta(\Theta)$: distribution to which $a_i \in A_i^\sharp$ is a best-response with $\sigma(\theta, a_i)$ being a probability assigned to $\theta \in \Theta$ by $\sigma(a_i)$
- $A_i^{\sharp,1}$: $A_i^{\sharp,1} \subseteq A_i^\sharp$ where each element $a_i \in A_i^{\sharp,1}$ is a best-response to some degenerate distribution.

B Example 4

In this section, we demonstrate how to solve the sender's optimization problem in Example 4—of course, by using standard software, one can obtain the same conclusion.

With $p_S^0 = \frac{8}{10}$, $p_1^0 = \frac{7}{10}$ and $p_2^0 = \frac{6}{10}$, the sender maximizes

$$\frac{1}{5}\left[4(w_a + z_a - \varepsilon y_a) + (w_b + z_b - \varepsilon y_b)\right]$$

subject to

$$\begin{aligned} (w_b + x_b) &\leq \tfrac{7}{3}(w_a + x_a) \\ \tfrac{7}{3}(y_a + z_a) &\leq (y_b + z_b) \end{aligned}$$

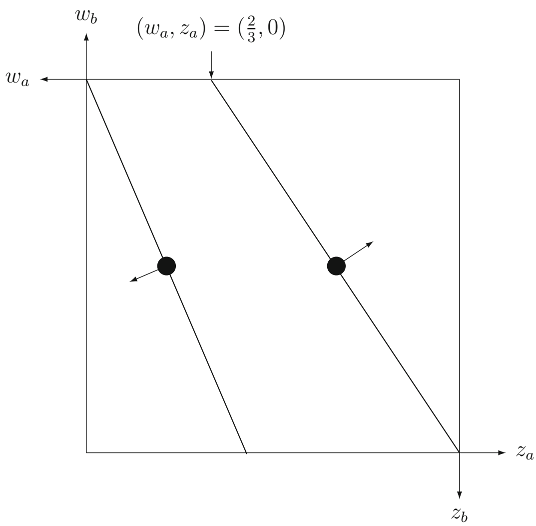


Fig. 7 Best scenario for sender: $w_a + z_a = 1$ and $w_b + z_b = 1$

and

$$\begin{aligned} \tfrac{3}{2}(w_a + y_a) &\leq (w_b + y_b) \\ (x_b + z_b) &\leq \tfrac{3}{2}(x_a + z_a). \end{aligned}$$

First, since y_a and x_b appear only on the left-hand sides, we have $y_a = 0$ and $x_b = 0$, the latter of which implies $y_b = 1 - (w_b + z_b)$. We can then rewrite (with rescaling) the objective function as well as the constraints as

$$4(w_a + z_a) + (1 - \varepsilon)(w_b + z_b) - \varepsilon$$

subject to

$$\begin{matrix} 3w_b \leq 7(1 - z_a) \\ 7z_a \leq 3(1 - w_b) \end{matrix} \; or \; \begin{matrix} 7z_a + 3w_b \leq 7 \\ 7z_a + 3w_b \leq 3 \end{matrix} \Rightarrow 7z_a + 3w_b \leq 3$$

and

$$\begin{matrix} 3w_a \leq 2(1 - z_b) \\ 2z_b \leq 3(1 - w_a) \end{matrix} \; or \; \begin{matrix} 3w_a + 2z_b \leq 2 \\ 3w_a + 2z_b \leq 3 \end{matrix} \Rightarrow 3w_a + 2z_b \leq 2$$

Even though there are still three parameters for each state, we can visualize how to solve this optimization problem.

First, let $w_a + z_a = 1$ (and thus $x_a = 0$) and $w_b + z_b = 1$ (and thus $y_b = 0$). Note that given the sender's objective function, this is the best scenario for her. Figure 7 shows the corresponding box diagram; (i) the bottom-left origin corresponds to the first inequality, $7z_a + 3w_b \leq 3$, with z_a on the horizontal axis and w_b on the vertical axis, and (ii) the top-right origin corresponds to the second inequality, $3w_a + 2z_b \leq 2$, with w_a on the horizontal axis and z_b on the vertical axis. Note that the length of the horizontal axis is $w_a + z_a$ and the length of the vertical axis is equal to $w_b + z_b$,

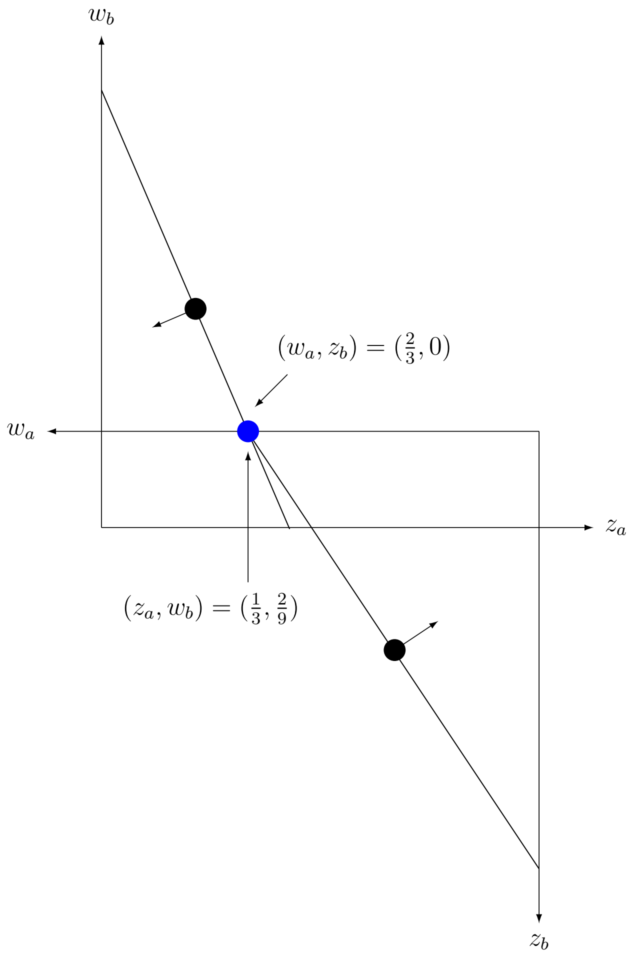


Fig. 8 Alternative scenario: $w_a + z_a = 1$ and $w_b + z_b = \frac{2}{9}$

implying that the sender's objective in this box diagram is to stretch both sides as much as possible. That $w_a + z_a = 1$ and $w_b + z_b = 1$ implies that each side of the box diagram in Fig. 7 is equal to one. As Fig. 7 shows, the combination of $w_a + z_a = 1$ and $w_b + z_b = 1$ does not satisfy the inequalities above.

Next, consider a case in which while $w_a + z_a = 1$ (and thus $x_a = 0$) is maintained, we let $w_b + z_b < 1$ (and thus $y_b > 0$) so that the two inequalities above are satisfied, as shown in Fig. 8. This can be seen as moving the top-right origin downwards so that two lines, which have different slopes, just touch each other. The sender's corresponding strategy is $(w_a, x_a, y_a, z_a) = (\frac{2}{3}, 0, 0, \frac{1}{3})$ and $(w_b, x_b, y_b, z_b) = (\frac{2}{9}, 0, \frac{7}{9}, 0)$.

Then, the question is whether this is the sender's optimal strategy. The only direction the sender can choose is to increase $w_b + z_b$ (the length of the vertical axis) while decreasing $w_a + z_a$ (the length of the horizontal axis)—still maintaining two lines touching each other in the same manner. Given the sender's objective function, the marginal rate of substitution between these two values is

$$-\frac{d(w_b + z_b)}{d(w_a + z_a)} = \frac{4}{1 - \varepsilon} > 4$$

meaning that the sender can give up one unit of $w_a + z_a$ with a compensation of a greater than four unit increase in $w_b + z_b$. Note however (the absolute value of) the slope of $7z_a + 3w_b = 3$ (or $w_b = -\frac{7}{3}z_a + 1$) is $\frac{7}{3} < 4$, implying that such a compensation is unfeasible. This means that $(w_a, x_a, y_a, z_a) = (\frac{2}{3}, 0, 0, \frac{1}{3})$ and $(w_b, x_b, y_b, z_b) = (\frac{2}{9}, 0, \frac{7}{9}, 0)$ is indeed the solution for the sender's optimization problem.

References

Adler I (2013) The equivalence of linear programs and zero-sum games. Int J Game Theory 42(1):165–177

Alonso R, Câmara O (2016a) Bayesian persuasion with heterogeneous priors. J Econ Theory 165:672–706

Alonso R, Câmara O (2016b) Persuading voters. Am Econ Rev 106(11):3590–3605

Aumann RJ, Maschler MB (1995) Repeated games with incomplete information. MIT Press, Cambridge

Battigalli P (1997) On rationalizability in extensive games. J Econ Theory 74(1):40–61

Battigalli P, Siniscalchi M (2003) Rationalization and incomplete information. Adv Theor Econ 3(1)

Bergemann D, Morris S (2016) Bayes correlated equilibrium and the comparison of information structures in games. Theor Econ 11(2):487–522

Bergemann D, Morris S (2019) Information design: a unified perspective. J Econ Lit 57(1):44–95

Brocas I, Carrillo JD (2007) Influence through ignorance. Rand J Econ 38(4):931–947

Chan J, Gupta S, Li F, Wang Y (2019) Pivotal persuasion. J Econ Theory 180:178–202

Crawford VP, Sobel J (1982) Strategic information transmission. Econometrica 50(6):1431–1451

Dufwenberg M, Stegeman M (2002) Existence and uniqueness of maximal reductions under iterated strict dominance. Econometrica 70(5):2007–2023

Eddie D, Marciano S (2015) Epistemic game theory. In: Young HP, Zamir S (eds) Handbook of game theory with economic applications, vol 4. Elsevier, Amsterdam, pp 619–702

Farrell J, Gibbons R (1989) Cheap talk with two audiences. Am Econ Rev 79(5):1214–1223

Goltsman M, Pavlov G (2011) How to talk to multiple audiences. Games Econ Behav 72(1):100–122

Hernández P, Neeman Z (2021) How Bayesian persuasion can help reduce illegal parking and other socially undesirable behavior. Am Econ J Microecon (**forthcoming**)

Hörner J, Skrzypacz A (2016) Selling information. J Political Econ 124(6):1515–1562

Kamenica E, Gentzkow M (2011) Bayesian persuasion. Am Econ Rev 101(6):2590–2615

Kolotilin A (2018) Optimal information disclosure: a linear programming approach. Theor Econ 13(2):607–635

Kolotilin A, Li M, Mylovanov T, Zapechelnyuk A (2017) Persuasion of a privately informed receiver. Econometrica 85(6):1949–1964

Lipnowski E, Mathevet L (2018) Disclosure to a psychological audience. Am Econ J Microecon 10(4):67–93

Mangasarian OL (1979) Uniqueness of solution in linear programming. Linear Algebra Appl 25:151–162

Miura S, Yamashita T (2020) Maximal miscommunication. Econ Lett 188:108962

Monderer D, Shapley LS (1996) Potential Games. Games Econ Behav 14(1):124–143

Ostrovsky M, Schwarz M (2010) Information disclosure and unraveling in matching markets. Am Econ J Microecon 2(2):34–63

Rayo L, Segal I (2010) Optimal information disclosure. J Political Econ 118(5):949–987

Shimoji M (2016) Rationalizable Persuasion. University of York, Discussion Paper

Shimoji M, Watson J (1998) Conditional dominance, rationalizability, and game forms. J Econ Theory 83(2):161–195

Sobel J (2013) Giving and receiving advice. In: Acemoglu D, Arellano M, Dekel E (eds) Advances in economics and econometrics, Tenth World Congress, Volume I, economic theory. Cambridge University Press, New York, pp 305–341

Tamura W (2018) Bayesian persuasion with quadratic preferences. Working Paper

Taneva I (2019) Information design. Am Econ J Microecon 11(4):151–185

Ui T (2009) Bayesian potentials and information structures: team decision problems revisited. Int J Econ Theory 5(3):271–291

van Heumen R, Peleg B, Tijs S, Born P (1996) Axiomatic characterizations of solutions for Bayesian games. Theory Decis 40(2):103–129

Wang Y (2013) Bayesian persuasion with multiple receivers. Working Paper