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Bounding the benefit of adaptivity in quantum metrology using the relative fidelity

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Protocols for discriminating between a pair of channels or for estimating a channel parameter can often be aided by adaptivity or by entanglement between the probe states. For large numbers of channel uses, this can make it difficult to bound the best possible performance for such protocols. In this paper, we introduce a quantity that we call the relative fidelity of a given pair of channels and a pair of input states to those channels. Constraining the allowed input states to all pairs of states whose fidelity is greater than some minimum “input fidelity” and minimising this quantity over the valid pairs of states, we get the minimum relative fidelity for that input fidelity constraint. We are then able to lower bound the fidelity between the possible output states of any protocol acting on one of two possible channels in terms of the minimum relative fidelity. This allows us to bound the performance of discrimination and parameter estimation protocols, as well as to rule out perfect discrimination for certain pairs of channels, even using the most general, adaptive protocols. By finding a continuity bound for the relative fidelity, we are also able to provide a simple proof that the quantum Fisher information (QFI) of the output of an N -use protocol is no more than N^2 times the one-shot QFI.

I. INTRODUCTION

Many physical processes can be modelled as quantum channels, so determining the identity of an unknown channel and estimating a parameter encoded in that channel are important tasks in the field of quantum information. The quantum fidelity [1–3] between the possible output states of channel discrimination or parameter estimation protocols give an indication of the distinguishability of those output states. By bounding the quantum fidelity between the possible output states of a protocol, we can bound the performance of both quantum channel discrimination [4–16] and quantum metrology [10, 17–23].

In order to achieve the minimum possible output fidelity, we must allow the protocols to be adaptive. This means that the output from a previous use of the unknown channel can affect the input to a subsequent channel use. In an example with two discrete-variable channels, it was shown that adaptive schemes can beat non-adaptive schemes for channel discrimination [24].

Here, we present a new measure on a pair of channels and a pair of input states to those channels called the relative fidelity. If we constrain the allowed input states to have a fidelity greater than or equal to some minimum “input fidelity” and then minimise the relative fidelity over the allowed states (for a fixed pair of channels), we get the minimum relative fidelity for that channel pair and that input fidelity. We can use this quantity to formulate lower bounds on the minimum fidelity between output states for any adaptive protocol. If the minimum relative fidelity is constant (i.e. has no dependence on the input fidelity), the optimal protocol not

only does not require adaptivity but also does not require entanglement between the input states for each channel use. If the minimum relative fidelity for a pair of channels is never equal to zero for any input fidelity, we are guaranteed that there exists no discrimination protocol - adaptive or otherwise - that allows that pair of channels to be perfectly distinguished. By finding continuity bounds on the minimum relative fidelity, we show that the maximum quantum Fisher information (QFI) for an N -use protocol is no more than N^2 times the maximum one-shot QFI (which was previously only known to hold asymptotically in N).

II. BOUNDING THE MINIMUM FIDELITY BETWEEN THE OUTPUTS OF A PROTOCOL

Suppose we have a black box containing a channel, \mathcal{C} , drawn from a set of two possible channels, $\{\mathcal{C}_1, \mathcal{C}_2\}$, both of which have input dimension d . Our task is to achieve the minimum fidelity between the two possible outputs of a fixed protocol that involves N uses of \mathcal{C} . This protocol can be adaptive, meaning that the input for a channel use can depend on the output from every previous channel use. Such protocols can be represented as quantum combs [10, 25] and are the most general strategies allowed by quantum mechanics.

Lower bounding the minimum fidelity between protocol outputs allows us to upper bound the distinguishability of the channels \mathcal{C}_1 and \mathcal{C}_2 . The minimum probability of error in discriminating between a pair of states is bounded by the Fuchs-van der Graaf inequality [26]:

$$p_{\text{err}} \geq \frac{1 - \sqrt{1 - F^2}}{2}, \quad (1)$$

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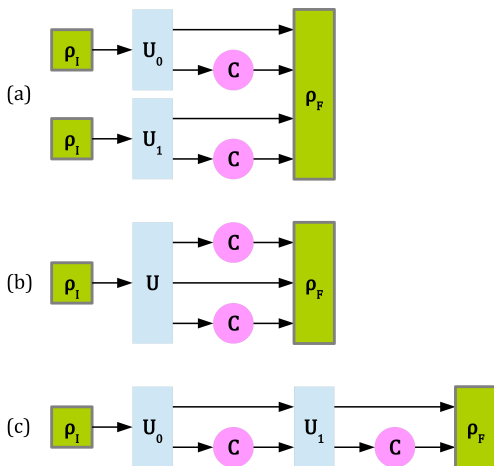


FIG. 1. Setups for different types of protocols involving two channel uses. In all cases, there is a channel-independent initial state, ρ_I , which undergoes state preparation by means of some unitary and is subject to a total of two channel uses (labelled C), resulting in a final state, ρ_F . In (a), state preparation is carried out in such a way that ρ_F takes tensor product form (there is no entanglement between the probe states for each channel use). U_0 prepares the first probe state and U_1 prepares the second. In (b), the initial state preparation, U , can result in entanglement between probe states, but there is no feedback between the output of one channel use and the input of the other. (b) defines the set of parallel protocols. In (c), we have a fully adaptive strategy, with U_0 carrying out the initial state preparation and U_1 describing any arbitrary processing on the state prior to the second channel use. Note that the protocols in (a) are a subset of those in (b), which are themselves a subset of those in (c).

where F is the fidelity between the states. Consequently, the minimum fidelity between the output states of any N -use protocol lower bounds the error probability for any such protocol that discriminates between the channels.

Alternatively, suppose we have a family of channels parametrised by a variable θ , \mathcal{C}_θ , and can find an analytical expression for the minimum fidelity between the outputs of an N -use protocol acting on channels \mathcal{C}_θ and $\mathcal{C}_{\theta+\Delta_\theta}$, in terms of θ . We can use this expression to upper bound the achievable QFI. The QFI is a crucial quantity in quantum metrology, because it appears in the quantum Cramér–Rao bound (QCRB), which bounds the variance of parameter estimation [17]. The QCRB states that

$$\text{var}(\tilde{\theta}) \geq \frac{1}{\text{QFI}_N(\tilde{\theta})}, \quad (2)$$

where $\tilde{\theta}$ is an unbiased estimator of θ , so by upper bounding the QFI of any possible protocol output state, we lower bound the variance of an unbiased estimator of θ .

Fig. 1 illustrates, for two channel uses, the different types of protocols that we consider. In the simplest case, we have independently prepared probe states (with no shared entanglement), each of which passes through the channel, resulting in a final state that takes tensor prod-

uct form. Each probe can have idler systems (systems which do not pass through the channel, but may be entangled to the states which do). The second case is similar, but entanglement between the probe states is allowed. The final case is fully adaptive. Note that the diagram uses unitaries, but these could be replaced by channels or measurements by tracing over some of the systems that the unitaries act on. Doing so cannot improve the discriminative power of the protocols. We are interested in classifying for which channel discrimination problems there are protocols in (c) that are more powerful than all protocols in (a).

A. Defining relative fidelity

The fidelity between a pair of states is defined by

$$F(\rho_1, \rho_2) = \text{Tr} \left[\sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right]. \quad (3)$$

We define the output fidelity, F_{out} , between \mathcal{C}_1 and \mathcal{C}_2 , for a given pair of input states, σ_1 and σ_2 , as the fidelity between the channel outputs. In other words

$$F_{\text{out}}(\sigma_1, \sigma_2) = F(\mathcal{I} \otimes \mathcal{C}_1[\sigma_1], \mathcal{I} \otimes \mathcal{C}_2[\sigma_2]), \quad (4)$$

where the identity operator acts on the idler modes. Let F_{con} be the minimum output fidelity for constant input

$$F_{\text{con}} = \inf_{\sigma \in \mathcal{D}(d^2)} F_{\text{out}}(\sigma, \sigma), \quad (5)$$

with $\mathcal{D}(d^2)$ being the space of density operators of dimension d^2 . Note that F_{con} can be efficiently computed via semidefinite programming [27].

Let us consider how the fidelity between output states evolves at each stage of the protocol. We define F_N as the fidelity between the possible output states after N channel uses, considering the most general case shown in Fig. 1(c). Similarly, F_i , with $i < N$, is the fidelity between the possible output states that would be obtained if the protocol were terminated prematurely immediately prior to the $(i+1)$ -th channel use. Fidelity is non-decreasing over trace-preserving operations, so the fidelity between output states can only be reduced by a use of the channel \mathcal{C} . It is immediate from the multiplicativity of fidelity over tensor products that we can always reduce the output fidelity by a multiplicative factor of at least F_{con} at each channel use, i.e. we can always choose some input to the i -th channel use such that

$$F_i \leq F_{\text{con}} F_{i-1}. \quad (6)$$

We can often do better than this, even for parallel strategies (strategies that do not involve adaptivity). For certain pairs of channels, the fidelity between output states can be multiplied by a factor smaller than F_{con} each time the channel is applied to one part of the probe state. An example would be discriminating between unitaries using

some initially entangled, multipartite probe state, such as the GHZ state (this would be an example of a protocol in (b), from Fig. 1). Such protocols are able to perfectly discriminate between two unitaries after a finite number of uses [28, 29].

A natural question is “how much can the fidelity between output states be reduced by in a single channel use?” Equivalently, we can ask “what is the minimum achievable value of $\frac{F_{i-1}}{F_i}$?” To address this question, we introduce the “relative fidelity”, F_R , which we define as

$$F_R(\sigma_1, \sigma_2) = \frac{F_{\text{out}}(\sigma_1, \sigma_2)}{F(\sigma_1, \sigma_2)}. \quad (7)$$

The relative fidelity is the ratio between the output fidelity and the input fidelity for a pair of states, σ_1 and σ_2 . Note that this can be either greater than or less than (or equal to) one, depending on the choice of states. In order for F_R to be well-defined for all input states, we define

$$F_R(\sigma_1, \sigma_2) = \lim_{\delta \rightarrow 0} F_R(\sigma_1, (1 - \delta)\sigma_2 + \delta\sigma_1) \quad (8)$$

for orthogonal σ_1 and σ_2 (since otherwise we would have both the numerator and the denominator equal to zero).

We now define the quantity

$$F_{R,\min}(f) = \inf_{\{\sigma_1, \sigma_2\} \in \{\mathcal{D}: F(\sigma_1, \sigma_2) \geq f\}} F_R(\sigma_1, \sigma_2), \quad (9)$$

where $\{\mathcal{D} : F(\sigma_1, \sigma_2) \geq f\}$ is the set of all pairs of density matrices with a fidelity $\geq f$. $F_{R,\min}(f)$ gives the maximum amount that the output fidelity of a protocol can be reduced by with a single channel use (the minimum factor that it can be multiplied by), so long as the fidelity before that channel use was $\geq f$.

If $F_{R,\min}(f) < F_{\text{con}}$, this does not necessarily mean adaptivity is required for the optimal protocol. $F_{R,\min}(f)$ may not be achievable by any protocol. For instance, the channels could output mixed states, but the inputs giving $F_{R,\min}(f)$ could be pure. Alternatively, a protocol may be non-adaptive but have entanglement between the probe states (as for the GHZ probe). This is still non-adaptive, despite the shared input state, because the output of one channel use never affects the input to another. In fact, for unitary channel discrimination, it is known that a non-adaptive strategy with entanglement between the probe states can be used instead of an adaptive strategy in order to achieve optimal discrimination [29].

The opposite is not true. If $F_{R,\min}(F_{\text{con}}^{N-1}) = F_{\text{con}}$ then no N -use protocol can benefit from adaptivity or entanglement between the probe states (note that $F_{R,\min}(F_{\text{con}}^{N-1})$ is never more than F_{con} , since we can always choose σ_1 and σ_2 to both be σ_{min} , the state that achieves F_{con}). If $F_{R,\min}(0) = F_{\text{con}}$, then adaptivity (and entanglement between probe states) is never required for the optimal protocol, and we can write the following inequality for any N -use protocol

$$F_N \geq F_{\text{con}}^N. \quad (10)$$

Otherwise, we can write the lower bound:

$$F_N \geq F_{N-1} F_{R,\min}(F_{N-1}). \quad (11)$$

Numerically, this can be calculated recursively, starting from a single channel use. The lower bound on F_N can then be substituted into Eq. (1) to lower bound the error probability for discrimination between a pair of channels.

Suppose we want to distinguish between those channel pairs which can be perfectly discriminated after finite uses (such as unitaries or the channels in Ref. [24]) and those which can never be perfectly discriminated after a finite number of channel uses, even if adaptivity or entangled probes can reduce the error probability (such as classical channels [24, 30, 31]). For perfect discrimination we require a trace norm between protocol outputs of one and hence a fidelity of zero. For such a fidelity to be achievable by some finite-use discrimination protocol, it must be the case that the fidelity is non-zero before a particular channel use and is zero afterwards. Hence, there must exist some non-zero input fidelity for which the minimum relative fidelity is zero, i.e. there must exist some threshold input fidelity, $f_T > 0$, such that

$$F_{R,\min}(f) = 0 \quad (12)$$

for all $f \leq f_T$. If, on the other hand, we have

$$F_{R,\min}(0) = \lim_{f \rightarrow 0} F_{R,\min}(f) > 0, \quad (13)$$

then no discrimination protocol with a finite number of channel uses can perfectly distinguish between \mathcal{C}_1 and \mathcal{C}_2 . One application of the relative fidelity is therefore to identify such channel pairs.

One question, when calculating $F_{R,\min}(f)$, is: what is the maximum dimension of σ_1 and σ_2 ? In the case of a constant input, a pure state is optimal, due to the superadditivity of fidelity, and the maximum required dimension of the input is d^2 . For $f < 1$, it is not immediately obvious, but we can again show that $F_{R,\min}(f)$ can always be achieved by a pair of pure input states with dimension d^2 . The proof of this is given in Appendix A.

A key property that the relative fidelity lacks (but that fidelity has) that may make the task of finding the minimum relative fidelity more difficult is concavity. See the supplementary MATLAB files for a counterexample based on a pair of amplitude damping channels.

III. CONTINUITY PROPERTIES

Suppose we know the minimum relative fidelity for some input fidelity, f , and wish to bound its value for some other input fidelity, f' . We can do so using the continuity bound

$$F_{R,\min}(f) \geq \frac{1}{f} \left(1 - \left(\sqrt{1 - f' F_{R,\min}(f')} + (\sqrt{1 - f} - \sqrt{1 - f'})^2 \right) \right), \quad (14)$$

where $f < f'$. This recursively defines $F_{R,\min}(f)$ in terms of $F_{R,\min}(f')$. See Appendix B for the proof of this bound. A scheme for numerically finding the minimum relative fidelity for any pair of channels, based on an output fidelity continuity bound, is given in Appendix C.

Setting $f' = 1$, in Eq. (14), we lower bound $F_{R,\min}(f)$ for any channel. We can write

$$F_{\text{con}} \geq F_{R,\min}(f) \geq \frac{1 - (\sqrt{1 - F_{\text{con}}} + \sqrt{1 - f})^2}{f}, \quad (15)$$

where F_{con} is, as before, the minimum output fidelity for constant channel inputs. Whilst Eq. (15) is not (necessarily) tight, it gives an ultimate bound on the benefit of adaptivity for any channel. The lower bound corresponds to the best possible output fidelity scaling that any protocol can achieve, adaptive or otherwise.

We may instead choose to express this bound in terms of the minimum output fidelity for a given input fidelity:

$$F_{\text{out},\min}(f) \geq 1 - (\sqrt{1 - F_{\text{con}}} + \sqrt{1 - f})^2. \quad (16)$$

This can also be expressed in terms of the minimum output fidelity of an N -use protocol as

$$F_N \geq 1 - N^2(1 - F_{\text{con}}). \quad (17)$$

This can be verified by substituting in the right-hand side of Eq. (17) as f in Eq. (16).

The N -use QFI is given by

$$\text{QFI}_N(\theta) = \frac{8(1 - F_N(\theta, \theta + d\theta))^2}{d\theta^2}. \quad (18)$$

We can write

$$1 - F_N(\theta, \theta + d\theta) \leq N^2(1 - F_{\text{con}}(\theta, \theta + d\theta)), \quad (19)$$

with equality if, for every θ and $\theta + d\theta$, we can achieve the N -use output fidelity given in Eq. (17) (with F_{con} depending on the choice of θ). Using Eq. (18), we write

$$\text{QFI}_N(\theta) \leq \frac{8N^2(1 - F_{\text{con}}(\theta, \theta + d\theta))^2}{d\theta^2} = N^2\text{QFI}_1(\theta), \quad (20)$$

where QFI_1 is the one-shot QFI. Consequently, we have shown (via the QCRB) that the variance scales with the inverse square of the number of channel uses. This was previously known to hold asymptotically (in the number of channel uses) [11, 32], but we here show that the bound is more general and that it also holds for a finite number of channel uses. We have therefore recovered the Heisenberg scaling as the maximum possible scaling with the number of channel uses, as expected [20].

IV. SCALING WITH INPUT FIDELITY

There are three possible scenarios for the scaling of $F_{R,\min}$ with the input fidelity.

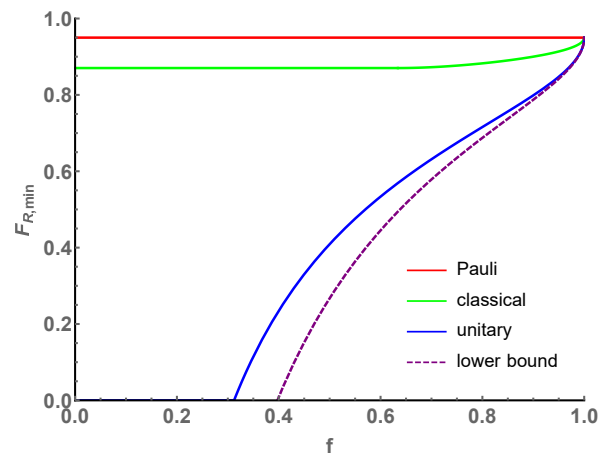


FIG. 2. The different ways in which the minimum relative fidelity, $F_{R,\min}$, could scale with the input fidelity, f . The three continuous lines represent specific examples of channel pairs demonstrating different types of scaling, whilst the dashed line is the lower bound in Eq. (15). All three examples have $F_{\text{con}} = 0.95$. The top line represents a pair of Pauli channels, for which neither adaptivity nor entanglement between the probes can reduce the minimum fidelity between output states. It corresponds to the upper bound in Eq. (15). The middle line represents a pair of classical channels, for which adaptivity or entanglement between probes may reduce the output fidelity but perfect discrimination is still impossible. The bottom line represents a pair of unitary channels for which adaptivity or entanglement between probes can reduce the output fidelity and perfect discrimination is possible.

The first is where $F_{R,\min}$ is constant and equal to F_{con} . This corresponds to channel discrimination problems for which neither adaptivity nor entanglement between probe states are required for optimal discrimination (and perfect discrimination is not possible). Discrimination between Pauli channels is a known example of such a problem [33].

Secondly, there are cases in which $F_{R,\min}$ decreases with the input fidelity and eventually goes to zero. In these situations, it may be the case that adaptivity or entanglement between probe states (or both) are required for optimal discrimination, and perfect discrimination may be possible. An example of such a situation is discrimination between two unitaries [28, 29].

Finally, for some discrimination tasks, $F_{R,\min}$ may decrease from F_{con} but then tend to a non-zero value at an input fidelity of zero (so that $0 < F_{R,\min}(0) < F_{\text{con}}$). This implies that adaptivity or entanglement between probe states may be required for optimal discrimination, but that perfect discrimination is not possible. Entanglement-breaking (classical) channels are an example of this scenario [24, 30, 31].

By investigating simple example cases for each of these scenarios and analytically finding the minimum relative fidelity, we can confirm that the minimum relative fidelity acts as expected in these situations. Fig. 2 shows the rela-

tive fidelity scaling for each of the example cases. These examples are discussed in more detail in Appendix D, and demonstrate how the relative fidelity can be used to bound how the error probability for channel discrimination scales with the number of channel uses and to identify situations in which adaptivity and entanglement between probe states cannot help with channel discrimination.

V. CONCLUSION

In this paper, we introduced a quantity that we call the relative fidelity of a pair of channels, which is the ratio between the input and the output fidelity for a pair of states. By minimising this quantity, subject to a constraint on the input fidelity, we lower bound the fidelity between outputs of an N -use channel discrimination or parameter estimation protocol. If the minimum relative fidelity is non-zero for every input fidelity, the pair of channels can never be perfectly discriminated.

The minimum relative fidelity for a given pair of channels (acting on d -dimensional inputs) and a given input fidelity constraint can always be achieved by a pair of pure states, each with dimension no more than d^2 . We also found a minimum relative fidelity continuity bound. By doing so, we lower bounded the output fidelity for any adaptive protocol discriminating between any pair of channels. As a result, we demonstrated that the QFI for an N -use protocol is no more than N^2 times the maximum

one-shot QFI. Whilst this was previously known to hold asymptotically [11, 32], the proof presented here is valid for any finite number of channel uses.

The minimum relative fidelity is a quantity that could prove useful for providing ultimate bounds on the performance of channel discrimination or parameter estimation protocols. Because finding it only requires a maximisation over two d^2 -dimensional pure qudit states for a given input fidelity, it should be easier, in many cases, to bound the performance of a protocol in this way than to optimise over all possible adaptive protocols (as represented by the set of all d^{2N} -dimensional quantum combs). Although the result of a full optimisation would be more precise than a bound based on the minimum relative fidelity, minimisation over the set of quantum combs involves $\sim d^{4N}$ free parameters (since each comb can be represented by a d^{2N} by d^{2N} Choi matrix), meaning that such a minimisation quickly becomes difficult for large N . The minimum relative fidelity allows the formulation of bounds - both analytical or numeric - for discrimination problems involving large numbers of channel uses, for which the true minimum output fidelity is computationally intractable to calculate.

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Appendix A: Proof the minimum relative fidelity can be achieved by a pair of pure states with dimension d^2

It is possible to prove that the minimum relative fidelity in any given domain is achieved by a pure state by using an alternative definition of the fidelity. The fidelity between any two states is the maximum absolute value of the overlap between purifications. This can be written as

$$F(\rho_1, \rho_2) = \max_{|\rho'_1\rangle, |\rho'_2\rangle} |\langle \rho'_1 | \rho'_2 \rangle|, \quad (\text{A1})$$

where $|\rho'_{1(2)}\rangle$ is a purification of $\rho_{1(2)}$. Note that, for pure states,

$$F(\rho_1, \rho_2) = |\langle \rho'_1 | \rho'_2 \rangle|. \quad (\text{A2})$$

We therefore know that, for any pair of states, there exists some purification that does not decrease (or increase) the fidelity between channel outputs. Starting from the input states σ_1 and σ_2 , let us replace them with the purifications that maximise their overlap, $|\sigma'_1\rangle$ and $|\sigma'_2\rangle$. These new input states have the same input fidelity. Since tracing over modes can never decrease the fidelity between a pair of states, the new input states also achieve the same or a lower output fidelity. Thus, $|\sigma'_1\rangle$ and $|\sigma'_2\rangle$ obtain a relative fidelity that is lower than or equal to that obtained by σ_1 and σ_2 . Consequently, pure input states obtain the minimum relative fidelity for given input fidelity.

Now suppose σ_1 is pure. Let us write

$$\sigma_1 = (\sigma_1)_{SI}, \quad (\text{A3})$$

where S labels the system that passes through the channel and I labels the idler modes (with unbounded dimension). There exists some unitary U , acting only on the idler modes, such that we obtain

$$U\sigma_1 U^\dagger = (\sigma'_1)_{SI'} \otimes |0\rangle\langle 0|_{I''}, \quad (\text{A4})$$

where I' labels a subset of the idler modes with dimension d and I'' labels some other subset of the idler modes with unbounded dimension. If U is applied to σ_2 as well, neither the fidelity of the input states nor the output fidelity will be affected, and so the relative fidelity will be unchanged. We call the obtained state σ'_2 . We now write

$$\sqrt{(\sigma'_1)_{SI'} \otimes |0\rangle\langle 0|_{I''}} = \left(\sqrt{\sigma'_1}\right)_{SI'} \otimes |0\rangle\langle 0|_{I''}. \quad (\text{A5})$$

Now we note that

$$(\sigma''_2)_{SI'} = \frac{1}{\alpha} \langle 0|_{I''} (\sigma'_2)_{SI'I''} |0\rangle_{I''} \quad (\text{A6})$$

is a state on the d^2 -dimensional system SI' , where α is a normalising factor. α is given by

$$\alpha = \text{Tr}[\langle 0|_{I''} (\sigma'_2)_{SI'I''} |0\rangle_{I''}]. \quad (\text{A7})$$

If σ_2 is pure then σ''_2 will also be pure. Consequently, we can write

$$\begin{aligned} F(\sigma_1, \sigma_2) &= \text{Tr} \left[\sqrt{\sqrt{\sigma_1} \sigma_2 \sqrt{\sigma_1}} \right] \\ &= \text{Tr} \left[\sqrt{\left(\sqrt{\sigma'_1} \otimes |0\rangle\langle 0| \right) \sigma'_2 \left(\sqrt{\sigma'_1} \otimes |0\rangle\langle 0| \right)} \right] \\ &= \text{Tr} \left[\sqrt{\sqrt{\sigma'_1} \langle 0|_{I''} (\sigma'_2)_{SI'I''} |0\rangle_{I''} \sqrt{\sigma'_1}} \right] \\ &= \sqrt{\alpha} F(\sigma'_1, \sigma''_2). \end{aligned} \quad (\text{A8})$$

Then, since the channels are trace-preserving, we can also write

$$F_{\text{out}}(\sigma_1, \sigma_2) = \sqrt{\alpha} F_{\text{out}}(\sigma'_1, \sigma''_2). \quad (\text{A9})$$

Therefore,

$$F_R(\sigma_1, \sigma_2) = F_R(\sigma'_1, \sigma''_2), \quad (\text{A10})$$

and so the relative fidelity of the dimension unbounded states σ_1 and σ_2 is the same as that of some other pair of states, σ'_1 and σ''_2 , which each have a dimension of d^2 .

Combining the two results, we can see that the minimum relative fidelity, $F_{R,\min}(f)$, can always be obtained by a pair of pure input states with dimension d^2 .

Although we do not guarantee that the states that achieve $F_{R,\min}(f)$ (which we will call $|\sigma_{1,\min}\rangle$ and $|\sigma_{2,\min}\rangle$) have an input fidelity equal to f (only that it is $\geq f$), we can easily construct a pair of pure states, $|\sigma'_{1,\min}\rangle$ and $|\sigma'_{2,\min}\rangle$, that achieve $F_{R,\min}(f)$ and have an input fidelity of f . Specifically, we can define

$$|\sigma'_{1,\min}\rangle = |\sigma_{1,\min}\rangle \otimes |0\rangle, \quad (\text{A11})$$

$$|\sigma'_{2,\min}\rangle = |\sigma_{2,\min}\rangle \otimes \left(\frac{f}{f'} |0\rangle + \sqrt{1 - \left(\frac{f}{f'}\right)^2} |1\rangle \right), \quad (\text{A12})$$

where the final qubit simply gives a constant multiplicative factor of $\frac{f}{f'}$ to both the input and the output fidelity. These states also achieve the minimum possible output fidelity for a given input fidelity.

Appendix B: Proof of the minimum relative fidelity continuity bound

We start by finding a continuity bound for relative fidelity, before using it to find the continuity bounds for minimum relative fidelity. Given a pair of density matrices, $\{\sigma_1, \sigma_2\}$, with a relative fidelity of $F_R(\sigma_1, \sigma_2)$, we wish to bound the relative fidelity of a nearby pair of density matrices, $\{\sigma'_1, \sigma'_2\}$. By nearby, we mean that the Bures distances, $d_B(\sigma_1, \sigma'_1)$ and $d_B(\sigma_2, \sigma'_2)$, between σ_1 and σ'_1 and between σ_2 and σ'_2 are known. We define

$$\delta(\sigma_1, \sigma'_1, \sigma_2, \sigma'_2) = \frac{1}{\sqrt{2}} (d_B(\sigma_1, \sigma'_1) + d_B(\sigma_2, \sigma'_2)), \quad (\text{B1})$$

$$d_B(\rho_1, \rho_2) = \sqrt{2} \sqrt{1 - F(\rho_1, \rho_2)}. \quad (\text{B2})$$

Starting from the triangle inequalities for the Bures distance, we can derive similar relationships for the fidelity. The triangle inequalities tell us that

$$d_B(\sigma'_1, \sigma'_2) \leq d_B(\sigma_1, \sigma_2) + \sqrt{2} \delta(\sigma_1, \sigma'_1, \sigma_2, \sigma'_2), \quad (\text{B3})$$

$$d_B(\sigma'_1, \sigma'_2) \geq d_B(\sigma_1, \sigma_2) - \sqrt{2} \delta(\sigma_1, \sigma'_1, \sigma_2, \sigma'_2). \quad (\text{B4})$$

We can therefore derive

$$F(\sigma'_1, \sigma'_2) \geq 1 - \left(\sqrt{1 - F(\sigma_1, \sigma_2)} + \delta \right)^2, \quad (\text{B5})$$

$$F(\sigma'_1, \sigma'_2) \leq 1 - \left(\sqrt{1 - F(\sigma_1, \sigma_2)} - \delta \right)^2. \quad (\text{B6})$$

Note that Eqs. (B5) and (B6) apply to any two pairs of states $\{\sigma_1, \sigma_2\}$ and $\{\sigma'_1, \sigma'_2\}$. In order to lower bound the relative fidelity, $F_R(\sigma'_1, \sigma'_2)$, we must upper bound the input fidelity, $F(\sigma'_1, \sigma'_2)$, and lower bound the output fidelity, $F_{\text{out}}(\sigma'_1, \sigma'_2)$.

The upper bound on the input fidelity comes directly from Eq. (B6). We now note that

$$d_B(\sigma_1, \sigma'_1) \geq d_B(\mathcal{I} \otimes \mathcal{C}_1[\sigma_1], \mathcal{I} \otimes \mathcal{C}_1[\sigma'_1]), \quad (\text{B7})$$

$$d_B(\sigma_2, \sigma'_2) \geq d_B(\mathcal{I} \otimes \mathcal{C}_2[\sigma_2], \mathcal{I} \otimes \mathcal{C}_2[\sigma'_2]), \quad (\text{B8})$$

from the monotonicity of the Bures distance. Therefore, we can write the lower bound

$$F_{\text{out}}(\sigma'_1, \sigma'_2) \geq 1 - \left(\sqrt{1 - F_{\text{out}}(\sigma_1, \sigma_2)} + \delta \right)^2. \quad (\text{B9})$$

Consequently, we can write the continuity bound:

$$F_R(\sigma'_1, \sigma'_2) \geq \frac{F_{\text{out}}(\sigma_1, \sigma_2) - 2\delta \sqrt{1 - F_{\text{out}}(\sigma_1, \sigma_2)} + \delta^2}{F(\sigma_1, \sigma_2) + 2\delta \sqrt{1 - F(\sigma_1, \sigma_2)} - \delta^2}. \quad (\text{B10})$$

This gives us a way of numerically finding $F_{R,\min}(f)$. We know that $F_{R,\min}(f)$ will be achieved by some pair of pure, d^2 -dimensional qudit states that have a fidelity greater than or equal to f . Further, the set of pairs of density matrices that have a fidelity greater than or equal to f , $\{\mathcal{D}(d^2) : F(\sigma_1, \sigma_2) \geq f\}$, is convex. In other words, given two pairs of states, $\{\sigma_1^{(A)}, \sigma_2^{(A)}\}$ and $\{\sigma_1^{(B)}, \sigma_2^{(B)}\}$, both of which lie in $\{\mathcal{D}(d^2) : F(\sigma_1, \sigma_2) \geq f\}$, any convex combination of the pairs also lies in $\{\mathcal{D}(d^2) : F(\sigma_1, \sigma_2) \geq f\}$. We can write

$$F(p\sigma_1^{(A)} + (1-p)\sigma_1^{(B)}, p\sigma_2^{(A)} + (1-p)\sigma_2^{(B)}) \geq f, \quad (\text{B11})$$

where we have used the concavity of the Bures fidelity. The set of pairs of pure states with fidelity greater than or equal to f forms part of the boundary of this convex set and is therefore compact. Therefore, by taking a finite number of samples from the set of pairs of pure states with fidelity greater than or equal to f , bounding the Bures distance between the samples, and using Eq. (B10) to lower bound the relative fidelity of any point between the samples, we can numerically lower bound the relative fidelity for any non-zero f (for $f = 0$, this method will not give a non-trivial bound on $F_{R,\min}(0)$; instead $F_{R,\min}(0)$ should be found analytically, as the limit of a sequence). By increasing the number of samples, we can tighten this lower bound, which will tend towards the true value of $F_{R,\min}(f)$ asymptotically with the number of samples. This concept is explored in more detail in Appendix C, where we present an algorithm for a numerical method to calculate $F_{R,\min}(f)$.

We now bound the behaviour of $F_{R,\min}(f)$ as a function of f . First, let us note that any pair of states, $\{\sigma_1, \sigma_2\}$, with a fidelity of f has a distance from some different pair of states, $\{\sigma'_1, \sigma'_2\}$, with a fidelity greater than or equal to f' that is upper bounded by

$$\delta(\sigma_1, \sigma'_1, \sigma_2, \sigma'_2) \leq \left| \sqrt{1-f} - \sqrt{1-f'} \right|. \quad (\text{B12})$$

Let us suppose that $f < f'$ and stipulate that $\{\sigma_1, \sigma_2\}$ are states that achieve a relative fidelity of $F_{R,\min}(f)$. Combining Eqs. (B9) and (B12), we have

$$F_{\text{out}}(\sigma_1, \sigma_2) \geq 1 - \left(\sqrt{1 - F_{\text{out}}(\sigma'_1, \sigma'_2)} + (\sqrt{1-f} - \sqrt{1-f'}) \right)^2. \quad (\text{B13})$$

Then, noting that

$$F_{\text{out}}(\sigma'_1, \sigma'_2) \geq f' F_{R,\min}(f'), \quad (\text{B14})$$

we can write the continuity bound in Eq. (14), which we repeat here for clarity:

$$F_{R,\min}(f) \geq \frac{1}{f} \left(1 - \left(\sqrt{1 - f' F_{R,\min}(f')} + (\sqrt{1-f} - \sqrt{1-f'}) \right)^2 \right).$$

Appendix C: Algorithm for numerically finding the minimum relative fidelity

Due to the continuity relation given in Eq. (B10), we can numerically bound the minimum relative fidelity (for a given, finite input fidelity, f) from below by sampling sufficiently many points. One way of doing this would be to parametrise the set of pairs of pure states with an input fidelity greater than or equal to f and then to evenly sample the parameters, calculating both the relative fidelity for that set of parameters and the maximum distance between any pair of pure states (with input fidelity greater than or equal to f) and the nearest sampled state. This would, however, be very inefficient. Instead, we present an algorithmic method that converges to the true value of $F_{R,\min}(f)$ as the number of samples increases.

We begin by recalling that the output fidelity is a concave function that can be minimised by a pair of states of the form given in Eqs. (A11) and (A12). This means that the minimum output fidelity is always achievable by a minimisation over all pairs of $2d^2$ -dimensional qudit states. In fact, we can constrain the final qubit of the first state to be $|0\rangle\langle 0|$, as per Eqs. (A11) and (A12), resulting in a minimisation over a d^2 -dimensional qudit state and a $2d^2$ -dimensional qudit state and reducing the dimension of the problem by $3d^4$; for simplicity, we do not

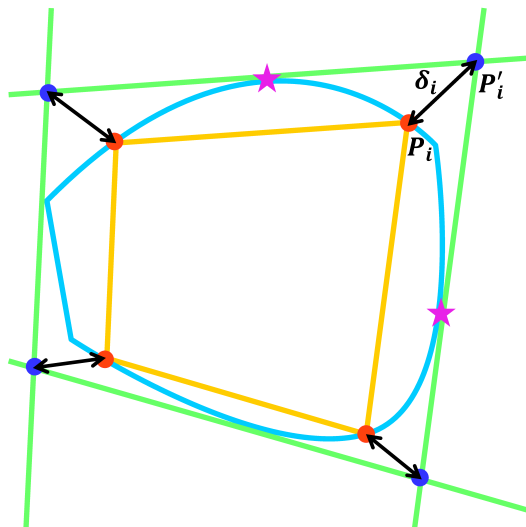


FIG. 3. Illustration of the algorithm for numerically finding the minimum relative fidelity. This diagram shows how the process would work in two dimensions (although, even in the qubit channel case, the actual dimension of the set we are minimising over is much more than two). The pale blue outline is the surface of the set, \mathcal{S} , of valid points (pairs of valid density matrices with a pairwise fidelity greater than f). We make no assumptions about the geometry of this set, other than that it is convex (meaning that the method works even if the surface of the set is not smooth, as in this diagram). The red dots on the surface of \mathcal{S} are points, $\{P_i\}$, for which we know the value of the output fidelity. The planes (lines) joining each known point to its neighbours define a polygon \mathcal{R} within the set \mathcal{S} , the surface of which is given by the orange lines. The green lines that are tangential to the set \mathcal{S} are also parallel to the orange lines and intersect each other at the points $\{P'_i\}$, which are represented by dark blue dots. Finally, we define the polygon \mathcal{Q} as the region enclosed by the green lines. \mathcal{Q} contains \mathcal{S} , meaning there are no points in \mathcal{S} that are not also in \mathcal{Q} . Consequently, no point in \mathcal{S} is more distant from \mathcal{R} (in terms of the trace norm between the density matrices comprising the two points) than the most distant point in \mathcal{Q} . In fact, the points $\{P'_i\}$ are the most distant points from \mathcal{R} in \mathcal{Q} . For each known point P_i , we must deduct an error cost, based on the distance δ_i , from the value of the output fidelity at that point, in order to lower bound the minimum output fidelity. We can now detail the update rule. Suppose the point labelled P_i on the diagram is the one that gives rise to the lower bound on the minimum output fidelity. Then, the candidate points (represented by the pink stars) are points at which the green lines that pass through point P'_i touch \mathcal{S} . Whichever of these is most distant from P_i is picked and added to the set of known points. We recalculate the points $\{P'_i\}$ and the distances $\{\delta_i\}$ accordingly.

do this here, but the algorithm can be trivially changed accordingly. Expanding Eq. (B9) in terms of δ , we get

$$|F_{\text{out}}(\sigma'_1, \sigma'_2) - F_{\text{out}}(\sigma_1, \sigma_2)| \leq 2\delta - \delta^2. \quad (\text{C1})$$

Note that this continuity relation, whilst not tight, is written purely in terms of δ .

We will refer to a pair of Hermitian, $2d^2$ -dimensional,

unit trace, square matrices as a point. Consider the set of pairs of pure states with an input fidelity equal to f . Let us call the convex hull of this set (the set of all convex combinations of such points) \mathcal{S} . \mathcal{S} is the convex set of points that we must minimise the output fidelity (a concave function) over. The dimension of \mathcal{S} is $D = 2(4d^4 - 1)$, in that this is the minimum number of real coordinates required to map each pair of density matrices to a unique displacement from the origin in coordinate space. If A is the matrix pair $\{\sigma_1, \sigma_2\}$, we define

$$F_{\text{out}}(A) = F_{\text{out}}(\sigma_1, \sigma_2). \quad (\text{C2})$$

We can define a hyperplane in \mathcal{S} as the set of points, P , that can be written as

$$P = P_0 + \sum_{i=1}^{D-1} p_i V_i, \quad (\text{C3})$$

where P_0 is a point and the V_i are pairs of Hermitian, $2d^2$ -dimensional, trace zero, square matrices, which play the role of vectors. Summation, in this context, means adding the first matrices of each pair to each other and then doing the same for the second matrices of each pair. The set $\{V_i\}$ therefore defines the hyperplane. The plane can equivalently be defined as the set of all points for which $\text{Tr}[NP]$ has a constant value, where N is a pair of Hermitian, $2d^2$ -dimensional, trace zero, square matrices, which we call the normal to the hyperplane. We can find N from $\{V_i\}$ by solving the $D - 1$ simultaneous equations

$$\text{Tr}[NV_i] = 0 \quad \forall i \in (1, D - 1). \quad (\text{C4})$$

Any point on the boundary of \mathcal{S} , P_{bound} , can be written as

$$P_{\text{bound}} = \arg \max_{P \in \mathcal{S}} \text{Tr}[NP], \quad (\text{C5})$$

for some choice of N (this can be seen from the convexity of \mathcal{S}). The plane defined by this N is a tangent to the set \mathcal{S} .

Now note that a set of points, $\mathcal{P} = \{P_i\}$, define a polygon, the interior of which is comprised of all points expressible as

$$P = \sum_i p_i P_i, \quad (\text{C6})$$

where the $\{p_i\}$ define a convex combination (i.e. they are all non-negative and sum to 1) and the index i ranges over the labels of all of the points in \mathcal{P} . The surface of this polygon is made up of hyperplanes and, so long as all of the points are linearly independent (no point can be written as a convex combination of the others), each point on a given face can be written as a convex combination of only $D - 1$ of the points in \mathcal{P} . Within a polygon, a concave function is minimised at one of the vertices, so the minimum value of F_{out} over the polygon will be equal to

$$\min_{P \in \mathcal{P}} F_{\text{out}}(P).$$

We can now briefly outline an algorithm to numerically find $F_{\text{out},\min}(f)$ (and hence $F_{R,\min}(f)$). The basic idea is to find the minimum output fidelity for a finite number of points on the boundary of \mathcal{S} . By doing so, we find the minimum over a (polygonal) subset of \mathcal{S} , which we call \mathcal{R} . We then upper bound the distance (δ in Eq. (C1)) between any point in \mathcal{S} and the nearest point in \mathcal{R} . We do this by finding a different convex, polygonal set, \mathcal{Q} , which surrounds \mathcal{S} (in the sense that every point in \mathcal{S} is in \mathcal{Q}) and finding the distance from each vertex of \mathcal{R} to the corresponding vertex of \mathcal{Q} . Finally, we use Eq. (C1) to lower bound the minimum output fidelity for any point in \mathcal{S} . Since \mathcal{S} includes all pairs of pure states of dimension $2d^2$, we can therefore also lower bound the minimum relative fidelity by dividing the minimum output fidelity by f . Fig. 3 provides a visualisation of how the algorithm works (albeit for only two dimensions).

1. Pick $D + 1$ initial points (pairs of states) from the set of pairs of states with an input fidelity equal to f . The initial set of points (which we will call $\mathcal{P} = \{P_i\}$) can be chosen in any way, so long as the points are linearly independent. All of the points in \mathcal{P} will lie on the boundary of \mathcal{S} . \mathcal{P} defines the polygon \mathcal{R} (each point is a vertex), the interior of which is comprised of all convex combinations of the chosen points.
2. Calculate the output fidelity for each point in \mathcal{P} . Write $F_{\text{out},i} = F_{\text{out}}(P_i)$. From the concavity of the output fidelity, the minimum output fidelity for any point in \mathcal{R} is given by $\min_i F_{\text{out},i}$.
3. Let \mathcal{F} be the set of faces of \mathcal{P} (that is, the set of hyperplanes that make up the surface of \mathcal{R}). Initially, there will be $D + 1$ faces (the same as the number of points in \mathcal{P}). Each F_i can be expressed as in Eq. (C3) by setting $P_0 = P_j$ and the V “vectors” to $P_j - P_k$, where the j and k refer to some arbitrary labelling of the points that form the vertices of F_i . For each face, F_i , in \mathcal{F} , find the normal to it, N_i . Some degree of care is required in choosing the sign of N_i so that it points outwards from the interior of \mathcal{R} . Then, for each N_i , find

$$\mu_i = \max_{P \in \mathcal{S}} \text{Tr}[N_i P], \quad (\text{C7})$$

$$P_{\text{max},i} = \arg \max_{P \in \mathcal{S}} \text{Tr}[N_i P] \quad (\text{C8})$$

This is a maximisation of a linear function over a convex set and can be done by, for instance, semidefinite programming.

4. We now define a new set of hyperplanes, \mathcal{F}' , comprised of hyperplanes, F'_i , that are parallel to the faces F_i but that pass through the points $P_{\text{max},i}$. These hyperplanes are therefore tangential to the set \mathcal{S} . We can then define the set, \mathcal{P}' , of points at which each hyperplane intersects with each of its $D - 1$ nearest neighbours. These points then define

a new polygon, \mathcal{Q} , that surrounds the set \mathcal{S} (since each of its faces are tangential to the set). No point in \mathcal{S} can be more distant from a point in \mathcal{F} than the vertices of \mathcal{Q} (i.e. the points in \mathcal{P}'). Thus, by finding the distance between each point P_i and the corresponding point P'_i , we can upper bound δ at each point. One complication is that the Bures distance is not a valid distance metric outside of \mathcal{S} . However, we can get around this by using the trace norm and then bounding the Bures distance using the Fuchs van der Graaf relations.

5. For each pair $\{A, B\}$ in \mathcal{P} and corresponding pair $\{A', B'\}$ in \mathcal{P}' , calculate

$$\delta_i = \sqrt{\|A - A'\|} + \sqrt{\|B - B'\|}. \quad (\text{C9})$$

Next, calculate the cost function,

$$c_i = 2\delta_i - \delta_i^2. \quad (\text{C10})$$

Finally, for each point in \mathcal{P} , deduct c_i from $F_{\text{out},i}$. The smallest value of $c_i - F_{\text{out},i}$ gives a lower bound on $F_{\text{out},\min}(f)$, whilst the smallest value of $F_{\text{out},i}$ gives an upper bound on $F_{\text{out},\min}(f)$.

6. We now detail the update rule for finding new bounds. The point giving rise to the current lower bound on $F_{\text{out},\min}(f)$, which we will label P_1 for ease, touches D faces. For each of these faces, we know the point at which a hyperplane parallel to them is tangential to \mathcal{S} . Pick the one that is furthest from P_1 and add it to \mathcal{P} . This will add a face to \mathcal{F} and change $D - 1$ of the existing faces. Repeat steps 3 to 5 for each of the faces affected.
7. Repeat step 6 until the desired level of precision is achieved. Divide by the input fidelity, f , to obtain bounds on $F_{R,\min}(f)$.

Since each iteration can only improve the lower bound on $F_{\text{out},\min}(f)$ (or leave it unchanged) and \mathcal{R} will model \mathcal{S} increasingly well as the number of points increases, it is evident that the bound obtained using this algorithm will converge to the true value asymptotically in the number of iterations. We have no guarantee, however, about the rate of convergence.

Appendix D: Minimum relative fidelity for specific channel pair examples

First, we consider discrimination between the identity channel and a Pauli channel. Specifically, we consider the channel that applies the Pauli-X operator with probability p and applies the identity operator with probability $1 - p$.

Consider the input states $|\phi_1\rangle$ and $|\phi_2\rangle$. The output fidelity is

$$F_{\text{out}}^X = \sqrt{(1-p)|\langle\phi_1|\phi_2\rangle|^2 + p|\langle\phi_1|X|\phi_2\rangle|^2}, \quad (\text{D1})$$

where X is the Pauli-X operator. Since the input fidelity is $|\langle\phi_1|\phi_2\rangle|$, we can write

$$F_R^X = \sqrt{(1-p) + p \frac{|\langle\phi_1|X|\phi_2\rangle|^2}{|\langle\phi_1|\phi_2\rangle|^2}}. \quad (\text{D2})$$

Setting both $|\phi_1\rangle$ and $|\phi_2\rangle$ to $|0\rangle$ gives $|\langle\phi_1|X|\phi_2\rangle| = 0$ and so the minimum relative fidelity is given by

$$F_{R,\min}^X = \sqrt{1-p}, \quad (\text{D3})$$

which is always independent of the input fidelity. This tells us that discrimination protocols for this pair of Pauli channels cannot benefit from adaptivity. This is as expected because all Pauli channels are jointly teleportation covariant [33].

Next, we apply our method to discrimination between unitary channels. In this case, we expect that the minimum relative fidelity will go to zero for some non-zero input fidelity, because unitary channels are perfectly discriminable after finite channel uses [28, 29]. We consider unitaries of the form

$$U(\theta) = X^\theta, \quad (\text{D4})$$

where θ is a real parameter. Since we are considering a pair of unitaries, $U(\theta_1)$ and $U(\theta_2)$, the quantity of interest is $\theta_1 - \theta_2$, since the relative fidelity for a pair of channels is unchanged by a unitary applied before the channels (so any pair $\{\theta_1, \theta_2\}$ with the same value of $\theta_1 - \theta_2$ will have the same minimum relative fidelity for any given input fidelity). Thus, for simplicity, we consider unitaries $U(0)$ (the identity channel) and $U(\theta)$.

The eigenvalues of a unitary, U , all have magnitude 1, and so can be expressed as $\exp(i\phi_j^U)$, where j is a label for the eigenvalue that runs from 1 to the dimension of the unitary, d . Suppose the eigenvalues are ordered such that $\phi_j^U \leq \phi_{j+1}^U$ (with all of the ϕ_j confined to the range $[0, 2\pi)$). For ease, define $\phi_{\min}^U = \phi_1^U$ and $\phi_{\max}^U = \phi_d^U$. From Ref. [27], we know that F_{con} for a unitary and the identity channel is given by

$$F_{\text{con}}^U = \cos\left(\frac{\phi_{\max}^U - \phi_{\min}^U}{2}\right), \quad (\text{D5})$$

so long as $F_{\text{con}}^U \geq 0$.

Note that the minimum relative fidelity can be expressed as

$$F_{R,\min}^U = \frac{1}{f} \min_{|\psi\rangle} \min_u |\langle\psi|u(\mathcal{I} \otimes U)|\psi\rangle|, \quad (\text{D6})$$

where the identity acts on any idler modes and u is a different unitary that obeys the constraint

$$|\langle\psi|u|\psi\rangle| \geq f. \quad (\text{D7})$$

This is because we can define the input state for the unitary as $|\psi\rangle$ and then express the input state for the identity channel as $u^\dagger|\psi\rangle$. Using Eq. (D5), we can write

$$\cos\left(\frac{\phi_{\max}^u - \phi_{\min}^u}{2}\right) \geq f. \quad (\text{D8})$$

The eigenvalues of the product of two unitaries, $u(\mathcal{I} \otimes U)$, are constrained by [34, 35]

$$\phi_{\min}^{u(\mathcal{I} \otimes U)} \geq \phi_{\min}^u + \phi_{\min}^U, \quad \phi_{\max}^{u(\mathcal{I} \otimes U)} \leq \phi_{\max}^u + \phi_{\max}^U. \quad (\text{D9})$$

For $U(\theta)$, as defined in Eq. (D4), we have

$$\phi_{\min}^U = 0, \quad \phi_{\max}^U = \pi\theta. \quad (\text{D10})$$

Hence, we can write

$$F_{R,\min}^U \geq \frac{1}{f} \cos\left(\frac{\pi\theta}{2} + \arccos(f)\right), \quad (\text{D11})$$

where $\arccos(f)$ takes its principle value. If we let u also be of the form in Eq. (D4), the inequalities in Eq. (D9) become equalities, so we can write

$$F_{R,\min}^U = \frac{1}{f} \cos\left(\frac{\pi\theta}{2} + \arccos(f)\right), \quad (\text{D12})$$

where we assume the right-hand side is non-negative (if not, we set it to 0). This corresponds to an output fidelity bound, after N channel uses, of

$$F_N^U \geq \cos\left(\frac{N\pi\theta}{2}\right) \quad (\text{D13})$$

for $N < \theta^{-1}$ (and zero for $N > \theta^{-1}$).

Finally, we consider a pair of entanglement-breaking, qubit channels, $\mathcal{C}_1^{\text{EB}}$ and $\mathcal{C}_2^{\text{EB}}$, defined by a measurement along one of a pair of axes followed by a rotation. Specifically, $\mathcal{C}_i^{\text{EB}}$ consists of the positive operator-valued measurement

$$M(\theta_i) = \{|\phi(\theta_i)\rangle\langle\phi(\theta_i)|, \mathcal{I} - |\phi(\theta_i)\rangle\langle\phi(\theta_i)|\}, \\ |\phi(\theta)\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle,$$

followed by the rotation given by

$$R(\theta_i) = \begin{pmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & \cos(\theta_i) \end{pmatrix}.$$

Once again, it is only the difference in rotation angle, $\Delta\theta = \theta_1 - \theta_2$, that matters, since we will get the same minimum relative fidelity for any value of θ_1 (so long as $\Delta\theta$ is fixed). We can therefore set $\theta_1 = 0$ without loss of generality.

This pair of channels is of interest because we expect adaptivity to be of benefit (this can be numerically confirmed; see the supplementary Mathematica files), but we also expect that the minimum relative fidelity will never go to zero (except in the case where $|\Delta\theta| = \frac{n}{4}\pi$, for integer n), because no non-orthogonal input states will result in orthogonal output states. This is in line with the fact that discrimination strategies between classical channels can benefit from adaptivity but adaptivity cannot improve the asymptotic rate of decay of the discrimination

error probability [31]. It is also known that if two classical channels are not perfectly discriminable after a single channel use, they will not be perfectly discriminable after any finite number of channel uses [24].

Since the channels are entanglement-breaking, we only need to consider single qubit inputs (without idlers), because idlers cannot decrease the relative fidelity in this case. This reduces the difficulty of the minimisation.

The minimum relative fidelity if the same state is used for both channels is given by

$$F_{\text{con}}^{\text{EB}} = \frac{1}{2} \sqrt{2 + \cos(2\Delta\theta) + \cos(6\Delta\theta)}. \quad (\text{D14})$$

As expected, $F_{\text{con}}^{\text{EB}} \rightarrow 1$ as $\Delta\theta \rightarrow 0$. Minimising the relative fidelity over all pairs of single qubit states (with no constraint on the input fidelity), we find, under the numerically verifiable assumption that we can set one of the input states to $|0\rangle$, that the minimum relative fidelity is given by

$$F_{R,\min}^{\text{EB}}(0) = \frac{|\cos(\Delta\theta) + \cos(3\Delta\theta)|}{\sqrt{2 + 2\cos(2\Delta\theta) + \cos(4\Delta\theta)}}. \quad (\text{D15})$$

This is less than $F_{\text{con}}^{\text{EB}}$ (except for at certain values of $\Delta\theta$), meaning that discrimination protocols can benefit from adaptivity, but only goes to zero for six different values of $\Delta\theta$ (in the range $0 \leq \Delta\theta < 2\pi$). It is interesting to note that $F_{R,\min}^{\text{EB}}(0) \not\rightarrow 1$ as $\Delta\theta \rightarrow 0$. In fact, $F_{R,\min}^{\text{EB}}(0) \rightarrow \frac{2}{\sqrt{5}}$. This may initially seem strange, as the channels are identical for $\Delta\theta = 0$, however, as the separation between the channels, $\Delta\theta$, becomes infinitesimal, the input fidelity required in order to achieve the minimum relative fidelity becomes infinitesimal too. The input fidelity between the optimal states achieving the minimum relative fidelity is given by

$$f_{\text{opt}}^{\text{EB}} = \frac{1 - \cos(2\Delta\theta)\cos(4\Delta\theta)}{|\sin(\Delta\theta)|\sqrt{4 + 2\cos(2\Delta\theta) - 2\cos(6\Delta\theta)}}, \quad (\text{D16})$$

which approaches zero as $\Delta\theta \rightarrow 0$. Thus, as $\Delta\theta$ becomes small, the number of previous channel uses required in order to have sufficiently separated input states to achieve the minimum possible relative fidelity, $F_{R,\min}^{\text{EB}}(0)$, with the next channel use becomes large. Another feature of interest is that $F_{R,\min}^{\text{EB}}(0) = F_{\text{con}}^{\text{EB}} \neq 0$ for $\Delta\theta = \frac{(1+2x)\pi}{8}$, where x is an integer. This shows that adaptivity does not have any benefit for these parameter values. We can bound the output fidelity of any adaptive protocol as

$$F_N^{\text{EB}} \geq F_{R,\min}^{\text{EB}}(0)^N. \quad (\text{D17})$$

To generate Fig. 2, Eqs. (D3), (D12), and (D15) were used, with the channel parameters (p , θ , and $\Delta\theta$) chosen such that $F_{\text{con}} = 0.95$ for all three.

See the supplementary Mathematica files for further details.