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Jordan permutation groups and limits of D-relations

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Abstract. We construct via Fraïssé amalgamation an ω-categorical structure whose automorphism group is an infinite oligomorphic Jordan primitive permutation group preserving a "limit of D-relations". The construction is based on a semilinear order whose elements are labelled by sets carrying a D-relation, with strong coherence conditions governing how these D-sets are inter-related.

1 Introduction

A transitive permutation group G on a set X is a *Jordan group* if there is $Y \subset X$ with $|Y| > 1$ such that the pointwise stabiliser $G_{(X \setminus Y)}$ is transitive on Y, together with a non-degeneracy condition which essentially says that this transitivity does not arise just from the degree of transitivity of G on X. It follows already from work of Jordan in 1871 that *finite* primitive Jordan groups are 2-transitive, and this led to a full classification of such permutation groups by Neumann in [24], with related work around the same time by Kantor in [19], and by Cherlin, Harrington and Lachlan in [14]. For infinite permutation groups, the supply of examples of Jordan groups is much richer – for example, $\mathrm{Aut}(\mathbb{Q}, <)$ is a primitive but not 2-transitive Jordan group (any proper non-empty open interval is a Jordan set). Other examples include the supergroups of $\mathrm{Aut}(\mathbb{Q}, <)$ in $\mathrm{Sym}(\mathbb{Q})$ – the ones which are closed in the topology of pointwise convergence on $\mathrm{Sym}(\mathbb{Q})$ were classified by Cameron in [11]. Many other examples arise as automorphism groups of the "tree-like" structures explored in [6]. In addition, there are examples, suggested by finite permutation group theory, consisting of projective and affine groups in their natural actions. Other examples of this type include automorphism groups of saturated strongly minimal sets (or more generally regular types) arising in model theory. There are also Jordan groups which are *highly transitive* (that is, k-transitive for all k), but little work has been done on these – the focus has been on Jordan groups which arise as automorphism groups of non-trivial first-order structures.

A structure theory for infinite primitive Jordan groups has emerged. In 1985, Neumann [24] classified the primitive Jordan permutation groups with cofinite

Jordan sets. Primitive Jordan groups with a proper *primitive* Jordan set (that is, the pointwise stabiliser of the complement acts *primitively* on the set) were classified by Adeleke and Neumann in [5] – here "classified" means that it was shown that any such group preserves a relational structure of one of a list of types. Finally, in [3], the following theorem was proved – see Definition 2.6 and other definitions in Section 2.

Theorem 1.1. *Let G be a primitive but not highly transitive Jordan group on an infinite set X. Then G preserves on X a structure of one of the following kinds:*

(i) *a Steiner system (possibly with infinite blocks),*

(ii) *a linear order, circular order, linear betweenness relation, or separation relation,*

(iii) *a semilinear order, general betweenness relation, C-relation, or D-relation,*

(iv) *a limit of Steiner systems, general betweenness relations, or D-relations.*

The examples of types (i) and (iv) do not have a proper primitive Jordan set so did not arise in [5], and those of type (i) include projective and affine groups, and some constructions arising from strongly minimal sets and regular types. The structures of type (ii) are essentially those classified by Cameron in [11], and those of type (iii) are described by Adeleke and Neumann in [6] – in particular, the relational structures are axiomatised and well-understood.

The examples of type (iv) are more mysterious, and are the focus of this paper. We do not give the definition of a limit of Steiner systems, but for a limit of D-relations (or of general betweenness relations), see Definition 2.6 below and the remark following it. There is an example of an infinite Jordan group preserving a limit of Steiner systems given by Adeleke in [1]. This is developed further by Johnson in [18], who gives for every $k \geq 2$ a construction of a k-transitive but not $(k+1)$-transitive example. An example of an infinite primitive Jordan group preserving a limit of betweenness relations is given by Bhattacharjee and Macpherson in [8]. The group acts on an ω-categorical structure which is built by a Fraïssé construction. Another example of an infinite Jordan permutation group preserving a limit of betweenness relations is given by Adeleke in his work [2] (work which, despite its later publication date, was done much earlier than [8], in the early 1990s, and which inspired [8] and the present work). Adeleke in [2] also gives an example of an infinite primitive Jordan permutation group preserving a limit of D-relations.

Recall that a countably infinite first-order structure is *ω-categorical* if it is determined up to isomorphism by its cardinality and its first-order theory. By the Ryll-Nardzewski theorem, this is equivalent to its automorphism group being *oligomor-*

phic, that is, having finitely many orbits on k-tuples for all k. The group preserving a limit of betweenness relations constructed by Adeleke in [2] is not oligomorphic, but the group constructed in [8] is. It is expected, but not verified, that the group preserving a limit of D-relations constructed by Adeleke [2] is not oligomorphic. Adeleke and Macpherson, in the end of their paper [3], posed the problem of explicitly classifying *oligomorphic* primitive Jordan permutation groups, and also asked whether it is possible for an infinite primitive oligomorphic Jordan permutation group to preserve a limit of betweenness relations or D-relations. With Theorem 1.2 below, together with that in [8], a positive answer has now been found in both cases. Furthermore, in the Adeleke paper [2], the Jordan group is built as a direct limit of an increasing chain of permutation groups, but no invariant relational structure is made explicit. In our construction here, the Jordan group is the automorphism group of a relational structure which can reasonably be claimed to be a "new" treelike structure, essentially distinct from those occurring in Theorem 1.1 (iii) or described in [6].

Our main theorem is the following. The overall strategy of the proof of Theorem 1.2 is analogous to that in [8], but there are significant differences.

Theorem 1.2. *There is an ω-categorical structure M whose automorphism group is a primitive Jordan group which preserves a limit of D-relations but does not preserve a structure of types* (i)*,* (ii)*, or* (iii) *of Theorem* 1.1.

Some introductory background is given in Section 2. In Section 3, we build a class of finite structures, each of which is essentially a finite lower semilinear order with vertices labelled by finite graph-theoretic unrooted trees, with coherence conditions. These are viewed as structures in a relational language with relations L, L', S, S', Q, R. We describe possible one-point extensions of such structures, prove an amalgamation theorem, and thereby obtain by Fraïssé's theorem a countably infinite ω-categorical structure M (the "Fraïssé limit"). In Section 4, we describe in detail the structure M and its automorphism group. On M, the relations L', S', Q, R (whose role is to ensure that an appropriate form of the amalgamation property holds) can be defined without parameters in terms of the ternary relation L and quaternary relation S. We show that there is in M an associated interpretable dense lower semilinear order (a meet semilattice), with vertices labelled by dense "D-sets" (in place of graph-theoretic trees), again with coherence conditions. Adapting an iterated wreath product construction described by Cameron in [12] which is based on Hall's wreath power, we show in Section 5 that $\mathrm{Aut}(M)$ is a Jordan group with a "pre-direction" as a Jordan set. Using this, we find many other Jordan sets. Finally, we prove that the Jordan group $G = \mathrm{Aut}(M)$ preserves a limit of D-relations, our main result.

We believe that our construction, and its companion in [8], opens the possibility to give a much more enlightening description of type (iv) in Theorem 1.1 by requiring that there is an invariant combinatorial structure satisfying certain explicit axioms. The constructions may also have interest for other test questions on homogeneous and ω-categorical structures, and may be open for further generalisation. This is explored briefly in Section 6.

We conclude with some remarks concerning the wider context and motivation. Structural results on Jordan groups have had a number of applications. First, Cherlin, Harrington and Lachlan in [14] used structural results on finite Jordan groups in model theory to classify ω-categorical *strictly minimal sets*, and thereby to develop a powerful structure theory for ω-categorical ω-stable structures – this paper was fundamental to the development of geometric stability theory in model theory. Neumann [24] used essentially the same result to describe primitive permutation groups on a countably infinite set which have no countable orbits on the set of infinite co-infinite subsets. The paper [4] uses results on primitive Jordan groups with primitive proper Jordan sets to obtain structural results on primitive groups on an uncountable set which contain a non-identity element of "small" support. This is analogous to the result of Wielandt that an infinite primitive permutation group with a non-identity element of finite support contains the finite alternating group, and Macpherson and Praeger in [21] used the full structure theory for primitive Jordan groups to show that a primitive permutation group realising a certain cycle type (a single infinite cycle, finitely many and at least one non-trivial finite cycles, and infinitely many fixed points) must be highly transitive. Several authors have used Jordan groups to show that certain automorphism groups are "maximal-closed" in the symmetric group: Kaplan and Simon [20] showed that $\mathrm{AGL}_n(\mathbb{Q})$ (for $n \geq 2$) and $\mathrm{PGL}_n(\mathbb{Q})$ (for $n \geq 3$) are maximal closed; Bradley-Williams in [10] described the closed supergroups of the automorphism group of certain semilinear orders, and Bodirsky and Macpherson [9] exhibited an uncountable *non-oligomorphic* maximal-closed permutation group acting on a countable set.

Semilinear orders, C-relations, general betweenness relations, and D-relations can naturally be viewed as "treelike". The classification in [5] of infinite primitive Jordan groups with primitive proper Jordan sets suggests that these are the only treelike structures. However, we would argue that the highly symmetric structure M constructed in Theorem 1.2 (and its cousin in [8]) involves all the above structures, but its automorphism group does not preserve any of the above structures, and thus it can claim to be a new treelike structure. This makes it potentially interesting in other ways – see for example Problem 6.7 below.

The methods in the paper are mainly combinatorial and permutation group-theoretic. We assume familiarity with some basic concepts from model theory such as relational structures, amalgamation and Fraïssé limits, and ω-categoricity, but give some explanation – see e.g. Theorem 2.1, the start of Section 3.2, and Theorem 3.17.

2 Definitions

Throughout the paper, we shall denote by (G, X) a permutation group G acting on a set X, and we say X is a *G-space*. We write x^g for the image of the element $x \in X$ under $g \in G$. For $Y \subset X$, the *setwise stabiliser* of Y in G is denoted by $G_{\{Y\}}$, and the *pointwise stabiliser* of Y in G is $G_{(Y)}$. The stabiliser of a point $x \in X$ is denoted by G_x. For a natural number k, a G-space X is said to be *k-transitive* if G is transitive on the set of ordered k-subsets of X; it is *k-primitive* if it is k-transitive, and in addition, for any distinct $x_1, \ldots, x_{k-1} \in X$, the stabiliser $G_{x_1,\ldots,x_{k-1}}$ acts primitively on $X \setminus \{x_1, \ldots, x_{k-1}\}$. If G is transitive on the set of *unordered* k-subsets of X, then it is called *k-homogeneous*. If G is k-transitive (respectively, k-homogeneous) on X for every $k \in \mathbb{N}$, then G is said to be *highly transitive* (respectively, *highly homogeneous*).

A group G acting on a set X is said to be *oligomorphic* if G has finitely many orbits on X^k, the set of all k-tuples of X, for every natural number k. For more about oligomorphic groups, see [13]. A structure M is *ω-categorical* if M is countably infinite and any countable structure N which satisfies the same first-order theory as M is isomorphic to M. The connection between these two notions lies in the following theorem.

Theorem 2.1 (Ryll-Nardzewski 1959, Engeler 1959, Svenonius 1959). *Let M be a countably infinite first-order structure. Then M is ω-categorical if and only if* $\mathrm{Aut}(M)$ *is oligomorphic on M.*

Definition 2.2. Let $Y \cup Z$ form a partition of a transitive G-space X with $|Z| > 1$. If the pointwise stabiliser $G_{(Y)}$ of Y in G is transitive on Z, then Z is called a *Jordan set* for (G, X) and Y is called a *Jordan complement*. The Jordan set Z is *improper* if, for some $k \in \mathbb{N}$, (G, X) is $(k+1)$-transitive and $|Y| = k$ – so in the improper case, the Jordan property is just a consequence of the degree of transitivity; it is *proper* otherwise. We say that Z is a *primitive Jordan set* if $G_{(Y)}$ is primitive on Z, and an *imprimitive* Jordan set otherwise. A *Jordan group* is a transitive permutation group with a proper Jordan set.

The following definition is taken from [5]. The subsequent lemma is heavily used in the classification results in [3,5] since many arguments apply properties of the family of all Jordan sets, or of an orbit on Jordan sets.

Definition 2.3. We fix the following terminology.

(a) A *typical pair* is a pair of subsets Y_1, Y_2 of X such that $Y_1 \not\subseteq Y_2$, $Y_2 \not\subseteq Y_1$, and $Y_1 \cap Y_2 \neq \emptyset$.

(b) A family of sets $\{Y_i : i \in I\}$ will be said to be *connected* if, for any $i, i' \in I$, there exists $j_0, \ldots, j_l \in I$ such that $j_0 = i$, $j_l = i'$, and $Y_{j_{r-1}} \cap Y_{j_r} \neq \emptyset$ for all $1 \leq r \leq l$.

Lemma 2.4. *The following statements hold.*

(i) [5, Lemma 3.2] *Suppose that* (G, X) *is a transitive* G*-space and* $\{Z_i : i \in I\}$ *is a connected system of Jordan sets. Then* $\bigcup_{i \in I} Z_i$ *is a Jordan set for* (G, X).

(ii) [5, Lemma 3.1] *The union of any typical pair of Jordan sets is a Jordan set.*

We now introduce, very briefly, some of the relational structures that arise in this paper. For further details, see [6].

First, recall the following, adopting the conventions of [3]. Let $n \in \mathbb{N}$ with $n > 1$. Then an n*-Steiner system* on X is a family $\mathcal{B}$ of subsets of X called *blocks*, all of the same size (possibly infinite), such that any n distinct elements of X lie in a unique block. We shall assume n-Steiner systems to be *non-trivial* in the sense that there is more than one block, and blocks have size greater than n. Jordan groups arising from projective and affine groups in their natural actions preserve Steiner systems.

Recall also that a *separation relation* (see Cameron [11]) is the natural arity 4 relation induced on a circularly ordered set indicating that two elements lie in distinct segments with respect to two other elements.

Let $(X, \leq)$ be a partially ordered set. Then X is said to be a *(lower) semilinearly ordered* set if, for any a in X, the set $\{x \in X : x \leq a\}$ is totally ordered by $\leq$, any two elements have a common lower bound, but the set X itself is not totally ordered. Given a lower semilinear order $(X, \leq)$, let $p \in X$ and put $Y_p := \{x \in X : x > p\}$. Define an equivalence relation E_p on Y_p, putting

$$xE_p y \iff \exists z(p < z \leq x \wedge p < z \leq y).$$

Then E_p is preserved by $(\mathrm{Aut}(X, \leq))_p$. The equivalence classes of the equivalence relation E_p at the point p are called the *cones* at p.

From now on, by *semilinear order*, we always mean a *lower* semilinear order.

Definition 2.5. A quaternary relation $D(x, y; z, w)$ on X is a *D-relation* if, for all $x, y, z, w \in X$, (D1)–(D4) hold.

(D1) $D(x, y; z, w) \Rightarrow D(y, x; z, w) \wedge D(x, y; w, z) \wedge D(z, w; x, y)$.

(D2) $D(x, y; z, w) \Rightarrow \neg D(x, z; y, w)$.

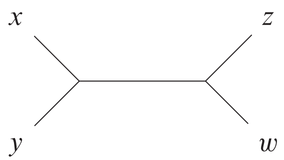

Figure 1. $D(x, y; z, w)$

(D3) $D(x, y; z, w) \Rightarrow (\forall a \in X)(D(a, y; z, w) \vee D(x, y; z, a))$.

(D4) $(x \neq z \wedge y \neq z) \Rightarrow D(x, y; z, z)$.

We say it is a *proper* D-set if in addition (D5) holds.

(D5) $(x, y, z \text{ distinct}) \Rightarrow (\exists t)(t \neq z \wedge D(x, y; z, t))$.

The D-set is said to be *dense* if

(D6) we have

$$D(x, y; z, w) \implies (\exists a \in X)(D(a, y; z, w) \wedge D(x, a; z, w) \\ \wedge D(x, y; a, w) \wedge D(x, y; z, a)).$$

Where D is clear from the context, we often refer to the D-set X.

We remark that if T is a finite graph-theoretic tree, then there is a D-relation on the set of leaves of T: put $D(x, y; z, w)$ if $x = y \notin \{z, w\}$ or $z = w \notin \{x, y\}$ or x, y, z, w are distinct and the path from x to y is disjoint from the path from z to w. The correspondence between finite D-sets and such trees follows from [12, Proposition 3.1 and Section 9], noting that the author uses different notation for the relation D. If T is an infinite tree, then there is a similar definition of a D-relation on the set of *ends* of T. The intuition is that the relation $D(x, y; z, w)$ captures the relation among leaves/ends (or "directions") depicted in Figure 1.

There are further treelike structures that we mention without detail, as they play a more peripheral role here; for example, a *general betweenness relation* is, informally, a ternary relation $B(x; y, z)$ on a set X which expresses that x lies on the unique path between y and z (we usually omit the word "general"). If $(X, \leq)$ is a lower semilinear order, then one can define a general betweenness relation B on X, putting $B(x; y, z)$ for any $x, y, z \in X$ if one of the following holds:

(i) $y \geq x \wedge \neg(z \geq x)$;

(ii) $z \geq x \wedge \neg(y \geq x)$;

(iii) $x = \text{glb}\{y, z\}$, where glb denotes the *greatest lower bound* (if it exists).

If T is an unrooted graph-theoretic tree, then there is a natural general betweenness relation on its vertices – $B(x; y, z)$ holds if and only if x lies on the yz-geodesic. It is easy to imagine an analogous relational structure with a notion of betweenness where edges are replaced, for example, by the real interval $[0, 1]$, and indeed, an $\mathbb{R}$-tree carries a natural general betweenness relation defined via geodesics as above – but a set with a general betweenness relation does not in general have any automorphism-invariant metric.

A *C-relation* is a ternary relation which can be viewed as describing the behaviour of the maximal chains of a semilinear order $(X, \leq)$: if x, y, z are maximal chains of X, then $C(x; y, z)$ holds if $x \cap y = x \cap z \subset y \cap z$. Much more detail, including axioms, can be found in [6], and there is an overview also in [3]. Note that if (Y, C) satisfies the axioms of a C-relation, then there is a lower semilinear order $(X, \leq)$ such that Y can be identified with a "dense" set of maximal chains of X with C interpreted as above – here the density means that every $a \in X$ lies in some maximal chain of Y.

If B is a general betweenness relation on X, then there is a concept of *direction* of (X, B), analogous to an *end* of a graph, and corresponding D-relation on the set of directions (see [6, Section 16 and Theorem 23.2]). Conversely, if D is a D-relation on X, then it is possible to interpret a general betweenness relation in the structure (X, D) – see [6, Theorem 25.3]. If (X, C) is a C-relation, then there is a natural D-relation on the set of elements of X: for example,

$$(\forall x, y, z, w \in X)(D(x, y; z, w) \iff (C(x; z, w) \wedge C(y; z, w)) \\ \vee (C(z; x, y) \wedge C(w; x, y))).$$

See [6, Theorems 23.4 and 23.5].

We now introduce the definition of a *limit of D-relations*, to give meaning to Theorem 1.1 (iv) and Theorem 1.2. We have reversed the ordering on J compared to presentations given previously – this seems to fit more naturally with our construction.

Definition 2.6 ([3, Definition 2.1.9]). If (G, X) is an infinite Jordan group, we say that G preserves a *limit of D-relations* if there are a linearly ordered set $(J, \leq)$ with no least element, a chain $(Y_i : i \in J)$ of subsets of X and chain $(H_i : i \in J)$ of subgroups of G with $Y_i \supset Y_j$ and $H_i > H_j$ whenever $i < j$ such that the following hold:

(i) for each i, $H_i = G_{(X \setminus Y_i)}$, and H_i is transitive on Y_i and has a unique non-trivial maximal congruence σ_i on Y_i;

(ii) for each i, $(H_i, Y_i/\sigma_i)$ is a 2-transitive but not 3-transitive Jordan group preserving a D-relation;

(iii) $\bigcup(Y_i : i \in J) = X$;

(iv) $(\bigcup(H_i : i \in J), X)$ is a 2-primitive but not 3-transitive Jordan group;

(v) $\sigma_j \supseteq \sigma_i|_{Y_j}$ if $i < j$;

(vi) $\bigcap(\sigma_i : i \in J)$ is equality in X;

(vii) $(\forall g \in G)(\exists i_0 \in J)(\forall i < i_0)(\exists j \in J)(Y_i^g = Y_j \wedge g^{-1}H_i g = H_j)$;

(viii) for any $x \in X$, G_x preserves a C-relation on $X \setminus \{x\}$.

The notion of preserving a limit of general betweenness relations is essentially the same, but with a general betweenness relation replacing the D-relation in (ii). Note that we do not define limits of Steiner systems since the concept is not used here.

3 Trees of D-sets

In this section, we construct the ω-categorical structure M whose automorphism group is shown in Sections 4 and 5 to preserve a limit of D-relations. The structure M is a Fraïssé limit of a class of finite structures ("trees of D-sets"). These, informally, may be viewed as rooted lower semilinear orders with each vertex labelled by a finite D-relation (so essentially by a finite graph-theoretic tree) with additional coherence conditions. We first introduce the key concept of a finite *tree of D-sets*.

Notation. Let $(T, \leq)$ be a finite lower semilinear order with a root ρ. Label each vertex ν of T by a finite D-set $D(\nu)$ with a D-relation D_ν defined on $D(\nu)$. We view $D(\nu)$ as the set of leaves of a finite unrooted tree $\overline{D(\nu)}$ (in the graph-theoretic sense) without dyadic vertices (vertices of degree two), and with D_ν defined in the natural way on $D(\nu)$ described in Section 2. We refer to vertices of $\overline{D(\nu)}$ as *nodes*, and those of degree at least three are called *ramification points*, with the set of these denoted by $\mathrm{Ram}(\overline{D(\nu)})$. If the ramification point r lies on the geodesics between any two of the distinct nodes x, y, z, we write $r = \mathrm{ram}(x, y, z)$, and any three distinct leaves of $\overline{D(\nu)}$ determine a unique ramification point such that the xy-path, the xz-path, and the yz-path all pass through $\mathrm{ram}(x, y, z)$. By a *successor* of a vertex $\nu \in T$, we mean a vertex $\mu \in T$ such that $\nu < \mu \wedge \neg\exists\lambda(\nu < \lambda < \mu)$; we write $\mathrm{succ}(\nu)$ for the set of successors of ν. For each $r \in \mathrm{Ram}(\overline{D(\nu)})$, there is an equivalence relation E_r on $D(\nu)$ such that two leaves w_1, w_2 of $D(\nu)$ are E_r-equivalent if the unique paths from r to w_1 and from r to w_2 have at least two common nodes (or equivalently, if the unique w_1w_2-path of $\overline{D(\nu)}$ does not pass through r). The E_r-classes will be called *branches* at r. For each $r \in \mathrm{Ram}(\overline{D(\nu)})$, one of the branches at r will be distinguished and called the *special branch at* r. This device enables us to define a ternary relation L and also ensures that the Fraïssé limit M which we construct has a strictly decreasing family of subsets as in Definition 2.6.

We shall use Roman letters $x, y, z, w, u, v, \ldots$ for leaves of a D-set, and letters r, r', r'' or $r_1, r_2, \ldots$ for the ramification points. Greek letters $\alpha, \nu, \mu, \ldots$ refer to the vertices of the tree, while we retain the letter ρ for the root. This notation will persist in Section 4, where everything is infinite and the labelling graph-theoretic trees are replaced by general betweenness relations.

The relationship between the various D-sets is governed by certain maps f_ν and $g_{\omega\nu}$ as in [8]. For each $\nu \in T$, we assume there is a fixed bijection

$$f_\nu\colon \mathrm{succ}(\nu) \to \mathrm{Ram}(\overline{D(\nu)})$$

from the set of successors of the vertex ν in T to the set of ramification points of the D-set $D(\nu)$. For $r \in \mathrm{Ram}(\overline{D(\nu)})$, if $\omega = f_\nu^{-1}(r)$, then there is a bijection $g_{\omega\nu}$ from the D-set $D(\omega)$ to the set of *non-special* branches at r (in the D-set $D(\nu)$).

Let $\nu_0, \ldots, \nu_m$ be vertices of the semilinear order T such that $\nu_0 < \cdots < \nu_m$. Then $(\nu_0, \ldots, \nu_m)$ is a *chain of successors* if we have $\nu_{i+1} \in \mathrm{succ}(\nu_i)$ for each $i \in \{0, \ldots, m-1\}$. Given the chain $(\nu_0, \ldots, \nu_m)$, there is a map $g_{\nu_m\nu_0}$, which we define by induction, that maps each leaf of the D-set $D(\nu_m)$ to a union of branches at the ramification point $f_{\nu_0}(\nu_1)$ of $\overline{D(\nu_0)}$. Let $a \in D(\nu_m)$, and define

$$g_{\nu_m\nu_0}(a) := \{x \in D(\nu_0) : \exists y \in g_{\nu_m\nu_{m-1}}(a)(x \in g_{\nu_{m-1}\nu_0}(y))\}.$$

The structure above, consisting of the labelled semilinear order and the maps f_ν and $g_{\mu\nu}$, will be called a (finite) *tree of D-sets*, and we use symbols τ, τ' to denote such structures, and refer to T as the *structure tree* of τ. We have not yet described how to parse a tree of D-sets as a first-order structure.

The diagram in Figure 2 is an example of a tree of D-sets. Note that, in $D(\rho)$, as indicated by the arrows, x lies in the special branch at r' and u lies in the special branch at r, and in $D(\nu)$, $\bar{x}$ lies in the special branch at r_1 and also in the special branch at r_1'.

Let τ, τ' be two trees of D-sets. An *isomorphism between trees of D-sets* is an isomorphism between the corresponding two lower semilinear orders

$$\phi\colon (T, \leq) \to (T', \leq)$$

together with, for every vertex $\nu \in T$, a graph isomorphism ψ_ν from $\overline{D(\nu)}$ to $\overline{D(\phi(\nu))}$. The maps ψ_ν are required to map the special branch at any ramification point r to the special branch at $\psi_\nu(r)$, and to commute with the maps f_ν and $g_{\omega\nu}$.

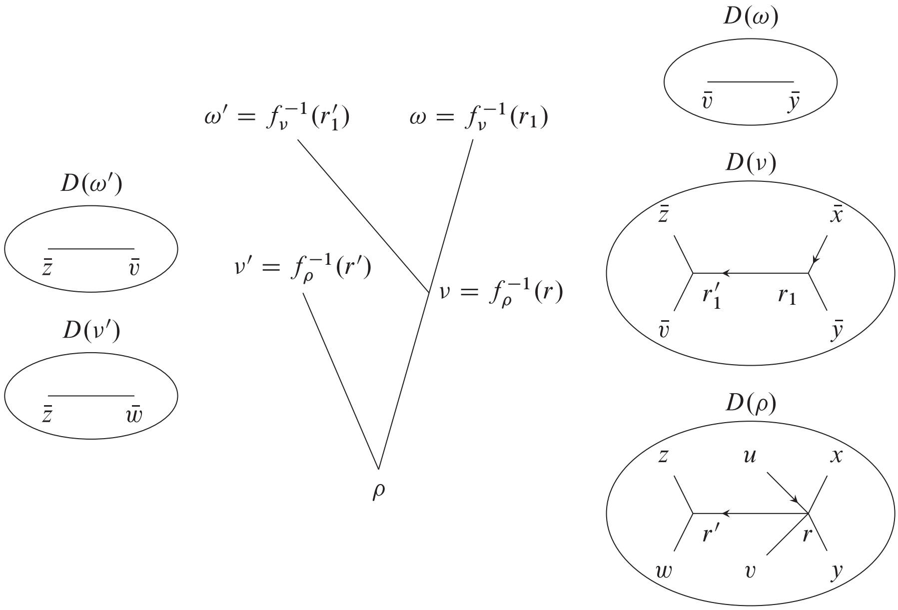

Figure 2. Tree of D-sets

If τ is a tree of D-sets with vertices $\mu < \nu$ of the structure tree, then we say that the D-set $D(\nu)$ *omits* the element $u \in D(\mu)$ if there is no $x \in D(\nu)$ such that $u \in g_{\nu\mu}(x)$. If ν is an immediate successor of μ, this means that u lies in the special branch of the ramification point $f_\mu(\nu)$ of $\overline{D(\mu)}$.

We shall view a finite tree of D-sets τ as a first-order structure in a language $\mathscr{L}$ which has a ternary relation L, two quaternary relations L' and S, a 5-ary relation S', a 6-ary relation R and a 7-ary relation Q. The relations L and S capture the special branch structure and the D-relations. The other relations are auxiliary; they ensure that there is a robust notion of instances of S or L "happening in the same D-set" and that an appropriate form of amalgamation holds. In the Fraïssé limit, these other relations will be $\emptyset$-definable without parameters in S and L. The universe of the $\mathscr{L}$-structure τ will be the domain of the root D-set of τ (i.e. the universe is the set $D(\rho)$, where ρ as usual denotes the root of the structure tree), and the relations are interpreted on $D(\rho)$ as follows.

(i) $L(x; y, z)$ holds in τ if

(a) either x, y, z lie in distinct branches at node r of the root D-set $D(\rho)$, and the branch containing x is special at r (Figure 3),

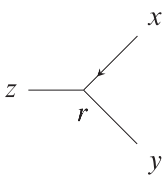

Figure 3. $L(x; y, z)$

(b) or there is a D-set $D(\nu)$ with a ramification point r, and leaves $\bar{x}, \bar{y}, \bar{z}$ lying in distinct branches at r with $\bar{x}$ lying in the special branch at r, such that $x \in g_{\nu\rho}(\bar{x})$, $y \in g_{\nu\rho}(\bar{y})$, $z \in g_{\nu\rho}(\bar{z})$.

We say in (a) that $D(\rho)$ *witnesses* $L(x; y, z)$, and in (b) that $D(\nu)$ *witnesses* $L(x; y, z)$. For example, in Figure 2, $D(\rho)$ witnesses $L(x; z, w)$ and $D(\nu)$ witnesses $L(x; z, v)$. We use the semi-colon to distinguish the special branch in the first argument, while there is symmetry between the other two arguments.

(ii) Let $x, y, z, w \in D(\rho)$ be distinct. Then the relation $S(x, y; z, w)$ holds, written $\tau \models S(x, y; z, w)$, if one of the following holds.

(a) In the root D-set, with universe denoted $D(\rho)$ and D-relation D_ρ, we have $D_\rho(x, y; z, w)$.

(b) x, y, z, w lie in distinct non-special branches at node r of $D(\rho)$, and there is some vertex $\nu \geq f_\rho^{-1}(r)$ such that $D(\nu)$ contains distinct $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ such that $D_\nu(\bar{x}, \bar{y}; \bar{z}, \bar{w})$ holds in $D(\nu)$, and $x \in g_{\nu\rho}(\bar{x})$, $y \in g_{\nu\rho}(\bar{y})$, $z \in g_{\nu\rho}(\bar{z})$, $w \in g_{\nu\rho}(\bar{w})$.

We say in (a) that $D(\rho)$ *witnesses* $S(x, y; z, w)$, and in (b) that $D(\nu)$ *witnesses* $S(x, y; z, w)$.

Note. (1) It is easy to see that, given a tree of D-sets τ and $x, y, z \in D(\rho)$, the relation $L(x; y, z)$ can be witnessed in at most one D-set of τ; to see this, observe that if it is witnessed in D-sets $D(\mu)$ and $D(\nu)$, then it cannot happen that $\mu < \nu$ or that μ and ν are incomparable. Likewise, if $x, y, z, w \in D(\rho)$, then $S(x, y; z, w)$ is witnessed in at most one D-set of τ. We omit the details.

(2) The relation S captures the D-relations; note that, unlike with D (see axiom (D4)), with S, we do not allow equality among its parameters. We use the semi-colon to reflect the symmetry between the first two arguments and the last two.

(iii) $L'(x; y, z; u)$ holds in τ if, in the D-set $D(\nu)$ witnessing $L(x; y, z)$, the element u is omitted, that is, there is no $\bar{u} \in D(\nu)$ with $u \in g_{\nu\rho}(\bar{u})$. We then say that the D-set $D(\nu)$ *witnesses* the relation $L'(x; y, z; u)$. We use the first semi-colon as in L above, and the second one to distinguish the omitted element.

(iv) $S'(x, y; z, w; t)$ holds in τ if in the D-set $D(\nu)$ witnessing $S(x, y; z, w)$, the element t is omitted (in the same sense as in (iii)). Here the D-set $D(\nu)$ witnesses the relation $S'(x, y; z, w; t)$. Again, the second semi-colon indicates that the last argument is distinguished.

(v) $Q(x, y; z, w : p; q, s)$ holds in τ if there is some D-set in which the relations $S(x, y; z, w)$ and $L(p; q, s)$ are both witnessed. We interpret this as $S(x, y; z, w)$ and $L(p; q, s)$ *happen in the same D-set* (which *witnesses* $Q(x, y; z, w : p; q, s)$).

(vi) $R(x; y, z : p; q, s)$ holds in τ if there is some D-set in which the relations $L(x; y, z)$ and $L(p; q, s)$ are both witnessed. Again, we interpret this as saying that the two L-relations *happen in the same D-set* (and again, this D-set *witnesses* $R(x; y, z : p; q, s)$).

Remark 3.1. (1) By the definition above, the relations L', S' cannot be witnessed in the root D-set. Regarding the need for L', S', Q, R, observe their role in the proofs of Lemma 3.9, Lemma 3.10 (and the paragraph following its proof – these lemmas underpin the amalgamation), Remark 3.4, and Lemma 4.7.

(2) When we say that one of the above relations holds in the tree of D-sets τ, we mean it is witnessed in some D-set of τ. We may thus view a finite tree of D-sets as an $\mathscr{L}$-structure whose universe is the set of leaves of the root D-set. From now on, we use symbols like $A, B, C, E, \dots$ (rather than τ, τ') for such finite $\mathscr{L}$-structures, and write τ_A for the corresponding tree of D-sets (which will be seen in Lemma 3.7 to be determined up to isomorphism by A); we denote the structure tree of τ_A by T_A. Also, we write $A < B$ if A is an $\mathscr{L}$-substructure of B in the sense of model theory. Occasionally, we write $L\{x, y, z\}$, as an abbreviation for $L(x; y, z) \vee L(y; x, z) \vee L(z; x, y)$, and we may say that $L\{x, y, z\}$ is *witnessed* in a specific D-set.

Let $\mathscr{D}$ be the collection of all finite $\mathscr{L}$-structures arising from finite trees of D-sets as described above. If $A \in \mathscr{D}$, we write A also for its universe (which will be the set $D(\rho)$, with corresponding graph-theoretic tree $\overline{D(\rho)}$). In line with earlier notation, the D*-relation* on the root D-set is denoted D_ρ, or D_ρ^A if the underlying structure A is unclear. A D*-set* $D(\nu)$ of A may be denoted $D^A(\nu)$ if A is unclear from the context.

Lemma 3.2. *Let $A \in \mathscr{D}$, and let x, y, z be distinct elements of A. Then,*

(i) *if x, y, z are distinct elements of A, then $L\{x, y, z\}$ holds in A, and*

(ii) *any substructure A' of A of size at most 3 lies in $\mathscr{D}$.*

Proof. (i) Let μ be a vertex of the structure tree of A maximal such that there are distinct $\bar{x}, \bar{y}, \bar{z} \in D(\mu)$ with $x \in g_{\mu\rho}(\bar{x})$, $y \in g_{\mu\rho}(\bar{y})$, and $z \in g_{\mu\rho}(\bar{z})$. Then, by maximality, one of $\bar{x}, \bar{y}, \bar{z}$ lies in the special branch at the ramification point $\mathrm{ram}(\bar{x}, \bar{y}, \bar{z})$, and hence $D(\mu)$ witnesses $L\{x, y, z\}$.

(ii) Suppose first $|A'| \leq 2$. Then the elements of A' do not determine any ramification point of $\overline{D(\rho)}$ (where $D(\rho)$ is the root D-set of A). It follows that the structure on A' corresponds to a tree of D-sets with just one vertex, with the corresponding D-set consisting just of the elements of A' (with an edge between them if $|A'| = 2$). If $|A'| = 3$ with $A' = \{x, y, z\}$, then as in (i), we may suppose $A \models L(x; y, z)$. Now A' is a structure arising from a tree of D-sets with two vertices ρ (the root) and its successor ν. Here $\overline{D(\rho)}$ has a ramification point r joined to just the three leaves x, y, z with x special, and $D(\nu)$ consists of just an edge joining the two nodes $g_{\nu\rho}^{-1}(y)$ and $g_{\nu\rho}^{-1}(z)$. □

By Remark 3.4 below, we cannot replace 3 by 4 in Lemma 3.2 (ii).

Lemma 3.3. *Let $A \in \mathscr{D}$, and let ρ be the root of T_A.*

(i) *The relation D_ρ on $D(\rho)$ satisfies the following: for all $x, y, z, w \in D(\rho)$,*

$$\begin{aligned} D_\rho(x, y; z, w) \iff & [(((x = y) \vee (z = w)) \wedge \{x, y\} \cap \{z, w\} = \emptyset) \\ & \vee (x, y, z, w \textit{ are all distinct} \wedge S(x, y; z, w) \\ & \wedge (\forall t)(\neg S'(x, y; z, w; t)))]. \end{aligned}$$

(ii) *If $x, y, z \in A$, then $L(x; y, z)$ is witnessed in $D(\rho)$ if and only if*

$$A \models L(x; y, z) \wedge \forall t \neg L'(x; y, z; t).$$

Proof. (i) We may suppose that x, y, z, w are distinct.

($\Rightarrow$) Suppose $D_\rho(x, y; z, w)$. Then $S(x, y; z, w)$ holds, witnessed in the root D-set. As $D(\rho)$ contains all elements of A and is the only D-set witnessing $S(x, y; z, w)$, it follows that $A \models (\forall t)\neg S'(x, y; z, w; t)$.

($\Leftarrow$) Suppose $S(x, y; z, w) \wedge (\forall t)(\neg S'(x, y; z, w; t))$ holds. For a contradiction, we will assume that the D-set $D(\nu)$ witnessing $S(x, y; z, w)$ is not the root. Then there is a D-set $D(\mu)$ with $\mu < \nu$ in which x, y, z, w lie in distinct branches $\bar{x}, \bar{y}, \bar{z}, \bar{w}$ respectively at a ramification point r. As $S(x, y, z, w)$ is witnessed in a higher D-set, none of x, y, z, w lies in a special branch at r. Since each ramification point has a special branch, some $t \in A$ lies in the special branch at r. Then $S'(x, y; z, w; t)$ holds, which is a contradiction.

(ii) This is immediate. □

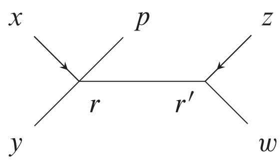

Figure 4. ρ_C

Remark 3.4. The collection $\mathscr{D}$ does not have the hereditary property, that is, it is not closed under substructure. For example, consider $C \in \mathscr{D}$ with elements x, y, z, w, p. Let $r = \mathrm{ram}(x, y, z)$, $r' := \mathrm{ram}(x, z, w)$, and suppose $S(x, y; z, w)$ holds, and $L(x; y, z) \wedge L(x; y, w) \wedge L(x; y, p)$ hold at r, all in the root D-set $D(\rho_C)$ as in Figure 4. Let $\nu := f_{\rho_C}^{-1}(r)$, and in $D(\nu)$, suppose that the relation $L(p; y, z)$ is witnessed in the unique ramification point r''. Also, let $\nu' := f_{\rho_C}^{-1}(r')$ and $\nu_1 := f_\nu^{-1}(r'')$. The two labelling D-sets $D(\nu')$ and $D(\nu_1)$ each have just two elements. Let A be the $\mathscr{L}$-substructure of C induced on $C \setminus \{x\}$. Then $A \notin \mathscr{D}$; indeed, if $A \in \mathscr{D}$, then by Lemma 3.3 (i), $S(p, y; z, w)$ must be witnessed in the root D-set of A and p must be in the special branch at $\mathrm{ram}(p, y, z)$ in this D-set, so we have $A \models Q(p, y; z, w : p; y, z)$, contradicting the fact that

$$C \models \neg Q(p, y; z, w : p; y, z).$$

Consider an $\mathscr{L}$-structure $A \in \mathscr{D}$ with structure tree T_A with root ρ, and let ν be a successor of ρ. We define an $\mathscr{L}$-structure A_ν whose domain is the set of leaves of the D-set $D(\nu)$, that is, the set of non-special branches of $\overline{D(\rho)}$ at $f_\rho(\nu)$. To define the relations of A_ν, suppose first that $\bar{a}, \bar{b}, \bar{c} \in A_\nu$ are distinct, and let $a \in g_{\nu\rho}(\bar{a})$, $b \in g_{\nu\rho}(\bar{b})$, and $c \in g_{\nu\rho}(\bar{c})$. Then $A_\nu \models L(\bar{a}; \bar{b}, \bar{c})$ if and only if $A \models L(a; b, c)$. It is easily checked that this is well-defined, i.e. independent of the choice of a, b, c. The relations S, L', S', Q, R are defined similarly on A_ν. If $r = f_\rho(\nu)$, we also sometimes write A_ν as A_r. If μ is any vertex of T_A, then T_μ is the subtree of T_A induced on $\{\sigma \in T_A : \sigma \geq \mu\}$. The structure tree of A_ν is just T_ν.

Given a structure tree T, we define the height $h(T)$ of T to be the number of vertices in the longest path from a leaf of T to the root ρ. If $A \in \mathscr{D}$, we put $h(A) := h(T_A)$. Using the preceding paragraph and the lemma below, we often argue inductively on $h(T)$.

Lemma 3.5. *Let $A \in \mathscr{D}$ with structure tree T_A with root ρ with a successor ν. Then the following hold.*

(i) *A_ν is isomorphic to a substructure of A.*

(ii) *$A_\nu \in \mathscr{D}$, with structure tree T_ν.*

(iii) $h(T_\nu) < h(T_A)$.

(iv) $|A_\nu| < |A|$.

Proof. All parts are elementary, and we omit the details. □

Remark 3.6. It follows from the last lemma that the above construction can be iterated for a successor of ν. Thus, inductively, for any vertex $\mu \neq \rho$ of the structure tree T_A of A, there is a corresponding $\mathscr{L}$-structure A_μ, and all parts of Lemma 3.5 hold with μ replacing ν. This is a convenient tool for inductive arguments.

Proposition 3.7. *Suppose that τ_1, τ_2 are trees of D-sets with corresponding $\mathscr{L}$-structures A_1, A_2, and let $\chi\colon A_1 \to A_2$ be an isomorphism of $\mathscr{L}$-structures. Then χ induces an isomorphism $\phi\colon \tau_1 \to \tau_2$ of trees of D-sets.*

Proof. Let T_i be the structure tree of τ_i for $i = 1, 2$. We apply induction on $h(T_1)$. Let ρ_1, ρ_2 be the roots of T_1, T_2 respectively, and put $\phi(\rho_1) = \rho_2$.

For the base case, suppose $h(T_1) = 1$. Then T_1 has just the root ρ_1, and $\overline{D(\rho_1)}$ has no ramification points, so has at most two leaves. Thus A_1 consists of a set of size at most 2 with none of the $\mathscr{L}$-relations holding, and as χ is an isomorphism, the same holds for A_2. Hence T_2 has one vertex ρ_2, and χ induces a unique isomorphism $T_1 \to T_2$ and $D(\rho_1) \to D(\rho_2)$.

For the inductive step, suppose $m := h(T_1) \geq 2$. By Lemma 3.3, χ determines an isomorphism of D-structures $D(\rho_1) \to D(\rho_2)$ which preserves also the special branch structure. This extends to a unique graph isomorphism (which we denote by $\bar{\chi}$) $\overline{D(\rho_1)} \to \overline{D(\rho_2)}$ taking the ramification points of $\overline{D(\rho_1)}$ to the ramification points of $\overline{D(\rho_2)}$.

For each $r \in \mathrm{Ram}(\overline{D(\rho_1)})$, put $\phi(f_{\rho_1}^{-1}(r)) = f_{\rho_2}^{-1}(\bar{\chi}(r))$ to obtain a bijection $\mathrm{succ}(\rho_1) \to \mathrm{succ}(\rho_2)$. Let ν_1 be the successor of ρ_1 corresponding to r and ν_2 the successor of ρ_2 corresponding to $\bar{\chi}(r)$. We claim that χ induces an isomorphism ϕ_ν from the $\mathscr{L}$-structure A_{ν_1} to the $\mathscr{L}$-structure A_{ν_2}. Indeed, $\bar{\chi}$ gives a bijection $D(\nu_1) \to D(\nu_2)$, and the fact that it is an isomorphism of $\mathscr{L}$-structures follows from the definition of the A_{ν_i}.

As $h(A_{\nu_1}) < h(A)$ (by Lemma 3.5 (iii)), it follows by induction that χ induces a unique isomorphism from the tree of D-sets corresponding to A_{ν_1} to that corresponding to A_{ν_2}. This holds for all successors of ρ_1, and the result follows. □

3.1 One-point extensions

Fix an $\mathscr{L}$-structure $A \in \mathscr{D}$. As a step towards proving the amalgamation property for $\mathscr{D}$, we will specify the possible forms of a one-point extension $E = A \cup \{e\}$

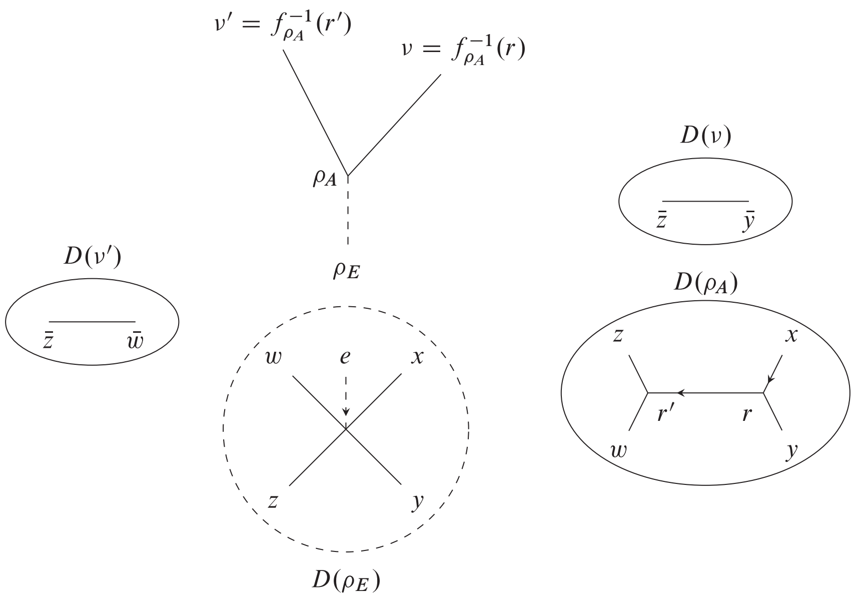

Figure 5. One-point extension: Type I

of A such that $E \in \mathscr{D}$ and A is an $\mathscr{L}$-substructure of E. We first describe some one-point extensions.

Type I (Star-like). To obtain T_E, which is the structure tree on the $\mathscr{L}$-structure E, from T_A, we add a new root ρ_E under the root ρ_A of T_A so that $D(\rho_E)$ is a star with one ramification point (the centre) and non-special branches, each containing a single leaf, in one-to-one correspondence with the leaves in the root D-set $D(\rho_A)$ of A, and a special branch with single leaf e. We shall use the word *star* to describe a tree of this form (a node connected to a finite collection of leaves). See Figure 3 for an example.

Since there is only one ramification point in $D(\rho_E)$, it will have the form $f_{\rho_E}(\rho_A)$, where ρ_A is the immediate successor of ρ_E. The D-set $D(\rho_E)$ is a star whose centre is $f_{\rho_E}(\rho_A)$, where the branches (here identified with leaves) are of the form $g_{\rho_A\rho_E}(x)$, x a leaf in $D(\rho_A)$. The relations on A will also hold in $E := A \cup \{e\}$. Thus, if $a, b, c \in A$ and $A \models L(a; b, c)$, then $E \models L(a; b, c)$; however, this is not witnessed in the root D-set of E, and indeed, $E \models L'(a; b, c; e)$. Likewise, if $a, b, c, d \in A$ and $A \models S(a, b; c, d)$, then $E \models S'(a, b; c, d; e)$.

Type II. Here we assume that the structure tree roots for the two structures A and E are the same, denoted by ρ, and we add the new leaf e to the existing D-set

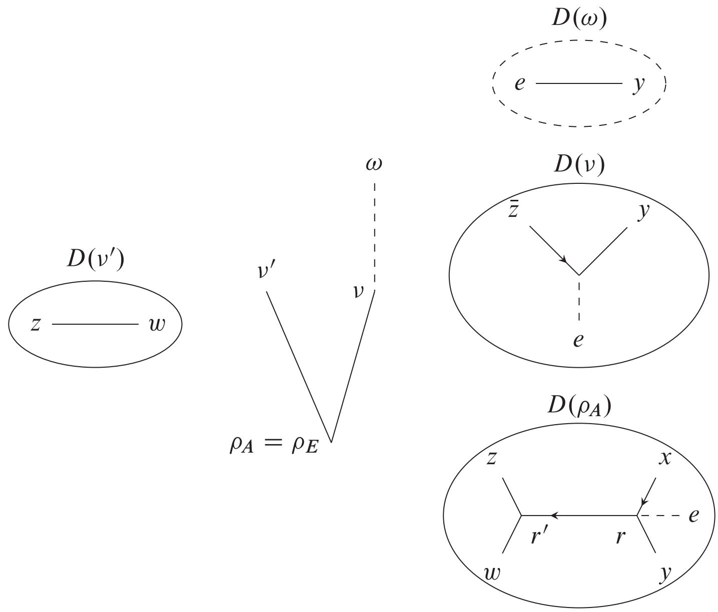

Figure 6. One point extension: Type II (a)

$D^A(\rho)$ of the root ρ of T_A to obtain the root D-set $D^E(\rho)$ of E. We can do this in two ways.

(a) Add a new leaf e adjacent to an existing ramification point r in $\overline{D^A(\rho)}$, with the branch of e at r non-special. So the D-set $D^E(\nu)$ corresponding to that ramification point (i.e. with $\nu = f_\rho^{-1}(r)$) gains a new leaf not in $D^A(\nu)$, namely $g_{\nu\rho}^{-1}(e)$. This process iterates through the structure tree, so the definition is inductive on $|A|$.

(b) Create a new ramification point by adding a node on the midpoint of an existing edge in $\overline{D^A(\rho)}$, then add a leaf e at this node. Here we consider two cases:

(i) e is the unique leaf of the special branch at this new ramification point;

(ii) e is not in the special branch at this ramification point.

In both cases, a new successor is added to the structure tree, but the D-set labelling the new successor has just two endpoints, and hence there are no modifications higher in the structure tree.

The following lemma is almost immediate, and we omit the proof.

Lemma 3.8. *If $A \in \mathscr{D}$ and E is a one-point extension of A of Type I or Type II, then $E \in \mathscr{D}$.*

Lemma 3.9. *If $A, E \in \mathscr{D}$ with $A < E$, and $a, b, c, d \in A$ are all distinct elements, then $D_\rho^E(a, b; c, d) \to D_\rho^A(a, b; c, d)$.*

Proof. As a, b, c, d are distinct, Lemma 3.3 yields

$$\begin{aligned} D_\rho^E(a, b; c, d) &\implies S(a, b; c, d) \wedge (\forall t \in E)\neg S'(a, b; c, d; t) \\ &\implies S(a, b; c, d) \wedge (\forall t \in A)\neg S'(a, b; c, d; t) \\ &\implies D_\rho^A(a, b; c, d). \end{aligned}$$ □

The next two lemmas yield that *all* one-point extensions $A < E$ with $A, E \in \mathscr{D}$ have Type I or Type II.

Lemma 3.10. *Suppose $A, E \in \mathscr{D}$ with $A < E$, and there is no $e \in E$ such that $A \cup \{e\}$ is a Type I extension of A. Then the root D-relation D_ρ^A of A and the relation $D_\rho^E(A)$ induced on A by the root D-relation D_ρ^E of E are the same.*

Note. We do not here assume that $|E \setminus A| = 1$.

Proof. Let $a, b, c, d \in A$, and assume $D_\rho^E(A)(a, b; c, d)$. Then $D_\rho^E(a, b; c, d)$. We may suppose that a, b, c, d are distinct. By Lemma 3.9, $D_\rho^A(a, b; c, d)$.

Conversely, let $a, b, c, d \in A$ be distinct, and suppose that $D_\rho^A(a, b; c, d)$ but $\neg D_\rho^E(A)(a, b; c, d)$. Let r, r' be as in Figure 7 in $\overline{D^A(\rho)}$.

As $A \models S(a, b; c, d)$ and this is not witnessed in the root D-set of E, it follows from Lemma 3.3 that there is $e \in E \setminus A$ such that $E \models S'(a, b; c, d; e)$. We have the picture in Figure 8 in $\overline{D^E(\rho)}$, with e special at the shown ramification point s.

This picture is a star, and we assume that $A \cup \{e\}$ is not a Type I extension of A. This means that there must be some $x \in A$ (hence in E) witnessing that $A \cup \{e\}$ is not a Type I extension of A, in particular, ensuring that the elements of $A \cup \{e\}$

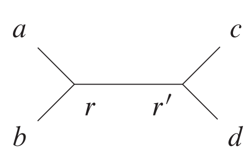

Figure 7. $D_\rho^A(a, b; c, d)$

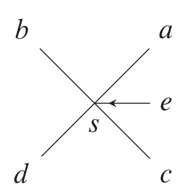

Figure 8. D_ρ^E

do not form a star at s in $\overline{D^E(\rho)}$. We consider the various possible positions of x with respect to a, b, c, d, e in $D^E(\rho)$.

Case (1). Suppose x lies in the same branch at s as c (with a similar argument if x lies in the same branch as a, b, or d). Since

$$S(a, d; c, x) \wedge (\forall w \in E)\neg S'(a, d; c, x; w)$$

holds in E and hence in A, x must lie in the same branch as c at r' in $\overline{D^A(\rho)}$. Put $r'' := \mathrm{ram}(x, c, d)$ (in $\overline{D^A(\rho)}$). Let $x' \in A$ be in the special branch at r''. We assume for convenience that $A \models L(x'; c, d)$ (other cases being similar). Now $Q(a, b; c, d : x'; c, d)$ and $Q(a, d; c, x : x'; c, d)$ both hold in A, and hence both hold in E, so $S(a, d; c, x)$ and $L(x'; c, d)$ are witnessed in the same D-set of E, which must be the root D-set (as $S(a, d; c, x)$ holds there). Likewise, $S(a, b; c, d)$ and $L(x'; c, d)$ are witnessed in the same D-set of E, so $S(a, b; c, d)$ is witnessed in the root D-set of E, a contradiction.

Case (2). If x is in the same branch at s as the special branch containing e in E, then we will see $S'(a, b; c, d; x)$ holds in E and hence in A. This is impossible, since we have $D^A_\rho(a, b; c, d)$, so $A \models S(a, b; c, d) \wedge (\forall t)\neg S'(a, b; c, d; t)$.

Case (3). Suppose neither of (1), (2) holds, but that (to ensure $A \cup \{e\}$ is not a Type I extension of A) there exists $x' \in A$ in the same branch as x at s in $D^E(\rho)$ (distinct from the branches containing a, b, c, d, e). Since $S(x, x'; u, v)$ (for any distinct $u, v \in \{a, b, c, d\}$) holds in the root D-set of E, the same holds in A. Let t be the unique ramification point of $\overline{D^A(\rho)}$ of form $\mathrm{ram}(x, x', u)$ for all $u \in \{a, b, c, d\}$. Also, let $x'' \in A$ lie in the special branch at t. We shall suppose $L(x''; x, a)$ (there are other similar cases, if say $x'' = x$). Now

$$Q(a, b; c, d : x''; x, a)$$

holds in A and hence in E, as does

$$Q(u, v; x, x' : x''; x, a)$$

for any distinct $u, v \in \{a, b, c, d\}$. Since such relations $S(u, v; x, x')$ are witnessed in the root D-set of E, so is $L(x''; x, a)$ and hence so also is $S(a, b; c, d)$, a contradiction. □

Similarly, it is readily seen, if $A, E \in \mathcal{D}$ with $A < E$, and there is no $e \in E \setminus A$ such that $A < A \cup \{e\}$ is of Type I, and $a, b, c \in A$ are distinct, then $L(a; b, c)$ is witnessed in $D^A(\rho)$ if and only if it is witnessed in $D^E(\rho)$.

Lemma 3.11. *If $A, E \in \mathcal{D}$ and E is a one-point extension of A with $E = A \cup \{e\}$, then the extension is of Type* I *or of Type* II.

Proof. Assume that the extension is not of Type I. By Lemma 3.10, we have $D_\rho^E(A) = D_\rho^A$, and the root D-set $D^A(\rho)$ of A is a substructure of $D^E(\rho)$ (in the language of D-relations), and hence we can identify $D^A(\rho)$ with a subset of $D^E(\rho)$. Furthermore, for $a, b, c \in A$, $L(a; b, c)$ is witnessed in $D^A(\rho)$ if and only if it is witnessed in $D^E(\rho)$ by the above paragraph. Thus, either e is added (in $D^E(\rho)$) as a new non-special leaf to an existing ramification point r of $D^A(\rho)$, or e is added on a new ramification point r' of an edge of $D^A(\rho)$. To prove the extension is of Type II, we consider the following cases.

Case (i). Suppose that e is added as the unique leaf of a new non-special branch at an existing ramification point r of $D^A(\rho)$. By induction, as $|A_r| < |A|$, $A_r < E_r$ is an extension of Type I or II (recall the notation A_r, E_r from before Lemma 3.5). It follows that $A < E$ is a Type II extension.

Case (ii). Suppose that e is added on a new ramification point r' of an edge of $D^A(\rho)$. In this case, for E, ρ_A obtains a new successor $\rho_{r'}$ whose D-set has size 2. The structure is otherwise unchanged, and E is an extension of A of Type II (b).

□

Lemma 3.12. *Let $A < E$ with $A, E \in \mathscr{D}$. Then there is an element $e \in E \setminus A$ such that $A \cup \{e\} \in \mathscr{D}$.*

Proof. We may suppose there is no $e \in E \setminus A$ such that $A < A \cup \{e\}$ is a Type I extension. Thus the relation D_ρ^E induces D_ρ^A on $D^A(\rho)$ by Lemma 3.10.

Suppose there is an edge of $D^A(\rho)$ such that E has a ramification point r on the edge and there is $e \in E \setminus A$ and $a, b \in A$ such that a, b, e lie in distinct branches at r. We may suppose (by careful choice of e) that one of a, b, e lies in the special branch at r in $D^E(\rho)$. In this case, $A \cup \{e\} \in \mathscr{D}$, a one-point extension of A of Type II (b).

Suppose the configuration described in the last paragraph does not occur. Since D_ρ^E induces D_ρ^A, there is a ramification point r of $D^A(\rho)$ and some $e \in E \setminus A$ lying in a new non-special branch at r of $D^E(\rho)$, with e adjacent to r in $D^E(\rho)$. Arguing inductively on $|A|$, we may choose e here (among elements of E lying in a new branch at r) so that if $E' := A \cup \{e\}$, then $E'_r \in \mathscr{D}$ and $A_r < E'_r$ is an extension of Type I or II. Then $E' \in \mathscr{D}$ and is a one-point extension of A of Type II (a). □

The next lemma enables us to reduce proving amalgamation to the special case of amalgamating one-point extensions.

Lemma 3.13. *Assume $A < E$ with $A, E \in \mathscr{D}$. Then we may enumerate $E \setminus A$ as $\{e_1, e_2, \dots, e_n\}$ so that, for each $i = 1, \dots, n$, if E_i is the $\mathscr{L}$-substructure of E on $A \cup \{e_1, \dots, e_i\}$, then $E_i \in \mathscr{D}$.*

Proof. Let $n := |E \setminus A|$. We prove by induction on m that, for all $m < n$, there are distinct $e_1, \ldots, e_m \in E \setminus A$ such that, for each $i = 0, \ldots, m$, the $\mathscr{L}$-structure E_i induced on $A \cup \{e_1, \ldots, e_i\}$ lies in $\mathscr{D}$ (where $E_0 = A$).

The base case $m = 0$ is trivial. Assume that the result holds for m. Then, by Lemma 3.12, there is some $e \in E \setminus E_m$ such that $E_m \cup \{e\} \in \mathscr{D}$. Put $e_{m+1} := e$. □

3.2 Amalgamation property

Fraïssé's amalgamation technique builds a countable structure M as a union of a sequence of finite structures, each itself an amalgam of substructures. We say that a class $\mathscr{C}$ has the *amalgamation property* if, when $A, E_1, E_2 \in \mathscr{C}$ and $f_i : A \to E_i$ are embeddings (for $i = 1, 2$), there is $D \in \mathscr{C}$ and embeddings $g_i : E_i \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$. We say that $\mathscr{C}$ has the *amalgamation property for one-point extensions* if the above holds whenever $|E_1 \setminus f_1(A)| = |E_2 \setminus f_2(A)| = 1$.

Lemma 3.14. *Let $\mathscr{C}$ be a class of finite structures, and suppose that the following hold:*

(i) *the class $\mathscr{C}$ has the amalgamation property for one-point extensions;*

(ii) *for any $A, E \in \mathscr{C}$ with $A < E$, we may write $E \setminus A = \{e_1, \ldots, e_n\}$ so that if E_i is the induced substructure of E on $A \cup \{e_1, \ldots, e_i\}$ (for each $i = 1, \ldots, n$), then $E_i \in \mathscr{C}$.*

Then the class $\mathscr{C}$ has the amalgamation property.

Proof. See the last three paragraphs of the proof of [8, Lemma 3.7]. □

Lemma 3.15. *The class $\mathscr{D}$ has the amalgamation property.*

Proof. We will prove the amalgamation property for one-point extensions, and the result then follows from Lemmas 3.13 and 3.14. Assume $A < E_1$ and $A < E_2$ with $A, E_1, E_2 \in \mathscr{D}$ such that $E_1 \setminus A = \{e_1\}$ and $E_2 \setminus A = \{e_2\}$. We may assume that e_1 and e_2 are distinct, and we want to define a structure E on $E_1 \cup E_2$, inducing each E_i, in such a way that $E \in \mathscr{D}$. Let T_i be the structure tree corresponding to E_i with root ρ_i, where $i = 1, 2$. We will consider three cases.

Case (i). Suppose that E_1 and E_2 are Type I extensions of A. Then, in the structure tree of E, place the root ρ_2 beneath the root ρ_1 in such a way that e_2 is special in $D^E(\rho_2)$ (i.e. in the special branch at the unique ramification point) with e_1 non-special, and in $D^E(\rho_1)$, the element e_1 is special. Here E is a Type I extension of E_1.

Case (ii). Suppose that one of the E_i, say E_1 is of Type I, and E_2 is of Type II. Then define the structure tree of E by placing the root ρ_1 under ρ_2 so that $D^E(\rho_1)$ is a star in which e_1 is special and e_2 is not.

Case (iii). Suppose that E_1 and E_2 are of Type II over A. Then we will consider the following four sub-cases.

(1) Assume that e_1, e_2 are added to the same ramification point r of $\overline{D(\rho_A)}$ to get E_1, E_2 respectively. Keep them distinct in E. Then neither of e_1, e_2 is special at r in the root D-sets $D^{E_1}(\rho)$ and $D^{E_2}(\rho)$. In the root D-set of E, e_1, e_2 will lie in non-special branches at r. Then, higher up, two new end-points are added to the same D-set $D(f_{\rho_E}^{-1}(r))$, and we finish inductively since $|A_r| < |A|$, so $(E_1)_r$ and $(E_2)_r$ can be amalgamated over A_r.

(2) Suppose that e_1 and e_2 are added to distinct ramification points r_1 and r_2 of $\overline{D(\rho_A)}$. Then, again, when building E, a leaf will be added to each of the D-sets corresponding to these ramification points. The structures E_{r_1} and $(E_1)_{r_1}$ will be isomorphic, and E_{r_2} will be isomorphic to $(E_2)_{r_2}$.

(3) Suppose that the branch e_1 is added to an old ramification point r of $\overline{D(\rho_A)}$, and e_2 creates a new ramification point s, i.e. $A < E_1$ is of Type II (a), and $A < E_2$ has Type II (b). Then a new successor $f_{\rho_E}^{-1}(s)$ has trivial D-set in E (i.e. with only 2 elements joined by an edge), and $D(f_{\rho_E}^{-1}(r))$ is isomorphic to $D(f_{\rho_{E_1}}^{-1}(r))$.

(4) Assume that both e_1 and e_2 create new ramification points, that is, both give Type II (b) extensions. Then keep these ramification points distinct in E. Hence $\overline{D(\rho_E)}$ will have two new ramification points (compared to $\overline{D(\rho_A)}$), and ρ_E has two new successors with labelling D-sets of just two elements. □

Lemma 3.16. *The class $\mathscr{D}$ has the joint embedding property.*

Proof. Take two finite structures $A, B \in \mathscr{D}$ with n, m points respectively. Consider their structure trees T_A and T_B with roots ρ_A, ρ_B respectively. Build a new tree T with root ρ such that $D(\rho)$ contains two ramification points r and r' with $n+1$ branches at r, and $m+1$ branches at r', with special branches as shown in Figure 9. The resulting structure E will have E_r isomorphic to A and $E_{r'}$ isomorphic to B. □

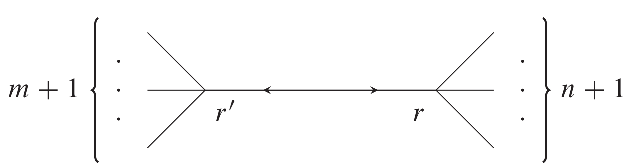

Figure 9

The class $\mathscr{D}$ is not closed under substructure (see Remark 3.4). However, the following theorem now follows from Lemmas 3.15 and 3.16 by a fairly standard version of Fraïssé's theorem (see for example [15, Theorem 2.10]). The approach is also described in [17], with the resulting Fraïssé limit described as *weakly homogeneous*.

Theorem 3.17. *There is a unique countable $\mathscr{L}$-structure M with the following properties.*

(i) *M is a union of a chain of members of $\mathscr{D}$.*

(ii) *Every member of $\mathscr{D}$ embeds in M.*

(iii) *If $A, E \in \mathscr{D}$ and $A < E$ and $f : A \to M$ is an embedding of $\mathscr{L}$-structures, then there is an embedding $g : E \to M$ which extends f.*

(iv) *Every isomorphism between substructures of M which lie in $\mathscr{D}$ extends to an automorphism of M.*

For the rest of this paper, M will denote the structure given by this theorem. We will refer to condition (iv) of the theorem as *semi-homogeneity* of M, and frequently just say that an automorphism exists, or that a tuple of M has a given finite extension in M, "by semi-homogeneity".

3.3 Oligomorphicity of M

In this section, we will show that the automorphism group of M is oligomorphic and hence by Theorem 2.1 (the Ryll-Nardzewski theorem) that M is ω-categorical.

To ensure that oligomorphicity of $\mathrm{Aut}(M)$ follows from Lemma 3.17, we need to eliminate situations such as the following. Suppose it happened that M has finite substructures E_i (for $i \in \mathbb{N}$) in the class $\mathscr{D}$, and suppose $|E_1| < |E_2| < |E_3| < \cdots$ and that E_i is a substructure of M of smallest size subject to lying in $\mathscr{D}$ and containing a_i, b_i. Then the pairs (a_i, b_i) all lie in distinct orbits of $\mathrm{Aut}(M)$ on M^2. Our next lemma eliminates this possibility. First we note the following lemma, a standard result easily proved by induction.

Lemma 3.18. *Let T be a graph-theoretic tree with n leaves, where $n \geq 3$. Then T has at most $n-2$ ramification points.*

Lemma 3.19. *Define $f : \mathbb{N} \to \mathbb{N}$ by*

$$f(n) = (n-2) + (n-2)(n-3) + \cdots + (n-2)(n-3)\cdots 2 = \sum_{i=1}^{n-3} \frac{(n-2)!}{i!}.$$

Then, for every finite $A \subset M$, there is $F \in \mathscr{D}$ with $A \leq F \leq M$ and $|F| \leq f(|A|)$.

Proof. By Theorem 3.17, A lies in a finite substructure E of M lying in $\mathscr{D}$. We aim to choose F inside E, of minimal size. Let ρ be the root of the structure tree of E, let $D^E(\rho)$ be the corresponding D-set, let $D^A(\rho)$ be the induced D-set structure on A, and let $\overline{D^E(\rho)}$, $\overline{D^A(\rho)}$ be the corresponding tree structures. Let $n := |A|$. We shall build F as the union of a finite sequence $A = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq E$. We may suppose that E is chosen minimally, that is, there is no proper substructure E' of E with $E' \in \mathscr{D}$ and $A \leq E' < E$.

Let $|A| = n$. We have $|\mathrm{Ram}(\overline{D^A(\rho)})| \leq n-2$ (by Lemma 3.18). We form F_1 by adding to A, for each ramification point r of $\overline{D^A(\rho)}$ such that the special branch of E at r contains no member of A, a member of that special branch. Then $|F_1| \leq |A| + n - 2$, and F_1 contains a special branch at each such ramification point r of $\overline{D^A(\rho)}$ and has no additional ramification points.

Next, for each such ramification point r of $\overline{D^A(\rho)}$, let σ be the corresponding successor in the structure tree of E. (We note here that, by minimality of E, it cannot happen that the elements of A all lie in distinct non-special branches at the same ramification point r of $\overline{D^E(\rho)}$, and thus indeed $|D^A(\sigma)| < |A| = n$.) There are at most $n-2$ such σ, and the D-set $D^E(\sigma)$ of E contains at most $n-1$ elements with representatives in A, giving a D-set $D^A(\sigma)$ of size at most $n-1$, so with at most $n-3$ ramification points. We build F_2 to ensure that it contains a special branch at each ramification point of $D^A(\sigma)$ for each σ. This requires adding at most $(n-2)(n-3)$ points to F_1, so $|F_2| \leq |F_1| + (n-2)(n-3)$.

We iterate this process. In order to build F_3 from F_2, we consider the at most $(n-2)(n-3)$ ramification points of F_2 (of D-sets of successors of ρ) and the corresponding $(n-2)(n-3)$ vertices λ of height 3 in the structure tree of E. Each D-set $D^E(\lambda)$ contains at most $(n-2)$ elements with representatives in A, so the corresponding D-set $D^A(\lambda)$ has at most $(n-4)$ ramification points.

Continuing this process, we find that, for F_i, each D-set of height i (where ρ has height 1) has at most $n-1-i$ ramification points and that, for $j \leq i$, each D-set of F_i at height j has a special branch at each ramification point. Thus, putting $F := F_{n-3}$, we find that F has a special branch at each ramification point of each D-set, so $F \in \mathscr{D}$. Finally, we see inductively that, for each i, $|F_i| = |F_{i-1}| + (n-2)(n-3)\ldots(n-(i+1))$. Thus $|F| \leq f(|A|)$, and the result follows. □

Corollary 3.20. *The structure M given in Theorem 3.17 is ω-categorical and has oligomorphic automorphism group.*

Proof. Since the language $\mathscr{L}$ is finite, there are finitely many members of $\mathscr{D}$ of any given finite size, and the result is immediate from Lemma 3.19 and Theorems 2.1 and 3.17. □

4 Analysing the Fraïssé limit

Throughout this section, we let M be the structure built in Section 3 and put $G = \mathrm{Aut}(M)$. In the previous section, we defined finite trees of D-sets. Here we show that M itself can be viewed as a "tree of D-sets". We have to construct the "structure tree" of M – in the language of model theory, we *interpret* it in M. It will be a dense semilinear order without maximal or minimal elements, so in particular, there will be no notion of "root" or of "successor". The vertices of the structure tree are labelled by classes of an equivalence relation on the set of triples from M satisfying L, with the equivalence relation determined by the relation symbol R. Vertices of the structure tree are labelled by corresponding D-sets. Given a vertex $\langle xyz \rangle$ of the structure tree, the elements of the corresponding D-set are equivalence classes of a further equivalence relation E_{xyz} defined on a subset J_{xyz} of M.

4.1 Automorphisms of M

In Lemma 4.2, we collect some basic symmetry properties of $G = \mathrm{Aut}(M)$. As the language $\mathscr{L}$ consists of six relations, it is convenient first to show that the relations L', S', Q, R are $\emptyset$-definable in M in terms of L, S.

Lemma 4.1. *The following statements hold:*

$$\begin{aligned}
M \models (\forall x, y, z, w)(L'(x; y, z; w) \\
&\leftrightarrow [L(x; y, z) \wedge L(w; y, z) \wedge L(w; x, z) \wedge L(w; x, y) \\
&\qquad \wedge \neg S(x, w; y, z)]), \qquad (4.1)\\
M \models R(x; y, z : p; q, s) \\
&\leftrightarrow [L(x; y, z) \wedge L(p; q, s) \\
&\qquad \wedge (\forall t)(L'(x; y, z; t) \Leftrightarrow L'(p; q, s; t))], \qquad (4.2)\\
M \models S'(x, y; z, w; t) \\
&\leftrightarrow \bigwedge_{u,v \in \{x,y,z,w\} \wedge u \neq v} R(t; x, y : t; u, v) \\
&\qquad \wedge \bigwedge_{u,v,s \in \{x,y,z,w\} \wedge L(u;v,s)} \neg R(t; x, y : u; v, s) \\
&\qquad \wedge S(x, y; z, w), \qquad (4.3)
\end{aligned}$$

$$\begin{aligned} M \models Q(x, y; z, w : p; q, s) \\ \leftrightarrow [S(x, y; z, w) \wedge L(p; q, s) \\ \wedge (\forall t)(S'(x, y; z, w; t) \Leftrightarrow L'(p; q, s; t))]. \end{aligned} \tag{4.4}$$

Proof. (4.1) ($\Rightarrow$) Suppose that $L'(x; y, z; w)$ holds in M. Pick a finite substructure $A \in \mathscr{D}$ such that $x, y, z, w \in A < M$. Then there is a D-set of A containing x, y, z, w with w lying in the special branch at $r = \mathrm{ram}(x, y, z)$, the ramification point (so all of x, y, z, w lie in distinct branches at the same ramification point r of this D-set). We may assume that this D-set is the root D-set $D(\rho)$ of A. So $L(w; y, z) \wedge L(w; x, z) \wedge L(w; x, y)$ are witnessed in this root D-set. Then the labelling D-set of the vertex $f_\rho^{-1}(r)$ (or one above it) witnesses $L(x; y, z)$ and omits w, and clearly $A \models \neg S(x, w; y, z)$, so $M \models \neg S(x, w; y, z)$.

($\Leftarrow$) In a finite structure $A \in \mathscr{D}$ with $x, y, z, w \in A < M$, suppose that

$$L(x; y, z) \wedge L(w; y, z) \wedge L(w; x, z) \wedge L(w; x, y) \wedge \neg S(x, w; y, z).$$

We aim to show $M \models L'(x; y, z; w)$. We may suppose (by choosing A as small as possible) that the root D-set $D(\rho)$ of A is the only one containing x, y, z, w as distinct elements, that is, in any higher D-set of A, either some of these will be omitted, or some element corresponds to a union of branches of the root D-set containing more than one of x, y, z, w.

Suppose first that $S(x, y; z, w)$ is witnessed in $D(\rho)$ (the argument is similar if $S(x, z; y, w)$ is witnessed in $D(\rho)$). Let $r_1 = \mathrm{ram}(x, y, z)$ and $r_2 = \mathrm{ram}(x, z, w)$. See Figure 10. Since $L(w; x, y)$, we see that x (and y) cannot be special at r_1. And since $L(x; y, z)$, we see that w cannot be special at r_1. Thus some other direction u (as depicted) must be special at r_1. Then, since z and w are identified in $f_\rho^{-1}(r_1)$, we cannot have $L(x; y, z) \wedge L(w; x, y)$, a contradiction.

Thus x, y, z, w all lie in different branches at the same ramification point r of $D(\rho)$. We may suppose further (by the minimality of the choice of A) that one of x, y, z, w is special at r. Since $L(w; y, z) \wedge L(w; x, z) \wedge L(w; x, y)$, this must

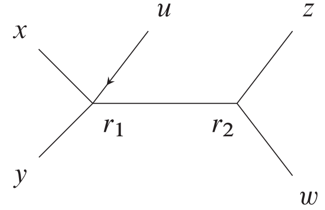

Figure 10

be w, with $L(x; y, z)$ witnessed in a higher D-set of A. Thus $A \models L'(x; y, z; w)$, so $M \models L'(x; y, z; w)$.

(4.2) ($\Rightarrow$) Suppose that $M \models R(x; y, z : p; q, s)$, and let $A \in \mathscr{D}$ be any finite substructure of M containing x, y, z, p, q, s. Then $A \models R(x; y, z : p; q, s)$, so from the way R was defined, $L(x; y, z)$ and $L(p; q, s)$ must be witnessed in the same D-set of A. It follows immediately that

$$A \models (\forall t)(L'(x; y, z; t) \Leftrightarrow L'(p; q, s; t)).$$

Since this holds for any such A, it holds in M.

($\Leftarrow$) Suppose that M satisfies

$$L(x; y, z) \wedge L(p; q, s) \wedge (\forall t)(L'(x; y, z; t) \Leftrightarrow L'(p; q, s; t)),$$

and let $A \in \mathscr{D}$ be a finite substructure of M containing x, y, z, p, q, s. Then, as $M \models L(p; q, s) \wedge L(x; y, z)$, these L-relations are witnessed in distinct comparable D-sets of A (here comparability is with respect to the structure tree ordering), or incomparable D-sets of A, or in the same D-set of A.

If $L(x; y, z)$ and $L(p; q, s)$ are witnessed in distinct comparable D-sets of A, say $L(p; q, s)$ below $L(x; y, z)$, then there is some $t \in A$ with

$$A \models L'(x; y, z; t) \wedge \neg L'(p; q, s; t),$$

a contradiction.

Suppose that $L(p; q, s)$ and $L(x; y, z)$ are witnessed in incomparable D-sets of A. Then we may suppose (replacing A by a substructure if necessary) that, in the root D-set $D(\rho)$ of A, there are distinct ramification points r_1 and r_2 such that x, y, z lie in distinct branches at r_1 and p, q, s lie in distinct branches at r_2. We now see that, for all possible choices of special branches at r_1 and r_2, $A \models (\exists t)\neg(L'(x; y, z; t) \Leftrightarrow L'(p; q, s; t))$, again a contradiction.

Thus $L(p; q, s)$ and $L(x; y, z)$ are witnessed in the same D-set of A, so we have $A \models R(x; y, z : p; q, s)$, and hence $M \models R(x; y, z : p; q, s)$, as required.

(4.3) ($\Rightarrow$) Assume $M \models S'(x, y; z, w; t)$, and let $A \in \mathscr{D}$ be a substructure of M containing x, y, z, w in distinct non-special branches of some ramification point r of the root D-set, and $t \in A$ in a special branch at r. As $A \models S'(x, y; z, w; t)$, there is a D-set of A witnessing $S(x, y; z, w)$ and omitting t. In particular, we have $M \models S(x, y; z, w)$.

In the root D-set of A, we see that $\bigwedge_{u,v \in \{x,y,z,w\} \wedge u \neq v} R(t; x, y : t; u, v)$ holds in A (and hence in M). Also, $L(t; x, y)$ is witnessed in the root D-set of A, which cannot witness $L(u; v, s)$ for $u, v, s \in \{x, y, z, w\}$. Thus

$$A \models \bigwedge_{u,v,s \in \{x,y,z,w\} \wedge L(u;v,s)} \neg R(t; x, y : u; v, s),$$

and hence this holds also in M.

($\Leftarrow$) Assume, for a contradiction, that the right-hand side holds and that

$$M \models \neg S'(x, y; z, w; t).$$

Then there is a finite $A \in \mathscr{D}$ with $x, y, z, w, t \in A \leq M$ and $A \models \neg S'(x, y; z, w; t)$. As $M \models S(x, y; z, w)$, there is a D-set of A witnessing $S(x, y; z, w)$ and containing t. Careful analysis of the possible positions of t, and possible choices of special branches, shows that

$$\bigwedge_{u,v \in \{x,y,z,w\} \wedge u \neq v} R(t; x, y : t; u, v) \wedge \bigwedge_{u,v,s \in \{x,y,z,w\} \wedge L(u;v,s)} \neg R(t; x, y : u; v, s)$$

cannot hold in A, so cannot hold in M.

(4.4) This is similar to (4.2). □

It follows from Lemma 4.1 that G is the automorphism group of the reduct of M to the language with just the relation symbols L and S.

Lemma 4.2. *The group G in its action on M is*

(i) 3-*homogeneous,*

(ii) 2-*transitive,*

(iii) *primitive,*

(iv) 2-*primitive.*

Proof. (i) Let $A = \{x, y, z\}$ and $A' = \{x', y', z'\}$ be 3-element subsets of M. Observe that the induced structures on A and A' lie in $\mathscr{D}$ since any 3-element substructure of any member of $\mathscr{D}$ lies in $\mathscr{D}$ (Lemma 3.2 (ii)), and M is a union of a chain of members of $\mathscr{D}$. By Lemma 3.2 (i), we have $A \models L\{x, y, z\}$ and $A' \models L\{x', y', z'\}$. Without loss of generality, assume $A \models L(x; y, z)$ and $A' \models L(x'; y', z')$. It is easily seen that the map $g : A \to A'$ with $(x, y, z)^g = (x', y', z')$ is an isomorphism. Hence, by Theorem 3.17, g extends to some element of G.

(ii) Suppose $x, y, x', y' \in M$ with $x \neq y$ and $x' \neq y'$. Let A be the induced structure on $\{x, y\}$, and A' that on $\{x', y'\}$. Then $A, A' \in \mathscr{D}$ by Lemma 3.2 (ii), and the map $g : A \to A'$ given by $(x, y)^g = (x', y')$ is an isomorphism. By Theorem 3.17, g extends to an element of G, as required.

(iii) This follows from (ii).

(iv) Since G is 2-transitive, it remains to check that, for $a \in M$, the group G_a is primitive on $M \setminus \{a\}$. Using semi-homogeneity, we see that if $M \models L(a; b, c)$, then G contains an element g with $(a, b, c)^g = (a, c, b)$. It follows using 3-homo-

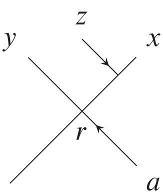

Figure 11

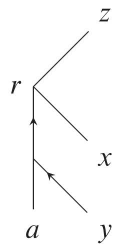

Figure 12

geneity that G_a has 3 orbits on ordered pairs of distinct elements from $M \setminus \{a\}$, namely

$$\{(x, y) : M \models L(a; x, y)\},$$
$$\{(x, y) : M \models L(x; a, y)\},$$
$$\{(x, y) : M \models L(y; a, x)\}.$$

We must show that there is no proper non-trivial G_a-congruence on $M \setminus \{a\}$. So, for the fixed a, it suffices to show the following are not equivalence relations.

(a) $E_a(x, y) \Leftrightarrow L(a; x, y) \vee x = y$. This relation is not transitive. Indeed, assume $L(a; x, y) \wedge L(a; y, z)$. Working in a finite substructure of M lying in $\mathscr{D}$, we may choose a, y, x to be in distinct branches distinct at a ramification point r with a in the special branch, and z lying in the same branch as x as in Figure 11. Therefore, $\neg L(a; x, z) \wedge x \neq z$, so E_a is not a transitive relation.

(b) $F_a(x, y) \Leftrightarrow L(x; a, y) \vee x = y$. This relation is not symmetric since we have $L(x; a, y) \to \neg L(y; a, x)$ (in any finite substructure and hence in M). The same argument also eliminates the relation $L(y; a, x) \vee x = y$.

(c) $F'_a(x, y) \Leftrightarrow L(x; a, y) \vee L(y; a, x) \vee x = y$. This is not transitive as, in the configuration in Figure 12, we have $F'_a(x, y) \wedge F'_a(y, z) \wedge \neg F'_a(x, z)$. □

4.2 Construction

In this section, we aim to recover a notion of structure tree for M, using the relations L and S. First, define $K^* := \{(x, y, z) \in M^3 : M \models L(x; y, z)\}$. Recall that if $A \in \mathscr{D}$, then $A \models R(x; y, z : p; q, s)$ if and only if $L(x; y, z)$ and $L(p; q, s)$ are witnessed in the same D-set of A. It follows that R defines an equivalence relation on K^*, and the structure tree of M will have universe K^*/R, with a semilinear order induced from substructures of M lying in $\mathscr{D}$ – see Lemma 4.10. We refer to the equivalence classes of R on K^* as *vertices* and denote the R-class containing (x, y, z) as $\langle xyz \rangle$.

We show next that each vertex $\langle pqs\rangle$ corresponds to a subset J_{pqs} of M and that there is a natural $G_{\langle pqs\rangle}$-invariant equivalence relation E_{pqs} on J_{pqs}. We then show in Lemma 4.7 that the quotient J_{pqs}/E_{pqs} carries a definable D-relation, thereby giving the claimed "tree of D-sets" structure for M.

Definition 4.3. Let $p, q, s \in M$ with $M \models L(p; q, s)$. Define

$$J_{pqs} := \{j : R(p; q, s : j; q, s) \vee R(p; q, s : p; j, s) \vee R(p; q, s : p; j, q)\}.$$

Lemma 4.4. *Let $x, y, z, p, q, s \in M$ with $M \models L(x; y, z) \wedge L(p; q, s)$. Then*

(i) $M \models R(x; y, z : p; q, s) \Leftrightarrow J_{xyz} = J_{pqs}$.

(ii) $J_{xyz} = J_{pqs} \Leftrightarrow \langle xyz\rangle = \langle pqs\rangle$.

Proof. (i) ($\Rightarrow$) Assume $M \models R(x; y, z : p; q, s)$. To show that $J_{xyz} \subseteq J_{pqs}$, let $b \in J_{xyz}$, so we want $b \in J_{pqs}$. There is a finite $\mathscr{L}$-substructure $A < M$ containing x, y, z, p, q, s, b with $A \in \mathscr{D}$. As R is a symbol of $\mathscr{L}$, $A \models R(x; y, z : p; q, s)$, so $L(x; y, z)$ and $L(p; q, s)$ are witnessed in the same D-set of A. But $b \in J_{xyz}$, so we may suppose that $R(x; y, z : b; y, z)$ holds (other cases are similar). Thus we may suppose $L(x; y, z)$ and $L(p; q, s)$ are witnessed in the root D-set of A. By considering possible configurations in A, we see

$$R(p; q, s : b; q, s) \vee R(p; q, s : p; b, s) \vee R(p; q, s : p; b, q),$$

so $b \in J_{pqs}$.

($\Leftarrow$) Assume, for a contradiction $J_{xyz} = J_{pqs}$ and $\neg R(x; y, z : p; q, s)$. Then there is a finite substructure $A \in \mathscr{D}$ such that $x, y, z, p, q, s \in A < M$. Since we have $A \models \neg R(x; y, z : p; q, s)$, $L(x; y, z)$ and $L(p; q, s)$ are witnessed in different D-sets of A. We suppose first that these D-sets are comparable, so (without loss of generality) there is $t \in A$ such that $A \models L'(x; y, z; t)$ and t lies in the D-set of A witnessing $L(p; q, s)$. We see easily that $t \in J_{pqs} \setminus J_{xyz}$. On the other hand, if $L(p; q, s)$ and $L(x; y, z)$ happen in two incomparable D-sets, then a lower D-set contains p, q, s in distinct branches at a ramification point, r say, with t in the special branch and x, y, z in distinct branches at another ramification point, r' say, with t' in the special branch. There are several possible configurations to consider, but in each case, we find $J_{xyz} \neq J_{pqs}$.

(ii) This is immediate from (i). $\square$

Definition 4.5. Define a relation E_{pqs} on J_{pqs} such that $uE_{pqs}v$ holds if and only if

$$\begin{aligned}(\forall m)(\forall n)[(R(p; q, s : m; n, u) &\leftrightarrow R(p; q, s : m; n, v))\\ \wedge (R(p; q, s : u; m, n) &\leftrightarrow R(p; q, s : v; m, n))].\end{aligned}$$

Observe that if $J_{xyz} = J_{x'y'z'}$, then $R(x; y, z : x'; y', z')$ by Lemma 4.4, and it follows that $E_{xyz} = E_{x'y'z'}$. Indeed, if $uE_{xyz}v$, then

$$(\forall m)(\forall n)[(R(x; y, z : m; n, u) \leftrightarrow R(x; y, z : m; n, v)) \\ \wedge (R(x; y, z : u; m, n) \leftrightarrow R(x; y, z : v; m, n))],$$

so (as R is an equivalence relation on K^*)

$$(\forall m)(\forall n)[(R(x'; y', z' : m; n, u) \leftrightarrow R(x'; y', z' : m; n, v)) \\ \wedge (R(x'; y', z' : u; m, n) \leftrightarrow R(x'; y', z' : v; m, n))],$$

so $uE_{x'y'z'}v$. Also, E_{xyz} is an equivalence relation on J_{xyz}, and is invariant under $G_{\{J_{xyz}\}}$.

Definition 4.6. We define a quotient structure as follows.

(i) Given J_{xyz} and E_{xyz}, define R_{xyz} to be the quotient J_{xyz}/E_{xyz}, so elements of R_{xyz} are E_{xyz}-classes of elements of M. We use the notation $[m]$ to refer to the element of R_{xyz} containing the element $m \in M$ (when the underlying equivalence relation E_{xyz} is clear). We call such objects $[m]$ *directions* when viewed as elements of R_{xyz}, and *pre-directions* when viewed as subsets of M. We shall refer to the subset J_{xyz} of M as a *pre-D-set*.

(ii) Let $[u], [v], [t], [s] \in R_{xyz}$. Write

$$D_{xyz}([u], [v]; [t], [s]) \iff ([u] = [v] \wedge [u] \notin \{[s], [t]\}) \\ \vee ([t] = [s] \wedge [t] \notin \{[u], [v]\}) \\ \vee Q(u, v; t, s : x; y, z).$$

By considering finite substructures, it can be checked that any such subset $[m]$ of M, as in (i) above, is a direction of a *unique* set R_{xyz}.

Lemma 4.7. *The following statements hold.*

(i) *The relation D_{xyz} is well-defined on R_{xyz}.*

(ii) *If $\langle xyz\rangle = \langle x'y'z'\rangle$, then $D_{xyz} = D_{x'y'z'}$.*

(iii) *The structure (R_{xyz}, D_{xyz}) is a dense proper D-set.*

(iv) *The relation D_{xyz} is $G_{\{J_{xyz}\}}$-invariant.*

Proof. (i) Suppose that $[u], [v], [s], [t] \in R_{xyz}$ are distinct, and $u' \in [u]$, $v' \in [v]$, $s' \in [s]$, and $t' \in [t]$. We must show $D_{xyz}(u, v; s, t) \leftrightarrow D_{xyz}(u', v'; s', t')$, so suppose $M \models D_{xyz}(u, v; s, t)$. We may assume $M \models Q(u, v; s, t : x; y, z)$, so must show $M \models Q(u', v'; s', t' : x; y, z)$. Choose any large $A \in \mathscr{D}$ with $A \leq M$ containing $x, y, z, u, u', v, v', s, s', t, t'$, so $A \models Q(u, v; s, t : x; y, z)$.

By the definition of E_{xyz} and using symmetry conditions on the variables in R, we have, in M and hence in A,

$$(\forall m \forall n)(R(x; y, z : u; m, n) \leftrightarrow R(x; y, z : u'; m, n)),$$
$$(\forall m \forall n)(R(x; y, z : m; v, n) \leftrightarrow R(x; y, z : m; v', n)),$$
$$(\forall m \forall n)(R(x; y, z : m; n, t) \leftrightarrow R(x; y, z : m; n, t')).$$

Consider the D-set $D(\nu)$ of A witnessing $S(u, v; s, t)$ and $L(x; y, z)$. If u, u' correspond to distinct leaves of $D(\nu)$, then there is some $w \in A$ so that $L\{u, u', w\}$ is witnessed in $D(\nu)$, say with $L(u; u', w)$ witnessed in $D(\nu)$. Thus we have $A \models R(x; y, z : u; u', w)$, so $A \models R(x; y, z : u'; u', w)$, which is clearly impossible. Thus u, u' correspond to the same leaf of $D(\nu)$, and likewise, v, v' correspond to the same leaf of $D(\nu)$, as do s, s', and also t, t'. It follows that

$$A \models Q(u', v'; s', t' : x; y, z),$$

so $M \models Q(u', v'; s', t' : x; y, z)$, as required.

(ii) As $\langle xyz \rangle = \langle x'y'z' \rangle$, we have $R(x; y, z : x'; y', z')$. So if $D_{xyz}(u, v; t, s)$, then $M \models Q(u, v; t, s : x; y, z)$, so any sufficiently large finite substructure A of M satisfies $R(x; y, z : x'; y', z') \wedge Q(u, v; t, s : x; y, z)$, so satisfies

$$Q(u, v; t, s : x'; y', z'),$$

so $M \models Q(u, v; t, s : x'; y', z')$. Hence $D_{x'y'z'}(u, v; t, s)$ holds.

(iii) We want to show that conditions (D1)–(D6) of Definition 2.5 hold. Axioms (D1), (D2), (D3), and (D4) follow immediately from corresponding conditions on S, inherited via Q. For (D5), suppose that $[u], [v], [t] \in R_{xyz}$ are distinct. Pick $A \in \mathscr{D}$ with $x, y, z, u, v, t \in A < M$. We may suppose that $L(x; y, z)$ is witnessed in the root D-set of A. By semi-homogeneity, A has a Type II (b) extension $A < A' = A \cup \{s\}$ such that $S(u, v; t, s)$ is witnessed in the root D-set of A', with $A' < M$. Then $M \models Q(u, v; t, s : x; y, z)$, and we have $D_{xyz}(u, v; t, s)$.

The argument is similar for (D6). Suppose

$$[u], [v], [t], [s] \in R_{xyz} \quad \text{with } D_{xyz}([u], [v]; [t], [s]),$$

and for convenience, we suppose them distinct. Pick

$$A \in \mathscr{D} \quad \text{with } x, y, z, u, v, t, s \in A < M.$$

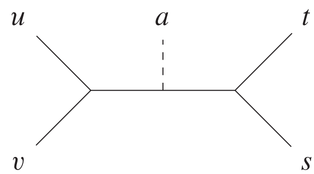

Figure 13

Then $A \models Q(u, v; t, s : x; y, z)$, and we may suppose $L(x; y, z)$ and $S(u, v; t, s)$ are witnessed in the root D-set of A. By semi-homogeneity A has a Type II (b) extension $A < A' = A \cup \{a\} < M$, as depicted in Figure 13.

Then

$$A \models S(a, v; t, s) \wedge S(u, a; t, s) \wedge S(u, v; a, s) \wedge S(u, v; t, a),$$

all witnessed in the root D-set of A, so in M, we have (putting $D = D_{xyz}$ and arguing via Q)

$$D([a], [u]; [t], [s]) \wedge D([u], [a]; [t], [s]) \wedge D([u], [v]; [a], [s]) \wedge D([u], [v]; [t], [a]),$$

as required.

(iv) This follows immediately from (ii) and Lemma 4.4 (ii). □

Lemma 4.8. *The following statements hold.*

(i) *If $L(p; q, s)$ holds in M, then there are $x, y, z, w \in M$ such that*

$$M \models Q(x, y; z, w : p; q, s).$$

(ii) *If $S(x, y; z, w)$ holds in M, then there are $p, q, s \in M$ such that*

$$Q(x, y; z, w : p; q, s).$$

Proof. (i) First observe that the induced $\mathscr{L}$-structure on $\{p, q, s\}$ lies in $\mathscr{D}$. Pick $A < M$ with $A \in \mathscr{D}$, containing distinct elements $p', q', s', x', y', z', w'$ such that the relations $L(p'; q', s')$ and $S(x', y'; z', w')$ are witnessed in the root D-set of A. Then $A \models Q(x', y'; z', w' : p'; q', s')$, so $M \models Q(x', y'; z', w' : p'; q', s')$. By 3-homogeneity (Lemma 4.2 (i)), there is $g \in G$ with $(p', q', s')^g = (p, q, s)$. Put $x := x'^g$, $y := y'^g$, $z = z'^g$, $w := w'^g$. Then $M \models Q(x, y; z, w : p; q, s)$, as required.

(ii) Similar to (i). □

The notions of *ramification point* and *branch* (at a ramification point), as introduced at the start of Section 3 for finite D-sets, make sense also for infinite D-sets, interpreted in the obvious way. We do not define them formally here, but refer to [6]. A D-set (R, D) determines a corresponding general betweenness relation (see [6, Theorem 25.3]), whose elements form the set $\mathrm{Ram}(R)$ of ramification points of R. If $r \in \mathrm{Ram}(R)$, then r corresponds to a *structural partition* of R (in the terminology of [6]), whose *sectors* are the branches at r (see [6, Section 24]).

We shall say that $L(p; q, s)$ is *witnessed* in R_{xyz} if $M \models R(x; y, z : p; q, s)$, and likewise that $S(u, v; t, s)$ is witnessed in R_{xyz} if $M \models Q(u, v; t, s : x; y, z)$. If $L(p; q, s)$ is witnessed in R_{xyz}, and the E_{xyz}-classes of p, q, s lie in distinct branches at the ramification point r, we say that the branch at r containing p/E_{xyz} is the *special branch* at r. Let $\mathrm{Ram}(R_{xyz})$ denote the set of ramification points of R_{xyz}. If $a_1, \ldots, a_n \in R_{xyz}$ (for $n \geq 3$) lie in distinct branches at $r \in \mathrm{Ram}(R_{xyz})$, we write $r = \mathrm{ram}(a_1, \ldots, a_n)$.

We next show that the set K^*/R carries a natural invariant partial order (a dense semilinear order without endpoints) and so plays the role of *structure tree* for M. The ordering is induced from finite substructures of M, using the next lemma. If $A \in \mathscr{D}$ with structure tree T_A and $x, y, z \in A$, then we write $\langle xyz \rangle_A$ for the vertex μ of T_A such that $D(\mu)$ witnesses $L(x; y, z)$. We also write J^A_{xyz} for the subset of A corresponding to $\langle xyz \rangle_A$ in A (the analogue of J_{xyz} in M); also, E^A_{xyz} denotes the equivalence relation on J^A_{xyz}, defined as in Definition 4.5 (so the E^A_{xyz}-classes are the leaves of $D(\mu)$).

Lemma 4.9. *Let $A, B \in \mathscr{D}$ with $A \leq B$, and $x, y, z, u, v, w \in A$ with*

$$A \models L(x; y, z) \wedge L(u; v, w).$$

(i) *We have $\langle xyz \rangle_A \leq \langle uvw \rangle_A$ if and only if $\langle xyz \rangle_B \leq \langle uvw \rangle_B$.*

(ii) *If $p, q \in A$, then*

$$p, q \in J^A_{xyz} \iff p, q \in J^B_{x,y,z} \quad \text{and} \quad pE^A_{xyz}q \iff pE^B_{xyz}q.$$

(iii) *If $A \leq M$ and $p, q \in A$, then*

$$p, q \in J^A_{xyz} \iff p, q \in J_{xyz} \quad \text{and} \quad pE^A_{xyz}q \iff pE_{xyz}q.$$

Proof. By Lemma 3.13, we may suppose that B is a one-point extension of A. Parts (i) and (ii) then hold by inspection of the different kinds of one-point extension, and (iii) follows from (ii). □

Since M is a union of a directed system of members of $\mathscr{D}$, it follows that there is a well-defined relation $\leq$ on K^*/R, with $\langle xyz \rangle \leq \langle uvw \rangle$ if and only if there is $A \in \mathscr{D}$ with $x, y, z, u, v, w \in A \leq M$ such that $\langle xyz \rangle_A \leq \langle uvw \rangle_A$. Since finite structure trees are lower semilinearly ordered, it follows that $(K^*/R, \leq)$ is also a lower semilinear order. The following lemma justifies us in viewing it as the structure tree of M. (As pointed out by Bradley-Williams and Truss, the definition of the corresponding ordering given in [8, Definition 5.2 (C)] – which in the notation here would be $\langle xyz\rangle \leq \langle uvw\rangle \Leftrightarrow J_{xyz} \supseteq J_{uvw}$ – is incorrect, but the version given in (i) below works in [8] too.)

Lemma 4.10. *The following statements hold.*

(i) *If $\langle uvw\rangle, \langle abc\rangle \in K^*/R$, then $\langle uvw\rangle \leq \langle abc\rangle$ if and only if*

$$\forall x, y \in J_{abc}(\neg xE_{abc}y \to (x, y \in J_{uvw} \wedge \neg xE_{uvw}y)).$$

In particular, $\leq$ on K^*/R is interpretable over $\emptyset$ in M and so is $\mathrm{Aut}(M)$*-invariant.*

(ii) *If $\langle xyz\rangle, \langle uvw\rangle \in K^*/R$ are comparable with respect to $\leq$, then we have $\langle xyz\rangle \leq \langle uvw\rangle \Leftrightarrow J_{xyz} \supseteq J_{uvw}$.*

(iii) *If $\langle xyz\rangle, \langle pqs\rangle$ are incomparable elements of K^*/R, then there is a vertex $\langle abc\rangle$ such that $\langle abc\rangle = \inf\{\langle xyz\rangle, \langle pqs\rangle\}$, so $(K^*/R, \leq)$ is a meet-semilattice.*

(iv) *The semilinear order $(K^*/R, \leq)$ has no maximal or minimal elements and is dense, that is, satisfies $\forall u\forall v(u < v \to \exists w(u < w < v))$.*

Proof. (i) For the left-to-right direction, suppose $\langle uvw\rangle \leq \langle abc\rangle$ and $x, y \in J_{abc}$ satisfy $\neg xE_{abc}y$. Let $A \in \mathscr{D}$ with $x, y, a, b, c, u, v, w \in A \leq M$. By the definition of $\leq$, we have $\langle uvw\rangle_A \leq \langle abc\rangle_A$, and by Lemma 4.9 (iii), $x, y \in J^A_{abc}$ with $\neg xE^A_{abc}y$. It follows easily that $x, y \in J^A_{uvw}$ with $\neg xE^A_{uvw}y$ and thus

$$x, y \in J_{uvw} \wedge \neg xE_{uvw}y.$$

For the converse, suppose that $\langle uvw\rangle$ and $\langle abc\rangle$ are incomparable in K^*/R. Pick $A \in \mathscr{D}$ with $a, b, c, u, v, w \in A \leq M$. Then $\langle abc\rangle_A$ and $\langle uvw\rangle_A$ are incomparable in the structure tree of A. If μ is the infimum of these vertices in T_A and $r = \mathrm{ram}(a, b, c)$ and $s = \mathrm{ram}(u, v, w)$ (in $\overline{D(\mu)}$), then $r \neq s$, so two of a, b, c, say a, b, are in the same branch at s. Thus $aE^A_{uvw}b$, and so $aE_{uvw}b$, but clearly, $a, b \in J_{abc}$ and $\neg aE_{abc}b$.

(ii) This follows easily from (i).

(iii) Choose $A \in \mathscr{D}$ with $x, y, z, p, q, s \in A \leq M$. Then $\langle xyz\rangle_A$ and $\langle pqs\rangle_A$ are incomparable by Lemma 4.9, and there are $a, b, c \in A$ with $L(a; b, c)$ such that $\langle abc\rangle_A = \inf\{\langle xyz\rangle_A, \langle pqs\rangle_A\}$. This property of being the infimum is preserved in one-point extensions of A and hence holds in M.

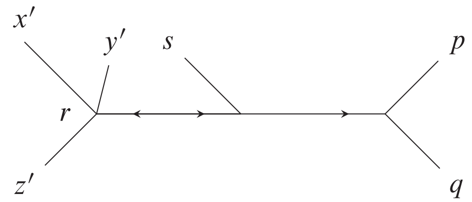

Figure 14. D_{ρ_A}

(iv) We first show that the semilinear order $(K^*/R, \leq)$ has no least element. Let $\langle xyz\rangle \in K^*/R$. Let B be a minimal substructure of M in $\mathscr{D}$ containing x, y, z, so $B = \{x, y, z\}$. Choose a structure $A \in \mathscr{D}$ containing x', y', z', p, q, s as depicted in Figure 14 (in the root D-set) so that $L(x'; y', z')$ is witnessed in the successor D-set $D(\mu)$ of the root $D(\rho)$ of A corresponding to the ramification point r, and p, q, s are as shown in $D(\rho)$. Let $\phi\colon A_r \to B$ be the isomorphism $(x', y', z') \mapsto (x, y, z)$.

We may suppose $A \leq M$. By semi-homogeneity, ϕ extends to some $g \in G$. Clearly, $\langle spq\rangle < \langle x'y'z'\rangle$, and it follows that $\langle s^g p^g q^g\rangle < \langle xyz\rangle$, as required. A similar argument shows that K^*/R has no greatest element under $\leq$.

The argument for density is a similar application of semi-homogeneity. Assume $\langle xyz\rangle < \langle pqs\rangle$. We may find a finite substructure A of M containing x, y, z, p, q, s such that $L(x; y, z)$ is witnessed in the root D-set $D(\rho)$ of A, which has a ramification point r at which p, q, s lie in distinct non-special branches. Using semi-homogeneity, we may suppose that there are $l, m, n \in A$ such that p, q, s, l, m, n all lie in distinct non-special branches at r, that $L(l; m, n)$ is witnessed in the successor $D(\mu)$ corresponding to r and that p, q, s lie in distinct non-special branches at a ramification point of $D(\mu)$. It follows that $\langle xyz\rangle_A < \langle lmn\rangle_A < \langle pqs\rangle_A$, and hence $\langle xyz\rangle < \langle lmn\rangle < \langle pqs\rangle$, as required. □

Our next task is to identify analogues for M of the maps f_μ and $g_{\mu\nu}$ for members of $\mathscr{D}$. The map $f_{\langle xyz\rangle}$ determines a bijection between the set of cones of $(K^*/R, \leq)$ at $\langle xyz\rangle$ and $\mathrm{Ram}(R_{xyz})$, and this is the context of the following lemma.

Lemma 4.11. *Suppose $x, y, z, p, q, s \in M$ and $\langle xyz\rangle < \langle pqs\rangle$.*

(i) *Then $E_{xyz}|_{J_{pqs}}$ refines E_{pqs}.*

(ii) *Let p, q, s lie in distinct branches at the ramification point r of R_{xyz}, and let $u, v \in J_{pqs}$ be E_{pqs}-inequivalent. Then u, v lie in distinct branches at r.*

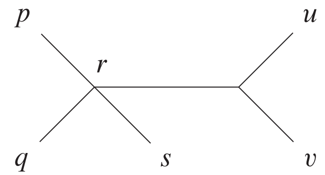

Figure 15

Proof. (i) This is immediate from Lemma 4.10 (i).

(ii) We prove the contrapositive, so suppose we have in R_{xyz} a diagram such as in Figure 15.

Using Q and R, we see that if $A \in \mathscr{D}$ with $x, y, z, p, q, s, u, v \in A \leq M$, then the D-set of A witnessing $L(x; y, z)$ has the same picture. Thus the D-set of A witnessing $L(p; q, s)$ has u, v in the same leaf, and it follows via the definition of E_{pqs} that $u E_{pqs} v$ holds. □

Now suppose $\langle xyz \rangle < \langle pqs \rangle$. Then, by the last lemma, p, q, s are inequivalent modulo E_{xyz}, so there is a ramification point r of R_{xyz} such that the E_{xyz}-classes of p, q, s lie in distinct branches at r. Put $f_{\langle xyz \rangle}(\langle pqs \rangle) = r$.

Lemma 4.12. *The following statements hold.*

(i) *In the above notation, the value of $f_{\langle xyz \rangle}(\langle pqs \rangle)$ depends only on the cone at $\langle xyz \rangle$ containing $\langle pqs \rangle$.*

(ii) *$f_{\langle xyz \rangle}$ determines a bijection between the set of cones at $\langle xyz \rangle$ and the set of ramification points of R_{xyz}.*

Proof. (i) We show the first part via the following three claims.

Claim 1. *If $a, b \in J_{pqs}$ are inequivalent modulo E_{pqs}, then they lie in distinct branches at r.*

Proof. This is immediate from Lemma 4.11 (ii). ◇

Claim 2. *If $\langle xyz \rangle < \langle pqs \rangle < \langle p'q's' \rangle$, then $f_{\langle xyz \rangle}(\langle pqs \rangle) = f_{\langle xyz \rangle}(\langle p'q's' \rangle)$.*

Proof. In this situation, p', q', s' are inequivalent modulo $E_{p'q's'}$ and hence modulo E_{pqs} (Lemma 4.11 (i)) so lie in distinct branches at r (by Claim 1). ◇

Claim 3. *If $\langle pqs \rangle$ and $\langle p'q's' \rangle$ are incomparable but in the same cone at $\langle xyz \rangle$, then $f_{\langle xyz \rangle}(\langle pqs \rangle) = f_{\langle xyz \rangle}(\langle p'q's' \rangle)$.*

Proof. Pick p'', q'', s'' with

$$\langle xyz \rangle < \langle p''q''s'' \rangle < \langle pqs \rangle \quad \text{and} \quad \langle xyz \rangle < \langle p''q''s'' \rangle < \langle p'q's' \rangle.$$

By Claim 2, $f_{\langle xyz \rangle}(\langle pqs \rangle) = f_{\langle xyz \rangle}(\langle p''q''s'' \rangle) = f_{\langle xyz \rangle}(\langle p'q's' \rangle)$. ◇

Part (i) follows.

(ii) To see that $f_{\langle xyz\rangle}$ is surjective, let $r \in \mathrm{Ram}(R_{xyz})$ and choose $p, q, s \in M$ such that, modulo E_{xyz}, they lie in distinct non-special branches at r. Then we have $\langle xyz\rangle < \langle pqs\rangle$ by considering finite substructures of M. It follows that $f_{\langle xyz\rangle}(\langle pqs\rangle) = r$.

For injectivity, suppose $\langle pqs\rangle, \langle p'q's'\rangle \in K^*/R$, with $\langle xyz\rangle < \langle pqs\rangle, \langle p'q's'\rangle$. Suppose that there is a finite $A \in \mathscr{D}$ with $x, y, z, p, q, s, p', q', s' \in A < M$ such that p, q, s and p', q', s' meet at the same ramification point in the D-set of A in which $L(x; y, z)$ holds. Then this holds in any $A' \in \mathscr{D}$ with $A < A' < M$ (e.g. consider a sequence of one-point extensions between A and A'). It follows by semi-homogeneity that there are $u, v, w \in M$ with

$$\langle xyz\rangle < \langle uvw\rangle, \quad \langle uvw\rangle < \langle pqs\rangle, \quad \langle uvw\rangle < \langle p'q's'\rangle,$$

so $\langle pqs\rangle$ and $\langle p'q's'\rangle$ lie in the same cone of K^*/R at $\langle xyz\rangle$. □

Lemma 4.13. *Let $\langle xyz\rangle < \langle pqs\rangle$, and let $[m]$ be a pre-direction of R_{pqs} and $r := f_{\langle xyz\rangle}(\langle pqs\rangle) \in \mathrm{Ram}(R_{xyz})$. Then there is a unique set t of branches of R_{xyz} at r such that $[m] = \bigcup\bigcup t$.*

Proof. To clarify the notation, observe that $\bigcup t$ is a set of directions of R_{xyz} and that $\bigcup\bigcup t$ is the corresponding subset of M.

Consider a finite structure $A \in \mathscr{D}$ with $x, y, z, p, q, s, m \in A \leq M$. Let μ, ν be the vertices of the structure tree of A whose D-sets witness $L(p; q, s)$ and $L(x; y, z)$ respectively, and let ρ label the root D-set of A, so $\rho < \nu < \mu$. Let r be the ramification point of $\overline{D(\nu)}$ corresponding to the cone at ν containing μ (so we use the same symbol r in M and A). Now m is a leaf of $D(\mu)$ and $g_{\mu\nu}(m) = \bigcup\{t_1, \ldots, t_n\}$ for some branches $t_1, \ldots, t_n$ of $D(\nu)$ at r. Each t_i is a set of leaves of $D(\nu)$, say

$$\begin{aligned} t_1 &= \{u_1^{(1)}, \ldots, u_{m_1}^{(1)}\} \\ &\vdots \\ t_n &= \{u_1^{(n)}, \ldots, u_{m_n}^{(n)}\}, \end{aligned}$$

so that

$$\bigcup_{i=1}^{n} t_i = t_1 \cup \cdots \cup t_n = \{u_1^{(1)}, \ldots, u_{m_1}^{(1)}, \ldots, u_1^{(n)}, \ldots, u_{m_n}^{(n)}\}.$$

Now the subset of A corresponding to m is $g_{\mu\rho}(m)$ which equals

$$\bigcup\{g_{\nu\rho}(u_1^{(1)}), \ldots, g_{\nu\rho}(u_{m_1}^{(1)}), \ldots, g_{\nu\rho}(u_1^{(n)}), \ldots, g_{\nu\rho}(u_{m_n}^{(n)})\}.$$

Since this holds for any $A' \in \mathscr{D}$ with $A < A' < M$, and branches of M at r arise as unions of branches of the corresponding finite structures, the result follows. $\square$

Define $g_{\langle pqs\rangle\langle xyz\rangle}([m]) = t$ where $[m] = \bigcup\bigcup t$ as above. So $g_{\langle pqs\rangle\langle xyz\rangle}$ is a map from the directions of R_{pqs} to the power set of the set of non-special branches at r. From the definition, we see that if $[m] \neq [m']$, then

$$g_{\langle pqs\rangle\langle xyz\rangle}([m]) \cap g_{\langle pqs\rangle\langle xyz\rangle}([m']) = \emptyset.$$

Lemma 4.14. *The map $g_{\langle xyz\rangle\langle pqs\rangle}$ depends only on $\langle xyz\rangle, \langle pqs\rangle$, not on the choice of x, y, z, p, q, s.*

Proof. The point essentially is that, in any finite structure

$$A \in \mathscr{D} \quad \text{with } x, y, z, p, q, s, m \in A < M,$$

the set t of branches depends just on the direction of m in the D-set witnessing $L(p; q, s)$ and on the map $g_{\mu\nu}$, where μ codes in A the D-set witnessing $L(p; q, s)$, and ν the D-set witnessing $L(x; y, z)$. $\square$

Proposition 4.15. *The following statements hold.*

(i) *The group $G_{\langle xyz\rangle}$ is transitive on* $\mathrm{Ram}(R_{xyz})$.

(ii) (a) *The stabiliser $G_{\langle xyz\rangle} = G_{\{J_{xyz}\}}$ induces a transitive group on the subset J_{xyz} of M.*

(b) *The group G is transitive on the semilinear order $(K^*/R, \leq)$.*

(iii) *The group $G_{\{J_{xyz}\}}$ induces a 2-transitive group on the set of directions of R_{xyz}, i.e. is transitive on the set of pairs of distinct directions.*

(iv) *The group G is transitive on the set $\mathscr{X}$, where $\mathscr{X} = \bigcup R_{xyz}$, the union of all the sets of directions in the structure M.*

(v) *The group $G_{\{J_{xyz}\}}$ is transitive on the set of non-special branches of R_{xyz}, and for each $r \in$ $\mathrm{Ram}(R_{xyz})$ *and branch U at r, the group $G_{\{J_{xyz}\},U}$ induces a transitive group on U.*

(vi) *The equivalence relation E_{xyz} is the unique maximal $G_{\{J_{xyz}\}}$-congruence on J_{xyz}.*

Proof. (i) Assume r, r' are two ramification points of R_{xyz} and x, y, z and p, q, s are triples lying in distinct branches at r, r' respectively with $L(p; q, s)$ witnessed in R_{xyz}. We want to find some

$$g \in G_{\langle xyz \rangle} \quad \text{such that} \quad r^g = r'.$$

Now $R(x; y, z : p; q, s)$ holds. By 3-homogeneity, there is

$$g \in G \quad \text{such that} \quad (x, y, z)^g = (p, q, s).$$

By Lemma 4.4, $J_{xyz} = J_{pqs}$ so $\langle xyz \rangle = \langle pqs \rangle$, so $g \in G_{\langle xyz \rangle}$, and g preserves the D-relation on R_{xyz} (Lemma 4.7 (iv)), so $r^g = r'$.

(ii) (a) Let $u \in J_{xyz}$, so

$$R(x; y, z : u; y, z) \vee R(x; y, z : x; u, z) \vee R(x; y, z : x; y, u)$$

holds. Let $r := \mathrm{ram}(x, y, z)$. To show the transitivity, we want to find $g \in G_{\langle xyz \rangle}$ such that $u^g = x$. There are three cases to consider.

Case (1). If u is in the same branch as x at r in R_{xyz}, then $L(x; y, z)$, $L(u; y, z)$ are witnessed in R_{xyz}, and hence $\langle xyz \rangle = \langle uyz \rangle$. By semi-homogeneity of G, there is $g \in G$ such that $(u, y, z)^g = (x, y, z)$. As $J_{xyz} = J_{uyz}$, by Lemma 4.4 (ii) we have $g \in G_{\langle xyz \rangle}$.

Case (2). If x, y, z, u are in distinct branches at a ramification point r, then

$$L(x; y, z) \wedge L(x; u, y) \wedge L(x; u, z)$$

holds, and by semi-homogeneity, there exists w in the same branch as u at r such that $L(u; w, z)$ is witnessed at $r' := \mathrm{ram}(u, w, z)$ in the same D-set R_{xyz}, i.e. $R(x; y, z : u; w, z)$ holds. Therefore, $L(x; y, z) \wedge L(u; w, z)$ holds, so there is $g \in G$ such that $(u, w, z)^g = (x, y, z)$. By Lemma 4.4, we have $g \in G_{\langle xyz \rangle}$.

Case (3). Suppose that u is in the same branch as z (the argument is the same if u is in the same branch as y). If u is special at $\mathrm{ram}(u, y, z)$, then there is some $g \in G$ fixing y, z and taking u to x, and as $R_{xyz} = R_{uyz}$, the D-set is fixed by g, so $g \in G_{\langle xyz \rangle}$. Otherwise, by semi-homogeneity, there is some w such that $L(u; w, y)$ is witnessed in R_{xyz}. Again, there is $g \in G$ with $(u, w, y)^g = (x, y, z)$, as required.

(ii) (b) Let $\langle xyz \rangle, \langle pqs \rangle \in (K^*/R, \leq)$. Then $M \models L(x; y, z) \wedge L(p; q, s)$, so by semi-homogeneity, there is $g \in G$ with $(p, q, s)^g = (x, y, z)$. Then we have $\langle pqs \rangle^g = \langle xyz \rangle$.

(iii) Let $[p] \neq [q]$ be distinct directions of R_{xyz} with

$$[p] = p/E_{xyz}, \quad [q] = q/E_{xyz},$$

and put $[x] = x/E_{xyz}$, $[y] = y/E_{xyz}$. It suffices to show there is $g \in G_{\{J_{xyz}\}}$ with $([x], [y])^g = ([p], [q])$. Choose $s \in M$ such that $R(x; y, z : p; q, s)$ holds – this exists by semi-homogeneity. Using 3-homogeneity (Lemma 4.2 (i)), there is $g \in G$ with $(x, y, z)^g = (p, q, s)$. Since $R(x; y, z : p; q, s)$ holds, g fixes J_{xyz} setwise, so g preserves E_{xyz}, and so it fixes R_{xyz} setwise, and clearly, we have $([x], [y])^g = ([p], [q])$.

(iv) This follows from (ii) (b) and (iii).

(v) Let U be the branch containing y at $r := \mathrm{ram}(x, y, z)$ in R_{xyz}, and let $L(p; q, s)$ be witnessed in R_{xyz}. Put $r' := \mathrm{ram}(p, q, s)$, and let V be the branch at r' containing q. It suffices to show that there is $g \in G_{\{J_{xyz}\}}$ with $V^g = U$. But this is immediate – we may choose any g with $(p, q, s)^g = (x, y, z)$, as exists by semi-homogeneity.

In order to prove the second assertion, with x, y, z, r, U as above, let $w \in U$. Then $L(x; w, z)$ is witnessed in R_{xyz}, and by semi-homogeneity, there is $g \in G$ with $(x, y, z)^g = (x, w, z)$. Clearly, g fixes U setwise.

(vi) Maximality of E_{xyz} follows immediately from 2-transitivity of $G_{\{J_{xyz}\}}$ on $R_{xyz} = J_{xyz}/E_{xyz}$, and this was proved in (iii).

It remains to prove that E_{xyz} is the *unique* maximal $G_{\{J_{xyz}\}}$-congruence. To see this, suppose E^* is a $G_{\{J_{xyz}\}}$-congruence on J_{xyz} and $E^* \not\subset E_{xyz}$. Without loss of generality, we may suppose xE^*y. Let $x' \in J_{xyz}$ with $xE_{xyz}x'$. Then $L(x; y, z) \wedge L(x'; y, z)$. It follows by semi-homogeneity that there is a $g \in G$ with $(x, y, z)^g = (x', y, z)$. Then $J^g_{xyz} = J_{xyz}$, and as $y^g = y$, g fixes $E^*(y)$ setwise, so as xE^*y, we have $x'E^*y$. Thus $x/E_{xyz} \subset E^*(y)$. Hence $E_{xyz} \subset E^*$, and it follows that E^* is universal, as required. □

5 Proof of the main theorem

In this section, we prove that $G = \mathrm{Aut}(M)$ is a Jordan group preserving a limit of D-relations. The main work is in Section 5.1, where we first show that every pre-direction is a Jordan set and then use Lemma 2.3 to identify other Jordan sets. We then show that G does not preserve on M any structure of types (i)–(iii) in Theorem 1.1. Finally, in Section 5.2, we prove that G satisfies the requirements of Definition 2.6 to obtain our main result, Theorem 1.2.

5.1 Finding a Jordan set

Recall first that $T = (K^*/R, \leq)$ is a lower semilinear order and meet semilattice. We refer to it as the *structure tree* of M.

Definition 5.1. A subset $\hat{U}$ of M is said to be a *pre-branch* if there are $x, y, z \in M$ with $L(x; y, z)$ and a branch U in R_{xyz} such that

$$\hat{U} = \{u \in M : [u] \in U\} = \bigcup\{[u] : [u] \in U\},$$

that is, $\hat{U}$ is the union of all E_{xyz}-classes in the single branch U at some ramification point r of the D-set (R_{xyz}, D_{xyz}). In this situation, we say $\hat{U}$ is a pre-branch at the ramification point r.

Given a D-set R_i of M, we put $\hat{R}_i = \bigcup R_i$, the union of the pre-directions (see Definition 4.6) corresponding to directions of R_i, so $\hat{R}_i \subset M$, and in the notation of Section 4, $\hat{R}_{xyz} = J_{xyz}$.

Fix a direction $[n]$ of M (so $n \in M$). Let $\langle pqn \rangle$ be the unique vertex of the structure tree of M, whose D-set R_{pqn} has $[n]$ as a direction (the uniqueness is noted after Definition 4.6); then $p, q \notin [n]$. Note that $L(p; q, n)$ is witnessed in this D-set. Define

$$I := \{i \in T : i < \langle pqn \rangle\},$$

and for each $i \in I$, let R_i be the D-set indexed by i. Let D_i denote the corresponding D-relation D_{xyz}, where $R_i = R_{xyz}$. Then I carries a total order $<$ induced from T, where $i < j \Leftrightarrow \hat{R}_j \subset \hat{R}_i$. For each $i \in I$, let $r_i = f_i(\langle pqn \rangle)$, the ramification point of R_i corresponding to the cone (of the structure tree) at i containing $\langle pqn \rangle$. By Lemma 4.14, there is a set S_i of branches at r_i such that $g_{\langle pqn \rangle i}([n]) = \bigcup \bigcup S_i$.

Since our goal is to show that $[n]$ is a Jordan set for G, we consider the induced structure on $[n]$, viewed as a pre-direction, i.e. as a subset of M. First, for each $i \in I$, there is an equivalence relation F_i on $[n]$ defined by

$$d_1 F_i d_2 \iff d_1, d_2 \text{ lie in the same pre-branch of } \hat{R}_i \text{ at } r_i.$$

Also, for each $i \in I$, let E_i be the equivalence relation E_{xyz} (restricted to $[n]$), where $R_i = R_{xyz}$.

Lemma 5.2. *Let $i, j \in I$ with $i < j$. Then $E_i \subset F_i \subset E_j \subset F_j$.*

Proof. Take a particular pre-branch at r_i in $\hat{R}_i$ lying in $[n]$, say $\hat{U}_i$. By the definition of the relation F_i, the pre-branch $\hat{U}_i$ is an F_i-class. It is clear that the relation of being in the same pre-direction of $\hat{R}_i$ refines F_i, so $E_i \subset F_i$. Similarly, we have $E_j \subset F_j$. To show that $F_i \subset E_j$, we see by Lemma 4.13 that if $[m]$ is a pre-direction for some R_j, where $j \in I$ with $j > i$, then $[m]$ is a union of pre-branches of $\hat{R}_i$ at r_i. □

Lemma 5.3. *Given an F_i-class $\hat{U}_i$, the intersection of the E_j-classes containing $\hat{U}_i$ (for $j > i$) is just $\hat{U}_i$.*

Proof. We want to show that $F_i = \bigcap_{j>i} E_j$. It is clear from Lemma 5.2 that $F_i \subseteq \bigcap_{j>i} E_j$. Conversely, suppose $u, v \in [n]$ with $\neg u F_i v$. We want to find $j \in I$ with $j > i$ such that $\neg u E_j v$. Let $a \in M$ lie in the special branch of R_i at r_i (so $a \notin [n]$). Consider a finite structure $A \in \mathcal{D}$ containing elements $a', u', v', w', s', t', p', q', n'$ in distinct branches at a ramification point r at the root D-set (with a' special) such that there is an $\mathcal{L}$-isomorphism

$$(a', u', v', p', q', n') \to (a, u, v, p, q, n).$$

We choose A so that, also in a higher D-set $D(\nu)$ (below the one $D(\mu)$ witnessing $L(p'; q', n')$), we have $L(w'; s', t')$, with u', v', w', s', t' again in distinct branches at the ramification point determined by the cone at ν containing μ. We may suppose $A \leq M$. By semi-homogeneity, there is $g \in G$ with

$$(a', u', v', p', q', n')^g = (a, u, v, p, q, n).$$

The relation $L(w'^g; s'^g, t'^g)$ will be witnessed in a D-set R_j with $j > i$, and we have $\neg u E_j v$. (The role of p, q, p', q' here is to ensure $j \in I$, that is, $j < \langle pqn \rangle$.) □

Lemma 5.4. *Let $u, v_1, \ldots, v_m$ be distinct elements of $[n]$. Then there is a greatest $i \in I$ such that u is E_i-inequivalent to each of $v_1, \ldots, v_m$, and for such i, the element u will be F_i-equivalent to at least one v_j with $j \in \{1, \ldots, m\}$.*

Proof. Find $i_0 \in I$ with $i_0 < \langle pqn \rangle$ containing elements $p, q, u, v_1, \ldots, v_m$ all lying in distinct branches at the ramification point r_{i_0} of the D-set R_{i_0}. Consider finite $A \leq M$ with $A \in \mathcal{D}$ and $p, q, u, v_1, \ldots, v_m$ lying in distinct branches at a ramification point of the root D-set.

By considering the structure of A, we see that there is i with $i_0 < i < \langle pqn \rangle$ such that, at r_i, u is in the same branch as at least one of the v_i, but in a distinct direction to each. (Working in A, consider the D-sets in the structure tree between the root and the D-set R_{pqn}, and the corresponding ramification points; there will be a least D-set such that u lies in the same branch as some v_j at the relevant ramification point.) □

Definition 5.5. For each $i \in I$, define a relation C_i on $\bigcup S_i$ (so on the set of directions of R_i lying in the branches of S_i) as follows: if $[x], [y], [z] \in \bigcup S_i$, then $C_i([z]; [x], [y]) \leftrightarrow D_i([x], [y]; [z], [w])$ for any direction $[w]$ of R_i lying outside $\bigcup S_i$.

It is easily seen that, for each $i \in I$, C_i induces a C-relation on each F_i-class of $[n]$ (considered as a branch at r_i, i.e. modulo E_i). Furthermore, C_i is invariant under $G_{(M \setminus \bigcup\bigcup S_i)}$, or under the subgroup of G stabilising both $[n] \subset M$ and $\hat{R}_i$ setwise.

Lemma 5.6. *Let g be a permutation of M which is the identity on $M \setminus [n]$ and, for each $i \in I$, preserves the equivalence relation E_i, the relations L and S on $[n]$, and the C-relation induced by C_i on each F_i-class of $[n]$ (modulo E_i). Then $g \in G$.*

Proof. By Lemma 5.3 and the assumption that g preserves the relations E_i, it follows that g preserves each $F_i|_{[n]}$ and hence each F_i. By Lemma 4.1, it is enough to show that g preserves L and S on M. Below, in Part (A), we show that g preserves L, and in Part (B), that it preserves S.

Part (A). To prove that $g \in G$ preserves L, we consider four cases.

Case (I). If $x, y, z \notin [n]$, then as g is the identity on $M \setminus [n]$, we have

$$L(x; y, z) \leftrightarrow L(x^g; y^g, z^g),$$

and likewise for the other orderings of $\{x, y, z\}$.

Case (II). Let $x \in [n]$, $y, z \in M \setminus [n]$. Let R be the D-set in which $L\{x, y, z\}$ is witnessed with x, y, z lying in distinct branches at the ramification point r of R. We need to show that the map $xyz \mapsto x^g yz$ preserves L. We will consider the possible cases based on where the D-set R witnessing $L\{x, y, z\}$ could be.

Sub-case (1). Assume that the D-set R is $R_{\langle pqn\rangle}$, and let $L\{x, y, z\}$ hold, witnessed in R. Now x, x^g lie in the same element of R, and y, z lie in two other distinct elements of R, fixed by g. It is therefore immediate that $xyz \mapsto x^g yz$ preserves L.

Sub-case (2). Assume that the D-set $R = R_i$ is lower than $R_{\langle pqn\rangle}$ (so $i \in I$). Now $L(x; y, z)$ cannot be witnessed in this D-set at r_i because $x \in [n]$, so x cannot lie in the special branch at r_i. However, possibly $L(y; x, z)$ is witnessed at r_i (likewise $L(z; x, y)$), and then, since $x^g \in \bigcup\bigcup S_i$ (as $x^g \in [n]$), the relation $L(y; x^g, z)$ holds.

Suppose that, as in Figure 16, $L(x; y, z)$ holds in R_i with x special at another ramification point r_i' which is not a ramification point of directions in $\bigcup S_i$. Then, again, because $x^g \in \bigcup\bigcup S_i$ and since x and x^g lie in the same branch at r_i', we get $L(x^g; y, z)$. Similarly, if $L(y; x, z)$ holds at r_i' (likewise for $L(z; x, y)$), then x and x^g will be in the same branch at r_i', and it is readily seen that $L(y; x^g, z)$ holds. It cannot happen that $L\{x, y, z\}$ is witnessed at r_i' within $\bigcup\bigcup S_i$ since y, z are not in $\bigcup\bigcup S_i$, so would lie in the same branch at such r_i'.

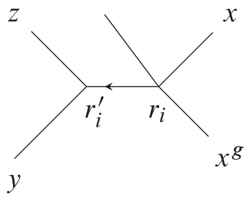

Figure 16

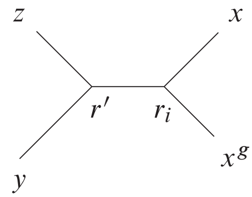

Figure 17

Sub-case (3). Assume that the D-set R is higher than $R_{\langle pqn\rangle}$. Now the direction containing x in R contains the whole of $[n]$, so is fixed by g, as are y and z. It follows that x, y, z and x^g, y, z satisfy the same L-relation.

Sub-case (4). Suppose the D-set R corresponds to the vertex k of the structure tree with k incomparable with $\langle pqn\rangle$. Let $i = \inf\{\langle pqn\rangle, k\}$, so $i \in I$. We may suppose that the cone of k at i (in the structure tree) corresponds to the ramification point r' of R_i; then $r' \neq r_i$. Since $L\{x, y, z\}$ is witnessed in R, x, y, z lie in distinct non-special branches at r'. Hence, as $y, z \notin \bigcup\bigcup S_i$, it follows that r' cannot be a ramification point of $\bigcup S_i$, and we have, for example, Figure 17 in R_i. Now, x, x^g lie in the same branch at r', so in the same pre-direction of R, so the same L-relation holds among x, y, z and x^g, y, z.

Case (III). Let $x, y \in [n]$ and $z \in M \setminus [n]$.

Sub-case (1). Suppose that the D-set R is $R_{\langle pqn\rangle}$. The relation $L\{x, y, z\}$ is not witnessed in $R_{\langle pqn\rangle}$ because x and y are in the same direction in $R_{\langle pqn\rangle}$.

Sub-case (2). Suppose that the D-set R is lower than $R_{\langle pqn\rangle}$, say $R = R_i$ for some $i \in I$. If $\neg x F_i y$, then the relation $L(x; y, z)$ or $L(y; x, z)$ cannot be witnessed at r_i because neither x nor y can be special at r_i. If $L(z; x, y)$ holds at r_i, then $L(z; x^g, y^g)$ is witnessed in R_i (for x^g, y^g are in distinct branches at r_i because F_i is preserved on $[n]$).

If $x F_i y$ and $L(x; y, z)$ is witnessed at r (Figure 18), then we want to see that $L(x^g; y^g, z)$ holds (and likewise if $L(y; x, z)$ holds). There is $t \in [n]$ such that $t F_i x \wedge t F_i y$ and $L(x; y, t)$ holds. Since g preserves L on elements of $[n]$, we get $L(x^g; y^g, t^g)$, and then

$$L(x; y, z) \leftrightarrow L(x; y, t) \quad \text{and} \quad L(x^g; y^g, z^g) \leftrightarrow L(x^g; y^g, t^g)$$

as g preserves C_i. Also, $L(x; y, t) \leftrightarrow L(x^g; y^g, t^g)$ as $x, y, t \in [n]$, so

$$L(x; y, z) \leftrightarrow L(x; y, t) \leftrightarrow L(x^g; y^g, t^g) \leftrightarrow L(x^g; y^g, z^g).$$

Sub-case (3). Suppose that the D-set R is higher than $R_{\langle pqn\rangle}$. Then $L\{x, y, z\}$ cannot be witnessed in R because x, y are in the same direction of R.

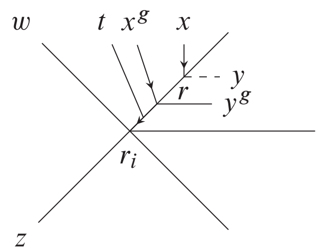

Figure 18

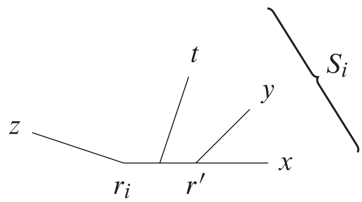

Figure 19

Sub-case (4). Assume R is the D-set of the vertex k incomparable with $\langle pqn \rangle$, and put $i = \inf\{k, \langle pqn \rangle\}$. Then the cone of k at i corresponds to a ramification point r' of R_i distinct from r_i, and as x, y, z lie in distinct branches of R_i at r', we must have that r' is a ramification point of S_i, as in Figure 19.

Choose t as depicted, in the same branch as z at r' and the same branch as x at r_i. As g preserves L on $[n]$ and the C-relation on F_i-classes (considered modulo E_i), we have

$$L(x; y, z) \iff L(x; y, t) \iff L(x^g; y^g, t^g) \iff L(x^g; y^g, z),$$

and likewise for other permutations of $\{x, y, z\}$.

Case (IV). If $x, y, z \in [n]$, then $L(x; y, z) \leftrightarrow L(x^g; y^g, z^g)$ follows immediately by the hypothesis that g preserves L on $[n]$.

Part (B). To prove that g preserves S, we argue as in Part (A). □

Lemma 5.7. *Each pre-direction* $[n]$ *is a Jordan set of* G.

Proof. To show this, we define a group $K \leq G$ which is transitive on $[n]$ and fixes the complement $M \setminus [n]$. We construct K as an iterated wreath product of groups of automorphisms of C-relations. The argument is similar to the proof of [8, Proposition 5.6], but there is an imprecision there: the map χ below is not defined precisely in [8], leading to problems with the proof of [8, Claim 8]. The approach given here works in [8] too.

Write $[n] = \{u_i : i \in \omega\} \subset M$. For each $u \in [n]$ and $i \in I$, put

$$[u]_i := \{x \in M : x F_i u\}$$

(the pre-branch of M at r_i containing u), and define $A^i(u) := [u]_i/E_i$ (the branch at r_i containing u). In particular, for each $i \in I$, let $V_i := [u_0]_i/E_i$, so we have $V_i = A^i(u_0)$. Let $e_i := u_0/E_i \in V_i$, where u_0/E_i denotes the E_i-class of u_0.

Define

$$\Omega := \left\{ f : I \to \bigcup_{i \in I} V_i \,:\, f(i) \in V_i \text{ for all } i,\ \mathrm{supp}(f) \text{ finite} \right\},$$

where $\mathrm{supp}(f) = \{i \in I : f(i) \neq e_i\}$.

We aim to find a system of maps $\phi^i_V : V \to V_i$, where $i \in I$ and V ranges through branches $A^i(u)$ for $u \in [n]$. Given such maps, define $\chi : [n] \to \Omega$ by

$$\chi(u)(i) = \phi^i_{A^i(u)}(u/E_i) \quad \text{for all } u \in [n] \text{ and } i \in I.$$

We shall define the maps ϕ^i_V so that χ is a bijection. The definition of χ is inductive, done in parallel with the definition of the ϕ^i_V. As the base case, define $\chi(u_0)$ so that $\chi(u_0)(i) = e_i$ for all $i \in I$.

Suppose that $\chi(u_0), \ldots, \chi(u_{k-1})$ have been defined. We may suppose that each map $\phi^i_{A^i(u_l)}$ has been defined for all $l < k$ and all $i \in I$.

Let $i(k)$ be the largest $i \in I$ such that u_k is E_i-inequivalent to u_l for each $l < k$. This exists by Lemma 5.4, and by that lemma, there is some $l < k$ such that $u_l F_{i(k)} u_k$, so $u_l F_i u_k$, that is, $A^i(u_l) = A^i(u_k)$ for all $i \geq i(k)$. Now, by assumption, $\phi^i_{A^i(u_l)}$ has been defined for all $i \in I$, so $\phi^i_{A^i(u_k)}$ has been defined for all $i \geq i(k)$, but not for $i < i(k)$. For $i < i(k)$, choose $g_i \in G$ such that

$$(A^i(u_k))^{g_i} = V_i \quad \text{and} \quad (u_k/E_i)^{g_i} = e_i$$

(this exists since, by Proposition 4.15 (v), G is transitive on the set of non-special branches and induces a transitive group on each branch). Then put

$$\phi^i_{A^i(u_k)}(u/E_i) = (u/E_i)^{g_i}$$

for all $i < i(k)$ and u with $u F_i u_k$. Observe that the maps $\phi^i_{A^i(u_l)}$ are now defined for all $l \leq k$ and all $i \in I$.

Claim 1. *With the maps $\phi^i_{A^i(u)}$ so defined, we have $\chi(u_k) \in \Omega$ for each $k \in \omega$.*

Proof. This is by induction on k. It is immediate that $\chi(u_0) \in \Omega$, so assume it holds for all $l < k$. By construction, as $\phi^i_{A^i(u_k)}$ is a bijection $[u_k]_i/E_i \to V_i$, we have $\chi(u_k)(i) \in V_i$. We must show $\mathrm{supp}(\chi(u_k))$ is finite. There is $l < k$ such that, for $i \geq i(k)$, $\chi(u_k)(i) = \chi(u_l)(i)$, so

$$\mathrm{supp}(\chi(u_k)) \cap \{j \in I : j \geq i(k)\} = \mathrm{supp}(\chi(u_l)) \cap \{j \in I : j \geq i(k)\},$$

so by induction is finite. By construction, $\chi(u_k)(i) = e_i$ for all $i < i(k)$, and the claim follows. $\diamond$

Claim 2. *χ: $[n] \to \Omega$ is a bijection.*

Proof. To show that χ is injective, suppose $l < k$. We must show $\chi(u_l) \neq \chi(u_k)$. Pick i such that $u_k F_i u_l, \neg u_k E_i u_l$. Then $[u_k]_i = [u_l]_i$, but $[u_k]_i/E_i \neq [u_l]_i/E_i$, so as $A^i(u_k) = A^i(u_l)$,

$$\chi(u_k)(i) = \phi^i_{A^i(u_k)}(u_k/E_i) \neq \phi^i_{A^i(u_l)}(u_l/E_i) = \chi(u_l)(i).$$

To see surjectivity, suppose for a contradiction that χ is not surjective, and let $f \in \Omega \setminus \mathrm{Range}(\chi)$ have minimal support, with $\mathrm{supp}(f) = \{i_1, \ldots, i_t\}$, where $i_1 < \cdots < i_t$. Define $f' \in \Omega$, where $f'(i_1) = e_{i_1}$, and $f'(j) = f(j)$ for all $j \neq i_1$.

By minimality of $\mathrm{supp}(f)$, there is $u \in [n]$ with $\chi(u) = f'$. Let $v = f(i_1) \in V_{i_1}$, and let k be least such that u_k lies in the E_{i_1}-class $(\phi^{i_1}_{A^{i_1}(u)})^{-1}(v)$.

To obtain a contradiction and thereby to prove surjectivity, it suffices to prove $\chi(u_k) = f$. Certainly,

$$\chi(u_k)(i_1) = \phi^{i_1}_{A^{i_1}(u_k)}(u_k/E_{i_1}) = v = f(i_1).$$

For $j > i_1$, we have $\chi(u_k)(j) = \phi^j_{A^j(u_k)}(u_k/E_j) = \phi^j_{A^j(u)}(u/E_j) = f(j)$. Also, $i(k) \geq i_1$, for otherwise, there is $l < k$ such that $u_l E_{i_1} u_k$, contradicting minimality of k. Hence $\chi(u_k)(j) = \phi^j_{A^j(u_k)}(u_k/E_j) = (u_k/E_j)^{g_j} = e_j = f(j)$ for all $j < i_1$, so indeed, $\chi(u_k)(j) = f(j)$ for all j. $\diamond$

For each $i \in I$, let H_i be the group induced by $G_{\{V_i\}}$ on V_i. For each triple (i, g, h), where $i \in I$, g: $(i, \infty) \to \bigcup_{j>i} V_j$ with $g(j) \in V_j$ for all j, and $h \in H_i$, define the function $x(i, g, h)$: $\Omega \to \Omega$ as follows:

$$f^{x(i,g,h)}(j) = \begin{cases} f(i)^h & \text{if } j = i \text{ and } f|_{(i,\infty)} = g, \\ f(j) & \text{otherwise.} \end{cases}$$

Now define K, the *generalised wreath product*, to be the subgroup of $\mathrm{Sym}(\Omega)$ generated by permutations $x(i, g, h)$ with i, g, h as above. By [16, Lemma 1], the group K is transitive on Ω. Thus K has an induced transitive action on $[n]$, given by $u^x = \chi^{-1}((\chi(u))^x)$ for all $x \in K$, $u \in U$. (Note that we use Cameron's notation for the permutation group K, as was also used in [8].) We extend this action to the whole of M by putting $v^x = v$ for all $v \notin [n]$.

Claim 3. *In this action, K is a subgroup of* $\mathrm{Aut}(M)$.

Proof. It suffices to show that elements $x(i, g, h)$ as above are automorphisms of M, and for this, we use Lemma 5.6. First, observe the following sub-claim.

Sub-claim 1. *For* $u, v \in [n]$, *and* $i \in I$, $uE_iv \Leftrightarrow \chi(u)(j) = \chi(v)(j)$ *for all* $j \geq i$.

Proof. If uE_iv, then $A^j(u) = A^j(v)$ for all $j \geq i$, so

$$\chi(u)(j) = \phi^j_{A^j(u)}(u/E_j) = \phi^j_{A^j(v)}(v/E_j) = \chi(v)(j)$$

for all $j \geq i$. Conversely, if $\neg uE_iv$, then by Lemma 5.4, there is $j \geq i$ such that uF_jv and $\neg uE_jv$. Then $A^j(u) = A^j(v)$, so

$$\chi(u)(j) = \phi^j_{A^j(u)}(u/E_j) \neq \phi^j_{A^j(v)}(v/E_j) = \chi(v)(j),$$

as required. $\triangleleft$

Since $x(i', g, h)$ acts as a permutation in the single coordinate i', in its action on Ω, it is clear that, for $u, v \in [n]$ and $i \in I$, we have $\chi(u)(j) = \chi(v)(j)$ for all $j \geq i$ if and only if $\chi(u)^{x(i',g,h)}(j) = \chi(v)^{x(i',g,h)}(j)$ for all $j \geq i$. Thus uE_iv if and only if $u^{x(i',g,h)}E_iv^{x(i',g,h)}$, so the maps $x(i', g, h)$ preserve all the equivalence relations E_i. Thus, by Lemma 5.3, the maps $x(i', g, h)$ also preserve all the F_i.

For $u, v, w \in [n]$, put

$$\sigma(u, v, w) = \mathrm{Max}\{i : u/E_i, v/E_i, w/E_i \text{ are all distinct}\},$$
$$\mu(u, v, w) = \mathrm{Max}\{i : u/E_i, v/E_i, w/E_i \text{ are not all equal}\}.$$

Then $\mu(u, v, w) \geq \sigma(u, v, w)$, and $\mu(u, v, w) = \sigma(u, v, w)$ if and only if there is i (namely $\sigma(u, v, w)$) such that u, v, w are F_i-equivalent but not E_i-equivalent.

Suppose $\mu(u, v, w) = \sigma(u, v, w) = i$. Let C_i be as in Definition 5.5 with the invariance properties noted there, and note that C_i induces a C-relation on V_i. Then, since the map $\phi^i_{A^i(u)}$ is induced by an element of G, we have

$$C_i(u/E_i; v/E_i, w/E_i) \leftrightarrow C_i(\phi^i_{A^i(u)}(u/E_i); \phi^i_{A^i(v)}(v/E_i), \phi^i_{A^i(w)}(w/E_i)).$$

It follows that, under the assumption $\mu(u, v, w) = \sigma(u, v, w) = i$, the fact that $C_i(u/E_i; v/E_i, w/E_i)$ holds depends just on $\chi(u)(i)$, $\chi(v)(i)$, $\chi(w)(i)$. Similarly, the fact that $L(u; v, w)$ holds depends just on $\chi(u)(i)$, $\chi(v)(i)$, $\chi(w)(i)$. And if u, v, w, z are all F_i-equivalent but E_i-inequivalent, the fact that $S(u, v; w, z)$ holds depends just on $\chi(u)(i)$, $\chi(v)(i)$, $\chi(w)(i)$, and $\chi(z)(i)$. We call this phenomenon *tail-independence*.

Sub-claim 2. *The group* K *preserves the* C*-relation induced by* C_i *on the branches at* r_i.

Proof. Suppose u, v, w lie in the same F_i-class but distinct E_i-classes, so

$$\mu(u, v, w) = \sigma(u, v, w) = i,$$

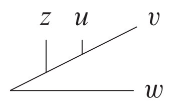

Figure 20

and assume $C_i(u; v, w)$ holds in this branch. Let $x = x(i', g, h) \in K$. If $i' > i$, then

$$\chi(u)(i) = \chi(u^x)(i), \quad \chi(v)(i) = \chi(v^x)(i), \quad \chi(w)(i) = \chi(w^x)(i),$$

so $C_i(u^x; v^x, w^x)$ by tail-independence. If $i = i'$, then $C_i(u^x; v^x, w^x)$ holds since the action of x in the i-th coordinate is induced by an element of G^{V_i} which preserves the C-relation on V_i. If $i' < i$, then $C_i(u^x; v^x, w^x)$ holds by tail-independence. $\triangleleft$

Sub-claim 3. *The group K preserves the L-relation and S-relation on the branches at r_i. That is, if $\mu(u, v, w) = \sigma(u, v, w) = i$, then for $x \in K$, we have*

$$L(u; v, w) \iff L(u^x; v^x, w^x),$$

and similarly for S.

Proof. This is similar to Sub-claim 2. $\triangleleft$

Sub-claim 4. *The group K preserves L on $[n]$.*

Proof. Let $u, v, w \in [n]$ be distinct with $L(u; v, w)$. By Sub-claim 3, we may suppose $i = \sigma(u, v, w) < \mu(u, v, w)$. Thus two of u, v, w are F_i-equivalent and the other F_i-inequivalent to these. We suppose $u F_i v$ and $\neg u F_i w$ (the other cases are similar). Pick $z \in A^i(u)$ with $C_i(z; u, v)$, as shown in Figure 20. Then, for $x \in K$,

$$L(u; v, w) \iff L(u; v, z) \xLeftrightarrow{\text{by Sub-claim 3}} L(u^x; v^x, z^x) \iff L(u^x; v^x, w^x)$$

(since x preserves the relations E_j, F_j and C). $\triangleleft$

Sub-claim 5. *The group K preserves S on $[n]$.*

Proof. Let $u, v, w, z \in [n]$ be distinct. Let i be greatest such that u/E_i, v/E_i, w/E_i, z/E_i are distinct. Then at least two of u, v, w, z are F_i-equivalent. If all are F_i-equivalent, then K preserves any S-relation among these by Sub-claim 3. If just three of u, v, w, z are F_i-equivalent, then K preserves any S-relation among them by the proof of Sub-claim 4. If say $u F_i v$ and $\neg u F_i w \wedge \neg u F_i z$, then as K preserves F_i, if $x \in K$, we have $u^x F_i v^x \wedge \neg u^x F_i w^x \wedge w^x F_i z^x$. We now see $S(u, v; w, z) \wedge S(u^x, v^x; w^x, z^x)$, as required. $\triangleleft$

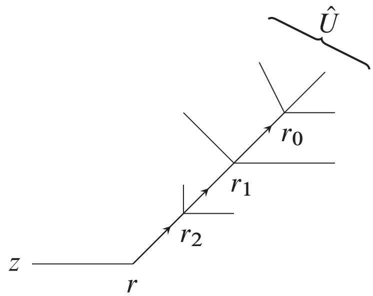

Figure 21

By the sub-claims, the conditions of Lemma 5.6 are satisfied, completing the proof of Claim 3 $\diamond$

It follows that $[n]$ is a Jordan set for G. $\square$

Remark 5.8. In the above proof, in view of the way the group K is built from the groups H_i, it follows that, for each r_i and branch V_i at r_i, the group $G_{(M\setminus[n]),\{V_i\}}$ induces the whole group induced by $G_{\{V_i\}}$ on V_i.

Proposition 5.9. *Each pre-branch is a Jordan set for G in its action on M.*

Proof. Let R be a D-set of M, and let U be a branch of R at a ramification point r, with corresponding pre-branch $\hat{U} \subset M$. Pick z lying in a pre-branch at r other than $\hat{U}$. We may choose a sequence $(r_i : i \in \mathbb{N})$ of ramification points which is coinitial in U, that is, for each ramification point r' in U, there is $i \in \mathbb{N}$ such that, for all $j \geq i$, r_j lies between r and r'. We may suppose in addition that r_{i+1} lies between r_i and r for each i and that z lies in the special branch at r_i for each i (Figure 21). For each i, there is a union T_i of pre-branches at r_i which is a pre-direction of a higher D-set. We may choose the T_i so that, for each i, r_i is a ramification point of one of the branches of T_{i+1}.

It follows that $T_i \subseteq T_{i+1}$ for each i and that $\bigcup_{i\in\mathbb{N}} T_i = \hat{U}$. Since pre-directions are Jordan sets by Lemma 5.7, each T_i is a Jordan set, so $\hat{U}$ is a Jordan set by Lemma 2.4. $\square$

Recall from Definition 4.6 that, given a D-set R of M, the corresponding *pre-D-set* is the union of the predirections of R.

Lemma 5.10. *Each pre-D-set $\hat{R}_i$ is a Jordan set for G.*

Proof. Consider two distinct ramification points r_1, r_2 of R. Let U_{r_1} be the branch at r_1 which includes r_2, and let U_{r_2} be the branch at r_2 containing r_1. We know by

Proposition 5.9 that the corresponding pre-branches are Jordan sets and they form a typical pair; hence, by Lemma 2.4, their union is a Jordan set and is the whole pre-D-set. □

Lemma 5.11. *Let $s \in M$. Then there is a G_s-invariant C-relation on $M \setminus \{s\}$.*

Proof. Consider all the pre-D-sets that contain s and the pre-branches $\hat{U}$ in these pre-D-sets that do not contain s, with the property that s lies in the special branch at the ramification point at which U is a branch. Let $\mathcal{K}$ be this collection of pre-branches. The elements of this collection are all Jordan sets (by Proposition 5.9). Now we check that $\mathcal{K}$ satisfies [3, Lemma 2.2.2 (i)–(v)], applied to G_s acting on $M \setminus \{s\}$.

(i) and (ii) are trivial, that is, each element of $\mathcal{K}$ has size greater than 1, and $\mathcal{K}$ is G_s-invariant.

(iii) $\mathcal{K}$ has no typical pair (Definition 2.3 (a)). First, suppose that $\hat{U}, \hat{V} \in \mathcal{K}$ are pre-branches of the same D-set. Since $\hat{U}, \hat{V}$ both omit the element s of this D-set, it is immediate that $\hat{U}, \hat{V}$ do not form a typical pair.

Next, suppose $\hat{U}, \hat{V} \in \mathcal{K}$ are pre-branches of distinct but comparable D-sets R and R' respectively with R' below R. We may suppose that R lies in a cone of the structure tree corresponding to the ramification point r of R' and that V is a branch at the ramification point r' of R'. If $r = r'$, then $\hat{U}$ is a union of pre-branches at r' omitting s, so contains $\hat{V}$ or is disjoint from $\hat{V}$. If r lies in the branch at r' containing s, then again, $\hat{U}$ either contains $\hat{V}$ or $\hat{U} \cap \hat{V} = \emptyset$. If r is a ramification point lying in $\hat{V}$, then $\hat{U} \subset \hat{V}$. And if r lies in a branch at r' other than V or that containing s, then $\hat{U} \cap \hat{V} = \emptyset$.

Finally, suppose that $\hat{U}$ and $\hat{V}$ are pre-branches of D-sets R_1, R_2 labelling incomparable vertices ν_1, ν_2 of the structure tree. Let $\mu := \inf\{\nu_1, \nu_2\}$, and let R be the D-set of μ, and suppose R_i corresponds to the ramification points r_i of R for $i = 1, 2$. Thus $\hat{U}$ and $\hat{V}$ correspond to unions of pre-branches at r_1 and r_2 respectively of R, omitting s. Note that s lies in both R_1 and R_2, and hence also in R. If, say, r_2 is a ramification point of U, then r_1 is not a ramification point of V, (otherwise, $s \in \hat{U} \cup \hat{V}$), and $\hat{V} \subset \hat{U}$; likewise with r_1, r_2 reversed. Alternatively, r_2 is not a ramification point of $\hat{U}$, and r_1 is not a ramification point of $\hat{V}$, and in this case, $\hat{U} \cap \hat{V} = \emptyset$.

(iv) We must show that, given distinct $u, v \in M \setminus \{s\}$, there is a member of $\mathcal{K}$ containing u, v. Choose a D-set R such that the pre-D-set $\hat{R}$ contains u, v, s in distinct pre-directions. There is a ramification point r at R such that s lies in the special pre-branch at r, and u, v lie in the same other pre-branch $\hat{U}$ at r. Then $\hat{U} \in \mathcal{K}$, and it contains u, v.

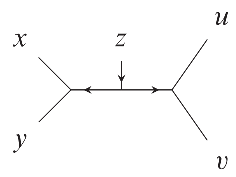

Figure 22

(v) We show that, given distinct $u, v \in M \setminus \{s\}$, there is a member of $\mathscr{K}$ containing u but not v. Choose a D-set R such that $\hat{R}$ contains u, v, s in distinct pre-directions, meeting at ramification point r. There is a ramification point r' in the branch at r containing u such that the branch at r' containing s is special. Let $\hat{U}$ be the pre-branch at r' containing u. Then $\hat{U} \in \mathscr{K}$, and it contains u but not v.

Now define a ternary relation C_s such that, for every $x, y, z \in M \setminus \{s\}$, the relation $C_s(x; y, z)$ holds if and only if $(\exists U \in \mathscr{K})(y, z \in U \wedge x \notin U)$. Then C_s is a G_s-invariant C-relation by [3, Lemma 2.2.2]. □

Lemma 5.12. *There is no G-invariant separation relation on M.*

Proof. Choose a configuration in M as depicted in Figure 22, in some D-set. By semi-homogeneity, there is $g \in G$ inducing $(x)(y)(z)(uv)$. It is easily seen that a permutation of M with such cycle structure cannot preserve a separation relation on M. □

Lemma 5.13. *There is no G-invariant Steiner system on M.*

Note. We use the idea of [8, proof of Lemma 6.5].

Proof. For a contradiction, suppose there is a G-invariant Steiner n-system on M. Let $s_1, \dots, s_{n+1}$ be distinct elements of a block $\mathfrak{B}$ of the Steiner system. Since we may choose a D-set in which all s_i lie in different branches at a ramification point, there is a pre-branch V containing s_{n+1} and omitting $s_1, \dots, s_n$. Let $t \in V$. Since V is a Jordan set, there is $g \in G_{(M \setminus V)}$ with $s_{n+1}^g = t$. As g fixes $s_1, \dots, s_n$, it fixes setwise the unique block $\mathfrak{B}$ containing $s_1, \dots, s_n$, so as $s_{n+1} \in \mathfrak{B}$, also $t \in \mathfrak{B}$, that is, $V \subseteq \mathfrak{B}$.

Let s^* be an element of $M \setminus \mathfrak{B}$ (hence not in V), and let $\mathfrak{B}'$ be the block containing $s_1, \dots, s_{n-2}, s_{n+1}, s^*$. As $|\mathfrak{B}'| \geq n+1$, there is $s^{**} \in \mathfrak{B}'$ distinct from $s_1, \dots, s_{n-2}, s_{n+1}, s^*$ with $s^{**} \notin \mathfrak{B}$, so as $V \subseteq \mathfrak{B}$, then we have $s^{**} \notin V$. But $s_1, \dots, s_{n-2}, s^*, s^{**}$ are all in $\mathfrak{B}'$, so determine $\mathfrak{B}'$. So, as $s_{n+1} \in V \cap \mathfrak{B}'$, by the above argument using the Jordan property of V, we obtain $V \subseteq \mathfrak{B}'$. So we have $V \subseteq \mathfrak{B} \cap \mathfrak{B}'$, a contradiction as V is infinite and $|\mathfrak{B} \cap \mathfrak{B}'| = n - 1$. □

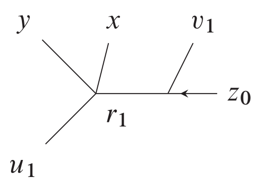

Figure 23. The D-set R_1

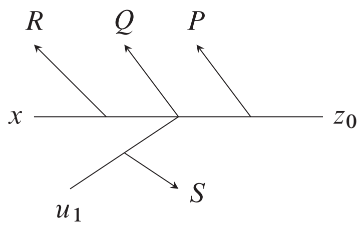

Figure 24

Lemma 5.14. *There is no G-invariant D-relation on M.*

Proof. Suppose for a contradiction that there is a G-invariant D-relation D defined on M. Fix $x, y, z_0 \in M$. Find $u_1 \in M \setminus \{x, y, z_0\}$ with $D(u_1, z_0; x, y)$. Note that, in the argument below, we should not confuse D with the various D-sets in M coded by the structure tree.

Find a D-set R_1 of M containing u_1, z_0, x, y in distinct branches at the same ramification point r_1, and pick $v_1 \in M$ lying in the pre-branch at r_1 containing z_0, with $L(z_0; v_1, x)$ witnessed in this D-set. See Figure 23. Let $z_1 \in M \setminus \hat{R}_1$. Choose $h_1, k_1 \in G_{z_0,z_1}$ with $(x, v_1)^{h_1} = (v_1, x)$ and $(u_1, v_1)^{k_1} = (v_1, u_1)$ – these exist by semi-homogeneity.

In the D-relation on M, consider the regions P, Q, R, S as depicted in Figure 24 (here $x \in R$, $u_1 \in S$, $z_0 \in P$).

Let $\mathrm{supp}\langle h_1, k_1\rangle$ denote the set of elements of M moved by some element of the subgroup $\langle h_1, k_1\rangle$ of G generated by h_1 and k_1. If say $v_1 \in R$, then we see that $R \cup S \subseteq \mathrm{supp}(k_1) \subseteq \mathrm{supp}\langle h_1, k_1\rangle$. If $v_1 \in S$, then

$$R \cup S \subseteq \mathrm{supp}(h_1) \subseteq \mathrm{supp}\langle h_1, k_1\rangle.$$

If $v_1 \in Q$, then $R \subseteq \mathrm{supp}(h_1) \subseteq \mathrm{supp}\langle h_1, k_1\rangle$, and $S \subseteq \mathrm{supp}(k_1) \subseteq \mathrm{supp}\langle h_1, k_1\rangle$. Finally, if $v_1 \in P$, then $R, S \subseteq \mathrm{supp}(h_1) \subseteq \mathrm{supp}\langle h_1, k_1\rangle$. Thus, wherever v_1 lies, $R \cup S \subseteq \mathrm{supp}\langle h_1, k_1\rangle$, so as h_1, k_1 fix z_1, so $z_1 \notin R \cup S$. Thus $z_1 \in P \cup Q$. Since $D(u_1, z_0; x, y)$, $y \in R$, so we have the picture in Figure 25.

Figure 25

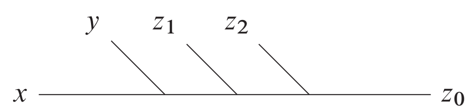

Figure 26

Now we iterate this argument with (z_0, x, z_1) in place of (z_0, x, y). Pick

$$u_2 \in M \setminus \{x, z_0, z_1\} \quad \text{with } D(u_2, z_0; x, z_1).$$

Find a D-set R_2 of M containing u_2, z_0, x, z_1 in distinct branches at the same ramification point r_2, and pick $v_2 \in M$ lying in the pre-branch at r_2 containing z_0, with $L(z_0; v_2, x)$ witnessed in this D-set. Let $z_2 \in M \setminus \hat{R}_2$. By semi-homogeneity, there are $h_2, k_2 \in G_{z_0,z_2}$ with $(x, v_2)^{h_2} = (v_2, x)$, $(u_2, v_2)^{k_2} = (v_2, u_2)$. Let $x, z_0, u_2, P', Q', R', S'$ replace x, z_0, u_1, P, Q, R, S above. Then we see that $z_2 \in P' \cup Q'$, and thus the D-relation on M satisfies the picture in Figure 26. Observe that, as $z_1 \notin \hat{R}_1$ and $z_2 \notin \hat{R}_2$, we have

$$L(z_1; x, z_0) \wedge L(z_2; x, z_0) \wedge L(z_2; x, z_1) \wedge L(z_2; z_0, z_1).$$

Thus, by semi-homogeneity, there is $g \in G_{z_1,z_2}$ inducing (x, z_0). Such g does not preserve the D-relation on M, a contradiction. □

5.2 Proof of the main theorem

In this section, we show that $G = \mathrm{Aut}(M, L, S)$ is an infinite primitive Jordan group preserving a limit of D-relations (Definition 2.6).

We may view M as an $\mathscr{L}$-structure, or as a structure in just the language with symbols L and S, since, by Lemma 4.1, the other $\mathscr{L}$-symbols are ∅-definable in terms of L and S.

Let $\hat{R}$ be a pre-D-set with D-set R, let $H := G_{(M \setminus \hat{R})}$, and let E be the equivalence relation on $\hat{R}$ corresponding to being in the same direction (the equivalence relation identified in Definition 4.5). Let D be the induced D-relation on $R = \hat{R}/E$.

Lemma 5.15. *In the above notation,*

(i) *H preserves E and the relation D,*

(ii) *H is transitive on $\hat{R}$,*

(iii) *H is 2-transitive but not 3-transitive on R,*

(iv) *E is the unique maximal H-congruence on $\hat{R}$.*

Proof. (i) H preserves E as $H < G_{\{M\setminus\hat{R}\}}$, which preserves E as noted after Definition 4.5. Also, the assertion that H preserves D follows from Lemma 4.7 (iv).

(ii) This follows from Lemma 5.10.

(iii) Fix $x_0 \in \hat{R}$, and let x_0/E denote the E-class of x_0. We show that $H_{x_0/E}$ is transitive on $R\setminus\{x_0/E\}$. Let u, v be E-inequivalent elements of $R\setminus\{x_0\}$. Choose a ramification point r such that there is a branch U at r containing u/E, v/E and omitting x_0/E. It is known that $\hat{U}$ is a Jordan set (pre-branches are Jordan sets), so there is $g \in G_{(M\setminus\hat{U})} < H$ with $u^g = v$, and hence $(u/E)^g = v/E$. However, H is not 3-transitive, for if $u, v, w \in \hat{R}$ and they meet at a ramification point r with $L(u; v, w)$, there is no element of H inducing $(u/E, v/E)(w/E)$.

(iv) The invariance of E follows from (i), and its maximality from (iii). For the uniqueness, suppose E^* is an H-congruence on $\hat{R}$ and there are $u, v \in \hat{R}$ with $\neg uEv$ and uE^*v. Since pre-directions are Jordan sets, for $v' \in \hat{R}$ if $v'Ev$, there is $g \in H$ fixing $M\setminus(v/E)$ pointwise with $v^g = v'$. As $u^g = u$, g fixes $E^*(u)$, so vE^*v', so $v/E \subset v/E^*$, so E^* contains E properly, hence is universal by maximality of E. □

Theorem 5.16. *G preserves a limit of D-relations on M.*

Proof. Let $G = \mathrm{Aut}(M)$. Then G is an infinite Jordan group acting on M. Let T be the structure tree of M (so $T = (K^*/R, \leq)$, as identified in Lemma 4.10). Let J be a maximal chain from T. Then J is linearly ordered by $\leq$. Let R_j be the D-set indexed by j for $j \in J$. Then, by Lemma 4.10 (ii), for $i, j \in J$, we have $i < j \Leftrightarrow \hat{R}_j \subset \hat{R}_i$. Thus $(\hat{R}_j : j \in J)$ is a strictly increasing chain of subsets of M, where the ordering under inclusion is the reverse of that induced from the index set J. Let $\hat{R}_j$ be the pre-D-set corresponding to R_j, let $H_j := G_{(M\setminus\hat{R}_j)}$, and let E_j be the unique maximal H_j-congruence on $\hat{R}_j$ as in Lemma 5.15 (iv). Then $\{H_j : j \in J\}$ is an increasing chain of subgroups of G, with the ordering under inclusion reversed from that of J. We must check conditions (i)–(viii) in Definition 2.6.

(i) This follows from (ii) and (iv) in Lemma 5.15.

(ii) This is (i) and (iii) in Lemma 5.15. Note that, since pre-branches and pre-directions are Jordan sets of G, branches are Jordan sets of each (H_j, R_j), so the latter are Jordan groups.

(iii) It is clear that $\bigcup(\hat{R}_i : i \in J) = M$.

(iv) Let $H := \bigcup_{j\in J} H_j$. Then H is a Jordan group on M since each R_j is a Jordan set for H. The group G is not 3-transitive since it preserves the relation L (and $L(u; v, w) \to \neg L(v; u, w)$); hence H is not 3-transitive.

To show that H is 2-primitive on M, first observe a point from Lemma 5.7. In the proof of that lemma (see also Remark 5.8), if $[n]$ is a pre-direction of the

D-set labelled by the vertex j_n, then for each $j < j_n$, there are a D-set R_j and ramification point r_j such that $[n] = \bigcup\bigcup S_j$ for some set S_j of branches at r_j. It follows from that proof that, for each branch $U \in S_j$ at r_j, the pointwise stabiliser of the complement of $[n]$ induces G^U on U.

Now let $x_0 \in M$, and let ρ be a non-trivial H_{x_0}-congruence on $M \setminus \{x_0\}$. We must show that ρ is universal. Pick distinct $u, v \in M \setminus \{x_0\}$ with $u \neq v$. Choose $j \in J$ such that x_0, u, v lie in distinct pre-directions of R_j. Let $\mathfrak{B}$ be the ρ-class containing u. For a contradiction, we suppose that ρ is not universal, so may suppose that $\mathfrak{B}$ does not contain each pre-direction of R_j other than that of x_0. In particular, by (iii) and Lemma 5.7, it follows that $\mathfrak{B}$ is a proper subset of $\hat{R}_j$ omitting at least two pre-directions, including that of x_0.

Let r be a ramification point of R_j such that u, v lie in the same pre-branch $\hat{U}$ at r, and x_0 in a different pre-branch. Let C be the C-relation induced on the corresponding branch U at r. Suppose there are distinct $u', v', w' \in \hat{U}$ such that $C(u'/E_j; v'/E_j, w'/E_j)$ and $u' \rho w'$. Let V be the largest branch in U containing v', w' and omitting u'. Then the pre-branch $\hat{V}$ is a Jordan set, so there is $g \in G_{(M\setminus\hat{V})} < H_{x_0}$ with $(u', w')^g = (u', v')$. As g fixes u', it follows that $v' \rho w'$. Thus $\mathfrak{B} \cap \hat{R}_j$ is a pre-branch of R_j, the union of a nested sequence of pre-branches of R_j, or a union of more than one pre-branch at some fixed vertex. By choosing j sufficiently low in the structure tree, we may assume that the last one holds, i.e. $\mathfrak{B} \cap \hat{R}_j$ is the union of more than one pre-branch at a ramification point r_j of R_j.

Pick a ramification point r^* of R_j such that elements of $\mathfrak{B}$ and x_0 lie in distinct pre-branches at r^* with the pre-branch containing elements of $\mathfrak{B}$ non-special, and that containing x_0 special. There is a pre-direction $[n]$ which is a union of pre-branches at r^* including the pre-branch $\hat{V}$ at r^* containing $\mathfrak{B}$, and excluding that containing x_0. Now, by the observation above (i.e. Remark 5.8), since $G_{(M\setminus[n])} \leq H$, H induces the full group G^V on V. In particular, using semi-homogeneity there is a ramification point r between r^* and r_j such that H_{x_0} contains an element h with $u^h = u$ and $r_j^h = r$. It follows that $\mathfrak{B}^h \supset \mathfrak{B}$, contradicting that $\mathfrak{B}$ is a block of H_{x_0}.

(v) $E_j|_{\hat{R}_i} \subseteq E_i$ if $i > j$, by Lemma 5.2.

(vi) We claim that $\bigcap(E_i : i \in J)$ is equality. Let $u, v \in M$ be distinct. By 2-transitivity of G, there is a D-set R such that u, v lie in distinct directions of R. Choose $j \in J$ such that the corresponding D-set R_j labels a vertex of the structure tree below that of R. Then u, v lie in distinct directions of R_j, so $\neg u E_j v$.

(vii) Given $g \in G$, choose an initial segment I of J which lies in the common part of J and J^g. Let $i_0 \in I^{g^{-1}} \subseteq J^{g^{-1}} \cap J$. Then, for any $i < i_0$, we have $i^g < i_0^g$ and so $i^g \in I$. Thus $i^g = j$ for some $j \in J$. Hence $g^{-1}H_i g = H_j$ and $R_i^g = R_j$.

(viii) This is by Lemma 5.11. □

Theorem 5.17. *There are a ternary relation L and a quaternary relation S on a countably infinite set M such that if $G := \mathrm{Aut}(M, L, S)$, then G is oligomorphic, 3-homogeneous, 2-primitive but not 3-transitive or 4-homogeneous on M, and is a Jordan group preserving a limit of D-relations on M, and not preserving any of the structures of types* (i)–(iii) *in Theorem* 1.1.

Proof. This is by Corollary 3.20, Lemmas 4.2, 5.12, 5.13, 5.14 and Theorem 5.16. The group G is not 4-homogeneous as some but not all quadruples satisfy S under some ordering. Note that G cannot preserve a linear or circular order or a linear betweenness relation since it does not preserve a separation relation, G cannot preserve a C-relation since it does not preserve a D-relation, and G cannot preserve a semilinear order or general betweenness relation since it is 2-primitive. □

6 Further questions

We have a number of questions around the construction in this paper, its companion in [8], and the exact statement of Theorem 1.1 which was proved in [3]. We also have questions concerning the flexibility of our construction, and how it fits in the developing theory of homogeneous and ω-categorical structures.

Problem 6.1. Axiomatise a concept of (L, S)-*structure*. The idea here is to identify a set, probably finite, of axioms for a ternary relation L and quaternary relation S, from which the basic combinatorics of Sections 2 and 3 can be derived. In particular, it should be possible from the axioms to interpret in any (L, S)-structure a semilinear order (the "structure tree"), a family of D-sets in bijection with the vertices of the semilinear order, a concept of special branch at a ramification point of a D-set, the maps f_ν associating cones at the vertex ν of the structure tree with ramification points of the associated D-set $D(\nu)$, and the corresponding maps $g_{\mu\nu}$. There is need for an analogous axiomatisation of the corresponding ternary relation (also denoted by L) in [8] – there is an initial discussion of this in the last section of that paper. This should also be done for limits of Steiner systems.

Problem 6.2. Sharpen Theorem 1.1 above (the main result of [3]), and its proof there, so that, in case (iv), the notion of limit of betweenness or D-relations (and possibly of Steiner systems) is replaced by the concept identified in Problem 6.1. At the very least, it should be possible to replace the total order I in Definition 2.6 by an *invariant* semilinear order, with a corresponding modification of the proof of Theorem 1.1.

Problem 6.3. Clarify the connection between a limit of D-relations and a limit of general betweenness relations. For example, is the structure constructed in [8] interpretable in the structure constructed in this paper (a question asked by Peter Cameron)? Note that any D-relation interprets a general betweenness relation.

Problem 6.4. Show that the group $G = \mathrm{Aut}(M)$ constructed in Section 3 does not preserve a limit of betweenness relations or Steiner systems on M. Is it maximal-closed, i.e. maximal subject to being a closed proper subgroup of $\mathrm{Sym}(M)$?

In his PhD thesis [10], David Bradley-Williams initiated a construction of a limit of betweenness relations based on a discrete rather than a dense semilinear order. This has been developed much further in joint work in preparation of Bradley-Williams and Truss.

Problem 6.5. Carry out a programme analogous to that of Bradley-Williams and Truss, but for limits of D-relations rather than betweenness relations. Can the betweenness relations and D-sets in these structures be replaced by other kinds of relational structures?

Problem 6.6. With $G = \mathrm{Aut}(M)$ as in this paper, let $n_k(G)$ be the number of orbits of G on the set of k-element subsets of M. Find the asymptotic growth rate of the sequence $(n_k(G))$.

Regarding the last problem, we know by the main theorem of [23] that $n_k(G)$ is bounded below by an exponential function. There are very few known examples of oligomorphic primitive permutation groups for which the growth is bounded above exponentially. Most of these examples are associated with treelike structures.

There is a well-known connection between valued fields and treelike structures. For example, given a field F equipped with a non-trivial valuation map $v: F \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group, there is a C-relation on F, invariant under addition and multiplication by non-zero elements, given by

$$C(x; y, z) \iff (v(x-y) < v(y-z));$$

see for example [22]. The well-known graph-theoretic tree on which $\mathrm{SL}_2(\mathbb{Q}_p)$ acts, as described by Serre [25, Chapter II], is associated with this. There is a D-relation on the projective line $\mathrm{PG}_1(F)$ defined by putting $D(x, y; z, w)$ if and only if the cross ratio $[x, y; z, w]$ lies in $1 + \mathcal{M}$, where $\mathcal{M}$ is the maximal ideal of the corresponding valuation ring – see [6, Theorem 30.4]. It is also well known that the set of all valuations on a field is lower semilinearly ordered under reverse inclusion of the corresponding valuation rings. This suggests the following problem.

Problem 6.7. Show that the structure M in this paper, or more generally an (L, S)-structure as in Problem 6.1, "lives" on a field, in the sense that the structure tree can be identified with a set of valuation rings of the field.

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