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# Shakedown analysis and its application in pavement and railway engineering

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**Abstract**

This paper has been prepared in memory of Professor Scott Sloan. It first presents a brief review of the development of shakedown analysis methods for pavement and railway engineering problems. In particular, it describes a new lower-bound method using the concept of critical residual stress fields and an upper-bound method using a nonlinear programming technique, which were developed by the authors, and then extended and applied to solve various shakedown problems in pavement and railway engineering. Moreover, this paper summarises and compares shakedown solutions for pavement and railway engineering problems, whilst highlighting the key factors that influence the shakedown limits. In addition, this paper proposes a simple, unified shakedown limit equation for pavements and railways under repeated moving surface loads. The equation includes three terms, which represent the resistances from cohesion, self-weight of the underlying soil, and self-weight of any superficial rigid layers, respectively, in a format analogous to Terzaghi's bearing capacity equation. Numerical results indicate that the coefficient in the cohesion term $N_c^{sd}$ depends on the soil friction angle; while the coefficient in the self-weight term $N_\gamma^{sd}$ is controlled by the soil friction angle and a dimensional factor $\gamma a/c$. Values of $N_c^{sd}$ and $N_\gamma^{sd}$ for a typical rolling point contact problem are also presented and interpreted, which explain the different contribution ratios from the soil self-weight to the shakedown limits of pavement and railway problems.

**Keywords:** Shakedown; Pavement; Railway; Lower bound; Upper bound; Unified equation

## 1 Introduction

The concept of shakedown was initially proposed some 80 years ago (e.g. Bleich, 1932; Melan, 1938; Prager, 1948; Koiter, 1960). It states that when an elastic-plastic structure is subjected to a repeated or variable load and the load level is higher than yield but lower than a 'shakedown limit', the structure will gradually adapt itself to the cyclic load and respond purely elastically to the following load cycles, leading to no further exhibition of plastic strain. Otherwise, if the load level is higher than the shakedown limit, the structure will eventually fail due to alternative plasticity or excessive permanent deformation. Shakedown phenomena of pavements and railways have been observed by researchers through road tests and field data analyses (e.g. Sharp and Booker, 1984; Radovsky and Murashina, 1996; Xiao et al., 2016). Many conventional cyclic triaxial tests (e.g. Lekarp and Dawson, 1998; Werkmeister et al., 2001) or cyclic hollow cylinder tests (e.g. Qian et al., 2016; Xiao et al., 2018; Wang et al., 2020a) have demonstrated the shakedown behaviours of soils, granular materials and asphalt mixtures. Moreover, by using wheel tracking tests, a series of validation experiments for shakedown theory (Juspi, 2007; Brown et al., 2008; Ravindra, 2008; Ravindra and Small, 2008; Chazallon et al., 2009; Brown et al., 2012) has been carried out for single-layered or multi-layered pavement systems with different types of soils and granular materials. Very recently, Liu et al. (2020) further validated the shakedown concept for bituminous pavement structures by conducting a series of wheel tracking tests on a layered bituminous pavement structure.

A key task of applying shakedown concept to practical problems is to determine the shakedown limit. Compared with conventional step-by-step analysis where the full stress and strain history is required (e.g. Wang and Yu, 2013a; Liu et al., 2016; Liu et al., 2017), shakedown analysis using either static (Melan, 1938) or kinematic (Koiter, 1960) shakedown theorem directly calculates the lower or upper bound of the shakedown limit thus attracts lots of attention. The application of the theoretical shakedown analysis in the design of steel rails was recognised more than 40 years ago (e.g. Doyle, 1980). Later, Sharp and Booker (1984) introduced the shakedown theory to pavement problems considering that wheels roll and slide on the surface of a cohesive-frictional material. Following that, a number of shakedown analysis methods for pavement and railway engineering problems were proposed (e.g. Ponter et al., 1985; Collins and Cliffe, 1987; Raad et al., 1988; Collins and Boulbibane, 1998; Yu and Hossain, 1998; Shiau and Yu, 2000; Yu, 2005; Boulbibane and Ponter, 2006; Li and Yu, 2006; Krabbenhøft et al., 2007; Nguyen et al., 2008; Zhao et al., 2008; Wang, 2011; Yu and Wang, 2012; Qian et al., 2017; Costa et al., 2018; Rahmani and Binesh, 2018; Wang et al., 2018a; Wang et al., 2018b; Liu et al., 2020).

One objective of this paper is to provide a review of the recent development of shakedown analysis methods for pavement and railway applications, with a brief description of establishing two recent shakedown analysis methods. The other objective of the paper is to summarise the key influencing factors for shakedown limits of pavements and railways, and finally to propose a new, simple, unified equation for shakedown limits of soils under repeated moving surface loads.

## 2 Shakedown theorems

The classical static shakedown theorem of Melan (1938) states that an elastic-perfectly plastic structure will shakedown under repeated/cyclic loads if a time-independent self-equilibrated residual stress field can be found, that when combined with the load-induced elastic stress field, does not violate the yield condition anywhere in the structure. This can be expressed as:

$$f(\lambda\boldsymbol{\sigma}_{ij}^{e} + \boldsymbol{\sigma}_{ij}^{r}) \leq 0 \qquad \text{Eq. 1}$$

where $\boldsymbol{\sigma}_{ij}^{e}$ is the elastic stress field induced by a unit load $p$; $\lambda$ is a dimensionless load multiplier; $\boldsymbol{\sigma}_{ij}^{r}$ is the time-independent self-equilibrated residual stress field; and $f \leq 0$ is the yield criterion for the material.

The classical kinematic shakedown theorem of Koiter (1960) states the structure will not shakedown under repeated or cyclic loads if any kinematically admissible plastic strain rate cycle $\dot{\boldsymbol{\varepsilon}}_{ij}^{p*}(\boldsymbol{t})$, body force $\boldsymbol{X_i}(\boldsymbol{t})$ and surface load $\boldsymbol{p_i}(\boldsymbol{t})$ within prescribed limits can be found for which:

$$\int_0^T \left(\int_V \boldsymbol{X}_i\dot{\boldsymbol{u}}_i^* dV + \int_{S_p} \boldsymbol{p}_i\dot{\boldsymbol{u}}_i^* dS\right) dt > \int_0^T \int_V \boldsymbol{\sigma}_{ij}^* \dot{\boldsymbol{\varepsilon}}_{ij}^{p*} dV dt \qquad \text{Eq. 2}$$

where $S_p$ is structure surface where external loads are specified; $V$ is structure volume; $T$ is time period for one loading cycle; $\dot{\boldsymbol{u}}_i^*$ and $\boldsymbol{\sigma}_{ij}^*$ are the velocity and the state of stress on the yield surface which are associated with the kinematically admissible plastic strain rate field $\dot{\boldsymbol{\varepsilon}}_{ij}^{p*}$. Alternatively, the upper-bound shakedown theorem can be formulated as follows when omitting the body force:

$$\lambda_{sd} \int_0^T \int_{S_p} \boldsymbol{p}_{0i}\dot{\boldsymbol{u}}_i^* dS dt \leq \int_0^T \int_V \boldsymbol{\sigma}_{ij}^* \dot{\boldsymbol{\varepsilon}}_{ij}^{p*} dV dt \qquad \text{Eq. 3}$$

where $p_{0i}$ is a unit load, $\lambda_{sd}$ is the shakedown load multiplier.

It should be noted that the two classical shakedown theorems were proved based on the assumption that the materials satisfy Drucker's stability conditions. Therefore, they are only valid for materials obeying an associated plastic flow rule. Nevertheless, it has been demonstrated by researchers (e.g. Maier, 1969; Pycko and Maier, 1995; Boulbibane and Weichert, 1997) that the shakedown theorems can be extended to consider the materials following a non-associated flow rule if the plastic potential function $g = 0$ is convex and contained by the yield surface.

Considering a rapid loading process, the dynamic shakedown concept was first introduced by Ceradini (1969). Based on additional assumptions that plastic deformations are instantaneous and the yield surface is fixed, Ceradini (1980) presented the proofs of the lower-bound and upper-bound dynamic shakedown theorems for elastic-perfectly plastic bodies subjected to a rapid loading process. The dynamic shakedown theorems take forms similar to the classical ones but use a fictitious dynamic response.

In general, shakedown analysis using the static or kinematic shakedown theorems obtains the shakedown limit through searching for a critical point or a critical failure mechanism. It leads to either a lower bound or an upper bound to the true shakedown limit, because the static shakedown theorem only satisfies internal equilibrium equations, yield criterion, and stress boundary conditions, while the kinematic shakedown theorem only satisfies compatibility, plastic flow rule and displacement boundary conditions.

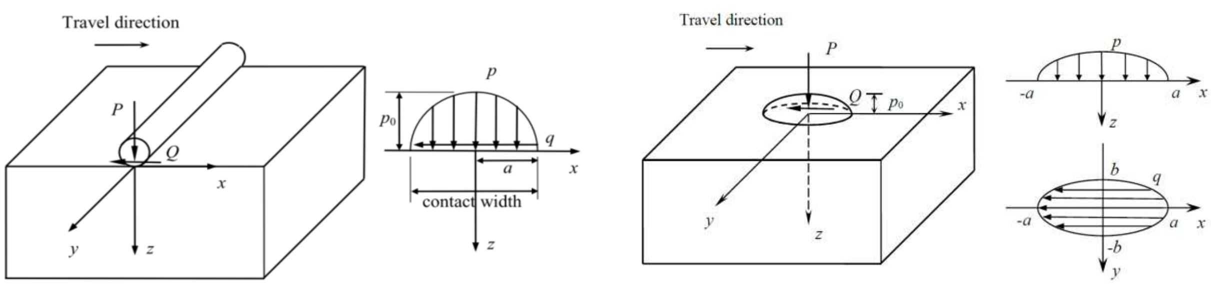
# 3 Shakedown analysis methods for pavements and railways

## 3.1 Simplification of problems

Shakedown of an elastic-plastic half space subjected to rolling and sliding contact is fundamental to the analyses of pavement and railway engineering problems. Two basic rolling and sliding contact problems are line and point contacts. The line contact problem assumes a simple situation in which the load is applied over a certain contact width by a long roller, whilst the point contact problem considers a surface contact loading limited to a circular, an elliptical, or a rectangular contact area. If the normal load is denoted as $P$, its tangential counterpart $Q$ can be expressed as $\mu P$, where $\mu$ is a frictional coefficient. It is usually assumed that the normal and shear stresses over the contact area are correlated by the frictional coefficient, and they distributed in a Hertz, Trapezoid or uniform shape. For instance, using the Hertz formulation, the distributions of normal and shear stresses are depicted in Figure 1, in which the $x$-axis is the travel direction.

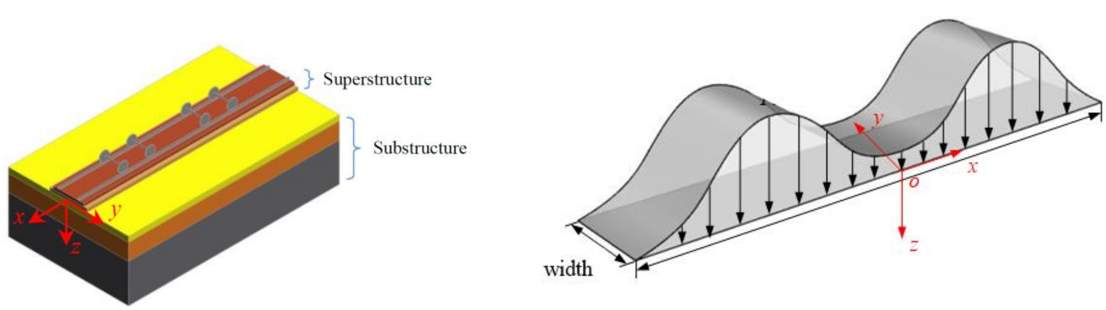
The Hertz load distribution has a maximum compressive pressure at the center $p_0 = 2P/\pi a$ for the line contact problem, and $p_0 = 3P/2\pi ab$ for the point contact problem.

For railway problems and some pavement situations, the stresses due to neighbouring wheels overlap within some regions in the ground. Therefore, shakedown analysis needs to be conducted considering that several wheel loads move simultaneously on the surface (e.g. Collins et al., 1993a; Wang et al., 2018a). Furthermore, the analysis of railway problems needs to take the effect of superstructure components into account. This can be done by converting a set of neighbouring train wheel loads as well as the influence of the superstructure into a distributed pressure on the top of the substructure (e.g. Liu et al., 2018; Wang et al., 2018a; Wang et al., 2020b), as shown in Figure 2.




(a) Line contact problem (b) Point contact problem

Figure 1: Rolling and sliding contact problems




(a) Train wheels loads and slab track structure (b) Converted pressure on substructure

Figure 2: Conversion of train loads into a pressure

## 3.2 Static shakedown analysis methods

### *3.2.1 Recent development*

Over a number of years, shakedown analysis methods based on the static shakedown theorem were developed for the line contact problem (e.g. Sharp and Booker, 1984; Sharp, 1985; Raad et al., 1988; Raad et al., 1989a, 1989b; Raad and Weichert, 1995; Radovsky and Murashina, 1996; Yu and Hossain, 1998; Boulbibane et al., 2000; Shiau and Yu, 2000; Krabbenhøft et al., 2007; Nguyen, 2007; Nguyen et al., 2008; Zhao et al., 2008). Because of the two-dimensional nature of the line contact problem, the solutions can only be considered as approximate ones. In recent years, efforts have been devoted to develop shakedown analysis methods considering more realistic loading situations in pavement and railway engineering. Based on the consideration, the problem becomes three-dimensional, and therefore it is much more difficult to derive relevant shakedown solutions. Shiau (2001) extended the linear programming technique of Yu and Hossain (1998) to solve the three-dimensional pavement shakedown problem. Although some reasonable results were obtained, it was found that the size of the linear programming problem became prohibitively large when a finer mesh was applied in the three-dimensional case. Nguyen et al. (2008) utilised an interior-point method to solve pavement shakedown problems considering a rounded Mohr-Coulomb or a von Mises yield criterion. Both two-dimensional and three-dimensional rolling and sliding contact problems were analysed, but the numerical results for the three-dimensional case cannot be yet considered reliable.

To efficiently solve the three-dimensional problem, Yu (2005) obtained an analytical necessary condition for shakedown limits of a cohesive-frictional half-space under a moving point load. Wang (2011) and Yu and Wang (2012) developed a rigorous lower-bound shakedown analysis method by deriving two critical residual stress fields, and thus reducing the problem to a simple mathematical optimisation problem. The analytical solutions of elastic stresses in a half-space under an elliptical Hertz load was used, so that the results can be used to benchmark numerical shakedown limits. The critical residual stress fields obtained in this method were also found in agreement with the numerical results obtained through a step-by-step finite element analysis and a mesh-free method for both line contact and point contact problems (Wang and Yu, 2013a; Liu et al., 2018; Rahmani and Binesh, 2018). Considering the cross-anisotropic behaviour of soils and pavement materials, Wang and Yu (2014) developed a shakedown analysis method to allow the variation of elastic and plastic material properties with direction.

Based on the lower-bound shakedown analysis method of Yu and Wang (2012), a number of extensions and applications for pavement and railway problems were conducted. In the field of pavement engineering, the method was utilised to solve pavement design problems (Wang and Yu, 2013b) and to compare with the analytical design approach in the UK for flexible road pavements (Wang et al., 2016). Liu et al. (2016) extended the method to examine the influence of dilation angle on lower-bound shakedown limits of pavements. By introducing a numerical approach which calculates the traffic-induced dynamic elastic stress field, Qian et al. (2017) studied the influence of traffic speed on shakedown limits, and the work was further extended to consider the effects of cross-anisotropic materials (Qian et al., 2018) and frictional coefficient (Qian et al., 2020). Since the properties of asphalt mixtures are highly dependent on temperature, Liu et al. (2020) proposed a temperature-dependent shakedown approach to obtain shakedown limits of asphalt pavements. In the field of railway engineering, although the shakedown solutions of steel rails have been studied for many years (e.g. Doyle, 1980; Bower and Johnson, 1991; Kapoor and Williams, 1994; Dyson et al., 1999; Ringsberga et al., 2005; Hasan, 2019), it was not until very recently that the shakedown analysis method of Yu and Wang (2012) was extended to allow the calculation of shakedown solutions for railway structures. Considering a typical slab track for high-speed railways, Wang et al. (2018a) investigated the dynamic shakedown limits of slab track substructures and revealed the relation between the dynamic shakedown limits and critical speeds. Liu et al. (2018) focused on the influence of a depth-dependent stiffness modulus on shakedown limits. Costa et al. (2018) considered at-rest stresses in the ground and adopted a 2.5D approach to calculate dynamic elastic stresses; and therefore the influences of train geometry, track stiffness, and soil improvement were examined for a slab track system. There were also some efforts in applying the shakedown analysis method of Yu and Wang (2012) in determining the shakedown limits of ballasted railways (e.g. Zhuang et al., 2019); however their analyses did not consider the spaced sleepers thus violated one basic assumption of the rolling and sliding contact problems that the residual stresses should be independent of the travel direction.

*3.2.2 Static shakedown analysis based on critical residual stress fields*

According to Yu and Wang (2012), for the three-dimensional rolling and sliding contact problems, symmetry and other considerations impose some constraints on the residual stresses: (1) the residual stresses must be independent of $x$; (2) $\sigma_{zz}^r$ and $\sigma_{xz}^r$ must be zero; (3) the residual stress itself must satisfy the yield condition, i.e., $f(\lambda\sigma_{ij}^r) \leq 0$. A detailed analysis of the possible residual stresses can be found in Yu and Wang (2012) and Yu (2005). The validity of the residual stresses has also been verified in the numerical studies of Shiau (2001) and Liu et al. (2017). Assuming that the material is described by the Mohr-Coulomb model and the critical planes are *x-z* planes, the static shakedown theorem (Eq. 1) can be rewritten as follows:

$$f = (\sigma_{xx}^r + \mathrm{M})^2 + \mathrm{N} \leq 0 \qquad \text{Eq. 4}$$

where $\sigma_{xx}^r$ is time-independent and self-equilibrated; $\mathrm{M} = \lambda\sigma_{xx}^e - \lambda\sigma_{zz}^e + 2\tan\phi\,(c - \lambda\sigma_{zz}^e \tan\phi)$; $\mathrm{N} = 4(1 + \tan^2\phi)[(\lambda\sigma_{xz}^e)^2 - (c - \lambda\sigma_{zz}^e \tan\phi)^2]$; $c$ and $\phi$ are cohesion and friction angle of the material, respectively.

Based on the conditions for residual stresses and the shakedown condition Eq. 4, Yu and Wang (2012) found that the actual residual stress field must lie within a region determined by two critical stress fields $\sigma_{xx-l}^r$ (denoted as 'maximum smaller root') and $\sigma_{xx-u}^r$ (denoted as 'minimum larger root'):

$$\sigma_{xx-l}^r = \max_{z=j}^{-\infty \leq x \leq \infty}\left(-\mathrm{M}_i - \sqrt{-\mathrm{N}_i}\right) \qquad \text{Eq. 5}$$

$$\sigma_{xx-u}^r = \min_{z=j}^{-\infty \leq x \leq \infty}\left(-\mathrm{M}_i + \sqrt{-\mathrm{N}_i}\right) \qquad \text{Eq. 6}$$

where $i$ represents a point in the half-space; and $j$ represents a depth. When the applied load is at the shakedown limit, the actual residual stress at a depth $z = j$ must be no smaller than $\sigma_{xx-l}^r$ and no higher than $\sigma_{xx-u}^r$, and the critical point of the half-space is located at the depth where the two critical residual stresses just intersect.

By substituting Eq. 5 or Eq. 6 into Eq. 4, the shakedown problem can be expressed as the following mathematical formulation:

$$\lambda_{sd} = \max(\lambda)$$
$$\text{s. b.}\begin{cases} f\left(\sigma_{xx}^{r}\left(\lambda\sigma^{e}\right), \lambda\sigma^{e}\right) \leq 0 \text{ for all points} \\ \sigma_{xx}^{r}\left(\lambda\sigma^{e}\right) = \sigma_{xx-l}^{r} \text{ or } \sigma_{xx-u}^{r} \end{cases} \qquad \text{Eq. 7}$$

Since the elastic stress fields $\sigma^e$ and the critical residual stress fields all depend on the load multiplier $\lambda$, Eq. 7 can be easily solved by using the procedure suggested in Wang (2011) and Yu and Wang (2012).

## 3.3 Kinematic shakedown analysis methods

### *3.3.1 Recent development*

Based on Koiter's kinematic shakedown theorem, Ponter et al. (1985) obtained shakedown limits of a cohesive half-space under a point contact loading. Two distinct failure modes were considered in the analysis: incremental collapse and alternating plasticity. Considering that the rate of plastic working per unit length on the slip line is the product of cohesion with the tangential velocity jump, Collins and Cliffe (1987) presented upper-bound shakedown solutions of a cohesive-frictional half-space under a moving two-dimensional or three-dimensional surface load. This method was extended to two-layered pavements by using a different shape of the slip channel (Collins et al., 1993a; Collins et al., 1993b), and then it was improved by introducing rut failure mechanisms (Collins and Boulbibane, 1998, 2000; Boulbibane et al., 2005).

Recently, the linear matching method, originally proposed for limit and shakedown analyses of metal structures under static or cyclic load (e.g. Ponter and Carter, 1997; Ponter and Engelhardt, 2000; Chen and Ponter, 2005; Ponter et al., 2006), has been applied to the pavement shakedown problem (Boulbibane and Ponter, 2005, 2006). The basic idea of this method is that the stress and strain fields for the nonlinear material behaviours may be simulated by the solution of linear problems where linear moduli vary with time and space. Li and Yu (2006) proposed a nonlinear programming approach for kinematic shakedown analysis of cohesive-frictional materials. It was further extended to consider a general yield condition with a non-associated plastic flow rule (Li, 2009). The approach has been applied to solve pavement shakedown problems with materials following an associated or a non-associated plastic flow rule. The approach of Li and Yu (2006) will be briefly explained in the following subsection. Apart from that, the work of Ponter et al. (1985) could also be very useful in terms of understanding the kinematic shakedown analysis.

*3.3.2 Kinematic shakedown analysis using a nonlinear programming approach*

By applying the principle of virtual work, the kinematic shakedown theorem Eq. 3 can be written as:

$$\lambda_{sd} \int_0^T \int_V \boldsymbol{\sigma}_{ij}^e \dot{\boldsymbol{\varepsilon}}_{ij}^{p*} dV dt \leq \int_0^T \int_V \boldsymbol{\sigma}_{ij}^* \dot{\boldsymbol{\varepsilon}}_{ij}^{p*} dV dt \qquad \text{Eq. 8}$$

in which $\boldsymbol{\sigma}_{ij}^e$ is the linear elastic stress response to external actions. Eq. 8 can be re-expressed as the following problem using the mathematical programming theory (Li and Yu, 2006):

$$\lambda_{sd} = \min_{\varepsilon_k^p, \Delta u} \int_0^T \int_V \boldsymbol{\sigma}_{ij}^* \dot{\boldsymbol{\varepsilon}}_{ij}^{p*} dV dt$$

$$\text{s.b.} \begin{cases} \int_0^T \int_V \boldsymbol{\sigma}_{ij}^e \dot{\boldsymbol{\varepsilon}}_{ij}^{p*} dV dt = 1 & \\ \Delta\varepsilon_{ij}^p = \int_0^T \dot{\boldsymbol{\varepsilon}}_{ij}^{p*} dt = \frac{1}{2}\left(\Delta\boldsymbol{u}_{i,j} + \Delta\boldsymbol{u}_{j,i}\right) & \text{in } V \\ \Delta u_i = \int_0^T \dot{\boldsymbol{u}}_i dt & \text{in } V \\ \Delta u_i = 0 & \text{on } S_u \end{cases} \qquad \text{Eq. 9}$$

in which $\Delta\varepsilon_{ij}^p$ and $\Delta u_i$ are cumulative plastic strain and displacement fields at the end of one loading cycle over one time cycle $[0, T]$, respectively; and $S_u$ is displacement boundary.

Many widely used yield criteria for cohesive-frictional materials can be expressed as:

$$F(\boldsymbol{\sigma}) = \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{Q} - 1 = 0 \qquad \text{Eq. 10}$$

in which $\boldsymbol{\sigma}$ is the stress vector; $F(\boldsymbol{\sigma})$ defines a yield function in terms of strength parameter; $\boldsymbol{P}$ and $\boldsymbol{Q}$ are coefficient matrices and vector which are related to strength properties of the material.

By using the general yield equation and a specific plastic flow rule, the plastic strain rate field can be obtained. For the case with an associated plastic flow rule, it can be written as:

$$\dot{\boldsymbol{\varepsilon}}^p = 2\dot{\eta}\boldsymbol{P}\boldsymbol{\sigma} + \dot{\eta}\boldsymbol{Q} \qquad \text{Eq. 11}$$

where $\dot{\eta} = \sqrt{\frac{(\dot{\boldsymbol{\varepsilon}}^p)^{\mathrm{T}}\boldsymbol{P}^{-1}\dot{\boldsymbol{\varepsilon}}^p}{4+\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q}}}$.

Then the plastic dissipation power can be obtained as follows:

$$\boldsymbol{\sigma}_{ij}^{e}\dot{\boldsymbol{\varepsilon}}_{ij}^{p*} = \frac{1}{2\dot{\eta}}(\dot{\boldsymbol{\varepsilon}}^{p})^{\mathrm{T}}\boldsymbol{P}^{-1}\dot{\boldsymbol{\varepsilon}}^{p} - \frac{1}{2}(\boldsymbol{Q})^{\mathrm{T}}\boldsymbol{P}^{-1}\dot{\boldsymbol{\varepsilon}}^{p} \qquad \text{Eq. 12}$$

Following the technique of König (1987), it is assumed that if a structure reaches a state of shakedown under any load vertices, it will shake down under the whole load domain. The admissible plastic strain cycles on the vertices $\boldsymbol{P}_k$ $(k = 1, 2, \dots, l)$ can generate plastic strain increment:

$$\boldsymbol{\varepsilon}_{k}^{p*} = \int_{\tau_k} \dot{\boldsymbol{\varepsilon}}^{p*}\mathrm{d}t \qquad \text{Eq. 13}$$

The cumulative plastic strain at the end of one loading cycle can be calculated from:

$$\Delta\boldsymbol{\varepsilon}^{p*} = \sum_{k=1}^{l}\boldsymbol{\varepsilon}_{k}^{p*} \qquad \text{Eq. 14}$$

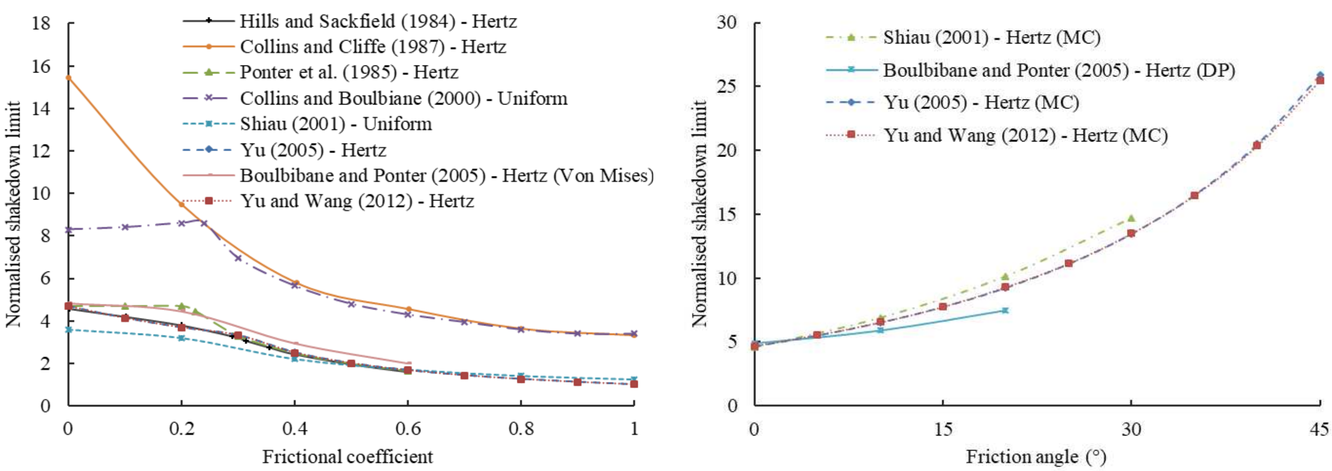
By substituting Eq. 12 and Eq. 14 into Eq. 9, the shakedown analysis becomes an optimisation problem with several equality constrains (Eq. 15) which can be solved according to the technique in Li and Yu (2006).

$$\lambda_{sd} = \min_{\varepsilon_k^p, \Delta u} \sum_{k=1}^{l} \int_V \left( \frac{1}{2\dot{\eta}}(\boldsymbol{\varepsilon}_k^{p*})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}_k^{p*} - \frac{1}{2}(\boldsymbol{Q})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}_k^{p*} \right) dV$$
$$\text{s.b.} \begin{cases} \sum_{k=1}^{l} \int_V (\sigma_k^e)^{\mathrm{T}} \varepsilon_k^{p*}\, dV = 1 & \\ \Delta\varepsilon_{ij}^{p} = \sum_{k=1}^{l} \boldsymbol{\varepsilon}_k^{p*} = \frac{1}{2}\left(\Delta\boldsymbol{u}_{i,j} + \Delta\boldsymbol{u}_{j,i}\right) & \text{in } V \\ \Delta u_i = 0 & \text{on } S_u \end{cases} \qquad \text{Eq. 15}$$

## 4 Shakedown solutions for pavement problems

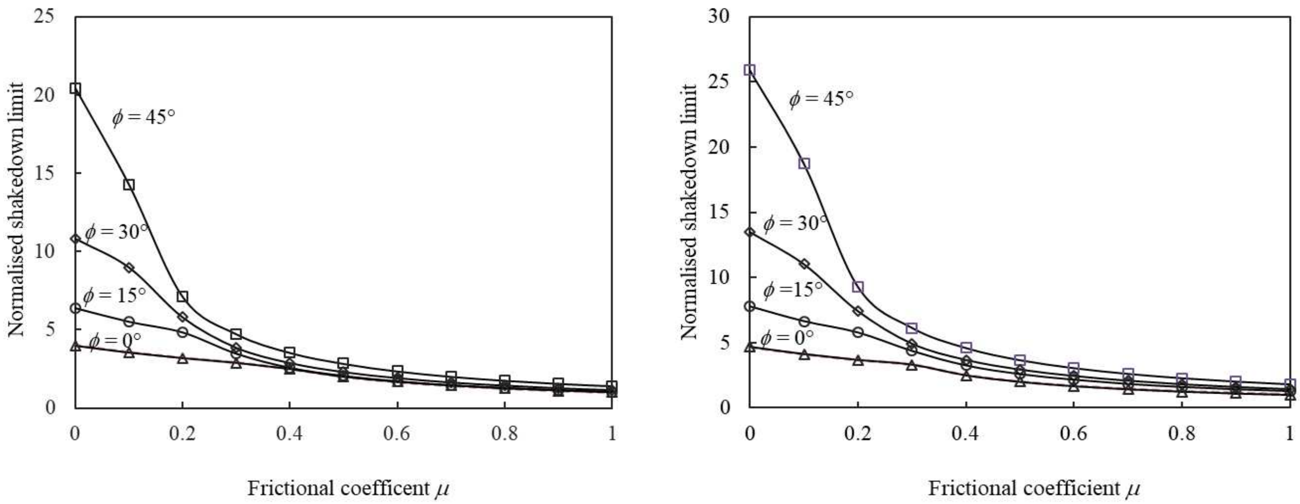
### 4.1 Effects of cohesion, friction angle, and frictional coefficient

Three-dimensional shakedown analysis of pavements usually assumes that the contact loading is limited within a circle of radius $a$. It was found the shakedown limit is always proportional to the material cohesion $c$ when omitting material self-weight. Hence, the shakedown solution is normally represented by a normalised shakedown limit $\lambda_{sd}p/c$. Figure 3 exhibits a comparison of existing lower bound shakedown solutions (Hills and Sackfield, 1984; Shiau, 2001; Yu, 2005; Yu and Wang, 2012) and upper bound shakedown solutions (Ponter et al., 1985; Collins and Cliffe, 1987; Collins and Boulbibane, 2000; Boulbibane and Ponter, 2005) for such a three-dimensional situation. For the case with cohesive materials, the upper-bound shakedown limits are usually higher than the lower-bound shakedown limits. The upper-bound solutions of Ponter et al. (1985) agree well with the lower-bound solutions for cases $\mu = 0$ and $\mu \geq 0.3$, which implies true shakedown limits. The static shakedown limit always decreases with increasing frictional coefficient, whereas the kinematic shakedown limit barely changes between $\mu = 0$ and $\mu = 0.2$. As for cohesive-frictional materials, it is clear that the static and kinematic shakedown limits are raised by increasing friction angle. The lower-bound shakedown solutions for Mohr-Coulomb materials are close, and they are slightly higher than Boulbibane and Ponter (2005)'s kinematic solutions, because an incompressible non-associated flow and a Drucker-Prager material was used instead. Figure 4 demonstrates and compares the interactive effect of friction angle and frictional coefficient on the normalised lower-bound shakedown limits for both line contact and point contact problems (Wang, 2011). When the frictional coefficient is relatively small, the critical point is located at a depth below the pavement surface; otherwise, failure initiates at the surface.




(a) Cohesive material (b) Cohesive-frictional material, $\mu = 0$

Figure 3: Comparison of shakedown limits for a point contact problem



(a) Line contact problem (b) Point contact problem

Figure 4: Comparison of static shakedown limits for line and point contact problems

### 4.2 Effect of elongated contact area

For the cases with an elliptical contact area, if the area of the contact loading does not change, a reduction of the aspect ratio $b/a$ (refers to Figure 1b) leads to a rise of the normalised shakedown limit; meanwhile, the critical point tends to move towards the surface (Yu and Wang, 2012).

### 4.3 Effect of stiffness ratio

In flexible pavements, upper layers normally have higher values of stiffness modulus than lower layers. If a quasi-static situation is considered, the normalised shakedown limit is affected by the stiffness ratios rather than the absolute values of the stiffness modulus. Moreover, there always exists an optimum stiffness ratio that provides the maximum resistance (i.e. highest shakedown limit), which represents the movement of the critical point from one layer to another (Sharp and Booker, 1984; Shiau, 2001; Wang and Yu, 2013b).

### 4.4 Effect of layer thickness

Increasing the thickness of an upper layer usually raises shakedown limit; however, when the thickness exceeds a certain value, the shakedown limit barely changes with the layer thickness or the stiffness ratio. This is because the failure is totally controlled by the strength properties of the upper layer (Shiau and Yu, 2000; Boulbibane et al., 2005; Wang and Yu, 2013b); and the corresponding layer thickness can be considered as the depth of influence of the contact loading.

### 4.5 Effect of anisotropy

For cases with cross-anisotropic materials, the elastic parameters which mainly affect shakedown limits are stiffness modulus ratio $E_v/E_h$ and shear modulus ratio $G_{vh}/G_h$ (the subscripts $v, h$ represent vertical and horizontal directions, respectively). When the material surface is critical, the shakedown limit mainly depends on the shear modulus ratio; otherwise, it is dominated by the stiffness modulus ratio (Wang and Yu, 2014).

For the plastic part, it is normally assumed that cohesion of the material changes with the direction. With rising cohesion ratio $c_v/c_h$, the normalised shakedown limit $\lambda P/c_h$ increases until a maximum value is reached, which is controlled by the subsurface failure mode. For any cases with $c_v/c_h$ > 1, Wang and Yu (2014) found that a peak shakedown limit exists at a frictional coefficient $\mu > 0$. This is different from isotropic solutions which are always largest at $\mu = 0$ (i.e. normal loading only).

### 4.6 Effect of plastic flow rule

For cohesive-frictional materials, a dilation angle $\psi$ ($0 \leq \psi < \phi$) is often used to describe the plastic potential. Both upper-bound (Li and Yu, 2006) and lower-bound (Liu et al., 2016) solutions suggested that the pavement shakedown limits are reduced due to the decrease of the dilation angle.

### 4.7 Effect of temperature

At a higher temperature, stiffness moduli and cohesions of asphalt mixtures are lower; consequently, the shakedown limit of each pavement layer is reduced, especially for the asphaltic layer (Liu et al., 2020). The shakedown approach is most applicable for asphalt mixtures in which the aggregate skeleton takes most of the stresses.

## 5 Shakedown solutions for railway problems

Shakedown analysis of railway problems is more complex than pavement problems since it requires a careful consideration and analysis of the superstructure. Take a typical Rheda 2000 single track system as an example (Figure 2), its superstructure includes rails, pads, fastening systems, slab, and concrete base; while its substructure contains an anti-frozen layer, a prepared subgrade, and subsoil. In railway problems, the sleepers are spaced at a certain interval, so that the residual stresses should be periodic along the travel direction, especially in the region close to the sleepers. Despite of that, for typical slab tracks, the stresses at or below the surface of the substructure has become independent of the locations of the sleepers due to high rigidity of the slab and the concrete base. Based on this fact, shakedown analysis was performed on slab track substructures (Liu et al., 2018; Wang et al., 2018a; Liu and Wang, 2019). It was considered that the superstructure components act together as a single infinite Euler-Bernoulli beam, while the supporting substructure behaves as a Winkler's foundation, so that the axle loads and the superstructure can be converted into a moving pressure directly acting on the top of the substructure, as shown in Figure 2. Alternatively, shakedown analysis of slab tracks can be carried out by directly modelling a track-ground system (Costa et al., 2018). In this section, several additional factors that greatly affect the shakedown limits of railways will be discussed.

## 5.1 Effect of at-rest stress

In railway problems, the load-influenced depth is much greater than that in pavements. For instance, an influencing depth of 7m was obtained in Tang et al. (2015) for a typical railway structure. Therefore, the effect of at-rest stress cannot be neglected in the shakedown analysis of railways. It was noted by Costa et al. (2018) that the shakedown limit could be underestimated if the at-rest stresses were not taken into account. Despite of that, it is still unclear of the contribution proportion of the self-weight on the shakedown limit.

## 5.2 Effect of depth-dependent stiffness

One typical feature of the clayey subsoil is that its stiffness increases with depth. Liu et al. (2018) found an increment of stiffness could have two effects on the shakedown limit. First, the pressure distribution on the substructure becomes relatively uneven, leading to higher stresses close to the surface thus a smaller shakedown limit of the surface layer. Second, the layer stiffness ratios are changed, causing more stresses transferred to the subsoil thus a smaller shakedown limit of the subsoil.

### 5.3 Effect of train speed

Train speed is critical in the design of high-speed railways. Wang et al. (2018a) revealed the dynamic shakedown limit of a slab track substructure depends on a velocity factor $\alpha = V/V_{cr}$ ($V$ is train velocity; $V_{cr}$ is the critical velocity) rather than the absolute value of train velocity. An attenuation factor $\eta$ was introduced to relate the dynamic shakedown limit $\lambda_{sd}^{d}P$ with the static shakedown limit $\lambda_{sd}^{s}P$:

$$\lambda_{sd}^{d}P = \eta\lambda_{sd}^{s}P \qquad \text{Eq. 16}$$

With increasing velocity factor, the change of the shakedown limit from the static solution also depends on the friction angle of subsoil $\phi_3$ (Figure 5a); however, it was barely affected by the layers above the subsoil (Wang et al., 2021). Therefore, a fitting equation is proposed:

$$\eta = \begin{cases} 1 & \text{for } \alpha \leq 0.1 \\ (1-\eta_{cr}) \times \sqrt[n]{1-\left(\frac{\alpha-0.1}{0.9}\right)^{n}} + \eta_{cr} & \text{for } 0.1 < \alpha \leq 1 \end{cases} \qquad \text{Eq. 17}$$

where $n$ is a coefficient depending on the friction angle of subsoil, the value of which can be obtained from Figure 5b; $\eta_{cr}$ is the attenuation factor at the critical velocity, the value of which can be taken as 0 in most design situations (exception occurs when the stiffness of subsoil is extremely low compared to the stiffness of the upper layers).

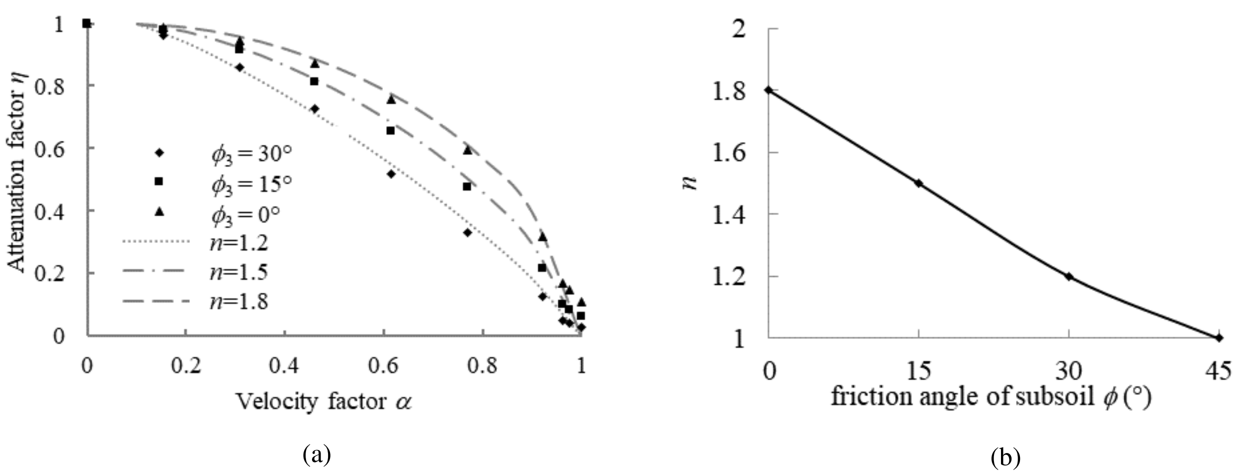



Figure 5: Influences of velocity factor and friction angle of subsoil on attenuation factor

## 6 A unified shakedown limit equation for rolling and sliding problems

As reviewed in the previous two sections, many studies have been devoted to obtain shakedown limits for pavements and railways, and to analyse the influences of various factors. Many of the studies were dedicated to very specific cases and there have been no general equations that would consider various contributing factors. It should be noted that although the shakedown solutions vary significantly for different problems, they share some common trends and key factors. In this section, a simple, unified equation for the shakedown limits of cohesive-frictional materials under repeated moving surface loads is proposed, which is applicable to both pavement and railway problems. This equation aims to bring together the effects of several basic and key factors.

### 6.1 A unified shakedown limit equation

Fundamentally, the shakedown limit is the bearing capacity of the material under the action of a repeated moving surface load. A natural choice of the format of the unified equation would be the one that is similar to the classical Terzaghi's bearing capacity equation (Eq. 18).

$$q_{ult} = N_c c + N_q q_0 + N_\gamma \gamma a \qquad \text{Eq. 18}$$

where $N_c$, $N_q$, and $N_\gamma$ give the resistances due to the material cohesion $c$, the overburden stress $q_0$, and the self-weight of the material, respectively; $\gamma$ is the unit weight of the soil; and $a$ is a half width of the contact area considering a strip footing. This equation is basic, concise and powerful, each term of which has a clear physical meaning. For pavements and railways under moving surface loads, the shakedown limit of the rolling and sliding contact problem (or called 'cyclic capacity'), can be expressed in an analogous format:

$$q_{ult}^{sd} = N_c^{sd} c + N_q^{sd} q_0 + N_\gamma^{sd} \gamma a \qquad \text{Eq. 19}$$

where $N_c^{sd}$ and $N_\gamma^{sd}$ stand for the resistances from the cohesion and the self-weight of the underlying material, respectively; $N_q^{sd}$ represents the resistance from overburden stress due to the self-weight of the structural components above the cohesive-frictional materials. Notice that the overburden term only applies if there exists one or more layers of rigid materials on the top of the cohesive-frictional materials, such as slabs and concrete base on the top of the substructure in a slab track, or a concrete layer above a granular layer in a pavement structure.

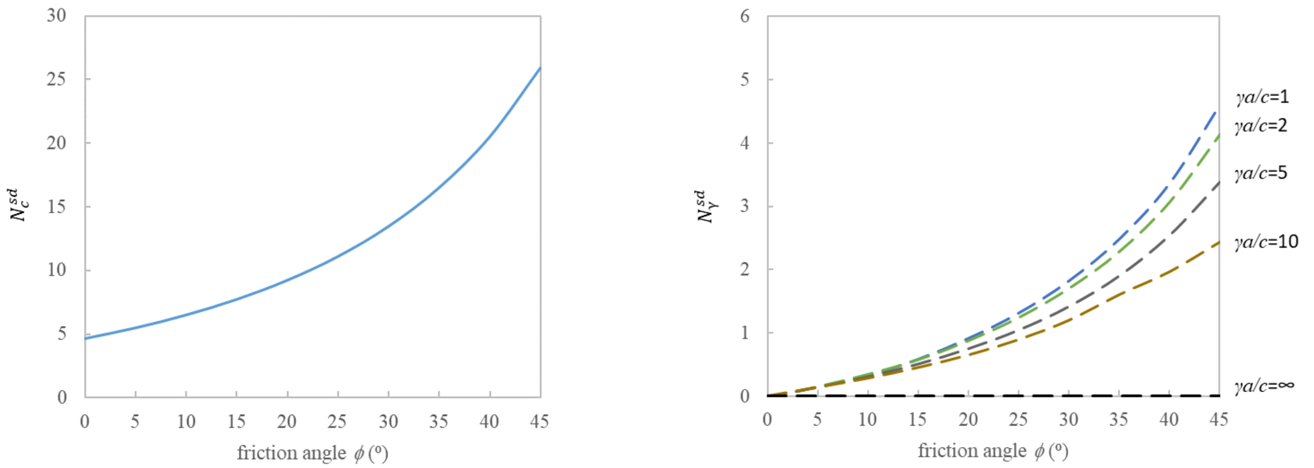
Similar to Terzaghi's bearing capacity equation, by including a set of correction coefficients in the expression, Eq. 19 can also be extended to consider the effects of other factors, such as the shape of the contact area, the distribution of the pressure, the horizontal component of the load, anisotropic soil and so on. Consequently, it can be readily used to estimate the shakedown limit (or cyclic capacity) of various pavements and railways under traffic loads.

### 6.2 $N_c^{sd}$ and $N_\gamma^{sd}$ for a rolling point contact problem

For one basic rolling point contact problem in which a homogenous and isotropic half-space is subjected to a moving Hertz contact loading limited within a circle of radius $a$, the values of the coefficients $N_c^{sd}$ and $N_\gamma^{sd}$ are presented in Figure 6 as examples. Those coefficients were calculated based on the lower-bound shakedown analysis method of Yu and Wang (2012). $N_c^{sd}$ is a function of the material friction angle $\phi$. It is obtained by applying zero self-weight in the calculation of the shakedown solutions. $N_\gamma^{sd}$ is a function of the friction angle $\phi$ and a dimensionless factor $\gamma a/c$. It is obtained by deducting the contribution of cohesion from the obtained shakedown limit, as follows:

$$N_\gamma^{sd} = (q_{ult}^{sd} - N_c^{sd} c)/\gamma a \quad \text{Eq. 20}$$

As can be seen, $N_\gamma^{sd}$ is only one fifth of $N_c^{sd}$ at its maximum. Since typical asphalt mixtures have high values of cohesion, say 200~1000kPa (Liu et al, 2020), the contribution of the self-weight term will be very small compared to the cohesion term. This explains why the shakedown limits of asphalt pavements are barely affected by the self-weight.



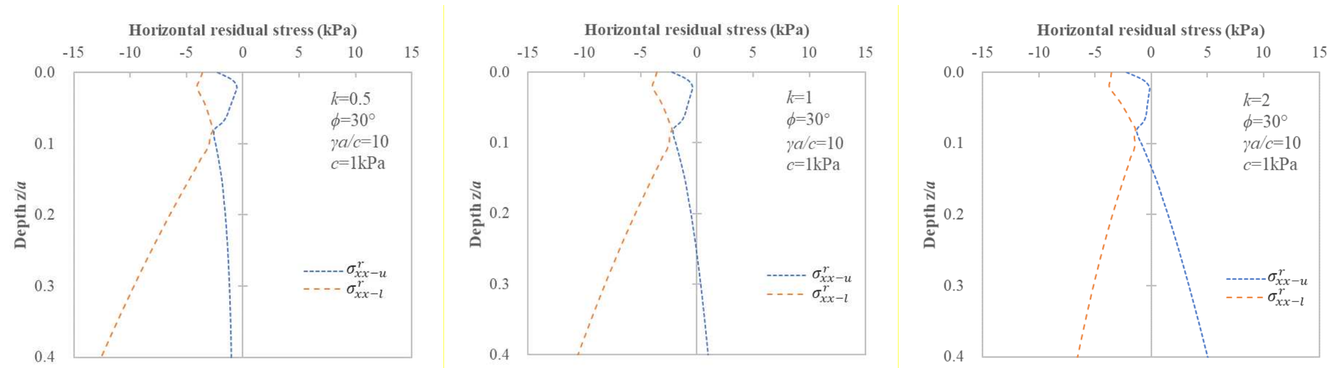

(a) $N_c^{sd}$ (b) $N_\gamma^{sd}$

Figure 6: Shakedown limit coefficients for a rolling point contact problem

When $c$ is equal to zero, no resistance can be provided by the self-weight of the material (i.e., $N_\gamma^{sd} = 0$ when $\gamma a/c =$ infinite), and the first term of Eq. 19 is also zero, so that the shakedown limit is zero for this problem. This means, theoretically speaking, purely frictional soils will always fail if it is directly under a repeated moving load.

It can also be deduced from Eq. 19 and Figure 6 that, for a certain soil, the size of the contact area only affects the resistance from the self-weight of the soil. The influence of the enlarged contact area on the shakedown capacity is competitively affected by increasing $a$ and decreasing $N_\gamma^{sd}$. When $a$ is very large, its effect on $N_\gamma^{sd}$ becomes very small; and therefore the shakedown limit tends to increase proportionally with $a$. This also explains that the slab track problems have much larger shakedown limits compared to the pavement problems. Apart from that, the overburden stress due to self-weight of superstructures in slab tracks also contribute to the high shakedown limit.

It should be noted that, the theoretical shakedown limit will not be affected by the lateral earth pressure coefficient $k$. Despite of that, the two critical residual stress fields (Eq. 5 and Eq. 6) at the shakedown limit are changed by $k$, as shown in Figure 7. As explained in Yu and Wang (2012) and Wang and Yu (2013a), when the applied load is at the shakedown limit, the actual residual stress field must lie within the region bracketed by the two critical residual stress fields (i.e. $\sigma_{xx-l}^r \leq \sigma_{xx}^r \leq \sigma_{xx-u}^r$); and the critical point is located at the depth where the two fields just meet. It can be seen from Figure 7, the critical depth is not changed by the lateral earth pressure coefficient $k$, but the residual stress at the critical depth (i.e. critical residual stress) varies. A relatively small $k$ value gives rise to a smaller (i.e. more compressive) critical residual stress and therefore yields an identical shakedown solution. This indicates that soils always tend to deform in a way that facilitate structural shakedown.




(a) $k = 0.5$ (b) $k = 1$ (c) $k = 2$

Figure 7: Distributions of critical residual stresses at shakedown limit $q_{ult}^{sd} = 25.4$kPa

## 7 Concluding remarks

In this paper, a review of the recent development of the shakedown analysis methods for pavement and railway applications was presented, including a brief recall of one static shakedown analysis based on critical residual stress fields and one kinematical shakedown analysis using a nonlinear programming approach. Comparison of the shakedown limits from different methods for a rolling and sliding point contact problem showed generally good agreements. It was found that the shakedown limits of pavements are mainly affected by the form of the contact loading, material properties, layer thicknesses, and temperature. For railway problems, the influences of at-rest stress, subsoil stiffness variation, and velocity factor could become significant.

A novel contribution of the paper is the introduction of a simple, unified equation for shakedown limit (or cyclic capacity) of soils under repeated moving surface loads. This equation can be applied in a similar manner to Terzaghi's bearing capacity equation, to provide different shakedown capacities for various pavement or railway problems. Different from Terzaghi's equation, the coefficient $N_\gamma^{sd}$ in the self-weight term depends on not only the soil friction angle but also the dimensionless factor $\gamma a/c$, and it is much smaller than the coefficient $N_c^{sd}$ in the cohesion term. The influence of a rising contact area on the shakedown limit is competitively controlled by the increase of $a$ and the decrease of $N_\gamma^{sd}$. The lateral earth pressure coefficient $k$ does not influence the shakedown limit and the critical depth, but it changes the critical residual stress fields. Finally, this equation can be extended to consider other factors by incorporating correction coefficients, so that the shakedown or cyclic capacities for different pavement and railway problems can be determined.

It should be noted that the existing shakedown analyses were usually conducted by assuming that material strength parameters do not change with time or the number of loading cycles. Based on this assumption, the influence of various factors could be thoroughly investigated, that contributes to the selection of design alternatives. In reality, shakedown phenomena observed in pavements or railways should be attributed to two mechanisms: one is the structural shakedown due to the build-up of residual stresses in the structure without any change of its inner structure, which could be interpreted as the pure increase of contact forces among soil particles; the other is the change of the soil inner structure due to cyclic loads or time, which may advantage (e.g., densification of sand) or disadvantage (e.g., degradation of clay) the soil strength and thus increase or decrease the shakedown limit. This paper focused on the structural shakedown due to the first mechanism. The influence of the second mechanism needs to be further explored for different types of soils.

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