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Optimal Investment and Abandonment Decisions for Projects with Construction Uncertainty[⋆]

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Abstract

A model of investment in projects with uncertain construction time is presented in which both the cash-flows upon completion and construction progress are modeled as diffusions. The decision maker has three sequential options: (i) to start construction, (ii) to abandon the project during construction, and (ii) to start operations or abandon after completion. Existence of a solution and some of its qualitative properties are proved. A Markov-chain approximation is used to numerically approximate the solution is introduced. The model is applied to a real-world railway project for which it is found that it does not represent value for money (VfM) and is unlikely to do so in the next decade. A comparative statics analysis reveals that the presence of an abandonment option can increase project value substantially and can speed up investment. Finally, it is argued that the commonly-used benefit-to-cost ratio (BCR) is not an appropriate measure to summarize a project's VfM. Instead, the value ratio (VR) is suggested as an alternative.

Keywords: Finance, Infrastructure investment, Cost-benefit analysis, Real options

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1. Introduction

It is a well-established stylized fact that the majority of large-scale (infrastructure) construction projects are finished over budget and behind schedule. Pohl and Mihaljek (1992), for example, show that there tends to be a divergence between *ex ante* and *ex post* project evaluations, especially when construction times are long and uncertain. In particular, appraisal estimates tend to be too optimistic in the sense that the reported benefit-to-cost ratio (BCR) is too high. A study by Flyvbjerg et al. (2002), using data on 258 transportation infrastructure projects worth US$90 billion, shows that almost 9 out of 10 projects have higher costs than estimated and that the average cost overrun is 28%. For rail projects this increases to 45%. The same authors, in Flyvbjerg et al. (2004), expand on these results and find evidence that cost overruns are more prominent the longer the implementation phase of the project.

Even though these facts are well-known, the problem seems nowhere near closer to being solved. Some high-profile recent examples are the Elbphilharmonie concert hall in Hamburg, Germany, the Olkiluoto 3 nuclear plant in Finland, railways running between the cities of Dali, Ruili and Baoshan in southwestern province of Yunnan, China, and the Crossrail project in London, UK. A particularly interesting example is Berlin Brandenburg Airport, which, after an initially planned opening date of October 2011 finally opened for commercial traffic and in the middle of a pandemic on 31 October 2020. The airport is now estimated to cost €10.3 bn, on an initial budget of €2.83 bn.[1]

This presents a challenge for the OR community to develop methods that allow managers (i) to better appreciate the interplay between the uncertain evolution of free cash flows and construction costs, (ii) to value the flexibility of the option to abandon a project before construction has finished, and (iii) to integrate this flexibility in the optimal timing of initial investment. In this paper, a model is developed that addresses these questions from a real options perspective.[2] Such a perspective allows for new managerial insights into both the *timing* dimensions of investment and abandonment decisions and their consequences for the assessment of a project's *value*.

While the engineering profession continues to work on improving the methods used for cost-benefit analysis, typically these models are not explicitly dynamic.[3] Therefore, a common approach

[1] Source: `https://en.wikipedia.org/wiki/Berlin_Brandenburg_Airport`; last accessed 18 December 2020.

[2] The real options approach has found many applications in OR over the last few decades. To appreciate the broad applicability of the approach the reader is referred to, e.g., Huisman and Kort (2003), who study investment in technological innovations with strategically interacting firms; Bøckman et al. (2008), who study investment timing and capacity choice for small hydropower projects; Décamps et al. (2005), who study optimal investment timing under incomplete information; Hsu and Lambrecht (2007), who study preemptive patenting; Truong and Trück (2016) who emphasize the value of timing flexibility in climate-adaptation investment; Lukas et al. (2017), who study optimal investment and capacity decisions under uncertainty over a product's life-cycle; Angelis et al. (2017), who study optimal investment timing with stochastic costs; Hellmann and Thijssen (2018), who study strategic investment in the presence of ambiguity; or Lukas and Thiergart (2019), who explore how cash incentives (subsidies) influence optimal investment decisions. Thijssen et al. (2004), explore the value of information on investment timing in situations where the decision-maker learns about her economic environment. This approach was later extended to study the issue of voluntary disclosure of information by listed firms in Delaney and Thijssen (2015).

[3] See, for example, Mills (2001), Molenaar (2005), and Touran and Lopez (2006).

is to report the results of such cost-benefit analyses by the so-called "Benefit-to-Cost Ratio" (BCR), where the expected present value of the profits is divided by the expected present value of costs. The project is deemed to be "value for money" (VfM) if the BCR exceeds a pre-determined threshold (Treasury, 2018). For example, in the UK, the following classification is used (the table is taken from Department for Transport, 2015):

VfM category	Implied by BCR
Very high	greater than 4
High	between 2 and 4
Medium	between 1.5 and 2
Low	between 1 and 1.5
Poor	between 0 and 1
Very poor	less than 0

However, actual revenues and costs are uncertain and are accrued at different points in time, in the sense that revenues can not be earned before construction has finished, which, in many cases, is quite far in the future. There are numerous ways for the revenues and construction costs to be higher or lower than expected. The BCR is not a good measure for the evaluation of projects in the presence of both revenue and cost uncertainty, because a construction time delay affects the BCR in two ways. First the construction costs rise because construction takes a longer period. Secondly, because revenues will be generated at a later point in time, these revenues are worth less because of additional discounting. In addition, using project-independent BCR thresholds is not helpful for the justification of a decision to start construction on a particular project, because it ignores the fact that the optimal investment threshold for projects with different risk characteristics will be different; this is one of the basic insights from the real-options approach to capital budgeting (Dixit and Pindyck, 1994). So, ideally, the BCR threshold should be project-dependent, which makes it more difficult to compare BCRs across projects. In fact, the notion of BCR itself is problematic, because in order to value the revenues, one has to take construction time into account, which implies that benefits and costs cannot easily be separated.

Another issue with the BCR is that it ignores the fact that, during the construction process, the decision could be taken that the project should be abandoned. Examples of large-scale projects that were abandoned are the World Islands in Dubai, the Wonderland theme park in Chenzhuang, China, Marble Hill nuclear power plant in the USA, and Ryugyong hotel in Pyongyang, North Korea. In this paper I embed, within a construction project's initial valuation, the option to abandon construction. It is to be expected that the presence of such an option creates value, because it allows for a ceiling to the losses that can be incurred due to construction delays and cost over-runs. The flip side is, however, that such abandonment options are costly. For example, as was reported in *The Washington Post* on 22 September 2020, the companies building the Purple

Line – a light-rail project in the US state of Maryland – had stopped construction amid disputes with the state about cost overruns. Packing up the project required the remaining workers to secure 16 miles of construction sites — partly built bridges, a tunnel and miles of ripped up roads — through several counties. At that time Maryland transit officials said they were still trying to reach a settlement with the project's concessionaire over what is claimed are about $800 million in delay-related cost overruns.[4]

There is, therefore, a need for the development of a VfM measure that (i) is explicitly dynamic, (ii) does not rely on a separation of benefits and costs, (iii) incorporates the value of the option to abandon the project, and (iv) has a project-independent VfM threshold. In the context of a real-options model with both construction costs and revenues, I introduce the *value ratio* (VR) as an easy-to-interpret and straightforwardly implementable VfM measure that satisfies all four criteria. The VR is defined as the present value of starting construction today relative to the project's value under the *optimal* investment timing decision. Under this measure, a project is value for money whenever the VR exceeds 1.

A second contribution of the paper is that revenues as well as construction progress are modeled as two, possibly correlated, Brownian motion-driven stochastic processes. While it is very common to model revenues in this way (see, e.g., Dixit and Pindyck, 1994), it is much less common to do so for construction uncertainty in this way. In fact, to the best of my knowledge, only Thijssen (2015) has used this approach. An important advantage of this way of co-modeling construction and revenue uncertainty is that the value of the project becomes the solution to a two-dimensional free-boundary problem, because both sources of uncertainty are genuinely integrated.

Since such problems have no known analytical solution, I use a Markov-chain approximation to apply the model to the High Speed 2 (HS2) project. This is a proposed high-speed railway to be built in the United Kingdom between London and Birmingham (first phase) and then further north to Manchester and Leeds (second phase). It is found that, given the characteristics of HS2, the current revenue projections are nowhere near good enough to justify building the first phase of HS2, unless the wider economic benefits are much higher than predicted. A comparative statics analysis shows that the presence of an abandonment option is particularly valuable in situations where the revenue volatility is high, the expected construction speed is low, or the discount rate is high.

From a managerial perspective, the most important conclusion of the application is that the presence of an abandonment option, while increasing value, may not have a substantial effect on the initial investment decision. The reason for this result is that the optimal investment trigger is typically so high (due to very high expected construction costs) that the abandonment option

[4] See `https://www.washingtonpost.com/local/trafficandcommuting/purple-line-construction-stops-as-builders-prepare-to-quit/2020/09/22/47de2236-fcfc-11ea-9ceb-061d646d9c67_story.html`; last accessed 14 February 2021.

is highly unlikely to be exercised. However, if expected revenues decrease sufficiently during the construction period, then the abandonment option becomes much more valuable. This suggests that for the initial valuation of a large infrastructure project, one can rely on a simpler model without abandonment option. Once the project has started, however, both the construction's progression as well as the (present value of the) project's expected free cash flows need to be taken into account, because the construction phase is when the abandonment option can be highly valuable.

In the existing literature on real options, there are many papers that incorporate time-to-build, usually in one of the following ways. First, Bar–Ilan and Strange (1996) and Alvarez and Keppo (2002) consider a model of investment under uncertainty where the time to build is deterministic. They find that an increase in the investment lag increases the investment threshold and, thus, delays investment. Sarkar and Zhang (2013) show that this result can be reversed if the project is sufficiently reversible and/or has a high enough growth rate. An alternative approach to investment lag is introduced by Majd and Pindyck (1987) who model the *remaining capital expenditure to completion* (RCEC) as a state variable and allow the decision-maker (DM) to vary construction intensity. In their model the evolution of the RCEC is deterministic and they find that the optimal construction intensity policy is of the bang-bang type: either construct at the maximum intensity or don't construct at all. Schwartz and Moon (2000) and Hsu and Schwartz (2008) extend this approach to the case where the RCEC evolves stochastically over time and apply it to R&D projects in the pharmaceutical industry (Schwartz and Moon, 2000) and the design of research incentives (Hsu and Schwartz, 2008). In another contribution, Pindyck (1993) distinguishes between technical uncertainty and input cost uncertainty for the construction process, but assumes that the value of the finished project is known and fixed, *ex ante*. A slightly different approach is taken by Grenadier and Weiss (1997), who do not model time-to-build as such, but analyse a firm's optimal investment decision in an environment where there are technological improvements arriving according to a Poisson process. They assume that firms who adopt current best technologies are better placed to benefit from future innovations. That kind of learning about future profitability is also present in this paper in the sense that construction progress provides updates of the stochastic discount factor that is applied to future revenues.

The present paper adds to the OR literature by studying the presence of abandonment options during construction of large-scale investment projects. Due to its use of a two-dimensional underlying stochastic process, it is related to Nunes and Pimentel (2017). However, while their model admits an analytical solution through a reduction in dimensionality (due to an absence of fixed sunk costs), the analysis in this paper relies on a combination of analytical and numerical methods. Importantly, by its development of an easy-to-explain VfM measure, the paper takes up the challenge posed by Ghosh and Troutt (2012), who emphasize the importance of clear communication between (real options) modelers and practitioners.

The paper is organized as follows. In Section 2 the two models are introduced: in Section 2.1 the model without an abandonment option and in Section 2.2 the model with an abandonment option. In Section 3, the value ratio as a VfM measure is introduced. Section 4 introduces a Markov-chain approximation method to numerically solve the two models, which is then applied to the case of High Speed rail in the UK in Section 5. Some concluding remarks are in Section 6. All proofs of mathematical propositions are given in the Appendix.

2. The model

The valuation model is a simple representation of the benefits and costs of constructing, for example, a piece of infrastructure like a railway, a motorway or an airport. Throughout, construction of a railway is used as an example, but the model is more widely applicable. It is assumed that a finished and operational railway yields stochastically evolving revenues with constant operating costs. Construction costs are assumed to be constant over time, but construction progress is stochastic. The decision maker in this paper is assumed to be a private concessionaire, whose objective is to maximize the present value of the discounted stream of all future cash flows pertaining to the project. The model's main ingredients are:

1. a (stochastic) construction process $(X_t)_{t\geq 0}$, $X_0 = 0$, that has to reach a value $x^* > 0$ before revenues can be earned;
2. a constant construction cost flow $c > 0$, incurred while construction takes place;
3. a perpetual (stochastic) revenue flow $(Y_t)_{t\geq 0}$, accruing after construction has finished and operations have commenced;
4. a perpetual constant operational cost flow $\eta > 0$, incurred after construction has finished and operations have commenced;
5. a perpetual constant cost flow $\kappa \in [0, \eta)$, incurred after construction has finished, but before the finished project is exploited.
6. a constant discount rate $r > 0$;
7. a sunk cost of exercising the option to abandon the project, $K > 0$, either during construction or after construction but before exploitation; it is assumed that $0 < rK < \min\{\kappa, c\}$.[5]

As is typical in the real options literature, the stochastic processes in this paper are all Markovian. Therefore, uncertainty is modeled on a state space $E = \mathbb{R} \times (0, \infty)$, with typical element $(x, y) \in E$. Here x represents the state of construction and y represents the annual revenues from an operational railway. Let Ω be the set of all continuous functions mapping $[0, \infty)$ into E and let $\mathscr{F}$ be its Borel σ-field. On the measurable space $(\Omega, \mathscr{F})$, define a probability measure P and the corresponding expectations operator E on $L^1(\Omega, \mathscr{F}, \mathsf{P})$.

[5] The model can easily be extended to allow for abandonment after exploitation as well, at a considerable technical cost for little economic insight, because the abandonment option after exploitation will be heavily discounted. Therefore, this option will be ignored.

On the probability space $(\Omega, \mathscr{F}, \mathsf{P})$ define two correlated standard Brownian motions B^X and B^Y, with $\mathsf{E}[\mathrm{d}B^X \mathrm{d}B^Y] = \rho \mathrm{d}t$, for some $\rho \in (-1, 1)$. The process $X \triangleq (X_t)_{t \geq 0}$ is the unique strong solution to the stochastic differential equation

$$\mathrm{d}X = \mu_1 \mathrm{d}t + \sigma_1 \mathrm{d}B^X, \quad X_0 = 0, \quad \mathsf{P} - \text{a.s.,},$$

and the process $Y \triangleq (Y_t)_{t \geq 0}$ is the unique strong solution to the stochastic differential equation

$$\mathrm{d}Y = \mu_2 Y \mathrm{d}t + \sigma_2 Y \mathrm{d}B^Y, \quad Y_0 = 1, \quad \mathsf{P} - \text{a.s,}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 > 0$. Dynamic revelation of information is modeled by the natural filtration $\mathbf{F} = (\mathscr{F}_t)_{t \geq 0}$, augmented by the P-null sets.

For any $(x, y) \in E$ one can now define the corresponding process that start at (x, y) as

$$X^x \triangleq x + X, \quad \text{and} \quad Y^y \triangleq yY,$$

respectively. Note that, for any $t \geq 0$, it holds that

$$X_t^x = x + \mu_1 t + \sigma_1 B_t^X, \quad \text{and} \quad Y_t^y = y e^{(\mu_2 - \sigma_2^2/2)t + \sigma_2 B^Y t},$$

so that X^x and Y^y are solutions to the SDEs

$$\mathrm{d}X^x = \mu_1 \mathrm{d}t + \sigma_1 \mathrm{d}B^X, \quad X_0^x = x, \quad \text{and} \quad \mathrm{d}Y^y = \mu_2 Y \mathrm{d}t + \sigma_2 Y \mathrm{d}B^Y, \quad Y_0^y = y,$$

respectively. To ensure finite valuations I make the usual assumption that $r > \mu_2$. For this model the state space can be restricted to $\mathcal{E} = (-\infty, x^*] \times (0, \infty) \subset E$.

While the assumption of a geometric Brownian motion (GBM) to model revenues is standard in the real options literature, the assumption of an arithmetic Brownian motion (ABM) to model construction progress is non-standard. An undesirable consequence of using an ABM is that the noise around mean construction is symmetric. However, it is not unrealistic to assume that X can decrease, i.e. that already constructed structures are removed again. For example, in 2015 during the construction of Brandenburg airport in Berlin, 600 fire protection walls had to be replaced because they were built out of aerated concrete blocks that provided insufficient fire protection.[6] Alternatively one could think of X as a latent variable, where periods of decreasing values represent construction problems that need to be solved first and prevent further progress. Recall that the completion of construction is given by the first hitting time of x^* (from below), which is denoted here by τ_*^x, i.e.

$$\tau_*^x = \inf \{ t \geq 0 \mid X_t^x \geq x^* \}.$$

To keep the notation manageable I will drop the superscript whenever no confusion is possible.

Throughout I have in mind a private concessionaire who takes investment, abandonment, and operationalization decisions. This decision maker (DM) has a sequence of options, which, together, comprise the *project*:

[6] Source: `https://en.wikipedia.org/wiki/Berlin_Brandenburg_Airport`; last accessed 20 December 2020.

1. the option to start construction;
2. after construction has started, the option to abandon construction before it is finished;
3. after construction has finished (conditional on not having abandoned), the option to start operating the railway.

In the remainder, I refer to the second and third options as the *project under construction* and *finished project*, respectively. In the next subsection, we will first study a slightly simpler problem, in which there is no abandonment option. Subsequently, I will add the abandonment option.

2.1. The model without abandonment option

In this subsection, the model without an abandonment option is briefly analysed. If the abandonment option is unavailable, then the decision to invest is irreversible. This implies that construction costs will be incurred until the stopping time τ_*^x. However, after construction has finished, the DM has the option not to exploit the finished project straightaway. At that time, the value of the project only depends on the state of uncertain revenues, $y > 0$, and equals

$$g(y) \triangleq \sup_{\tau} \mathsf{E}\left[-\kappa \int_0^{\tau} e^{-rt}\mathrm{d}t + \int_{\tau}^{\infty} e^{-rt}(Y_t^y - \eta)\mathrm{d}t\right].$$

Applying standard techniques (see, for example, Dixit and Pindyck, 1994) and imposing the boundary condition $g(0) = -\kappa/r$ lead to the following result:

$$g(y) = \begin{cases} -\frac{\kappa}{r} + \left(\frac{y}{\tilde{y}}\right)^{\beta_1}\left[\frac{\tilde{y}}{r-\mu_2} - \frac{\eta-\kappa}{r}\right] & \text{if } y < \tilde{y} \\ \frac{y}{r-\mu_2} - \frac{\eta}{r} & \text{if } y \geq \tilde{y}, \end{cases}$$

where

$$\tilde{y} = \frac{\beta_1}{\beta_1 - 1}\frac{r-\mu_2}{r}(\eta - \kappa),$$

is the threshold that optimally triggers operations (when hit from below) and $\beta_1 > 1$ is the positive root of the quadratic equation

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\sigma_2^2\beta(\beta-1) + \mu_2\beta - r = 0. \tag{1}$$

At a point in time after construction has started, but before it has finished, the value of the project under construction, at state $(x, y) \in \mathcal{E}$, is given by

$$f(x,y) \triangleq \mathsf{E}\Big[-c\int_0^{\tau_*} e^{-rt}\mathrm{d}t + e^{-r\tau_*}g(Y_{\tau_*}^y)\Big]. \tag{2}$$

The optimal time to invest is then obtained by solving the optimal stopping problem

$$v(y) \triangleq \sup_{\tau} \mathsf{E}\Big[e^{-r\tau}f\left(0, Y_\tau^y\right)\Big]. \tag{3}$$

The value of the option to invest depends on X only through the present value (PV) of the construction costs and the fraction by which the present value of the expected revenues is discounted over the (uncertain) construction period.

The solution to (3) can be obtained by applying the usual value-matching and smooth-pasting conditions, which leads to the conclusion that the optimal time to invest is the first-hitting time (from below) of the unique threshold $\tilde{y}^*$ that solves

$$\beta_1 f(0,\tilde{y}^*) = \tilde{y}^* f'_y(0,\tilde{y}^*). \tag{4}$$

The solution to (3) is given in the following proposition, whose proof can be found in Appendix A.

Proposition 1. *The function f is affine, increasing in y, and increasing in x. Equation (4) has a unique solution, which is the optimal investment trigger. The value of the project is*

$$v(y) = \begin{cases} \left(\frac{y}{\tilde{y}^*}\right)^{\beta_1} f(0,\tilde{y}^*) & \text{if } y < \tilde{y}^* \\ f(0,y) & \text{if } y \geq \tilde{y}^*. \end{cases}$$

2.2. The model with exit option

I now turn to the model where the DM can decide (irreversibly) to abandon the project at a sunk cost $K > 0$. As per usual, I start with the final option and work backward. After construction has finished, i.e. when the process X has reached x^*, the DM has a "two-sided" option either to exploit the finished project at some stage, or to abandon the project altogether. So, after construction has finished, the value of the project only depends on the state of uncertain revenues, $y > 0$, but equals

$$G(y) \triangleq \sup_{\tau} \mathsf{E}\left[-\kappa \int_0^{\tau} e^{-rt}\mathrm{d}t + e^{-r\tau} \max\left\{\frac{Y_\tau^y}{r-\mu_2} - \frac{\eta}{r}, -K\right\}\right]. \tag{5}$$

The following result is fairly standard and is proved in Appendix B.

Proposition 2. *There exists a unique pair (y_A, y_I), with $y_A < \tilde{y} < y_I$, such that (5) is solved by the optimal stopping time $\tau_G = \inf\{\, t \geq 0 \mid Y_t^y \notin (y_A, y_I)\,\}$, with value function*

$$G(y) = \begin{cases} -K & \text{if } y \leq y_A \\ -\frac{\kappa}{r} + \check{\nu}_y(y_A,y_I)\left(\frac{\kappa}{r} - K\right) + \hat{\nu}_y(y_A,y_I)\left(\frac{y_I}{r-\mu_2} - \frac{\eta-\kappa}{r}\right) & \text{if } y_A < y < y_I \\ \frac{y}{r-\mu_2} - \frac{\eta}{r} & \text{if } y \geq y_I \end{cases},$$

where

$$\check{\nu}_y(y_A,y_I) = \frac{y_I^{\beta_1} y^{\beta_2} - y^{\beta_1} y_I^{\beta_2}}{y_I^{\beta_1} y_A^{\beta_2} - y_A^{\beta_1} y_I^{\beta_2}}, \quad \hat{\nu}_y(y_A,y_I) = \frac{y^{\beta_1} y_A^{\beta_2} - y_A^{\beta_1} y^{\beta_2}}{y_I^{\beta_1} y_A^{\beta_2} - y_A^{\beta_1} y_I^{\beta_2}},$$

and $\beta_2 < 0$ is the negative root of the quadratic equation $\mathscr{Q}(\beta) = 0$. The function G is convex.

After construction has started, but before it has finished, the value of the project at state $(x,y) \in \mathcal{E}$ with $x < x^*$ is given by

$$\begin{aligned} F(x,y) &\triangleq \sup_{\tau \leq \tau_*} \mathsf{E}\Big[-c\int_0^{\tau} e^{-rt}\mathrm{d}t + e^{-r\tau}\left(1_{\tau<\tau_*}(-K) + 1_{\tau=\tau_*}G(Y_{\tau_*}^y)\right)\Big] \\ &= -\frac{c}{r} + \sup_{\tau\leq\tau_*} \mathsf{E}\Big[e^{-r\tau}\left(\frac{c}{r} + 1_{\tau<\tau_*}(-K) + 1_{\tau=\tau_*}G(Y_{\tau_*}^y)\right)\Big]. \end{aligned} \tag{6}$$

There is no known analytical solution to this problem; Section 4 provides a Markov-chain approximation to tackle this problem numerically.

From the general theory of controlled diffusion process (cf., Krylov, 1980) it is known that (6) has a solution and that the state space can be split in a *continuation region*, $\mathcal{C}$, and a *stopping region*, $\mathcal{D}$. When in state $(x, y) \in \mathcal{E}$, the DM will continue with the project if the present value of the expected free cash flows from a finished project net of the still-to-be-incurred expected construction costs is higher than the cost of using the abandonment option. So, the continuation and stopping regions are the sets

$$\mathcal{C} = \{(x, y) \in \mathcal{E} | F(x, y) > -K\}, \quad \text{and} \quad \mathcal{D} = \{(x, y) \in \mathcal{E} | F(x, y) = -K\},$$

respectively. The following proposition establishes some properties of the value function, F, and the boundary separating the continuation and stopping regions, $\partial\bar{\mathcal{C}}$. Its proof can be found in Appendix C.

Proposition 3. *The continuation region $\mathcal{C}$ and the stopping region $\mathcal{D}$ are separated by a boundary $x \mapsto b(x)$. The value function F and the boundary b have the following properties:*

1. *for all $x < x^*$, it holds that*

$$b(x) = \inf \{ y \in \mathbb{R}_{++} \mid F(x, y) > -K \},$$

   i.e.

$$\partial\bar{\mathcal{C}} = \{ (x, y) \in \mathcal{E} \mid y = b(x) \};$$

2. *the mapping $y \mapsto F(x, y)$ is increasing and convex;*
3. *the mapping $x \mapsto F(x, y)$ is non-decreasing;*
4. *the mapping $x \mapsto b(x)$ is non-increasing and continuous.*

The optimal time to invest is now obtained by solving the optimal stopping problem

$$V(y) \triangleq \sup_{\tau} \mathsf{E} \left[e^{-r\tau} F\left(0, Y_{\tau}^{y}\right) \right]. \tag{7}$$

As before, the value of the option to invest depends on X only through the PV of the construction costs and the fraction by which the PV of the expected revenues is discounted over the (uncertain) construction period. The solution to (7) can be obtained by applying the usual value-matching and smooth-pasting conditions. In particular, the optimal time to invest is the first-hitting time (from below) of a threshold y^*, which solves

$$\beta_1 F(0, y^*) = y^* F_y'(0, y^*). \tag{8}$$

Again, this threshold has to be found numerically; see Section 4. The solution to (7) is given in the following proposition, whose proof can be found in Appendix D.

Proposition 4. *If the mapping $x \mapsto F(x, y)$ is convex and $\rho = 0$, then the solution to (7) is given by the stopping time $\tau^* = \inf \{ t \geq 0 \mid Y_t \geq y^* \}$, where $y^* > 0$ is the unique solution to (8). The value function is*

$$V(y) = \begin{cases} \left(\frac{y}{y^*}\right)^{\beta_1} F(0, y^*) & \text{if } y < y^* \\ F(0, y) & \text{if } y \geq y^* \end{cases}.$$

If $x \mapsto F(x, y)$ is not convex and/or $\rho \neq 0$, then it can be seen from the proof that the result still holds under the following, less easily interpreted, condition:

$$\frac{1}{2}\sigma_1^2 F_{xx}''(x, y) + \rho\sigma_1\sigma_2 y F_{xy}''(x, y) + \mu_1 F_x'(x, y) \geq 0, \quad \text{all } (x, y) \in \mathcal{E}.$$

3. Value for Money and the Value Ratio

When it comes to judging whether or not a project represents "value for money" (VfM) the commonly-used benefit-to-cost ratio (BCR) is not very useful. To see this, suppose that construction time is not uncertain, but equal to some time $T > 0$, and that the DM has no abandonment and no mothballing options. That is, we are in the setting of Section 2.1, but with $\tau_* = T$ a.s., and $g(y) = y/(r - \mu_2) - \eta/r$. Then the value of the project, at state $(x, y) \in \mathcal{E}$, is

$$\begin{aligned} f(x,y) &= \mathsf{E}\left[-c\int_0^T e^{-rt}\mathrm{d}t\right] + e^{-rT}\left(\frac{\mathsf{E}[Y_T^y]}{r-\mu_2} - \frac{\eta}{r}\right) \\ &= \underbrace{-\frac{c}{r}\left(1 - e^{-rT}\right)}_{\text{costs}} + \underbrace{e^{-rT}\left(\frac{ye^{\mu_2 T}}{r-\mu_2} - \frac{\eta}{r}\right)}_{\text{benefits}}, \end{aligned}$$

so that the BCR is well-defined:

$$BCR(y) = e^{-rT}\frac{ye^{\mu_2 T}/(r-\mu_2) - \eta/r}{-(c/r)(1-e^{-rT})}.$$

The "target" BCR, i.e. the BCR above which the project can be deemed to represent value for money, is then given by

$$BCR^* = e^{-rT}\frac{y^* e^{\mu_2 T}/(r-\mu_2) - \eta/r}{-(c/r)(1-e^{-rT})}. \tag{9}$$

When the DM has an abandonment option, the fixed time T gets replaced by a stopping time τ, the value f becomes dependent on the co-movement of X and Y, because the benefits now depend on $\mathsf{E}[Y_\tau^y]$. This implies that benefits and costs can no longer be split and the BCR can no longer be computed.

Alternatively one could use the *value ratio* as a VfM measure, which, for all $y > 0$, I define as the ratio of the values of immediate investment and optimal investment, i.e. (in the model with abandonment option),

$$VR(y) \triangleq \frac{F(0,y)}{V(y)}. \tag{10}$$

Investment is optimal once this ratio reaches 1. Note that both the numerator and denominator evolve with the revenue-related state variable y. Also note that the VR can be negative, because it is possible that $F(0, Y_0) < 0$.

The VR measure has, as an additional advantage, that its VfM "trigger" value is always 1. As is clear from (9), the "trigger" BCR value is different for every application, which makes it less useful as a measure to compare VfMs across different projects.

A disadvantage of the VR measure is that it only tells us something about value destruction in cases where it is not yet optimal to invest in the project, i.e. when $y < y^*$. When $y > y^*$ there is value destruction because the project should optimally already have been invested in. This is, however, a typical feature of any real option model and not particular to this definition. It is entirely driven by the fact that $V(y) = F(0, y)$ on $[y^*, \infty)$.

4. A Markov-Chain Approximation for Numerical Implementation

The optimal stopping problem (6) has no known analytical solution. As a consequence, the first-order conditions (4) and (8) can also not be solved analytically. Therefore, I provide in this section some details of a Markov-chain approximation scheme.

The numerical implementation of the optimal stopping problem (6) uses the Markov chain approximation method as described in, e.g., Kushner (1997) and Kushner and Dupuis (2001). I will focus here on the optimal stopping problem of the project with abandonment option after construction has started, i.e.

$$F(x,y) = \sup_{\tau \leq \tau_*} \mathsf{E}\left[-\int_0^\tau e^{-rt} c \mathrm{d}t + e^{-r\tau}\left(1_{\tau<\tau_*}(-K) + 1_{\tau=\tau_*} G(Y^y_{\tau_*})\right)\right]. \tag{11}$$

For numerical stability it is desirable to use the transformation $z \triangleq \log(y)$, so that (11) can be rewritten as

$$\hat{F}(x,z) \triangleq \sup_{\tau \leq \tau_*} \mathsf{E}\left[-\int_0^\tau e^{-rt} c \mathrm{d}t + e^{-r\tau}\left(1_{\tau<\tau_*}(-K) + 1_{\tau=\tau_*} G(\exp(Z^z_{\tau_*}))\right)\right], \tag{12}$$

which has the property that

$$F(x,y) = \hat{F}(x, \log(y)), \quad \text{all } (x,y) \in \mathcal{E}.$$

Note that a straightforward application of Itō's lemma gives that

$$\mathrm{d}Z = (\mu_2 - \sigma_2^2/2)\mathrm{d}t + \sigma_2 \mathrm{d}B^Y.$$

The basic idea behind the scheme is to replace the continuous-time stochastic process (X,Z) by a discrete-time Markov chain (X^h, Z^h), where $h > 0$ is the step-size on a grid

$$\mathscr{G} \triangleq [\underline{x}, \overline{x}] \times [\underline{z}, \overline{z}]$$

for some $\underline{x} < \overline{x}$ and $\underline{z} < \overline{z}$ that ensure that $\mathscr{G}$ is large enough to produce an accurate approximation to $\hat{F}$. It turns out that choosing $\underline{x} = -150$, $\underline{z} = -10$, $\overline{x}$ as the smallest multiple of h in excess of $\underline{x}$ such that $\overline{x} \geq x^*$, and $\overline{z}$ as the smallest multiple of h in excess of $\underline{z}$ such that $\overline{z} \geq \log(10y_I)$ leads to results that are insensitive to an enlargement of the truncated state space.[7] Any point (x,z) on the grid is then such that

$$x \in \mathcal{N}^h_x \triangleq \{\underline{x}, \underline{x}+h, \dots, x^* - h, \overline{x}\}, \quad \text{and} \quad z \in \mathcal{N}^h_z \triangleq \{\underline{z}, \underline{z}+h, \dots, \overline{z} - h, \overline{z}\}.$$

The next step consists of establishing that the solution to the optimal stopping problem (6) can be found by solving a Hamilton–Jacobi–Bellman (HJB) equation. To establish this result, denote the characteristic operator of the process (X,Y) on the space of twice continuously differentiable functions, C^2, by $\mathscr{L}$, i.e.

$$\mathscr{L}\varphi \triangleq \frac{1}{2}\sigma_1^2\varphi''_{11} + \frac{1}{2}\sigma_2^2 y^2 \varphi''_{22} + \rho\sigma_1\sigma_2 y \varphi''_{12} + \mu_1\varphi'_1 + \mu_2 y \varphi'_2, \quad \text{all } \varphi \in C^2.$$

The following free-boundary problem gives sufficient conditions for a solution to (11).

[7] Recall that y_I is the exploitation trigger upon completion of the project with abandonment option.

Proposition 5. *Suppose that $\mathcal{C} \subseteq \mathcal{E}$ and $\varphi \in C^2(\mathcal{C})$ solve the free-boundary problem*

$$\begin{cases} \varphi > -K, & \text{on } \mathcal{C} \\ \varphi = -K, & \text{on } \mathcal{E} \setminus \mathcal{C} \\ \mathscr{L}\varphi - r\varphi - c = 0, & \text{on } \mathcal{C} \\ \mathscr{L}\varphi - r\varphi - c \leq 0, & \text{on } \mathcal{E} \setminus \mathcal{C} \\ \varphi(x^*, y) = G(y), & \text{all } y > 0, \end{cases} \tag{13}$$

then φ solves (11) *and the optimal stopping time is the first exit time from $\mathcal{C}$.*

The proof of Proposition 5 can be found in Appendix E. From this result it follows that a solution to the HJB equation,

$$\hat{F}(x, z) = \max\left\{1_{x<x^*}(-K) + 1_{x=x^*}G(y) + \hat{F}(x, z), \hat{\mathscr{L}}\hat{F}(x, z) + (1-r)\hat{F}(x, z) - c\right\}, \tag{14}$$

needs to be found. Here $\hat{\mathscr{L}}$ is the characteristic operator of (X, Z), i.e. for all $\varphi \in C^2$ it holds that

$$\hat{\mathscr{L}}\varphi \triangleq \frac{1}{2}\sigma_1^2\varphi_{11}'' + \frac{1}{2}\sigma_2^2\varphi_{22}'' + \rho\sigma_1\sigma_2\varphi_{12}'' + \mu_1\varphi_1' + (\mu_2 - \sigma_2^2/2)\varphi_2'.$$

Substituting the finite-difference approximations

$$\begin{aligned}
\varphi_1'(x, z) &\approx \begin{cases} [\varphi(x+h, z) - \varphi(x, z)]/h & \text{if } \mu_1 \geq 0 \\ [\varphi(x, z) - \varphi(x-h, z)]/h & \text{if } \mu_1 < 0 \end{cases} \\
\varphi_2'(x, z) &\approx \begin{cases} [\varphi(x, z+h) - \varphi(x, z)]/h & \text{if } \mu_2 - \sigma_2^2/2 \geq 0 \\ [\varphi(x, z) - \varphi(x, z-h)]/h & \text{if } \mu_2 - \sigma_2^2/2 < 0 \end{cases} \\
\varphi_{11}''(x, z) &\approx [\varphi(x+h, z) - 2\varphi(x, z) + \varphi(x-h, z)]/h^2 \\
\varphi_{22}''(x, z) &\approx [\varphi(x, z+h) - 2\varphi(x, z) + \varphi(x, z-h)]/h^2, \\
\varphi_{12}''(x, z) &\approx [2\varphi(x, z) + \varphi(x+h, z+h) + \varphi(x-h, z-h)]/(2h^2) \\
&\quad - [\varphi(x+h, z) + \varphi(x-h, z) + \varphi(x, z+h) + \varphi(x, z-h)]/(2h^2), \quad \text{if } \rho \geq 0, \text{ and} \\
\varphi_{12}''(x, z) &\approx [2\varphi(x, z) + \varphi(x+h, z-h) + \varphi(x-h, z+h)]/(2h^2) \\
&\quad + [\varphi(x+h, z) + \varphi(x-h, z) + \varphi(x, z+h) + \varphi(x, z-h)]/(2h^2), \quad \text{if } \rho < 0,
\end{aligned}$$

into the HJB equation (14), defining the notation $a^+ \triangleq \max\{0, a\}$ and $a^- \triangleq \max\{-a, 0\}$, and rearranging gives a discretization of the characteristic operator $\hat{\mathscr{L}}$:

$$\begin{aligned}
\hat{\mathscr{L}}^h\varphi^h(x, z) \triangleq\ & p(x+h, z|x, z)\varphi^h(x+h, z) + p(x, z+h|x, z)\varphi^h(x, z+h) \\
& + p(x-h, z|x, z)\varphi^h(x-h, z) + p(x, z-h|x, z)\varphi^h(x, z-h) \\
& + 1_{\rho\geq 0}\left[p(x+h, z+h|x, z)\varphi^h(x+h, z+h) + p(x-h, z-h|x, z)\varphi^h(x-h, z-h)\right] \\
& + 1_{\rho<0}\left[p(x+h, z-h|x, z)\varphi^h(x+h, z-h) + p(x-h, z+h|x, z)\varphi^h(x-h, z+h)\right].
\end{aligned}$$

Here,

$$
\begin{aligned}
p^h(x \pm h, z|x,z) &= \frac{(\sigma_1^2 - |\rho|\sigma_1\sigma_2)/2 + h\mu_1^+}{Q^h(x,z)}, \\
p^h(x, z \pm h|x,z) &= \frac{(\sigma_2^2 - |\rho|\sigma_1\sigma_2)/2 + h\left(\mu_2 - \sigma_2^2/2\right)^+}{Q^h(x,z)}, \\
p^h(x+h, z+h|x,z) = p^h(x-h, z-h|x,z) &= \frac{\rho^+\sigma_1\sigma_2}{2Q^h(x,z)}, \\
p^h(x+h, z-h|x,z) = p^h(x-h, z+h|x,z) &= \frac{\rho^-\sigma_1\sigma_2}{2Q^h(x,z)},
\end{aligned}
$$

and

$$
Q^h(x,z) = \sigma_1^2 + \sigma_2^2 - |\rho|\sigma_1\sigma_2 + |\mu_1|\, h + \left|\mu_2 - \tfrac{1}{2}\sigma_2^2\right| h,
$$

are the transition probabilities of an approximating Markov chain on the grid $\mathcal{N}_x^h \times \mathcal{N}_z^h$. These probabilities are non-negative if $\rho = 0$ or if

$$
|\rho| < \frac{\sigma_1}{\sigma_2} < |\rho|^{-1}, \quad \text{when } \rho \neq 0.
$$

The HJB equation (14) can now be replaced by the discrete-time approximation

$$
\hat{F}^h(x,z) = (T\hat{F}^h)(x,z), \tag{15}
$$

where the operator T is given by

$$
\begin{aligned}
(T\hat{F}^h)(x,z) := \max\Big\{ & 1_{x<x^*}(-K) + 1_{x=x^*} G(e^z) + \hat{F}^h(x,z), (1 - r\Delta^h(x,z))\mathscr{L}^h\hat{F}^h(x,z) \\
& - c\Delta t^h(x,z)\Big\}.
\end{aligned} \tag{16}
$$

Given a grid $\mathcal{N}_x^h \times \mathcal{N}_z^h$, the time discretization is chosen such that this discrete-time approximation (15) converges weakly to the optimal stopping problem (12). It turns out (see, e.g., Kushner and Dupuis, 2001) that a sequence of time points with (potentially state-dependent) time intervals of length

$$
\Delta t^h(x,z) \triangleq h^2/Q^h(x,z),
$$

achieves this.

Using Blackwell's theorem (Aliprantis and Border, 2006, Theorem 3.53), it is easy to see that the operator T in (15) is a contraction mapping.[8] From the Banach fixed point theorem (Aliprantis and Border, 2006, Theorem 3.48) it then follows that problem (15) has a unique fixed point. Consequently, repeated application of T leads to convergence to the fixed point $\hat{F}^h$, which then acts as the approximation to the value function F.

[8] Blackwell's sufficient conditions for an operator T to be a contraction are

1. *Monotonicity*: $T\hat{F}^h \leq T\hat{H}^h$, whenever $\hat{F}^h \leq \hat{H}^h$, and
2. *Discounting*: there exists some $\beta \in (0,1)$ such that for all $a \geq 0$ it holds that $[T(\hat{F}^h + a)](x,z) \leq [T\hat{F}^h](x,z) + \beta a$.

To implement this scheme, I start with an initial guess $\hat{F}_0^h$ on the grid $\mathcal{N}_x^h \times \mathcal{N}_z^h$. The initial guess is taken by a linear interpolation over the grid of the following boundary conditions.

1. When construction has finished, i.e. when $x = x^*$, the project's value is equal to the option to exploit the project. That is, for all $z \in \mathbb{R}$ it holds that

$$\hat{F}(x^*, z) = G\left(e^z\right).$$

2. If y falls below the exit trigger y_A at the time that construction is finished, the DM will decide to terminate the project straightaway and incur the sunk costs K. That implies that $b(x^*) = y_A$. Since b is non-increasing it is known that the boundary condition

$$\hat{F}(x, z) = -K, \quad \text{all } \underline{x} < x \leq x^* \text{ and } z \leq \log(y_A),$$

can be used. This, in turn, implies that one can take $\underline{z} = \log(y_A)$.

3. As y gets unboundedly large, the option to abandon loses its value, because the probability of y decreasing sufficiently to exercise that option is very small indeed. This observation is implemented by assuming that, for every $\underline{x} < x < x^*$, it holds that

$$\hat{F}\left(x, e^{\overline{z}}\right) = -\left(1 - e^{\alpha_1(x - x^*)}\right)\frac{c}{r} + e^{\alpha_1(x - x^*)}\left(\frac{e^{\overline{z}}}{r - \mu_2} - \frac{\eta}{r}\right),$$

where $\alpha_1 > 0$ is the positive root of the quadratic equation

$$\mathscr{R}(\alpha) \equiv \frac{1}{2}\sigma_1^2\alpha^2 + \mu_1\alpha - r = 0. \tag{17}$$

4. At $\underline{x}$ I impose a linear interpolation of the values at $\underline{z}$ and $\overline{z}$.

From this initial guess I extract an initial guess of the continuation region, $\mathcal{C}_0$, by extracting all points $(x, z) \in \mathcal{N}_x^h \times \mathcal{N}_z^h$ for which $\hat{F}_0^h(x, z) > 1_{x<x^*}(-K) + 1_{x=x^*}G(e^z)$. I then compute a new iteration, $\hat{F}_1^h$ by applying the operator T, i.e.

$$\hat{F}_1^h \triangleq T\hat{F}_0^h.$$

This procedure is repeated until the change between $\hat{F}_n^h$ and $\hat{F}_{n-1}^h$ (in the sup norm) falls below 1.

To find the value function f for the project without abandonment option, I follow the same procedure, but with the operator $\tilde{T}$, defined by

$$(\tilde{T}\hat{f}^h)(x, z) \triangleq (1 - r\Delta^h(x, z))\mathscr{L}^h\hat{f}^h(x, z) - c\Delta t^h(x, z).$$

Note that the HJB equation for the function f is

$$\mathscr{L}f - rf - c = 0, \quad \text{on } E.$$

This follows from the Feynman-Kac theorem, rather than from Bellman's optimality principle. The boundary conditions imposed on f are the same as those imposed on F, but with the second one replaced by the following.

2'. If $y = 0$, then the project will never be exploited after construction has finished. Therefore, the DM will only incur construction costs until construction is completed and mothballing costs forever after, i.e. for all $x < x^*$ it holds that

$$\lim_{z \to -\infty} \hat{f}(x, z) = -\left(1 - e^{\alpha_1 (x - x^*)}\right) \frac{c}{r} - e^{\alpha_1 (x - x^*)} \frac{\kappa}{r},$$

where $\alpha_1 > 0$ is the positive root of the equation $\mathscr{R}(\alpha) = 0$; cf. (17). I implement this boundary by imposing it at $\underline{z}$.

The final problem is to determine the value functions v and V. Here, I discuss the procedure for approximating V; the procedure for v is identical. Even though there is an implicit equation for the threshold,

$$\beta_1 F(0, y^*) = y^* F'_y(0, y^*),$$

and an analytical form for the value function in the continuation region,

$$V(y) = \left(\frac{y}{y^*}\right)^{\beta_1} F(x, y),$$

standard methods to solve for y^* cannot be used, because only approximations to F at points $(x, z) \in \mathcal{N}_x \times \mathcal{N}_z$ are available. I, therefore, first rewrite the first-order condition in terms of z,

$$\beta_1 \hat{F}(0, e^z) = \hat{F}'_z(0, z), \tag{18}$$

and then replace $\hat{F}'_z$ by its backward difference for $z \in \mathcal{N}_z$.[9] A search over $\mathcal{N}_z$ then gives the value z^* that solves (18) most closely.

5. Illustration: High-Speed Rail in the UK

In this section, the model will be applied to a proposed high-speed rail project in the UK, so-called HS2, connecting, in the first phase, London to Birmingham and in a second phase connecting Birmingham to Leeds and Manchester. The main reason to build HS2 is improving the connectivity between the north and the south of the UK. When completed a train journey from London to Manchester will only take 68 minutes instead of the current 128 minutes; see Department for Transport (2013, Figure 11). In the following, I am particularly interested in exploring how much value is created by the addition of an abandonment option.

5.1. Base-case model

I will, as much as possible, base the analysis on a 2013 report into the economic case for HS2, published as Department for Transport (2013). This report estimates the present value (in 2011 prices) of benefits of this rail link to be £28 bn (this includes £4.3 bn in wider economic benefits),

[9] I use a backward difference here because, while $\hat{F}$ will be C^1 on the boundary of the continuation region, it will in general not be C^2.

whereas the (present value) of construction costs are estimated to be £15.65 bn. Operating costs are estimated to have a present value of £8.2 bn. The report includes capital spending such as replacement of rolling stock, etc., which will be ignored here. The report then provides a benefit-to-cost ratio (BCR) of 1.7, which renders this a "medium value" project in government parlance (see the table in Section 1).

Since this estimate of the BCR is obtained without using an explicitly dynamic model, the parameters for the model described here have to be calibrated and "guesstimated" based on the information provided. The estimate of the (present value of the) construction costs are reported with the upper bound of a 95% prediction interval of £21.4 bn, and an estimated time to completion of 8 years. Since we model the construction of track is modeled as (the supremum process of) an arithmetic Brownian motion, the volatility of this process can be found to be $\sigma_1 = 9.94$ miles p.a.[10] The drift of the process is $\mu_1 = 150/8 = 18.75$ miles p.a., since Birmingham is 150 miles from London. This also suggests a terminal value $x^* = 150$. The construction cost flow is inferred to be $c = £2.24$ bn p.a..

To capture the effect of construction uncertainty on likely delays, I compute, for $x < x^*$, the smallest time T_α such that $\mathsf{P}(M^x_{T_\alpha} \geq x^*) = \alpha$, for some $\alpha \in (0,1)$, using the formula (see, e.g., Harrison, 2013, Section 1.9)

$$\mathsf{P}(M^x_{T_\alpha} \geq x^*) = \Phi\left(\frac{\mu_1 T_\alpha - (x^* - x)}{\sigma_1 \sqrt{T_\alpha}}\right) + e^{2\mu_1(x^*-x)/\sigma_1^2}\Phi\left(\frac{(x - x^*) - \mu_1 T_\alpha}{\sigma_1\sqrt{T_\alpha}}\right), \tag{19}$$

where

$$M^x_t \triangleq \sup_{0 \leq s \leq t} X^x_s.$$

In the HS2 case we get, for example, $T^0_{0.95} = 10.68$ years. That is, with probability 95%, construction will be finished within 10.68 years. Similarly, with probability 95%, construction will take at least 5.79 years. While this may look like an implausibly low confidence bound, it is a consequence of the symmetric distribution implied by the ABM process.

The discount rate used in the report is 3.5%, which is transformed to the continuous rate $r = 0.0344$. The present value of the benefits of the railway is estimated to be £28 bn. No clear growth rate of revenues is mentioned in the report, so it will be assumed here that $\mu_2 = 0.022$, which is the report's assumed growth rate of passenger numbers. A present value of £8.2 bn for operating costs leads to a constant operating cost flow of $\eta = £0.28$ bn p.a. With an estimated construction time of 8 years, this would lead to a PV of operations after construction has finished and assuming that operations start immediately of $Y_8/(r - \mu_2) - \eta/r$. Discounted back to $t = 0$, i.e. over 8 years, the authors assume this value to be equal to £8.2bn. This then implies that

[10] I assume that the report's authors used a normal distribution and that the reported 95% prediction interval for the PV of construction costs is symmetric around the point estimate. I then transform the upper bound into an upper bound of constructed track, UB, over $T = 8$ years through multiplying by x^*/I. I then estimate σ_1 by $(UB - x^*)/(1.96\sqrt{T})$.

Description	Parameter	Source	Value
construction duration	T	report	8 years
present value of revenues	$PV_R(Y_0)$	report	£28.94 bn
present value of operating costs	$e^{-rT}\frac{\eta}{r}$	report	£8.2 bn
present value of construction costs	$PV_I(\hat{x})$	report	£15.65 bn
expected construction time	$\mathsf{E}_{\hat{x}}[\tau(x^*)]$	report	8 years
discount rate	r	report	3.44%
construction state at completion	x^*	inferred	150 miles
current revenues (p.a.)	Y_0	inferred	£313.8 mln
construction costs (p.a.)	c	inferred	£2.24 bn
operating costs (p.a.)	η	inferred	£0.28 bn
mothballing costs (p.a.)	κ	assumed	£0.14 bn
Sunk abandonment costs	K	assumed	£1 bn
expected construction (p.a.)	μ_1	inferred	18.75 miles
expected revenue growth rate	μ_2	assumed	2.20%
construction volatility	σ_1	inferred	9.94 miles
revenue volatility	σ_2	assumed	20%
correlation coefficient	ρ	assumed	0

Table 1: Data for the base-case of a numerical analysis of the HS2 project.

$Y_8 = £0.3742$ bn. Since $\mathsf{E}(Y_t^{Y_0}) = Y_0 e^{\mu_2 t}$, this implies that $Y_0 = e^{-8\mu_2} = £0.3138$ bn. The volatility of revenues accruing from HS2 is taken to be $\sigma_2 = 0.2$.[11]

Finally, I assume that the mothballing costs are $\kappa = \eta/2 = £0.14$ bn p.a. and that the sunk abandonment costs are $K = £1$ bn. These parameter values are summarized in Table 1 below. In the computations I follow the Markov-chain approximation from Section 4 on a 791×684 grid, with $h = 0.38$, $\underline{x} = -150$ and $\overline{y} = 260$. For the base-case parameters it holds that

$$\alpha_1 = 18.26 \cdot 10^{-4}, \quad \beta_1 = 1.26, \quad \text{and} \quad \beta_2 = -1.36.$$

The left-hand panel of Figure 1 shows the value functions of the project at the moment of construction completion for the models without and with abandonment option, i.e. the functions g (in green) and G (in blue). The corresponding NPV functions are depicted by dashed lines. It is clear that the abandonment option provides a floor under the losses that exploitation of the project can generate, which increases the project's value. For the base-case model, without abandonment option the exploitation trigger is $\tilde{y} = £244.6$ mln, whereas the abandonment and exploitation triggers in the model with abandonment option are $y_A = £40.7$ mln and $y_I = 259.4$ mln, respectively. Note that, as expected, $y_A < \tilde{y} < y_I$.

The right-hand panel of Figure 1 shows the value functions at $t = 0$. The NPV of immediate construction $f(0, \cdot)$ (the green dashed line). Given the estimate of Y_0 derived above, it is found that the current value of immediate investment is $f(0, Y_0) = £761$ mln. The NPV of immediate investment in the model with abandonment option (the blue dashed line) gives the Markov chain approximation to $F(0, \cdot)$. The optimal exercise boundary is depicted in Figure 2. It is found that

[11] This figure is not given in the report, but is commensurate with values used regularly in the literature, see Dixit and Pindyck (1994).

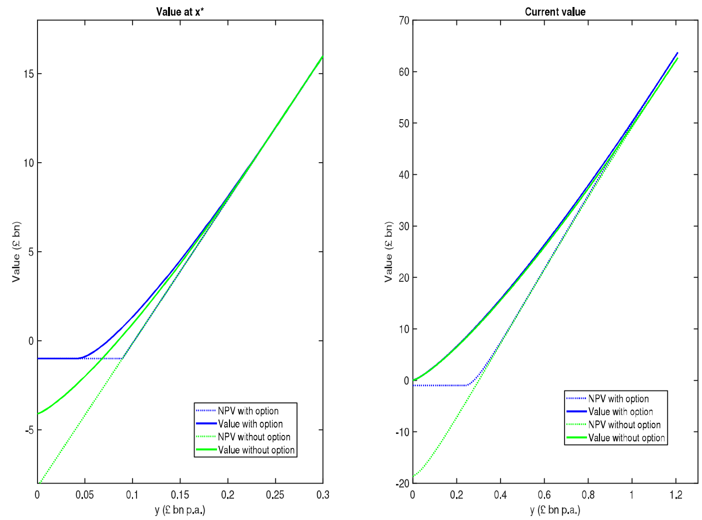

Figure 1: Value functions at x^* (left-panel) and $t = 0$ (right-panel). The green lines depict value functions for the model without abandonment option, whereas the blue lines denotes values for the model with abandonment option.

the current NPV of the project is $F(0, Y_0) = £1.68$ bn. This shows that an abandonment option creates significant value.

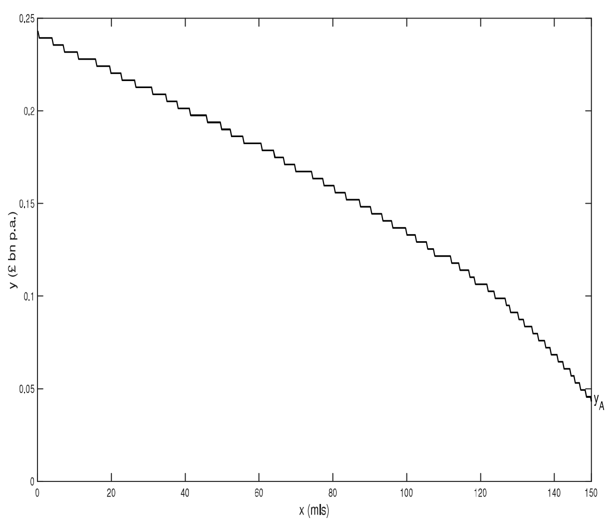

Figure 2: Markov-chain approximation of the optimal abandonment boundary $x \mapsto b(x)$ during the construction phase. The trigger at x^* is equal to the optimal abandonment trigger after construction has finished, y_A.

The right-hand panel of Figure 1 also depicts (solid lines) the optimal value of the project for both the model without (green) and the model with (blue) abandonment option. In the former model the optimal investment trigger is $\tilde{y}^* = £1.13$ bn, whereas in the latter model it is $y^* = £1.15$ bn. So in the base case, investment is optimal only when the current estimate of annual revenues of exploitation are so high that it is very unlikely that the abandonment option will ever be exercised.

This explains why the values with and without abandonment option are not very different: $v(Y_0) =$ £11.52 bn and $V(Y_0) =$ £11.71 bn. In both models, the probability that investment will be optimal in the next 10 years is around 4.3%. The value ratio (cf. (10)), however, is quite different: 0.0660 without abandonment option and 0.1435 with abandonment option. This reflects the fact that the NPV of immediate investment is much higher in the model with abandonment option. For example in the model with abandonment option, the NPV of immediate investment represents 14% of the optimal value.

The analysis presented so far may give the impression that the abandonment option has no value at all, but that conclusion would be incorrect. Once construction has started it is, of course, possible (albeit unlikely) that the exercise boundary gets approached. At that time, the abandonment option increases in value.

In Figure 3 I have plotted the difference between the value of the project once construction has started in the model with abandonment option, F, and in the model without abandonment option, f. As can be seen, the abandonment option is very valuable value near the exercise boundary, but vanishes fairly rapidly for higher values of y. This shows that, while the abandonment option may not affect the original investment decision very much, it can create a lot of value during the construction phase and should, therefore, not be ignored.

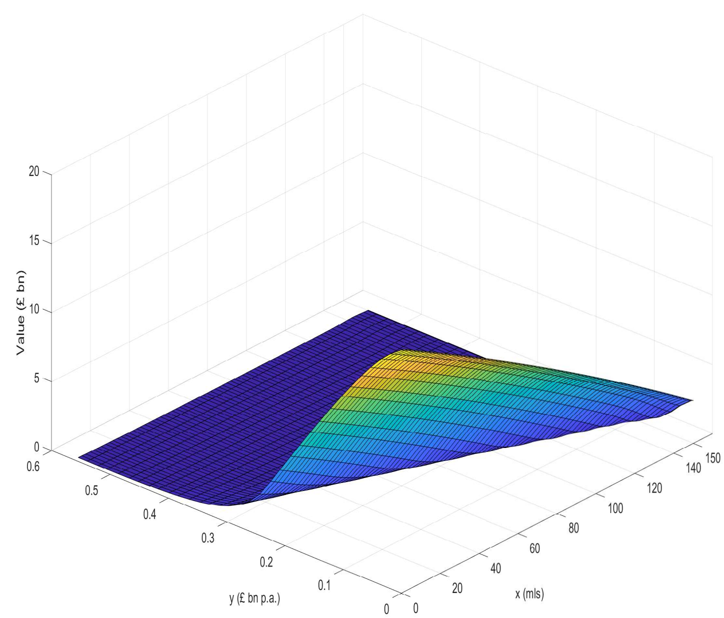

Figure 3: Markov-chain approximation of the value of the abandonment option.

5.2. Comparative statics

In order to understand the sensitivity of the results obtained for the base-case scenario, a comparative statics analysis is conducted in this section for several parameters.

Revenue volatility. The sensitivity of the results with respect to the volatility of revenues, σ_2, is depicted in Figure 4. This parameter does not feature in government assessments of the HS2 project and, thus, had to be assumed. As can be seen, for low values of σ_2, the project values with or without abandonment option are quite similar, as are the optimal investment triggers and the probability of investment being optimal anytime in the next 10 years. For larger values of σ_2, however, the abandonment option has substantial value. In particular, while the optimal investment trigger is, as expected, increasing in σ_2, the probability that investment will be optimal at any time in the next 10 years is *increasing* for the project with abandonment option, whereas it is first increasing and then decreasing for the project without the abandonment option. Also note that as expected, the mothballing region upon finishing construction is expanding in σ_2. Here, too, the abandonment option creates value. As can be seen it also leads to later commercialization than when the option to abandon is not available. This is the flip side of the abandonment option: because it is valuable one should be less willing to give it u, even by commercialization.

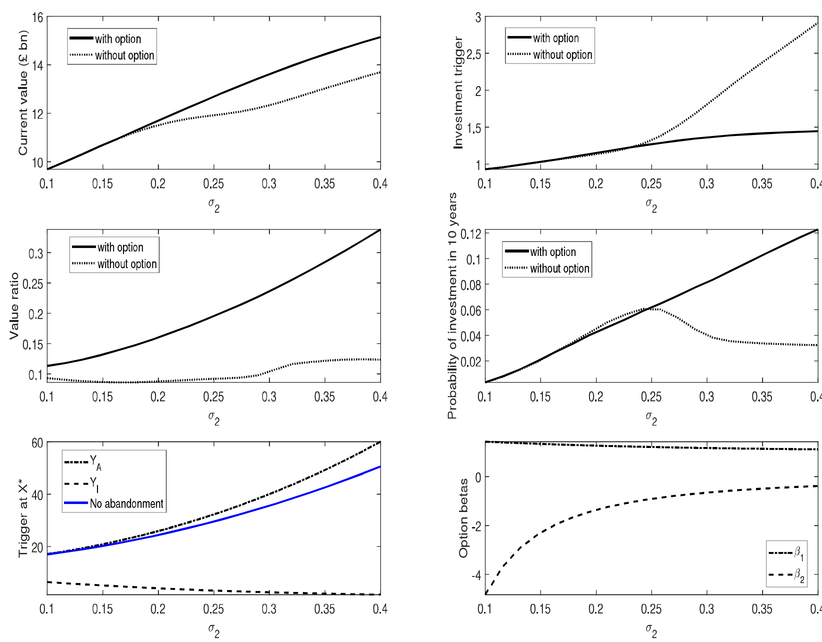

Figure 4: Value at $t = 0$ (top-left panel), optimal investment trigger (top-right panel), Value ration (middle-left panel), probability of investment in 10 years (middle-right panel), investment triggers upon completion in the model with abandonment option (bottom-left panel), and β_1 (bottom-right panel) as a function of σ_2. All other parameter values are as in the base-case scenario.

Revenue trend. The sensitivity of the results with respect to the trend in revenues, μ_2, is depicted in Figure 8 in Appendix F. Here we see that for reasonable values of the expected revenue growth rate the project values, investment triggers and investment probabilities with or without abandonment option are very close together. The only real effect is on the value ratio, which is substantially lower for the project without abandonment option with lower values for the growth rate. This happens because the NPV of immediate investment is much lower when there is no abandonment option. As expected, the optimal investment trigger is decreasing and the value, value ratio and probability of investment are all increasing in μ_1.

Construction standard deviation. The sensitivity of the results with respect to the standard deviation in annual rail construction (in miles), σ_1, is depicted in Figure 9 in Appendix F. Here the value of the project with abandonment option is increasing, whereas the value of the project without abandonment option is decreasing. This happens because in the latter case the investment decision has more downside risk that is not capped by the abandonment option, while both project have the same upside of a riskier construction process. Both projects' optimal investment triggers are fairly insensitive to the risk in the construction process (there is some numerical noise in the trigger for the project with abandonment option), which is not surprising given that I have assumed that the DM is risk neutral. Note that the trigger for the project with abandonment option is decreasing for larger values of σ_1 because then the upside becomes valuable enough for the DM to invest earlier. Also note that the trigger for the project without abandonment option lies below than the trigger for the project with such an option, although not by much. The intuition for this result lies in the fact that the commercialization trigger at completion is higher for the project with abandonment option. While the option adds value, it also makes it more valuable to wait, which is reflected by a higher investment trigger.

Expected annual construction. The sensitivity of the results with respect to the expected annual rail construction (in miles), μ_1, is depicted in Figure 5. As before, we see that the abandonment option adds value, although that value is eroded by a higher expected construction speed. This is not surprising, because if construction is expected to finish sooner, then there is less construction risk over time (because the variance of miles constructed is linear in time), which makes the abandonment option less valuable. Also notice here that the optimal investment trigger with abandonment option can be either higher or lower than without this option. This happens because of two opposing effects. On the one hand, the project with abandonment option has a higher commercialization trigger at completion, which increases the optimal investment trigger relative to the project without abandonment option. This effect dominates for higher values of μ_1, i.e. when construction is expected to be faster. On the other hand, for lower values of μ_1, i.e. when construction is expected to take longer, this effect is discounted away more strongly, while the project without abandonment option becomes more risky due to a longer construction time. This makes the floor under construction losses presented by the abandonment option more valuable and allows for earlier investment.

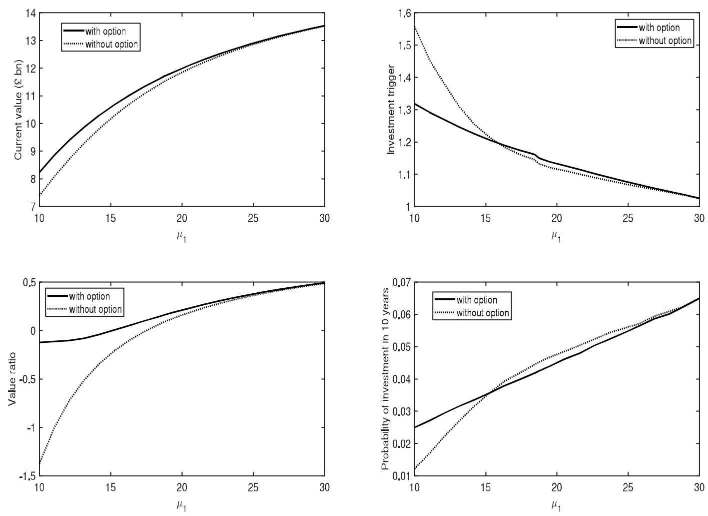

Figure 5: Value at $t = 0$ (top-left panel), optimal investment trigger (top-right panel), Value ration (bottom-left panel), and the probability of investment in 10 years (bottom-right panel) as a function of μ_1. All other parameter values are as in the base-case scenario.

Discount rate. The effect of the discount rate, r, is shown in Figure 6. It is clear that the project's value, optimal investment trigger and investment probability are all highly sensitive to this parameter. Especially for larger values of r, the abandonment option allows for earlier triggering of the investment compared to the project without abandonment option. The intuition for this result is that with a higher discount rate the PV of a finished railway line is lower, an effect that is partly offset by the increased value of the option to abandon.

Construction size. Project values, investment triggers, and investment probabilities are, unsurprisingly, also very sensitive to the size of the size of the construction project as parameterized by x^*. Here too though, the differences between the projects with and without abandonment options are quite small. This can be seen in Figure 10 in Appendix F.

Construction costs. From Figure 11 in Appendix F it is clear that higher construction costs reduce value and delay the likelihood of investment. Here it can be seen, however, that the value erosion due to higher construction costs is less for the project with abandonment option. This happens because the floor on costs that the abandonment option gives is more valuable when construction costs are higher.

Sunk abandonment costs. So far it has been assumed that the abandonment costs are independent of the amount of track that has been constructed before the decision to abandon is taken. It is perhaps more reasonable to assume that the abandonment costs are increasing in the mileage

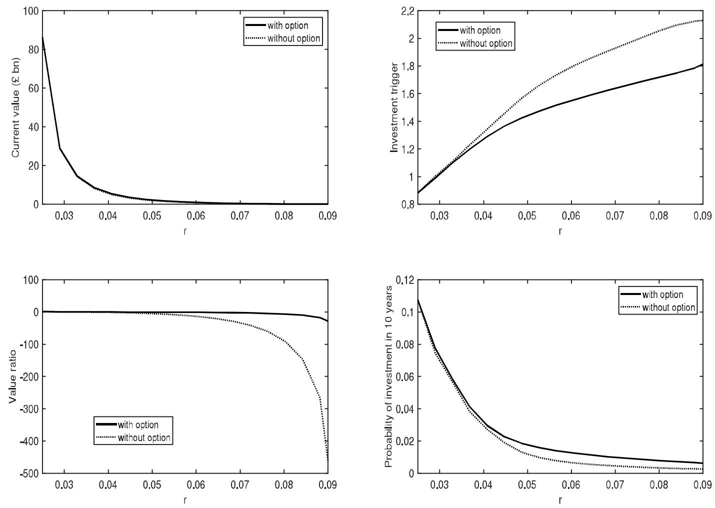

Figure 6: Value at $t = 0$ (top-left panel), optimal investment trigger (top-right panel), Value ration (bottom-left panel), and the probability of investment in 10 years (bottom-right panel) as a function of r. All other parameter values are as in the base-case scenario.

of track already constructed. So, instead one can think of a per-mile sunk cost of abandoning/decommissioning the track. For this comparative statics analysis I use the following piece-wise linear sunk cost function (with a slight abuse of notation):

$$K(x) = \begin{cases} K_0 & \text{if } x \leq 0 \\ K_0 + \frac{K - K_0}{x^*} x & \text{if } 0 < x < x^* \\ K & \text{if } x \geq x^*, \end{cases}$$

for some constant $K_0 \in [0, K]$. From Figure 12 in Appendix F it can be seen that varying K_0 does not have a significant impact on project value, nor on investment triggers and investment probabilities. The intuition for this result is that the sunk costs are low in that region of the state space where it is unlikely that the abandonment option is going to be used, i.e. in the early stages of construction when x is small. There is a small effect on the value ratio, which is mainly due to a decrease in the NPV of immediate investment.

Wider economic benefits. As a final exercise, I compute the additional wider economic benefits that are required to make immediate investment in the project just optimal in the model without abandonment option. Note that the present value of revenues already include £4.3 bn of wider economic benefits. Figure 13 in Appendix F shows some comparative statics for the wider economic benefits (WEBs). These are added to the estimated present value of direct benefits upon construction being finished. This value is then transformed into an estimate of Y_0 in the same way as above. That is, the estimate of current (p.a.) total revenues accruing from a fully finished and exploited railway are increasing in the wider economic benefits. It is clear from the analysis that the presence of the abandonment option does not change the estimate of the current value of the project much, even for larger values of WEBs. The value ratios, too, stay very close together as do the probabilities of investment being optimal within the next 10 years. These probabilities only start diverging somewhat when WEBs exceed approximately £140 bn. This is due to the fact that the optimal investment trigger starts *increasing* in WEBs for the project without abandonment option. The intuition for this somewhat counter-intuitive result is that at any time the DM has to balance two counter-balancing effects. First, higher WEBs lead to a higher "prize" (in expectation) after construction has finished. Therefore, the DM has an incentive to invest sooner. On the other hand, higher WEBs also allow the DM to insure herself more cheaply (because the discounted benefits at the time construction is expected to finish are higher) against the possibility of higher than expected construction costs, which give the DM an incentive to invest later. In the model with an abandonment option the latter effect is less important, because the abandonment itself already provides insurance against this possibility.

Conclusion. In conclusion, the presence of an abandonment option is particularly valuable in situations where the revenue volatility is high, the expected construction speed is low, or the discount rate is high.

6. Concluding Remarks

The main focus of this paper has been to develop a model to value projects with substantial levels of uncertainty around construction costs and to assess the value of an embedded abandonment option. The model should help decision makers and/or policy makers in framing investment and abandonment decisions for projects with long and uncertain construction times.

Some analytical properties of the model were explored and a Markov-chain approximation scheme to implement the model numerically was developed. When applied to the case of building the HS2 railway line in the UK, it can be concluded that the HS2 project should not be executed at this point in time and that the probability that it should start in the next 10 years is very small indeed. For this case I find that the abandonment option is particularly valuable in situations where the revenue volatility is high, the expected construction speed is low, or the discount rate is high.

The typical value-for-money (VfM) measure that is used in project evaluation, the BCR, is not appropriate when construction duration uncertainty is present. This uncertainty links the benefits and costs of a project in such a way that they can not be separated and, thus, their ratio cannot be computed in any useful way. I, therefore, propose the use of the "value ratio" (VR), which is the ratio of a project's NPV of immediate investment over the value if the optimal investment strategy is used. An additional feature of the VR is that its "trigger value" (i.e., the value beyond which investment is optimal) is always 1, regardless of the project under consideration. This makes the VR an attractive measure from a practical point of view, because it allows for a VfM comparison across a variety of projects. In addition, while the VR requires the use of real options theory, the VR itself can easily be explained to non-experts and should, thus, be of use to OR practitioners.

The model presented in this paper describes the (sequence of) decisions taken by a private concessionaire. This ignores welfare effects to the wider economy in the decision making process. In the comparative statics section, a first step towards including social benefits through the "wider economic benefits" is taken, but a full exploration of these issues will be left to future research.

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