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Efficiency, Stability, and Commitment in Senior Level Job Matching Markets[*]

Ning Sun[†] and Zaifu Yang[‡]

Abstract: We propose a senior level job matching model with multiple heterogeneous incumbents and entrants. An agent (firm or worker) can be committed or uncommitted (i.e., free). A committed agent is bound by her commitment and cannot unilaterally dissolve her partnership unless her partner agrees to do so. A free agent can make decisions independently. Every agent has preferences over multiple contracts and tries to find her best possible partner with contract. We examine the problem of how to match workers and firms as well as possible and to set free as many committed agents as possible without violating any commitments. We show the existence of stable and (strict) core matchings through a constructive procedure and obtain a lattice result for such outcomes and incentive compatibility results for the procedure.

Keywords: Job matching, core, stability, commitment, procedure, incentive.

JEL classification: C71, C78, D47, J12.

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[†]N. Sun, Institute for Social and Economic Research, Nanjing Audit University, Nanjing 211815, China; `sunning@nau.edu.cn`.

[‡]Z. Yang, Department of Economics and Related Studies, University of York, York YO10 5DD, UK; `zaifu.yang@york.ac.uk`.

1 Introduction

This paper introduces a new model of senior level job matching market with commitments. Following the classic work of Gale and Shapley (1962) substantial progress has been made in understanding entry level labor markets in which all participants are new entries and firms try to hire workers to fill their positions and workers search for jobs; see e.g., Roth and Sotomayor (1990). However, there is not much study of senior level job matching markets. We attempt to fill a gap by investigating such a market. The market under consideration consists of multiple heterogeneous incumbents and entrants. An incumbent can be a firm or a worker who has an initial partner (worker or firm) and an entrant can also be a firm or a worker. A free agent can make decisions independently. A committed agent cannot unilaterally dissolve her partnership but can do so only with her partner's consent.

Commitments exist in various forms and contexts and can influence people's behavior and affect the performance of the system involved. They can be imposed by law, by custom, by contract, by convention, or by morality. For instance, universities with a tenure track system are committed to their tenured faculty members in the sense that they generally cannot fire a tenured professor unless she/he is willing to leave. But, a tenured professor can move rather freely without facing this kind of commitment constraint. The civil service system is another typical example of similar nature where government employees are free but their employers are committed. US national football league guaranteed contracts are commitments of clubs to their players. Yet another prominent example of opposite nature is the non-compete clause or the restrictive covenants in contract law under which an employee agrees to commit herself to her position for a certain period of time. In all these cases, commitment lies on one side. The mutual consent divorce law (see e.g., Voena 2015) permits divorce only when both husband and wife agree to it. Under this law, both husband and wife are committed to each other and can divorce if both consent to do so. This is a typical case of two-sided commitment. In some professions involving highly sensitive matters, a job contract may explicitly require an employee to commit to the position for at least a certain period of time. In this environment, commitment is two-sided with one side being explicit and the other being implicit. Schelling (1956, 1960) is the first to study how players may use commitments as tactics to advance their interest in bargaining or negotiation. Commitment has been further studied in repeated games, contracts, fiscal and monetary policies, etc; see e.g., Laffont and Martimort (2002), Mailath and Samuelson (2006), and Lucas and Stokey (1983). The meaning of commitment may, however, vary from one situation to another.

In our model every firm has preferences over workers with multiple contracts and the prospect of not hiring any one and every worker has preferences over firms with multiple contracts and the prospect of being unemployed. Following the two-sided matching models with wages and relevant job characteristics given by Crawford and Knoer (1981), Kelso and Crawford (1982), Hatfield and Milgrom (2005), and Ostrovsky (2008), we represent the relation governing every worker and every firm by contracts. Every contract signifies a relationship between a firm and a worker and specifies the amount of remuneration to be paid by the firm, and the service to be supplied by the worker in return for the payment. There can be multiple contracts between every firm and every worker. When a firm and a worker haggle over contracts and reach an agreement, only one contract will be taken by both sides. A free agent makes decisions independently, while a committed agent cannot unilaterally dissolve her partnership to find a better alternative without the consent of her partner. Every agent tries to find her best possible partner with contract. The central issue is how to match workers and firms as well as possible and to set free as many committed agents as possible without contravening their commitments.

Our analysis focuses on what can be a reasonable and natural solution to this new matching problem and how to design a mechanism for finding the solution. In the two-sided matching literature, the notion of (pairwise) stability is the most widely used solution stemming from Gale and Shapley (1962); see also Roth and Sotomayor (1990). We need to adapt this concept to our setting which contains not only chains but also cycles induced by the presence of commitments. The existing literature typically deals with settings without cycles. In our model we call a firm and a worker a pair if they are matched by a common contract. A matching is a collection of contracts with their corresponding pairs and singles. A matching state consists of a matching and an associated set of committed agents. We say that a matching state is stable if it is individually rational and not blocked by any of its chains or cycles. Our second solution concerns the fundamental concept of core from game theory and equilibrium theory; see e.g., Gillies (1959), Scarf (1967), Shapley and Scarf (1974), and Predtetchinski and Herings (2004). The core of an economic model is the set of outcomes that can be achieved collectively by the entire group of market participants but cannot be improved upon by any agent or coalition of agents, acting by themselves. Its analogue in the current model is introduced to accommodate the initial matching and its associated set of committed agents. The core does not coincide with the set of stable matchings, in a sharp contrast with the Gale-Shapley model for which both sets are identical. Our first major Theorem 1 establishes the existence of at least one stable matching state whose matching is in the strict core. This outcome is called a stable core matching state. At this outcome every committed agent must be committed at the initial state and there can be less committed agents than at the initial state.

A key step in our analysis is to develop a market mechanism for finding a stable core matching state and thus to give a constructive proof of our Theorem 1. This mechanism to be called *the Hybrid Procedure* is a novel blend of two generalizations of the deferred acceptance (DA) procedure of Gale and Shapley (1962) and the top trading cycle (TTC) method of Shapley and Scarf (1974). It should be noted that neither of the two generalizations is sufficient to reach a stable core matching state but the Hybrid Procedure always finds it. In our generalized DA procedure, free workers propose their most-preferred contracts to their corresponding firms while any firm which receives any proposals rejects all proposals but her favorite one subject to the constraint that if she is a committed firm and receives the initial contract proposed by her initial partner worker, she should rank this proposal above any other proposal. A committed worker can become free during the process if his initial partner has been provisionally matched to another worker. In our modified TTC procedure, each worker in an almost committed set will propose his favorite contract among all his mutually relatively acceptable contracts to its associated firm and each firm in the almost committed set points to her initially matched worker. The Hybrid Procedure implements sequentially the generalized DA procedure and the modified TTC procedure and produces a stable core matching state as its final outcome. In the process some committed agents will be set free and so the stable core matching state contains fewer committed agents than the initial matching state.

We also examine the important issues concerning the structure of the family of stable matching states and the strategic behavior of market participants in the Hybrid Procedure. We show that the family of stable matching states with a common commitment set is a lattice, generalizing the classical lattice result of Gale and Shapley (1962). We obtain incentive compatibility results saying that it is a dominant strategy for agents in a certain group to act truthfully when facing the Hybrid Procedure, extending the well-known result of Dubins and Freedman (1981) for the Gale-Shapley marriage matching. Furthermore, we derive results for the case of job-specific commitments.

Blum et al. (1997) are the first to examine a senior level one-to-one labor market with incumbents and new entries. Each firm has preferences over all workers and herself and every worker has preferences over all firms and himself. All agents are free and have no commitment. They show that when the market is destabilized, it can regain stability in the sense of Gale and Shapley by a decentralized process of offers and acceptances. Our model contains theirs as a special case. Our model, motivation, solution and procedure differ from theirs. First, our model incorporates contracts which contain explicitly salaries and other characteristics concerning each job and each worker and which can be negotiated between firms and workers. This feature allows us to examine real life competitive markets with monetary transfers; see Crawford and Knoer (1981), Kelso and Crawford (1982), Crawford (1991), Hatfield and Milgrom (2005), Ostrovsky (2008), and Hatfield et al. (2013). Second, we introduce commitments which enable us to handle a variety of practical situations and make our model markedly different from theirs. Third, while their stability is the traditional one concerning only pairs, our stability involves chains and circles. Our solution is the intersection of stability and (strict) core and our process is totally different from theirs.

Abdulkadiroğlu and Sönmez (1999) study a house allocation model such as college dormitories or subsidized public houses where there are both existing tenants and new applicants. They propose a modification of the TTC method that is individually rational, Pareto-efficient, and strategy-proof. Sun and Yang (2016) consider a marriage matching model in which there are many single men and single women and married couples. Married couples are two-sided committed and can divorce under the mutual consent divorce law.

We show the existence of a stable and core matching between men and women via a constructive procedure. Diamantoudi et al. (2015) investigate the role of commitment in a dynamic matching model. Their commitment differs crucially from ours in that if both parties are committed to each other, they stay together permanently. Combe et al. (2017) examine a problem of teacher assignment in their independent study. Their model can be seen as a senior level market with only incumbents. Their teacher optimal block exchange algorithm modifies the TTC method, is strategy proof for teachers, and yields a two-sided maximal outcome.[1] Their model and solutions are different from ours.

This paper is organized as follows. Section 2 introduces the model and basic concepts. Section 3 presents the procedures and stable and core matching existence theorems. Section 4 establishes lattice and incentive compatibility results. Section 5 discusses the case of job-specific commitments. Section 6 concludes. Several proofs are given in the appendix.

2 Model and Basic Concepts

Consider a senior level job matching labor market with many heterogeneous incumbents and entrants. Let $W = \{w_1, \cdots, w_s\}$ denote the set of all workers and $F = \{f_1, \cdots, f_t\}$ the set of all firms. We use the English letters f, w, f_1, w_1, x, etc to denote agents (workers or firms) and A, B, etc to denote sets of agents. We will take several steps to give a full description of the model. Throughout the paper we treat any worker as male and any firm as female. When we talk about a generic agent which can be a firm or worker, we treat the agent as female. We assume that every firm hires at most one worker and every worker can work for at most one firm.[2]

Relationships between firms and workers are governed by bilateral contracts. Contracts

[1] Their model implicitly implies that teachers and schools are two-sided committed in our sense. In their model each school can hire multiple teachers. A two-sided maximal matching that means individual rationality and Pareto-efficiency for both teachers and schools is not necessarily in the (strict) core nor (strongly) stable in our current model.

[2] This assumption has been used in Koopmans and Beckmann (1957), Gale and Shapley (1962), Shapley and Shubik (1971), Shapley and Scarf (1974), Crawford and Knoer (1981), Demange et al. (1986), Blum et al. (1997), Abdulkadiroğlu and Sönmez (1999), Chung (2000), Andersson and Svensson (2014), etc.

will be represented by the Greek letters α, β, α_1, etc. A contract α usually has several components including a firm and a worker and what service the worker should provide to the firm and what the worker should get from the firm in return for the service. The firm involved in contract α is denoted by α_F and the worker is denoted by α_W. This way of describing contracts is inspired by Hatfield and Milgrom (2005) and Ostrovsky (2008). In particular, if a worker w does not work for any firm or a firm f does not hire any worker, this state of standing alone will be represented by a simple contract $\alpha = \{w\}$ or $\alpha = \{f\}$. For simplicity, we also use w and f to denote these two special contracts. If a firm has multiple contracts with a worker, then every contract has its own terms and conditions and differs from any other. There are no contracts between any two firms or workers.

Let Σ be the set of all possible but finite contracts given exogenously. Subsets of Σ will be denoted by Ω, Ψ, etc. Given a subset Ψ of Σ, an agent $x \in F \cup W$ and a subset S of $F \cup W$, let $\Psi(x)$ represent the set of contracts in Ψ in which x is involved, and let $\Psi(S) = \cup_{x \in S} \Psi(x)$. Clearly, $\Sigma(x)$ contains the obvious contract $\alpha = \{x\}$ for every agent x.

Every worker w has strict preferences over all his possible contracts in $\Sigma(w)$. This relationship will be represented by $\succeq_w$ and $\succ_w$. If $\alpha_1 \succeq_w \alpha_2$ for $\alpha_1, \alpha_2 \in \Sigma(w)$, we say that he likes contract α_1 at least as well as contract α_2. If $\alpha_1 \succ_w \alpha_2$ for $\alpha_1, \alpha_2 \in \Sigma(w)$, then he prefers contract α_1 to contract α_2. Similarly, every firm f has strict preferences over all her possible contracts in $\Sigma(f)$. This relationship will be represented by $\succeq_f$ and $\succ_f$.

A set Ψ of contracts in Σ is said to be *a matching* if the set $\Psi(x)$ contains exactly one contract for every agent $x \in F \cup W$. If a contract α in a matching Ψ involves a worker w and a firm f, i.e., $\Psi(w) = \Psi(f) = \{\alpha\}$, then w and f are called *a partner of each other* or *matched with contract*; if a contract in a matching Ψ involves only one agent x, then the agent is said to be *self-matched* or *a single*. Traditionally, a one-to-one mapping μ from the set $F \cup W$ onto itself is also called a *matching* if $\mu(\mu(x)) = x$ for every $x \in F \cup W$, $\mu(f) \neq f$ implies $\mu(f) \in W$ for every $f \in F$, and $\mu(w) \neq w$ implies $\mu(w) \in F$ for every $w \in W$. Clearly, every matching Ψ (of contracts) determines a unique matching (of one-to-one mapping), denoted by μ_Ψ or simply by μ.

2.1 Commitment and Matching State

In the classic marriage matching model of Gale and Shapley (1962), marriages are not binding on either side. In reality, however, marriages, or partnerships between workers and firms, or contracts between two parties can be binding on both sides, or one side. In other words, one or two parties involved may have commitments to their partner. We will incorporate this important feature into our matching model. For a given matching Ψ, let $P(\Psi) = \{x \in W \cup F \mid \mu_\Psi(x) \neq x\}$ denote the set of agents who have a partner under Ψ. An agent x in $P(\Psi)$ is said to be *committed* if she cannot unilaterally dissolve the relation with her partner $\mu_\Psi(x)$ but can do so with her partner's consent. In our context, $\mu_\Psi(x) = y$ would agree if doing so does not make y worse off. This also implies that even if two parties are committed to each other, they can still dissolve their partnership as long as both consent to do so. Let $C(\Psi) = \{x \in P(\Psi) \mid x \text{ is committed}\}$ be the set of committed individuals under Ψ. Any agent in $W \cup F$ but not in $C(\Psi)$ is said to be *free or uncommitted*. A free agent can make decisions independently. Let $V(\Psi) = (W \cup F) \setminus C(\Psi)$ represent the set of all free agents. A free individual $x \in P(\Psi)$ can unilaterally rescind her relationship with her partner $\mu_\Psi(x)$. A matching state consists of a matching Ψ and its set of committed agents $C(\Psi)$ and is denoted by $(\Psi, C(\Psi))$. In the Gale-Shapley model, for any matching Ψ we have $C(\Psi) = \emptyset$ and $V(\Psi) = W \cup F$.

Commitments introduced here are context-dependent, effective, and general. In addition to those typical cases mentioned in Section 1 they can also cover the most general case where some pairs of worker and firm are two-sided committed, some workers are committed but their partners are not, some firms are committed but their partners are not, and some agents are free. In Section 5 we will discuss job-specific commitments which are simple and special but still quite common.

Note that in some environments commitment can be imbedded in a contract and analyzed. In the current paper we treat commitment as an independent constraint in order to deal with very general and complex situations. This modelling has several obvious advantages. Firstly, it allows us to accommodate various environments with flexibility and effectiveness. Secondly, it makes it possible to analyze how commitment can influence the behaviour of agents and the outcome. Thirdly, as committed agents and free agents play quite different roles, it is necessary to treat commitment as an independent factor in order to have a clear understanding of its effect.

In contrast to entry level matching models where all agents are entrants and there is no initial matching between any firm and any worker, our senior level matching model comprises many incumbents and also many entrants. Incumbents may be committed to their initial partners and every agent may be associated with multiple contracts. To reflect this situation, let $(\Psi^0, C(\Psi^0))$ denote the initial matching state, which is exogenously given. An agent $x \in W \cup F$ is said to be *an incumbent* if $\mu_{\Psi^0}(x) \neq x$. An incumbent must initially match someone from the opposite group. For an incumbent x, we say that x is an *initial partner* of $\mu_{\Psi^0}(x)$ and vice versa. An agent $x \in W \cup F$ is *an entrant or a single* if $\mu_{\Psi^0}(x) = x$. Singles can be new entries or experienced agents. For instance, a single worker can be a person who just attains a professional qualification and starts to find a job, or an experienced worker who just lost or quitted his job. For convenience, let $\mu^0 = \mu_{\Psi^0}$ denote the initial matching, $P^0 = P(\Psi^0)$ denote the set of all incumbents, $C^0 = C(\Psi^0)$ denote the set of all committed incumbents, and $V^0 = V(\Psi^0)$ be the set of all free or uncommitted agents at $(\Psi^0, C(\Psi^0))$. Observe that if an incumbent $x \in P^0$ prefers remaining single to her initial partnership $\Psi^0(x)$, i.e., $x \succ_x \Psi^0(x)$, this agent wants to dissolve this partnership unconditionally and her partner $\mu^0(x)$ can act freely. Therefore, for any incumbent $x \in P^0$ with $x \succ_x \Psi^0(x)$, we can naturally assume that her parter $\mu^0(x)$ is free, i.e., $\mu^0(x) \in V^0$. Let $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$ represent the current model.

Clearly, entrants and free agents will be more likely than committed agents to disturb and reshape the market. Every individual tries to find the best possible partner for herself and committed individuals strive to find better partners but have to comply with their commitment constraints. Naturally, commitment constraints whether imposed upon or chosen by individuals will certainly restrict the choices of the individuals who face the constraints. In a society, *a state which is stable when every individual makes her choice under few or no constraints is intrinsically more desirable than a state which is stable when every individual makes her choice under many constraints.* It will therefore be beneficial to free as many individuals as possible from the shackles of their commitment constraints. With this in mind we study the problem of how to match workers and firms as well as possible and to set free as many committed agents as possible without contravening their commitments. In particular, we attempt to address two fundamental questions: Given a matching model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, what can be a feasible and desirable outcome in terms of stability, efficiency and commitment? and how can such an outcome be achieved by a market mechanism?

2.2 Feasibility

With respect to the market $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, we say that a contract $\alpha \in \Sigma(x)$ is *acceptable* to an agent $x \in W \cup F$ if $\alpha \succeq_x x$ and that the contract α is *relatively acceptable* to the agent if $\alpha \succeq_x \Psi^0(x)$. Given a matching Ψ, an agent $x \in W \cup F$ is said to be *rematched* if the contract $\Psi(x)$ is different from the initial contract $\Psi^0(x)$. Observe that rematching allows x to have the same partner, i.e., $\mu_\Psi(x) = \mu^0(x)$.

As the market starts with the initial state and will be reshaped, one may wonder what outcome could possibly emerge. The following definition introduces feasible matchings. Feasibility is a basic condition that a proposed solution should satisfy.

Definition 1 *A matching Ψ is feasible if every free agent $x \in V^0$ gets at least an acceptable contract (i.e., $\Psi(x) \succeq_x x$), if every agent x with a committed partner $\mu^0(x) \in C^0$ gets at least a relatively acceptable contract (i.e., $\Psi(x) \succeq_x \Psi^0(x)$), and if every committed agent $x \in C^0$ gets at least an acceptable contract (i.e., $\Psi(x) \succeq_x x$) or a relatively acceptable contract (i.e., $\Psi(x) \succeq_x \Psi^0(x)$).*

Observe that a feasible matching Ψ is defined by comparing with the initial matching state (Ψ^0, C^0) and has to respect the commitment constraints. The definition is intuitive and easy to understand. We have the following observation.

Lemma 1 *At a feasible matching, an agent with a committed initial partner gets an acceptable and relatively acceptable contract.*

2.3 Stability

(Pairwise) stability introduced by Gale and Shapley (1962) is the most widely used solution concept in the two-sided matching literature. Their stability concept, however, cannot be directly applied to our current model due to the presence of commitments. To see this, let us consider the following example.

Example 1 *There are three workers w_0, w_1, w_2 and two firms f_1, f_2. We consider the simplest case in which there is at most one contract between every worker and every firm. We can therefore represent the preferences of each individual in the standard way as follows:*

$$\succ_{w_0}: \quad f_1, \quad w_0$$

$$\succ_{w_1}: \quad f_2, \quad f_1, \quad w_1 \qquad\qquad \succ_{f_1}: \quad w_0, \quad w_2, \quad w_1, \quad f_1$$

$$\succ_{w_2}: \quad f_1, \quad f_2, \quad w_2 \qquad\qquad \succ_{f_2}: \quad w_1, \quad w_2, \quad f_2$$

Here we read that w_0 prefers f_1 to standing alone or himself. All other firms which are not listed on his preferences are worse than standing alone.

Assume that we have a matching μ given by $\mu(w^0) = w^0$, $\mu(w_1) = f_1$, and $\mu(w_2) = f_2$. Note that *because μ is a matching, it implies $\mu(f_1) = w_1$ and $\mu(f_2) = w_2$ so we can omit them and will always do so.* With respect to this matching we consider the following scenarios: Without commitment and with different commitments.

Case 1: Without any commitment, the pairs of worker and firm $\{w_0, f_1\}$, $\{w_1, f_2\}$ and $\{w_2, f_1\}$ would obviously form blocking pairs to the matching μ in the sense of Gale and Shapley, as doing so would make every individual of every blocking pair better off than they are at matching μ.

Case 2: Under commitment $C(\mu) = \{w_1, w_2, f_1\}$, the pair $\{w_0, f_1\}$ cannot form a blocking pair to the matching μ because f_1 is committed to w_1 and cannot leave w_1 unilaterally. Nevertheless, it will be perfectly possible for w_0, f_1, w_1, f_2 to form a chain so that w_0 matches f_1, and w_1 matches f_2. As w_0 and f_2 are free, they can take the initiative to form this blocking chain so that everyone on the chain gets better off. Observe that f_1 is committed to w_1 and can dissolve his partnership with w_1 as w_1 finds a better replacement f_2 of f_1. Similarly, it is easy to see that $\{w_1, f_2\}$ and $\{w_2, f_1\}$ cannot form blocking pairs to μ as w_1 is committed to f_1 and w_2 is committed to f_2.

Case 3: Under commitment $C'(\mu) = \{w_1, w_2, f_1, f_2\}$, the pairs $\{w_0, f_1\}$, $\{w_1, f_2\}$ and $\{w_2, f_1\}$ cannot form blocking pairs to the matching μ because f_1 is committed to w_1 and cannot leave w_1 unilaterally, w_1 is committed to f_1 and cannot leave f_1 unilaterally and f_2 is committed to w_2 and cannot leave w_2 unilaterally. The same reason applies to $\{w_2, f_1\}$.

Nevertheless, it will be perfectly plausible for w_1, f_2, w_2 and f_1 to form a cycle so that w_1 matches f_2, and w_2 matches f_1, as doing so will make everyone on the cycle better off without violating any commitment.

This discussion motivates us to introduce concepts of blocking chain and blocking cycle as a necessary and plausible generalization of the Gale-Shapley concept of blocking pair.

Given a matching state $(\Psi, C(\Psi))$, a *chain* of the state is an ordered sequence of an even number of distinct agents $\overrightarrow{X} = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ with $K \geq 1$ such that $x_1, y_K \in V(\Psi)$ and $\mu_\Psi(y_k) = x_{k+1}$ for every $k = 1, 2, \cdots, K-1$. x_1 and y_K are called *end agents* and they are free. A *cycle* of the state is an ordered sequence of an even number of distinct agents $\overrightarrow{X} = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ with $K \geq 1$ such that $\mu_\Psi(y_k) = x_{k+1}$ for every $k = 1, 2, \cdots, K$, where x_{K+1} becomes x_1 by convention. In particular, any pair (x_1, y_1) with $\mu_\Psi(y_1) = x_1$ can be looked as a cycle. For a chain or cycle $\overrightarrow{X} = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ we will use $A(\overrightarrow{X}) = \{x_1, y_1, x_2, y_2, \cdots, x_K, y_K\}$ to denote the set of all agents in $\overrightarrow{X}$. A chain $\overrightarrow{X} = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ of $(\Psi, C(\Psi))$ is said to be *minimal* if $A(\overrightarrow{X}) \backslash \{x_1, y_K\} \subset C(\Psi)$, i.e., all but the end agents x_1 and y_K have commitment to their partners at μ_Ψ. It is clear that every chain contains at least one minimal chain.[3] A cycle of a matching state $(\Psi, C(\Psi))$ is a *pure cycle* if it does not contain any chain.

A matching state $(\Psi, C(\Psi))$ is *blocked by a free individual* $x \in V(\Psi)$ if $x \succ_x \Psi(x)$. It is *individually rational* if it is not blocked by any of its free individuals. A matching state $(\Psi, C(\Psi))$ is *blocked by a chain or cycle* $\overrightarrow{X} = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$, if for every $k = 1, \cdots, K$, there is a mutually acceptable contract α_k to x_k and y_k such that $\alpha_k \succ_{x_k} \Psi(x_k)$ and $\alpha_k \succ_{y_k} \Psi(y_k)$, i.e., agents x_k and y_k prefer contract α_k to their respective contract at Ψ. By definition, all members in a blocking chain of a matching state will be better off than in the matching state and therefore will have incentives to deviate from the matching state. The end members are free and may initialize this blocking. The same is true for all members in a blocking cycle except that there are no end members in the cycle. So

[3]For any chain $\overrightarrow{X} = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ of a matching state $(\Psi, C(\Psi))$, we can find one of its minimal sub-chain as follows. We first find some $j \in \{1, 2, \cdots, K\}$ such that $y_j \in V(\Psi)$ and $y_k \notin V(\Psi)$ for all $k = 1, \cdots, j-1$, and next find some $i \in \{1, 2, \cdots, j\}$ such that $x_i \in V(\Psi)$ and $x_k \notin V(\Psi)$ for all $k = i+1, \cdots, j$. Then, we can show $(x_i, y_i, \cdots, x_j, y_j)$ is a minimal chain of $(\Psi, C(\Psi))$.

blocking chains can be more easily formed than blocking cycles, as the latter do not have free end members to take the initiative. For our model it can be shown (see the appendix) that if an individually rational matching state is blocked by a group of agents, it must be blocked by a chain or by a cycle. So it suffices to focus on chains and cycles.

Definition 2 *A matching state $(\Psi, C(\Psi))$ is chain stable if it is not blocked by any of its free individuals or any of its chains. It is stable if it is not blocked by any of its free individuals or any of its chains or cycles.*

Our chain stability is similar to the chain stability introduced by Ostrovsky (2008) for his general vertical supply chain model but our motivation and context are different from his. In particular, unlike his model which contains chains and no cycles, chains and cycles in our model are induced due to the presence of commitments. Our notion of stability is defined with respect to not only chains but also cycles and thus strengthens the notion of (chain) stability defined with respect to chains only. Chain stability is immune to all possible blocking chains. Stability enhances chain stability by also excluding any possibility of blocking cycles. We have the following observation.

Lemma 2 *A matching state $(\Psi, C(\Psi))$ is chain stable if and only if it is not blocked by any of its free agents or any of its minimal chains. A matching state is stable if and only if it is not blocked by any of its free agents or any of its minimal chains or pure cycles.*

Note that for any two matching states $(\Psi, C(\Psi))$ and $(\Psi, C'(\Psi))$ satisfying $C(\Psi) \subseteq C'(\Psi)$, every chain of $(\Psi, C'(\Psi))$ is also a chain of $(\Psi, C(\Psi))$, a pure cycle of $(\Psi, C'(\Psi))$ may also contain a chain of $(\Psi, C(\Psi))$. Therefore, if $(\Psi, C(\Psi))$ is a (chain) stable matching state, then every matching state $(\Psi, C'(\Psi))$ with $C(\Psi) \subseteq C'(\Psi)$ is also (chain) stable. The converse may not be true.

Let us go back to Example 1. It is easy to verify that $(\mu, C(\mu))$ is not chain stable, because (w_0, f_1, w_1, f_2) is a blocking chain. $(\mu, C'(\mu))$ is chain stable but not stable, because (w_1, f_2, w_2, f_1) is a blocking cycle. Nevertheless, the matching state $(\mu^1, C(\mu^1))$ is stable, where $\mu^1(w_0) = w_0$, $\mu^1(w_1) = f_2$, $\mu^1(w_2) = f_1$, and $C(\mu^1) = \{f_1, f_2\}$.

When applying the notion of stability to the current market $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, we will show through a procedure that there exists a stable matching state $(\Psi, C(\Psi))$

with a feasible matching Ψ and the set of committed agents $C(\Psi)$ being a subset of the initial commitment set C^0. The stable matching state $(\Psi, C(\Psi))$ is generated from the initial matching state (Ψ^0, C^0) and sets free some committed agents without violating any commitments. Note that our stability is defined with respect to a given matching state which is not directly related to the initial matching state.

2.4 Core

The notion of core is one of the most fundamental solution concepts in game theory and equilibrium theory and can be seen as a generalization of the venerable Edgeworth's contract curve. We will adapt this solution to our current model. It will be helpful to recall from Shapley and Scarf (1974) that "The core consists of those outcomes of the game that are feasible, and that cannot be improved upon by any individual or coalition of individuals." There are three key elements about the core. The first is the initial state (including certain rights) of every agent and feasibility, the second is the proposed solution-the core-a set of states being immune to all possible coalition improvements, and thirdly trade takes place, i.e., agents exchange their goods, only after a core state is reached and agreed by all agents through certain procedures, negotiations or bargaining.

Analogously, the current market has the initial state (Ψ^0, C^0). Imagine that "a social planner" proposes a matching Ω to all workers and firms as a possible (strict) core matching. Obviously Ω must be a feasible matching. The proposal Ω is a (strict) core matching if it is robust against any possible coalition improvement. At this moment Ω is just a proposal under consideration, agents are still at the initial state (Ψ^0, C^0) and they are each thinking about who by herself or which coalition of agents by themselves can possibly improve upon the proposal Ω, i.e., a feasible counter-proposal against the proposal Ω. Clearly, a free agent can act independently or form freely a coalition with any other free agents when she considers how to improve her situation at Ω. However, if an agent has an initial partner and is committed, she cannot act unilaterally but has to ask her partner to cooperate with her when she contemplates whether she can initiate a coalition or accept an invitation to form a coalition in order to improve her position at Ω, because a committed agent cannot abandon her partner without the consent of her partner. This means that unlike Shapley and Scarf (1974) or Gale and Shapley (1962), not every coalition in our current market is permissible or feasible. Note that only after the proposed solution Ω is accepted by all agents, initial partnerships may cease to exist and new contracts can be signed. Now we are ready to introduce permissible coalitions and our solution concept. A nonempty subset S of the set $W \cup F$ of workers and firms is called a coalition. $W \cup F$ itself is called the grand coalition. A coalition S is said to be *permissible* if $x \in S \cap C^0$ implies her partner $\mu^0(x) \in S$. We say that a coalition S *improves upon a matching* Ψ of the grand coalition $W \cup F$ if there exists a matching $\Phi \subseteq \Sigma(S)$ among workers and firms from the coalition alone such that everyone x in S weakly prefers $\Phi(x)$ to $\Psi(x)$ and at least one agent $y \in S$ prefers $\Phi(y)$ to $\Psi(y)$. A coalition S *strongly improves upon a matching* Ψ if there exists a matching $\Phi \subseteq \Sigma(S)$ among workers and firms from the coalition alone such that every agent x in S prefers $\Phi(x)$ to $\Psi(x)$.

Definition 3 *In a matching model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, a feasible matching Ψ is in the strict core, called a strict core matching if it cannot be improved upon by any permissible coalition. It is in the core if it cannot be strongly improved upon by any permissible coalition.*

It should be noted that because the number of permissible coalitions is smaller than the number of all coalitions of $W \cup F$, it appears more likely to enlarge the core, but it is not the case, because it is less likely for firms and workers to form feasible partnerships and feasible matchings (thus to shrink the core) due to their commitments.

Observe that the set of all agents in a chain or a cycle of the initial matching state (Ψ^0, C^0) is a permissible coalition for the initial matching state. A feasible matching Ψ is *improved upon by a chain or cycle* $\overrightarrow{X} = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ of the initial state (Ψ^0, C^0) if for every $k = 1, \cdots, K$, there is a contract $\alpha_k \in \Sigma(x_k) \cap \Sigma(y_k)$ such that $\alpha_k \succeq_{x_k} \Psi(x_k)$ and $\alpha_k \succeq_{y_k} \Psi(y_k)$, and for some $k = 1, \cdots, K$, it holds $\alpha_k \succ_{x_k} \Psi(x_k)$ or $\alpha_k \succ_{y_k} \Psi(y_k)$. $\overrightarrow{X}$ will be called *an improvement chain or cycle* of Ψ.

Let us check if there is any core matching in Example 1. Suppose that the initial matching state (μ^0, C^0) is given by $\mu^0(w_0) = w_0$, $\mu^0(w_1) = f_1$, $\mu^0(w_2) = f_2$, and $C^0 = \{w_1, w_2, f_1, f_2\}$. So w_1, w_2, f_1, and f_2 are incumbents and committed, and w_0 is an entrant.

In this example, there is a unique strict core matching μ given by $\mu(w_0) = w_0$, $\mu(w_1) =$

f_2, and $\mu(w_2) = f_1$. At μ, no agent gets worse off and w_1, w_2, f_1, and f_2 all get strictly better off. The initial partners $\{w_1, f_1\}$ and $\{w_2, f_2\}$ get dissolved and rematched.

It is easy to see that the matching state $(\mu, C(\mu))$ with $C(\mu) = \{f_1, f_2\} \subset C^0$ is stable, whereas the matching state $(\mu, \hat{C}(\mu))$ with $\hat{C}(\mu) = \{w_1, w_2, f_2\}$ is not even chain stable, because (w_0, f_1) forms a blocking chain for the matching state.

Note that different from $(\mu, C(\mu))$, $(\mu^2, C(\mu^2))$ is also a stable matching state, where $\mu^2(w_0) = f_1$, $\mu^2(w_1) = f_2$, $\mu^2(w_2) = w_2$, and $C(\mu^2) = \emptyset$. But μ^2 is not a core matching as w_2 becomes a single at μ^2 and is worse off than her initial state at (μ^0, C^0) and thus μ^2 is not a feasible matching. This is in contrast to the model of Gale and Shapley (1962) for which core and stability are identical.

Consider now a different scenario. If agents had no commitment and thus could act freely, then μ^2 above would be the unique stable matching in the sense of Gale and Shapley (1962). In this case w_2 would get fired and be worse off.

The above discussion shows that commitment can affect the behavior of agents as well as the outcome considerably.

Lemma 3 *If a feasible matching Ψ is improved upon by a permissible coalition S, then it must be improved upon by a chain or cycle of the initial matching state (Ψ^0, C^0).*

The next lemma follows immediately from Lemma 3.

Lemma 4 *A feasible matching Ψ is in the strict core if it cannot be improved upon by any chain or cycle of the initial state (Ψ^0, C^0).*

While stability introduced previously is motivated from a noncooperative viewpoint, the core ensures (Pareto) efficiency but also reflects a kind of stability from a cooperative viewpoint. The solution found by our Hybrid Procedure will meet all requirements for feasibility, stability, and core.

3 Procedures and Existence Results

In this section we present two existence results. Our first result (Lemma 5) establishes the existence of a chain stable matching state with a feasible matching Ψ in which every committed agent at Ψ must be committed at the initial matching Ψ^0. This implies that the number of committed agents at this stable state can be reduced and thus some committed agents at Ψ^0 will be set free. We prove this result through our first procedure which is a generalization of the deferred acceptance (DA) procedure of Gale and Shapley (1962). Lemma 5 will be used in the proof of our key existence Theorem 1. This theorem shows that there exists at least one stable matching state $(\Omega, C(\Omega))$ in which the matching Ω is in the strict core and the family of committed agents $C(\Omega)$ is a subset of the initial commitment set C^0. We establish this theorem through our Hybrid Procedure, which is a combination of our first procedure and our second procedure-a generalization of the top trading cycle (TTC) method from Shapley and Scarf (1974).

We first introduce the following generalization of the deferred acceptance (DA) procedure of Gale and Shapley (1962). Dubins and Freedman (1981) have shown the strategic property of the DA procedure. Kojima and Manea (2010) have axiomatized the DA procedure. Erdil and Ergin (2017) have permitted indifference in every agent's preferences and proposed a procedure by using stable worker improvement chains and cycles to find a worker-optimal stable matching. All these papers deal with entry level markets.

Workers Proposing Deferred Acceptance (WP-DA) Procedure

- At the beginning, every committed worker in C^0 is provisionally matched to his original partner under their initial contract.

- Then at every step, any worker who is currently free[4] and has not made any proposal or who was rejected previously proposes his best-liked contract among those which are acceptable to him and which he has not yet proposed, to its associated firm. Every firm rejects all proposed unacceptable contracts and also rejects all but her most-preferred contract among those proposed acceptable contracts she has received, as well as the contract she is provisionally matched with, subject to the constraint that if she is committed in C^0 and has received her initial contract at Ψ^0 proposed by

[4] In this procedure, we say that a worker w is *currently free*, if he is free in V^0 or his partner $\mu^0(w)$ has being provisionally matched with a new contract rather than her initial contract at Ψ^0.

her initial partner, she should treat this contract as her unique favourite one and be provisionally matched to the worker with the contract. Any worker whose proposed contract has not been rejected is provisionally matched with his proposed contract.

- When there is no new proposal from any worker, we then arrive at a matching state as follows: The matching Ψ consists of all those contracts that remain in force and are currently provisionally matched with some firm and some worker, and of those trivial contracts involving only single agents who are currently not provisionally matched with any other agent. Let W_C denote the set of those workers who are committed at Ψ^0 and have never become free in the procedure, and $F_C = \{f \in C^0 \cap F \mid \Psi(f) = \Psi^0(f)\}$ denote the set of those committed firms which keep the same partner with the same contract as the initial state $(\Psi^0, C(\Psi^0))$. Then let $C(\Psi) = W_C \cup F_C$ be the set of committed agents at Ψ. This yields the matching state $(\Psi, C(\Psi))$.

Several remarks are in order. Firstly, in this procedure, any committed worker in C^0 can make a proposal only if his initial partner has received some proposal and been provisionally matched with a new contract rather than her initial contract at Ψ^0. Any committed firm in C^0 cannot reject her initial partner if the proposed contract is her initial contract at Ψ^0. Secondly, free workers have no restriction of proposing their most preferred contracts to their associated firms nor have free firms any restriction of tentatively accepting their received favorite proposals. Thirdly, when a committed firm receives a proposal of a better contract than her initial contract, she will reject her initial contract and let her initial partner free if her initial partner is committed. However, when her initial partner regardless of being committed or free later comes to propose to her, she will accept her initial partner's proposal (i.e., initial contract) and reject all other proposals including the one she provisionally holds. Finally, a committed worker becomes free when he is set free in the procedure, and a committed firm becomes also free when she is finally rematched with a new contract rather than her initial contract. Namely, agents x who are committed at (Ψ^0, C^0) and are rematched at Ψ (i.e., $\Psi(x) \neq \Psi^0(x)$) are set free at $(\Psi, C(\Psi))$. As $W_C \subset C^0 \cap W$ and $F_C \subset C^0 \cap F$, $C(\Psi) = W_C \cup F_C$ is a subset of C^0.

Analogously one can introduce the firms proposing deferred acceptance (FP-DA) procedure. To facilitate a better understanding of the above procedure, we illustrate it by the following example that is simple but shows several essential features of the procedure.

Example 2 *There are six workers* w_1*,* w_2*,* $\cdots$*,* w_6*, and six firms* f_1*,* f_2*,* $\cdots$*,* f_6*.* w_6 *and* f_6 *are new entries but the rest are incumbents, and the initially matched pairs are* $\{f_1, w_1\}$*,* $\{f_2, w_2\}$*,* $\{f_3, w_3\}$*,* $\{f_4, w_4\}$*, and* $\{f_5, w_5\}$*. We have the initial matching* $\mu^0(w_1) = f_1$*,* $\mu^0(w_2) = f_2$*,* $\mu^0(w_3) = f_3$*,* $\mu^0(w_4) = f_4$*,* $\mu^0(w_5) = f_5$*,* $\mu^0(w_6) = w_6$*, and* $\mu^0(f_6) = f_6$ *and the set of committed agents* $C^0 = \{f_1, f_2, f_3, f_5, w_1, w_2, w_3, w_4, w_5\}$*. Observe that the incumbent firm* f_4 *is free. Each agent's preferences are given by*

$$\begin{array}{llllllll} \succ_{w_1}: & f_2, & f_1, & w_1 & \qquad \succ_{f_1}: & w_6, & w_2, & w_1, \; f_1 \\ \succ_{w_2}: & f_1, & f_2, & w_2 & \qquad \succ_{f_2}: & w_5, & w_1, & w_2, \; f_2 \\ \succ_{w_3}: & f_6, & f_3, & w_3 & \qquad \succ_{f_3}: & w_6, & w_3, & f_3 \\ \succ_{w_4}: & f_5, & f_4, & w_4 & \qquad \succ_{f_4}: & w_1, & w_5, & w_4, \; f_4 \\ \succ_{w_5}: & f_2, & f_4, & f_5, \; w_5 & \qquad \succ_{f_5}: & w_4, & w_5, & f_5 \\ \succ_{w_6}: & f_1, & f_3, & w_6 & \qquad \succ_{f_6}: & w_3, & f_6 & \end{array}$$

The steps of the procedure is given as follows: Step 1: Free worker w_6 proposes to f_1 who will set w_1 free and be provisionally matched to w_6. Now w_1 is free. Step 2: w_1 proposes to f_2 who will let w_2 free and be provisionally matched to w_1. Now w_2 is free. Step 3: w_2 proposes to f_1. f_1 declines w_2 and is provisionally matched to w_6. Step 4: w_2 proposes to his initial partner f_2. Observe that by the rule of the procedure f_2 has to reject w_1 and is provisionally matched to w_2 because f_2 is committed to w_2. Step 5: w_1 proposes to his initial partner f_1. f_1 has to reject w_6 and is provisionally matched to w_1 because f_1 is committed to her initial partner w_1. Step 6: w_6 proposes to f_3. f_3 sets w_3 free and is provisionally matched to w_6. Now w_3 is free. Step 7: w_3 proposes to f_6 who will be provisionally matched to w_3.

Step 8: Now there is no new proposal. The procedure terminates with a chain stable matching state $(\Psi, C(\Psi))$ where the matching $\Psi = \mu$ is given by

$$\mu(w_1) = f_1,\ \mu(w_2) = f_2,\ \mu(w_3) = f_6,\ \mu(w_4) = f_4,\ \mu(w_5) = f_5,\ \mu(w_6) = f_3,$$

and the set of committed agents is $C(\Psi) = \{f_1, f_2, f_5, w_4, w_5\}$. In the process, w_1, w_2, and w_3 have been set free, and also f_3 has been set free as f_3 is rematched to w_6 instead of

w_3. So in total we have now seven free agents f_3, f_4, f_6, w_1, w_2, w_3, and w_6. Recall that initially we have only three free agents f_4, f_6, and w_6.

We have the following lemma on the existence of chain stable matching state.

Lemma 5 *For the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, the WP-DA procedure finds a chain stable matching state $(\Psi, C(\Psi))$, where Ψ is a feasible matching and the commitment set $C(\Psi)$ is a subset of the initial commitment set C^0.*

We now turn to present a modification of the top trading cycle (TTC) method of Shapley and Scarf (1974) which will be an integral part of our Hybrid Procedure for finding a stable matching state with a strict core matching. Ma (1994) has given an axiomatic characterization of the TTC procedure. Abdulkadiroğlu and Sönmez (1999, 2003) have adapted the procedure to a house allocation model with both existing tenants and new applicants, and to the school choice. Combe et al. (2017) have transformed the TTC procedure to the block exchange and teacher optimal block exchange algorithms in order for teachers and schools to improve their welfare.

A key feature of the TTC procedure and its variants is to generate (top trading) cycles in which agents can get better offers or positions and thus improve their welfare. In our modification this feature will be maintained but we need to apply this modified procedure to *an almost committed set* which is defined next. This modified procedure will be implemented only after our modified DA procedure is executed. A subset of partners $S \subseteq P^0$ is called an *almost committed set* if for every $x \in S$ it holds $\mu^0(x) \in S$ and $\{x, \mu^0(x)\} \cap C^0 \neq \emptyset$. It follows that if an incumbent x is in an almost committed set, then her partner $\mu^0(x)$ must be also in the almost committed set and at least one of the two agents must be committed. Clearly, every almost committed set S and its complementary set $(W \cup F) \setminus S$ are both permissible coalitions for the initial matching state (Ψ^0, C^0). The following procedure will be applied to any given almost committed set S. In the procedure, every worker in the set S points to the firm involved in his favourite contract among all his mutually relatively acceptable contracts in $\Sigma(S)$, and each firm f in S points to her partner $\mu^0(f)$ in S.

Workers Proposing TTC (WP-TTC) Procedure

- Every worker in the almost committed set S points to the firm involved in his favourite contract among all his mutually relatively acceptable contracts in $\Sigma(S)$[5], and each firm in S points to her worker partner under μ^0.

- There exists at least one directed cycle[6]. In each directed cycle, match every worker to his pointed firm under the worker's favorite mutually relatively acceptable contracts between them. All such matched work-firm pairs leave the market.

- Repeat this process to every almost committed set formed by the remaining agents, until no agent is left. Let Π be the matching consisting of all matched contracts.

Similarly we can have the firms proposing top trading cycle (FP-TTC) procedure.

Next we introduce a Hybrid Procedure of the WP-DA and WP-TTC procedures for finding a stable matching state with a strict core matching.

The Hybrid Procedure

Step 1: Apply the WP-DA procedure to the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$ and give a chain stable matching state $(\Psi, C(\Psi))$. Let $S = \{x \in P^0 \mid \Psi(x) = \Psi^0(x) \text{ and } \{x, \mu^0(x)\} \cap C^0 \neq \emptyset\}$ be the almost committed set. Then all agents outside the set S (i.e., singles and rematched agents in Ψ) leave the market.

Step 2: Apply the WP-TTC procedure to the set S and generate a matching Π.[7]

Step 3: Based on matchings Ψ and Π, construct a matching Ω by

$$\Omega(x) = \begin{cases} \Psi(x) & \text{if } x = (W \cup F) \setminus S, \\ \Pi(x) & \text{if } x \in S. \end{cases}$$

[5]In a decentralized market where each individual knows only their own preferences but not others, a worker can find such a contract as follows: He first proposes his favourite contract among all his relatively acceptable contract in $\Sigma(S)$. If his proposal is not relatively acceptable to his proposed firm, then his proposal will be rejected. Repeat this process to the remaining contracts that he has not yet proposed until he will not be rejected.

[6]We say a cycle $(\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$ of the initial state $(\mu^0, C(\mu^0))$ with mutually relatively acceptable pairs $(\bar{w}_k, \bar{f}_k)$ $(k = 1, \cdots, K)$ is a *directed cycle* if each $\bar{w}_k$ points to $\bar{f}_k$ and each $\bar{f}_k$ points to $\bar{w}_{k+1} = \mu^0(\bar{f}_k)$ for $k = 1, \cdots, K$, where $\bar{w}_{K+1}$ denotes $\bar{w}_1$.

[7]If the set S is empty, let the matching Π be empty.

Set $C(\Omega) = W_C \cup \{f \in S \cap F \mid \mu^0(f) \in W_V\}$, where W_V is the set of those workers who are free at the state $(\Psi, C(\Psi))$ given at Step 1, and W_C is the set of those workers who are committed at the state $(\Psi, C(\Psi))$. The procedure stops with the matching state $(\Omega, C(\Omega))$.

In order to have a better understanding of the Hybrid Procedure, we explain how some committed incumbents will maintain their commitment and how other committed incumbents can be freed from their commitment. Observe that the almost committed set $S = \{x \in P^0 \mid \Psi(x) = \Psi^0(x) \text{ and } \{x, \mu^0(x)\} \cap C^0 \neq \emptyset\}$ is the family of agents who have gone through the WP-DA procedure but still retained the same contracts as they had at the initial state (Ψ^0, C^0), and who had partners at (Ψ^0, C^0) and are committed to their partners or whose partners are committed. We can decompose the set S into two disjoint subsets S_1 and S_2 as follows. Let $S_1 = \{x \in S \mid \{x, \mu^0(x)\} \cap W_V \neq \emptyset\}$ denote the set of agents in S who did not rematch in the WP-DA procedure but had opportunities to participate in the rematching process. Clearly, $f \in S_1 \cap F$ implies $f \in C^0$, i.e., every firm in S_1 is a committed agent in C^0. Let $S_2 = \{x \in S \mid \{x, \mu^0(x)\} \cap W_C \neq \emptyset\}$ denote the set of agents in S who did not have any opportunity to rematch in the WP-DA procedure. Also $w \in S_2 \cap W$ implies $w \in C^0$, i.e., every worker in S_2 is a committed agent in C^0. Then, we have $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$. Let $S_3 = (W \cup F) \setminus S$. S_3 contains those agents who leave the market at the end of Step 1 of the Hybrid Procedure. Clearly S_1, S_2 and S_3 form a partition of $W \cup F$.

Note that $C(\Omega) = (S_2 \cap W) \cup (S_1 \cap F)$. So at $(\Omega, C(\Omega))$, only workers in S_2 and firms in S_1 will maintain their commitment but all other agents are free. This means that at $(\Omega, C(\Omega))$ workers in S_1 who are committed at the initial state (Ψ^0, C^0) are set free, so are firms in S_2 who are committed at the initial state (Ψ^0, C^0), and so are agents in S_3 who are committed at the initial state (Ψ^0, C^0). This implies that at the final matching state $(\Omega, C(\Omega))$ only necessary commitments will be kept and other can be removed. Clearly, $C(\Omega)$ is a subset of C^0. We will show that $(\Omega, C(\Omega))$ is stable and Ω is in the strict core.

By assumption that $\mu^0(x) \in V^0$ for every x with $x \succ_x \Psi^0(x)$, we see that $\Psi^0(w) \succ_w w$ for every worker $w \in S_1$ and $\Psi^0(f) \succ_f f$ for every firm $f \in S_2$. Recall that $S_2 \cap W \subseteq C(\Psi)$, i.e., no worker $w \in S_2$ has become free in the WP-DA procedure at Step 1 of the Hybrid

Procedure. For every worker $w \in S_1$ and every firm $f \in S_2$ there is no mutually acceptable contract α to them such that $\alpha \succ_w \Psi^0(w) \succ_w w$ and $\alpha \succ_f \Psi^0(f) \succ_f f$. Otherwise firm f would have received the proposal α from worker w and set his partner $\mu^0(f) \in S_2$ free. This means that there is no mutually relatively acceptable contract α to f and w. As a result, in the WP-TTC procedure at Step 2 of the Hybrid Procedure, no worker in S_1 points to a firm in S_2. That is, every directed cycle is contained either in S_1 or in S_2 and cannot be across both S_1 and S_2. So every matched pair from every directed cycle must be either in S_1 or in S_2.

We use again Example 2 to illustrate the Hybrid Procedure. The detailed steps are given as follows:

Step 1: The WP-DA procedure runs and generates the matching state $(\Psi, C(\Psi))$ where $\Psi = \mu$ is given by

$$\mu(w_1) = f_1,\ \mu(w_2) = f_2,\ \mu(w_3) = f_6,\ \mu(w_4) = f_4,\ \mu(w_5) = f_5,\ \mu(w_6) = f_3,$$

and $C(\Psi) = \{f_1, f_2, f_5, w_4, w_5\}$. We get the almost committed set

$$S = \{f_1, f_2, f_4, f_5, w_1, w_2, w_4, w_5\}.$$

f_3 hires w_6 and f_6 hires w_3 and they leave the market as they are rematched. We apply the WP-TTC procedure to the set S.

Step 2: w_1 points to f_2, w_2 to f_1, w_4 to f_5, and w_5 to f_2, while f_1 points to w_1, f_2 to w_2, f_4 to w_4, and f_5 to w_5. We have a directed cycle (w_1, f_2, w_2, f_1) from which two matched pairs $\{w_1, f_2\}$ and $\{w_2, f_1\}$ are generated. They leave the market. Step 3: w_4 points to f_5, and w_5 to f_4, while f_4 points to w_4, and f_5 to w_5. We have a directed cycle (w_4, f_5, w_5, f_4) from which two matched pairs $\{w_4, f_5\}$ and $\{w_5, f_4\}$ are made. They leave the market. Step 4: We obtain the matching $\pi(w_1) = f_2$, $\pi(w_2) = f_1$, $\pi(w_4) = f_5$, and $\pi(w_5) = f_4$.

Step 5: We construct the final matching $\Omega = \nu$ with

$$\nu(w_1) = f_2,\ \nu(w_2) = f_1,\ \nu(w_3) = f_6,\ \nu(w_4) = f_5,\ \nu(w_5) = f_4,\ \nu(w_6) = f_3,$$

and a set of committed agents $C(\Omega) = \{f_1, f_2, w_4, w_5\}$. Thus, we obtain a stable matching state $(\Omega, C(\Omega))$ with a strict core matching Ω.

In this example, we have $W_C = \{w_4, w_5\}$ and $W_V = \{w_1, w_2, w_3, w_6\}$ from $(\Psi, C(\Psi))$. Moreover, we have $S_1 = \{f_1, f_2, w_1, w_2\}$, $S_2 = \{f_4, f_5, w_4, w_5\}$, and $S_3 = \{f_3, f_6, w_3, w_6\}$ forming a partition of $F \cup W$.

We are now ready to establish the following major existence theorem.

Theorem 1 *For the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, the Hybrid Procedure finds a stable matching state $(\Omega, C(\Omega))$, where Ω is a strict core matching and the commitment set $C(\Omega)$ is a subset of the initial commitment set C^0.*

Proof: It is easy to see that the Hybrid Procedure terminates with $(\Omega, C(\Omega))$ in a finite number of steps. We first show that Ω is a strict core matching. In Step 1 of the Hybrid Procedure, by Lemma 5 the WP-DA procedure produces a chain stable matching state $(\Psi, C(\Psi))$. Ψ is a feasible matching. In Step 2 of the Hybrid Procedure, the WP-TTC procedure generates a matching Π. At Π every agent in the almost committed set S is matched to a partner through a mutually relatively acceptable contract. Observe that $\Psi^0(x) \succ_x x$ for every free agent $x \in S \cap V^0$ (or else $\mu^0(x) \in V^0$ as well) and also for every agent x with $\mu^0(x) \in C^0$. $\Pi(x)$ is acceptable to every free agent $x \in S \cap V^0$ and also to every agent x with $\mu^0(x) \in C^0$. So the matching Ω given at Step 3 of the Hybrid Procedure is feasible. If $S = \emptyset$, then $\Omega = \Psi$ and $C(\Omega) = C(\Psi) = \emptyset$. And $\Omega = \Psi$ is stable also in the sense of Gale and Shapley and is in the strict core.

Now consider the general case of $S \neq \emptyset$. Suppose to the contrary that Ω is not in the strict core. By Lemma 3, Ω must be improved upon by a chain or cycle of the initial state (Ψ^0, C^0). Suppose that Ω could be improved upon by a chain $\overrightarrow{X} = (\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$ of (Ψ^0, C^0). This means that (i) for each $k = 1, \cdots, K$, there is a contract $\alpha_k \in \Sigma(\bar{w}_k) \cap \Sigma(\bar{f}_k)$ such that $\alpha_k \succeq_{\bar{w}_k} \Omega(\bar{w}_k)$ and $\alpha_k \succeq_{\bar{f}_k} \Omega(\bar{f}_k)$; and (ii) there is some $\bar{k} = 1, \cdots, K$ such that $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}})$ or $\alpha_{\bar{k}} \succ_{\bar{f}_{\bar{k}}} \Omega(\bar{f}_{\bar{k}})$. It follows from strict preferences that $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}})$ and $\alpha_{\bar{k}} \succ_{\bar{f}_{\bar{k}}} \Omega(\bar{f}_{\bar{k}})$. We have the following possibilities:

(i) $A(\overrightarrow{X}) \not\subset S_3$, or else $(\bar{w}_{\bar{k}}, \bar{f}_{\bar{k}})$ is a blocking pair of the matching state $(\Psi, C(\Psi))$.

(ii) $A(\overrightarrow{X}) \not\subset S_1$, because there is no firm in S_1 who is free at the initial state.

(iii) $A(\overrightarrow{X}) \not\subset S_2$, because there is no worker in S_2 who is free at the initial state. Furthermore, $A(\overrightarrow{X}) \cap S_2 = \emptyset$, because for any worker $w \in S_1 \cup S_3$ and any firm $f \in S_2$ there is no contract $\alpha \in \Sigma(w) \cap \Sigma(f)$ such that $\alpha \succeq_w \Psi(w)$ and $\alpha \succeq_f \Psi(f) \succeq_f \Psi^0(f)$.

We will further show that $A(\overrightarrow{X}) \not\subset S_1 \cup S_3$. Suppose to the contrary that $A(\overrightarrow{X}) \subset S_1 \cup S_3$. Then, because of $A(\overrightarrow{X}) \not\subset S_1$ and $A(\overrightarrow{X}) \not\subset S_3$ we have $A(\overrightarrow{X}) \cap S_1 \neq \emptyset$ and $A(\overrightarrow{X}) \cap S_3 \neq \emptyset$. From $S_1 \cap F \subset C(\Omega) \subset C^0$, we see $\bar{f}_K \in S_3$. Thus, there is some $k = 1, \cdots, K$ such that $\bar{w}_k \in S_1 \cap W \subset V(\Psi)$ and $\bar{f}_k \in S_3 \subset V(\Psi)$. From $\alpha_k \neq \Omega(\bar{w}_k)$ and $\alpha_k \neq \Omega(\bar{f}_k)$, we further have $\alpha_k \succ_{\bar{w}_k} \Omega(\bar{w}_k) \succeq_{\bar{w}_k} \Psi^0(\bar{w}_k) = \Psi(\bar{w}_k)$ and $\alpha_k \succ_{\bar{f}_k} \Omega(\bar{f}_k) = \Psi(\bar{f}_k) \succeq_{\bar{f}_k} \bar{f}_k$. Note that $\Psi^0(\bar{w}_k) \succeq_{\bar{w}_k} \bar{w}_k$, or else by assumption his initial parter $\mu^0(\bar{w}_k) \in S_1 \cap F \subset C_0$ is in V_0, yielding a contradiction. This implies that $(\bar{w}_k, \bar{f}_k)$ is a blocking pair of $(\Psi, C(\Psi))$. We have proved that Ω cannot be improved upon by any chain of the initial state (Ψ^0, C^0).

Suppose that Ω could be improved upon by a cycle $\overrightarrow{X} = (\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$ of (Ψ^0, C^0). This means that (i) for each $k = 1, \cdots, K$, there is a contract $\alpha_k \in \Sigma(\bar{w}_k) \cap \Sigma(\bar{f}_k)$ such that $\alpha_k \succeq_{\bar{w}_k} \Omega(\bar{w}_k)$ and $\alpha_k \succeq_{\bar{f}_k} \Omega(\bar{f}_k)$; and (ii) there is some $\bar{k} = 1, \cdots, K$ such that $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}})$ or $\alpha_{\bar{k}} \succ_{\bar{f}_{\bar{k}}} \Omega(\bar{f}_{\bar{k}})$. It follows from strict preferences that $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}})$ and $\alpha_{\bar{k}} \succ_{\bar{f}_{\bar{k}}} \Omega(\bar{f}_{\bar{k}})$. Obviously, $A(\overrightarrow{X}) \not\subset S_3$, or else $(\bar{w}_{\bar{k}}, \bar{f}_{\bar{k}})$ is a blocking pair of $(\Psi, C(\Psi))$.

We claim that $A(\overrightarrow{X}) \not\subset S_1$. Suppose to the contrary that $A(\overrightarrow{X}) \subset S_1$. Observe that for every agent $x \in S_1$, agents x, $\mu^0(x)$, $\Pi(x)$ and $\mu^0(\Pi(x))$ must be matched and removed at the same round in the WP-TTC procedure. Let A_t denote the set of all agents in S_1 matched and removed at round t of the WP-TTC procedure. Assume that each worker $\bar{w}_k$ is removed at round t_k, i.e., $\bar{w}_k \in A_{t_k}$. Then $\bar{f}_k = \mu^0(\bar{w}_{k+1})$ is removed at round t_{k+1}. Recall that α_k is mutually relatively acceptable to $\bar{w}_k$ and $\bar{f}_k$, and $\alpha_k \succeq_{\bar{w}_k} \Omega(\bar{w}_k)$, the WP-TTC procedure matches $\bar{w}_k \in A_{t_k}$ to his pointed firm under the worker's favorite mutually relatively acceptable contracts in $\Sigma(\cup_{s \geq t_k} A_s)$. We see that $\alpha_k \in \Sigma(\cup_{s \leq t_k} A_s)$ and $\bar{f}_k \in \cup_{s \leq t_k} A_s$. Therefore, $\bar{f}_k$ and $\bar{w}_{k+1} = \mu^0(\bar{f}_k)$ must be removed no later than $\bar{w}_k$. Thus, it must hold $t_k \geq t_{k+1}$ for all $k = 1, 2, \cdots, K$, where $K + 1$ denotes 1. This implies $t_1 = t_2 = \cdots = t_K$. However, $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}}) = \Pi(w_{\bar{k}})$ implies that $\alpha_{\bar{k}} \in \Sigma(\cup_{s < t_{\bar{k}}} A_s)$ and $t_{\bar{k}} > t_{\bar{k}+1}$. This contradiction shows that $A(\overrightarrow{X}) \not\subset S_1$.

Using exactly the same argument (for proving $A(\overrightarrow{X}) \not\subset S_1$) above, we can show that $A(\overrightarrow{X}) \not\subset S_2$. Moreover, $A(\overrightarrow{X}) \cap S_2 = \emptyset$, because for any worker $w \in S_1 \cup S_3$ and any firm

$f \in S_2$ there is no contract $\alpha \in \Sigma(w) \cap \Sigma(f)$ such that $\alpha \succeq_w \Psi(w)$ and $\alpha \succeq_f \Psi(f) \succeq_f \Psi^0(f)$.

Finally, we will show that $A(\overrightarrow{X}) \not\subset S_1 \cup S_3$. Suppose to the contrary that $A(\overrightarrow{X}) \subset S_1 \cup S_3$. Recall that $A(\overrightarrow{X}) \not\subset S_1$ and $A(\overrightarrow{X}) \not\subset S_3$. Then, $A(\overrightarrow{X}) \cap S_1 \neq \emptyset$ and $A(\overrightarrow{X}) \cap S_3 \neq \emptyset$. Thus, there is some $k = 1, \cdots, K$ such that $\bar{w}_k \in S_1 \cap W \subset V(\Psi)$ and $\bar{f}_k \in S_3 \cap F \subset V(\Psi)$. We can again show that $(\bar{w}_k, \bar{f}_k)$ is a blocking pair of $(\Psi, C(\Psi))$, leading to a contradiction.

We have so far proved that Ω cannot be improved upon by any chain or cycle of the initial state (Ψ^0, C^0). Consequently, Ω must be a strict core matching of the market.

It remains to show that $(\Omega, C(\Omega))$ is a stable matching state. By Lemma 2, it suffices to show that $(\Omega, C(\Omega))$ is individually rational and not blocked by any its minimal chain or pure cycle. First, note from the WP-DA procedure that $\Omega(x) = \Psi(x)$ is acceptable to every agent $x \in S_3$. For every $w \in S_1$, we have $\Omega(w) \succeq_w \Psi^0(w) \succeq_w w$, i.e., $\Omega(w)$ is acceptable to him. Similarly, $\Omega(f)$ is acceptable to every firm $f \in S_2$. Thus, $\Omega(x)$ is acceptable to every free agent $x \in V(\Omega)$, and hence $(\Omega, C(\Omega))$ is individually rational. Recall we have previously shown that $C(\Omega)$ is a subset of C^0 after the description of the Hybrid Procedure.

Next, note that at matching state $(\Omega, C(\Omega))$ there are no mutually committed partners. Therefore, a minimal chain of $(\Omega, C(\Omega))$ must be a free pair of worker and firm (w, f). Suppose to the contrary that $(\Omega, C(\Omega))$ is blocked by a free pair of worker and firm (w, f). Then, $w \in S_3 \cup S_1$ and $f \in S_3 \cup S_2$, and there is a mutually acceptable contract α to them such that $\alpha \succ_w \Omega(w)$ and $\alpha \succ_f \Omega(f)$. Thus, if $\{w, f\} \subset S_3$, then (w, f) is a blocking pair of the matching state $(\Psi, C(\Psi))$. If $w \in S_3, f \in S_2$, then $\alpha \succ_w \Omega(w) = \Psi(w)$ and $\alpha \succ_f \Omega(f) \succeq_f \Psi^0(f) = \Psi(f)$, and (w, f) is a blocking pair of the chain stable matching state $(\Psi, C(\Psi))$, yielding a contradiction. Similarly, we can show (w, f) is a blocking pair of $(\Psi, C(\Psi))$ for $w \in S_1$ and $f \in S_2 \cup S_3$. So this concludes that $(\Omega, C(\Omega))$ is chain stable.

Finally suppose that $(\Omega, C(\Omega))$ is blocked by a pure cycle $\overrightarrow{X} = (\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$ of $(\Omega, C(\Omega))$. We first see that $A(\overrightarrow{X}) \cap S_3 = \emptyset$, or else $\overrightarrow{X}$ becomes a blocking chain of $(\Omega, C(\Omega))$. Moreover, cycle $\overrightarrow{X}$ cannot be across both S_1 and S_2, or else it becomes a blocking chain of $(\Omega, C(\Omega))$ as well. Thus, it is only possible $A(\overrightarrow{X}) \subset S_1$ or $A(\overrightarrow{X}) \subset S_2$.

Assume that $A(\overrightarrow{X}) \subset S_1$. Let A_t denote the set of all agents in S_1 matched and removed at round t of the WP-TTC procedure. Suppose that each worker $\bar{x}_k$ is removed at round t_k, i.e., $\bar{w}_k \in A_{t_k}$. Then $\bar{f}_k = \Pi(\bar{w}_{k+1})$ is removed at round t_{k+1}. Note that for each $k = 1, \cdots, K$, there is a mutually acceptable contract α_k to $\bar{w}_k$ and $\bar{f}_k$ such that $\alpha_k \succ_{\bar{w}_k} \Omega(\bar{w}_k) \succeq_{\bar{w}_k} \Psi^0(\bar{w}_k)$ and $\alpha_k \succ_{\bar{f}_k} \Omega(\bar{f}_k) \succeq_{\bar{f}_k} \Psi^0(\bar{f}_k)$, the WP-TTC procedure matches $\bar{w}_k \in A_{t_k}$ to his pointed firm under the worker's favorite mutually relatively acceptable contracts in $\Sigma(\cup_{s \geq t_k} A_s)$. We see that $\alpha_k \in \Sigma(\cup_{s<t_k} A_s)$ and $\bar{f}_k \in \cup_{s<t_k} A_s$. Therefore, $\bar{f}_k$ and $\Pi(\bar{f}_k) = \bar{w}_{k+1}$ must have been removed earlier than $\bar{w}_k$ i.e., $t_k > t_{k+1}$. This yields a contradiction that $t_1 > t_2 > \cdots > t_{K-1} > t_K > t_1$. Similarly, we can show that $A(\overrightarrow{X}) \not\subset S_2$. Thus, we have proved that $(\Omega, C(\Omega))$ cannot be blocked by any pure cycle.

In summary we have shown that $(\Omega, C(\Omega))$ is individually rational and cannot be blocked by any minimal chain or pure cycle. By Lemma 2, $(\Omega, C(\Omega))$ is stable. □

4 Lattice and Incentive Results

It is well-known for the Gale-Shapley marriage market that when all men and women have strict preferences, the set of stable matchings is a lattice. Dubins and Freedman (1981) have shown that it is optimal for every man to act truthfully in the face of the deferred acceptance procedure with men proposing provided that women are honest.

We will examine to what extent lattice and incentive results can be obtained for our current market. We first discuss the lattice property of stable matching states. For any two matchings Ψ and Φ, and any agent $x \in W \cup F$, let

$$\Psi(x) \vee_x \Phi(x) = \begin{cases} \Psi(x) & \text{if } \Psi(x) \succeq_x \Phi(x) \\ \Phi(x) & \text{otherwise} \end{cases} \text{ and } \Psi(x) \wedge_x \Phi(x) = \begin{cases} \Psi(x) & \text{if } \Phi(x) \succeq_x \Psi(x), \\ \Phi(x) & \text{otherwise.} \end{cases}$$

Then we define two sets $\Psi \vee_W \Phi = \Psi \wedge_F \Phi$ and $\Psi \wedge_W \Phi = \Psi \vee_F \Phi$ of contracts in Σ as

$$\Psi \vee_W \Phi = \Psi \wedge_F \Phi = \{\Psi(w) \vee_w \Phi(w) \,|\, w \in W\} \cup \{\Psi(f) \wedge_f \Phi(f) \,|\, f \in F\},$$

$$\Psi \wedge_W \Phi = \Psi \vee_F \Phi = \{\Psi(w) \wedge_w \Phi(w) \,|\, w \in W\} \cup \{\Psi(f) \vee_f \Phi(f) \,|\, f \in F\}.$$

Note that $\Psi \vee_W \Phi$ and $\Psi \wedge_W \Phi$ may not be matchings. For the traditional matching models (i.e., $C(\Psi) = C(\Phi) = \emptyset$), if Ψ and Φ are stable matchings, then they are not only matchings but also stable. In our current market, even if Ψ and Φ under some commitments $C(\Psi)$ and $C(\Phi)$ are (chain) stable matching states, their join and meet $\Psi \vee_W \Phi$ and $\Psi \wedge_W$ may not yield a matching. To see this, let us revisit Example 1 in Section 2. Recall that $(\mu, C(\mu))$ and $(\mu^1, C(\mu^1))$ are chain stable matching states, where $\mu(w_0) = w_0$, $\mu(w_1) = f_1$, $\mu(f_1) = w_1$, $\mu(w_2) = f_2$, $\mu(f_2) = w_2$, and $C(\mu) = \{w_1, w_2, f_1, f_2\}$, $\mu^1(w_0) = w_0$, $\mu^1(w_1) = f_2$, $\mu^1(f_2) = w_1$, $\mu^1(w_2) = f_1$, $\mu^1(f_1) = w_2$, and $C(\mu^1) = \{f_1, f_2\}$. We see $\mu(w_1)\vee_{w_1}\mu^1(w_1) = f_2$ and $\mu(f_2)\wedge_{f_2}\mu^1(f_2) = w_2 \neq w_1$. So $\mu\vee_W\mu^1$ is not a matching. However, when two stable matching states share a common commitment set, we have the following lattice theorem generalizing the traditional one. Clearly, the conclusion holds true also for chain stability.

Theorem 2 *Let $(\Psi, C(\Psi))$ and $(\Phi, C(\Phi))$ be stable matching states. If $C(\Psi) = C(\Phi)$ and $\Psi(x) = \Phi(x)$ for every $x \in C(\Psi)$, then $\Psi\vee_W\Phi$ and $\Psi\wedge_W\Phi$ are both matchings. Furthermore, matching states $(\Psi\vee_W\Phi, C(\Psi))$ and $(\Psi\wedge_W\Phi, C(\Psi))$ are both stable.*

That the operations $\vee_W$ and $\wedge_W$ ($\vee_F$ and $\wedge_F$) each generate a stable matching state from a pair of stable matching states implies that the family of stable matching states forms a lattice. From Theorems 1 and 2 we immediately have

Corollary 1 *The market model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$ has a nonempty lattice of stable matching states.*

We turn now to the strategic question. Specifically, we want to know how we should expect individual agents to act in the face of our proposed procedure. It is already known from Ostrosky (2008, p.914) that in general incentive compatibility cannot be achieved in a setting involving chains. The presence of both chains and cycles in the current market makes it even harder to obtain such a result. Nevertheless, we will show that it is a dominant strategy for "certain individuals" to behave honestly in the face of our Hybrid Procedure. As in Dubins and Freedman (1981), Roth and Sotomayor (1990), and Hatfield and Milgrom (2005), it is common to examine the behavior of agents on one side of the market who make proposals by assuming that agents on the other side are honest.

To approach the issue, we need to analyze our proposed procedures. With regard to the WP-DA procedure, let $P_w(t)$ denote the set of those contracts which are just proposed by worker w at step t. Let $P(t) = \cup_{w\in W} P_w(t)$ be the union of the sets $P_w(t)$ of all workers. Furthermore, let $R_f(t)$ denote the set of proposed contracts which are rejected by firm

f at step t and may include the initial contract. Let $R(t) = \cup_{f \in F} R_f(t)$ be the union of the sets $R_f(t)$ of all firms. If $P(T) = \emptyset$ or $R(T) = \emptyset$ at some step T, then the WP-DA procedure terminates. Therefore this sequence generated by the WP-DA procedure can be represented by an ordered list $DA =< (P(1), R(1)), \cdots, (P(T), R(T)) >$, where $P(t) \neq \emptyset$ and $R(t) \neq \emptyset$ for all $t = 1, \cdots, T-1$, but $P(T) = \emptyset$ or $R(T) = \emptyset$. Observe that the outcome of the WP-DA procedure is determined by the set of all proposals $\cup_{t=1,\cdots,T} P(t)$ and the set of all rejections $\cup_{t=1,\cdots,T} R(t)$.

In the current market, every worker takes only one job and every firm hires only one worker and all agents have strict preferences over contracts. This means that bilateral substitutability and irrelevance of rejected contracts are satisfied; see Hirata and Kasuya (2014). The above WP-DA procedure is order-independent. As a result, when investigating the strategic behavior of any given worker $w^* \in W$, we can always assume that worker w^* makes his first proposal only at the last step in the sense that the procedure will end without his participation if he does not act. So from now on we can focus on the following WP-DA procedure which generates a WP-DA sequence $DA(w^*) =< (P(1), R(1)), \cdots, (P(T), R(T)) >$ satisfying the three conditions: (i) there is a step t^* such that $P_{w^*}(t) = \emptyset$ for all $t = 1, \cdots, t^*-1$; (ii) for every $t = t^*, \cdots, T-1$, there are two contracts α^t and β^t such that $P(t) = \{\alpha^t\}$, $R(t) = \{\beta^t\}$, $\alpha_W^{t^*} = w^*$, $\alpha_F^t = \beta_F^t$, and $\alpha_W^{t+1} = \beta_W^t$; (iii) there is a contract α^T such that $P(T) = \{\alpha^T\}$ and $\alpha_W^T = \beta_W^{T-1}$ if $P(T) \neq \emptyset$. If $P(T) = \emptyset$, let α_W^T equal $\emptyset$. Let $DA_1(w^*) =< (P(1), R(1)), \cdots, (P(t^*-1), R(t^*-1)) >$ denote the first part of the sequence before worker w^* takes part in the procedure. Let $DA_2(w^*) =< (P(t*), R(t^*)), \cdots, (P(T), R(T)) >=< (\alpha^{t^*}, \beta^{t^*}), \cdots, (\alpha^T, \emptyset) >$ represent the second part of the sequence after worker w^* participates in the procedure. Observe that at every step $t = t^*, \cdots, T$, only one worker makes a proposal and only the firm who receives the proposal can make a rejection. And when no worker makes a proposal or no firm rejects, the procedure stops.

Lemma 6 *Take any worker $w^* \in W$. Let Ψ' and Ψ'' be the matchings generated by the above WP-DA procedure for the markets $(W, F, \Psi^0, C^0, \succ')$ and $(W, F, \Psi^0, C^0, \succ'')$, respectively, where $\succ'_x = \succ''_x$ for all agents x except worker w^* and $\Psi'(w^*)$ is the first choice for worker w^* in $\succ''_{w^*}$. Then we have $\Psi''(w^*) = \Psi'(w^*)$.*

The following result generalizes the classical result of Dubins and Freedman (1981) without commitment to the current market with commitment and holds true for all workers.

Lemma 7 *When facing the WP-DA procedure, it is a dominant strategy for every worker to behave truthfully.*

We also know from Roth (1982) that when facing the WP-TTC procedure, it is a dominant strategy for every worker to act truthfully. However, when facing the Hybrid Procedure, i.e., the combination of the WP-DA procedure and the WP-TTC procedure, it is possible for some worker to gain by acting strategically. Let us illustrate this point through the next example.

Example 3 *There are three workers w_0, w_1, w_2 and three firms f_0, f_1, f_2. Consider the simplest case in which there is at most one contract between every worker and every firm. The preferences of all individuals are given below:*

$$\succ_{w_0}: \ f_1, \ w_0 \qquad\qquad \succ_{f_0}: \ w_1, \ f_0$$

$$\succ_{w_1}: \ f_2, \ f_0, \ f_1, \ w_1 \qquad\qquad \succ_{f_1}: \ w_0, \ w_2, \ w_1, \ f_1$$

$$\succ_{w_2}: \ f_1, \ f_2, \ w_2 \qquad\qquad \succ_{f_2}: \ w_1, \ w_2, \ f_2$$

The initial matching state (μ^0, C^0) is given by $\mu^0(w_0) = w_0$, $\mu^0(w_1) = f_1$, $\mu^0(w_2) = f_2$, $\mu^0(f_0) = f_0$, and $C^0 = \{w_1, w_2, f_1, f_2\}$.

When all agents act honestly, the Hybrid Procedure yields the matching state $(\Omega, C(\Omega))$, where $\Omega = \mu$, $\mu(w_0) = f_1$, $\mu(w_1) = f_0$, $\mu(w_2) = f_2$, and $C(\mu) = \{f_2\}$.

Now assume that all agents except worker w_1 behave honestly in the Hybrid Procedure, while worker w_1 acts instead according to the preferences $\succ_{w_1}: f_2, f_1, w_1$. Then the Hybrid Procedure yields the matching state $(\Omega', C(\Omega'))$, where $\Omega' = \mu'$, $\mu'(w_0) = w_0$, $\mu'(w_1) = f_2$, $\mu'(w_2) = f_1$, $\mu'(f_0) = f_0$, and $C(\mu') = \{f_2\}$. Clearly, $\mu'(w_1) = f_2 \succ_{w_1} f_0 = \mu(w_1)$ and worker w_1 gains in benefit by acting strategically rather than honestly.

This example shows that in the Hybrid Procedure an incumbent worker who has a committed initial partner and could participate in the WP-TTC procedure would possibly gain by manipulation if he should have the full knowledge of other agents and be confident that others would be honest.

Now we are ready to establish a basic incentive compatibility result for the Hybrid Procedure. By Lemma 7 and the strategy-proofness of TTC mechanism, we see that in the Hybrid Procedure, for any worker who can only influence either the WP-DA procedure or exclusively the WP-TTC procedure, it is a dominant strategy to act truthfully. Clearly, when facing the Hybrid Procedure, it is a dominant strategy for every entrant worker to behave honestly as he has no chance to participate in the WP-TTC procedure. We will show that acting truthfully is also a dominant strategy for every worker w who has a free initial partner f. We need to consider two cases. Firstly, when w is also free, then both w and f are free. In the Hybrid Procedure, these agents will immediately leave the market after the WP-DA procedure. By Lemma 7 it is a dominant strategy for the worker to act truthfully. Secondly, when w is committed, there are two possibilities. The first possibility is that when his partner f receives a better proposal in the WP-DA procedure-the first phase of the Hybrid Procedure, f dissolves her partnership with w and w becomes free. In this case, both w and f will immediately leave the market after the WP-DA procedure. By Lemma 7 it is a dominant strategy for the worker to act truthfully. The second possibility is that when worker w's partner f does not receive any proposal better than the one with worker w in the WP-DA procedure-the first phase of the Hybrid Procedure, then w will never become free thus having no influence in the WP-DA procedure and both w and f go through the WP-TTC procedure-the second phase of the Hybrid Procedure. In this case, it is a dominant strategy for worker w to behave honestly facing the WP-TTC procedure. The above discussion has proved the following result.

Theorem 3 *When facing the Hybrid Procedure, for every worker who is an entrant or has a free initial partner, it is a dominant strategy to act truthfully.*

This theorem extends the classical result of Dubins and Freedman (1981) but cannot cover those incumbent workers whose initial parters are committed, because these workers may be able to manipulate when they have the information of the other agents, as shown in the example above. Nevertheless, it is important to know that in practice as long as the number of participants is relatively large, it would be extremely difficult for any participant to manipulate in order to make a profitable gain even in two-sided job matching markets (see Kojima and Pathak 2009).

5 Job-Specific Commitments

We examine a market where commitments are job-specific. In other words, commitments are inherent properties of the jobs or positions offered by firms, regardless of whether these jobs are currently occupied or not. In the market, jobs or positions are exogenously classified into commitment required positions and no-commitment required ones, committed positions and non-committed ones. Many sensitive positions such as defense or security related ones are commitment required jobs in the sense that any person who accepts such a job must agree to be committed to the job. In fact, non or less sensitive jobs can be commitment required jobs as well, such as those jobs which require costly training or are less popular but offer particular benefits. Commitment required jobs demand workers taking these positions to be committed. Committed positions require that firms be committed to their hired workers like many tenured positions and civil service positions.

We will represent this market by $\mathcal{M} = (W, F = F_{cr} \cup F_{nc} = F_{tp} \cup F_{tt}, \Psi^0, \succ)$ and show how it fits in with the model in Section 2. Here F_{cr} is the set of firms which offer commitment required positions. F_{nc} is a set of firms whose positions are not commitment required, namely, F_{nc} is the complement of F_{cr} in F. So F_{cr} and F_{nc} form a partition of F. F_{tp} is the set of firms whose positions are committed to employees such as tenured positions. F_{tt} is the set of firms whose positions are not committed like non-tenured or tenure-track ones, i.e., F_{tt} is the complement of F_{tp} in F. Note that we use firms, jobs or positions interchangeably.

To put the current market into the framework given in Section 2, we need to know what are matching states here. Recall that we define every matching state in a general but also economical and effective way. That is, the commitment set $C(\Psi)$ of every matching Ψ records only those commitments currently in force at Ψ, i.e., those agents who are currently matched at Ψ (i.e., workers have taken positions and firms have hired workers at Ψ) and are committed. More precisely, $C(\Psi) = \{w \in W \mid \mu_\Psi(w) \in F_{cr}\} \cup \{f \in F_{tp} \mid \mu_\Psi(f) \in W\}$, where μ_Ψ is the one-to-one mapping or matching from $W \cup F$ onto itself induced by

Ψ. Because commitments are job-specific and fixed here, every matching state $(\Psi, C(\Psi))$ (including the initial matching state (Ψ^0, C^0)) and its committed set $C(\Psi)$ can be easily determined by Ψ, F_{cr} and F_{tp}. To see this, consider the following example.

Example 4 *Reconsider Example 1 which has three workers w_0, w_1 and w_2, two firms f_1 and f_2. Their preferences are given in Section 2. Suppose now that $F_{cr} = \{f_1\}$, $F_{nc} = \{f_2\}$, $F_{tp} = \{f_1, f_2\}$, and $F_{tt} = \emptyset$, and that the initial matching $\Psi^0 = \mu^0$ is given by $\mu^0(w_0) = w_0$, $\mu^0(w_1) = f_1$, and $\mu^0(w_2) = f_2$.*

From μ^0, F_{cr} and F_{tp} we get the initial commitment set $C^0 = \{w_1, f_1, f_2\}$ and the initial matching state (Ψ^0, C^0). Applying the Hybrid Procedure to this example gives a strict core matching μ given by $\mu(w_0) = w_0$, $\mu(w_1) = f_2$, and $\mu(w_2) = f_1$ and a stable matching state $(\mu, C(\mu))$ with $C(\mu) = \{f_1, f_2\}$. Using μ, F_{cr} and F_{tp} again we obtain the commitment set $\hat{C}(\mu) = \{w_2, f_1, f_2\}$. Because $C(\mu)$ is a subset of $\hat{C}(\mu)$, $(\mu, \hat{C}(\mu))$ is also stable. We see $(\mu, \hat{C}(\mu))$ has an unnecessary or redundant commitment constraint on w_2.

Compared with context-dependent commitments discussed in Section 2 job-specific commitments are simple so every matching state can be easily obtained from its matching, F_{cr} and F_{tp}. Recall that stability is defined with respect to matching states and core is defined by comparing with the initial matching state. So both solutions are at the ready for the current market. How does the Hybrid Procedure apply here? The Procedure starts with the initial matching state (Ψ^0, C^0). All agents in C^0 are committed and follow the same rule as described in the Procedure. Every agent x in $F \cup W$ but not in C^0 (i.e., x is initially a single) acts as a free agent in the Procedure.

We can now establish the following result. Note that in the theorem although both $(\Omega, C(\Omega))$ and $(\Omega, \hat{C}(\Omega))$ are stable, $C(\Omega)$ contains fewer commitment constraints than $\hat{C}(\Omega)$ (i.e., $C(\Omega) \subset \hat{C}(\Omega)$). This reveals an important feature of our solution concept and the Hybrid Procedure that they can achieve not only stability and efficiency but also reduce the number of commitment constraints, as in a society a state which is stable under few or no constraints is inherently more desirable than a state which is stable under a lot of constraints. In particular, the Procedure can identify which commitments are necessary or essential and which are not.

Theorem 4 *For the model $\mathcal{M} = (W, F = F_{cr} \cup F_{nc} = F_{tp} \cup F_{tt}, \Psi^0, \succ)$, starting with the initial matching state (Ψ^0, C^0) the Hybrid Procedure generates a stable matching state $(\Omega, C(\Omega))$ with a strict core matching Ω and $C(\Omega) \subseteq C^0$. Furthermore, $C(\Omega)$ is a subset of $\hat{C}(\Omega)$ derived from Ω, F_{cr}, and F_{tp}, and $(\Omega, \hat{C}(\Omega))$ is also stable.*

In our profession, it is widely observed that if a tenured professor moves to another university, his new post is usually a tenured position. One may wonder if this property can be maintained in our Procedure. The following gives a positive answer.

Proposition 1 *If a person who has a tenured (i.e., committed) position at Ψ^0 ranks every tenured position above every non-tenured (i.e., non-committed) position, he will receive a tenured position at the strict core matching generated by the Hybrid Procedure.*

It is worth pointing out that our current model and the procedure can handle another interesting case as well: It is not unusual to observe that a tenured professor from a less prestigious university may be willing to give up his current job to accept a tenure track position at a prestigious university.

6 Conclusion

Entry level two-sided matching markets have been extensively studied since Gale and Shapley (1962). In such a market all firms and workers are new entries and they each try to find a best possible partner to match. In this paper we have developed a senior level two-sided matching model with commitments. There are many heterogeneous incumbents and new entries. A free agent makes her decisions independently, while a committed agent is bound by her commitment and cannot unilaterally dissolve her partnership unless her partner agrees to do so. Every agent has preferences over multiple contracts and tries to find her best possible partner with contract. Every contract specifies its terms and conditions between a firm and a worker. There are multiple different contracts between every firm and every worker. When a firm reaches a deal with a worker, they are both agreed on a common contract to implement. Going beyond traditional ones, our model covers a variety of practical job matching settings such as tenure track systems, civil service systems and non-compete clauses in contract law, etc.

We have examined the question of how to match workers and firms with contracts as well as possible and to free as many committed agents as possible without violating their commitments. Two basic and independent solutions-stability and core-are introduced. Our first major result shows that there exists at least one stable matching state whose matching is in the strict core, and moreover the family of committed agents in this matching state is contained by the family of committed agents at the initial matching state and therefore some committed agents at the initial state will be freed in this final state. We have proposed a market mechanism-the Hybrid Procedure-for finding this solution. The procedure is a novel combination of two generalizations of the deferred acceptance (DA) procedure and the top trading cycle (TTC) method. We have shown that neither of the two generalizations suffices to find a desired solution but the Hybrid Procedure guarantees to discover one. Furthermore, we have established a lattice theorem for stable matching states with a common commitment set and two incentive compatibility results for the Hybrid Procedure and examined job-specific commitments.

The current paper has focused on a senior level one-to-one job matching model. It would be interesting to extend the current arguments and results to the more general many-to-one setting. The entry level many-to-one matching models include Kelso and Crawford (1982), Hatfield and Milgrom (2005), Ostrovsky (2008), Kojima and Pathak (2009), and Hatfield et al. (2013) among others. Fu et al. (2017) introduce job security into the model of Kelso and Crawford (1982) and examine an allocation problem when firms hire new workers but have to keep all their previously employed workers permanently.

We hope that the current study has shed some new insights into senior level job matching markets.

The Appendix

Proof of Lemma 1: By definition, $\Psi(x)$ is relatively acceptable to x when her partner $\mu^0(x)$ is committed. By the assumption that $\mu^0(x) \in V^0$ for every x with $x \succ_x \Psi^0(x)$, we know by negation that $\mu^0(x) \in C^0$ implies $\Psi^0(x) \succeq_x x$. Then x cannot be made worse than being single. So $\Psi(x)$ must be both acceptable and relatively acceptable to x when $\mu^0(x)$ is committed. □

Recall that in the marriage matching model of Gale and Shapley (1962), it is sufficient to concentrate on individuals and pairs of man and woman, because if a matching is blocked by a group of men and women, it must be blocked by some individual or some pair. In the more general supply chain model of Ostrovsky (2008), it suffices to focus on chains, as the model does not contain any cycle. Analogously, in our current model which contains chains and cycles, it will be sufficient to concentrate on chains and cycles. We will show this point here. A matching state $(\Psi, C(\Psi))$ is said to *be blocked by a group of agents* A if the following two conditions are satisfied: (i) if an agent $x \in A \cap C(\Psi)$ is committed, her partner must be also a member of the group A, i.e., $\mu_\Psi(x) \in A$; (ii) there exists a matching $\Phi \subseteq \Sigma(A)$ among agents from A alone such that for every $x \in A$, the contract $\Phi(x)$ is acceptable and $\Phi(x) \succ_x \Psi(x)$. Then we have the following result.

Proposition 2 *If an individually rational matching state is blocked by a group of agents, it must be blocked by a chain or by a cycle.*

Proof: Let $(\Psi, C(\Psi))$ be an individually rational matching state. In the first case $A \setminus C(\Psi) \neq \emptyset$, pick any free agent in $A \setminus C(\Psi)$ as x_1. Since $(\Psi, C(\Psi))$ is individually rational, we have $\mu_\Phi(x_1) \neq x_1$. Let $y_1 = \mu_\Phi(x_1) \in A$. If y_1 is free, then (x_1, y_1) is a blocking chain (pair) of $(\Psi, C(\Psi))$. If y_1 is committed, then his partner $\mu_\Psi(y_1)$ at matching Ψ is in A. Let $x_2 = \mu_\Psi(y_1)$. If $x_2 = x_1$ then (x_1, y_1) is a blocking cycle of $(\Psi, C(\Psi))$. If $x_2 \neq x_1$, we can define y_2 in exactly the same way. Since there are only finite number of agents in A, we can iteratively find a blocking chain or cycle $(x_1, y_1, x_2, y_2, \cdots, x_k, y_K)$ in group A. In the second case $A \subset C(\Psi)$, pick any agent in A as x_1. Then, we can use the above method to find a blocking cycle $(x_1, y_1, \cdots, x_k, y_K)$ of $(\Psi, C(\Psi))$ in group A. □

Proof of Lemma 3: By definition, there exists a matching Φ among agents from the coalition S alone such that every agent x in S weakly prefers $\Phi(x)$ to $\Psi(x)$ and at least one agent $y \in S$ prefers $\Phi(y)$ to $\Psi(y)$. Using these two matchings Ψ^0 and Φ, we define a directed bipartite graph $G = (S, E)$ on S by setting

$$\begin{aligned} E &= \Big\{(w, f) \,|\, w \in W \cap S, f \in F \cap S, \Phi(w) = \Phi(f)\Big\} \\ &\quad \cup \Big\{(f, w) \,|\, w \in W \cap S, f \in F \cap S, \Psi^0(w) = \Psi^0(f)\Big\}. \end{aligned}$$

That is, there is a directed arc from a worker $w \in S$ to a firm $f \in S$ if and only if they are partners under Φ, and there is a directed arc from a firm $f \in S$ to a worker $w \in S$ if and only if they are partners under Ψ^0.

Choose any individual $y \in S$ such that $\Phi(y) \succ_y \Psi(y)$. Note that in this graph G every vertex's degree is less than or equal to 2. Let G' denote the component (the maximal connected subgraph) of G which contains y. Then, G' is a directed chain or cycle. If G' is a directed cycle, then it is an improvement cycle of Ψ. If G' is a directed chain, then every its end vertex must be one of the following three cases: (i) a single agent under Ψ^0; (ii) a free incumbent in the initial state (Ψ^0, C^0) whose initial partner is not in S; (iii) a committed agent in the initial state (Ψ^0, C^0) who becomes a single under Φ. Consider Case (iii). In this case for an end agent $x \in C^0 \cap S$ with $x = \Phi(x) \succeq_x \Psi(x)$, it satisfies $\mu^0(x) \in V^0 \cap S$. This is because $\Psi^0(x) \succ_x x$ implies $\Psi^0(x) \succ_x x = \Phi(x) \succeq_x \Psi(x)$. By definition of feasible matching we see $\mu^0(x) \in V^0$. Otherwise, from $x \succ_x \Psi^0(x)$ and by assumption we see $\mu^0(x) \in V^0$. In all these three cases we can find an improvement chain of Ψ contained in the directed chain G'. So we can always find an improvement chain or cycle of Ψ contained in G'. □

Proof of Lemma 5: The WP-DA procedure will generate a matching state $(\Psi, C(\Psi))$ in a finite number of rounds because there is only a finite number of contracts, and no worker proposes one contract to any firm more than once. In the procedure every worker who has become free proposes only acceptable contracts, and every firm rejects all unacceptable contracts that she has received except that she is a committed firm and has received her initial contract proposed by her initial partner. So no agent is provisionally matched with a new but unacceptable contract. By assumption that $\mu^0(x) \in V^0$ for every incumbent x with $x \succ_x \Psi^0(x)$, the matching Ψ is feasible and $(\Psi, C(\Psi))$ is individually rational.

We will show that $(\Psi, C(\Psi))$ is chain stable. Suppose to the contrary that the matching state $(\Psi, C(\Psi))$ is blocked by one of its minimal chain $(\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$. Then, for each $k = 1, \cdots, K$, there exists a mutually acceptable contract α_k to $\bar{w}_k$ and $\bar{f}_k$ such that $\alpha_k \succ_{\bar{w}_k} \Psi(\bar{w}_k)$ and $\alpha_k \succ_{\bar{f}_k} \Psi(\bar{f}_k)$, i.e., both $\bar{w}_k$ and $\bar{f}_k$ prefer contract α_k to their contracts under Ψ. Since $\bar{w}_1 \in W_V$ and $\alpha_1 \succ_{\bar{w}_1} \Psi(\bar{w}_1)$, worker $\bar{w}_1$ must have previously proposed the contract α_1 to firm $\bar{f}_1$. If $K = 1$, then $\bar{f}_1$ is a free firm or a rematched firm. In both cases, $\bar{f}_1$ should not have rejected the proposal α_1 from worker $\bar{w}_1$, yielding a contradiction. In the case of $K \geq 2$, a minimal chain implies that $\bar{f}_1$ and $\bar{w}_2$ are in $C(\Psi)$. Thus, $\bar{f}_1 \in C^0$ and $\Psi(\bar{f}_1) = \Psi^0(\bar{f}_1)$, and so $\bar{w}_2 = \mu_\Psi(\bar{f}_1) = \mu^0(\bar{f}_1)$. Note that $\alpha_1 \succ_{\bar{f}_1} \Psi(\bar{f}_1) = \Psi^0(\bar{f}_1)$. Thus α_1 is both acceptable and relatively acceptable to firm $\bar{f}_1$. This implies that firm $\bar{f}_1$ should have freed her initial partner $\bar{w}_2 = \mu^0(\bar{f}_1)$ at some step in the procedure, namely, $\bar{w}_2$ must be in $V(\Psi)$, leading to a contradiction as well.

This shows that $(\Psi, C(\Psi))$ cannot be blocked by any minimal chain. By Lemma 2, $(\Psi, C(\Psi))$ is chain stable. □

Proof of Theorem 2: Let $(\Psi, C(\Psi))$ and $(\Phi, C(\Phi))$ be stable matching states. By assumption, we have $\mu_\Psi(x) = \mu_\Phi(x)$ for every $x \in C(\Psi) = C(\Phi)$. Let $\hat{C}(\Psi) = \{\mu_\Psi(x) \mid x \in C(\Psi)\}$ be the set of partners of committed agents in the matching state $(\Psi, C(\Psi))$ and $\hat{C}(\Phi) = \{\mu_\Phi(x) \mid x \in C(\Phi)\}$ the set of partners of committed agents in the matching state $(\Phi, C(\Phi))$. Clearly $\hat{C}(\Psi) = \hat{C}(\Phi)$. We also have $(\Psi \vee_W \Phi)(x) = \Psi(x) = \Phi(x)$ for every $x \in C(\Psi) \cup \hat{C}(\Psi) = C(\Phi) \cup \hat{C}(\Phi)$.

Consider the set $\hat{A} = (W \cup F) \setminus (C(\Psi) \cup \hat{C}(\Psi))$. With respect to the truncating agents in $\hat{A}$ and the corresponding contracts in $\Sigma(\hat{A})$, we obtain a standard matching model with contract with agents $\hat{A}$ and contracts $\Sigma(\hat{A})$. It is easy to check that $\Psi \cap \Sigma(\hat{A})$ and $\Phi \cap \Sigma(\hat{A})$ are stable matchings. So $(\Psi \vee_W \Phi) \cap \Sigma(\hat{A})$ is also a stable matching in the truncated model. This means that for every $x \in \hat{A}$, $(\Psi \vee_W \Phi)(x)$ is a singleton. We have proved that, for every agent $x \in W \cup F$, $(\Psi \vee_W \Phi)(x)$ contains exactly one contract. So $\Psi \vee_W \Phi$ is a matching.

We first show that the matching state $(\Psi \vee_W \Phi, C(\Psi))$ is chain stable. It follows easily from the assumption that $(\Psi \vee_W \Phi, C(\Psi))$ is individually rational. Suppose by way of contradiction that $(\Psi \vee_W \Phi, C(\Psi))$ is blocked by a minimal chain $\overrightarrow{X} = (w_1, f_1, \cdots, w_K, f_K)$. Then, for every $k = 1, \cdots, K$, there is a mutually acceptable contract α_k to w_k and f_k such that $\alpha_k \succ_{w_k} (\Psi \vee_W \Phi)(w_k)$ and $\alpha_k \succ_{f_k} (\Psi \vee_W \Phi)(f_k)$. Observe that $\{f_1, w_2, \cdots, f_{K-1}, w_K\} \subseteq C(\Psi) = C(\Phi)$ and $\{w_1, f_K\} \subseteq V(\Psi) = V(\Phi)$. For every $k = 1, \cdots, K-1$, we have

$\alpha_{k+1} \succ_{w_{k+1}} (\Psi \vee_W \Phi)(w_{k+1}) = \Psi(w_{k+1}) = \Phi(w_{k+1})$, and $\alpha_k \succ_{f_k} (\Psi \vee_W \Phi)(f_k) = \Psi(f_k) = \Phi(f_k)$. For the free worker w_1, it holds that $\alpha_1 \succ_{w_1} \Psi(w_1)$ and $\alpha_1 \succ_{w_1} \Phi(w_1)$. Thus, if $(\Psi \vee_W \Phi)(f_K) = \Psi(f_K)$, then $\overrightarrow{X}$ is a blocking chain of the stable matching state $(\Psi, C(\Psi))$. Otherwise, $\overrightarrow{X}$ is a blocking chain of the stable matching state $(\Psi, C(\Phi))$. This yields a contradiction. We have shown that $(\Psi \vee_W \Phi, C(\Psi))$ is chain stable.

Next, suppose to the contrary that $(\Psi \vee_W \Phi, C(\Psi))$ is blocked by a pure cycle $\overrightarrow{X} = (w_1, f_1, \cdots, w_K, f_K)$. Note that if there are some free worker and free firm in a cycle, then this cycle must contain some chain. Therefore, in a pure cycle $\overrightarrow{X} = (w_1, f_1, \cdots, w_K, f_K)$, all workers or all firms must be in $C(\Psi) = C(\Phi)$. This implies that $\Psi(x) = \Phi(x) = \Psi(x) \vee_W \Phi(x)$ for every agent x in this blocking cycle. Thus, $\overrightarrow{X}$ is also a blocking cycle of the stable matching states $(\Psi, C(\Psi))$ and $(\Phi, C(\Phi))$, yielding a contradiction. This concludes that $(\Psi \vee_W \Phi, C(\Psi))$ is stable.

Analogously, one can prove that $(\Psi \vee_F \Phi, C(\Psi))$ is also stable. By definition, because $\Psi \wedge_W \Phi = \Psi \vee_F \Phi$, then $(\Psi \wedge_W \Phi, C(\Psi))$ must be stable. □

Proof of Lemma 6: Let $DA'(w^*) =< (P'(1), R'(1)), \cdots, (P'(T'), R'(T')) >$ and $DA''(w^*) =< (P''(1), R''(1)), \cdots, (P''(T''), R''(T'')) >$ denote the corresponding WP-DA sequences for $(W, F, \Psi^0, C^0, \succ')$ and $(W, F, \Psi^0, C^0, \succ'')$. Note that $P''(t) \subset \cup_{t'=1}^{T'} P'(t')$ for all $t = 1, \cdots, T''$. We therefore have $\Psi''(w^*) = \Psi'(w^*)$. □

Proof of Lemma 7: Firstly, consider any worker who has never become free in the process. Such a worker clearly has no opportunity to make a proposal and therefore has no influence. Obviously, acting honestly is an optimal strategy for such a worker. Secondly, consider any worker w^* who will make proposals in the process. Assume that worker w^* acts according to preferences $\succ'_{w^*}$ which may be different from his true preferences $\succ_{w^*}$ but all other agents act according to any fixed (strict) preferences $\succ'_{-w^*}$. Let Ψ and Ψ' be the matchings generated by the WP-DA procedure for the markets $(W, F, \Psi^0, C^0, \succ_{w^*}, \succ'_{-w^*})$ and $(W, F, \Psi^0, C^0, \succ'_{w^*}, \succ'_{-w^*})$, respectively. By Lemma 6, we can further assume that $\Psi'(w^*)$ is the first choice for worker w^* in $\succ'_{w^*}$. We will show that $\Psi(w^*) \succeq_{w^*} \Psi'(w^*)$.

Let $DA(w^*) =< (P(1), R(1)), \cdots, (P(T), R(T)) >$ and $DA'(w^*) =< (P'(1), R'(1)), \cdots, (P'(T'), R'(T')) >$ denote the corresponding sequences generated by the WP-DA procedure for $(W, F, \Psi^0, C^0, \succ_{w^*}, \succ'_{-w^*})$ and $(W, F, \Psi^0, C^0, \succ'_{w^*}, \succ'_{-w^*})$, respectively. It follows from the WP-DA procedure that $DA_1(w^*) = DA'_1(w^*)$. In the sub-sequence $DA_2(w^*)$, worker w^* acts according to his true preferences $\succ_{w^*}$. But in the sub-sequence $DA'_2(w^*)$, worker w^* makes the proposal $\Psi'(w^*)$ and finally gets it by Lemma 6. Thus, in $DA'_2(w^*)$, there are two possible cases: The firm which receives w^*'s proposal and accepts it did not receive any proposal previously; and the firm which receives w^*'s proposal and accepts it has to reject her provisionally accepted contract from some worker. In the latter case, there exists another worker $\hat{w}$ at the end of the sequence $DA'_2(w^*)$ who either leaves the market with no job or makes a proposal to some firm which accepts it as its final contract.

Consider when worker w^* acts truthfully, namely, the sub-sequence $DA_2(w^*)$. If $DA_2(w^*)$ ends before worker w^* making the proposal $\Psi'(w^*)$, clearly we have $\Psi(w^*) \succ_{w^*} \Psi'(w^*)$. Next consider the case in which worker w^* makes the proposal $\Psi'(w^*)$ in $DA_2(w^*)$. There are two cases as indicated previously. In the first case, the firm which receives w^*'s proposal and accepts it did not receive any proposal previously. Thus the sequence $DA(w^*)$ ends immediately and so $\Psi(w^*) = \Psi'(w^*)$. In the second case, we will show that this proposal $\Psi'(w^*)$ is not rejected by firm $\Psi'_F(w^*)$ in $DA_2(w^*)$.

Suppose to the contrary that the proposal $\Psi'(w^*)$ is rejected by firm $\Psi'_F(w^*)$ at step $\bar{t}$ in $DA_2(w^*)$. We will show by induction that in each step $t = 1, \cdots, T'$ of $DA'(w^*)$, it satisfies $P'(t) \subseteq \cup_{t'=1}^{\bar{t}-1} P(t')$ and so $\cup_{t=1}^{T'} P'(t) \subseteq \cup_{t'=1}^{\bar{t}-1} P(t')$. Recall that all other agents act according to exactly the same preferences $\succ'_{-w^*}$ in both $DA(w^*)$ and $DA'(w^*)$, $DA_1(w^*) = DA'_1(w^*)$, and at the first step $t = t^*$ in $DA'_2(w^*)$ worker w^* makes the proposal $\Psi'(w^*)$, i.e., $P'(t^*) = P'_{w^*}(t^*) = \{\Psi'(w^*)\}$. Thus, we first have $P'(t) \subseteq \cup_{t'=1}^{\bar{t}-1} P(t')$ for all $t = 1, \cdots, t^*$. Now assume by induction that $P'(t) \subseteq \cup_{t'=1}^{\bar{t}-1} P(t')$ for all $t = 1, \cdots, \hat{t}$, where $\hat{t} \in \{t^*, \cdots, T'-1\}$. Assume that worker w' is rejected by firm f' at step $\hat{t}$ in $DA'_2(w^*)$, and worker w' makes a proposal α' at step $\hat{t}+1$. This proposal α' must be in $\cup_{t'=1}^{\bar{t}-1} P(t')$, or else firm f' should have received some proposal not in $\cup_{t'=1}^{\bar{t}-1} P(t')$. Then, $P'(\hat{t}+1) \subseteq \cup_{t'=1}^{\bar{t}-1} P(t')$. Thus, by induction we know $\cup_{t=1}^{T'} P'(t) \subseteq \cup_{t'=1}^{\bar{t}-1} P(t')$. This implies that in $DA_2(w^*)$ the worker $\hat{w}$ (who either leaves the market with no job or makes a proposal to some firm which accepts it as its final contract at the end of $DA'_2(w^*)$) has left the market with no job or has found a job from a firm who does not reject any one before step $\bar{t}$. In other words, the sequence $DA_2(w^*)$

must have ended before step $\bar{t}$, contradicting the hypothesis. This shows that worker w^*'s proposal $\Psi'(w^*)$ is not rejected by firm $\Psi'_F(w^*)$. Thus $\Psi(w^*) = \Psi'(w^*)$. □

Proof of Theorem 4: The first statement follows immediately from Theorem 1. Next, recall that the Hybrid Procedure splits all workers and firms in $W \cup F$ into S_1, S_2 and S_3. S_3 contains all agents who leave the market immediately after the WP-DA procedure-the first phase of the Hybrid Procedure. $S_1 \cup S_2 = S$ is the almost committed set formed by the remaining agents. The Procedure generates the matching state $(\Omega, C(\Omega))$. Let μ_Ω be the one-to-one mapping or matching from $F \cup W$ onto itself induced by Ω. It is easy to see that $S_1 \cap F \subset F_{tp}$, $S_2 \cap F \subset F_{cr}$, $\mu_\Omega(f) \in S_1 \cap W$ for every firm $f \in S_1$, and $\mu_\Omega(w) \in S_2 \cap F$ for every worker $w \in S_2$. Thus, we have $S_2 \cap W \subset \{w \in W \,|\, \mu_\Omega(w) \in F_{cr}\}$ and $S_1 \cap F \subset \{f \in F_{tp} \,|\, \mu_\Omega(f) \in W\}$. Recall that $C(\Omega) = (S_2 \cap W) \cup (S_1 \cap F)$. We therefore have $C(\Omega) \subset \{w \in W \,|\, \mu_\Omega(w) \in F_{cr}\} \cup \{f \in F_{tp} \,|\, \mu_\Omega(f) \in W\} = \hat{C}(\Omega)$. Finally, because $(\Omega, C(\Omega))$ is a stable matching state and $\hat{C}(\Omega)$ contains $C(\Omega)$, $(\Omega, \hat{C}(\Omega))$ is obviously a stable matching state. See also the discussion right after Lemma 2. □

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