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Statistical Mechanical Analysis for Unweighted and Weighted Stock Market Networks

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Abstract

Financial markets are time-evolving complex systems containing different financial entities, such as banks, corporations and institutions that interact through transactions and respond to external economic and political events. They can be conveniently represented as a network structure. In this paper, we analyse the unweighted and weighted market networks from a statistical mechanical perspective. In particular, we propose a novel thermodynamic analogy to characterise the dynamic structural properties of time-evolving networks. The intricate pattern of edge connections in the network is modelled by using a heat bath analogy in which particles occupy the energy states according to the Boltzmann distribution. According to this analogy the occupation of the energy states is determined by the temperature of the heat bath, and the spectrum of energy states of the network is determined by the number of nodes and edges. For unweighted networks, the binary representation of the elements in the adjacency matrix can be modelled as a statistical ensemble, using the corresponding partition function to compute thermodynamic network characterisations. For weighted networks, on the other hand, the derived thermodynamic quantities together with their distribution of fluctuations identify the salient structure in the network evolu-

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tion. We conduct experiments on time-evolving stock exchanges using data for the S&P500 Index Stock Exchanges over the past decade. The thermodynamic characterisations provide an excellent framework to identify epochs in which there is significant variance in network structure during financial crises induced by economic and political events.

Keywords: Stock Market Networks, Thermodynamic Characterisations, Statistical Mechanics

1. Introduction

Trading in financial markets is usually associated with transactions that involve the exchange of assets, and reflect the interactions between different financial entities. However, this method of trading is difficult to accurately explain or predict when the market is influenced by an external economic or political event that has a long-term impact on the global economy [1]. Since 2008, there is a growing literature on applying ideas from statistical mechanics and thermodynamics, sometimes referred to as econophysics, to explore what happens in the proximity of critical market events [2, 3]. Generally speaking, the time-evolving financial market can be regarded as a network in which the nodes represent financial entities or individuals and the edges capture how the correlations in their performance evolve with the time [4, 5]. The graph-based methods, often known as complex networks for large-scale datasets, provide a number of impressive applications in the financial and business markets. As a specific example, for stock markets the volatility in stock prices is taken as the main representative of market activity and the trading relationships between the financial entities [6]. The corresponding thermodynamic characteristics of the network structure reflect the variations in trading activity as the market network evolves.

To quantify financial market networks, sophisticated tools developed in complex network theory have been deployed to analyse the topological and structural properties of the underlying time-evolving networks [7]. Most recent literature attempts to link structural changes to change in network function, and thus explain the intrinsic mechanisms underlying the financial market [8, 9]. For example, considering the long-term impact of the financial crises on the global economy, or studying whether the entropic features of stock markets identify the relevant nature of a crisis [10]. Recently, ideas from statistical mechanics developed in the field of physics have provided a powerful tool for analysing the dynamics of network evolution [11]. This method attempts to describe the characteristics of stock market networks from a thermodynamic perspective [12]. The energy and entropy of the thermal system are derived from a partition function which represents states of the stock market network [13, 14]. For example, by interpreting the weighted network as a grand canonical ensemble it is possible to develop an analogy in which the edges to are mapped to particles in a statistical mechanical system. By transforming the original edge structure of a graph together with its adjacency matrix into the equivalent oriented line graph, the edge weights can be treated as generalized coordinates. These co-ordinates include energy and the magnetic moment, and can be used to derive theoretical models for reconstructing the network connections [15].

However, most of the methods reported in the literature focus on the networks which describe how the pattern of binary-valued edges evolve with time [16, 17]. There is little literature that discusses both unweighted and weighted networks, and their use in representing financial market [18]. Motivated by the need to fill this gap in the literature and to augment the methods available for understanding the evolution of financial market network dynamics, this paper proposes a novel representation of time-evolving networks from the perspective of statistical mechanics. We commence from the analogue of particles in a statistical mechanical system with the energy states of the edge configuration of the network populated according to the Boltzmann distribution [19]. The total energy of this system is related to the sum of the edge weights [15]. Following this approach, the entire network system can be represented by the configuration of edges at a fixed number of elements in the adjacency matrix [15], where the nodes in the network correspond to the financial entities under study.

Taking the simple case of an undirected and unweighted network, the thermodynamic microstates for each edge are associated with the binary state in the adjacency matrix. Introducing the concept of a statistical ensemble for network dynamics further enriches the interpretation of thermodynamic characteristics [20]. Distinct from the previous interpretation of weighted edges as particles in a grand canonical ensemble [15], here we commence from the definition of the entropy derived from the configuration probability on an unweighted graph. The resulting statistical mechanical model can then be extended to the case of weighted networks by considering the distribution of edge weights. In this way, we derive new expressions for the partition function and the derived thermodynamic quantities, and then use these to describe the time-evolving behaviour of financial market networks.

The motivation of this paper is to provide graph-based pattern recognition approaches to model the complex structural relations and to extract useful characterisations from the time-varying stock market networks. The statistical mechanical methods in complex networks hold out the novel potential as powerful tools for the description of dynamic networks [21, 22]. This derives the corresponding thermodynamic quantities for both unweighted and weighted stock market networks. The developed approach here contribute to the comprehensive analysis of the stock markets and detect anomalous behaviours.

The main contribution of this paper is to provide the novel thermodynamic analogy in which the particles correspond to network edges, and where the weighted edge connections explicitly explain the fundamental meaning of microstates in the network structure. In our previous work, we use the graph spectra and partition function to derive the network thermal quantities [11, 23]. This does not provide an explicit explanation of the physical meaning for the particles in the network system. Different from the previous analytical framework, here, we commence from the configuration of edges in the unweighted network to propose a new way to develop the fundamental concept of entropy and temperature in the network system. Starting with a simple binary state network [12, 24], we propose a new and more intuitive method for network description, i.e., the canonical ensemble, to analyse the thermal characteristics of the network structure. This can be further extended to the more complicated cases, such as weighted networks and directed networks, to derive the novel thermodynamic properties in network structure.

Table 1: A Table of Thermal Quantities with their Corresponding Meanings in Stock Market Networks

Thermal Quantities	Corresponding Meanings in Stock Market Networks
Thermodynamic system	stock market networks
Network edges	stock price interactions
Particles analogue	edges in the network are analogue to particles in the thermodynamic system
Total Energy	the total edge weights in the network
Energy	edge weight
Energy States/levels	discretized edge weights; binary values for unweighted network
Entropy	the logarithm of the number of edge combinations
Temperature	change of energy over the change of entropy $T = \partial U/\partial S$
Heat Bath	external environment outside the stock markets
Partition Function	exponential summation of all possible energy states
Canonical Ensemble	a group of stock market networks with a fixed number of nodes and various value of edge weights

Compared to existing wors, the advantages of our proposed method are to characterise the dynamic structural properties from the thermodynamic point of view. This proposes an efficient and accurate computation for the unweighted and weighted networks from the statistical mechanics . By analysing the probability configuration of edge connections, this work reveals the intricate mechanism in the network structure evolution [25]. The corresponding derived thermal quantities, such as entropy, energy and temperature, provide a sophisticate way to describe the network evolution. This brings a bridge to analyse the network structural characterisations between the microscope and macroscopic perspectives [26]. In particular to the application of the financial market, these derived thermodynamic quantities can effectively detect statistically significant observations on stock market networks [27]. This is more sensitive than other state-of-the-arts indicators to detect the salient features in the network structure. It provides an efficient way of identifying critical financial events during the evolution and can be regarded as a novel reference index to predict the financial crisis.

To better describe the statistical mechanical concepts in the financial markets, we provide a list of terms, as shown in Table1, a summary with the names in thermal physical background corresponding to the econophysic meaning of stock market networks. A more detailed description of individual quantities will be further developed in our theoretical analysis.

This paper is organised as follows. We first introduce the preliminary concepts of market network construction in Sec.II. Then, we derive the fundamental definition of temperature in networks. This provides a thermal analogy to analyse the network structure. In Sec.III, we begin with a simple case of unweighted networks. This introduces the concept of microstate and the corresponding binary state in the network system. Then, in Sec.IV, we describe networks as a canonical ensemble and make use of the concept of the partition function for calculating the thermodynamic quantities. In Sec.V, we explore the case of weighted networks. We propose two kinds of weight distribution and discuss the resulting weight fluctuations. Finally, we conduct the experiments on the stock market data with S&P500 Index to characterise and evaluate the time-varying network properties.

2. Thermodynamic Representation

2.1. Preliminaries

Let $G(V, E)$ be an unweighted network with a set of nodes V and a set of edges $E \subseteq |V| \times |V|$. The adjacency matrix A is defined as

$$A = \begin{cases} 1 & \text{if } (u, v) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

where (u, v) is a pair of nodes forming an edge in the network. The corresponding degree matrix K is diagonal, whose elements are the node degrees

$$K(u, u) = d_u = \sum_{v \in V} A_{uv} \tag{2}$$

For a weighted network G_w, the pair of nodes (u, v) is also associated with a weighting function which gives a real non-negative value $w(u, v)$ for each edge, i.e., $u \in V, v \in V$, and $u \neq v$. The weighted adjacency matrix A_w for a weighted network is given by

$$A_w(u, v) = \begin{cases} w(u, v) & \text{if } (u, v) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

where, for the undirected network, the weighting function is symmetric, i.e., $w(u, v) = w(v, u)$ for all pairs of nodes that $(u, v) \in E, u \neq v$.

2.2. Network Temperature

Commencing from this prescription of complex networks, we interpret the network as a thermal particle system. According to our analogy, the particles occupy the energy levels of the thermal system that is in equilibrium with a heat bath. These particles can occupy a set of energy states, and the probability of occupying a particular energy state at a given temperature is determined by the Boltzmann distribution.

We first consider an unweighted network. The number of nodes is fixed then the structure of the network is specified by a $|V| \times |V|$ adjacency matrix whose elements indicate the existence or otherwise of edges. Suppose that the probability of connecting a randomly selected pair of nodes with an edge is p. We can thus regard each edge as the outcome of a Bernoulli trial. In other words, an edge connection can be regarded as a binomial distribution

$$p(u,v) = p^{A(u,v)}(1-p)^{(1-A(u,v))} = \begin{cases} p & \text{if } A(u,v) = 1 \\ 1-p & \text{if } A(u,v) = 0 \end{cases} \tag{4}$$

where $A(u,v)$ is an element in the adjacency matrix A.

If the edges are the results of independent Bernoulli trials, then the probability of the observed edge set E can be written in the form

$$P(E) = \prod_{(u,v)} p(u,v), \text{ if } (u,v) \in |V| \times |V| \tag{5}$$

We can thus dichotomise the network into those pairs of nodes connected by an edge, and those that are not connected. With a binomial model of edge formation, the probability of the observed configuration of edges is

$$P(E) = \binom{|V|^2}{|E|} p^{|E|}(1-p)^{|V|^2-|E|} = ce^{-U/T} \tag{6}$$

where T is the temperature, c is a normalizing constant and U is the energy associated with the edges in the network.

The exponential function in Eq.(6) comes from the thermal equilibrium under the Boltzmann distribution [5]. The thermal system aims to occupy the state of highest multiplicity in terms of the entropy associated with the distribution of particles among different energy states. The exponential distribution results from this model by using the method of Lagrange multipliers [6].

According to our analogy, the total energy U is the sum of weights over the complete set of edges. For the unweighted network, the weight of an edge is set to unity. Thus, the definition of total energy in the network is given by

$$U = w|E| \tag{7}$$

where w is the weight for each edge, which we take as unity in the unweighted networks.

The conservation law applying, in this case, is that if the number of nodes $|V|$ is constant then the $|V| \times |V|$ adjacency matrix has a fixed number of locations (or traps) that putative edges can occupy. When the number of edges is determined, then the total energy of the network is also determined.

According to Boltzmann statistics, the entropy of a statistical mechanical system is proportional to the logarithm of the number of microstates, and the constant of proportionality is the Boltzmann constant k_B. To compute the entropy via this route, we introduce the multiplicity parameter W, to account for the number of ways for choosing the $|E|$ edge states from the available $|V|^2$ combinatorial possibilities. This is just the number of permutations for all edges obtained with the combinatorial formula

$$W(|V|,|E|) = \frac{(|V|^2)!}{|E|!(|V|^2 - |E|)!} \tag{8}$$

where the symbol of ! indicates a factorial.

According to Boltzmann's formula, the entropy is given by $S = k_B \ln W$, where k_B is the Boltzmann constant. In other words, the entropy is proportional to the logarithm of the number of edge combinations. Using Stirling's approximation $\log n! \approx n \log n - n$ for large n, we find

$$\begin{aligned} S &= k_B \ln W \qquad (9) \\ &= \log[(|V|^2)!] - \log(|E|!) - \log[(|V|^2 - |E|)!] \\ &= -|V|^2 k_B \left[\frac{|E|}{|V|^2} \ln \frac{|E|}{|V|^2} + (1 - \frac{|E|}{|V|^2}) \ln(1 - \frac{|E|}{|V|^2}) \right] \end{aligned}$$

The quantity $p = \frac{|E|}{|V|^2}$ is the probability of a connection between a pair of nodes and is the edge-density in the network. As a result

$$S = -|V|^2 k_B \left[p \ln + (1-p) \ln(1-p)\right] \tag{10}$$

Since p is the probability of an edge link between a pair of nodes, and $1-p$ the probability of a pair of nodes being unlinked, then the entropy is just the sum of the Shannon entropies of the total expected number of edges and non-edges.

With expressions for network energy and entropy to hand, for a network with a fixed volume (i.e. the number of nodes), the relationship between the incremental changes in energy and entropy is $dU = TdS$, where T is the temperature. Hence, the temperature T for the network configuration with a fixed number of nodes $|V|$ and edges $|E|$ is given by

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,E} = \frac{k_B}{w} \ln\left(\frac{|V|^2}{|E|} - 1\right) \tag{11}$$

In Eq.(11), if we exponentiate both left and right hand sides, the exponential factor is related to the average degree in the network

$$e^{w/k_B T} = \frac{|V|^2}{|E|} - 1 = \frac{|V|}{\langle d \rangle} - 1 \tag{12}$$

where $\langle d \rangle = |E|/|V|$ is the average node degree or the number of connections per node. $\langle \cdot \rangle$ denotes an ensemble average as the mean of a quantity following a function of possible states according to the Boltzmann distribution.

Thus, this gives the relationship between the average degree with the inverse temperature $1/T$

$$\langle d \rangle = \frac{|E|}{|V|} = \frac{|V|}{e^{w/k_B T} + 1} = \frac{|V|}{Z} e^{-\beta w} \tag{13}$$

where $\beta = 1/k_B T$ and Z is the partition function for each edge given by

$$Z = \sum_i e^{-\beta w_i} = e^{-\beta w} + 1 \tag{14}$$

Here, the edge is binary state in the partition function, i.e., $w_1 = 0$ and $w_2 = w$.

The standard deviation for the node degree can be regarded as the measure of fluctuation in global structure of the network. It is given by,

$$\sigma_d = \sqrt{\sum_{i=1}^{|V|} (d_i - \langle d \rangle)^2} = \sqrt{\sum_{i=1}^{|V|} \left(d_i - \frac{|V|}{Z} e^{-\beta w}\right)^2} \tag{15}$$

The degree fluctuation can be used to measure how the set of nodes constituting a graph deviate both from the mean value and the individual value at thermal equilibrium. The latter is controlled by the global temperature parameter.

3. Unweighted Networks

We commence from the edge configuration to develop the fundamental concepts of entropy and temperature in the network. This is the crucial ingredient of our method and it is discussed in detail in Section 2. Both the unweighted and weighted variants of our method rely fundamentally on the idea of edge configuration.

We make an analogy in which the edges are the particles in the thermal system [28]. A physical model which allows us to exploit this analogy is to regard the edges as $|E|$ impurity atoms trapped in a potential defined by the $|V| \times |V|$ adjacency matrix [5]. The non-zero entries in the adjacency matrix are the locations of these traps. Each edge exists in one of two energy states, either the ground state $w_1 = 0$ or the excited state $w_2 = w$.

The binary edge states are the microstates in the network, and they are specified by the set of occupation numbers $\{n_i\}$, where $n_i = 0$ or 1 depending on whether the ith edge is in its ground state or the excited state. The total energy of the network can be derived from the sum of these edge microstates by summing over all node degrees

$$U\left(\{n_i\}\right) = w \sum_{i=1}^{|E|} n_i \equiv w|E| \tag{16}$$

Thus, the macrostates of the network are specified by their total energy U, and their number of nodes $|V|$. This is consistent with Eq.(7) for the unweighted network. The difference is that instead of considering the weights on the edges, we regard the total energy in the network from an alternative point of view, i.e., energy states and occupation number. Here, the state for each edge is binary and represents whether or not the edge exists. The occupation number relates to the degree of each node. In this way, the physical meaning of energy is more related to the network structure in a more reasonable way.

The probability of the microstates can be derived from the joint probability distribution which in turn depends on the microstate energies. For a single particular edge in the microstate n_1, the unconditional probability is

$$p\left(n_1\right) = \sum_{\{n_2,\ldots,n_N\}} p\left(\{n_i\}\right) = \frac{W\left(U - n_1 w, |V|^2 - 1\right)}{W(U, |V|^2)} \tag{17}$$

where W is the binomial coefficient in Eq.(8), i.e. the number of ways of choosing $\frac{U}{n_1 w}$ excited states from among the available $|V|^2$ possible configurations. Once, a fractional amount of energy for a specific edge is allocated, the remaining energy to be distributed among the $|V|^2 - 1$ remaining configurations. Hence

$$\begin{aligned} p\left(n_1\right) &= \frac{W(U, |V|^2 - 1)}{W(U, |V|^2)} \\ &= \frac{(|V|^2 - 1)!}{|E|!\,(|V|^2 - |E| - 1)!} \cdot \frac{|E|!\left(|V|^2 - |E|\right)!}{(|V|^2)!} \\ &= 1 - \frac{|E|}{|V|^2} \end{aligned} \tag{18}$$

Since $|E|/|V|^2 = U/(|V|^2 w)$, the occupation probabilities at an inverse temperature β are

$$p(w_1 = 0) = \frac{1}{1 + e^{-\beta w}} = \frac{1}{Z} \tag{19}$$

where $Z = 1 + e^{-\beta w}$ is the partition function for the edge. For $p(w_2 = 1) = 1 - p(w_1 = 0) = |E|/|V|^2$, the probability for an edge to be in the excited state is

$$p(w_2 = 1) = \frac{e^{-\beta w}}{1 + e^{-\beta w}} = \frac{e^{-\beta w}}{Z} \tag{20}$$

Therefore, the probability for the binary states in the network relates directly to the temperature and the corresponding partition function.

4. Canonical Ensembles

The definition of the canonical ensemble in statistical mechanics represents a group of particle systems, each of which can exchange its energy with a large heat bath.

By applying the probabilistic methods to describe networks, the traditional approach is to measure or observe a property of a network several times without controlling the microscopic states to estimate the distribution of the observations. Gibbs introduced the concept of an ensemble [29]. This consists of a large number of virtual copies (sometimes infinitely many) of the system, each of which represents a possible state that the system might occupy. A statistical ensemble is thus a probability distribution for the state of the system. Each member of the ensemble represents a possible macrostate [29]. This provides a convenient way to calculate the thermodynamic properties of a network system.

Recall, the network of binary states defined in the last section is analogous to a thermal system forming a canonical ensemble. It can exchange energy until it is in thermal equilibrium with a large heat reservoir whose temperature remains constant. This means that the ensemble consists of a set of networks, each with an identical number of edges and nodes, but with different edge configurations and structural connections.

Additionally each edge has microstates. These are the binary states with occupation numbers n_i , where $n_i = 0$ or 1. The total energy of the network in Eq.(16) is conserved and the probabilities of microstates n_i in the canonical ensemble are given by

$$p(\{n_i\}) = \frac{1}{\mathcal{Q}} \exp\left[-\beta w \sum_{i=1}^{|E|} n_i\right] \tag{21}$$

where $\beta = 1/k_B T$ and $\mathcal{Q}$ is the partition function for the whole network system,

$$\mathcal{Q}(\beta, |E|) = \sum_{\{n_i\}} \exp\left[-\beta w \sum_{i=1}^{|E|} n_i\right] = (1 + e^{-\beta w})^{|E|} = Z^{|E|} \tag{22}$$

With these ingredients, a) the corresponding free energy is

$$F(T, |E|) = -k_B T \ln \mathcal{Q} = -|E| k_B T \ln\left[1 + e^{-w/(k_B T)}\right] \tag{23}$$

b) the corresponding entropy is

$$S = -\left(\frac{\partial F}{\partial T}\right)_E = \underbrace{|E| k_B \ln\left[1 + \mathrm{e}^{-\beta w}\right]}_{-F/T} + \left(\frac{|E| w}{T}\right) \frac{\mathrm{e}^{-\beta w}}{1 + \mathrm{e}^{-\beta w}} \tag{24}$$

and c) the average internal energy is

$$U = F + TS = \frac{|E| w}{1 + e^{w/(k_B T)}} \tag{25}$$

Alternatively, the average internal energy can also be obtained from the corresponding partition function

$$U = -\frac{\partial \ln \mathcal{Q}}{\partial \beta} = \frac{|E| w e^{-\beta w}}{1+e^{-\beta w}} = \frac{|E|}{Z} w e^{-\beta w} \tag{26}$$

Finally, we note that the heat capacity is given by

$$C = \left(\frac{\partial U}{\partial T}\right)_E = |E| \frac{k_B (\beta w)^2 e^{\beta w}}{[1+e^{-\beta w}]^2} = |E| k_B \left(\frac{\beta w}{Z}\right)^2 e^{\beta w} \tag{27}$$

The heat capacity is an interesting characteristic of a network structure whose temperature limits reflect the structural properties in the network. At high temperature, the heat capacity corresponds to a saturation state in which the edge states reach maximum occupation. At low temperature, the heat capacity vanishes because of the exponential term $\exp(-\beta w)$. This characterises the energy gap between the lowest state and the second-lowest state, which shows the numerical imbalance between the edge and the non-edge states.

Since the joint probability in Eq.(21) of each edge microstate is independent, we have

$$p(n_i) = \frac{e^{-\beta w n_i}}{1+e^{-\beta w}} \tag{28}$$

where $n_i = 0$ or 1.

This result is consistent with Eq.(19) and Eq.(20), which provides a more elaborate way to analyse the network in the canonical ensemble.

The total partition function of Eq.(22) covers the cases of both particles and non-particles. This relates to the Erdos-Renyi model and the probability of edge connection between nodes. However, the important difference is that it provides a new way to view this simple network model from the thermodynamic perspective. The canonical ensemble method introduces novel thermodynamic quantities related to the network structure. In Section 3 and Section 4, we consider the simple case that all edge connections have an equal probability. This lays the groundwork for the treatment of weighted networks where different edge connections have different probabilities.

5. Weighted Networks

A weighted network contains a weighting function for the edges. This function encodes information concerning the nature of the connections. Examples include the strength of the connection between a pair of nodes, the correlation between the distributions of node attributes, and pairwise node degree co-occurrence. By mapping the weighted edges to particles, the grand canonical ensemble framework has been used to develop statistical mechanical models that can be used to describe real weighted networks [15].

To determine the partition function of a network, we need to know the relevant energy levels so that we can label the states of the system. We take the edge weights to be analogous to the energy states in the previously discussed thermal network system [15]. For the weighted network, the microstates associated with each edge are no longer binary. We need to determine how many states lie within a given energy interval, and this leads us to define the density of states (DOS).

The distribution of weights is mapped to the density of microstates (DOS) in the networks. This is closely related to the degree distribution [30]. Here, we study two conservative distributions of DOS, namely, the exponential distribution and the power-law. This allows us to compute the thermal characteristics for weighted networks.

Extending the discrete form of partition function in Eq.(14),to accommodate the continuous distribution of weights present we compute the integral form of the partition function for the weighted network

$$Z_w = \int_0^\infty e^{-\beta w} D(w) dw \tag{29}$$

where $D(w)$ is the distribution function for the weights, here taken to be the density of states (DOS).

Here, we consider two typical examples of weight distributions, i.e., exponential and power-law distributions. These two distributions always appear in statistical physics known as "thermal" and "superthermal" parts.

5.1. Exponential Distribution

When the edge weights in the network follow an exponential distribution, the corresponding partition function is given by

$$Z_w^E = \int_0^\infty e^{-\beta w} k e^{\alpha w} dw \tag{30}$$

where the $D(w) = ke^{\alpha w}, \alpha > 0, k = 1/\alpha \geq 1$.

The integral form of the partition function converges when it satisfies the condition that $\beta > \alpha >= 0$. The resulting partition function in the case of the exponential distribution is,

$$Z_w^E = \frac{k}{\beta - \alpha}, (\beta > \alpha > 0) \tag{31}$$

giving a) the average internal energy

$$U_w^E = -\frac{\partial \ln Z_w^E}{\partial \beta} = \frac{1}{\beta - \alpha} \tag{32}$$

b) the entropy,

$$S_w^E = \log Z_w^E + \beta U_w^E = \log\left(\frac{k}{\beta - \alpha}\right) + \frac{\beta}{\beta - \alpha} \tag{33}$$

and c) the heat capacity

$$C_w^E = -\beta^2 \frac{\partial \ln U_w^E}{\partial \beta} = \frac{\beta^2}{(\beta - \alpha)^2} \tag{34}$$

All three thermal quantities depend on both the exponential parameter and the temperature.

5.2. Power-law Distribution

When the distribution of edge weights follows the power-law, the corresponding partition function can be written as

$$Z_w^P = \int_0^\infty e^{-\beta w} c w^\gamma dw = \frac{c\Gamma(\gamma + 1)}{\beta^{\gamma+1}} \tag{35}$$

where $D(w) = cw^\gamma, \gamma > 0, c \geq 1$ and $\Gamma(\gamma + 1)$ is the Gamma function that $\Gamma(\gamma + 1) = \gamma! = \int_0^\infty e^{-x} x^\gamma dx$.

This partition function gives the average internal energy as

$$U_w^P = -\frac{\partial \ln Z_w^P}{\partial \beta} = \frac{\gamma + 1}{\beta} \tag{36}$$

the entropy

$$S_w^P = \log Z_w^P + \beta U_w^P = \log\left(\frac{c\Gamma(\gamma+1)}{\beta^{\gamma+1}}\right) + \gamma + 1 \tag{37}$$

and the heat capacity

$$C_w^P = -\beta^2 \frac{\partial \ln U_w^P}{\partial \beta} = \gamma + 1 \tag{38}$$

While both the average energy and entropy depend on temperature and the power index, the heat capacity depends only on the power index and not temperature.

Therefore, the thermal quantities in the weighted network depend on the corresponding partition functions with the appropriately selected distribution of edge weights. For the exponential distribution, the corresponding thermodynamic quantities take on simple forms depending on both the exponential constant and temperature. However, in the power-law case, the corresponding energy and entropy present more complicated forms, depending on a Gamma function of the power-law parameter. Importantly, the heat capacity does not depend on the temperature.

5.3. Network Weight Fluctuation

For a weighted network in the canonical ensemble, the weights of edges can range between zero and infinity. The average internal energy is proportional to the average weight in the network. This can be expressed in terms of the thermodynamic equilibrium under the Boltzmann distribution as shown in Eq.(26)

$$U = \langle w \rangle |E| = -\frac{\partial \log Z}{\partial \beta} = \frac{\sum_s w_s e^{-\beta w_s}}{\sum_s e^{-\beta w_s}} \tag{39}$$

Holding the edge weight w_s constant, we take the partial derivative of the mean weights with respect to the parameter β to obtain

$$\frac{\partial U}{\partial \beta} = -\frac{\sum_s w_s^2 e^{-\beta w_s}}{\sum_s e^{-\beta w_s}} + \frac{\left[\sum_s w_s e^{-\beta w_s}\right]^2}{\left[\sum_s e^{-\beta w_s}\right]^2} = -\left\langle w^2 \right\rangle + \langle w \rangle^2 \tag{40}$$

where $\langle\cdot\rangle$ denotes an average. This give the definition of network weights fluctuation as

$$\langle\delta w\rangle^2 = \langle w^2\rangle - \langle w\rangle^2 = -\frac{\partial U}{\partial\beta} = k_B T^2 \frac{\partial U}{\partial T} = k_B T^2 C \tag{41}$$

where C is the heat capacity in Eq.(27). The relative root-mean-square fluctuation in terms of network weights is given by

$$\frac{\langle\delta w\rangle}{\langle w\rangle} = \sqrt{\frac{\langle w^2\rangle - \langle w\rangle^2}{\langle w\rangle^2}} = \frac{\sqrt{k_B T^2 C}}{\langle w\rangle} \sim \frac{1}{\sqrt{|E|}} \tag{42}$$

which means for large number of edges in the network, the relative root-mean-square fluctuation in the values of weights is negligible. Thus, in the thermodynamic limit the network has weights equal to, or almost equal to, the average weight.

For further understanding of the weight fluctuation, we treat the weight w as a continuous variable with the density of states $D(w)$. This gives the probability density $P(w)$ as

$$P(w)dw = \rho(w)D(w)dw \propto e^{-\beta w} D(w)dw \tag{43}$$

Thus, two factors influence the probability density for network weights. The first is the Boltzmann factor which exponentially decreases with w, and the second is the density of states $D(w)$ which monotonically increases with w. Therefore, there is an optimal value w^* which gives the maximum probability and that satisfies the condition

$$\frac{\partial}{\partial w}\left[e^{-\beta w} D(w)\right]\Big|_{w=w^*} = 0 \tag{44}$$

that is, by

$$\frac{\partial}{\partial w}\log[D(w)]\Big|_{w=w^*} = \frac{\partial}{\partial w}\log[e^{-\beta w}]\Big|_{w=w^*} = \beta \tag{45}$$

Recalling the definition of temperature in Eq.(11), the corresponding calculation gives optimal solution for the network weight

$$w^* = \frac{U}{|E|} = \langle w\rangle \tag{46}$$

This shows that the most probable value of network weight is identical to the average weight in a network, i.e. the mode and the mean are identical.

We can apply the Taylor expansion to further explore the behaviour of the logarithm of the probability density around the optimal solution $w^* =< w >$. The first two terms in the Taylor expansion are

$$\begin{aligned}\log\left[e^{-\beta w}D(w)\right] &= (-\beta w + S/k_B) + \frac{1}{2}\frac{\partial^2}{\partial w^2}\log\left[e^{-\beta w}D(w)\right]\Big|_{w=w^*} + \dots \\ &= -\beta(w - TS) - \frac{(w-w^*)^2}{2k_B T^2 C} + \dots\end{aligned} \tag{47}$$

As a result the probability density $P(w)$ has the following proportionality

$$P(w) \propto e^{-\beta w}D(w) \approx e^{-\beta(w-TS)}\exp\left[-\frac{(w-w^*)^2}{2k_B T^2 C}\right] \tag{48}$$

Leading us to conclude that the weights w are distributed according to a Gaussian distribution. The mean value is w^* and the dispersion is $\sqrt{k_B T^2 C}$. When the number of edges in the network is large, this results in an extremely sharp (narrow) distribution, so that the fluctuation of network weights is small.

The matrix $D(w)$ is the general representation of the density of states. It is related to the edge weight matrix $P(w)$ which is the probability density function for each edge. The relationship between the two matrices is given exactly by Eq.(43). A more detailed analysis of the edge weight variance around the mean is given by Eq.(48).

6. Experiment and Evaluation

In this section, we experiment with the methods developed in the earlier sections of this paper. Although our analysis applies to any time-evolving network, here we concentrate on the analysis of financial market data, and in particular stock market data.

6.1. Time-evolving Stock Market Networks

The stock market data comes from the Yahoo! financial dataset [31]. This dataset contains the closing prices of 485 companies in the S&P500 Index Stock

Exchanges. The stock closing prices are recorded over 2619 trading days, from the beginning of January 2010 to the end of June 2020. The 415 stocks that are present on all trading days are selected from this set. These company stocks are grouped into 11 industrial stock sectors, namely, Consumer Discretionary, Consumer Staples, Energy, Financials, Health Care, Industrials, Information Technology, Materials, Real Estate, Telecommunication Services and Utilities.

To represent the stock market data as a time-evolving network, we compute the logarithm of the return price and use this to describe the closing price of the stock over the trading period. We then make use of the cross-correlation coefficients for the closing price time series for the individual stock. According to our representation, each stock is represented by a labelled node in the network which is attributed by the closing price time series for that stock. In other words, each node represents a company whose stock is traded. The corresponding edges between pairs of nodes indicate statistical similarities between the closing process time series over a specified time window. This statistical similarity is characterised by using the time-serial logarithm of return prices over a particular period. In particular, we use the cross-correlation coefficients between the time series.

$$C_{ij}^{\Delta T}(t,\tau_{i,j}) = E[P_i(t) * P_j(t+\tau_{i,j})] = \frac{1}{\Delta t}\sum_{t'=0}^{\Delta t-1} P_i(t+t')P_j(t_{\tau_{i,j}}+t') \qquad (49)$$

where $P_i(t)$ is the closing price of the stock (node) indexed i at the day (time) indexed t, $\tau_{i,j}$ is a time-lag for the time series of nodes i and j and Δt the width of a sampling window interval in trading days.

We rank the value of cross-correlation coefficients to identify those pairs of nodes with the largest statistical similarity between stock closing price time series. The maximum value of the correlation function in Eq.(49) can well reflect the periodic components in the stock sequence. Thus, the correlation matrix M is the largest value of cross-correlation coefficients in C, which is also viewed as the edge weights in the weighted stock market network.

However, the correlation coefficient matrix M alone is not sufficient to represent the topology structure of financial networks. For instance, it does not fulfil the definition of axiomatic requirements of a metric. To overcome these problems, we apply the method to generate edges by considering the multivariate normal distribution of the covariance matrix [32]. To get the adjacency matrix A from the correlation matrix M, we assume the elements in M follow a multivariate normal distribution, i.e., $M \sim \mathcal{N}(0, \Sigma)$, where $Var[M_{ij}] = \Sigma_{(i,j),(i,j)} = 1$, $Cov[M_{ij}, M_{ik}] = \Sigma_{(i,j),(i,k)} = \rho$, and $Cov[M_{ij}, M_{kl}] = \Sigma_{(i,j),(k,l)} = 0$. The structure of the covariance matrix Σ is such that the edge weights connecting to a common node have covariance ρ, while the edge weights not connected by a common node have zero covariance.

The adjacency matrix of the unweighted network is obtained by setting a threshold $\xi \in \mathbb{R}$ and covariance $\rho \in [0, 1/2]$ to obtain the pairs of nodes with the most significant correlations as the edges in the network. The threshold ξ is setting with the inverse error function of the Gaussian distribution [32], i.e.,

$$\xi = \Phi^{-1}\left(1 - \frac{\langle d \rangle}{|V|^2 - 1}\right) \tag{50}$$

where $\Phi(\xi)$ is the normal distribution, $\langle d \rangle$ is the average degree in the network. Then, The corresponding adjacency matrix is given by [32]

$$A_{ij}(t) = \begin{cases} 1 & \text{if } M_{(i,j)} \geq \xi \\ 0 & \text{otherwise.} \end{cases} \tag{51}$$

where the constrain $\rho \in [0, 1/2]$ makes Σ positive semidefinite.

Finally, we sequentially slide the sampling window by steps of one trading day to generate a sequence of networks according to the stock market time. This yields a time-varying stock market network with a fixed number of 415 nodes and varying edge structure for each of the 2,619 trading days. The direction for each edge is determined by the sign of time lag. All these directions represent the trading in stock price between the companies.

6.2. Experimental Results

6.2.1. Unweighted Stock Market Networks

We first investigate the thermal quantities for the unweighted time-varying networks. This is useful as a means to analyse the network fluctuation and to detect the structural variance in network time series.

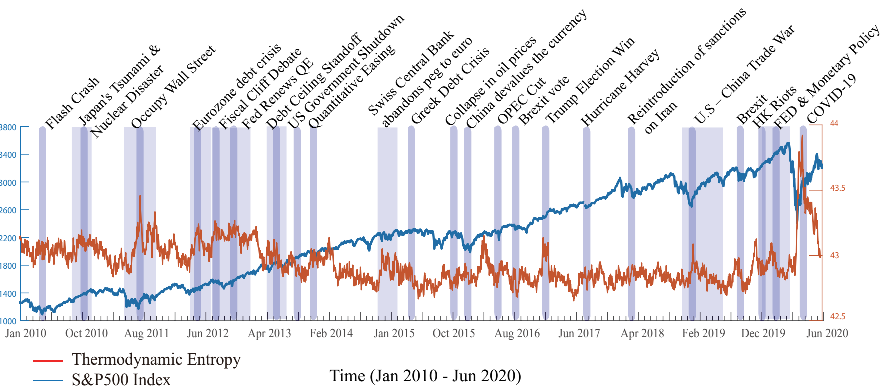

Figure 1: Thermodynamic entropy and S&P500 index in the recent ten years (January 2010 – June 2020). The critical financial events are indicated, namely, Flash Crash in 2010, Japan's Tsunami in 2011, Occupy Wall Street, European Debt Crisis, Fiscal Cliff Debate, Fed Renews QE, Debt Ceiling Standoff, US Government Shutdown in 2013, Quantitative Easing, Swiss Central Bank abandons peg to euro, Greek Debt Crisis, Collapse in oil prices, China devalues the currency, OPEC Cut, Brexit Vote, Trump Election Win, U.S – China Trade War, HK Riots, FED & Monetary Policy and COVID-19.

In Fig.1, we plot the thermodynamic entropy for the stock exchange networks and compare it with the value of the S&P500 market index over the decade covered by our data. The stock market networks undergo fluctuation during critical financial events listed in the caption of the figure. Compared with the traditional S&P500 market index, the sharp peaks in the thermodynamic entropy indicate significant changes in network structure during the different financial events. Examples include the Flash Crash in 2010, Japan's Tsunami in 2011, Occupy Wall Street, European Debt Crisis, Fiscal Cliff Debate, Fed Renews QE, Debt Ceiling Standoff, US Government Shutdown in 2013, Quantitative Easing, Swiss Central Bank abandons peg to euro, the Greek Debt Crisis, Collapse in oil prices, China devalues the currency, OPEC Cut, Brexit Vote, Trump Election Win, U.S – China Trade War, HK Riots, FED & Monetary Policy and COVID-19. Each peak of the thermal entropy can identify with changes in network structure due to the listed events or anomalies.

Fig.2 plots the four thermal quantities for the network time series. These are a) temperature,b) average internal energy, c) entropy and d) heat capacity. Similar to the thermodynamic entropy, the additional thermal quantities can identify anomalies in the time series due to critical financial events. Compared to the alternative thermal representations, entropy (yellow line in Fig.2) is more sensitive to the network variations. Taking the Flash Crash in 2010 as an

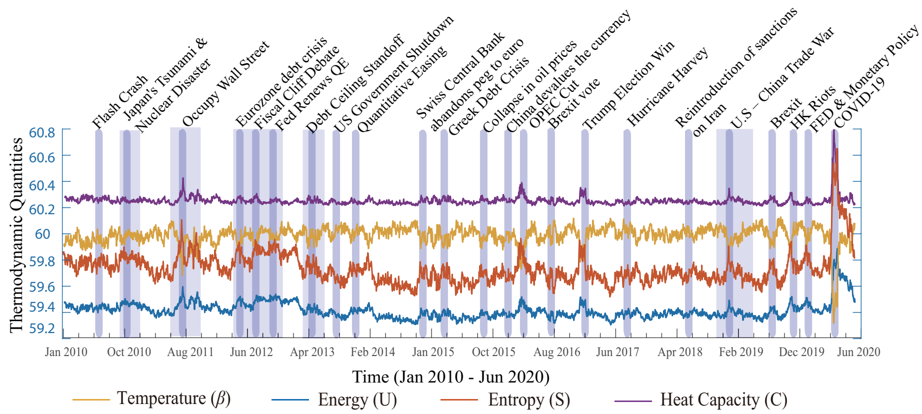

Figure 2: Four thermodynamic quantities for the time-evolving stock market networks in the last decade. (a) Blue line, temperature; (b) Red line, energy; (c) Yellow line, entropy; (d) Purple line, heat capacity.

example, entropy yields a larger fluctuation than the alternative three thermal quantities, undergoing a sharp downturn during the crisis. This is consistent with market behaviour during the crisis. Many investors lose trust in the market as stock prices collapse. Subsequently, numerous large corporations attempt to re-establish investor confidence to stimulate market activity. This results in significant fluctuation in the structure of the stock market, which potentially reflects the exchanges of investor information as measured by entropy. On the other hand, the thermal quantity of temperature is the least sensitive to the stock markets. It remains at a constant level compared to other thermodynamic representations. This is in line with the theoretical analysis that temperature is related to the number of nodes and edges in the network. For this stock market dataset, the number of nodes is fixed and the edge sets vary.

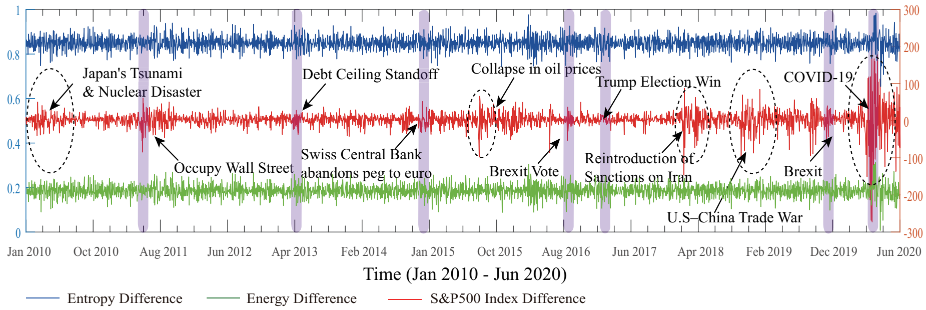

Figure 3: The thermodynamic entropy difference and energy difference in S&P500 Index Stock Data (2010–2020) for original financial networks. Critical financial events, i.e, Japan's Tsunami in 2011, Occupy Wall Street, Debt Ceiling Standoff, Swiss Central Bank abandons peg to euro, Brexit Vote, Trump Election Win, U.S–China Trade War and COVID-19.

Then, we explore the anomaly detection in the first-order difference of thermodynamic entropy and energy. A set of well-documented crisis periods have quantitatively identified the relationship between a financial crisis and network changes as shown in Fig.3. This plots the curve of the first-order difference in entropy and energy alongside critical financial events. Compared to the representation of thermodynamic quantities in Fig.2, the first difference of these variables are more clear to indicate the change of time and the significant financial events.

Furthermore, for each of the considered financial events, the detailed information around the period of the relevant crisis is represented with the corresponding thermodynamic quantities. Taking the Flash Crash in 2013 as an example, both the thermodynamic energy and entropy present a sharp trough and peak in the corresponding time series. This indicates a dramatic variation in network structure over a short time. On the other hand, the Eurozone Debt Crisis presents a broad span of thermal quantities. This implies that the related financial event has a persistent effect on the pattern of trades in the stock market. The three derived thermodynamic quantities are all efficient ways to capture the network structure during the different financial events.

Then, to explore the variations in network structure with the thermodynamic

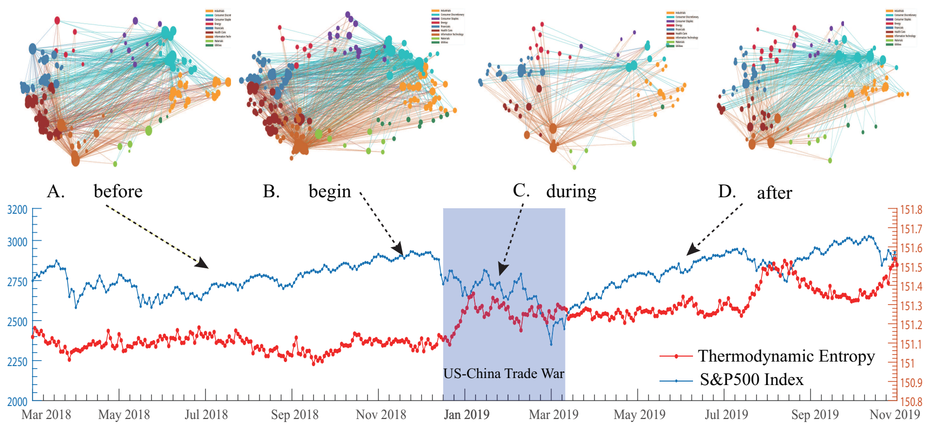

Figure 4: An example of critical financial events, i.e., US-China Trade War, with thermodynamic entropy and S&P500 Index. We illustrate the visualization of network structures during and around the US-China trade war.

quantities, we choose the period of the US-China trade war as an example to visualize the structural fluctuations with thermodynamic entropy around the critical financial events. This illustrates the corresponding thermal quantities can be used as indicators to identify instabilities in the stock markets. As shown in Fig.3, before the US-China trade war, the value of entropy remains at a constant low-level corresponding to a dense community structure or the connected components in the network. However, when the trade war began, the network structure undergoes a significant variation corresponding to the large fluctuation in the thermodynamic entropy. The connections between stock in the network become increasingly sparse during the trade war. Afterwards, the market recovers confidence with the density of network connection increasing. This is illustrated in Fig.4, we select four-time epochs to visualise the network structures A, B, C, and D. In the figure the node colour represents the density connections.

Finally, we compare two significant financial crises, namely, the 2008 Global Financial Crisis (GFC 2008)) and 2020 COVID-19 Crisis (C-19), to explore the different patterns of the stock market index and entropy. As shown in Fig.5(a), for GFC-2008 the market index a gentle decrease, commencing from the middle

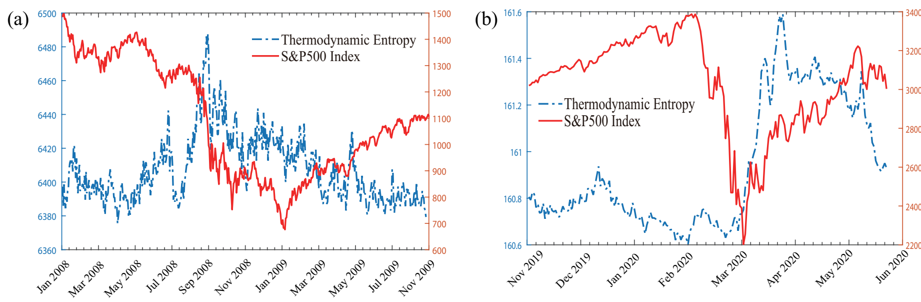

Figure 5: Comparisons between 2008 Crisis and COVID-19 with thermodynamic entropy and S&P500 Index. (a)2008 Global Financial Crisis (Jan 2008 - Dec 2009); (b) COVID-19 (Nov 2019 - Jun 2020))

of 2008 and ending early in 2009. There is a sharp fall in the market index in September 2008, which is reflected by a peak in the value of the thermodynamic entropy. During the entire period of crisis, the entropy undergoes significant local fluctuations. However, the peaks and troughs in the entropy suggest that the structural changes in the pattern of market trades (the inter-stock edges) occur when there are steep drops or increases in the market index. On the other side, although there is a sharp drop in the market index during C-19, the pattern of this financial crisis is different from GFC 2008. In Fig.5(b), the duration of the COVID-19 crisis is very short. It takes place over a period of one month. It manifests itself as an abrupt drop in the market index and a sudden jump in entropy. There are no significant fluctuations in the thermodynamic entropy. Therefore, the abrupt change of thermodynamic entropy represents changes in the trading pattern associated with the financial crisis, and its lack of fluctuations reflects the difference greater stability in the pattern of trading connections.

6.2.2. Weighted Stock Market Networks

For weighted stock market networks, we first investigate the network weight fluctuations. Fig.6 plots the average network weights and fluctuations and compares these to the value of the S&P500 Index. Both the average weights and their fluctuations exhibit the same trends as the S&P500 Index. Over the decade

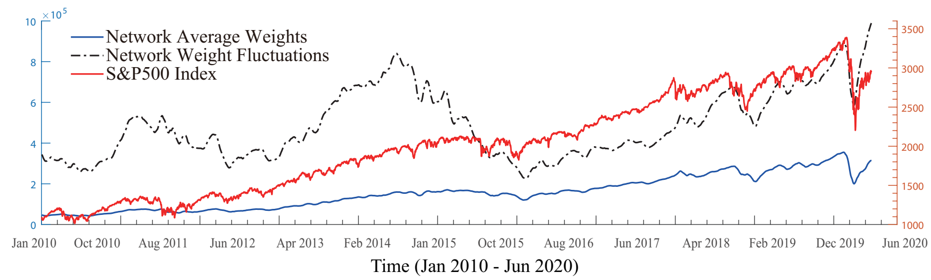

Figure 6: The stock market network average weights and fluctuations compared to S&P500 Index.

covered by the data, the index was more or less monotonically increasing except for the recent COVID-19 crisis, when there was an abrupt drop in share prices. The network weight fluctuations are inline with the stock market index, which is more sensitive as an indicator of significant financial crises since it smooths out small fluctuations.

Next, we apply a maximum likelihood to estimate the parameters for both the exponential and power-law distributions of the DOS in the weighted network models. Fig.7(a) and Fig.7(b) plot the histograms of the weight distribution in the selected stock market networks respectively. Both of the exponential and power-law distributions exist in the constructed weighted networks.

Furthermore, we explore the estimated parameters in both weight distributions. In Fig.7(c), the estimated exponential parameter α for the DOS fall into two main intervals between $[0.06, 0.08]$ and $[0.10, 0.18]$. This means that the most weighted networks in the stock market present an exponential distribution. In Fig.7(d), the power-law parameters γ are estimated to fall in the interval of $[0.2, 0.45]$, which is not consistent with the values expected from the theoretical analysis of the power-law distribution that the internal parameters are between $[2, 3]$. This show that most stock market networks do not strictly follow the power-law distribution for the weights[33].

Then, we plot the estimated parameters (i.e., α in the exponential distribution and γ in the power-law distribution) in two-dimensional space. This shows how the estimated parameters α-γ changes with time. As shown in Fig.8(a),

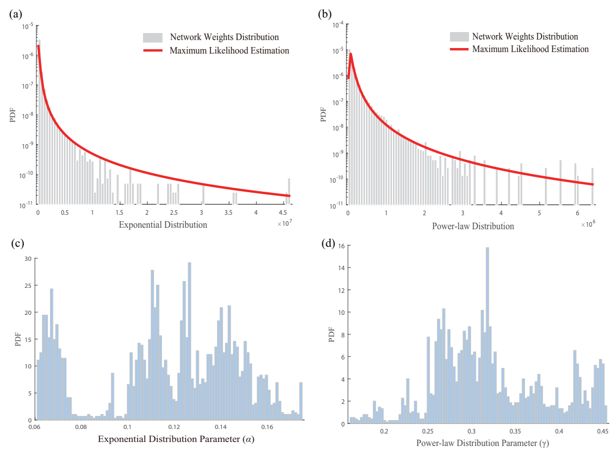

Figure 7: The exponential and power-law distributions of DOS in the weighted networks. (a) network weights in the exponential distribution; (b) network weights in the power-law distribution; (c) the histogram of estimated exponential parameter α for the DOS; (d) the histogram of power-law parameters γ for the DOS.

the critical financial crises are indicated on the network trajectory in the estimated α-γ parameter space, where the points are connected by a line according to their time of occurrence. The global crises are marked as different colour symbols against the background. Examples include the 2011 Japanese Tsunami (red triangles), Occupy Wall Street in 2011 (yellow squares), US Government Shutdown in 2013 (black circles), Swiss Central Bank abandons peg to euro in 2015, and Trump Electron Win in 2016 (purple left-pointing and blue right-point triangles, respectively), and 2019 US-China Trade War (green squares), Brexit in 2019 (purple diamonds), and COVID-19 in 2020 (black asterisks). The plots show the values of the estimated parameters from the exponential distribution and the power-law distribution change with the time for the evolving financial market network. Different financial crises occur at different locations in this α-γ parameter space. This highlights that the critical event structure of the network can be captured by the estimated parameters.

A similar pattern can be observed in Fig.8(b) which plots the average weights $\langle w \rangle$ and their fluctuations $\langle \delta w \rangle$ as a function of time. This shows a roughly

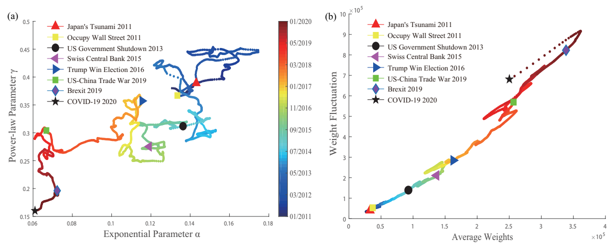

Figure 8: (a) The trajectory of estimated parameters α-γ space. (b) The trajectory of average weights $\langle w \rangle$ and their fluctuations $\langle \delta w \rangle$. Critical financial events: Japan's Tsunami in 2011 (red triangles), Occupy Wall Street in 2011 (yellow squares), US Government Shutdown in 2013 (black circles), Swiss Central Bank in 2015 and Trump Win Electron in 2016 (purple left-pointing and blue right-point triangles, respectively), and US-China Trade War in 2019 (green squares), Brexit 2019 (purple diamonds), and COVID-19 in 2020 (black asterisks).

linear correspondence between the average weights and weight fluctuations. The most significant difference is that the COVID-19 event presents a wider variety compared to other financial crises.

Therefore, the thermodynamic quantities for both the unweighted and weighted networks can be used to represent the time-evolving stock market networks. For the unweighted networks, the derived thermal characterisations indicate critical financial events. For the weighted networks, the corresponding weight distributions and fluctuations identify the detailed trading patterns and can be used to assess the potential risk to investors in different market sectors.

6.3. Evaluation of threshold sensitivity

In order to evaluate the effect of varying the threshold in Eq.(50), Fig.9 shows the relationship between the number of edges and the total weight for the different networks in the financial market as a function of time. Here, we set the cumulative probability determined by the inverse error function for the Gaussian distribution to be 95%, 75% and 50%, respectively. This generates corresponding threshold values of 0.4303, 0.5891 and 0.8788 for each network configuration. In Fig.9, the total number of edges for the generated time-evolving networks shows the same gross trends as the total network weight. There is sudden jump in both the number of edges and the total weight during 2015. The higher the value of the threshold, the fewer edges in the generated networks. After 2015 for low threshold values the number of edges in the stock market networks is quite volatile. However, by weighting the edges this volatility is to some extent regulated and the total weight varies more smoothly with time.

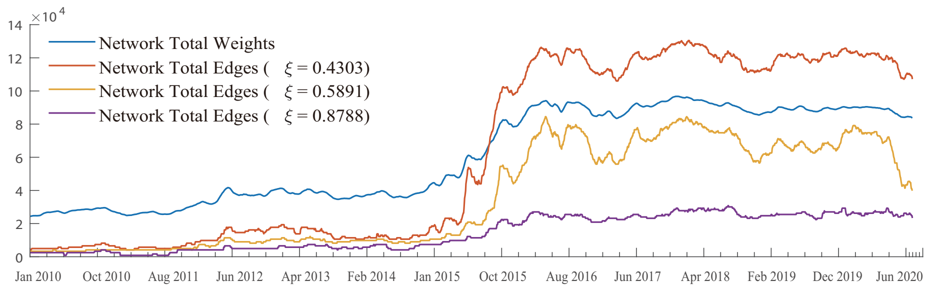

Figure 9: The effect of the threshold in Eq.(50) on the number of edges and the total weight.

Fig.1 and Fig.2 provide a qualitative performance for the derived thermal characterisations to detect financial events. To better quantitatively evaluate these results, we apply the regression in the time-evolving stock market networks to indicate the peaks of these fluctuations. Here, we regarded the critical financial detection as a binary classification problem, and provide more strict performance evaluations to compare with other stock market indicators, such as S&P500 Index, NYSE Arca International Market Index(ADR), Moving Average Convergence/Divergence(MACD), etc. Table2 shows the performance of these evaluations for several different indicators, respectively, coming from the numerical stock market analysis, network structural properties and thermodynamic quantities.

For the accuracy to detect the financial crisis, our derived thermodynamic entropy presents a high performance (82.47% in accuracy) compared to the other quantities. The accuracy of other thermodynamic indicators slightly drops to around 80%, but still keeping excellent to identify the critical financial events. For other stock indicators, S&P 500 Index shows a better result than other sophisticated stock market index. However, the simple network structural prop-

Table 2: Performance metrics for different stock market indicators

Indicator	Accuracy	Sensitivity	Specificity	Precision	Recall	F1 Score	G-mean
S&P500 Index	0.7852	1.0000	0.7146	0.5353	1.0000	0.6973	0.8454
ADR	0.5286	0.4855	0.5439	0.2731	0.4855	0.3496	0.5138
MACD	0.5265	0.4532	0.5351	0.1029	0.4532	0.1678	0.4925
Average Degree	0.4563	0.3224	0.4941	0.1526	0.3224	0.2071	0.3991
Average Shortest Path	0.4093	0.3157	0.4575	0.2305	0.3157	0.2665	0.3801
Temperature	0.7885	0.9561	0.7238	0.5718	0.9561	0.7156	0.8319
Energy	0.8144	0.9854	0.7452	0.6103	0.9854	0.7538	0.8569
Heat Capacity	0.8125	0.9752	0.7453	0.6128	0.9752	0.7526	0.8525
Entropy	0.8247	1.0000	0.7531	0.6235	1.0000	0.7681	0.8678

erties, such as the average degree and the average shortest path, do not present a strong ability to identify the fluctuated peaks in this case. Thus, the resulting method combined with the thermodynamic network characterisations can work as an efficient tool to detect stock market fluctuations related to critical financial events.

7. Conclusion

The analysis of time-evolving stock markets provides a reliable indicator for identifying the decline of stock value during the financial crisis. This study applies the tools in the complex networks to describe the dynamic stock exchanges with the inference of underlying financial activities and partnerships.

To this end, we have presented a novel statistical mechanical description for the time-evolving unweighted and weighted stock market networks. Commencing from the analogy of the network edges as the particles in the thermal system, we introduce the physical interpretation of temperature related to the network structure. We have explored the binary representation of the adjacency matrix and the description of the network connection as statistical canonical ensembles. We have also discussed the origins of the weight fluctuations in dynamic networks.

Experimental results for the S&P500 Index Stock Exchanges over the past decade reveal that the thermodynamic characterisations can effectively detect the statistically significant of observations on stock market networks. Four kinds of thermodynamic quantities, i.e., energy, entropy, temperature and heat capacity, are useful to provide an indicator to identify the financial crisis during the network evolution. In particular, the thermodynamic entropy is more sensitive than the other three quantities to detect the salient features in the network structure. This provides an efficient way of identifying critical financial events during evolution.

The work suggests a number of directions for further investigation. Novel methods for dynamic network construction provide new insights into stock market evolution. In addition, the unweighted adjacency matrix depends on the certain strategy of choosing a suitable threshold to the cross-correlation coefficients of the weighted networks. This provides additional developments about the relevance of thermodynamic characterisations with the parameters in the time-evolving network construction.

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