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Criteria for a direct sum of modules to be a multiplication module over noncommutative rings

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Abstract

We study multiplication modules. The rings are not assumed to be commutative. Several criteria with some applications are given for a direct sum of modules to be a multiplication module.

Key Words: multiplication module, multiplication ideal, multiplication ring, Dedekind domain.

Mathematics subject classification 2010: 13C05, 13E05, 13F05, 13F10, 16D10.

Contents

1 Introduction

In this paper, module means a left module and R is a ring with 1 (not necessarily commutative).

An R-module M is called a **multiplication module** if every submodule of M is equal to IM for some two-sided ideal I of the ring R. If M is a left ideal of R then M is called a **left multiplication ideal**.

An R-module M is a multiplication R-module iff $N = [N : M]M$ for every submodule N of M where $[N : M] := \mathrm{ann}_R(M/N)$ (Lemma 2.1). The class of all multiplication R-modules is denoted by $\mathrm{Mod}_m(R)$. In case R is a commutative ring, $\mathrm{Mod}_m(R)$ contains R, all cyclic R-modules, and all invertible ideals of R. For a noncommutative ring, a cyclic module is not necessarily a multiplication module.

For a ring R, let $\mathcal{I}(R)$ be the set of its ideals. The set $(\mathcal{I}(R), \subseteq)$ is a *partially ordered set* (a *poset*, for short). For an R-module M, let $\mathrm{Sub}_R(M)$ be the set of its submodules. The set $(\mathrm{Sub}_R(M), \subseteq)$ is a poset. The map $\mu_M : \mathcal{I}(R) \to \mathrm{Sub}_R(M)$, $I \mapsto IM$ respects inclusion. Clearly, an R-module M is a multiplication module iff the map μ_M is a surjection.

A ring R is said to be a **left multiplication ring** (resp., **a right multiplication ring**) if for every two ideals I and J of R such that $J \subseteq I$, there exists an ideal J' of R such that $J = J'I$ (resp., $J = IJ'$). A ring R is called a **multiplication ring** if it is a right and left multiplication ring. In case R is a commutative ring, *R is a multiplication ring iff $\mathcal{I}(R) \subseteq \mathrm{Mod}_m(R)$.* Examples of multiplication commutative rings are Dedekind domains, principal ideal domains and rings in which all ideals are idempotent.

Krull introduced the concept of a commutative multiplication ring in [12] as a generalization of the concept of Dedekind domain. Larsen and McCarty in [13, Theorem 9.21] proved that *if every prime ideal of a commutative ring R is a multiplication ideal then R is a multiplication ring.* In 1981, Barnard [7] presented the notion of multiplication modules over commutative rings. The *fundamental theorem of abelian group* could be described as *every finitely generated $\mathbb{Z}$-module is a direct sum of multiplication modules.* This result stimulated many authors to study properties of such class of modules over a commutative ring. For example, Barnard [7], P. F. Smith [15], D. D. Anderson[4], and Y. Alshaniafi and S. Singh [2]. The first systematic study of multiplication modules over commutative ring seems to start with El-bast and Smith in [10]. Besides proving many basic properties of multiplication modules over commutative ring, they gave several characterisations of such modules. In [14], Mott proved that a multiplication ring has finitely many minimal prime ideals iff it is a Noetherian ring. In 2019, T. Alsuraiheed and V. V. Bavula [3], presented a characterization of multiplication commutative rings with finitely many minimal prime ideals: *Each such ring is a finite direct product of rings $\prod_{i=1}^{n} D_i$ where D_i is either a Dedekind domain or an Artinian, local, principal ideal ring and vice versa.*

The *algebras of polynomial integro-differential operators* over a field K of characteriastic zero (introduced in [8]),

$$\mathbb{I}_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n, \int_1, \dots, \int_n \rangle,$$

have many interesting properties that are almost opposite to the ones of the algebra of polynomial differential operators $A_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$, the *Weyl algebra* (where ∂_i and $\int_i$ are the partial derivations and integrations with respect to the variable x_i). In particular, the algebras $\mathbb{I}_n$ are neither left nor right Noetherian and non-simple. Futhermore, the classical Krull dimension of the algebra $\mathbb{I}_n$ is n and *all ideals of $\mathbb{I}_n$ are idempotent ideals*, [8]. Therefore, the algebras $\mathbb{I}_n$ are multiplication rings (Corollary 3.2). This result motivated us to study the class of multiplication modules over noncommutative rings. We have to indicate that there are only very few results in the literatures about multiplication modules over noncommutative rings (see [16]).

Five criteria for a direct sum of modules to be a multiplication module over an arbitrary ring. Let R be a ring (not necessarily commutative). To formulate the first criterion (Theorem 1.4) we need to introduce the following concepts.

Definition 1.1. *We say that the* **intersection condition** *holds for a direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ of nonzero R-modules M_λ if for all submodules N of M,*

$$N = \bigoplus_{\lambda\in\Lambda} N \cap M_\lambda.$$

Definition 1.2. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geqslant 2$, $\mathfrak{a}_\lambda = \mathrm{ann}_R(M_\lambda)$ and $\mathfrak{a}'_\lambda = \cap_{\mu\neq\lambda}\mathfrak{a}_\mu$. We say that the* **orthogonality condition** *holds for the direct* *sum M if*

$$\mathfrak{a}'_\lambda M_\mu = \delta_{\lambda\mu} M_\mu \text{ for all } \lambda, \mu \in \Lambda$$

where $\delta_{\lambda\mu}$ is the Kronecker delta. Clearly, $\mathfrak{a}'_\lambda \neq 0$ for all $\lambda \in \Lambda$ (since all $M_\lambda \neq 0$). In particular, $\mathfrak{a}_\lambda \neq 0$ for all $\lambda \in \Lambda$.

Definition 1.3. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with* $\mathrm{card}(\Lambda) \geq 2$. *We say that the* **strong orthogonality condition** *holds for the direct sum M if for each set of R-modules $\{N_\lambda\}_{\lambda\in\Lambda}$ such that $N_\lambda \subseteq M_\lambda$, there is a set of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of R such that*

$$I_\lambda M_\mu = \delta_{\lambda\mu} N_\lambda \text{ for all } \lambda, \;\; \mu \in \Lambda.$$

The set of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ is called an **orthogonalizer** *of $\{N_\lambda\}_{\lambda\in\Lambda}$.*

In particular, the orthogonality condition holds for $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ iff the set of ideals $\{\mathfrak{a}'_\lambda\}_{\lambda\in\Lambda}$ is an orthogonalizer of $\{M_\lambda\}_{\lambda\in\Lambda}$. If the orthogonality condition holds for $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ and $\{I_\lambda\}_{\lambda\in\Lambda}$ is an orthogonalizer of $\{M_\lambda\}_{\lambda\in\Lambda}$ then $I_\lambda \subseteq \mathfrak{a}'_\lambda$ for all $\lambda \in \Lambda$.

Theorem 1.4 is the first criterion for a direct sum of modules to be a multiplication module which given via the intersection and strong orthogonality conditions.

Theorem 1.4. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with* $\mathrm{card}(\Lambda) \geq 2$. *Then M is a multiplication R-module iff the intersection and strong orthogonality conditions hold for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$. Furthermore, if $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a multiplication R-module then*

1. *the R-modules M_λ are multiplication modules, and*
2. *for each submodule N of M and each ideal I of R such that $N = IM$, $N \bigcap M_\lambda = IM_\lambda$ for all $\lambda \in \Lambda$.*

Before stating the second criterion, we need to introduce and discuss some concepts.

Definition 1.5. *Let N be an R-submodule of M. An ideal J of the ring R is called a* **compressor** *of N (in M) if $N = JM$.*

Any sums of compressors of N is a compressor of N. The set of all compressors of N (in M) is denoted by $\mathcal{I}(N, M)$. The set $\mathcal{I}(N, M)$ is a non-empty set iff $[N : M]$ is a compressor of N ($[N : M]M = N$), and in that case $[N : M]$ is the largest compressor of N. Notice that $[N : M]M \subsetneq N$, in general.

Let N be a submodule of a multiplication R-module M. Then the set

$$\mathcal{I}(N, M) := \{I \lhd R \,|\, IM = N\} \tag{CINM}$$

is a non-empty set which is closed under addition of ideals (if $I, J \in \mathcal{I}(N, M)$ then $I+J \in \mathcal{I}(N, M)$). The sum

$$I(N, M) = \sum_{I\in\mathcal{I}(N,M)} I \tag{CINM1}$$

is the *largest element* of the set $\mathcal{I}(N, M)$ (w.r.t. inclusion). Clearly, $I(N, M) = [N : M]$.

Let $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{N} = \{N_\lambda\}_{\lambda\in\Lambda}$ be sets of R-modules such that $N_\lambda \subseteq M_\lambda$ for all $\lambda \in \Lambda$. Let $\mathcal{I}(\mathcal{N}, \mathcal{M})$ be the set of all sets of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ such that $I_\lambda M_\mu = \delta_{\lambda\mu} N_\lambda$ for all $\lambda, \mu \in \Lambda$, i.e., the set contains all the orthogonalizers for $\mathcal{N}$. In general, the set $\mathcal{I}(\mathcal{N}, \mathcal{M})$ could be an empty set. In particular, if $\mathcal{M} = \{M\}$ and $\mathcal{N} = \{N\}$ then $\mathcal{I}(\mathcal{N}, \mathcal{M}) = \mathcal{I}(N, M)$.

Lemma 1.6. *Let $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules such that their direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a multiplication module. Then for every set of R-modules $\mathcal{N} = \{N_\lambda\}_{\lambda\in\Lambda}$ such that $N_\lambda \subseteq M_\lambda$ for all $\lambda \in \Lambda$, the set $\mathcal{I}(\mathcal{N}, \mathcal{M})$ is a non-empty set.*

Proof. The result follows from Theorem 1.4. □

Suppose that $\mathcal{I}(\mathcal{N}, \mathcal{M}) \neq \emptyset$. Then the set $\mathcal{I}(\mathcal{N}, \mathcal{M})$ is closed under addition (componentwise): if sets $\mathcal{I} = \{I_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{J} = \{J_\lambda\}_{\lambda\in\Lambda}$ belong to $\mathcal{I}(\mathcal{N}, \mathcal{M})$ then $\mathcal{I} + \mathcal{J} = \{I_\lambda + J_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N}, \mathcal{M})$. So, the sum in $\mathcal{I}(\mathcal{N}, \mathcal{M})$,

$$I(\mathcal{N}, \mathcal{M}) := \sum_{\mathcal{I}\in\mathcal{I}(\mathcal{N},\mathcal{M})} \mathcal{I}$$

is the *largest element* of the set $\mathcal{I}(\mathcal{N}, \mathcal{M})$ w.r.t. componentwise inclusion, i.e., $\mathcal{I} \subseteq \mathcal{J}$ iff $I_\lambda \subseteq J_\lambda$ for all $\lambda \in \Lambda$. The set $I(\mathcal{N}, \mathcal{M})$ is called the **largest orthogonalizer** in $\mathcal{I}(\mathcal{N}, \mathcal{M})$.

If the orthogonality condition holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ then $I(\mathcal{M}, \mathcal{M}) = \{\mathfrak{a}'\}$ where $\mathfrak{a}' = \mathrm{ann}_R(\bigoplus_{\mu\neq\lambda} M_\mu) = \bigcap_{\mu\neq\lambda} \mathrm{ann}_R(M_\mu)$ (Corollary 4.3).

Now, we can state the second criterion for a direct sum of modules to be a multiplication module.

Theorem 1.7. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geq 2$. Then M is a multiplication module iff*

1. *the R-modules M_λ, where $\lambda \in \Lambda$, are multiplication modules, and*
2. *for each submodule N of M, $\mathcal{I}(\mathcal{N}, \mathcal{M}) \neq \emptyset$ where $\mathcal{N} = \{N \cap M_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$, and $N = (\sum_{\lambda\in\Lambda} I_\lambda)M$ for all/some $\{I_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N}, \mathcal{M})$.*

We need to the following definition to introduce the third criterion.

Definition 1.8. *A set of ideals $\{\mathfrak{a}_\lambda\}_{\lambda\in\Lambda}$ of a ring R is called an* **orthogonal set of ideals** *of R if $\mathfrak{a}_\lambda\mathfrak{a}_\mu = 0$ for all $\lambda \neq \mu$.*

The next theorem is the third criterion for a direct sum of modules to be a multiplication modules given via orthogonal ideals.

Theorem 1.9. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geq 2$, $\mathfrak{a} := \mathrm{ann}_R(M)$ and $\overline{R} := R/\mathfrak{a}$. Then M is a multiplication module iff*

1. *the ring $\overline{R}$ contains a direct sum of nonzero orthogonal ideals $\mathfrak{a}' = \bigoplus_{\lambda\in\Lambda} \mathfrak{a}'_\lambda$ such that $M_\lambda = \mathfrak{a}'_\lambda M$ for all $\lambda \in \Lambda$, and*
2. *for each submodule N of M, $N = \mathfrak{b}'M$ for an ideal $\mathfrak{b}'$ of $\overline{R}$ such that $\mathfrak{b}' = \bigoplus_{\lambda\in\Lambda} \mathfrak{b}'_\lambda$ is a direct sum of ideals $\mathfrak{b}'_\lambda = \mathfrak{b}' \cap \mathfrak{a}'_\lambda$ of $\overline{R}$ for all $\lambda \in \Lambda$.*

Theorem 1.10 is the fourth criterion for a direct sum of modules to be a multiplication module given via the orthogonality and intersection conditions.

Theorem 1.10. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geq 2$. Then M is a multiplication module iff*

1. *the R-modules M_λ are multiplication modules where $\lambda \in \Lambda$,*
2. *the intersection condition holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, i.e., for any sumbodule N of M, $N = \bigoplus_{\lambda\in\Lambda}(N \cap M_\lambda)$, and*

3. *the orthogonality condition holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, i.e., for all $\lambda, \mu \in \Lambda$, $\mathfrak{a}'_\lambda M_\mu = \delta_{\lambda\mu} M_\mu$.*

Definition 1.11. *A submodule N of an R-module M is called an* $\mathrm{End}_R(M)$-stable submodule *(resp.,* $\mathrm{End}_R(M)$-invariant submodule*) if $f(N) \subseteq N$ (resp., $f(N) = N$) for every $0 \neq f \in \mathrm{End}_R(M)$.*

Definition 1.12. *We say that the* $\mathrm{End}_R(M)$**-stability condition** *holds for an R-module M if every submodule N of M is an $\mathrm{End}_R(M)$-stable submodule.*

Theorem 1.13 is the fifth criterion for a direct sum of modules to be a multiplication module given via the orthogonality and $\mathrm{End}_R(M)$-stability conditions.

Theorem 1.13. *Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules with $\mathrm{card}(\Lambda) \geq 2$. Then M is a multiplication module iff*

1. *the R-modules M_λ are multiplication modules where $\lambda \in \Lambda$,*
2. *every submodule N of M is an $\mathrm{End}_R(M)$-stable submodule, and*
3. *the orthogonality condition holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$.*

The proofs of the theorems/criteria above are given in Section 4. Section 5 contains many applications and results based on these criteria. In Section 2, some known results are generalized to noncommutative case. Section 3 provides some properties of the class of multiplication modules over an arbitrary ring that are used in the proofs of this paper.

2 Preliminaries

For an R-module M, let $\mathrm{Cyc}_R(M)$ be the set of its *cyclic* submodules. For an R-module M, we denote by $\mathrm{ann}_R(M)$ its annihilator. An R-module M is called *faithful* if $\mathrm{ann}_R(M) = 0$. For a submodule N of M, the set $[N : M] := \mathrm{ann}_R(M/N) = \{r \in R \,|\, rM \subseteq N\}$ is an ideal of the ring R that contains the *annihilator* $\mathrm{ann}_R(M) = [0 : M]$ of the module M. The set $\theta(M) := \sum_{C\in\mathrm{Cyc}_R(M)} [C : M]$ is an ideal of R. Clearly, $\mathrm{ann}_R(M) \subseteq \theta(M)$. If M is an ideal of R then $M \subseteq \theta(M)$.

Let M be an R-module. We denote by $\mathrm{Sub}_{\mathrm{R}}^{\oplus}(M)$ the set of the direct summands of M. $\mathrm{Epi}_R(M)$ and $\mathrm{Mon}_R(M)$ denote the set of all epimorphisms and monomorphisms from M to M, respectively. Clearly, $\mathrm{Epi}_R(M) \subseteq \mathrm{End}_R(M)$ and $\mathrm{Mon}_R(M) \subseteq \mathrm{End}_R(M)$. The group of automorphisms of an R-module M is denoted by $\mathrm{Aut}_R(M)$. Clearly, $\mathrm{Aut}_R(M) = \mathrm{Epi}_R(M) \cap \mathrm{Mon}_R(M)$.

In this section, several properties of the class of multiplication modules over a ring (**not necessarily commutative**) are provided. They are generalizations of some known results in the commutative case.

Lemma 2.1. *An R-module M is a multiplication module iff $N = [N : M]M$ for any submodule N of M.*

Proof. ($\Rightarrow$) Let N be a submodule of the multiplication module M. Then there exists an ideal I of R such that $N = IM$. Hence, $N = IM \subseteq [N : M]M \subseteq N$. This implies that $N = [N : M]M$.

(⇐) This is implication is obvious. □

Let N be a submodule of M. Then $[N:M]M \subseteq N$, and so we have a short exact sequence of modules

$$0 \to [N:M]M \to N \to N/[N:M]M \to 0.$$

Lemma 2.2. *An R-module M is a multiplication module iff $C = [C:M]M$ for every $C \in \mathrm{Cyc}_R(M)$.*

Proof. (⇒) Lemma 2.1.

(⇐) Suppose that $C = [C:M]M$ for every cyclic submodule C of M. Let N be a submodule of M and $I = \sum_{C\in \mathrm{Cyc}_R(N)}[C:M]$. Then

$$IM = \sum_{C\in \mathrm{Cyc}_R(N)}[C:M]M = \sum_{C\in \mathrm{Cyc}_R(N)} C = N.$$

Hence, M is multiplication module. □

Proposition 2.3. *Any homomorphic image of a multiplication module is a multiplication module.*

Proof. Let M be a multiplication R-module and $f : M \to N$ be an R-epimorphism. For each submodule K of N, $f^{-1}(K) = IM$ for some ideal I of R. Now, $K = f(f^{-1}(K)) = f(IM) = If(M) = IN$. Hence, N is multiplication module. □

Proposition 2.4. *Let M be an R-module. Then M is a multiplication R-module if the following two conditions hold:*

1. *$\cap_{\lambda\in\Lambda} I_\lambda M = (\cap_{\lambda\in\Lambda} I_\lambda)M$ for every non-empty set of ideals $\{I_\lambda \mid \lambda \in \Lambda\}$ of R, and*
2. *for any submodule N of M and an ideal I of R such that $N \subset IM$, there exists an ideal J of R such that $J \subset I$ and $N \subseteq JM$.*

Proof. Let N be a submodule of M and S be the set of ideals I' of R such that $N \subseteq I'M$. Clearly, $R \in S$. The ideal $A = \cap_{I'\in S} I'$ is the smallest element in S since, by condition 1, $N \subseteq \cap_{I'\in S} I'M = AM$. Now, suppose that $N \subset AM$, we seek a contradiction. Then, by condition 2, there exists an ideal B of R such that $B \subset A$ and $N \subseteq BM$. Therefore $B \in S$. This contradicts to the minimality of A and hence, $N = AM$, i.e., M is a multiplication module. □

Proposition 2.5 is a criterion for a module to be a multiplication module.

Proposition 2.5. *Let M be an R-module. Then M is a multiplication R-module iff for every nonzero submodule N of M, M/N is a multiplication module such that $[N:M] \nsubseteq \mathrm{ann}_R(M)$.*

Proof. (⇒) Let N be a nonzero submodule of M. Then $N = [N:M]M$ (Lemma 2.1). Since $N \neq 0$, we must have $[N:M] \nsubseteq \mathrm{ann}_R(M)$. By Proposition 2.3, the factor module M/N is a multiplication module.

(⇐) Let N be a nonzero submodule of the R-module M and $I = [N:M]$. By the assumption, $IM \neq 0$ (since $I \nsubseteq \mathrm{ann}_R(M)$) and M/IM is a multiplication module. So, by Lemma 2.1, $N/IM = [N/IM : M/IM](M/IM) = [N:M](M/IM) = 0$, and therefore $N = IM$. Hence, M is a multiplication module. □

Proposition 2.6. *Let M be a multiplication R-module. Then*

1. *If N and K are submodules of M such that $M/N \cong M/K$ then $N = K$.*

2. *If $f : M \to R$ is an R-homomorphism then for every $m \in M$, $f(m)M \subseteq Rm$.*

Proof. 1. Since M is a multiplication module and $M/N \cong M/K$, $N = [N : M]M = \operatorname{ann}_R(M/N)M = \operatorname{ann}_R(M/K)M = [K : M]M = K$.

2. Since M is a multiplication module, $Rm = IM$ for some ideal I of R. Now, $f(m)M \subseteq f(Rm)M = f(IM)M = If(M)M \subseteq IM = Rm$. Hence, $f(m)M \subseteq Rm$. $\square$

Proposition 2.7. *Let M be a semisimple R-module such that $[N : M] \nsubseteq \operatorname{ann}_R(M)$ for every simple submodule N of M. Then M is a multiplication module.*

Proof. Let K be a submodule of M. Since M is a semisimple module, $K = \bigoplus_{i\in I} M_i$ is a direct sum of simple submodules M_i of M. As M_i is simple and $[M_i : M] \nsubseteq \operatorname{ann}(M)$, $[M_i : M]M = M_i$. Now,

$$K = \sum_{i\in I} M_i = \sum_{i\in I}[M_i : M]M = (\sum_{i\in I}[M_i : M])M.$$

Hence, M is a multiplication module, by Lemma 2.1. $\square$

Let M be an R-module. Anderson in [4], defined the ideal $\theta(M) = \sum_{C\in \mathrm{Cyc}_R(M)}[C : M]$ where R is a commutative ring. In case I is an ideal of R, it is clear that $I \subseteq \theta(I)$.

Lemma 2.8. *Let M be a multiplication R-module. Then $M = \theta(M)M$.*

Proof. Since M is a multiplication R-module, $M = \sum_{C\in \mathrm{Cyc}_R(M)} C = \sum_{C\in \mathrm{Cyc}_R(M)}[C : M]M \subseteq (\sum_{C\in \mathrm{Cyc}_R(M)}[C : M])M = \theta(M)M$, by Lemma 2.2, and the lemma follows. $\square$

The next lemma provides a sufficient condition for a multiplication module to be finitely generated.

Lemma 2.9. *Let M be a multiplication R-module. If $\theta(M)$ is a finitely generated R-module then the R-module M is finitely generated.*

Proof. Since $\theta(M) = \sum_{C\in \mathrm{Cyc}_R(M)}[C : M]$ and the R-module $\theta(M)$ is finitely generated, $\theta(M) = \sum_{i=1}^{n} R\theta_i$ for some elements $\theta_i \in [C_i : M]$ where C_i are cyclic submodules of M. Now, by Lemma 2.8, $M = \theta(M)M = \sum_{i=1}^{n} R\theta_i M \subseteq \sum_{i=1}^{n} C_i \subseteq M$, i.e., $M = \sum_{i=1}^{n} C_i$ is a finitely generated R-module. $\square$

Corollary 2.10. *If R is a left Noetherian ring then every multiplication R-module is finitely generated.*

Proof. The corollary follows from Lemma 2.9. $\square$

3 Some properties of multiplication modules over noncommutative rings

In this section, we provide some properties and characterizations of multiplication modules over an arbitrary ring.

The algebras $\mathbb{I}_n$ form a subclass of the class of multiplication rings.

Recall that a ring R is a multiplication ring iff for every two ideals $I \subseteq J$ of R, $I = KJ = JK'$ for suitable ideals K and K' of R.

Lemma 3.1. *Let R be a ring such that all its ideals are idempotent ideals. Then R is a multiplication ring.*

Proof. Let I and J be ideals of R such that $J \subseteq I$. Then $J = J^2 \subseteq JI \subseteq J$, i.e., $J = JI$. Hence, I is a left multiplication module. Similarly, I is a right multiplication module, and hence, R is a multiplication ring. $\square$

Corollary 3.2. *The algebras $\mathbb{I}_n$, $n \geq 1$, of polynomial integro-differential operators over a field of characteristic zero are (left and right) multiplication rings.*

Proof. By [8, Corollary 3.3(3)], every ideal of all $\mathbb{I}_n$ is an idempotent ideal. So, the result follows from Lemma 3.1. $\square$

Incomparability of the annihilators.

Proposition 3.3. *Let M be a multiplication R-module and M_1, M_2 be R-modules such that $\mathrm{ann}_R(M_1) \subseteq \mathrm{ann}_R(M_2)$ and the direct sum of R-modules $M_1 \bigoplus M_2$ is an epimorphic image of the R-module M. Then $M_2 = 0$.*

Proof. Let $f : M \to M_1 \bigoplus M_2$ be an epimorphism and p_1, p_2 be the projections of the module $M_1 \bigoplus M_2$ onto M_1 and M_2, respectively. For $i = 1, 2$, let $f_i = p_i f$ and $K_i = \ker(f_i)$. Then $M/K_i \cong M_i$. So,

$$[K_1 : M] = \mathrm{ann}_R(M_1) \subseteq \mathrm{ann}_R(M_2) = [K_2 : M].$$

Since the R-module M is a multiplication module, we have $K_1 = [K_1 : M]M \subseteq [K_2 : M]M = K_2$, by Lemma 2.1. Let $k_2 \in M_2$. Then $(0, k_2) = f(m)$ for some element $m \in M$. Clearly, $f_1(m) = p_1(0, k_2) = 0$, i.e., $m \in K_1$. Since $K_1 \subseteq K_2$, $0 = f_2(m) = p_2 f(m) = p_2(0, k_2) = k_2$, i.e., $M_2 = 0$. $\square$

Corollary 3.4. *Let M be a multiplication R-module and M_1, M_2 be R-modules such that $\mathrm{ann}_R(M_1) = \mathrm{ann}_R(M_2)$ and the direct sum of R-modules $M_1 \bigoplus M_2$ is an epimorphic image of the R-module M. Then $M_1 = M_2 = 0$.* $\square$

Proof. The corollary follows from Proposition 3.3. $\square$

Proposition 3.5 shows that every multiplication module M does not admit a direct summand which is isomorphic to M.

Proposition 3.5. *Let R be a ring and M be a multiplication R-module. If $M \cong M \oplus N$ where N is an R-module then $N = 0$.*

Proof. Since $M \cong M \bigoplus N$, $\operatorname{ann}_R(M) = \operatorname{ann}_R(N) \cap \operatorname{ann}_R(M)$. So, $\operatorname{ann}_R(M) \subseteq \operatorname{ann}_R(N)$. Hence, by Proposition 3.3, $N = 0$. □

Ideals $\{\mathfrak{a}_i \,|\, i \in I\}$ of a ring R are called *incomparable* if $\mathfrak{a}_i \nsubseteq \mathfrak{a}_j$ for all distinct elements $i, j \in I$.

Corollary 3.6. *Let M be a multiplication R-module and the direct sum of nonzero R-modules $\bigoplus_{i\in I} M_i$ be an epimorphic image of M. Then the set of ideals $\{\operatorname{ann}_R(M_i) \,|\, i \in I\}$ are incomparable. In particular, all the ideals $\{\operatorname{ann}_R(M_i) \,|\, i \in I\}$ are distinct and the modules $\{M_i \,|\, i \in I\}$ are not pairwise isomorphic. In particular, $\operatorname{ann}_R(M_i) \neq 0$ for all $i \in I$.*

Proof. Let $\mathfrak{a}_i = \operatorname{ann}_R(M_i)$. Suppose that $\mathfrak{a}_i \subseteq \mathfrak{a}_j$ for some $i \neq j$. Then the direct sum $M_i \bigoplus M_j$ is an epimorphic image of M such that $\mathfrak{a}_i \subseteq \mathfrak{a}_j$. By Proposition 3.3, $M_j = 0$, a contradiction. □

Corollary 3.7. *Let a direct sum of R-modules $M = \bigoplus_{i\in I} M_i$ be a multiplication module with $\operatorname{card}(I) \geq 2$. Then the set of ideals $\{\operatorname{ann}_R(M_i) \,|\, i \in I\}$ are incomparable. In particular, none of the direct summands M_i is a faithful R-module, i.e., $\operatorname{ann}_R(M_i) \neq 0$ for all $i \in I$.*

Proof. The corollary follows from Corollary 3.6. □

For a module N and a set I, we denote by $N^{(I)}$ a direct sum of I copies of N.

Corollary 3.8. *Let M be a multiplication R-module. Then every nonzero factor module of M cannot be of the type $N^{(I)}$ for some nonzero R-module N and a set I of cardinality ≥ 2.*

Proof. This follows from Proposition 3.3. □

Corollary 3.9. *Let M be a nonzero multiplication R-module. If M is a free R-module then $M \cong R$.*

Proof. If M is free R-module, i.e., $M \cong R^{(I)}$ for some I then, by Corollary 3.8, the set I must be a single element and hence $M \cong R$. □

Let R be a ring and $\widehat{R}$ be the set of isomorphism classes of its simple modules. Let M be a semisimple R-module. Then $M = \bigoplus_{V\in\widehat{R}} M(V)$ where $M(V)$ is the sum of all simple submodules of M isomorphic to V. The module $M(V)$ is called the *isotypic component* of M corresponding to V, or, briefly, the V*-isotypic component* of M.

Corollary 3.10. *Let M be a multiplication R-module. Every semisimple factor module of M is a direct sum of non-isomorphic simple modules, (i.e., each isotypic component is a simple module) with incomparable annihilators.*

Proof. The corollary follows from Corollary 3.6. □

Corollary 3.11. *Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of R-modules where M_1 and M_2 are non-zero R-modules and $\mathfrak{a}_i = \operatorname{ann}_R(M_i)$ for $i = 1, 2$. If M is a multiplication module and either $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ or $\mathfrak{a}_2 \subseteq \mathfrak{a}_1$ then the short exact sequence is not split. In particular, $\mathrm{Ext}^1_\mathrm{R}(\mathrm{M}_2, \mathrm{M}_1) \neq 0$.*

Proof. If the short exact sequence were split then $M \cong M_1 \oplus M_2$. By Corollary 3.6, the ideals $\mathfrak{a}_1$ and $\mathfrak{a}_2$ would be incomparable as M is a multiplication module, a contradiction. □

4 Criteria for a direct sum of modules to be a multiplication module

In this section, the proofs of the five criteria stated in the Introduction are given for a direct sum of modules to be a multiplication module.

Proof of Theorem 1.4. ($\Rightarrow$) Since M is a multiplication module, for each submodule N of M there is an ideal I of R such that $N = IM = I(\bigoplus_{\lambda\in\Lambda} M_\lambda) = \bigoplus_{\lambda\in\Lambda} IM_\lambda \subseteq \bigoplus_{\lambda\in\Lambda} N\bigcap M_\lambda \subseteq N$, i.e., $N = \bigoplus_{\lambda\in\Lambda} N\bigcap M_\lambda$ and $N\bigcap M_\lambda = IM_\lambda$, and so, the intersection condition holds for M.

Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules such that $N_\lambda \subseteq M_\lambda$ for all $\lambda\in\Lambda$. Clearly, N_λ is a submodule of M. So, there exists an ideal I_λ of R such that $N_\lambda = I_\lambda M = \bigoplus_{\mu\in\Lambda} I_\lambda M_\mu$, and so $I_\lambda M_\mu = \delta_{\lambda\mu}N_\lambda$ for all λ, $\mu\in\Lambda$, i.e., the strong orthogonality condition holds.

($\Leftarrow$) Let N be a submodule of M. We have to show that $N = IM$ for some ideal I of R. By the intersection condition, $N = \bigoplus_{\lambda\in\Lambda} N_\lambda$ where $N_\lambda = N\bigcap M_\lambda \subseteq M_\lambda$. For the set $\{N_\lambda\}_{\lambda\in\Lambda}$, let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a set of ideals that satisfies the strong orthogonality condition ($I_\lambda M_\mu = \delta_{\lambda\mu}N_\mu$ for all $\lambda,\mu\in\Lambda$). Then $I = \sum_{\lambda\in\Lambda} I_\lambda$ is an ideal of R such that $IM = \sum_{\lambda,\mu\in\Lambda} I_\lambda M_\mu = \sum_{\lambda,\mu\in\Lambda}\delta_{\lambda\mu}N_\mu = \sum_{\lambda\in\Lambda} N_\lambda = \bigoplus_{\lambda\in\Lambda} N_\lambda = N$, as required. $\square$

Lemma 4.1. *Suppose that a direct sum of nonzero R-modules $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a multiplication module. Let N be a submodule of M, $\mathcal{N} = \{N_\lambda := N\bigcap M_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$. Then for all $\{I_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N},\mathcal{M})$, $N = (\sum_{\lambda\in\Lambda} I_\lambda)M$.*

Proof. $(\sum_{\lambda\in\Lambda} I_\lambda)M = \sum_{\lambda,\mu\in\Lambda} I_\lambda M_\mu = \sum_{\lambda,\mu\in\Lambda}\delta_{\lambda\mu}N_\lambda = \sum_{\lambda\in\Lambda} N_\lambda = N$, by Theorem 1.4. $\square$

The next theorem is an explicit description of the largest orthogonalizer $I(\mathcal{N},\mathcal{M})$ in $\mathcal{I}(\mathcal{N},\mathcal{M})$.

Theorem 4.2. *Suppose that a direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ of nonzero R-modules with $\mathrm{card}(\Lambda) \geq 2$ is a multiplication module. Let $\mathfrak{a}_\lambda = \mathrm{ann}_R(M_\lambda)$, $M'_\lambda := \bigoplus_{\mu\neq\lambda} M_\mu$ and $\mathfrak{a}'_\lambda = \mathrm{ann}_R(M'_\lambda) = \bigcap_{\mu\neq\lambda}\mathfrak{a}_\mu$. Let N be a submodule of M, $\mathcal{N} = \{N_\lambda := N\bigcap M_\lambda\}_{\lambda\in\Lambda}$ and $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$. Then $I(\mathcal{N},\mathcal{M}) = \{I_\lambda\}_{\lambda\in\Lambda}$ where $I_\lambda = I(N_\lambda, M_\lambda)\bigcap\mathfrak{a}'_\lambda$ for all $\lambda\in\Lambda$ (M_λ is a multiplication R-module as an epimorphic image of the multiplication R-module M, so $I(N_\lambda, M_\lambda)$ makes sense).*

Proof. Let $\mathcal{J} = \{J_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N},\mathcal{M})$. Then $J_\lambda \subseteq I_\lambda$ for all $\lambda\in\Lambda$, by the maximality of $I(\mathcal{N},\mathcal{M})$. By the very definition, $I_\lambda \subseteq I'_\lambda := I(N_\lambda, M_\lambda)\bigcap\mathfrak{a}'_\lambda$ for all $\lambda\in\Lambda$. To finish the proof of the theorem it suffices to show that $\{I'_\lambda\}_{\lambda\in\Lambda} \in \mathcal{I}(\mathcal{N},\mathcal{M})$. For all $\lambda\neq\mu$, $I'_\lambda M_\mu = 0$ (since $I'_\lambda \subseteq \mathfrak{a}_\mu$). Finally, $N_\lambda = I_\lambda M_\lambda \subseteq I'_\lambda M_\lambda \subseteq I(N_\lambda, M_\lambda)M_\lambda = N_\lambda$. Hence, $N_\lambda = I'_\lambda M_\lambda$, as required. $\square$

Proof of Theorem 1.7. ($\Rightarrow$) If M is a multiplication module then so is every module M_λ (since M_λ is an epimorphic image of M, Proposition 2.3), and so the condition 1 holds. The condition 2 holds, by Lemma 4.1.

($\Leftarrow$) Let N be a submodule of M. Since M_λ is a multiplication module for all $\lambda\in\Lambda$, $I(N_\lambda, M_\lambda)$ makes sense. Let $I'_\lambda = I(N_\lambda, M_\lambda)\cap\mathfrak{a}'_\lambda$ where $\mathfrak{a}'_\lambda = \mathrm{ann}_R(\bigoplus_{\mu\neq\lambda} M_\mu)$. Then $\{I'_\lambda\} \in \mathcal{I}(\mathcal{N},\mathcal{M})$, i.e. $\mathcal{I}(\mathcal{N},\mathcal{M}) \neq \emptyset$, and therefore, by condition 2, $N = (\sum_{\lambda\in\Lambda} I_\lambda)M$. Hence, M is a multiplication module. $\square$

The next corollary is an explicit description of the element $I(\mathcal{M},\mathcal{M})$.

Corollary 4.3. *Suppose that a direct sum of nonzero R-modules $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a multiplication module where $\mathrm{card}(\Lambda) \geq 2$, $\mathfrak{a}_\lambda = \mathrm{ann}_R(M_\lambda)$ and $\mathfrak{a} = \mathrm{ann}_R(M)$. Assume $\mathcal{M} = \{M_\lambda\}_{\lambda\in\Lambda}$. Then*

1. $I(\mathcal{M},\mathcal{M})=\{\mathfrak{a}'_\lambda\}_{\lambda\in\Lambda}$ *and* $\mathfrak{a}'_\lambda:=\cap_{\mu\neq\lambda}\mathfrak{a}_\mu\neq\cap_{\mu\in\Lambda}\mathfrak{a}_\mu=\mathrm{ann}_R(M)$ *for all* $\lambda\in\Lambda$.
2. *Let* $\pi:R\to\overline{R}=R/\mathfrak{a}$, $r\mapsto\overline{r}:=r+\mathfrak{a}$. *Then* $\sum_{\lambda\in\Lambda}\overline{\mathfrak{a}'_\lambda}=\bigoplus_{\lambda\in\Lambda}\overline{\mathfrak{a}'_\lambda}$ *in* $\overline{R}$ *and* $\overline{\mathfrak{a}'_\lambda}\neq 0$ *for all* $\lambda\in\Lambda$.
3. $M=\overline{\mathfrak{a}'}M$ *where* $\overline{\mathfrak{a}'}=\sum_{\lambda\in\Lambda}\overline{\mathfrak{a}'_\lambda}$. *In particular,* $\overline{\mathfrak{a}'_\lambda}M_\mu=\delta_{\lambda\mu}M_\mu$ *for all* $\lambda,\mu\in\Lambda$.
4. *For each submodule* N *of* M, $N=\mathfrak{b}'M$ *for some ideal* $\mathfrak{b}'$ *of* $\overline{R}$ *such that* $\mathfrak{b}'=\bigoplus_{\lambda\in\Lambda}\mathfrak{b}'_\lambda$ *is a direct sum of ideals* $\mathfrak{b}'_\lambda$ *of* $\overline{R}$ *such that* $\mathfrak{b}'_\lambda\subseteq\overline{\mathfrak{a}'_\lambda}$ *for all* $\lambda\in\Lambda$.

Proof. 1. By Theorem 4.2, $I(\mathcal{M},\mathcal{M})=\{I_\lambda\}_{\lambda\in\Lambda}$ where $I_\lambda=I(M_\lambda,M_\lambda)\cap\mathfrak{a}'_\lambda=R\cap\mathfrak{a}'_\lambda=\mathfrak{a}'_\lambda$. Since $\mathrm{card}(\Lambda)\geqslant 2$ and $M_\lambda\neq 0$ for all $\lambda\in\Lambda$, $\mathfrak{a}'_\lambda\neq 0$ for all $\lambda\in\Lambda$, by Theorem 1.4.

2. Suppose that the sum $\sum_{\lambda\in\Lambda}\overline{\mathfrak{a}'_\lambda}$ is not a direct sum. Then there is a nonzero element $a_\lambda\in\overline{\mathfrak{a}'_\lambda}$ that can be written as a sum $\sum_{\mu\neq\lambda}a_\mu$ for some elements $a_\mu\in\overline{\mathfrak{a}'_\mu}$. As the $\overline{R}$-module M is faithful, $0\neq a_\lambda M=a_\lambda M_\lambda=(\sum_{\mu\neq\lambda}a_\mu M_\lambda)=0$, a contradiction. Hence, the sum $\sum_{\lambda\in\Lambda}\overline{\mathfrak{a}'_\lambda}$ is a direct sum, and, by statement 1, $\overline{\mathfrak{a}'_\lambda}\neq 0$ for all $\lambda\in\Lambda$.

3. $\overline{\mathfrak{a}'}M=\sum_{\lambda,\mu\in\Lambda}\mathfrak{a}'_\lambda M_\mu=\sum_{\lambda,\mu\in\Lambda}\delta_{\lambda\mu}M_\lambda=\sum_{\lambda\in\Lambda}M_\lambda=M$, by statement 1.

4. This statement follows from Theorem 4.2. □.

Proof of Theorem 1.9. (⇒) This implication follows from Corollary 4.3.

(⇐) It suffices to show that the conditions of Theorem 1.4 are satisfied for the module $M=\bigoplus_{\lambda\in\Lambda}M_\lambda$ where $M_\lambda=\mathfrak{a}'_\lambda M$. Let N be a submodule of M. By condition 2, there is an ideal $\mathfrak{b}'$ such that $N=\mathfrak{b}'M=\bigoplus_{\lambda\in\Lambda}\mathfrak{b}'_\lambda M$ where $\mathfrak{b}'_\lambda=\mathfrak{b}'\cap\mathfrak{a}'_\lambda$. Let $N_\lambda=\mathfrak{b}'_\lambda M$. Then $N=\bigoplus_{\lambda\in\Lambda}N_\lambda\subseteq\bigoplus_{\lambda\in\Lambda}(N\cap M_\lambda)\subseteq N$, i.e., $N_\lambda=N\cap M_\lambda$, and so, the condition 1 of Theorem 1.4 holds. By the very definition of N_λ, $N_\lambda=\mathfrak{b}'_\lambda M=\mathfrak{b}'_\lambda M_\lambda$, and for all $\lambda\neq\mu$, $\mathfrak{b}'_\lambda M_\mu\subseteq\mathfrak{a}'_\lambda M_\mu=\mathfrak{a}'_\lambda\mathfrak{a}'_\mu M=0\cdot M=0$, i.e., $\mathfrak{b}'_\lambda M_\mu=0$. So, the condition 2 of Theorem 1.4 holds, as required. □

Proof of Theorem 1.10. (⇒) This implication follows from Theorem 1.4 and Corollary 4.3.

(⇐) Let N be a submodule of M. Then, by the intersection condition, $N=\bigoplus_{\lambda\in\Lambda}N_\lambda$ where $N_\lambda=N\cap M_\lambda$. Since M_λ is a multiplication module, $N_\lambda=J_\lambda M_\lambda$ for an ideal J_λ of R. Then we have $N_\lambda=J_\lambda\mathfrak{a}'_\lambda M_\lambda$, by the orthogonality condition. Let $I_\lambda=J_\lambda\mathfrak{a}'_\lambda$. Then the conditions 1 and 2 of Theorem 1.4 are satisfied since $\mathfrak{a}'_\lambda\mathfrak{a}'_\mu\subseteq\mathfrak{a}=\mathrm{ann}_R(M)$, and so, $I_\lambda M_\mu=I_\lambda I_\mu M_\mu\subseteq\mathfrak{a}'_\lambda\mathfrak{a}'_\mu M_\mu\subseteq\mathfrak{a}M=0$. Therefore the implication (⇐) follows from Theorem 1.4. □

The next corollary is a criterion for a direct sum of simple modules to be a multiplication module.

Corollary 4.4. *Let* $M=\bigoplus_{\lambda\in\Lambda}M_\lambda$ *be a direct sum of simple* R*-modules. We keep the notation of Corollary 4.3. Then* M *is a multiplication* R*-module iff for all* $\lambda,\mu\in\Lambda$, $\mathfrak{a}'_\lambda M_\mu=\delta_{\lambda\mu}M_\mu$.

Proof. (⇒) By Corollary 4.3, $I(\mathcal{M},\mathcal{M})=\{\mathfrak{a}'_\lambda\}_{\lambda\in\Lambda}$ where $\mathcal{M}=\{M_\lambda\}_{\lambda\in\Lambda}$, and the implication follows.

(⇐) Suppose that $\mathfrak{a}'_\lambda M_\mu=\delta_{\lambda\mu}M_\mu$ for all $\lambda,\mu\in\Lambda$. Clearly, the simple R-modules $\{M_\lambda\}_{\lambda\in\Lambda}$ are pairwise non-isomorphic. So, if N is a submodule of M then $N=\bigoplus_{\lambda\in\Lambda}N\cap M_\lambda$. Let $\mathrm{Supp}(N)=\{\lambda\in\Lambda\,|\,N\cap M_\lambda\neq 0\text{, i.e., }N\cap M_\lambda=M_\lambda\}$. Then

$$N=\bigoplus_{\lambda\in\mathrm{Supp}(N)}M_\lambda=\bigoplus_{\lambda\in\mathrm{Supp}(N)}\mathfrak{a}'_\lambda M_\lambda=(\sum_{\lambda\in\mathrm{Supp}(N)}\mathfrak{a}'_\lambda)M.\ \square$$

The next lemma introduces some properties of the endomorphisms ring of a multiplication module.

Lemma 4.5. *Let M be a multiplication R-module. Then*

1. $\mathrm{Epi}_R(M) = \mathrm{Aut}_R(M)$.

2. *The $\mathrm{End}_R(M)$-stability condition holds for the R-module M. In particular, if $f \in \mathrm{End}_R(M)$ then for all $g \in \mathrm{End}_R(M)$, $g(\mathrm{im}(f)) \subseteq \mathrm{im}(f)$.*

3. *If N is a submodule of M then N is an $\mathrm{Epi}_R(M)$-invariant submodule, i.e., for every $f \in \mathrm{Epi}_R(M)$, $f(N) = N$.*

4. *If $M = \bigoplus_{i\in I} M_i$ and $f \in \mathrm{Hom}_R(M, N)$ where N is an R-module then $f(M) = \bigoplus_{i\in I} f(M_i)$.*

5. *If $M = \bigoplus_{i\in I} M_i$ then $\mathrm{End}_R(M) = \prod_{i\in I} \mathrm{End}_R(M_i)$, i.e., the inclusion $\prod_{i\in I} \mathrm{End}_R(M_i) \subseteq \mathrm{End}_R(M)$ is an equality. In particular, $\mathrm{Hom}_R(M_i, M_j) = 0$ for all $i \neq j$; and $\mathrm{Aut}_R(M) \cong \prod_{i\in I} \mathrm{Aut}_R(M_i)$.*

6. *If R is a commutative ring and C is a cyclic submodule of the R-module M then for every $f \in \mathrm{End}_R(M)$, $f|_C = r_C : C \to C$, $m \mapsto rm$ for some element $r = r(f) \in R$. Futhermore, the ring $\mathrm{End}_R(M)$ is a commutative ring.*

Proof. 1. Let $f \in \mathrm{Epi}_R(M)$. Then $M/\ker(f) \cong M$. Since M is a multiplication module, $\ker(f) = [\ker(f) : M]M = \mathrm{ann}_R(M/\ker(f))M = \mathrm{ann}_R(M)M = 0$, and therefore $f \in \mathrm{Aut}_R(M)$.

2. Let N be a submodule of M and $f \in \mathrm{End}_R(M)$. Then $N = IM$ for some ideal I of R. So, $f(N) = f(IM) = If(M) \subseteq IM = N$, i.e., N is an $\mathrm{End}_R(M)$-stable submodule.

3. Let N be a submodule of M and $f \in \mathrm{Epi}_R(M)$. By statement 2, $f(N) \subseteq N$. By statement 1, there exists $g \in \mathrm{Aut}_R(M)$ such that $gf = fg = 1$, and therefore $N = 1N = g(f(N)) \subseteq f(N) \subseteq N$, i.e., $N = f(N)$. Hence, N is an $\mathrm{Epi}_R(M)$-invariant submodule.

4. By Theorem 1.4, $\ker(f) = \bigoplus_{i\in I}(\ker(f)\cap M_i)$. Hence $f(M) \cong M/\ker(f) = \bigoplus_{i\in I} M_i/\ker(f)\cap M_i \cong \bigoplus_{i\in I} f(M_i)$, and so $f(M) = \bigoplus_{i\in I} f(M_i)$.

5. Statement 5 follows from statement 2 since $f(M_i) \subseteq M_i$ for all $i \in I$ and $f \in \mathrm{End}_R(M)$.

6. Statement 6 follows from statement 2. □

Proof of Theorem 1.13. ($\Rightarrow$) It follows from Theorem 1.10 and Lemma 4.5.

($\Leftarrow$)In view of Theorem 1.10, it suffices to prove that the intersection condition holds for the direct sum $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$. For each $\lambda \in \Lambda$, let $j_\lambda : M \to M$ be a composition of the projection homomorphism $M \to M_\lambda$ and the inclusion homomorphism $M_\lambda \to M$. Clearly, $j_\lambda \in \mathrm{End}_R(M)$, and $N \subseteq \sum_{\lambda\in\Lambda}(N \cap M_\lambda) = \bigoplus_{\lambda\in\Lambda}(N \cap M_\lambda) \subseteq N$ since N is $\mathrm{End}_R(M)$-stable, i.e., $N = \bigoplus_{\lambda\in\Lambda}(N \cap M_\lambda)$, and so, the intersection condition holds. Hence, by Theorem 1.10, the R-module M is a multiplication module. □

5 Some applications

In this section, we will use the above mentioned criteria to introduce many properties of the class of multiplication modules over noncommutative rings.

The refinement condition.

Definition 5.1. *Let M be an R-module. We say that two of its decompositions into direct sum of submodules, $M = \bigoplus_{i\in I} M_i$ and $M = \bigoplus_{j\in J} N_j$, satisfy the* **refinement condition** *if $M = \bigoplus_{i\in I, j\in J} M_i \cap N_j$. We say that the module M satisfies the* ***refinement condition*** *if every two of its direct sum decomposition satisfy the refinement condition.*

Let M be an R-module and $E = \mathrm{End}_R(M)$. Then the R-module M is a direct sum of its submodules iff the identity map $1 : M \to M$, $m \mapsto m$ is a *sum of orthogonal idempotents*, that is $1 = \sum_{\lambda\in\Lambda} e_\lambda$ where $e_\lambda e_\mu = \delta_{\lambda\mu} e_\lambda$ and for every element $m \in M$, $e_\lambda m = 0$ for all but finitely many λ.

In more detail, if $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ then $1 = \sum_{\lambda\in\Lambda} e_\lambda$ where e_λ is the projection onto M_λ. Conversely, if $1 = \sum_{\lambda\in\Lambda} e_\lambda$ is a sum of orthogonal idempotents then $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ where $M_\lambda = e_\lambda M$. The infinite sum makes sense as it is applied to a finite sum of elements and the maps are projections.

Proposition 5.2 is a criterion for a module to satisfy the refinement condition.

Proposition 5.2. *Let M be an R-module and $E = \mathrm{End}_R(M)$. Then the R-module M satisfies the refinement condition iff for any two sums of orothognal idempotents in E, $1 = \sum_{i\in I} e_i$ and $1 = \sum_{j\in J} f_j$, $e_i f_j = f_j e_i$ for all $i \in I$ and $j \in J$.*

Proof. ($\Rightarrow$) Suppose that an R-module M satisfies the refinement condition. Let $1 = \sum_{i\in I} e_i$ and $1 = \sum_{j\in J} f_j$ be sums of orthogonal idempotents in E. Then $M = \bigoplus_{i\in I} M_i = \bigoplus_{j\in J} M_j$ where $M_i = e_i M$ and $M_j = f_j M$. Since the R-module satisfies the refinement condition, $M = \bigoplus_{i\in I, j\in J} M_i \cap M_j$, and $1 = \sum_{i\in I, j\in J} e_i f_j$ is the correspondent sum of orthogonal idempotents such that $e_i f_j = f_j e_i$ for all $i \in I$ and $j \in J$.

($\Leftarrow$) Suppose that $M = \bigoplus_{i\in I} M_i$ and $M = \bigoplus_{j\in J} M_j$ and $1 = \sum_{i\in I} e_i$, $1 = \sum_{j\in J} f_j$ are the correspondent sum of orthogonal idempotents. Since $e_i f_j = f_j e_i$ for all $i \in I$ and $j \in J$, $1 = \sum_{i\in I, j\in J} e_i f_j$ is a sum of orthogonal idempotents. Hence, $e_i f_j M = M_i \cap M_j$ and $M = \bigoplus_{i\in I, j\in J} (M_i \cap M_j)$. $\square$

Definition 5.3. *For an R-module M, let $\mathrm{Dec}(M) = \mathrm{Dec}_R(M)$ be the set of all its direct sum decompositions. We say that a direct sum decomposition $\bigoplus_{i\in I} M_i$ is* **finer** *than a direct sum decomposition $\bigoplus_{j\in J} N_j$ and write $\bigoplus_{i\in I} M_i \geq \bigoplus_{j\in J} N_j$ if $I = \coprod_{j\in J} I_j$ is disjoint union of non-empty subsets I_j such that for each $j \in J$, $N_j = \bigoplus_{i\in I_j} M_i$.*

The set $(\mathrm{Dec}(M), \geq)$ is a partially ordered set (a poset, for short). Let $\mathrm{maxDec}(M)$ be a set of maximal elements of $\mathrm{Dec}(M)$.

Definition 5.4. $\mathrm{ds.dim}(M) = \sup\{\mathrm{card}(I) \mid M = \bigoplus_{i\in I} M_i \in \mathrm{Dec}(M)\}$ *is called the* ***direct sum decomposition dimension****.*

If an R-module M satisfies the refinement condition and $\mathrm{maxDec}(M) \neq \emptyset$ then $\mathrm{maxDec}(M)$ contains a unique decomposition, say $\bigoplus_{i\in I} M_i$, and so, $\mathrm{ds.dim}(M) = \mathrm{card}(I)$.

Corollary 5.5. *Let M be a multiplication module such that $M = \bigoplus_{i\in I} M_i = \bigoplus_{j\in J} N_j$ and L be a submodule of M. Then*

1. *$M = \bigoplus_{i\in I, j\in J} M_i \cap N_j$, i.e., every multiplication module satisfies the refinement condition, and*

2. $L=\bigoplus_{i\in I, j\in J} L\cap M_i\cap N_j$.

Proof. By Theorem 1.4, $M_i=\bigoplus_{j\in I} M_i\cap N_j$ for all $i\in I$. Then $M=\bigoplus_{i\in I} M_i=\bigoplus_{i\in I, j\in J} M_i\cap N_j$, and statement 1 holds. Statement 2 follows from Theorem 1.4 and statement 1. □

Definition 5.6. *Let R be a ring. The **ideal uniform dimension** of R, $\mathrm{iu.dim}(R)$, is the supremum of cardinalities of sets I such that $\bigoplus_{i\in I}\mathfrak{a}_i$ is a direct sum of ideals of R.*

Proposition 5.7. *Let M be a multiplication R-module. Then $\mathrm{ds.dim}(M)\leq \mathrm{iu.dim}(R/\mathrm{ann}_R(M))$.*

Proof. The proposition follows from Corollary 4.3. □

Definition 5.8. *Let R be a ring. Then $\mathrm{m.dim}(R)=\sup\{\mathrm{ds.dim}(M)\,|\, M \text{ is a multiplication } R\text{-module}\}$ is called the **multiplication dimension**.*

Corollary 5.9. *Let R be a ring. Then $\mathrm{m.dim}(R)\leq \sup\{\mathrm{iu.dim}(R/\mathrm{ann}_R(M))\,|\, M \text{ is a multiplication } R\text{-module}\}\leq \sup\{\mathrm{iu.dim}(R/\mathrm{ann}_R(M))\,|\, M \text{ is an } R\text{-module}\}$.*

Proof. The corollary follws from Proposition 5.7. □

Every multiplication module is a unique direct sum of indecomposable modules. The next proposition shows that there is a unique decomposition of any multiplication module as a direct sum of indecomposable modules (if such decomposition exists).

Proposition 5.10. *Let M be a multiplication R-module. Suppose that $M=\bigoplus_{i\in I} M_i=\bigoplus_{j\in J} N_j$ are direct sums of indecomposable R-modules. Then there is a bijection $\sigma: I\rightarrow J$ such that $M_i=N_{\sigma(i)}$ for all $i\in I$.*

Proof. By Theorem 1.4, $M_i=\bigoplus_{j\in J} M_i\cap N_j$ for all $i\in I$. The module M_i is indecomposable. So, $M_i=N_{\sigma(i)}$ for a unique $\sigma(i)\in J$. If $i\neq i'$ then $\sigma(i)\neq\sigma(i')$, i.e., the map $\sigma: I\rightarrow J$, $i\mapsto \sigma(i)$, is an injection. By symmetry, there is an injection $\tau: J\rightarrow I$, $j\mapsto\tau(j)$, such that $N_j=M_{\tau(j)}$. Clearly, $\sigma\tau(j)=j$ and $\tau\sigma(i)=i$ for all $j\in J$ and $i\in I$. So, $\sigma=\tau^{-1}$, and the result follows. □

Definition 5.11. *The set of nonzero R-modules $\{M_\lambda\}_{\lambda\in\Lambda}$ where $\mathrm{card}(\Lambda)\geqslant 2$ is called* **homomorphically independent** *if $\mathrm{Hom}_R(M_\lambda, M_\mu)=0$ for all $\lambda\neq\mu$ where $\lambda,\mu\in\Lambda$.*

We say that a direct sum $M=\bigoplus_{\lambda\in\Lambda} M_\lambda$ **has enough complements** *if for every direct summand K of M, there is a subset $\Lambda'\subseteq\Lambda$ such that $M=\bigoplus_{\lambda\in\Lambda'} M_\lambda\bigoplus K$.*

Proposition 5.12. *Let $M=\bigoplus_{\lambda\in\Lambda} M_\lambda$ be a direct sum of nonzero R-modules where $\mathrm{card}(\lambda)\geqslant 2$. If M is a multiplication R-module then*

1. *The set $\{M_\lambda\}_{\lambda\in\Lambda}$ is homomorphically independent.*
2. *If M_λ is indecomposable and K is a nonzero direct summand submodule of M then either $M_\lambda\subseteq K$ or $M_\lambda\cap K=0$. Moreover, if $M=\bigoplus_{\lambda\in\Lambda} M_\lambda$ is direct sum of indecomposable modules then $\mathrm{Sub}_{\mathrm{R}}^{\oplus}(M)=\{\bigoplus_{\lambda\in\Lambda'} M_\lambda\,|\,\Lambda'\subseteq\Lambda\}$, i.e., M has enough complements.*
3. *If all R-modules M_λ, $\lambda\in\Lambda$, have direct sum with enough complements then the R-module M has enough complements.*

Proof. 1. It follows from Lemma 4.5.

2. If K is a nonzero direct summand submodule of M then there exists a submodule K' of M such that $M = K \bigoplus K'$. By Theorem 1.4, $M_\lambda = M_\lambda \cap K \bigoplus M_\lambda \cap K'$. So, either $M_\lambda \subseteq K$ or $M_\lambda \cap K = 0$ (since M_λ is indecomposable). Therefore, by Theorem 1.4, $K = \bigoplus_{\lambda\in\Lambda}(M_\lambda \cap K) = \bigoplus_{\lambda\in\Lambda'} M_\lambda$ where $\Lambda' \subseteq \Lambda$.

3. For every $\lambda \in \Lambda$, let $M_\lambda = \bigoplus_{i\in I_\lambda} N_i$. Clearly, $\bigoplus_{\lambda\in\Lambda}(\bigoplus_{i\in I_\lambda} N_i)$ is a direct sum decomposition of M. Let K be a direct summand of M, i.e., there exists a submodule K' of M such that $M = K \bigoplus K'$. By Theorem 1.4, $M_\lambda = M_\lambda \cap K \bigoplus M_\lambda \cap K'$, i.e., $M_\lambda \cap K$ is a direct summand of M_λ. So, by statement 2, $M_\lambda \cap K = \bigoplus_{i\in I'_\lambda\subseteq I_\lambda} N_i$. Therefore, by Theorem 1.4, $K = \bigoplus_{\lambda\in\Lambda}(K \cap M_\lambda) = \bigoplus_{\lambda\in\Lambda}(\bigoplus_{i\in I'_\lambda\subseteq I_\lambda} N_i)$, i.e., M has enough complements. □

Corollary 5.13. *Let $M = \bigoplus_{i\in I} M_i$ be a direct sum of nonzero R-modules where* $\operatorname{card}(I) \geqslant 2$. *If M is a multiplication R-module and N is an indecomposable submodule of M then $N \subseteq M_i$ for some $i \in I$.*

Proof. By Theorem 1.4, $N = \bigoplus_{i\in I}(N \cap M_i)$. Since N is an indecomposable submodule, $N = N \bigcap M_i$ for some $i \in I$, i.e., $N \subseteq M_i$ for some $i \in I$. □

Corollary 5.14. *Let $M = \bigoplus_{i\in I} M_i$ be a direct sum of indecomposable R-submodules of M where* $\operatorname{card}(I) \geqslant 2$. *If M is a multiplication module and N is an indecomposable direct summand of M then $N = M_i$ for some $i \in I$.*

Proof. Since N is a direct summand of M, there exists a submodule K of M such that $M = N \bigoplus K$. By Theorem 1.4, $N = \bigoplus_{i\in I}(N \bigcap M_i)$. It follows that there exists $i \in I$ such that $N = N \bigcap M_i$ (since N is indecomposable). Again, by Theorem 1.4, since $M = N \bigoplus K$ and M_i is a submodule of M, $M_i = (M_i \bigcap N) \bigoplus (M_i \bigcap K) = N \bigoplus (M_i \bigcap K)$. It follows that $M_i = N$ (since M_i is indecomposable and $N \neq 0$). □

Definition 5.15. *An R-module M satisfies the* **direct sum cancellation property** *if $M = K \bigoplus L = K \bigoplus L'$ where K, L and L' are R-modules then $L = L'$.*

Lemma 5.16. *Every multiplication module satisfies the direct sum cancellation property.*

Proof. Let M be a multiplication R-module such that $M = K \bigoplus L = K \bigoplus L'$ where K, L and L' are R-modules. Then, by Theorem 1.4, $L = (K \cap L) \oplus (L \cap L') = L \cap L'$ which follows that $L \subseteq L'$. Similarly, $L' \subseteq L$. □

Definition 5.17. *Let R be a ring. An R-module M satisfies the* **summand property** *if $K + L$ and $K \cap L$ are also direct summands of M for all direct summands K and L of M.*

Proposition 5.18. *Every multiplication module satisfies the summand property.*

Proof. Let K and L be direct summand submodules of M. Then $M = L \oplus L^* = K \oplus K^*$ for some submodule L^* and K^* of M. Since M is a multiplication module,

$$K = (K \cap L) \oplus (K \cap L^*),$$

by Theorem 1.4. Therefore $K \cap L$ is a direct summand submodule of K. Hence, $K \cap L$ is a direct summand submodule of M. Now, since $K + L$ is a submodule of $M = L \oplus L^*$, $K + L = L \oplus (K \cap L^*)$, by Theorem 1.4. Since K is a direct summand submodule of M and L^* is a submodule of M, $K \cap L^*$

is a direct summand submodule of L^*, by Theorem 1.4, i.e., $L^* = (K \cap L^*) \oplus K'$ where $K' = K^* \cap L^*$. So,

$$M = L \oplus L^* = L \oplus ((K \cap L^*) \oplus K') = (L \oplus (K \cap L^*)) \oplus K' = (K + L) \oplus K'.$$

Hence, $K + L$ is a direct summand submodule of M. Hence, M satisfies the summand property. $\square$

Let M be a multiplication module with a direct sum decomposition. The next corollary gives an intersection decomposition for every submodule of M.

Corollary 5.19. *Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ be a direct sum of nonzero R-modules where* $\mathrm{card}(I) \geqslant 2$. *If M is a multiplication module then for each submodule N of M, $N = \bigcap_{\lambda \in \Lambda} (N + M'_\lambda)$ where $M'_\lambda = \bigoplus_{\mu \neq \lambda} M_\mu$.*

Proof. By Theorem 1.4, $N = \bigoplus_{\lambda \in \Lambda} N \bigcap M_\lambda$. So, for every $\lambda \in \Lambda$, $N + M'_\lambda = N \cap M_\lambda \oplus M'_\lambda$, and the corollary follows. $\square$

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References

[1] M. Ali and D. J. Smith, Finite and infinite collections of multiplication modules, *Beitrage Algebra Geom*, **42** (2001) 557–573.

[2] Y. Alshaniafi and S. Singh, A companion ideal of a multiplication module, *Period. Math. Hungar.*, **46** (2003) 1-8.

[3] T. Alsuraiheed and V. V. Bavula, Characterization of multiplication commutative rings with finitely many minimal prime ideals. *Comm. Algebra*, **47** (2019), no. 11, 4533–4540.

[4] D. D. Anderson, Some remarks on multiplication ideals, *Math. Japon.,* **25** (1980) 463–469.

[5] D. D. Anderson and Y. Alshaniafi, Multiplication modules and the ideal $\theta(M)$, *Comm. Algebra*, **30** (7) (2002) 3383–3390.

[6] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Wesley Publishing Company, 1969.

[7] A. Barnard, Multiplication modules, *J. Algebra*, **71** (1981) 174–178.

[8] V. V. Bavula, The algebra of integro-differential operators on a polynomial algebra, *J. Lond. Math. Soc.,* **83** (2011) no. 2, 517–543. Arxiv:math.RA: 0912.0723.

[9] V. V. Bavula, The algebra of integro-differential operators on an affine line and its modules, *J. Pure Appl. Algebra*, **217** (2013) 495–529. Arxiv:math.RA: 1011.2997.

[10] Z. Elbast and P. Smith, Multiplication Modules, *Comm. Algebra*, **16** (1988) 755–779.

[11] M. Fontana, S.E. Kabbaj, B. Olberding and I. Swanson, Commutative Algebra Noetherian and Non-Noetherian Perspectives, *Springer, York*, 2010.

[12] W. Krull, Ideal Theory, New York, 1948.

[13] M. Larsen and P. McCarthy, Multiplicative Theory of Ideals, *Academic Press*, 1971.

[14] J. L. Mott, Multiplication rings containing only finitely many minimal prime ideals, *J. Sci. Hiroshima. UNIV. SER.,* **16** (1969) 73–83.

[15] P. F. Smith, Some Remarks on Multiplication Modules, *Arch. Math,* **50** (1988) 223–235.

[16] A. Tuganbaev, Multiplication Modules over Non-commutative rings, . *Sbornik. Mathematics. Russ. Acad. Sci.*, **194** (2007) 1837–1864.

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