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DIOPHANTINE APPROXIMATION ON CURVES AND THE DISTRIBUTION OF RATIONAL POINTS: CONTRIBUTIONS TO THE DIVERGENCE THEORY

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Dedicated to Basil Bernik on his 71st birthday

ABSTRACT. In this paper we develop an explicit method for studying the distribution of rational points near manifolds. As a consequence we obtain optimal lower bounds on the number of rational points of bounded height lying at a given distance from an arbitrary non-degenerate curve in $\mathbb{R}^n$. This generalises previous results for analytic non-degenerate curves. Furthermore, the main results are proved in the inhomogeneous setting. For $n \geq 3$, the inhomogeneous aspect is new even under the additional assumption of analyticity. Applications of the main distribution theorem also include the inhomogeneous Khintchine-Jarník type theorem for divergence for arbitrary non-degenerate curves in $\mathbb{R}^n$.

Key words and phrases: simultaneous Diophantine approximation on manifolds, metric theory, rational points near manifolds, Khintchine theorem, Jarník theorem, Hausdorff dimension, ubiquitous systems

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1. Introduction and statement of results

In this paper we establish an optimal statement concerning the distribution of rational points lying close to an arbitrary non-degenerate curve in $\mathbb{R}^n$. As a consequence we obtain a sharp lower bound for the number of such rational points. Motivated by applications to Diophantine approximation on non-degenerate manifolds, the corresponding results were obtained for planar curves in [5, 9, 11] and under the extra assumption of analyticity for submanifolds of $\mathbb{R}^n$ in [2]. Our motivation is in line with these previous works and we shall describe the applications to Diophantine approximation in §1.2 below. Removing the analyticity assumption within the statements of [2] is non-trivial, requiring new ideas and

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techniques – see Remark 1.2 below. In this paper, we complete this task in full in the case the manifold is a curve.

Recall that a real connected analytic submanifold of $\mathbb{R}^n$ is non-degenerate if and only if it is not contained in any hyperplane of $\mathbb{R}^n$ [17, p. 341]. The main covering result in [2], implies that for any analytic non-degenerate submanifold $\mathcal{M} \subset \mathbb{R}^n$ of dimension d and codimension $m = n - d$:

$$\#\Big\{\mathbf{p}/q \in \mathbb{Q}^n : \ 1 \leq q \leq Q, \ \mathrm{dist}(\mathbf{p}/q, \mathcal{M}) \leq \frac{\psi}{Q}\Big\} \geq C_1 \psi^m Q^{d+1} \tag{1.1}$$

for all sufficiently large Q and all real ψ satisfying

$$C_2 Q^{-1/m} < \psi < 1\,. \tag{1.2}$$

Here the symbol # stands for 'cardinality' and C_1 and C_2 are positive constants depending only on the manifold $\mathcal{M}$ and the dimension n of the space. Furthermore, it is shown [2, Theorem 7.1] that for analytic non-degenerate curves (1.2) can be relaxed to

$$C_2 Q^{-\frac{3}{2n-1}} < \psi < 1\,. \tag{1.3}$$

It is believed that the above results for analytic non-degenerate manifolds should hold for arbitrary non-degenerate manifolds. Indeed, this is the case for planar curves, see [5, 11]. In this paper we obtain complete results for non-degenerate curves in arbitrary dimensions. Moreover, we obtain an inhomogeneous extension of (1.1), which to date is only known in the case $n = 2$, see [9]. It is worth mentioning that the methods developed in this paper are up to a point applicable to arbitrary non-degenerate manifolds and we isolate "what needs to be done" to obtain the desired result beyond curves – see Remark 1.2 below.

Before we proceed with the statement of results, let us recall the definition of non-degeneracy in the non-analytic case. Firstly, a map $\mathbf{f} : \mathcal{U} \to \mathbb{R}^n$ defined on an open set $\mathcal{U} \subset \mathbb{R}^d$ is called *l-non–degenerate at* $\mathbf{x} \in \mathcal{U}$ if $\mathbf{f}$ is l times continuously differentiable on some sufficiently small ball centred at $\mathbf{x}$ and the partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ of orders up to l span $\mathbb{R}^n$. The map $\mathbf{f}$ is called *non–degenerate* at $\mathbf{x}$ if it is l-non–degenerate at $\mathbf{x}$ for some l; in turn a manifold $\mathcal{M} \subset \mathbb{R}^n$ is said to be non–degenerate at $\mathbf{y} \in \mathcal{M}$ if there is a neighbourhood of $\mathbf{y}$ that can be parameterised by a map $\mathbf{f}$ non-degenerate at $\mathbf{f}^{-1}(\mathbf{y})$. In general, non-degenerate manifolds are smooth sub-manifolds of $\mathbb{R}^n$ which are sufficiently curved so as to deviate from any hyperplane at a polynomial rate, see [1, Lemma 1(c)].

1.1. **Results for rational points near manifolds.** Throughout, $|X|$ is the Lebesgue measure of a measurable subset X of $\mathbb{R}$, $\|\cdot\|_2$ is the Euclidean norm and $\|\cdot\|_\infty$ is the supremum norm. In what follows, unless otherwise stated, all balls will be considered with respect to the supremum norm. Let $d, m \in \mathbb{N}$, $n = d + m$ and $\boldsymbol{f} = (f_1, \dots, f_m)$ be defined and continuously differentiable on a given fixed ball $\mathcal{U}$ in $\mathbb{R}^d$. The map $\boldsymbol{f}$ naturally gives rise to the d-dimensional manifold

$$\mathcal{M}_{\boldsymbol{f}} := \{(\mathbf{x}, \boldsymbol{f}(\mathbf{x})) \in \mathbb{R}^n : \mathbf{x} = (x_1, \ldots, x_d) \in \mathcal{U}\} \tag{1.4}$$

immersed in $\mathbb{R}^n$. By the Implicit Function Theorem, any smooth submanifold $\mathcal{M}$ of $\mathbb{R}^n$ can be (at least locally) defined in this manner; i.e. with a Monge parametrisation. Hence, in what follows, without loss of generality, we will work with a manifold $\mathcal{M}$ as in (1.4).

Given $0 < \psi < 1$, $Q > 1$, a ball $B \subset \mathcal{U}$ and $\boldsymbol{\theta} = (\boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \mathbb{R}^d \times \mathbb{R}^m$, consider the set

$$\mathcal{R}(Q, \psi, B, \boldsymbol{\theta}) := \left\{ (q, \mathbf{a}, \mathbf{b}) \in \mathbb{N} \times \mathbb{Z}^d \times \mathbb{Z}^m : \begin{array}{l} \frac{\mathbf{a}+\boldsymbol{\lambda}}{q} \in B\,,\ \frac{1}{2}Q < q \le Q\,, \\ \|q\boldsymbol{f}\big(\frac{\mathbf{a}+\boldsymbol{\lambda}}{q}\big) - \boldsymbol{\gamma} - \mathbf{b}\|_\infty < \psi \end{array} \right\}. \tag{1.5}$$

Also we define

$$\Delta(Q, \psi, B, \boldsymbol{\theta}, \rho) := \bigcup_{(q,\mathbf{a},\mathbf{b}) \in \mathcal{R}(Q,\psi,B,\boldsymbol{\theta})} B\big(\tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q}, \rho\big), \tag{1.6}$$

where $B(\mathbf{x}, \rho)$ denotes the ball in $\mathbb{R}^d$ centred at $\mathbf{x}$ and of radius ρ. Clearly, elements of $\mathcal{R}(Q, \psi, B, \boldsymbol{\theta})$ gives rise to shifted rational points

$$\big(\tfrac{a_1+\lambda_1}{q}, \ldots, \tfrac{a_d+\lambda_d}{q}, \tfrac{b_1+\gamma_1}{q}, \ldots, \tfrac{b_m+\gamma_m}{q}\big) \in \mathbb{R}^n \tag{1.7}$$

with denominators q in $[\frac{1}{2}Q, Q]$ that lie within the $2\psi/Q$-neighbourhood of $\mathbf{f}(B) \subset \mathcal{M}_{\boldsymbol{f}}$, where $\mathbf{f}(x) := (x, \boldsymbol{f}(x))$. Thus, an appropriate lower bound on the cardinality of $\mathcal{R}(Q, \psi, B, \boldsymbol{\theta})$ would yield (1.1).

The following covering result represents our main result. As we shall see, it immediately yields a sharp lower bound for the cardinality of $\mathcal{R}(Q, \psi, B, \boldsymbol{\theta})$ in the case the manifold $\mathcal{M}_{\boldsymbol{f}} \subset \mathbb{R}^n$ is a non-degenerate curve; that is to say $d = 1$ and $m = n-1$ in the above general discussion. It also enables us to establish the divergent part of the inhomogeneous Khintchine-Jarník type theorem for non-degenerate curves - this will be the subject of §1.2. In fact, as described in Remark 1.7 below, it leads to divergent measure theoretic results beyond curves.

Theorem 1.1. *Let $\boldsymbol{\theta} \in \mathbb{R}^n$, $\boldsymbol{f} = (f_1, \ldots, f_{n-1})$ be a map of one real variable such that $x \mapsto \mathbf{f}(x) := (x, \boldsymbol{f}(x))$ is non-degenerate at some point $x_0 \in \mathbb{R}$. Then, there exists a sufficiently small interval $\mathcal{U}$ centred at x_0 and constants $C_0, K_0 > 0$ (depending on n, $\mathbf{f}$ and x_0 only) such that for any subinterval $B \subset \mathcal{U}$ there is a constant Q_B, depending on n, $\boldsymbol{f}$ and B, only such that for any integer $Q \ge Q_B$ and any ψ satisfying*

$$K_0 Q^{-\frac{3}{2n-1}} \le \psi < 1 \tag{1.8}$$

we have

$$|\Delta(Q, \psi, B, \boldsymbol{\theta}, \rho)| \ge \tfrac{1}{2}|B| \tag{1.9}$$

where

$$\rho = \frac{C_0}{\psi^{n-1}Q^2}\,. \tag{1.10}$$

Trivially,

$$\big|\Delta(Q,\psi,B,\boldsymbol{\theta},\rho)\big| \le \sum_{(q,\mathbf{a},\mathbf{b})\in\mathcal{R}(Q,\psi,B,\boldsymbol{\theta})} \big|B(\tfrac{\mathbf{a}+\boldsymbol{\lambda}}{q},\rho)\big| \ \le\ \#\mathcal{R}(Q,\psi,B,\boldsymbol{\theta})\ 2\rho$$

and so the desired counting result is an immediate consequence of the theorem.

Corollary 1.2. *Let $\boldsymbol{\theta}$, $\mathcal{U}$, $\boldsymbol{f}$, x_0, B, C_0, ψ and Q be as in Theorem 1.1. Then*

$$\#\mathcal{R}(Q,\psi,B,\boldsymbol{\theta})\ \ge\ \frac{|B|}{4C_0}\ \psi^{n-1}Q^2\,. \tag{1.11}$$

Remark 1.1. The constant C_0 appearing in the above statements will be defined within (2.22) below and can be expressed explicitly in terms of certain parameters associated with $\mathbf{f}$ and x_0.

Remark 1.2. Lower and matching upper bounds for rational points near non-degenerate planar curves can be found in [5, 9, 11, 13, 14, 15]. In the homogeneous case (i.e. when $\boldsymbol{\theta}=\mathbf{0}$), the lower bound given by (1.11) is established in [2] for analytic non-degenerate curves embedded in $\mathbb{R}^n$. A key outcome of this paper is thus the removal of the analytic assumption, which is done upon introducing a new technique for detecting rational points near manifolds. This technique enables us to perform explicit analysis of the two conditions within the so-call quantitative non-divergence estimate of Kleinbock & Margulis (see Theorem KM in Section 3) that underpins the proof of our main result; namely Theorem 1.1. Note that condition (i) within Theorem KM comes for free if the non-degenerate manifold is analytic. This is the setup in [2] and thus no attempt is made to analyse it. Establishing condition (i) for non-degenerate manifolds represents a significant problem, which is addressed in this paper for curves. Indeed, it is possible to verify condition (ii) within Theorem KM for arbitrary non-degenerate manifolds and thus establishing condition (i) is the only barrier to proving our main results beyond curves.

1.2. **Simultaneous Diophantine approximation on manifolds.** Given a function $\Psi:(0,+\infty)\to(0,+\infty)$ and a point $\boldsymbol{\theta}=(\theta_1,\dots,\theta_n)\in\mathbb{R}^n$, let $\mathcal{S}_n(\Psi,\boldsymbol{\theta})$ denote the set of $\mathbf{y}=(y_1,\dots,y_n)\in\mathbb{R}^n$ for which there exists infinitely many $(q,\mathbf{p})=(q,p_1,\dots,p_n)\in\mathbb{N}\times\mathbb{Z}^n$ such that

$$\max_{1\le i\le n}|qy_i-\theta_i-p_i|<\Psi(q)\,.$$

If $\boldsymbol{\theta}=\mathbf{0}$ then the corresponding set $\mathcal{S}_n(\Psi):=\mathcal{S}_n(\Psi,\mathbf{0})$ is the usual homogeneous set of simultaneously Ψ-approximable points in $\mathbb{R}^n$. In the case Ψ is $\Psi_\tau:r\to r^{-\tau}$

with $\tau > 0$, let us write $\mathcal{S}_n(\tau, \boldsymbol{\theta})$ for $\mathcal{S}_n(\Psi, \boldsymbol{\theta})$ and $\mathcal{S}_n(\tau)$ for $\mathcal{S}_n(\tau, \mathbf{0})$. Recall that, by Dirichlet's theorem, $\mathcal{S}_n(\tau) = \mathbb{R}^n$ for $\tau \leq 1/n$.

As an application of our main result (Theorem 1.1) we have the following statement concerning the 'size' of the set of simultaneously Ψ-approximable points restricted to lie on a curve in $\mathbb{R}^n$.

Theorem 1.3. *Let $\boldsymbol{\theta} \in \mathbb{R}^n$ and $\Psi : (0, +\infty) \to (0, +\infty)$ be any monotonic function such that $q\Psi(q)^{(2n-1)/3} \to \infty$ as $q \to \infty$. Let $\mathcal{M}$ be any non-degenerate curve in $\mathbb{R}^n$. Then for any $s > 0$ we have that*

$$\mathcal{H}^s\big(\mathcal{S}_n(\Psi, \boldsymbol{\theta}) \cap \mathcal{M}\big) = \mathcal{H}^s(\mathcal{M}) \qquad \textit{when} \qquad \sum_{q=1}^{\infty} q^n \left(\tfrac{\Psi(q)}{q}\right)^{s+n-1} = \infty \,. \tag{1.12}$$

Furthermore, on letting

$$\tau(\Psi) := \liminf_{q \to \infty} \frac{-\log \Psi(q)}{\log q}$$

we have that

$$\dim\big(\mathcal{S}_n(\Psi, \boldsymbol{\theta}) \cap \mathcal{M}\big) \geq \min\left\{1, \ \frac{n+1}{\tau(\Psi)+1} - n + 1\right\}. \tag{1.13}$$

Remark 1.3. In the case $s < 1$ we have that $\mathcal{H}^s(\mathcal{M}) = \infty$ and thus Theorem 1.3 represents an analogue of Jarník's theorem [18]. When $s = 1$, Theorem 1.3 represent an analogue of Khintchine's theorem [19] for curves.

Remark 1.4. Note that for $s > 1$ we have that $\mathcal{H}^s(X) = 0$ for any $X \subset \mathbb{R}$ and (1.12) is trivial. Furthermore, in view of the assumption $q\Psi(q)^{(2n-1)/3} \to \infty$ as $q \to \infty$ we have that the divergence condition in (1.12) always holds for $s \leq \frac{1}{2}$. Thus, for $s \leq \frac{1}{2}$ we unconditionally have that $\mathcal{H}^s\big(\mathcal{S}_n(\Psi, \boldsymbol{\theta}) \cap \mathcal{M}\big) = \mathcal{H}^s(\mathcal{M}) = \infty$ for any approximating function Ψ in question. Thus the most 'interesting' case of Theorem 1.3 is that of $\frac{1}{2} < s \leq 1$.

Remark 1.5. Note that the condition $q\Psi(q)^{(2n-1)/3} \to \infty$ as $q \to \infty$ imposed on Ψ implies that $\tau(\Psi) \leq 3/(2n-1)$ and therefore the lower bound given by (1.13) is never less than $\frac{1}{2}$.

Remark 1.6. Theorem 1.3 was previously proved for planar curves, see [5, Theorem 3], [9, Theorem 1] and [11, Theorem 4]. For $n > 2$, Theorem 1.3 was previously established in the homogeneous case for non-degenerate curves that are additionally assumed to be analytic [2, Theorem 7.2]. Most recently, it was proved in [6], that if the stronger inequality $1/n \leq \tau(\Psi) < 1/(n-1)$ holds and the upper and lower orders of $1/\Psi$ coincide; i.e.

$$\limsup_{q \to \infty} \frac{-\log \Psi(q)}{\log q} = \liminf_{q \to \infty} \frac{-\log \Psi(q)}{\log q} \,,$$

then the lower bound dimension statement (1.13) is valid in the homogeneous case for arbitrary C^2 curves (including degenerate ones) in $\mathbb{R}^n$. To date, the complementary convergence theory for curves is only known in full when $n = 2$ – see [24]. For submanifolds of $\mathbb{R}^n$ of dimension ≥ 2, see [10, 16, 23] and references within for various convergence results and upper bound statements for Hausdorff dimension. For a general background to previous results and what one expects to be able to prove, see [8, §1.6].

Remark 1.7. Theorem 1.3 can be extended to sufficiently smooth non-degenerate submanifolds of $\mathbb{R}^n$ of any dimension. This would involve fibering the manifold into non-degenerate curves using suitable techniques such as those developed by Pyartly or a suitable generalisation of Sprindžuk's Fibering Lemma [3, §2.1] to non-analytic manifolds.

In short, Theorem 1.1 establishes a ubiquitous system of shifted rational points (1.7) near $\mathcal{M}_{\boldsymbol{f}}$. Ubiquity [4] is a well developed mechanism for proving divergence statements such as Theorem 1.3 above. In particular, the deduction of Theorem 1.3 from Theorem 1.1 follows the blue print presented in [5, §3] which develops an appropriate framework of ubiquitous systems close to a curve in $\mathbb{R}^n$. Indeed, the homogenous planar curves argument in fully presented in [5, §7]. The necessary modifications (for proving Theorem 1.3 from Theorem 1.1) are relatively obvious and essentially account for the shift in the numerators of the rational points to reflect the inhomogeneous nature of the problem under consideration. However, for the sake of completeness, we provide the details of the proof of Theorem 1.3 in the appendix.

Remark 1.8. Note that the $s = 1$ case of Theorem 1.3 cannot imply the $s < 1$ case via the Mass Transference Principle [7] since the approximating shifted rational points do not necessarily lie on the manifolds. Thus passing from the $s = 1$ to the $s < 1$ case requires sieving out certain shifted rational points and this is not a built in feature of the Mass Transference Principle. Having said that we should note that the Mass Transference Principle can be used instead of ubiquity to establish the $s < 1$ case of Theorem 1.3 in a similar way as it is used in [6] to obtain a dimension result. However, this would first require establishing an appropriate Khintchine type theorem replacing Theorem 4 in [6] and this would still require exploiting the ubiquity framework as in this paper.

2. Detecting rational points near manifolds

In this section we will not make any assumption concerning the dimension d of $\mathcal{M}$. Recall, that $m := n - d$ is the co-dimension of $\mathcal{M}$.

Without loss of generality, we assume that $\mathcal{M}$ is given by its Monge parameterisation (1.4) and that there exists a constant $M > 0$ such that

$$\max_{1\le k\le m}\ \max_{1\le i,j\le d}\ \sup_{\mathbf{x}\in\mathcal{U}}\left|\frac{\partial^2 f_k(\mathbf{x})}{\partial x_i\partial x_j}\right| \le M\,. \tag{2.1}$$

With this in mind, define the following m auxiliary functions of $\mathbf{x}=(x_1,\dots,x_d)$:

$$g_j := f_j - \sum_{i=1}^{d} x_i\frac{\partial f_j}{\partial x_i} \qquad (1\le j\le m) \tag{2.2}$$

and the following $(n+1)\times(n+1)$ matrix

$$G = G(\mathbf{x}) := \begin{pmatrix} g_1 & \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_d} & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ g_m & \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_d} & 0 & 0 & \dots & -1 \\ x_1 & -1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ x_d & 0 & \dots & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.3}$$

Next, given positive c, Q, ψ, let

$$g = g(c,Q,\psi) := \operatorname{diag}\Big\{\underbrace{\psi,\dots,\psi}_{m},\underbrace{(\psi^m Q)^{-1/d},\dots,(\psi^m Q)^{-1/d}}_{d}, cQ\Big\} \tag{2.4}$$

be a diagonal matrix. Finally, define the set

$$\mathcal{G}(c,Q,\psi) := \Big\{\mathbf{x}\in\mathcal{U} : \delta\big(g^{-1}G(\mathbf{x})\mathbb{Z}^{n+1}\big)\ge 1\Big\}, \tag{2.5}$$

where for a given lattice $\Lambda\subset\mathbb{R}^{n+1}$

$$\delta\big(\Lambda\big) := \inf_{\mathbf{v}\in\Lambda\setminus\{\mathbf{0}\}}\|\mathbf{v}\|_\infty\,. \tag{2.6}$$

Given a set $S\subset\mathbb{R}^d$ and a real number $\rho>0$, S^ρ will denote its 'ρ-interior'; that is the set of $\mathbf{x}\in S$ such that $B(\mathbf{x},\rho)\subset S$.

Lemma 2.1. *Let $Q,\psi>0$ be given and satisfy the following inequality*

$$\psi \ge Q^{-\frac{d+2}{2m+d}}\,. \tag{2.7}$$

Let $\mathcal{U}$ be a ball in $\mathbb{R}^d$ and let $\boldsymbol{f} = (f_1, \dots, f_m) : \mathcal{U} \to \mathbb{R}^m$ be a C^2 map such that (2.1) is satisfied for some $M > 0$. Let $c \in (0, 1]$, $\boldsymbol{\theta} := (\boldsymbol{\lambda}, \boldsymbol{\gamma}) \in \mathbb{R}^d \times \mathbb{R}^m$ and

$$\rho := \frac{1}{2c} \left(\psi^m Q^{d+1}\right)^{-1/d}. \tag{2.8}$$

Then, for any $\mathbf{x} = (x_1, \dots, x_d) \in \mathcal{G}(c, Q, \psi) \cap \mathcal{U}^\rho$ there exists an integer point $(q, a_1, \dots, a_d, b_1, \dots, b_m) \in \mathbb{Z}^{n+1}$ such that

$$2(n+1)Q \ < \ q \ < \ 4(n+1)Q, \tag{2.9}$$

$$|qx_i - a_i - \lambda_i| \ < \ \frac{n+1}{c} \left(\psi^m Q\right)^{-1/d} \qquad (1 \le i \le d), \tag{2.10}$$

and

$$\begin{aligned} &\left| q f_j \left(\frac{a_1 + \lambda_1}{q}, \dots, \frac{a_d + \lambda_d}{q} \right) - b_j - \gamma_j \right| \\ &\qquad < \ \left(1 + \frac{Md^2}{2c}\right) \frac{n+1}{c} \, \psi \qquad (1 \le j \le m). \end{aligned} \tag{2.11}$$

Remark 2.1. The fact that $\mathbf{x}$ is restricted to lie in $\mathcal{U}^\rho$ means that $B(\mathbf{x}, \rho) \subset \mathcal{U}$ and this ensures that the shifted rational point $\left(\frac{a_1+\lambda_1}{q}, \dots, \frac{a_d+\lambda_d}{q}\right)$ lies in $\mathcal{U}$. Indeed, once (2.9) and (2.10) are met, the associated shifted rational point lies in $B(\mathbf{x}, \rho)$ and hence in $\mathcal{U}$. It is not difficult to see from (2.7) that $\rho \to 0$ as $Q \to \infty$ uniformly in ψ and thus considering points $\mathbf{x}$ lying in $\mathcal{U}^\rho$ rather than $\mathcal{U}$ is not particularly restrictive. The homogeneous case of Lemma 2.1 can be viewed as an explicit generalisation of [2, Theprem 4.5]. It is this explicit nature of Lemma 2.1 that is crucial is removing the analyticity requirement from the results of [2]. Note that the explicit form of the matrix G was introduced in [6, Therorem 4], although its use was restricted to the homogeneous setup and to establish a dimension result only.

Proof. Fix any $\mathbf{x} \in \mathcal{G}(c, Q, \psi) \cap \mathcal{U}^\rho$ and consider the lattice

$$\Lambda := g^{-1} G(\mathbf{x}) \mathbb{Z}^{n+1}.$$

Let $\mu_1, \dots, \mu_{n+1}$ be the successive Minkowski minima of Λ with respect to the body

$$B := [-1, 1]^{n+1}.$$

By definition, μ_i is the infimum of all $x > 0$ such that $\operatorname{rank}(\Lambda \cap xB) \ge i$, where $xB := [-x, x]^{n+1}$. In particular, we have that $\mu_1 \le \ldots \le \mu_{n+1}$. By Minkowski's theorem on successive minima, we have that

$$\frac{2^{n+1}}{(n+1)!} \le \frac{\operatorname{Vol}(B)}{\operatorname{covol}(\Lambda)} \prod_{i=1}^{n+1} \mu_i \le 2^{n+1}.$$

Observe, on using (2.3) and (2.4), that the covolume of Λ is c^{-1} and that the volume of B is 2^{n+1}. Hence

$$c\prod_{i=1}^{n+1}\mu_i \le 1\,.$$

Further, by the assumption that $\mathbf{x}\in\mathcal{G}(c,Q,\psi)$, we have that $\mu_1\ge 1$. This follows from (2.5). Hence,

$$\mu_{n+1}\le c^{-1}\prod_{i=1}^{n}\mu_i^{-1}\le c^{-1}\,.$$

Therefore there exists a basis of Λ, say $\mathbf{v}_1,\dots,\mathbf{v}_{n+1}$, lying in $c^{-1}B$, that is

(2.12) $$\|\mathbf{v}_i\|_\infty \le c^{-1} \qquad (1\le i\le n+1).$$

Let

$$\boldsymbol{\omega} := (\omega_0,\omega_1,\dots,\omega_n)\in\mathbb{R}^{n+1}\,,$$

where

$$\omega_0 := 3(n+1)Q\,,$$

(2.13) $$\omega_i := \lambda_i+\omega_0 x_i \qquad (1\le i\le d)$$

and

(2.14) $$\omega_{d+j} := \gamma_j+\omega_0 f_j(\mathbf{x}) \qquad (1\le j\le m)\,.$$

Since $\mathbf{v}_1,\dots,\mathbf{v}_{n+1}$ are linearly independent, there exist unique real parameters $\eta_1,\dots,\eta_{n+1}$ such that

(2.15) $$-g^{-1}\,G(\mathbf{x})\,\boldsymbol{\omega} \;=\; \sum_{i=1}^{n+1}\eta_i\mathbf{v}_i\,.$$

Let $t_1,\dots,t_{n+1}$ be any collection of integers, not all zeros, such that

(2.16) $$|\eta_i-t_i|\le 1 \qquad (1\le i\le n+1).$$

The existence of such integers is obvious. Define

$$\mathbf{v} := \sum_{i=1}^{n+1} t_i\mathbf{v}_i\,.$$

Since the t_i's are integers and not all of them are zero, we have that $\mathbf{v}\in\Lambda\setminus\{\mathbf{0}\}$. Hence, by the definition of Λ, there exists a non-zero integer point $\mathbf{p}\in\mathbb{Z}^{n+1}$, which we will write as $(q,a_1,\dots,a_d,b_1,\dots,b_m)^t$, such that

$$\mathbf{v} = g^{-1}G(\mathbf{x})\,\mathbf{p}\,.$$

Then, using (2.12), (2.15) and (2.16), we find that

$$\begin{aligned}
\left\|g^{-1}G(\mathbf{x})(\mathbf{p}+\boldsymbol{\omega})\right\|_\infty &= \left\|g^{-1}G(\mathbf{x})\mathbf{p}+g^{-1}G(\mathbf{x})\boldsymbol{\omega}\right\|_\infty \\
&= \left\|\mathbf{v}+g^{-1}G(\mathbf{x})\boldsymbol{\omega}\right\|_\infty \\
&= \left\|\sum_{i=1}^{n+1} t_i\mathbf{v}_i - \sum_{i=1}^{n+1}\eta_i\mathbf{v}_i\right\|_\infty \\
&\le \sum_{i=1}^{n+1}|t_i-\eta_i|\cdot\|\mathbf{v}_i\|_\infty \\
&\le c^{-1}(n+1)\,.
\end{aligned} \tag{2.17}$$

Observe that the last coordinate of the vector

$$g^{-1}G(\mathbf{x})(\mathbf{p}+\boldsymbol{\omega}) \tag{2.18}$$

is $(cQ)^{-1}(q+\omega_0)$, which by (2.17) is $\le c^{-1}(n+1)$ in absolute value. Hence $|q+\omega_0|\le(n+1)Q$ and since $\omega_0:=3(n+1)Q$, inequalities (2.9) readily follow.

Furthermore, for $i\in\{1,\dots,d\}$ the $m+i$ coordinate of (2.18) is

$$\begin{aligned}
(\psi^mQ)^{1/d}\Big((q+\omega_0)x_i-(a_i+\omega_i)\Big) &\overset{(2.13)}{=} (\psi^mQ)^{1/d}\Big((q+\omega_0)x_i-(a_i+\lambda_i+\omega_0x_i)\Big)\\
&= (\psi^mQ)^{1/d}(qx_i-a_i-\lambda_i)\,.
\end{aligned}$$

By (2.17) again, we have that $|(\psi^mQ)^{1/d}(qx_i-a_i-\lambda_i)|\le c^{-1}(n+1)$, whence inequalities (2.10) follow.

It now remains to verify (2.11). With this in mind, for $j\in\{1,\dots,m\}$ the j-th coordinate of (2.18) equals

$$\psi^{-1}\left((q+\omega_0)g_j(\mathbf{x})+\sum_{i=1}^{d}(a_i+\omega_i)\frac{\partial f_j(\mathbf{x})}{\partial x_i}-(b_j+\omega_{d+j})\right)$$

and by (2.13) and (2.14) this is equivalent to

$$\psi^{-1}\left((q+\omega_0)g_j(\mathbf{x})+\sum_{i=1}^{d}(a_i+\lambda_i+\omega_0x_i)\frac{\partial f_j(\mathbf{x})}{\partial x_i}-\Big(b_j+\gamma_j+\omega_0f_j(\mathbf{x})\Big)\right)\,.$$

Now on using the expression for $g_j(\mathbf{x})$ from (2.2), we can simplify the above to

$$\psi^{-1}\left(qg_j(\mathbf{x})+\sum_{i=1}^{d}(a_i+\lambda_i)\frac{\partial f_j(\mathbf{x})}{\partial x_i}-b_j-\gamma_j\right)\,.$$

Once again, by (2.17) this is $\le c^{-1}(n+1)$ in absolute value and so it follows that

$$\left|qg_j(\mathbf{x}) + \sum_{i=1}^{d}(a_i+\lambda_i)\frac{\partial f_j(\mathbf{x})}{\partial x_i} - b_j - \gamma_j\right| < c^{-1}(n+1)\psi\,.$$

Using the expression for $g_j(\mathbf{x})$ given by (2.2), we obtain that

$$\left|qf_j(\mathbf{x}) + \sum_{i=1}^{d}(a_i+\lambda_i-qx_i)\frac{\partial f_j(\mathbf{x})}{\partial x_i} - b_j - \gamma_j\right| < c^{-1}(n+1)\psi\,. \tag{2.19}$$

We are now ready to establish (2.11). As already mentioned in Remark 2.1, it follows via (2.9) and (2.10) that for any point $\mathbf{x}\in\mathcal{U}^{\rho}$

$$\left(\tfrac{a_1+\lambda_1}{q},\dots,\tfrac{a_d+\lambda_d}{q}\right)\in\mathcal{U}.$$

Hence, on using Taylor's expansion to the second order followed by the triangle inequality, for any $j\in\{1,\dots,m\}$ we obtain that

$$\begin{aligned}
&\left|qf_j\left(\frac{a_1+\lambda_1}{q},\dots,\frac{a_d+\lambda_d}{q}\right) - b_j - \gamma_j\right| \\
&\qquad = \left|q\left(f_j(\mathbf{x}) + \sum_{i=1}^{d}\frac{\partial f_j(\mathbf{x})}{\partial x_i}\left(\frac{a_i+\lambda_i}{q}-x_i\right)\right.\right. \\
&\qquad\qquad + \left.\left.\sum_{i,l=1}^{d}\frac{\partial^2 f_j(\widetilde{\mathbf{x}})}{\partial x_i\partial x_l}\left(\frac{a_i+\lambda_i}{q}-x_i\right)\left(\frac{a_l+\lambda_l}{q}-x_l\right)\right) - b_j - \gamma_j\right| \\
&\qquad \le \left|qf_j(\mathbf{x}) + \sum_{i=1}^{d}(a_i+\lambda_i-qx_i)\frac{\partial f_j(\mathbf{x})}{\partial x_i} - b_j - \gamma_j\right| \\
&\qquad\qquad + \left|\frac{1}{q}\sum_{i,l=1}^{d}\frac{\partial^2 f_j(\widetilde{\mathbf{x}})}{\partial x_i\partial x_l}\left(a_i+\lambda_i-qx_i\right)\left(a_l+\lambda_l-qx_l\right)\right|
\end{aligned}$$

This together with (2.1), (2.10) and (2.19), implies that

$$\left|qf_j\left(\frac{a_1+\lambda_1}{q},\dots,\frac{a_d+\lambda_d}{q}\right)-b_j-\gamma_j\right|$$

$$\leq\ c^{-1}(n+1)\psi+\frac{1}{q}Md^2\left(c^{-1}(n+1)(\psi^mQ)^{-1/d}\right)^2$$

$$\overset{(2.9)}{\leq}\ c^{-1}(n+1)\psi+\frac{Md^2\left(c^{-1}(n+1)(\psi^mQ)^{-1/d}\right)^2}{2(n+1)Q}$$

$$\overset{(2.7)}{\leq}\left(1+\frac{Md^2}{2c}\right)c^{-1}(n+1)\psi.$$

This verifies (2.11) and thereby completes the proof of the lemma. □

We will make direct use of the following variant of Lemma 2.1.

Corollary 2.2. *Let* $c\in(0,1],M,\widetilde{Q},\widetilde{\psi}>0$ *be given such that*

$$\widetilde{\psi}\geq K_0\,\widetilde{Q}^{-\frac{d+2}{2m+d}}\quad\textit{with}\quad K_0\geq(4(n+1))^{\frac{d+2}{2m+d}}\left(1+\frac{Md^2}{2c}\right)\frac{n+1}{c}\,. \tag{2.20}$$

Let $\mathcal{U}$ *be a ball in* $\mathbb{R}^d$ *and let* $\boldsymbol{f}=(f_1,\dots,f_m):\mathcal{U}\to\mathbb{R}^m$ *be a* C^2 *map such that* (2.1) *is satisfied. Let* $\boldsymbol{\theta}=(\boldsymbol{\lambda},\boldsymbol{\gamma})\in\mathbb{R}^d\times\mathbb{R}^m$ *and let*

$$Q:=\frac{\widetilde{Q}}{4(n+1)}\ ,\qquad\psi:=\frac{\widetilde{\psi}}{\left(1+\frac{Md^2}{2c}\right)\frac{n+1}{c}}\ , \tag{2.21}$$

$$\rho:=\frac{1}{2c}\left(\psi^mQ^{d+1}\right)^{-1/d}=C_0(\widetilde{\psi}^m\widetilde{Q}^{d+1})^{-1/d}$$

where

$$C_0:=\frac{1}{2c}\left(\left(4(n+1)\right)^{d+1}\left(\left(1+\frac{Md^2}{2c}\right)\frac{n+1}{c}\right)^m\right)^{1/d}. \tag{2.22}$$

Then for any ball $B\subset\mathcal{U}$*, we have that*

$$\mathcal{G}(c,Q,\psi)\cap B^\rho\subset\Delta(\widetilde{Q},\widetilde{\psi},B,\boldsymbol{\theta},\rho)\,, \tag{2.23}$$

where $\Delta(\widetilde{Q},\widetilde{\psi},B,\boldsymbol{\theta},\rho)$ *is defined as in* (1.6) *.*

Proof. First, observe that (2.20) implies (2.7). Then if $\mathbf{x}\in\mathcal{G}(c,Q,\psi)\cap B^\rho$, it follows by Lemma 2.1 that there exists an integer point $(q,a_1,\dots,a_d,b_1,\dots,b_m)\in\mathbb{Z}^{n+1}$ satisfying (2.9)–(2.11). By (2.21), condition (2.9) translates into $\frac{1}{2}\widetilde{Q}<q<$

$\widetilde{Q}$; condition (2.11) translates into $\|q\boldsymbol{f}\big(\frac{\mathbf{a}+\boldsymbol{\lambda}}{q}\big)-\boldsymbol{\gamma}-\mathbf{b}\|_\infty<\widetilde{\psi}$; and condition (2.10) together with the fact that $q>\frac12\widetilde{Q}$ imply that $\mathbf{x}\in B(\frac{\mathbf{a}+\boldsymbol{\lambda}}{q},\rho)$. Thus, in view of (1.5) and (1.6), where Q and ψ are replaced with $\widetilde{Q}$ and $\widetilde{\psi}$, to complete the proof of the corollary it remains to show that $\frac{\mathbf{a}+\boldsymbol{\lambda}}{q}\in B$. This is trivially the case since $\mathbf{x}\in B^\rho$ and $\mathbf{x}\in B(\frac{\mathbf{a}+\boldsymbol{\lambda}}{q},\rho)$. $\square$

Note that within the above corollary ρ depends on the parameters $\widetilde{\psi}$ and $\widetilde{Q}$ and the constants C_0 defined by (2.22), but not on the ball B. Also note that in view of (2.20), we have that $\rho\to0$ as $\widetilde{Q}\to\infty$ and thus $B^\rho\to B$ as $\widetilde{Q}\to\infty$. Thus, in light of the above corollary our strategy for proving Theorem 1.1 will be to find a suitable constant $c>0$ and a ball $\mathcal{U}$ centred at a given point $\mathbf{x}_0$ such that for any ball $B\subset\mathcal{U}$ for all sufficiently large $Q\in\mathbb{N}$ the Lebesgue measure of $\mathcal{G}(c,Q,\psi)\cap B^\rho$ is at least a constant times the Lebesgue measure of B. In this paper we establish precisely such a statement in the case of non-degenerate curves.

Theorem 2.3. *Let $\boldsymbol{f}=(f_1,\dots,f_{n-1})$ be a map of one real variable such that $x\mapsto\mathbf{f}(x)=(x,\boldsymbol{f}(x))$ is non-degenerate at some point $x_0\in\mathbb{R}$. Fix any $\kappa>0$. Then, there exists a sufficiently small open interval $\mathcal{U}$ centred at x_0 and constants $c,K_0>0$ such that for any subinterval $B\subset\mathcal{U}$ there is a constant Q_B depending on n, $\mathbf{f}$, κ and B only such that for any integer $Q\ge Q_B$ and any $\psi\in\mathbb{R}$ satisfying* (1.8) *we have that*

$$|B\setminus\mathcal{G}(c,Q,\psi)|\le\kappa\,|B|\,. \tag{2.24}$$

Proof of Theorem 1.1 modulo Theorem 2.3. We shall use Corollary 2.2 with the same $\boldsymbol{f}$ and $\boldsymbol{\theta}$ as in the statement of Theorem 1.1. The fact that $\mathbf{f}$ is non-degenerate at x_0 implies that $\mathbf{f}$ is at least twice continuously differentiable on a sufficiently small neighborhood $\mathcal{U}$ of x_0. Hence the existence of the constant M satisfying (2.1) follows on taking $\mathcal{U}$ sufficiently small so that $\mathbf{f}$ is C^2 on the closure of $\mathcal{U}$. Shrink $\mathcal{U}$ further if necessary and choose $c,K_0>0$ such that the conclusions of Theorem 2.3 hold with $\kappa=\frac13$. In particular, let $B\subset\mathcal{U}$ be any subinterval, $Q>Q_B$ and ψ satisfy (1.8). Then we have that

$$|B\setminus\mathcal{G}(c,Q,\psi)|\le\tfrac13\,|B|\,. \tag{2.25}$$

Assuming without loss of generality that K_0 is at least as in (2.20), we observe that Corollary 2.2 is applicable. In particular, by (2.23), we have that

$$|\Delta(\widetilde{Q},\widetilde{\psi},B,\boldsymbol{\theta},\rho)|\ge|\mathcal{G}(c,Q,\psi)\cap B|-|B\setminus B^\rho|, \tag{2.26}$$

where $\widetilde{Q}$, $\widetilde{\psi}$ and ρ are defined by (2.21). It follows from the definition of B^ρ that $B\setminus B^\rho$ is a union of two intervals, each of length $\le\rho$. Therefore

$$|B\setminus B^\rho|\le2\rho.$$

The lower bound in (1.8) implies that $\psi^{n-1}Q^2 > K_0^{n-1}Q^{1/2}$. Hence, for any

$$Q \geq \left(\frac{12 \cdot C_0}{K_0^{n-1}|B|}\right)^2, \tag{2.27}$$

where C_0 is defined by (2.22), we have that $\rho \leq \frac{1}{12}|B|$ and consequently

$$|B \setminus B^\rho| < \frac{1}{6}|B|. \tag{2.28}$$

Thus, by (2.25), (2.26) and (2.28), we obtain that

$$|\Delta(Q, \psi, B, \boldsymbol{\theta}, \rho)| \geq \tfrac{1}{2}|B|$$

provided that

$$Q \geq Q_B^* := \max\left\{Q_B,\ \left(\frac{12 \cdot C_0}{K_0^{n-1}|B|}\right)^2\right\}.$$

This verifies (1.9) and completes the proof of Theorem 1.1 modulo Theorem 2.3. □

3. Quantitative non-divergence

In what follows we give a simplified account of the theory developed by Kleinbock and Margulis in [17] by restricting ourselves to functions of one variable. Let U be an open subset of $\mathbb{R}$, $f : U \to \mathbb{R}$ be a continuous function and let $C, \alpha > 0$. The function f is called (C, α)-*good on* U if for any open ball (interval) $B \subset U$ the following is satisfied

$$\forall\, \varepsilon > 0 \qquad \left|\left\{x \in B : |f(x)| < \varepsilon \sup_{x\in B}|f(x)|\right\}\right| \ \leq\ C\,\varepsilon^\alpha\,|B|. \tag{3.1}$$

Given $\lambda > 0$ and a ball $B = B(x_0, r) \subset \mathbb{R}$ centred at x_0 of radius r, λB will denote the 'scaled' ball $B(x_0, \lambda r)$. Given $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{R}^{n+1}$ we shall write $\|\mathbf{v} \wedge \ldots \wedge \mathbf{v}_r\|_\infty$ for the supremum norm of the multivector $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_r$. By definition, this is the maximum of the absolute values of the coordinates of $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_r$ in the standard basis. These coordinates are all the possible $r \times r$ minors of the matrix Γ composed of the vectors $\mathbf{v}_i$ as its columns, see [21]. Also, given an $(n+1) \times r$ matrix Γ, we will write $\|\Gamma\|_\infty$ for $\|\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_r\|_\infty$, where $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are the columns of Γ.

We will use the following slightly simplified version of [17, Theorem 5.2] due to Kleinbock and Margulis.

Theorem KM (Quantitative Non-Divergence)**.** *Let $n \in \mathbb{N}$, $C, \alpha > 0$ and $0 < \rho \leq 1/(n+1)$ be given. Let B be a ball in $\mathbb{R}$ and $h : 3^{n+1}B \to \mathrm{GL}_{n+1}(\mathbb{R})$ be given. Assume that for any linearly independent collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{Z}^{n+1}$*

(i) *the function* $x \mapsto \|h(x)\mathbf{v}_1 \wedge \ldots \wedge h(x)\mathbf{v}_r\|_\infty$ *is* (C,α)*-good on* $3^{n+1}B$*, and*

(ii) $\sup_{x\in B} \|h(x)\mathbf{v}_1 \wedge \ldots \wedge h(x)\mathbf{v}_r\|_\infty \geq \rho$.

Then for any $\varepsilon > 0$

$$\left|\Big\{x \in B : \delta\big(h(x)\mathbb{Z}^{n+1}\big) \leq \varepsilon\Big\}\right| \;\leq\; (n+1)C6^{n+1}\left(\frac{\varepsilon}{\rho}\right)^{\alpha} |B|\,, \tag{3.2}$$

where $\delta(\cdot)$ *is given by* (2.6).

We now bring to the forefront the role this theorem plays in establishing Theorem 2.3. In the case $d = 1$, $m = n-1$ the matrix G given by (2.3) has the following form:

$$G(x) = \begin{pmatrix} f_1 - xf_1' & f_1' & -1 & 0 & \ldots & 0 \\ f_2 - xf_2' & f_2' & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1} - xf_{n-1}' & f_{n-1}' & 0 & 0 & \ldots & -1 \\ x & -1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{3.3}$$

We define

$$h(x) := D\,g^{-1}G(x) \quad \text{where} \quad D := \operatorname{diag}\{c^{1/(n+1)},\ldots,c^{1/(n+1)}\} \tag{3.4}$$

and $g = g(c,Q,\psi)$ is as in (2.4). Then, by (2.5) it follows that

$$B \setminus \mathcal{G}(c,Q,\psi) = \Big\{x \in B : \delta\big(D\,g^{-1}G(x)\mathbb{Z}^{n+1}\big) < c^{1/(n+1)}\Big\}\,. \tag{3.5}$$

Therefore, the measure of $B \setminus \mathcal{G}(c,Q,\psi)$ can be estimated via (3.2) subject to verifying conditions (i) and (ii) of Theorem KM. These conditions involve the quantity $\|h(x)\mathbf{v}_1 \wedge \ldots \wedge h(x)\mathbf{v}_r\|_\infty$ which, by definition, is the maximum of the absolute values of the coordinates of $h(x)\mathbf{v}_1 \wedge \ldots \wedge h(x)\mathbf{v}_r$ in the standard basis. The coordinates run over all possible $r \times r$ minors of the matrix $h(x)\Gamma$, where Γ is composed of the vectors $\mathbf{v}_i$ as its columns. Hence,

$$\|h(x)\mathbf{v}_1 \wedge \ldots \wedge h(x)\mathbf{v}_r\|_\infty = \max_{I=\{i_1,\ldots,i_r\}\subset\{1,\ldots,n+1\}} \big|\det\big(h_I(x)\Gamma\big)\big|\,, \tag{3.6}$$

where $h_I(x)$ stands for the $r\times(n+1)$ matrix formed by the rows $i_1,\ldots,i_r$ of $h(x)$ and $\Gamma = (\mathbf{v}_1,\ldots,\mathbf{v}_r)$ is an $(n+1)\times r$ matrix over $\mathbb{Z}$ of rank r.

Remark 3.1. We emphasise that it is possible to verify condition (ii) within Theorem KM for arbitrary non-degenerate manifolds. We only need to make the restriction to curves in order to verify condition (i). Thus, as mentioned in Remark 1.2 establishing the latter for manifolds is the only barrier to proving our main results beyond curves.

Throughout the rest of the paper the set of integer $k \times r$ matrices over $\mathbb{Z}$ (respectively, over $\mathbb{R}$) will be denoted by $\mathrm{Mat}_{\mathbb{Z}}(k,r)$ (respectively, by $\mathrm{Mat}_{\mathbb{R}}(k,r)$). In turn, the subset of $\mathrm{Mat}_{\mathbb{Z}}(k,r)$ of full rank, that is of rank $\min\{k,r\}$, will be denoted by $\mathrm{Mat}^*_{\mathbb{Z}}(k,r)$, and the subset of $\Gamma \in \mathrm{Mat}_{\mathbb{R}}(k,r)$ with $\|\Gamma\|_\infty \geq 1$ will be denoted by $\mathrm{Mat}^*_{\mathbb{R}}(k,r)$. Observe that $\mathrm{Mat}^*_{\mathbb{Z}}(k,r) \subset \mathrm{Mat}^*_{\mathbb{R}}(k,r)$.

Given $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n+1\}$, let $G_I(x)$ denote the $r \times (n+1)$ matrix formed by the rows $i_1, \dots, i_r$ of $G(x)$. Define the function

$$\phi_{I,\Gamma}(x) := \det\big(G_I(x)\Gamma\big)\,.$$

Since $d = 1$, we have that

$$g = \mathrm{diag}\Big\{\underbrace{\psi, \dots, \psi}_{n-1}, (\psi^{n-1}Q)^{-1}, cQ\Big\}\,. \tag{3.7}$$

Then, for h given by (3.4), we have that

$$\det\big(h_I(x)\Gamma\big) = c^{\frac{r}{n+1}} \cdot \Phi_I \cdot \phi_{I,\Gamma}(x)\,, \tag{3.8}$$

where

$$\Phi_I = \begin{cases} \psi^{-r} & \text{if } n \notin I \text{ and } n+1 \notin I, \\ \psi^{n-r}Q & \text{if } n \in I \text{ and } n+1 \notin I, \\ (c\,\psi^{r-1}Q)^{-1} & \text{if } n \notin I \text{ and } n+1 \in I, \\ c^{-1}\psi^{n-r+1} & \text{if } n \in I \text{ and } n+1 \in I. \end{cases} \tag{3.9}$$

In view of (3.8) and (3.9) verifying conditions (i) and (ii) of Theorem KM for our choice of h is reduced to understanding the functions $\phi_{I,\Gamma}(x)$ for all possible choices of I and Γ. With this in mind we now state the main assertion regarding $\phi_{I,\Gamma}(x)$.

Proposition 3.1. *Let $\boldsymbol{f} = (f_1, \dots, f_{n-1})$ be a map of one real variable such that $x \mapsto \mathbf{f}(x) := (x, \boldsymbol{f}(x))$ is non-degenerate at some point $x_0 \in \mathbb{R}$. Then, there exists a sufficiently small open interval $\mathcal{U}$ centred at x_0, $l \in \mathbb{N}$ and a constant $C > 0$ satisfying the following. For any interval $B \subset \mathcal{U}$ there exists a constant $\rho_B > 0$ such that for any $1 \leq r \leq n$ and any $\Gamma \in \mathrm{Mat}^*_{\mathbb{Z}}(n+1, r)$ the following two properties are satisfied:*

(i) *for any* $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n+1\}$ *we have that*

$$|\phi_{I,\Gamma}| \quad is \quad \left(C, \frac{1}{2l-1}\right)\text{-}good\ on\ 3^{n+1}B, \tag{3.10}$$

(ii) *for some* $I = \{i_1, \dots, i_r\} \subset \big\{1, \dots, \max\{r, n-1\}\big\}$ *we have that*

$$\sup_{x \in B} |\phi_{I,\Gamma}(x)| \geq \rho_B. \tag{3.11}$$

As we shall see in §6 below, once armed with Proposition 3.1 it is not difficult to establish the desired Theorem 2.3.

4. Non-degenerate maps and (C, α)-good functions

In this section we collect together several statements regarding (C, α)-good functions that will be required during the course of establishing Proposition 3.1. Within this section we will use the notion $\mathbb{R}^n_1$ for the unit Euclidean sphere in $\mathbb{R}^n$, that is $\mathbb{R}^n_1 := \{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}|_2 = 1\}$.

We begin with the following basic lemma which is a direct consequence of Lemma 3.1 in [12] (see also [17, Lemma 3.1]).

Lemma 4.1. *Let* $V \subset \mathbb{R}$ *be open and* $C, \alpha > 0$. *If* $g_1, \dots, g_m$ *are* (C, α)*-good functions on* V *and* $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, *then* $\max_i |\lambda_i g_i|$ *is a* (C', α')*-good function on* V' *for every* $C' \geq C$, $0 < \alpha' \leq \alpha$ *and open subset* $V' \subset V$.

The next lemma is a straightforward consequence of the definition of non-degeneracy.

Lemma 4.2. *Let* $\boldsymbol{f} = (f_1, \dots, f_{n-1})$ *be a map of one real variable such that* $x \mapsto \mathbf{f}(x) := (x, \boldsymbol{f}(x))$ *is* l*-non-degenerate at some point* $x_0 \in \mathbb{R}$. *Then for any* r *indices* $1 \leq i_1 < \dots < i_r \leq n-1$ *the map* $x \mapsto (x, f_{i_1}(x), \dots, f_{i_r}(x))$ *is* l*-non-degenerate at* x_0.

We will be interested in three particular classes of functions associated with the map

$$\mathbf{f}(x) := (x, \boldsymbol{f}(x)) = (x, f_1(x), \dots, f_{n-1}(x)). \tag{4.1}$$

The first two are

$$\mathcal{F} := \{u_0 + \mathbf{u} \cdot \mathbf{f}(x) : \sum_{j=0}^{n} u_j^2 = 1\} \tag{4.2}$$

and

$$\mathcal{F}' := \{\mathbf{u}\cdot\mathbf{f}'(x) : \sum_{j=1}^{n} u_j^2 = 1\}\,, \tag{4.3}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ and the 'dot' represents the standard inner product. For these two classes we will will make use of the following statement.

Proposition 4.3. *Suppose that the map $x \mapsto \mathbf{f}(x)$ is l-non-degenerate at some point $x_0 \in \mathbb{R}$. Then there exists a constant $C > 0$ and a neighbourhood V of x_0 such that*

(a) *every function in $\mathcal{F}$ is $(C, \frac{1}{l})$-good on V;*

(b) *every function in $\mathcal{F}'$ is $(C, \frac{1}{l-1})$-good on V;*

(c) *for any interval $B \subset V$ there exists a constant $\rho_B > 0$ such that*

$$\inf_{f\in\mathcal{F}} \sup_{x\in B} |f(x)| \geq \rho_B \qquad \text{and} \qquad \inf_{f\in\mathcal{F}'} \sup_{x\in B} |f(x)| \geq \rho_B\,. \tag{4.4}$$

Proof. Parts (a) and (b) appear as Corollary 3.5 in [12]; see also [17, Proposition 3.4]. For part (c) we choose V sufficiently small so that $\mathbf{f}$ is non-degenerate everywhere on V. Then, note that the map $(u_0, \mathbf{u}) \mapsto \sup_{x\in B} |u_0 + \mathbf{u}\cdot\mathbf{f}(x)|$ is continuous and strictly positive. The latter is due to the linear independence of $1, x, f_1(x), \ldots, f_{n-1}(x)$ over $\mathbb{R}$ which in turn is a consequence of the non-degeneracy of $\mathbf{f}$ on $B \subset V$. Then

$$\inf_{(u_0,\mathbf{u})\in\mathbb{S}^n} \sup_{x\in B} |u_0 + \mathbf{u}\cdot\mathbf{f}(x)| := \rho_B > 0\,,$$

since we are taking the infimum of a positive continuous function over a compact set, namely $\mathbb{S}^n$, which is the unit (Euclidean) sphere in $\mathbb{R}^{n+1}$. This proves the first of the inequalities associated with (4.4). The proof of the second is similar once we make the observations that the non-degeneracy of $\mathbf{f}$ at x_0 implies the non-degeneracy of $\boldsymbol{f}' = (f_1', \ldots, f_{n-1}')$ at x_0. □

In what follows, given a map $\mathbf{g} = (g_1, g_2)$ of one real variable, the associated function

$$\widetilde{\nabla}\mathbf{g} := g_1 g_2' - g_1' g_2 \tag{4.5}$$

will be referred to as the *skew gradient of* $\mathbf{g}$. This notion was introduced in [12, §4] and the following statement concerning the skew gradient is a simplified version of Proposition 4.1 from [12].

Proposition 4.4. *Let U be an open interval, $x_0 \in U$ and let $\mathcal{G}$ be a family of C^l maps $\mathbf{g} : U \to \mathbb{R}^2$ such that*

$$\text{the family } \big\{g_i' : \mathbf{g} = (g_1, g_2) \in \mathcal{G},\ i = 1, 2\big\} \text{ is compact in } C^{l-1}(U). \tag{4.6}$$

Assume also that

$$\inf_{\mathbf{v}\in\mathbb{S}^1}\ \inf_{\mathbf{g}\in\mathcal{G}}\ \max_{1\le i\le l}|\mathbf{v}\cdot\mathbf{g}^{(i)}(x_0)|>0\,. \tag{4.7}$$

Then there exists a constant $C>0$ and a neighbourhood V of x_0 such that

(a) $|\widetilde{\nabla}\mathbf{g}|$ *is* $(C,\frac{1}{2l-1})$*-good on V for all $\mathbf{g}\in\mathcal{G}$,*

(b) *for every interval $B\subset V$ there exists $\rho_B>0$ such that for all $\mathbf{g}\in\mathcal{G}$*

$$\sup_{x\in B}|\widetilde{\nabla}\mathbf{g}(x)|\ge\rho_B\,.$$

The third class of functions that we will be interested in, associated with the map $\mathbf{f}$ defined by (4.1), is the class

$$\mathcal{G}:=\left\{\big(\mathbf{u}_1\cdot\mathbf{f}(x),u_0+\mathbf{u}_2\cdot\mathbf{f}(x)\big):\ \begin{matrix}\mathbf{u}_1,\mathbf{u}_2\in\mathbb{R}^n_1,\\ \mathbf{u}_1\cdot\mathbf{u}_2=0\end{matrix}\ \right\}. \tag{4.8}$$

In particular, we will make use of the following statement concerning the skew gradient of maps in $\mathcal{G}$. It follows on showing that Proposition 4.4 is applicable to the specific $\mathcal{G}$ given by (4.8).

Proposition 4.5. *Let U be an open interval, $x_0\in U$, $\mathbf{f}:U\to\mathbb{R}^n$ be l-non-degenerate at x_0 and let $\mathcal{G}$ be a family of C^l maps $\mathbf{g}:U\to\mathbb{R}^2$ given by* (4.8). *Then there exists a constant $C>0$ and a neighbourhood V of x_0 such that*

(a) *for every $\mathbf{g}\in\mathcal{G}$ the function $|\widetilde{\nabla}\mathbf{g}|$ is $(C,\frac{1}{2l-1})$-good on V;*

(b) *for any interval $B\subset V$ there exists a constant $\rho_B>0$ such that*

$$\inf_{\mathbf{g}\in\mathcal{G}}\ \sup_{x\in B}|\widetilde{\nabla}\mathbf{g}(x)|\ge\rho_B\,. \tag{4.9}$$

Proof. Since $\mathbf{f}$ is l-non-degenerate at x_0, there exists an open interval U centred at x_0 such that $\mathbf{f}$ is C^l on the closure of U. Hence the family of functions within (4.6), which is simply $\{\mathbf{u}\cdot\mathbf{f}':\mathbf{u}\in\mathbb{R}^n_1\}$, is compact in $C^{l-1}(U)$ due to the compactness of the unit sphere in $\mathbb{R}^n$. Thus the first hypothesis (4.6) is satisfied.

Next, given $\mathbf{g}=(\mathbf{u}_1\cdot\mathbf{f},u_0+\mathbf{u}_2\cdot\mathbf{f})\in\mathcal{G}$ and $\mathbf{v}=(v_1,v_2)\in\mathbb{R}^2_1$, we have that

$$\mathbf{v}\cdot\mathbf{g}=u_0'+\mathbf{u}'\cdot\mathbf{f}\,,$$

where

$$(u_0',\mathbf{u}')=v_1(0,\mathbf{u}_1)+v_2(u_0,\mathbf{u}_2)\,.$$

Since the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthonormal, we have that

$$\|\mathbf{u}'\|_2^2\ =\ v_1^2\|\mathbf{u}_1\|_2^2+v_2^2\|\mathbf{u}_2\|_2^2\ =\ v_1^2+v_2^2=1$$

and thus (4.7) immediately follows from the l-non-degeneracy of $\mathbf{f}(x)$ at x_0.

The upshot of the above is that the desired statements (a) and (b) of Proposition 4.5 now directly follow on applying Proposition 4.4. □

5. Proof of Proposition 3.1

The proof of Proposition 3.1 is split into several lemmas. We will use various properties of multi-vectors and their relations with linear subspaces, which can be found in [2, §3], [21] and [22]. We begin with an auxiliary statement that will be helpful for calculating $\phi_{I,\Gamma}(x)$.

Lemma 5.1. *Suppose that* $\Gamma = (\mathbf{v}_1, \dots, \mathbf{v}_r) \in \mathrm{Mat}^*_{\mathbb{R}}(n+1, r)$ *with* $1 \le r \le n$ *and* $\mathbf{v}_1, \dots, \mathbf{v}_r$ *denote the columns of* Γ. *Let* $I = \{i_1, \dots, i_r\} \subset \{1, \dots, n+1\}$ *and* $G_{i_1}(x), \dots, G_{i_r}(x)$ *be the corresponding rows of* $G(x)$. *Then for any collection* $\mathbf{a}_1, \dots, \mathbf{a}_{n+1-r} \in \mathbb{R}^{n+1}$ *of linearly independent rows such that* $\mathbf{a}_i \mathbf{v}_j = 0$ *for all* $i = 1, \dots, n+1-r$, $j = 1, \dots, r$ *and*

$$\|\mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_{n+1-r}\| = \|\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_r\| \tag{5.1}$$

we have that

$$\begin{aligned} |\phi_{I,\Gamma}(x)| &= \|G_{i_1}(x) \wedge \dots \wedge G_{i_r}(x) \wedge \mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_{n+1-r}\| \\ &= \left| \det\big(G_{i_1}(x), \dots, G_{i_r}(x), \mathbf{a}_1, \dots, \mathbf{a}_{n+1-r}\big) \right| . \end{aligned} \tag{5.2}$$

Furthermore, $\mathbf{a}_1, \dots, \mathbf{a}_{n+1-r}$ *can be taken to be integer if* $\Gamma \in \mathrm{Mat}^*_{\mathbb{Z}}(n+1, r)$.

Proof. Let

$$\mathbf{w} := \mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_r \tag{5.3}$$

and

$$\mathbf{m}(x) := G_{i_1}(x) \wedge \dots \wedge G_{i_r}(x). \tag{5.4}$$

Then, by the Laplace identity (see, for example, [2, Equation (3.3)] or [22, Lemma 5E]) it follows that

$$\phi_{I,\Gamma}(x) = \mathbf{m}(x) \cdot \mathbf{w} \,,$$

where the 'dot' represents the standard inner product on $\bigwedge^r(\mathbb{R}^{n+1})$. Let $\mathbf{w}^\perp$ be the Hodge dual of $\mathbf{w}$, see [2, §3] for its definition and properties. Since $\mathbf{w}$ is decomposable, so is $\mathbf{w}^\perp$. This means that

$$\mathbf{w}^\perp = \mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_{n+1-r} \tag{5.5}$$

for some linearly independent rows $\mathbf{a}_1, \dots, \mathbf{a}_{n+1-r}$ which form a basis of the linear subspace of $\mathbb{R}^{n+1}$ orthogonal to $\mathbf{v}_1, \dots, \mathbf{v}_r$. Equation (5.1) is a consequence of the Hodge operator being an isometry, see [2, §3.2]. Furthermore, it follows from [22, Lemma 5G] that it is possible to choose $\mathbf{a}_i$, for all $i = 1, \dots, n+1-r$, to be integer vectors in the case Γ is an integer matrix.

By duality (see [2, Equation (3.10)]), we have that

$$|\phi_{I,\Gamma}(x)| = |\mathbf{m}(x)\cdot\mathbf{w}| = |\mathbf{m}(x)\wedge\mathbf{w}^{\perp}|$$

whence (5.2) follows on substituting (5.4) and (5.5). □

Lemma 5.2. *With reference to Proposition 3.1*

- *Statement* (3.10) *holds if* $r = n-1$ *and* $I = \{1,\dots,n-1\}$,
- *Statement* (3.11) *holds if* $r = n-1$ *and* I *as above.*

Proof. Although, in the context of Proposition 3.1 we are ultimately interested in integer Γ, it will be necessary for the proof of Lemma 5.2 to consider Γ lying in the larger set $\mathrm{Mat}^*_{\mathbb{R}}(n+1,r)$. With this in mind, by Lemma 5.1 there exist linearly independent row-vectors $\mathbf{a}_1,\mathbf{a}_2\in\mathbb{R}^{n+1}$ such that

$$|\phi_{I,\Gamma}(x)| = |G_1(x)\wedge\ldots\wedge G_{n-1}(x)\wedge\mathbf{a}_1\wedge\mathbf{a}_2|.$$

For convenience, and in view of Lemma 5.1 without loss of generality, we take

$$\mathbf{a}_1 = (0,\mathbf{u}_1) \qquad\text{and}\qquad \mathbf{a}_2 = (u_0,\mathbf{u}_2)$$

such that $\mathbf{u}_1\cdot\mathbf{u}_2 = 0$, $\mathbf{u}_1\in\mathbb{R}^n_1$ and $\mathbf{u}_2\in\mathbb{R}^n$, where $\mathbf{u}_i = (u_{i,1},\dots,u_{i,n})$ for $i=1,2$. Thus, $|\phi_{I,\Gamma}(x)|$ is equal to the absolute value of the determinant of the following $(n+1)\times(n+1)$ matrix:

$$\Psi_x := \begin{pmatrix} f_1 - xf_1' & f_1' & -1 & 0 & \dots & 0\\ f_2 - xf_2' & f_2' & 0 & -1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ f_{n-1} - xf_{n-1}' & f_{n-1}' & 0 & 0 & \dots & -1\\ 0 & u_{1,1} & u_{1,2} & u_{1,3} & \dots & u_{1,n}\\ u_0 & u_{2,1} & u_{2,2} & u_{2,3} & \dots & u_{2,n} \end{pmatrix}, \tag{5.6}$$

To proceed, define the following auxiliary $(n+1)\times(n+1)$ matrix

$$\xi_x := \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0\\ x & 1 & 0 & 0 & \dots & 0\\ f_1 & f_1' & 1 & 0 & \dots & 0\\ f_2 & f_2' & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ f_{n-1} & f_{n-1}' & 0 & 0 & \dots & 1 \end{pmatrix} \tag{5.7}$$

Since $\det\xi_x = 1$, we have that

$$|\phi_{I,\Gamma}(x)| = |\det\Psi_x| = |\det(\Psi_x\xi_x)|. \tag{5.8}$$

On the other hand, we have that

$$\Psi_x\xi_x = \begin{pmatrix} 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 \\ \mathbf{u}_1\cdot\mathbf{f}(x) & \mathbf{u}_1\cdot\mathbf{f}'(x) & u_{1,2} & u_{1,3} & \dots & u_{1,n} \\ u_0+\mathbf{u}_2\cdot\mathbf{f}(x) & \mathbf{u}_2\cdot\mathbf{f}'(x) & u_{2,2} & u_{2,3} & \dots & u_{2,n} \end{pmatrix}, \tag{5.9}$$

where as usual $\mathbf{f}$ is given by (4.1).

Suppose for the moment that $\mathbf{u}_2=\mathbf{0}$. Then $u_0\neq 0$ and

$$|\phi_{I,\Gamma}(x)| = |u_0\mathbf{u}_1\cdot\mathbf{f}'(x)|,$$

which is a non-zero multiple of the absolute value of a function from the class $\mathcal{F}'$ defined by (4.3). Thus in this case we can apply Proposition 4.3 (together with Lemma 4.1) to deduce that $|\phi_{I,\Gamma}|$ is $(C,\frac{1}{2l-1})$-good on V for a suitably chosen constant $C>0$ and neighbourhood V of x_0. Now suppose that $\mathbf{u}_2\neq\mathbf{0}$. Then

$$\begin{aligned}|\phi_{I,\Gamma}(x)| &= \big|\widetilde{\nabla}\big(\mathbf{u}_1\cdot\mathbf{f}(x), u_0+\mathbf{u}_2\cdot\mathbf{f}(x)\big)\big| \\ &= \|\mathbf{u}_1\|_2\|\mathbf{u}_2\|_2\,\big|\widetilde{\nabla}\big(\|\mathbf{u}_1\|_2^{-1}\mathbf{u}_1\cdot\mathbf{f}(x), \|\mathbf{u}_2\|_2^{-1}u_0+\|\mathbf{u}_2\|_2^{-1}\mathbf{u}_2\cdot\mathbf{f}(x)\big)\big|.\end{aligned}$$

Note that

$$\big(\|\mathbf{u}_1\|_2^{-1}\mathbf{u}_1\cdot\mathbf{f}(x), \|\mathbf{u}_2\|_2^{-1}u_0+\|\mathbf{u}_2\|_2^{-1}\mathbf{u}_2\cdot\mathbf{f}(x)\big)\in\mathcal{G},$$

where $\mathcal{G}$ is the class of functions defined by (4.8). Hence, we can apply Proposition 4.5 (together with Lemma 4.1) to deduce that $|\phi_{I,\Gamma}|$ is $(C,\frac{1}{2l-1})$-good on V for a suitably chosen constant $C>0$ and neighbourhood V of x_0. This thereby completes the proof of the first part of the lemma.

We now turn our attention to the second part of the lemma. Since $\|\Gamma\|_\infty\geq 1$, the vector $\mathbf{w}$ defined by (5.3) satisfies $\|\mathbf{w}\|_2\geq 1$. By Propositions 4.3 and 4.5, it follows that for any ball $B\subset V$

$$\sup_{x\in B}|\phi_{I,\Gamma}(x)| = \sup_{x\in B}|\mathbf{m}(x)\cdot\mathbf{w}| = \|\mathbf{w}\|_2\sup_{x\in B}|\mathbf{m}(x)\cdot\mathbf{w}'|>0, \tag{5.10}$$

where $\mathbf{w}'=\mathbf{w}/\|\mathbf{w}\|_2$ is a unit decomposable multivector. Note that the set of decomposable unit multivectors $\mathbf{w}'\in\bigwedge^2(\mathbb{R}^{n+1})$ is compact and

$$\mathbf{w}\mapsto\sup_{x\in B}|\mathbf{m}(x)\cdot\mathbf{w}|$$

is strictly positive and continuous (see [2, p. 218]). Then taking the infimum in (5.10) over $\mathbf{w}'$ implies that the right hand side of (5.10) is bounded away from zero by a constant $\rho_B>0$. This completes the proof of the lemma. □

Lemma 5.3. *With reference to Proposition 3.1*

- *Statement* (3.10) *holds if* $r \le n-1$ *and* $I \subset \{1, \dots, n-1\}$,
- *Statement* (3.11) *holds if* $r \le n-1$ *for some* $I \subset \{1, \dots, n-1\}$.

Proof. With in the context of Proposition 3.1, we are given that $\Gamma \in \mathrm{Mat}^*_{\mathbb{Z}}(n+1, r)$. The fact that Γ is an integer is absolutely crucial in the proof of Lemma 5.3. In short, it allows us to make a reduction to a lower dimension statement to which Lemma 5.2 is applicable.

Fix any multiindex I such that $n \notin I$ and $n+1 \notin I$. Recall that the matrix $G(x)$ is defined by (3.3). Consider the auxiliary $(r+2) \times (r+2)$ matrix $\widetilde{G}(x)$ formed by the rows $i_1, \dots, i_r, n, n+1$ and columns $1, 2, i_1+2, \dots, i_r+2$ of $G(x)$. Also, consider the matrix $\widetilde{\Gamma}$ formed by the rows $1, 2, i_1+2, \dots, i_r+2$ of the matrix Γ. Observe that

$$\phi_{I,\Gamma}(x) = \det \widetilde{G}_{\widetilde{I}}(x)\widetilde{\Gamma}\,, \tag{5.11}$$

where $\widetilde{I} := \{1, \dots, r\}$. This is because when going from G_I to $\widetilde{G}_{\widetilde{I}}$ we simply cross out zero columns and so $G_I(x)\Gamma = \widetilde{G}_{\widetilde{I}}(x)\widetilde{\Gamma}$. Thus the desired properties of $\phi_{I,\Gamma}(x)$ can be investigated via the lower dimensional matrix $\widetilde{G}$, which has exactly the same structure as G. Indeed the matrix $\widetilde{G}$ is the analogue of G with the associated map $\mathbf{f}$ replaced by

$$\widetilde{\mathbf{f}}(x) \to (x, f_{i_1}(x), \dots, f_{i_r}(x))\,. \tag{5.12}$$

Note that by Lemma 4.2, the map $\widetilde{\mathbf{f}}$ is l-non-degenerate at x_0. With this in mind and without loss of generality assuming that $\phi_{I,\Gamma}$ is given by (5.11), we are in the position to prove the lemma.

Suppose to start with that $\widetilde{\Gamma}$ has rank $< r$. Then, we have that $\phi_{I,\Gamma}$ is identically zero and it follows that $|\phi_{I,\Gamma}|$ is (C,α)-good for any choice of C and α. Now suppose that $\widetilde{\Gamma}$ has rank exactly r, that is $\widetilde{\Gamma} \in \mathrm{Mat}^*_{\mathbb{Z}}(r+2, r)$. Then, on applying Lemma 5.2 with $\widetilde{\mathbf{f}}$ in place of $\mathbf{f}$, $\widetilde{\Gamma}$ in place of Γ, $r+1$ in place of n, $\widetilde{G}(x)$ in place of $G(x)$ and $\widetilde{I}$ in place of I, we complete the proof of the first part of Lemma 5.3. To verify the second part it remains to note that, since $\operatorname{rank}\Gamma = r$, there is always a choice of $I = \{i_1 < \dots < i_r\} \subset \{1, \dots, n-1\}$ such that $\widetilde{\Gamma}$ defined above has rank r. □

Lemma 5.4. *With reference to Proposition 3.1*

- *Statement* (3.10) *holds if* $r = n$ *and* $I = \{1, \dots, n\}$,
- *Statement* (3.11) *holds if* $r = n$ *and* I *as above.*

Proof. By Lemma 5.1, $|\phi_{I,\Gamma}(x)| = |\det \Psi_x|$, where

$$\Psi_x := \begin{pmatrix} f_1 - xf_1' & f_1' & -1 & 0 & \dots & 0 \\ f_2 - xf_2' & f_2' & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1} - xf_{n-1}' & f_{n-1}' & 0 & 0 & \dots & -1 \\ x & -1 & 0 & 0 & \dots & 0 \\ a_0 & a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix} \tag{5.13}$$

for some non-zero integer vector $(a_0, \dots, a_n)$. With ξ_x given by (5.7), we obtain that

$$\Psi_x\xi_x = \begin{pmatrix} 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 \\ 0 & -1 & 0 & 0 & \dots & 0 \\ a_0 + xa_1 + \sum_{j=1}^{n-1} a_{j+1}f_j(x) & a_1 + \sum_{j=1}^{n-1} a_{j+1}f_j' & a_2 & a_3 & \dots & a_n \end{pmatrix}.$$

Hence

$$|\phi_{I,\Gamma}(x)| = |\det \Psi_x\xi_x| = |a_0 + xa_1 + \sum_{j=1}^{n-1} a_{j+1}f_j(x)|$$

is a constant multiple of a function from the class $\mathcal{F}$ defined by (4.2). Since $(a_0, \dots, a_n)$ is a non-zero integer vector, the constant multiple in question is ≥ 1. Therefore, the lemma readily follows from Proposition 4.3 (parts (a) and (c)) together with Lemma 4.1. □

Lemma 5.5. *With reference to Proposition 3.1*

- *Statement* (3.10) *holds if* $r \leq n$, $n \in I$ *and* $n+1 \notin I$.

Proof. To start with observe that when $r = 1$, we necessarily have that $I = \{n\}$ and thus $G_I(x)\Gamma$ is either identically zero or is a non-zero linear function. In the former case it easily follows that $|\phi_{I,\Gamma}|$ is (C, α)-good for any C and α. In the latter case, $\phi_{I,\Gamma}$ it is a multiple of an element of the class $\mathcal{F}$ defined by (4.2) and so by Lemma 4.1, $|\phi_{I,\Gamma}|$ is $(C, \frac{1}{2l-1})$-good on some neighbourhood of x_0.

Without loss of generality, we assume that $r > 1$. Fix any multiindex I such that $n \in I$ and $n+1 \notin I$. Recall that the matrix $G(x)$ is defined by (3.3). Consider the auxiliary $(r+1)\times(r+1)$ matrix $\widetilde{G}(x)$ formed by the rows $i_1,\ldots,i_r,n+1$ and columns $1,2,i_1+2,\ldots,i_{r-1}+2$ of $G(x)$. Note that $i_r=n$. Also, consider the matrix $\widetilde{\Gamma}$ formed by the rows $1,2,i_1+2,\ldots,i_{r-1}+2$ of the matrix Γ. Observe that

$$\phi_{I,\Gamma}(x)=\det\widetilde{G}_{\widetilde{I}}(x)\widetilde{\Gamma}\,, \tag{5.14}$$

where $\widetilde{I}=\{1,\ldots,r\}$. Thus the desired properties of $\phi_{I,\Gamma}(x)$ can be investigated via the lower dimensional matrix $\widetilde{G}$, which has exactly the same structure as G. Indeed the matrix $\widetilde{G}$ is the analogue of G with the associated $\mathbf{f}$ replaced by

$$\widetilde{\mathbf{f}}(x)\to(x,f_{i_1}(x),\ldots,f_{i_{r-1}}(x))\,. \tag{5.15}$$

Note that by Lemma 4.2, the map $\widetilde{\mathbf{f}}$ is l-non-degenerate at x_0. With this in mind and without loss of generality assuming that $\phi_{I,\Gamma}$ is given by (5.14), we are in the position to prove the lemma.

Suppose to start with that $\widetilde{\Gamma}$ has rank $<r$. Then, we have that $\phi_{I,\Gamma}$ is identically zero and it follows that $|\phi_{I,\Gamma}|$ is (C,α)-good for any choice of C and α. Now suppose that $\widetilde{\Gamma}$ has rank exactly r, that is $\widetilde{\Gamma}\in\mathrm{Mat}^*_{\mathbb{Z}}(r+1,r)$. Then, on applying Lemma 5.4 with $\widetilde{\mathbf{f}}$ in place of $\mathbf{f}$, $\widetilde{\Gamma}$ in place of Γ, $r+1$ in place of n, $\widetilde{G}(x)$ in place of $G(x)$ and $\widetilde{I}$ in place of I, completes the proof of Lemma 5.5. □

Lemma 5.6. *With reference to Proposition 3.1*

- *Statement* (3.10) *holds if* $r=n$ *and* $I=\{1,\ldots,n-1,n+1\}$.

Proof. By Lemma 5.1, $|\phi_{I,\Gamma}(x)|=|\det\Psi_x|$, where

$$\Psi_x:=\begin{pmatrix} f_1-xf_1' & f_1' & -1 & 0 & \ldots & 0\\ f_2-xf_2' & f_2' & 0 & -1 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ f_m-xf_m' & f_m' & 0 & 0 & \ldots & -1\\ 1 & 0 & 0 & 0 & \ldots & 0\\ a_0 & a_1 & a_2 & a_3 & \ldots & a_n \end{pmatrix} \tag{5.16}$$

for some non-zero integer vector $(a_0,\ldots,a_n)$. With ξ_x given by (5.7), we obtain that

$$\Psi_x\xi_x = \begin{pmatrix} 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ a_0 + xa_1 + \sum_{k=1}^{n-1} a_{k+1}f_k & a_1 + \sum_{k=1}^{n-1} a_{k+1}f'_k & a_2 & a_3 & \dots & a_n \end{pmatrix}.$$

Hence

$$|\phi_{I,\Gamma}(x)| = |\det \Psi_x\xi_x| = |a_1 + \sum_{k=1}^{n-1} a_{k+1}f'_k(x)|$$

is either identically zero or a constant multiple of a function from $\mathcal{F}'$. In the latter case the claim of the lemma readily follow from Proposition 4.3(b) combined with Lemma 4.1. In the case $|\phi_{I,\Gamma}(x)|$ is identically zero the claim is trivial. Indeed, in that case $|\phi_{I,\Gamma}|$ is (C,α)-good for any choice of C and α. □

Lemma 5.7. *With reference to Proposition 3.1*

- *Statement* (3.10) *holds if* $r \le n$, $n \notin I$ *and* $n+1 \in I$.

Proof. To start with observe that when $r = 1$, then we necessarily have that $I = \{n+1\}$ and thus $G_I(x)\Gamma$ is a constant, hence $|\phi_{I,\Gamma}|$ is (C,α)-good for any C and α.

Without loss of generality, we assume that $r > 1$. Fix any multiindex I such that $n \notin I$ and $n+1 \in I$. Recall that the matrix $G(x)$ is defined by (3.3). Consider the auxiliary $(r+1)\times(r+1)$ matrix $\widetilde{G}(x)$ formed by the rows $i_1,\dots,i_{r-1},n,n+1$ and columns $1,2,i_1+2,\dots,i_{r-1}+2$ of $G(x)$. Note that $i_r = n+1$. Also, consider the matrix $\widetilde{\Gamma}$ formed by the rows $1,2,i_1+2,\dots,i_{r-1}+2$ of the matrix Γ. Observe that

$$\phi_{I,\Gamma}(x) = \det \widetilde{G}_{\widetilde{I}}(x)\widetilde{\Gamma}\,, \tag{5.17}$$

where $\widetilde{I} = \{1,\dots,r-1,r+1\}$. Thus the desired properties of $\phi_{I,\Gamma}(x)$ can be investigated via the lower dimensional matrix $\widetilde{G}$, which has exactly the same structure as G. Indeed the matrix $\widetilde{G}$ is the analogue of G with the associated $\mathbf{f}$ replaced by

$$\widetilde{\mathbf{f}}(x) \to (x, f_{i_1}(x),\dots,f_{i_{r-1}}(x))\,. \tag{5.18}$$

Note that by Lemma 4.2, the map $\widetilde{\mathbf{f}}$ is l-non-degenerate at x_0. With this in mind and without loss of generality assuming that $\phi_{I,\Gamma}$ is given by (5.17), we are in the position to prove the lemma.

Suppose to start with that $\widetilde{\Gamma}$ has rank $< r$. Then, we have that $\phi_{I,\Gamma}$ is identically zero and it follows that $|\phi_{I,\Gamma}|$ is (C,α)-good for any choice of C and α. Now suppose that $\widetilde{\Gamma}$ has rank exactly r, that is $\widetilde{\Gamma}\in \mathrm{Mat}^*_{\mathbb{Z}}(r+1,r)$. Then, on applying Lemma 5.6 with $\widetilde{\mathbf{f}}$ in place of $\mathbf{f}$, $\widetilde{\Gamma}$ in place of Γ, $r+1$ in place of n, $\widetilde{G}(x)$ in place of $G(x)$ and $\widetilde{I}$ in place of I, completes the proof of Lemma 5.7. □

Completion of the proof of Proposition 3.1. First of all note that Property (3.10) has already been established in Lemma 5.3 if $I\cap\{n,n+1\}=\emptyset$, in Lemma 5.5 if $I\cap\{n,n+1\}=\{n\}$ and in Lemma 5.7 if $I\cap\{n,n+1\}=\{n+1\}$. If $I\cap\{n,n+1\}=\{n,n+1\}$, then $\det\left(G_I(x)\Gamma\right)$ is readily seen to be constant, which is thus $(C,1/(2l-1))$-good. Therefore, for any $1\le r\le n$ and any $\Gamma\in\mathrm{Mat}^*_{\mathbb{Z}}(n+1,r)$ Property (i) holds for all choices of I.

Regarding Property (ii) of Proposition 3.1, if $r=n$ then it is established in Lemma 5.4 and if $r<n$ it is established in Lemma 5.3. □

Remark 5.1. The constants C and ρ_B that arise from the various lemmas proved in this section may in principle depend on the choice of I. However, since there are only finitely many different choices of I both the constants in question can be made independent of I. Indeed ρ_B has to be taken as the minimum while C has to be taken to be the maximum of all the possible values over all different choices of I. The fact that the maximum choice for C works for all I is a consequence of Lemma 4.1.

6. Proof of Theorem 2.3

Let h be given by (3.4) and let $x_0\in\mathbb{R}$ be such that $\mathbf{f}$ is non-degenerate at x_0. Then, by (3.6), (3.8), Proposition 3.1 and Lemma 4.1, there exists a neighbourhood $\mathcal{U}$ of x_0 such that for any collection of linearly independent integer points $\mathbf{v}_1,\dots,\mathbf{v}_r$ $(1\le r\le n)$ the map

$$x\mapsto \|h(x)\mathbf{v}_1\wedge\ldots\wedge h(x)\mathbf{v}_r\|_\infty \text{ is } (C,\tfrac{1}{2l-1})\text{-good on } 3^{n+1}\mathcal{U}$$

and

$$\sup_{x\in B}\|h(x)\mathbf{v}_1\wedge\ldots\wedge h(x)\mathbf{v}_r\|_\infty\ \ge\ c^{\frac{r}{n+1}}\,\rho_B\min_I\Phi_I\,,$$

where the minimum is taken over $I=\{i_1,\dots,i_r\}\subset\big\{1,\dots,\max\{r,n-1\}\big\}$ and Φ_I is given by (3.9). It follows from the definition of Φ_I, that for $r\le n$

$$\min_I\Phi_I\ \ge\ \min\{\psi^{-r},Q\}$$

and consequently

$$\sup_{x\in B}\|h(x)\mathbf{v}_1\wedge\ldots\wedge h(x)\mathbf{v}_r\|_\infty \;\ge\; c^{\frac{r}{n+1}}\,\rho_B\,\min\{\psi^{-r},Q\}\;\ge\;1$$

provided that $Q \ge Q_B$ for some sufficiently large Q_B and $\psi \le \psi_B$ for some sufficiently small ψ_B. If $r = n+1$, then trivially the map

$$x\mapsto \|h(x)\mathbf{v}_1\wedge\ldots\wedge h(x)\mathbf{v}_r\|_\infty = \|\mathbf{v}_1\wedge\ldots\wedge\mathbf{v}_r\|_\infty \ge 1$$

is constant and hence (C,α)-good for the same choice of α and some absolute constant $C > 0$.

The upshot of the above is that all the conditions of Theorem KM are met for any ball $B \subset \mathcal{U}$, some constants $C, \alpha > 0$ and $\rho = 1/(n+1)$. Therefore, by (3.2) and (3.5), we obtain that

$$|B\setminus\mathcal{G}(c,Q,\psi)| \;\le\; (n+1)\,C6^{n+1}\left(\frac{c^{1/(n+1)}}{1/(n+1)}\right)^{\alpha}|B|\,, \tag{6.1}$$

where $\alpha = \frac{1}{2l-1}$. The latter inequality implies (2.24) for a suitably chosen $c > 0$ that is independent of B. This thereby completes the proof of the theorem.

Appendix A. Deduction of Theorem 1.3 from Theorem 1.1

A.1. **Ubiquitous systems close to a curve in** $\mathbb{R}^n$ **.** In this subsection we recall the definitions are facts from [5, §§3.2,3.3]. Let I_0 be an interval in $\mathbb{R}$, $n \ge 2$ and $\mathcal{R} := (R_\alpha)_{\alpha\in\mathcal{J}}$ be a family of 'resonant' points R_α of $\mathbb{R}^n$ indexed by an infinite countable set $\mathcal{J}$. Let $\beta : \mathcal{J} \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a positive function on $\mathcal{J}$. Thus, the function β attaches a 'weight' β_α to the resonant point R_α. Further, let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ denote a function satisfying $\lim_{t\to\infty}\rho(t) = 0$ and is usually referred to as the *ubiquitous function*. Also $B(x, r)$ will denote the ball (or rather the interval) in $\mathbb{R}$ centred at x of radius r.

For a point R_α in $\mathcal{R}$, let $R_{\alpha,k}$ represent the k'th coordinate of R_α. Thus, $R_\alpha := (R_{\alpha,1}, R_{\alpha,2}, \ldots, R_{\alpha,n})$. Given a function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$, let $\mathcal{R}_{\mathcal{C}}(\Phi)$ denote the sub-family of resonant points R_α in $\mathcal{R}$ which are "Φ–close" to the curve $\mathcal{C} = \mathcal{C}_{\mathbf{f}} := \{(x, f_2(x), \ldots, f_n(x)) : x \in I_0\}$ where $\mathbf{f} = (f_1, \ldots, f_n) : I_0 \to \mathbb{R}^n$ is a continuous map with $f_1(x) = x$ and I_0 is an interval in $\mathbb{R}$. Thus,

$$\mathcal{R}_{\mathcal{C}}(\Phi) := (R_\alpha)_{\alpha\in\mathcal{J}_{\mathcal{C}}(\Phi)}\,,$$

where

$$\mathcal{J}_{\mathcal{C}}(\Phi) := \{\alpha\in\mathcal{J} : R_{\alpha,1}\in I_0,\ \max_{2\le k\le n}|f_k(R_{\alpha,1}) - R_{\alpha,k}| < \Phi(\beta_\alpha)\}\,. \tag{A.1}$$

We will also denote by $\mathcal{R}_1$ the family of first co-ordinates of the points in $\mathcal{R}_{\mathcal{C}}(\Phi)$; that is

$$\mathcal{R}_1 \;:=\; (R_{\alpha,1})_{\alpha\in J_{\mathcal{C}}(\Phi)}\quad.$$

By definition, $\mathcal{R}_1$ is a subset of the interval I_0. Finally, for $t \in \mathbb{N}$ let $\mathcal{J}_\mathcal{C}(\Phi, t) := \{\alpha \in \mathcal{J}_\mathcal{C}(\Phi) : \beta_\alpha \leq 2^t\}$ and assume that $\#\mathcal{J}_\mathcal{C}(\Phi, t)$ is always finite. The following appears as Definition 3 in [5].

Definition A.1 (Ubiquitous systems near curves)**.** *The system $(\mathcal{R}_\mathcal{C}(\Phi), \beta)$ is called locally ubiquitous with respect to ρ if there exists an absolute constant $\kappa > 0$ such that for any interval $B \subseteq I_0$*

$$\liminf_{t\to\infty} \left| \bigcup_{\alpha\in\mathcal{J}_\mathcal{C}(\Phi,t)} \left(B(R_{\alpha,1}, \rho(2^t)) \cap B \right) \right| \ \geq \ \kappa\, |B| \ . \tag{A.2}$$

Next, given an approximating function $\widetilde{\Psi}$, let $\Lambda(\mathcal{R}_\mathcal{C}(\Phi), \beta, \widetilde{\Psi})$ denote the set $x \in I_0$ for which the system of inequalities

$$\begin{cases} |x - R_{\alpha,1}| \ < \ \widetilde{\Psi}(\beta_\alpha) \\ \max\limits_{2\leq k\leq n} |f_k(x) - R_{\alpha,k}| \ < \ \widetilde{\Psi}(\beta_\alpha) + \Phi(\beta_\alpha) \end{cases} ,$$

is simultaneously satisfied for infinitely many $\alpha \in \mathcal{J}_\mathcal{C}(\Phi)$. The following lemma is merely a combination of Lemmas 3 and 4 established in [5] with some simplifications.

Lemma A.2. *Consider the curve $\mathcal{C} := \{(x, f_2(x), \ldots, f_n(x)) : x \in I_0\}$, where $f_2, \ldots, f_n$ are locally Lipshitz in a finite interval I_0. Let Φ and $\widetilde{\Psi}$ be approximating functions. Suppose that $(\mathcal{R}_\mathcal{C}(\Phi), \beta)$ is a locally ubiquitous system with respect to ρ and let $\widetilde{\Psi}$ be such that $\widetilde{\Psi}(2^{t+1}) \leq \frac{1}{2}\widetilde{\Psi}(2^t)$ for t sufficiently large. Then for any $s \in (0, 1]$ we have that*

$$\mathcal{H}^s\Big(\Lambda(\mathcal{R}_\mathcal{C}(\Phi), \beta, \widetilde{\Psi})\Big) \ = \ \mathcal{H}^s(I_0) \qquad \text{if} \qquad \sum_{t=1}^{\infty} \frac{\widetilde{\Psi}(2^t)^s}{\rho(2^t)} \ = \ \infty \ .$$

A.2. **Proof of Theorem 1.3.** In line with the setup of Theorem 1.3, let $\boldsymbol{\theta} = (\lambda_1, \gamma_1, \ldots, \gamma_{n-1}) \in \mathbb{R}^n$, $\Psi : (0, +\infty) \to (0, +\infty)$ be any monotonic function such that $q\Psi(q)^{(2n-1)/3} \to \infty$ as $q \to \infty$. Without loss of generality we may assume that $\Psi(q) < 1$ for all q. Let $\mathcal{M}$ be a non-degenerate curve in $\mathbb{R}^n$ and without loss of generality we will assume that $\mathcal{M} = \mathbf{f}(I_0)$ for some interval I_0 and a non-degenerate map $\mathbf{f} = (x, f_2, \ldots, f_n)$. Fix any point $x_0 \in I_0$ and let $\mathcal{U}$ be the interval around x_0 arising from Theorem 1.1 and K_0 and C_0 be the constants arising from Theorem 1.1. Without loss of generality we may assume that $\mathcal{U} = I_0$. Suppose that $0 < s \leq 1$ (as we noted above the case $s > 1$ is trivial) and suppose that

$$\sum_{q=1}^{\infty} q^n \left(\frac{\Psi(q)}{q} \right)^{s+n-1} \ = \ \infty \, . \tag{A.3}$$

With reference to §A.1, let

$$\mathcal{J} = \{(q, a_1, b_1, \ldots, b_{n-1}) : q \in \mathbb{N},\ a_1, b_1, \ldots, b_{n-1} \in \mathbb{Z}\},$$

$$R_\alpha = \left(\frac{a_1+\lambda_1}{q}, \frac{b_1+\gamma_1}{q}, \dots, \frac{b_{n-1}+\gamma_{n-1}}{q}\right) \qquad \text{and} \qquad \beta_\alpha = q$$

when $\alpha = (q, a_1, b_1, \dots, b_{n-1}) \in \mathcal{J}$. Furthermore, for $Q \in \mathbb{N}$ let $Q = 2^t$, $\psi = \frac{1}{4}\Phi(Q)$,

$$\rho(Q) = \frac{C_0}{(\frac{1}{4}\Psi(Q))^{n-1}Q^2}$$

and

$$\widetilde{\Psi}(Q) = \Phi(Q) := \frac{\Psi(Q)}{2Q}\,.$$

Now let $B \subset I_0$ be any interval. Then, in view of (1.5) and (A.1) we have that

$$\mathcal{R}(Q,\psi,B,\boldsymbol{\theta}) \subset \mathcal{J}_{\mathcal{C}}(\Phi,t)$$

and so in view of (1.6) we have that

$$\Delta(Q,\psi,B,\boldsymbol{\theta},\rho(Q)) \ \subset\ \textstyle\bigcup_{\alpha\in\mathcal{J}_{\mathcal{C}}(\Phi,t)} \big(B(R_{\alpha,1},\rho(2^t)) \cap B\big)\,. \tag{A.4}$$

Now, with the view to applying Theorem 1.1, let $Q \geq Q_B$ be sufficiently large and note that condition (1.8) in our case becomes

$$K_0 Q^{-\frac{3}{2n-1}} \leq \tfrac{1}{4}\Psi(Q) < 1\,. \tag{A.5}$$

This is true in view of our assumptions on Ψ; in particular, the fact that $q\Psi(q)^{(2n-1)/3} \to \infty$ as $q \to \infty$. Hence, by Theorem 1.1 and (A.4), we get that (A.2) holds with $\kappa = \frac{1}{2}$ and so $(\mathcal{R}_{\mathcal{C}}(\Phi),\beta)$ is locally ubiquitous with respect to ρ given above.

Now we wish to apply Lemma A.2. First of all observe that $\widetilde{\Psi}(2^{t+1}) \leq \frac{1}{2}\widetilde{\Psi}(2^t)$ for t since Ψ is monotonically decreasing and $\widetilde{\Psi}(Q) = \frac{\Psi(Q)}{2Q}$. Next observe that

$$\sum_{t=1}^{\infty} \frac{\widetilde{\Psi}(2^t)^s}{\rho(2^t)} \ =\ 2^{2-s-2n} C_0^{-1} \sum_{t=1}^{\infty} \Psi(2^t)^{s+n-1} 2^{(2-s)t}\,.$$

Since Ψ is monotonic, by Cauchy condensation test, this sum is divergent if and only if

$$\sum_{q=1}^{\infty} q^n \left(\frac{\Psi(q)}{q}\right)^{s+n-1} \ =\ \sum_{q=1}^{\infty} \Psi(q)^{s+n-1} q^{1-s} = \infty\,.$$

Thus, assuming the divergence condition in (1.12), we have the divergence condition in Lemma A.2 and conclude that

$$\mathcal{H}^s\Big(\Lambda(\mathcal{R}_{\mathcal{C}}(\Phi),\beta,\widetilde{\Psi})\Big) \ =\ \mathcal{H}^s(I_0)\,. \tag{A.6}$$

It remain to note that $\Lambda(\mathcal{R}_{\mathcal{C}}(\Phi),\beta,\widetilde{\Psi}) \subset \mathbf{f}^{-1}(\mathcal{S}_n(\Psi,\boldsymbol{\theta}))$ and that f is locally bi-Lipschitz and thus f preserves sets of zero/full/infinite Hausdorff measure. Thus, (A.6) implies (1.12) and this complete the proof of the main part of Theorem 1.3.

For the furthermore part of Theorem 1.3, let $\tau(\Psi)$ be as in Theorem 1.1 and

$$s := \frac{n+1}{\tau(\Psi)+1+\varepsilon} - n + 1 \in (0,1)$$

for arbitrary sufficiently small $\varepsilon > 0$. By the definition of $\tau(\Psi)$ we have that

$$\Psi(q) \geq q^{-\tau(\Psi)-\varepsilon}$$

for infinitely many $q \in \mathbb{N}$. Then,

$$\sum_{q=1}^{\infty} q^n \left(\frac{\Psi(q)}{q}\right)^{s+n-1} \geq \sum_{q=1}^{\infty} q^n (q^{-(\tau(\Psi)+1+\varepsilon})^{s+n-1} = \sum_{q=1}^{\infty} q^{-1} = \infty\,.$$

and hence

$$\mathcal{H}^s(\mathcal{S}_n(\Psi,\boldsymbol{\theta}) \cap \mathcal{M}) = \infty\,.$$

Therefore,

$$\dim\big(\mathcal{S}_n(\Psi,\boldsymbol{\theta}) \cap \mathcal{M}\big) \geq s = \min\left\{1,\ \frac{n+1}{\tau(\Psi)+1+\varepsilon} - n + 1\right\}$$

and on letting $\varepsilon \to 0$ we obtain (1.13). This competes the proof of the theorem.

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