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In this paper, we give an accessible procedure to construct QEC codes that not only completely avoid coherent phase errors, but also support fault-tolerant quantum computation. Namely, we concatenate stabilizer codes $\mathcal{C}_{Stab}$ with a length two repetition code $\mathcal{C}_{REP2}$, and apply a bit-flip on half of the qubits. We can also naturally interpret these codes within the codeword stabilized (CWS) framework[15,16], thereby extending the utility of CWS codes beyond a purely theoretical setting.

Amplitude damping (AD) errors model energy relaxation, and accurately describe errors in many physical systems. By concatenating the four-qubit AD code[17] with the dual-rail code[18], we construct an eight-qubit CE code that corrects a single AD error. We provide this code's QEC circuits (Figs. 3 and 4), and analyze its potential as a quantum memory under the AD noise model (Fig. 5).

Our work paves the way towards integrating CE codes with mainstream QEC codes. By doubling the number of qubits required, we make any quantum code immune against coherent phase errors. When coherent phase errors are a dominant source of errors, we expect CE codes to significantly reduce fault-tolerant overheads.

## RESULTS

### Hybridizing stabilizer and CE codes

Coherent phase errors can arise from the collective interaction of identical qubits with a classical field. Since the collective Hamiltonian of non-interacting identical qubits is proportional to $S^z = Z_1 + \cdots + Z_N$ where $Z_j$ flips the $j$th qubit's phase, we model coherent phase errors with unitaries of the form $U_\theta = \exp(-i\theta S^z)$. Here, $\theta$ depends on both the interacting field's magnitude and the qubits' energy levels.

Using any CE code, we can completely avoid coherent phase errors. This is because such codes must lie within an eigenspace of $S^z$, which is spanned by the computational basis states $|\mathbf{x}\rangle = |x_1\rangle \otimes \cdots \otimes |x_N\rangle$ for which the excitation number, given by the Hamming weight $\mathrm{wt}(\mathbf{x}) = x_1 + \cdots + x_N$ of $\mathbf{x}$, is constant. The simplest CE code is the dual-rail code[18], $\mathcal{C}_{KLM}$, with logical codewords $|0_{KLM}\rangle = |01\rangle$ and $|1_{KLM}\rangle = |10\rangle$.

[1]Department of Physics and Astronomy, University of Sheffield, Sheffield, UK. ✉email: oyingkai@gmail.com
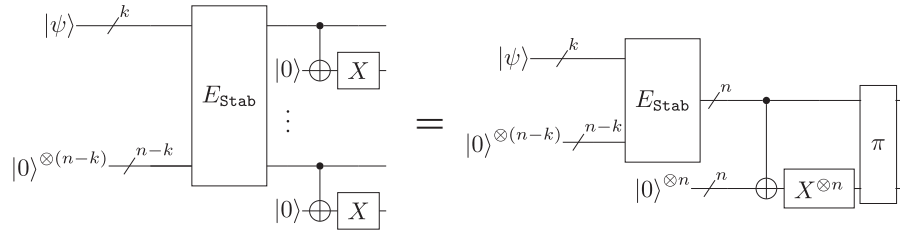
**Fig. 1 Encodings of $\mathcal{C}_{Stab,KLM}$ from the encoding $E_{Stab}$ of $\mathcal{C}_{Stab}$.** On the right side, CNOTs apply transversally to each pair of control and target qubits in the code blocks. The permutation $\pi$ maps the $j$th qubit in the first block of $n$ qubits to the $(2j-1)$th qubit and the $j$th qubit in the second block of $n$ qubits to the $(2j)$th qubit.



$$(a)\ L_{rep2}(U)$$
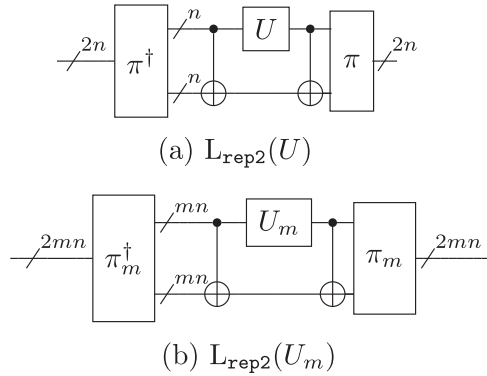


$$(b)\ L_{rep2}(U_m)$$

**Fig. 2 Logical operators for $\mathcal{C}_{Stab,REP2}$.** Given single-qubit and multi-qubit logical operators of $\mathcal{C}_{Stab}$ denoted by $U$ and $U_m$, respectively, we obtain corresponding logical operators for $\mathcal{C}_{Stab,REP2}$ in **a** and **b**, respectively. The permutation $\pi_m$ maps the $j$th qubit in the first block of $mn$ qubits to the $(2j-1)$th qubit and the $j$th qubit in the second block of $mn$ qubits to the $(2j)$th qubit.

However, $\mathcal{C}_{KLM}$ cannot correct any errors. Therefore, we concatenate it with an $[[n, k, d]]$ stabilizer code $\mathcal{C}_{Stab}$ to obtain a code $\mathcal{C}$ with encoding circuit given in Fig. 1. Then $\mathcal{C}$ is an $[[2n, k, d]]$ QEC code that is also impervious to coherent phase errors. Now, concatenating any state $\sum_{\mathbf{x}\in\{0,1\}^n} a_{\mathbf{x}}|\mathbf{x}\rangle \in \mathcal{C}_{Stab}$ with $\mathcal{C}_{KLM}$ yields $\sum_{\mathbf{x}\in\{0,1\}^n} a_{\mathbf{x}}|\varphi(\mathbf{x})\rangle$, where $\varphi((x_1, x_2, ..., x_{n-1}, x_n)) = (x_1, 1 - x_1, x_2, 1 - x_2, ..., x_{n-1}, 1 - x_{n-1}, x_n, 1 - x_n)$. Since $wt(\varphi(\mathbf{x})) = n$ for every $\mathbf{x} \in \{0, 1\}^n$, it follows that the concatenated state must be an eigenstate of $S^z$ with the same eigenvalue. Hence, $\mathcal{C}_{Stab,KLM}$ is a CE code, and therefore avoids coherent phase errors.

The code $\mathcal{C}_{Stab,KLM}$ is very similar to $\mathcal{C}_{Stab,REP2}$, which is $\mathcal{C}_{Stab}$ concatenated with a length two repetition code $\mathcal{C}_{REP2}$ that maps $|0\rangle$ to $|00\rangle$ and $|1\rangle$ to $|11\rangle$. Since $\mathcal{C}_{Stab,KLM} = R\mathcal{C}_{Stab,REP2}$ where $R = (I \otimes X)^{\otimes n}$, and $I$ and $X$ denote the identity and bit-flip operations on a qubit respectively, $\mathcal{C}_{Stab,KLM}$ is equivalent to $\mathcal{C}_{Stab,REP2}$ up to the Pauli rotation $R$ and we call $\mathcal{C}_{Stab,KLM}$ a rotated-stabilizer code.

We can also cast $\mathcal{C}_{Stab,KLM}$ within the CWS framework by deriving its word stabilizer and word operators. Since $\mathcal{C}_{Stab,KLM}$ and $\mathcal{C}_{Stab,REP2}$ are equivalent up to $R$, it suffices to derive $\mathcal{C}_{Stab,REP2}$'s word stabilizer and word operators. Namely, $\mathcal{C}_{Stab,KLM}$ and $\mathcal{C}_{Stab,REP2}$ have identical word stabilizers generated by the stabilizer and logical $Z$ operators of $\mathcal{C}_{Stab,REP2}$. Moreover, the word operators $w_1, ..., w_{2^k}$ of $\mathcal{C}_{Stab,REP2}$ are its logical $X$ operators and the word operators $\mathcal{C}_{Stab,KLM}$ are $Rw_1, ..., Rw_{2^k}$. We supply explicit constructs of the word stabilizer and word operators of $\mathcal{C}_{Stab,KLM}$ in "Methods" section.

The code $\mathcal{C}_{Stab,KLM}$ inherits its logical operators from the logical operators of $\mathcal{C}_{Stab,REP2}$. Given any single-qubit logical operator $U$ on $\mathcal{C}_{Stab}$, the corresponding unitary $L_{REP2}(U)$ on $\mathcal{C}_{Stab,REP2}$ is given in Fig. 2a. Then the corresponding logical operator on $\mathcal{C}_{Stab,KLM}$ is $\tilde{U} = RL_{REP2}(U)R$. Similarly, given an $m$-qubit logical operator $U_m$ on $\mathcal{C}_{Stab}$, the corresponding logical operator on $\mathcal{C}_{Stab,REP2}$ is $L_{REP2}(U_m)$ (Fig. 2b), and the corresponding logical operator on $\mathcal{C}_{Stab,KLM}$ is

$R^{\otimes m}L_{REP2}(U_m)R^{\otimes m}$. If $U$ is a tensor product of single-qubit Pauli gates, then $\tilde{U}$ is also a tensor product of single-qubit Pauli gates. Hence, if $\mathcal{C}_{Stab}$ has transversal gates comprising of single-qubit Paulis, then $\mathcal{C}_{Stab,KLM}$ also has corresponding transversal gates of the same form. If $U_m$ is a diagonal unitary in the computational basis, then $\tilde{U}_m = \pi_m^\dagger(U_m \otimes I^{\otimes nm})\pi_m$ is also the logical operator on $\mathcal{C}_{Stab,KLM}$.

To design error-correction procedures for $\mathcal{C}_{Stab,KLM}$, we leverage on the error-correction procedures of $\mathcal{C}_{Stab,REP2}$ and the interpretation that $\mathcal{C}_{Stab,KLM}$ is $\mathcal{C}_{Stab,REP2}$ with an effective $R$ error. We can extract the syndrome of a Pauli error $E$ acting on $\mathcal{C}_{Stab,KLM}$ by measuring eigenvalues of Pauli observables. These Pauli observables can be generators associated with $\mathcal{C}_{Stab,REP2}$'s stabilizer, and these generators are derived easily from the generators of $\mathcal{C}_{Stab}$; if $G_1, ..., G_{n-k}$ are $\mathcal{C}_{Stab}$'s stabilizer's generators, then $\overline{G}_1, ..., \overline{G}_{2n-k}$ generate $\mathcal{C}_{Stab,REP2}$'s stabilizer, where $\overline{G}_i = L_{REP2}(G_i)$ for $i = 1, ..., n - k$ and $\overline{G}_{n-k+j} = Z_{2j-1}Z_{2j}$ for $j = 1, ..., n$. We complete the QEC procedure by using measured eigenvalues of $\overline{G}_1, ..., \overline{G}_{2n-k}$ to estimate the Pauli error $E'$ that could have occurred, and reverse its effect.

The generator $\overline{G}_j$'s eigenvalue on $E|\psi\rangle$ for $|\psi\rangle \in \mathcal{C}_{Stab,KLM}$ when measured is $\theta_j = (-1)^{s_j}$ for some $s_j = 0, 1$. Here, $s_j = 0$ when $\overline{G}_j$ and $ER$ commute and $s_j = 1$ otherwise. Now, denote the eigenvalue of $\overline{G}_j$ on $R|\psi_{KLM}\rangle \in \mathcal{C}_{Stab,REP2}$ as $(-1)^{r_j}$ for some $r_j = 0, 1$. Whenever $E = I^{\otimes 2n}$, we have $\mathbf{r} \oplus \mathbf{s} = 0$ where $\mathbf{r} = (r_1, ..., r_{2n-k})$ and $\mathbf{s} = (s_1, ..., s_{2n-k})$. Using $\mathbf{r} \oplus \mathbf{s}$, we estimate the error $E'$ that could have occurred on $\mathcal{C}_{Stab,KLM}$. For this, we use any decoder $\mathtt{Dec}_{Stab,REP2}$ that maps a syndrome vector obtained from a corrupted state of $\mathcal{C}_{Stab,REP2}$ to an estimated Pauli error. Such a decoder $\mathtt{Dec}_{Stab,REP2}$ can be a maximum likelihood decoder[19,20] or a belief propagation decoder[21–23]. Explicitly, our code $\mathcal{C}_{Stab,KLM}$'s decoder has the form

$$Dec_{Stab,KLM}(\mathbf{s}) = Dec_{Stab,REP2}(\mathbf{r} \oplus \mathbf{s}), \tag{1}$$

and thereby inherits its performance from the decoder $\mathtt{Dec}_{Stab,REP2}$ on the stabilizer code $\mathcal{C}_{Stab,REP2}$.

Now let us introduce some terminology related to the decoding of stabilizer codes. Denoting the single-qubit Pauli operators as $I, X$, the phase-flip operator $Z$, and $Y = iXZ$, the set of $n$-qubit Pauli operators is $\{I, X, Y, Z\}^{\otimes n}$. Define $\mathrm{bin}(P) = (\mathbf{a}|\mathbf{b})$ as a $2n$-bit binary vector where $\mathbf{a} = (a_1, ..., a_n)$ and $\mathbf{b} = (b_1, ..., b_n)$ are $n$-bit binary vectors such that $P = wX^{a_1}Z^{b_1} \otimes \cdots \otimes X^{a_n}Z^{b_n}$ for some $w = \pm 1, \pm i$. Given any two Pauli matrices $P$ and $P'$ with binary representations $\mathrm{bin}(P) = (\mathbf{a}, \mathbf{b})$ and $\mathrm{bin}(P') = (\mathbf{a}', \mathbf{b}')$, their symplectic inner product[24] over $\mathbb{F}_2$ is defined to be $\langle \mathrm{bin}(P), \mathrm{bin}(P') \rangle_{sy} = \mathbf{a} \cdot \mathbf{b}' + \mathbf{a}' \cdot \mathbf{b}$.

To see how to decode our concatenated code, note that

$$r_j = \langle \mathrm{bin}(\overline{G}_j), \mathrm{bin}(R) \rangle_{sy},$$
$$s_j = \langle \mathrm{bin}(\overline{G}_j), (\mathrm{bin}(E) + \mathrm{bin}(R)) \rangle_{sy}, \tag{2}$$

By linearity of the inner product, it follows that $r_j \oplus s_j = \langle \mathrm{bin}(\overline{G}_j), \mathrm{bin}(E) \rangle_{sy}$. This shows that $(-1)^{r_j \oplus s_j}$ is equal to the eigenvalue of $G_j$ when measured on $R|\psi\rangle$, the latter of which is a state in $\mathcal{C}_{Stab,REP2}$, from which we can deduce (1).

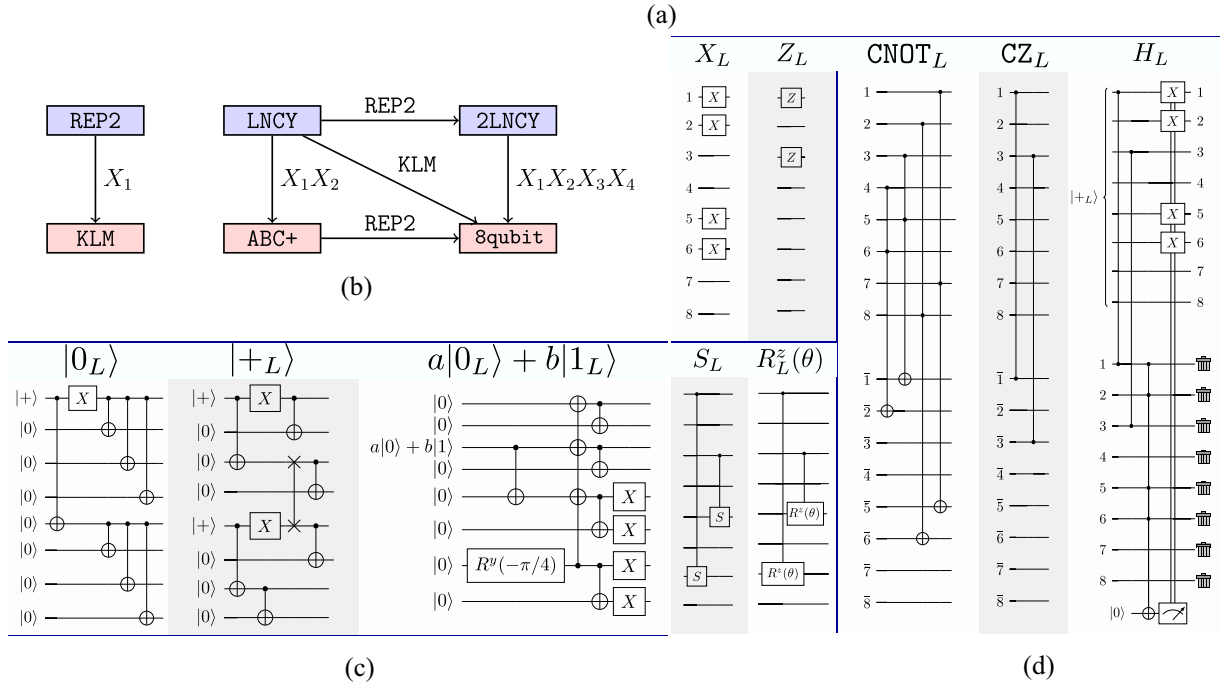| QEC code | Logical zero | Logical one | Type |
|----------|--------------|-------------|------|
| REP2 | $\lvert 00 \rangle$ | $\lvert 11 \rangle$ | STAB |
| KLM | $\lvert 01 \rangle$ | $\lvert 10 \rangle$ | CE |
| LNCY | $\lvert 0000 \rangle + \lvert 1111 \rangle$ | $\lvert 0101 \rangle + \lvert 1010 \rangle$ | STAB |
| 2LNCY | $\lvert 00000000 \rangle + \lvert 11111111 \rangle$ | $\lvert 00110011 \rangle + \lvert 11001100 \rangle$ | STAB |
| ABC+ | $\lvert 1100 \rangle + \lvert 0011 \rangle$ | $\lvert 0110 \rangle + \lvert 1001 \rangle$ | CE |
| 8qubit | $\lvert 11110000 \rangle + \lvert 00001111 \rangle$ | $\lvert 00111100 \rangle + \lvert 11000011 \rangle$ | CE |

(a)

(b)

(c)

(d)

**Fig. 3  An amplitude damping CE code, its relationship with other codes, encoding circuits and logical computations. a** A table of various CE and stabilizer codes. The logical codewords are listed without their normalization factors. REP2 is the two-qubit repetition code, KLM is the dual-rail code[18], LNCY code is the four-qubit AD code[17], [a] up to a permutation of qubits, ABC+ is a four-qubit CE code[6], 2LNCY is LNCY concatenated with REP2 and is a step to obtain our construct, and the 8qubit code is our eight-qubit code. **b** We depict the relationship between the codes in **a** pictorially. Here $X_j$ denotes a bit ip on the $j$th qubit. **c** State preparation circuits for $\mathcal{C}_{8qubit}$, such as $\lvert 0 \rangle$ and $\lvert +_L \rangle$ and the logical encoding of an arbitrary logical codestate. **d** Logical computations on $\mathcal{C}_{8qubit}$ are depicted. Here, $R^z(\theta) = e^{iZ\theta}$. The logical Hadamard is performed via logical gate-teleportation after preparing a logical $\lvert +_L \rangle$ ancilla. [a]The four-qubit AD code is also a subcode of the [[4,2,2]] code.

When stochastic errors evolve under the influence of $U_\theta = \exp(-i\theta S^z)$, their weight is preserved. First, note that

$$U_\theta = \prod_{j=1}^{N} \exp(-i\theta Z_j) = \exp(-i\theta Z)^{\otimes N}. \quad (3)$$

Then, for any $N$-qubit Pauli matrix $P = P_1 \otimes \cdots \otimes P_N$, we have that

$$\tilde{P} = U_\theta P U_\theta^\dagger = \bigotimes_{j=1}^{N} \exp(-i\theta Z) P_j \exp(i\theta Z). \quad (4)$$

When $P_j = I$ or $Z$, we clearly have $\exp(-i\theta Z)P_j\exp(i\theta Z) = P_j$. When $P_j = X$ or $Y$, we have $\exp(-i\theta Z)P_j\exp(i\theta Z) = \exp(-2i\theta Z)P_j$. For any value of $\theta$, $\exp(-2i\theta Z)X$ and $\exp(-2i\theta Z)Y$ are never the identity operator. Hence we can see that the weight of $\tilde{P}$ is identical to the weight of $P$. By performing stabilizer measurements, the error $\tilde{P}$ gets projected randomly onto some Pauli of weight equal to the weight of $P$, if this weight is no greater than half of the code's distance, it can be corrected according to the earlier-described decoding procedure.

We now show that $\mathcal{C}_{Stab,KLM}$ has a positive fault-tolerant threshold when $\mathcal{C}_{Stab}$ is a Calderbank–Shor–Steane (CSS) code[25,26] that encodes a single logical qubit and has transversal logical Pauli $I$, $X$, $Y$, and $Z$ gates given by $\bar{I} = I^{\otimes n}$, $\bar{X} = X^{\otimes n}$, $\bar{Y} = Y^{\otimes n}$ and $\bar{Z} = Z^{\otimes n}$, respectively. (also with transversal Hadamard.) First, $\mathcal{C}_{Stab,KLM}$ has transversal logical Pauli and controlled-not (CNOT) gates. Then

$\mathcal{C}_{Stab,REP2}$ has transversal logical $X$ and $Z$ gates given by $\bar{X}_{REP2} = \bar{X}^{\otimes 2} = X^{\otimes 2n}$ and $\bar{Z}_{REP2} = \pi(\bar{Z} \otimes \bar{I})\pi^\dagger$, respectively, and logical CNOT gate $\overline{\text{CNOT}}_{REP2}$ given by $2n$ transversal CNOT gates. Thus, $\mathcal{C}_{Stab,KLM}$ has its logical $X$ and $Z$ operators given by $\bar{X}_{KLM} = R\bar{X}_{REP2}R = X^{\otimes 2n}$ and $\bar{Z}_{KLM} = R\bar{Z}_{REP2}R = (-1)^n Z_{REP2}$, respectively. Furthermore, the logical CNOT gate of $\mathcal{C}_{Stab,KLM}$ has the form $\overline{\text{CNOT}}_{KLM} = (R \otimes R)\overline{\text{CNOT}}_{REP2}(R \otimes R) = \overline{\text{CNOT}}_{REP2}$. Second, since we can perform these transversal CNOTs and have stabilizers that correspond to a CSS code, we can measure syndromes and logical Paulis fault-tolerantly using Steane's method for CSS codes[27]. Relying on gate-teleportation techniques[28], we can implement all Clifford and non-Clifford gates fault-tolerantly. Since the fault-tolerant logical operations will have a finite number of circuit components, using the method of counting malignant combinations in extended rectangles[29] yields a positive fault-tolerant threshold for stochastic noise.

## An amplitude damping CE code

The simplest CE code that detects AD errors is the four-qubit $\mathcal{C}_{ABC+}$ code[6]. AD errors are introduced by an AD channel $\mathcal{A}_\gamma$, which has Kraus operators $A_0 = \lvert 0 \rangle\langle 0 \rvert + \sqrt{1-\gamma}\lvert 1 \rangle\langle 1 \rvert$ and $A_1 = \sqrt{\gamma}\lvert 0 \rangle\langle 1 \rvert$. These Kraus operators model the damping an excited state's amplitude and the relaxation of an excited state to the ground state with probability $\gamma$. While $\mathcal{C}_{ABC+}$ detects a single AD error, it cannot correct any AD errors. Other CE codes that can correct some AD
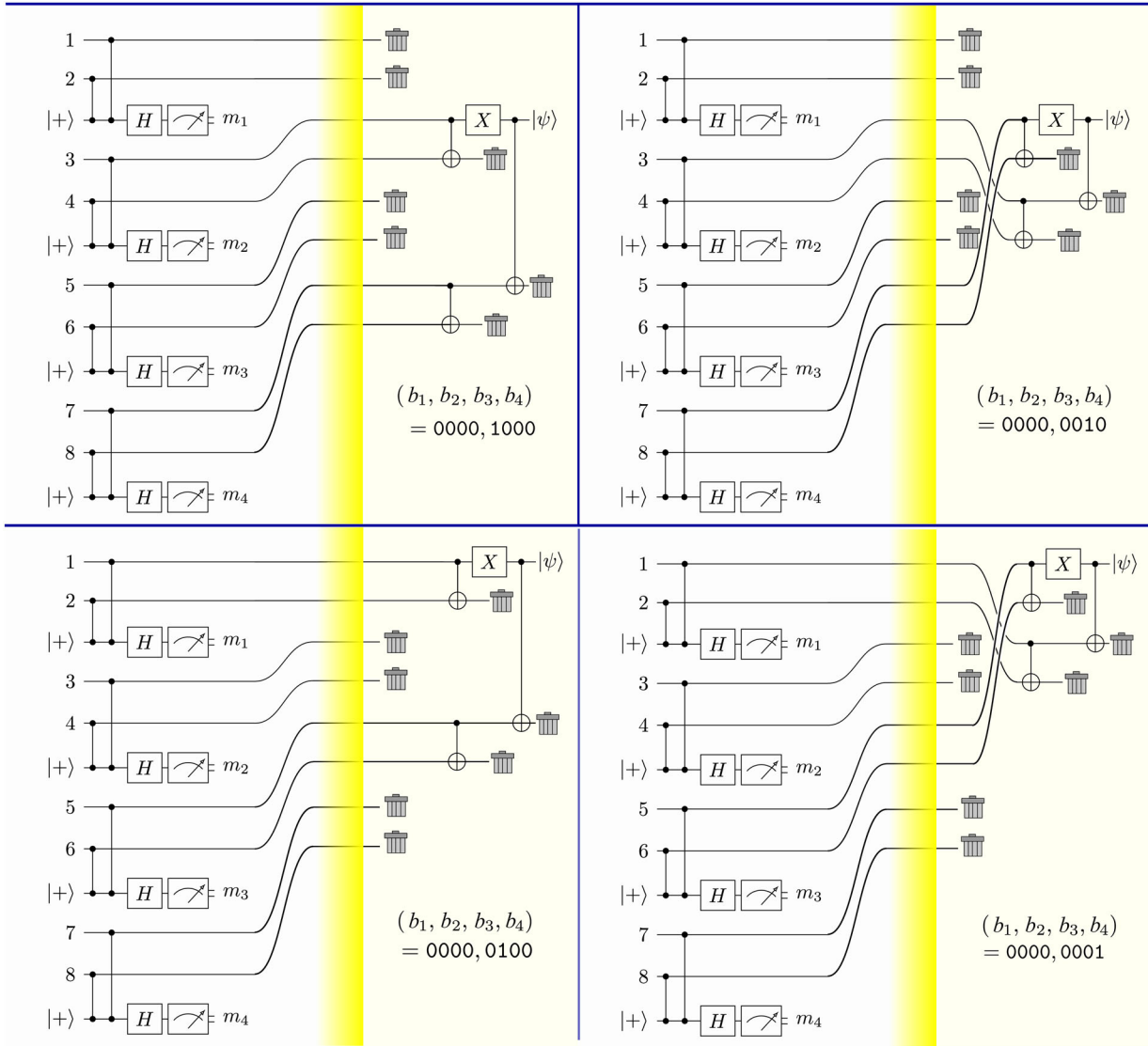
**Fig. 4 Syndrome extraction and decoding of $\mathcal{C}_{8qubit}$.** The syndrome vector is $\mathbf{b} = (b_1; b_2; b_4; b_4) = (1 - (m_1; m_2; m_3; m_4))/2$. If the Hamming weight of the syndrome vector is one, we can still correctly decode the logical qubit. For this, we discard four qubits and subsequently employ the same decoding circuit up to a permutation. If the Hamming weight of $\mathbf{b}$ is 0, we can use any of the above decoding circuits.

errors have been designed, but either have overly complicated encoding and QEC circuits[3], or lack explicit QEC circuits[5,7–10].

Here, we present a CE code that is the concatenation of the four-qubit AD code $\mathcal{C}_{LNCY}$[17] with $\mathcal{C}_{KLM}$, and permute the qubits to get $\mathcal{C}_{8qubit}$ with logical codewords

$$|0_L\rangle = (|11110000\rangle + |00001111\rangle)/\sqrt{2}$$
$$|1_L\rangle = (|00111100\rangle + |11000011\rangle)/\sqrt{2}. \qquad (5)$$

We elucidate the connection between $\mathcal{C}_{LNCY}$, $\mathcal{C}_{ABC+}$, $\mathcal{C}_{8qubit}$, $\mathcal{C}_{KLM}$, and $\mathcal{C}_{REP2}$ in Fig. 3b. We prove that $\mathcal{C}_{8qubit}$ corrects a single AD error by verifying that the Knill–Laflamme QEC criterion[30] holds with respect to the Kraus operators $K_1, ..., K_8$ and $A_0^{\otimes 8}$ where $K_a$ denotes an $n$-qubit operator that applies $A_1$ on the $a$th qubit and $A_0$ on each of the remaining qubits. The simplicity of $\mathcal{C}_{8qubit}$ allows for the direct construction of a simple error-correction strategy for AD errors, without referring to the properties of $\mathcal{C}_{LNCY}$, $\mathcal{C}_{ABC+}$, and $\mathcal{C}_{KLM}$.

In Fig. 3, we illustrate accessible constructs for $\mathcal{C}_{8qubit}$'s encoding circuits and logical computations. In Fig. 4, we give decoding procedures when an AD error is detected. We measure the eigenvalues $m_1, m_2, m_3,$ and $m_4$ of the respective operators $Z_1Z_2$, $Z_3Z_4, Z_5Z_6,$ and $Z_7Z_8$ to determine if any AD error has occurred.

Denoting $b_a = (1 - m_a)/2$ for $a = 1, ..., 4$, we have five correctible outcomes with respect to the syndrome vector $\mathbf{b} = (b_1, b_2, b_3, b_4)$. When $\mathbf{b} = \mathbf{0}$, the codespace is damped uniformly and no AD error has occurred. When $\mathbf{b}$ has a Hamming weight equal to one, each logical codeword is mapped to a unique product state, and we can ascertain that exactly one AD error must have occurred. When $b_a = 1$ and the other syndrome bits are zero, an AD error must have occurred on either the $(2a - 1)$th or the $(2a)$th qubit. Since the effect of an AD error on the $(2a - 1)$th and $(2a)$th qubit is identical, this makes $\mathcal{C}_{8qubit}$ a degenerate quantum code with respect to AD errors, and explains why $\mathcal{C}_{8qubit}$ has five correctible outcomes as opposed to nine if it were non-degenerate. The elegant structure of the four corrupted codespaces with a single AD error aids our construction of decoding circuits for $\mathcal{C}_{8qubit}$ (see details in the "Methods" section).

We illustrate $\mathcal{C}_{8qubit}$'s performance as a quantum memory assuming perfect encoding and decoding and that AD errors only occur during the memory storage. We calculate probabilities $\epsilon$ and $\epsilon_{base}$ of having uncorrectable AD errors occurring on $\mathcal{C}_{8qubit}$ and an unprotected qubit after $T$ applications of $\mathcal{A}_\delta^{\otimes 8}$ and $\mathcal{A}_\delta$ respectively. Since the transmissivity $(1 - \delta)$ of an AD channel $\mathcal{A}_\delta$
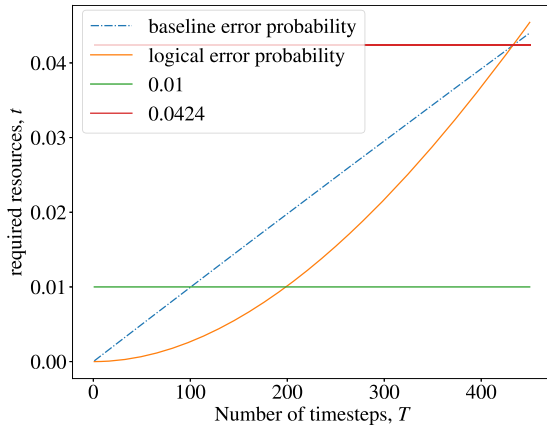
**Fig. 5 Failure probability of using $\mathcal{C}_{8qubit}$ and an unprotected qubit versus the number of timesteps when exposed to AD errors.** The baseline error probability is $\epsilon_{base}$ and the logical error probability is $\epsilon$. At each timestep, $\mathcal{A}_\delta$ afflicts each qubit with $\delta = 10^{-4}$. When the target failure probability is 0.01, using $\mathcal{C}_{8qubit}$ increases the number of timesteps $T$ from about 100 to 200. When the target failure probability is over 0.0424, there is no advantage in using $\mathcal{C}_{8qubit}$.

is multiplicative under composition, $(1 - \epsilon_{base}) = (1-\delta)^T$ and

$$\epsilon = 1 - (1 - \epsilon_{base})^8 - 8\epsilon_{base}(1 - \epsilon_{base})^7 \leq 28\epsilon_{base}^2. \quad (6)$$

Whenever $28\epsilon_{base}^2 \leq \epsilon_{base}$, it is advantageous to use $\mathcal{C}_{8qubit}$. Hence, whenever $T \leq T^*$, where

$$T^* = \frac{\log(27/28)}{\log(1 - \delta)}, \quad (7)$$

using $\mathcal{C}_{8qubit}$ is advantageous as compared to leaving a qubit unprotected (Fig. 5).

## DISCUSSION

When coherent phase errors occur more frequently than stochastic errors, we expect CE codes to outperform generic QEC codes. For future work, the numerical fault-tolerant thresholds of our codes can be calculated when the noise model is a convex combination of stochastic errors and coherent phase errors. In particular, the outer codes could be chosen to be surface codes[31–33], quantum LDPC codes[34,35] and Aliferis–Preskill concatenated codes for biased noise[36]. One can also study other choices for the inner codes in our construction to obtain concatenated codes with different structures and residing in different types of decoherence-free subspaces. For instance, we can consider other CE codes[37], quantum codes that avoid exchange errors[38–42], and quantum codes that avoid other different errors[4,5,8,18,43].

## METHODS
### Our CE code as a CWS code
Here, we derive the word stabilizer and word operators of our CE code $\mathcal{C}_{Stab,KLM}$. Now denote $S_{Stab}$ as the stabilizer of $\mathcal{C}_{Stab}$ and $G_1, \ldots, G_{n-k}$ as its generators. Then the operators $\mathsf{L}_{REP2}(G_i), Z_{2j-1}Z_{2j}$ generate $\mathcal{C}_{Stab,REP2}$'s stabilizer where $i = 1, \ldots, n-k$ and $j = 1, \ldots, n$. Denoting the logical $X$ and $Z$ operators of $\mathcal{C}_{Stab}$ as $\overline{X}_1, \ldots, \overline{X}_k$ and $\overline{Z}_1, \ldots, \overline{Z}_k$ respectively, the logical $X$ and $Z$ operators of $\mathcal{C}_{Stab,REP2}$ are given by $\mathsf{L}_{REP2}(\overline{X}_1), \ldots, \mathsf{L}_{REP2}(\overline{X}_k)$ and $\mathsf{L}_{REP2}(\overline{Z}_1), \ldots, \mathsf{L}_{REP2}(\overline{Z}_k)$, respectively. Since $\mathcal{C}_{Stab,REP2}$ is a stabilizer code, its word stabilizer of $\mathcal{C}_{Stab,REP2}$ is

$$W = \left\{ S_{Stab,REP2}^a \prod_{j=1}^k \mathsf{L}_{REP2}(\overline{Z}_j)^{z_j} : a, z_1, \ldots, z_k = 0, 1 \right\}. \quad (8)$$

Since the word stabilizer of $\mathcal{C}_{Stab,KLM}$ is identical to the word stabilizer of $\mathcal{C}_{Stab,REP2}$, the word stabilizer of $\mathcal{C}_{Stab,KLM}$ is then given by $W$.

Clearly, the word operators of $\mathcal{C}_{Stab,REP2}$ are generated by $\mathsf{L}_{REP2}(\overline{X}_1), \ldots, \mathsf{L}_{REP2}(\overline{X}_k)$. Hence, the word operators of $\mathcal{C}_{Stab,KLM}$ are

$$w_{(x_1, \ldots, x_k)} = R \prod_{j=1}^k \mathsf{L}_{REP2}(\overline{X}_j)^{x_j} \quad (9)$$

where $x_1, \ldots, x_k = 0, 1$.

### An amplitude damping CE code: additional details
We now explain the connection between the codes $\mathcal{C}_{LNCY}$, $\mathcal{C}_{ABC+}$, $\mathcal{C}_{8qubit}$, $\mathcal{C}_{KLM}$, and $\mathcal{C}_{REP2}$ as illustrated in Fig. 4b. Now recall that the four-qubit amplitude damping code[17] has logical codewords

$$|0_{LNCY}\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2} \quad (10)$$

$$|1_{LNCY}\rangle = (|1100\rangle + |0011\rangle)/\sqrt{2}. \quad (11)$$

Concatenating this with the dual-rail code $\mathcal{C}_{KLM}$ gives the code

$$|0_{LNCY,KLM}\rangle = (|01010101\rangle + |10101010\rangle)/\sqrt{2} \quad (12)$$

$$|1_{LNCY,KLM}\rangle = (|10100101\rangle + |01011010\rangle)/\sqrt{2}. \quad (13)$$

It is visually easier to work with a code if we collect the odd and even qubits in separate blocks of four qubits. We can achieve this by applying the permutation $\pi^\dagger$, which maps qubits 1, 3, 5, 7 to qubits 1, 2, 3, 4 and qubits 2, 4, 6, 8 to qubits 5, 6, 7, 8, to get our code with logical codewords

$$|0_L\rangle = (|00001111\rangle + |11110000\rangle)/\sqrt{2} \quad (14)$$

$$|1_L\rangle = (|11000011\rangle + |00111100\rangle)/\sqrt{2}. \quad (15)$$

Note that the above code can be obtained from the four-qubit code $\mathcal{C}_{ABC+}$ with logical codewords

$$|0_{ABC+}\rangle = (|0011\rangle + |1100\rangle)/\sqrt{2} \quad (16)$$

$$|1_{ABC+}\rangle = (|1001\rangle + |0110\rangle)/\sqrt{2}, \quad (17)$$

after concatenation with $\mathcal{C}_{REP2}$. Note that by concatenating $\mathcal{C}_{LNCY}$ with $\mathcal{C}_{REP2}$, we get a concatenated code $\mathcal{C}_{2LNCY} = \mathcal{C}_{LNCY} \circ \mathcal{C}_{REP2}$ with logical codewords

$$\begin{aligned} |0_{2LNCY}\rangle &= (|00000000\rangle + |11111111\rangle)/\sqrt{2}, \\ |1_{2LNCY}\rangle &= (|00110011\rangle + |11001100\rangle)/\sqrt{2}. \end{aligned} \quad (18)$$

Since the stabilizer code $\mathcal{C}_{2LNCY}$ is equivalent to $\mathcal{C}_{8qubit}$ up to a Pauli rotation given by $X^{\otimes 4} \otimes I^{\otimes 4}$, we can interpret $\mathcal{C}_{8qubit}$ as a rotated concatenated stabilizer code.

To encode an arbitrary single-qubit logical state into $\mathcal{C}_{8qubit}$, we concatenate the encoding circuits of $\mathcal{C}_{LNCY}$ and $\mathcal{C}_{REP2}$, and apply a Pauli rotation. Quantum circuits can be further simplified when encode the logical stabilizer states $|0_L\rangle$ and $|+_L\rangle = (|0_L\rangle + |1_L\rangle)/\sqrt{2}$.

To show that our QEC code spanned by $|0_L\rangle$ and $|1_L\rangle$, corrects single AD errors, it suffices to verify the Knill–Laflamme QEC conditions. In particular, we show that for $i, j = 0, 1$ and $a, b = 1, \ldots, 8$, we have $\langle i_L|K_a K_b|j_L\rangle = \delta_{i,j}\delta_{a,b}g_a$ for some real number $g_a$. Now let us explain the effects of correctible AD errors on $\mathcal{C}_{8qubit}$. Recall that the correctible AD errors are given by $K_0 = A_0^{\otimes 8}, K_1 = A_1 \otimes A_0^{\otimes 7}, K_2 = A_0 \otimes A_1 \otimes A_0^{\otimes 6}, \ldots, K_7 = A_0^{\otimes 6} \otimes A_1 \otimes A_0$, and $K_8 = A_0^{\otimes 7} \otimes A_1$. Then we can see the following.

1. $K_0|0_L\rangle = (1-\gamma)^2|0_L\rangle$
   $K_0|1_L\rangle = (1-\gamma)^2|1_L\rangle$.
2. $K_1|0_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|01110000\rangle$
   $K_1|1_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|01000011\rangle$.
3. $K_2|0_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|10110000\rangle$
   $K_2|1_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|10000011\rangle$.
4. $K_3|0_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|11010000\rangle$
   $K_3|1_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|00011100\rangle$.
5. $K_4|0_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|11100000\rangle$
   $K_4|1_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|00101100\rangle$.

6. $K_5|0_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|00000111\rangle$

    $K_5|1_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|00110100\rangle.$

7. $K_6|0_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|00001011\rangle$

    $K_6|1_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|00111000\rangle.$

8. $K_7|0_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|00001101\rangle$

    $K_7|1_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|11000001\rangle.$

9. $K_8|0_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|00001110\rangle$

    $K_8|1_L\rangle = \sqrt{\gamma}\sqrt{(1-\gamma)^3}|11000010\rangle.$

In the above, we can see that the effect of $K_{2j-1}$ is identical to $K_{2j}$ for $j = 1, \ldots, 4$. Hence there are only five unique correctible outcomes that correspond to the correctible errors $K_0, K_1, K_3, K_5$ and $K_7$. Each of these correctible outcomes are clearly orthogonal. Hence to perform quantum error correction, it suffices to rotate the orthogonal corrupted codespaces back to the original codespace.

Now, to extract the error syndrome, it suffices to measure the stabilizers $Z_{2j-1}, Z_{2j}$ for $j = 1, 2, 3, 4$. These stabilizer measurements leave the codespace afflicted with correctible AD errors unchanged, and measure the parity of the $(2j-1)$th and $(2j)$th qubits. We can then make the following decisions.

1. If the parity of the all blocks is even, then we can ascertain that no AD error has occured, which corresponds to the effect of the Kraus operator $K_0$.
2. If the parity of the first and second qubit is odd, while the parity of the remaining blocks is even, then we can ascertain that either $K_1$ or $K_2$ has occured.
3. If the parity of the third and fourth qubit is odd, while the parity of the remaining blocks is even, then we can ascertain that either $K_3$ or $K_4$ has occured.
4. If the parity of the fifth and sixth qubit is odd, while the parity of the remaining blocks is even, then we can ascertain that either $K_5$ or $K_6$ has occured.
5. If the parity of the seventh and eight qubit is odd, while the parity of the remaining blocks is even, then we can ascertain that either $K_7$ or $K_8$ has occured.

The structure of the corrupted codespaces allows us to decode them into a physical qubit by first discarding four qubits, and subsequently employing the same decoding circuit up to a permutation.

## DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

## CODE AVAILABILITY

The code used in this work is available from the corresponding author upon reasonable request.

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## AUTHOR CONTRIBUTIONS

Y.O. contributed to all aspects of the manuscript.

## COMPETING INTERESTS

The author declares no competing interests.

## ADDITIONAL INFORMATION

**Correspondence** and requests for materials should be addressed to Y.O.

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