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τ -EXCEPTIONAL SEQUENCES

ASLAK BAKKE BUAN AND BETHANY ROSE MARSH

Dedicated to the memory of Ragnar-Olaf Buchweitz

ABSTRACT. We introduce the notions of τ -exceptional and signed τ -exceptional sequences for any finite dimensional algebra. We prove that for a fixed algebra of rank n , and for any positive integer $t \leq n$, there is a bijection between the set of signed τ -exceptional sequences of length t , and (basic) ordered support τ -rigid objects with t indecomposable direct summands. If the algebra is hereditary, our notions coincide with exceptional and signed exceptional sequences. The latter were recently introduced by Igusa and Todorov, who constructed a similar bijection in the hereditary setting.

INTRODUCTION

An object in an abelian or triangulated category is said to be *exceptional* if its endomorphism algebra is a division ring and it has no self-extensions of any degree. Exceptional sequences are sequences of exceptional objects satisfying certain orthogonality conditions involving the vanishing of Hom and Ext-groups. They were first introduced in an algebraic geometry setting [4, 18, 19] (see also [31]). This motivated their consideration in the context of the representation theory of finite dimensional hereditary algebras (such as path algebras of quivers) [13, 30].

The length of an exceptional sequence of finite-dimensional modules over an algebra is at most the number of simple modules. If we have equality, the sequence is said to be *complete*.

Complete sequences always exist over hereditary algebras. Moreover, there is in this case a transitive braid group action [13, 30], making it possible to find all exceptional sequences and hence also all exceptional modules. Hence one can hope for a classification of such modules using combinatorial methods.

However, for an arbitrary finite-dimensional algebra, complete sequences do not always exist. For example, this is the case for the path algebra of an oriented two-cycle, modulo the square of its radical: only the projective indecomposable modules are exceptional. Since there are non-trivial morphisms between the projective modules, they cannot form an exceptional sequence, and hence no complete exceptional sequence exists

Key words and phrases. Finite-dimensional algebra, exceptional sequence, τ -exceptional sequence, τ -rigid module, τ -tilting theory, 2-term silting object, perpendicular category, Bongartz complement, cluster morphism category.

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for this algebra. There are however non-hereditary settings where exceptional sequences are of importance; see [28, 22].

Our primary aim in this article is to give an alternative approach to generalizing the definition of exceptional sequences from the hereditary to the general case, ensuring that *complete* sequences always exist for arbitrary finite dimensional algebras. Our work is firstly motivated by the signed exceptional sequences for hereditary finite dimensional algebras H which were recently introduced by Igusa and Todorov [25]. In this case, the projective objects appearing in the sequence can be signed. Such sequences were needed in order for the authors to define the *cluster morphism category* of H , whose objects are the finitely generated wide subcategories of $\text{mod } H$. The article [25] shows that for a hereditary algebra of finite representation type, the geometric realization of the cluster morphism category is an Eilenberg-MacLane space whose fundamental group coincides with the picture group of [24].

Signed exceptional sequences were needed to explain the composition and associativity of maps in the cluster morphism category. In particular it was shown that complete signed exceptional sequences are in bijection with ordered cluster-tilting objects in the cluster category [11] corresponding to H . These are known to be in bijection with ordered clusters in the corresponding (acyclic) cluster algebra [10, 12].

Secondly, Adachi, Iyama and Reiten [1], motivated by the behaviour of cluster categories and cluster-tilted algebras (or, more generally, 2-Calabi-Yau categories and 2-Calabi-Yau-tilted algebras), recently introduced τ -tilting theory for finite-dimensional algebras (note that some related ideas also appear in [16]). This included, in particular, the notions of τ -rigid modules and support τ -tilting objects. This theory has been explored in a number of subsequent papers.

Motivated by these ideas, we introduce the notion of a (signed) τ -exceptional sequence of modules over a finite dimensional algebra. Although a module M appearing in such a sequence is not exceptional in general, it does satisfy $\text{Ext}^1(M, M) = 0$ (using the Auslander-Reiten formula: see Corollary 5.3). We show that any finite dimensional algebra has a complete (signed or unsigned) τ -exceptional sequence. Furthermore, the complete signed τ -exceptional sequences are in bijection with ordered support τ -tilting objects (see Corollary 5.5), generalizing the result of Igusa and Todorov in the hereditary case. Our approach is very different from that in [25]. As well as dealing with the general case, an important ingredient in our proof is the correspondence [1] between τ -rigid modules and rigid 2-term complexes in the derived category.

In the case of a cluster-tilted algebra [9], there is a connection to cluster algebras, which we now describe. Fix a cluster-tilted algebra, i.e. an algebra of the form $\Lambda = \text{End}_{C_Q}(T)$, where Q is a finite acyclic quiver, C_Q is the corresponding cluster category, and T is a cluster-tilting object in C_Q . By [1], support τ -tilting Λ -modules are in bijection with cluster-tilting objects in C_Q . By [10, 12], cluster-tilting objects in C_Q are in bijection with clusters in the cluster algebra \mathcal{A}_Q [17] associated to Q . Both of these bijections induce bijections between the ordered versions. It follows from these results and the results proved here that there is a bijection between complete signed τ -exceptional sequences over Λ and ordered clusters in the cluster algebra \mathcal{A}_Q . We remark that if Λ is hereditary, (signed) τ -exceptional sequences and (signed) exceptional sequences coincide (although

in general they are different), and we obtain a bijection between complete signed exceptional sequences over Λ and ordered clusters in \mathcal{A}_Q ; in this case such a bijection was also obtained in [25].

A finite dimensional algebra is called τ -tilting finite [14] if there are only finitely many basic τ -tilting objects (up to isomorphism). Motivated by the cluster morphism categories mentioned above, in [8] we construct a natural category whose objects are the wide subcategories of the module category of a τ -tilting finite algebra. The construction relies heavily on the bijection established in this paper. After the first preprint-version of this paper and [8] appeared, Hanson and Igusa [21] used our results to show that a generalized version of the picture group can be understood in terms of the category of wide subcategories from [8].

The paper is organized as follows. In Section 1 we give some notation and background, and state the main result. In Section 2 we give some background and results concerning 2-term silting objects in the derived category, and their links to τ -rigid objects in module categories. In Sections 3 and 4 we prepare for the proof of the main result, which is then completed in Section 5. We conclude by giving some examples in Section 6.

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1. NOTATION AND MAIN RESULT

Let Λ be a basic finite dimensional algebra over a field k , and let $\text{mod } \Lambda$ denote the category of finite dimensional Λ -modules. Let $\mathcal{P}(\Lambda)$ denote the full subcategory of projective objects in $\text{mod } \Lambda$. Similarly, if \mathcal{X} is a subcategory of $\text{mod } \Lambda$, let $\mathcal{P}(\mathcal{X})$ denote the full subcategory of \mathcal{X} consisting of the Ext-projective objects in \mathcal{X} , i.e. the objects P in \mathcal{X} such that $\text{Ext}^1(P, X) = 0$ for all $X \in \mathcal{X}$.

For an additive category C and an object X in C , we denote by $\text{add } X$ the additive subcategory of C generated by X , i.e. the full subcategory of C whose objects are all direct summands of direct sums of copies of X . For a subcategory $\mathcal{X} \subseteq C$, we define $\mathcal{X}^\perp = \{Y \in C \mid \text{Hom}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\}$, and define ${}^\perp\mathcal{X}$ similarly.

If C is (skeletally small) and Krull-Schmidt, we denote by $\text{ind}(C)$ the set of isomorphism classes of indecomposable objects in C . For an object X , we write $\text{ind } X$ for $\text{ind}(\text{add}(X))$. For any basic object X in C , let $\delta(X)$ denote the number of indecomposable direct summands of X . We denote $\delta(\Lambda)$ by n .

If C is abelian and X is an object of C , we denote by $\text{Gen } X$ the full subcategory of C consisting of all objects which are factors of objects in $\text{add } X$.

In general, all subcategories considered are assumed to be full and closed under isomorphisms. All objects are taken to be basic where possible and considered up to isomorphism.

Let τ denote the Auslander-Reiten translate in $\text{mod } \Lambda$. We now recall notation and definitions of from [1]. Note that our definitions are slightly different, but clearly equivalent to the corresponding definitions in [1].

An object U in $\text{mod } \Lambda$ is called τ -rigid if $\text{Hom}(U, \tau U) = 0$. Let $D^b(\text{mod } \Lambda)$ denote the bounded derived category of $\text{mod } \Lambda$, with shift functor denoted by [1]. We consider $\text{mod } \Lambda$ as a full subcategory of $D^b(\text{mod } \Lambda)$ by regarding a module as a stalk complex concentrated in degree 0. Consider the full subcategory $C(\Lambda) = \text{mod } \Lambda \amalg \text{mod } \Lambda[1]$ of $D^b(\text{mod } \Lambda)$.

Definition 1.1. *The object $M \amalg P[1]$ in $C(\Lambda)$ is called support τ -rigid if*

- (i) *The object M lies in $\text{mod } \Lambda$ and satisfies $\text{Hom}(M, \tau M) = 0$, and*
- (ii) *The object P lies in $\mathcal{P}(\Lambda)$ and satisfies $\text{Hom}(P, M) = 0$.*

The object $M \amalg P[1]$ is called a support τ -tilting object if $\delta(M \amalg P[1]) = n$. Moreover, M is in this case called a support τ -tilting module or just a τ -tilting module if in addition $P = 0$.

We want to consider all possible orderings of such objects, in the following sense.

Definition 1.2. *For a positive integer t , an ordered t -tuple of indecomposable objects $(\mathcal{T}_1, \dots, \mathcal{T}_t)$ in $C(\Lambda)$ is called an ordered support τ -rigid object if $\amalg_{i=1}^t \mathcal{T}_i$ is a basic support τ -rigid object. If, in addition, $t = n$, then $(\mathcal{T}_1, \dots, \mathcal{T}_t)$ is called an ordered support τ -tilting object.*

Let U be an indecomposable τ -rigid Λ -module. Jasso [27] considered the category $J(U) = U^\perp \cap {}^\perp(\tau U)$, and, in particular, proved that $J(U)$ is equivalent to the module category of a finite-dimensional algebra Λ_U with $\delta(\Lambda_U) = \delta(\Lambda) - 1$. By [7, Cor. 3.22], [15, Theorem 3.28] $J(U)$ is an exact abelian (wide) subcategory of $\text{mod } \Lambda$ (see Proposition 4.2 for more details). We say that an object N in $J(U)$ is τ -rigid in $J(U)$ if the corresponding Λ_U -module is τ -rigid.

Note that a Λ -module which is not τ -rigid can be τ -rigid in $J(U)$. For example, let Q be the quiver

$$\begin{array}{ccc} & 2 & \\ \alpha \nearrow & & \searrow \beta \\ 1 & \xrightarrow{\gamma} & 3 \end{array}$$

and let $\Lambda = kQ/I$ where I is the ideal generated by the path $\beta\alpha$. Let S_i (respectively, I_i, P_i) be the simple (respectively, indecomposable injective, indecomposable projective) module associated to vertex i of Q . Then $\tau I_3 = P_1$ and I_3 is not τ -rigid in $\text{mod } \Lambda$. However, I_3 is projective in $J(S_2)$ (which is equivalent to the module category of a quiver of Dynkin type A_2), so I_3 is τ -rigid in $J(S_2)$. See Section 6.3 for more details.

For a projective object P , we let $J(P[1]) = J(P) = P^\perp$ and set $\Lambda_{P[1]} = \Lambda_P$. Then, for any indecomposable object \mathcal{U} in $C(\Lambda)$, we have an equivalence $F_{\mathcal{U}} : J(\mathcal{U}) \rightarrow \text{mod } \Lambda_{\mathcal{U}}$.

For a full subcategory \mathcal{Y} of $\text{mod } \Lambda$, we denote by $C(\mathcal{Y})$ the full subcategory $\mathcal{Y} \amalg \mathcal{Y}[1]$ of $C(\Lambda)$. Thus, for \mathcal{U} as above, we have the full subcategory $C(J(\mathcal{U}))$ of $C(\Lambda)$. Since $J(\mathcal{U})$ is a wide subcategory of $\text{mod } \Lambda$, we have that for $A, B \in J(\mathcal{U})$,

$$\text{Hom}_{D^b(\Lambda)}(A, B[1]) \cong \text{Ext}_\Lambda^1(A, B) \cong \text{Ext}_{J(\mathcal{U})}^1(A, B) \cong \text{Hom}_{D^b(\Lambda_U)}(F_{\mathcal{U}}(A), F_{\mathcal{U}}(B)[1]).$$

It follows that $C(J(\mathcal{U})) \simeq C(\text{mod } \Lambda_{\mathcal{U}})$.

We say that an object $N \amalg Q[1]$ in $C(J(\mathcal{U}))$ is *support τ -rigid in $C(J(\mathcal{U}))$* if the corresponding object in $C(\text{mod } \Lambda_{\mathcal{U}})$ is support τ -rigid, i.e. N is τ -rigid in $J(\mathcal{U})$ and Q is an object in $J(\mathcal{U})$ which is projective as an object in $J(\mathcal{U})$.

This allows us to define signed τ -exceptional sequences recursively as follows:

Definition 1.3. *For a positive integer t , an ordered t -tuple of indecomposable objects $(\mathcal{U}_1, \dots, \mathcal{U}_{t-1}, \mathcal{U}_t)$ in $C(\Lambda)$ is called a signed τ -exceptional sequence, if \mathcal{U}_t is a support τ -rigid object in $C(\Lambda)$ and $(\mathcal{U}_1, \dots, \mathcal{U}_{t-1})$ is a signed τ -exceptional sequence in $C(J(\mathcal{U}_t))$. A signed τ -exceptional sequence $(\mathcal{U}_1, \dots, \mathcal{U}_{t-1}, \mathcal{U}_t)$ in which all of the \mathcal{U}_i lie in $\text{mod } \Lambda$ is called a τ -exceptional sequence. A (signed) τ -exceptional sequence of length n is said to be complete.*

In the above example, (P_2, I_3, S_2) is a τ -exceptional sequence in $\text{mod } \Lambda$. Note that, although I_3 is not projective in $\text{mod } \Lambda$, it is projective in $J(S_2)$ and therefore can take a ‘sign’; in other words, $(P_2, I_3[1], S_2)$ is a signed τ -exceptional sequence in $\text{mod } \Lambda$ (see Section 6.3). This is the reason we take $\text{mod } \Lambda[1]$ as a summand in the definition of $C(\Lambda)$ above, rather than just the shift of the subcategory of projective Λ modules.

We can now state our main result.

Theorem 1.4. *Let Λ be a finite dimensional algebra. For each $t \in \{1, \dots, n\}$ there is a bijection between the set of ordered support τ -rigid objects of length t in $C(\Lambda)$ and the set of signed τ -exceptional sequences of length t in $C(\Lambda)$.*

For $t = n$, we obtain the following.

Corollary 1.5. *Let Λ be a finite dimensional algebra. Then there is a bijection between the set of ordered support τ -tilting objects in $C(\Lambda)$ and the set of complete signed τ -exceptional sequences in $C(\Lambda)$.*

We recall the following crucial result of [1] which shows that, given a τ -rigid module, there are two support τ -tilting objects containing the module as a direct summand, satisfying certain properties.

Theorem 1.6. [1, Section 2.3] *Let U be a τ -rigid module.*

- (a) *Up to isomorphism, there is a unique basic module $B[U]$ such that $B[U] \amalg U$ is a τ -tilting module and $\text{add}(B[U] \amalg U) = \mathcal{P}({}^{\perp}\tau U)$. Moreover, we have ${}^{\perp}\tau U = {}^{\perp}\tau(U \amalg B[U]) = \text{Gen}(U \amalg B[U])$. The equality ${}^{\perp}\tau U = {}^{\perp}\tau(U \amalg B[U])$ characterizes $B[U]$.*
- (b) *Up to isomorphism, there is a unique basic module $C[U]$ and a basic projective module Q such that $C[U] \amalg U \amalg Q[1]$ is a support τ -tilting object and $\text{add}(C[U] \amalg U) = \mathcal{P}(\text{Gen } U)$. In particular, we have $\text{add } Q = \mathcal{P}(\Lambda) \cap {}^{\perp}U$.*

Proof. Part (a), apart from the characterization, follows from [1, Theorem 2.10]. For the characterization, suppose that B satisfies ${}^{\perp}\tau U = {}^{\perp}\tau(U \amalg B)$ and that $U \amalg B$ is a basic τ -tilting module. By [1, Theorem 2.12(c)], we have ${}^{\perp}\tau(U \amalg B) = \text{Gen } U \amalg B$. By [1, Lemma 2.11], ${}^{\perp}\tau U$ is functorially finite, so $U \amalg B$ is Ext-projective in ${}^{\perp}\tau U$ by [1, Prop 2.9], and hence $B \amalg U$ is in $\text{add } B[U] \amalg U$ by assumption. Now, since $B \amalg U$ and $B[U] \amalg U$ are both basic τ -tilting modules, we must have that B and $B[U]$ are isomorphic.

For part (b), let $C[U]$ be defined so that $C[U] \amalg U$ is basic and $\text{add}(C[U] \amalg U) = \mathcal{P}(\text{Gen } U)$. Let Q be basic and such that $\text{add } Q = \mathcal{P}(\Lambda) \cap {}^{\perp}U$. Since $\text{Gen } U$ is functorially

finite, it follows from [1, Theorem 2.7] that $C[U] \amalg U$ is support τ -tilting. We have that $\text{Gen } U = \text{Gen } C[U] \amalg U$, since $C[U]$ is in $\text{Gen } U$. If P is a projective module it is easy to see that $\text{Hom}(P, M) = 0$ for a module M if and only if $\text{Hom}(P, N) = 0$ for all $N \in \text{Gen } M$. Hence, we have:

$$\mathcal{P}(\Lambda) \cap {}^\perp U = \mathcal{P}(\Lambda) \cap {}^\perp(\text{Gen } U) = \mathcal{P}(\Lambda) \cap {}^\perp(C[U] \amalg U)$$

and it follows that $C[U] \amalg U \amalg Q[1]$ is a support τ -tilting object. \square

The module $B[U]$ in Theorem 1.6 is known as the *Bongartz complement* of U , and we refer to $C[U]$ as the *co-Bongartz complement* of U . An important step towards our main theorem is the construction of explicit bijections between the indecomposable direct summands of $B[U]$ and the indecomposable direct summands of $C[U] \amalg Q[1]$.

2. 2-TERM RIGID OBJECTS

In this section we discuss 2-term silting objects in the derived category and their links to τ -rigid objects in the module category.

We denote by H^n the functor from $D^b(\text{mod } \Lambda)$ which maps a complex to its n th homology. We regard the bounded homotopy category of projectives $\mathcal{K} = K^b(\mathcal{P}(\Lambda))$ as a full subcategory of $D^b(\text{mod } \Lambda)$. An object \mathbb{U} in \mathcal{K} is said to be *rigid* if $\text{Hom}(\mathbb{U}, \mathbb{U}[i]) = 0$ for all $i > 0$, and *silting* if in addition it generates \mathcal{K} as a triangulated category (i.e. \mathbb{U} is not contained in any proper triangulated subcategory of \mathcal{K} closed under direct summands).

An object of the form

$$\cdots \rightarrow 0 \rightarrow P^{-1} \rightarrow P^0 \rightarrow 0 \rightarrow \cdots$$

in \mathcal{K} is called a *2-term object*. If $P^0 = 0$, we denote such an object by $P^{-1}[1]$. A 2-term object in \mathcal{K} which is rigid (respectively, silting) is called *2-term rigid* (respectively, *2-term silting*).

For a module U , we let \mathbb{P}_U denote the minimal projective presentation of U , considered as a 2-term object in \mathcal{K} . The following two lemmas are well-known.

Lemma 2.1. (a) *Let \mathbb{X} and \mathbb{Y} be 2-term objects in \mathcal{K} . Then H^0 induces an epimorphism $\text{Hom}_{\mathcal{K}}(\mathbb{X}, \mathbb{Y}) \rightarrow \text{Hom}(H^0(\mathbb{X}), H^0(\mathbb{Y}))$ with kernel consisting of the maps factoring through $\text{add } \Lambda[1] = \text{add}(\Lambda \rightarrow 0)$.*

(b) *Let X, Y, Z be in $\text{mod } \Lambda$. If any map in \mathcal{K} from \mathbb{P}_X to \mathbb{P}_Z factors through \mathbb{P}_Y , then any map from X to Z factors through Y .*

Proof. Part (a) is straightforward, and part (b) is a direct consequence of part (a). \square

Lemma 2.2. *Let X be in $\text{mod } \Lambda$. Then \mathbb{P}_X is indecomposable in \mathcal{K} if and only if X is an indecomposable module.*

Proof. Straightforward. \square

We also need the following facts from [1].

Lemma 2.3. [1, Section 3] *Let U, X be in $\text{mod } \Lambda$.*

(a) *$\text{Hom}(U, \tau X) = 0$ if and only if $\text{Hom}_{\mathcal{K}}(\mathbb{P}_X, \mathbb{P}_U[1]) = 0$. In particular, the module U is τ -rigid if and only if \mathbb{P}_U is rigid.*

- (b) If $U \amalg P[1]$ is a support τ -rigid object in $C(\Lambda)$, then $\mathbb{P}_U \amalg P[1]$ is a 2-term rigid object in \mathcal{K} , and it is 2-term silting if and only if $U \amalg P[1]$ is support τ -tilting.
- (c) Let \mathbb{U} be a 2-term rigid object in \mathcal{K} . Write \mathbb{U} in the form $\mathbb{P} \amalg Q[1]$, where Q is a projective module and \mathbb{P} has no direct summand of the form $Q'[1]$, where Q' is a projective module. Then $H^0(\mathbb{P}) \amalg Q[1]$ is support τ -rigid in $C(\Lambda)$. If \mathbb{U} is 2-term silting, then $H^0(\mathbb{P}) \amalg Q[1]$ is support τ -tilting in $C(\Lambda)$.
- (d) The constructions of (b) and (c) give a bijection between 2-term silting objects in \mathcal{K} and support τ -tilting objects in $C(\Lambda)$.

Note that by Lemma 2.3 any 2-term silting object in \mathcal{K} is, up to homotopy, of the form $\mathbb{P}_X \amalg Q[1]$, where \mathbb{P}_X is a minimal projective presentation for some Λ -module X . In particular, this implies the following.

Lemma 2.4. *Let \mathbb{U} be a 2-term silting object in \mathcal{K} . Then $H^0(\mathbb{U})$ is τ -tilting if and only if \mathbb{U} has no direct summand of the form $P \rightarrow 0$*

Proof. Since \mathbb{U} is 2-term silting, we have $\mathbb{U} \simeq \mathbb{P}_U \amalg P[1]$ for a support τ -tilting object $U \amalg P[1]$ in $C(\Lambda)$ with $H^0(\mathbb{U}) = U$. And we have $P = 0$ if and only if U is τ -tilting. \square

Parts (b) to (e) in the following Lemma are essentially contained in [2, Section 2], but we include proofs for convenience.

Lemma 2.5. *Let \mathbb{U} and \mathbb{X} be 2-term objects in \mathcal{K} satisfying*

- (i) $\mathbb{U} \amalg \mathbb{X}$ is rigid;
- (ii) $\text{add } \mathbb{U} \cap \text{add } \mathbb{X} = 0$.

Let

$$(1) \quad \mathbb{Y} \xrightarrow{\beta} \mathbb{U}' \xrightarrow{\alpha} \mathbb{X} \rightarrow$$

be a triangle in which α is a minimal right $\text{add } \mathbb{U}$ -approximation. Then the following hold.

- (a) The object \mathbb{Y} is a 2-term object if and only if the induced map $H^0(\alpha): H^0(\mathbb{U}') \rightarrow H^0(\mathbb{X})$ is an epimorphism.
- (b) The object $\mathbb{Y} \amalg \mathbb{U}$ is rigid.
- (c) The morphism $\mathbb{Y} \xrightarrow{\beta} \mathbb{U}'$ is a minimal left $\text{add } \mathbb{U}$ -approximation.
- (d) $\text{add } \mathbb{U} \cap \text{add } \mathbb{Y} = 0$.
- (e) The object $\mathbb{Y} \amalg \mathbb{U}$ is silting if and only if $\mathbb{X} \amalg \mathbb{U}$ is silting.
- (f) If $\mathbb{X} \amalg \mathbb{U}$ is 2-term silting and $H^0(\alpha)$ is an epimorphism, then $\mathbb{Y} \amalg \mathbb{U}$ is 2-term silting.

Proof. Note that we can assume that any differential appearing in an object in \mathcal{K} is radical, i.e. that its image is contained in the radical of its target. Let $\text{cone}(\alpha)$ be the mapping cone of α , which has the form:

$$\cdots \rightarrow 0 \rightarrow U^{-1} \rightarrow U^0 \amalg X^{-1} \rightarrow X^0 \rightarrow 0 \rightarrow \cdots$$

Then $\mathbb{Y} \simeq \text{cone}(\alpha)[-1]$. Clearly $\text{cone}(\alpha)[-1]$ is 2-term if and only if the map $U^0 \amalg X^{-1} \rightarrow X^0$ is an epimorphism, which is equivalent to $U^0 \rightarrow X^0$ being a (split) epimorphism, since the map $X^{-1} \rightarrow X^0$ is radical. This holds if and only if the induced map $H^0(\alpha): H^0(\mathbb{U}') \rightarrow H^0(\mathbb{X})$ is an epimorphism, giving (a).

We now prove (b). We apply $\text{Hom}(\mathbb{U}, _)$ to (1), obtaining the exact sequences

$$\text{Hom}(\mathbb{U}, \mathbb{U}'[i]) \rightarrow \text{Hom}(\mathbb{U}, \mathbb{X}[i]) \rightarrow \text{Hom}(\mathbb{U}, \mathbb{Y}[i+1]) \rightarrow \text{Hom}(\mathbb{U}, \mathbb{U}'[i+1]),$$

for all i . Since α is a right add \mathbb{U} -approximation, the map $\text{Hom}(\mathbb{U}, \mathbb{U}') \rightarrow \text{Hom}(\mathbb{U}, \mathbb{X})$ is an epimorphism. Hence $\text{Hom}(\mathbb{U}, \mathbb{Y}[i]) = 0$ for all $i > 0$.

Applying $\text{Hom}(\mathbb{X}, _)$ to (1), and considering the exact sequence

$$\text{Hom}(\mathbb{X}, \mathbb{X}[i-1]) \rightarrow \text{Hom}(\mathbb{X}, \mathbb{Y}[i]) \rightarrow \text{Hom}(\mathbb{X}, \mathbb{U}[i])$$

gives $\text{Hom}(\mathbb{X}, \mathbb{Y}[i]) = 0$ for all $i > 1$.

Applying $\text{Hom}(_, \mathbb{Y})$ to (1) gives an exact sequence

$$\text{Hom}(\mathbb{U}', \mathbb{Y}[i]) \rightarrow \text{Hom}(\mathbb{Y}, \mathbb{Y}[i]) \rightarrow \text{Hom}(\mathbb{X}, \mathbb{Y}[i+1]),$$

and hence $\text{Hom}(\mathbb{Y}, \mathbb{Y}[i]) = 0$ for all $i > 0$.

Finally, applying $\text{Hom}(_, \mathbb{U})$ to (1) gives an exact sequence

$$\text{Hom}(\mathbb{U}', \mathbb{U}[i]) \rightarrow \text{Hom}(\mathbb{Y}, \mathbb{U}[i]) \rightarrow \text{Hom}(\mathbb{X}, \mathbb{U}[i+1]),$$

and hence $\text{Hom}(\mathbb{Y}, \mathbb{U}[i]) = 0$ for all $i > 0$. This finishes the proof of (b).

For part (c) consider the exact sequence

$$\text{Hom}(\mathbb{U}', \mathbb{U}) \rightarrow \text{Hom}(\mathbb{Y}, \mathbb{U}) \rightarrow \text{Hom}(\mathbb{X}, \mathbb{U}[1]),$$

which is part of the long exact sequence obtained by applying $\text{Hom}(_, \mathbb{U})$ to (1). Since the last term vanishes, the first map is an epimorphism, and hence the map $\mathbb{Y} \xrightarrow{\beta} \mathbb{U}'$ is a left add \mathbb{U} -approximation. It is minimal since $\text{add } \mathbb{U} \cap \text{add } \mathbb{X} = 0$.

Part (d) follows directly from the minimality of α and part (c).

Part (e) follows from part (b) and the existence of the triangle (1), and part (f) is a direct consequence of parts (a) and (e). \square

3. EXCHANGE

Let U be a τ -rigid Λ -module. Recall from Theorem 1.6 the notation $B[U]$ for the Bongartz complement of U and $C[U] \amalg Q[1]$ for the co-Bongartz complement of U in $C(\Lambda)$. We will denote $C[U]$ by C in the sequel. The aim of this section is to give an explicit bijection between the indecomposable direct summands in these two complements of U .

Remark 3.1. Let $\mathbb{C}_Q = \mathbb{P}_C \amalg Q[1]$ be the 2-term rigid object in \mathcal{K} corresponding to the support τ -rigid object $C \amalg Q[1]$ in $C(\Lambda)$. By Lemma 2.3, we have that $\mathbb{C}_Q \amalg \mathbb{P}_U$ is a 2-term sifting object.

The next lemma, and also Lemma 3.4, appear in a more general context in [6]. We provide proofs adapted to our setting, for the convenience of the reader.

Lemma 3.2. Let U be a τ -rigid module, and consider the support τ -tilting object $C \amalg U \amalg Q[1]$ in $C(\Lambda)$, where $C \amalg Q[1]$ is the co-Bongartz complement of U . Let $\mathbb{C}_Q = \mathbb{P}_C \amalg Q[1]$ be the corresponding 2-term rigid object in \mathcal{K} . Let $\alpha: \mathbb{P}'_U \rightarrow \mathbb{C}_Q$ be a minimal right add \mathbb{P}_U -approximation of \mathbb{C}_Q , and complete it to a triangle:

$$\mathbb{Y} \xrightarrow{\beta} \mathbb{P}'_U \xrightarrow{\alpha} \mathbb{C}_Q \rightarrow$$

in \mathcal{K} . Then the following hold.

- (a) The map $H^0(\alpha)$ is a minimal right add U -approximation of $H^0(\mathbb{C}_Q) = C$.
- (b) The object \mathbb{Y} is 2-term.
- (c) The object $\mathbb{Y} \amalg \mathbb{P}_U$ is 2-term silting.
- (d) Let $B = H^0(\mathbb{Y})$. Then $B \amalg U$ is τ -tilting, and $\mathbb{Y} = \mathbb{P}_B$.
- (e) The map $\beta: \mathbb{Y} \rightarrow \mathbb{P}'_U$ is a minimal left add \mathbb{P}_U -approximation.

Proof. To prove (a), consider a map $\eta: U'' \rightarrow C$, with U'' in add U . Let $\tilde{\eta}: \mathbb{P}_{U''} \rightarrow \mathbb{P}_C$ be a map such that $H^0(\tilde{\eta}) = \eta$. Consider the map $\gamma = \begin{pmatrix} \tilde{\eta} \\ 0 \end{pmatrix}: \mathbb{P}_{U''} \rightarrow \mathbb{P}_C \amalg Q[1] = \mathbb{C}_Q$.

Since α is a right add \mathbb{P}_U -approximation, the map γ factors through α . But then $\eta = H^0(\gamma)$ factors through $H^0(\alpha)$. Using Lemma 2.1, we have that the minimality of α implies that $H^0(\alpha)$ also is minimal. This gives part (a).

Since C lies in Gen U , the map $H^0(\alpha)$ is an epimorphism, and part (b) then follows from Lemma 2.5(a).

Since $\mathbb{P}_U \amalg \mathbb{C}_Q$ is 2-term silting by Remark 3.1, part (c) now follows from Lemma 2.5(f).

We now prove part (d). We have that $\mathbb{Y}[1]$ is homotopic to the mapping cone of α . This mapping cone is

$$\cdots \rightarrow 0 \rightarrow P_{U'}^{-1} \rightarrow P_{U'}^0 \amalg P_C^{-1} \amalg Q \rightarrow P_C^0 \rightarrow 0 \rightarrow \cdots$$

where $U' = H^0(\mathbb{P}'_U)$ is in add U . Since the induced map $H^0(\alpha): U' \rightarrow C$ is an epimorphism, the map between the projective covers $\omega: P_{U'}^0 \rightarrow P_C^0$ is a (split) epimorphism. Then $P_{U'}^0 = W \amalg W'$, where $W = \ker \omega$ and ω induces an isomorphism $W' \simeq P_C^0$. We may replace the complex with an isomorphic one in which the image of the map $P_{U'}^{-1} \rightarrow P_{U'}^0$ is contained in W and the second component of the map $W \amalg W' \amalg P_C^{-1} \amalg Q \rightarrow P_C^0$ is an isomorphism, with the other components equal to zero.

Hence the complex

$$0 \rightarrow W' \xrightarrow{\simeq} P_C^0 \rightarrow 0$$

splits off as a direct summand of the above mapping cone complex. Therefore, in the homotopy category, it is isomorphic to the complex

$$(2) \quad \cdots \rightarrow 0 \rightarrow P_{U'}^{-1} \xrightarrow{\mu} W \amalg P_C^{-1} \amalg Q \rightarrow 0 \rightarrow \cdots$$

The map $P_{U'}^{-1} \rightarrow P_{U'}^0$ is right minimal, that is: no direct summands of $P_{U'}^{-1}$ are mapped to 0. Since the image of the map $P_{U'}^{-1} \rightarrow P_{U'}^0$ is contained in W , the complex (2) has no direct summands of the form $P \rightarrow 0$. In particular \mathbb{Y} is a (minimal) projective presentation of $\text{coker } \mu = H^0(\mathbb{Y})$. Applying Lemma 2.4 to $\mathbb{Y} \amalg \mathbb{P}_U$, we see that $H^0(\mathbb{Y} \amalg \mathbb{P}_U) = B \amalg U$ is a τ -tilting object.

Part (e) follows directly from Lemma 2.5(d). □

By Lemma 3.2(d), we can write the triangle in the statement of the lemma as:

$$(3) \quad \mathbb{P}_B \xrightarrow{\beta} \mathbb{P}'_U \xrightarrow{\alpha} \mathbb{C}_Q \rightarrow$$

Let $\mathbb{C}_Q = \amalg_i \mathbb{X}_i$ be a decomposition of \mathbb{C}_Q into indecomposable direct summands. For each i , consider a minimal right add \mathbb{P}_U -approximation $(\mathbb{P}'_U)_i \xrightarrow{\alpha_i} \mathbb{X}_i$. It is easy to check

that $\coprod_i \alpha_i$ is a minimal right add \mathbb{P}_U -approximation of \mathbb{C}_Q , and hence we may assume that $\alpha = \coprod_i \alpha_i$. Hence, we obtain for each i a triangle

$$(4) \quad \mathbb{P}_{B_i} \xrightarrow{\beta_i} (\mathbb{P}'_U)_i \xrightarrow{\alpha_i} \mathbb{X}_i \rightarrow$$

obtained by completing a minimal right add \mathbb{P}_U -approximation α_i of \mathbb{X}_i to a triangle. We now have $\beta = \coprod_i \beta_i$, $B = \coprod_i B_i$ and $\mathbb{P}_B = \coprod_i \mathbb{P}_{B_i}$.

Lemma 3.3. *With notation as above, the map $\mathbb{X}_i \mapsto \mathbb{P}_{B_i}$ is a bijection between the indecomposable direct summands of \mathbb{C}_Q and the indecomposable direct summands of \mathbb{P}_B .*

Proof. By Lemma 3.2 and the above discussion, each map $\mathbb{P}_{B_i} \xrightarrow{\beta_i} (\mathbb{P}'_U)_i$ is a minimal left add \mathbb{P}_U -approximation. \square

In particular we now have that the B_i are indecomposable by Lemma 2.2.

Lemma 3.4. *The Λ -module B is the Bongartz complement $B[U]$ of U ; in particular B is basic.*

Proof. We first prove that B is basic. Suppose that $B_i \simeq B_j$ for some $i \neq j$. Then also $\mathbb{P}_{B_i} \simeq \mathbb{P}_{B_j}$, and, by Lemma 3.3, we have $\mathbb{X}_i \simeq \mathbb{X}_j$. But \mathbb{C}_Q is basic, since by construction both C and Q are basic (see Theorem 1.6(b) and the remark afterwards). Hence $i = j$, and therefore B is basic.

By Theorem 1.6(a), the Bongartz complement $B[U]$ of U is characterized by the property that ${}^{\perp}\tau(B[U] \amalg U) = {}^{\perp}\tau U$. It is therefore sufficient to prove that ${}^{\perp}\tau U \subset {}^{\perp}\tau B$. That is, we need to prove that $\text{Hom}(X, \tau U) = 0$ implies that $\text{Hom}(X, \tau B) = 0$. By Lemma 2.3, this is equivalent to proving that $\text{Hom}_{\mathcal{K}}(\mathbb{P}_U, \mathbb{P}_X[1]) = 0$ implies that $\text{Hom}_{\mathcal{K}}(\mathbb{P}_B, \mathbb{P}_X[1]) = 0$.

For this, consider part of the long exact sequence obtained by applying $\text{Hom}_{\mathcal{K}}(\cdot, \mathbb{P}_X[1])$ to the triangle (3):

$$\text{Hom}((\mathbb{P}'_U, \mathbb{P}_X[1]) \rightarrow \text{Hom}(\mathbb{P}_B, \mathbb{P}_X[1]) \rightarrow \text{Hom}(\mathbb{C}_Q, \mathbb{P}_X[2]).$$

The last term vanishes since the complexes \mathbb{C}_Q and \mathbb{P}_X are both 2-term. Hence $\text{Hom}_{\mathcal{K}}(\mathbb{P}_U, \mathbb{P}_X[1]) = 0$ implies that $\text{Hom}_{\mathcal{K}}(\mathbb{P}_B, \mathbb{P}_X[1]) = 0$, as required.

This proves that ${}^{\perp}\tau U \subset {}^{\perp}\tau B$, and hence that ${}^{\perp}\tau(B \amalg U) = {}^{\perp}\tau U$, which implies that $B = B[U]$. \square

Recall that $\mathbb{C}_Q = \mathbb{P}_C \amalg Q[1]$, where $C = C[U]$ is the co-Bongartz complement of U . We now focus on the indecomposable direct summands $Q'[1]$ of \mathbb{C}_Q , where Q' is an indecomposable direct summand of Q .

Lemma 3.5. *Let \mathbb{X}_i be an indecomposable direct summand of \mathbb{C}_Q of the form $Q'[1]$, where Q' is an indecomposable direct summand of Q . Consider triangle (4). The induced map $H^{-1}(\mathbb{X}_i) \rightarrow H^0(\mathbb{P}_{B_i})$, that is $Q' \rightarrow B_i$, is a minimal left ${}^{\perp}\tau U$ -approximation, and hence also a left $\mathcal{P}({}^{\perp}\tau U)$ -approximation.*

Proof. Let M be in ${}^{\perp}\tau U$. Then, by Lemma 2.3, we have $\text{Hom}_{\mathcal{K}}(\mathbb{P}_U, \mathbb{P}_M[1]) = 0$. Apply $\text{Hom}_{\mathcal{K}}(\cdot, \mathbb{P}_M)$ to the triangle (4) and consider the exact sequence

$$\text{Hom}(\mathbb{P}_{B_i}, \mathbb{P}_M) \rightarrow \text{Hom}(\mathbb{X}_i, \mathbb{P}_M[1]) \rightarrow \text{Hom}((\mathbb{P}'_U)_i, \mathbb{P}_M[1])$$

Since the last term vanishes, every map $\mathbb{X}_i[-1] \rightarrow \mathbb{P}_M$ factors through $\mathbb{X}_i[-1] \rightarrow \mathbb{P}_{B_i}$. This means that every map from Q (regarded as a complex concentrated in degree 0) to \mathbb{P}_M factors through \mathbb{P}_{B_i} . Hence, by Lemma 2.1, the map $Q' \rightarrow B_i$ is a left ${}^\perp\tau U$ -approximation. This map is non-zero, since ${}^\perp\tau U$ is sincere by [1, Theorem 2.10]. It is therefore minimal, since B_i is indecomposable. Since B_i is in $\mathcal{P}({}^\perp\tau U)$ by Lemma 3.4, the last statement also follows. \square

Taking homology, the triangle (3) induces an exact sequence

$$H^{-1}(\mathbb{C}_Q) \rightarrow H^0(\mathbb{P}_B) \rightarrow H^0(\mathbb{P}'_U) \rightarrow H^0(\mathbb{C}_Q) \rightarrow 0.$$

Note that $C = H^0(\mathbb{C}_Q)$ is the co-Bongartz complement of U and $B = H^0(\mathbb{P}_B)$ is the Bongartz complement of U . Let $U' = H^0(\mathbb{P}'_U)$ and $H^{-1}(\mathbb{C}_Q) = \widetilde{Q}$, where Q is a direct summand of \widetilde{Q} . The sequence above becomes:

$$(5) \quad \widetilde{Q} \xrightarrow{\delta} B \xrightarrow{\mu} U' \xrightarrow{\gamma} C \rightarrow 0$$

Since the triangles (4) sum to the triangle (3), we also have that the sequence (5) is a direct sum of $n - \delta(U)$ exact sequences

$$(6) \quad Q_i \xrightarrow{\delta_i} B_i \xrightarrow{\mu_i} U'_i \xrightarrow{\gamma_i} C_i \rightarrow 0$$

where for each i either

- Case (i) C_i is non-zero and is an indecomposable direct summand of C (this happens when \mathbb{X}_i is the minimal projective presentation of a summand C_i of C), or
- Case (ii) Q_i is an indecomposable projective direct summand of Q , the map μ_i is an epimorphism and $C_i = 0$ (this happens when \mathbb{X}_i is the shift of an indecomposable projective direct summand of Q).

Lemma 3.6. *With notation as above, consider the exact sequence (6). Then we have the following:*

- (a) *The map μ_i is a minimal left add U -approximation.*
- (b) *In case (i), the map γ_i is a minimal right add U -approximation, while in case (ii), the map γ_i is the zero map and δ_i is a minimal left $\mathcal{P}({}^\perp\tau U)$ -approximation.*

Proof. Using the fact that each β_i in (4) is a minimal left add \mathbb{P}_U -approximation, in combination with Lemma 2.1(b), it follows that each μ_i is a left add U -approximation. Minimality follows from the fact that $\text{add } U \cap \text{add } C = 0$. This proves (a).

For (b), note that in case (i) the fact that γ_i is a minimal right add U -approximation follows from Lemma 3.2. In case (ii), the map γ_i must be zero as $C_i = 0$. The fact that δ_i is a minimal left $\mathcal{P}({}^\perp\tau U)$ -approximation follows from Lemma 3.5. \square

Let us summarize our findings.

Proposition 3.7. *Let U be a τ -rigid module. Let B be the Bongartz complement of U and C the co-Bongartz complement of U , with corresponding support τ -tilting object $C \amalg U \amalg Q[1]$ such that $\text{add}(C \amalg U) = \mathcal{P}(\text{Gen } U)$, as in Theorem 1.6.*

Then there is a triangle

$$(7) \quad \mathbb{P}_B \xrightarrow{\beta} \mathbb{P}'_U \xrightarrow{\alpha} \mathbb{C}_Q \rightarrow,$$

where $\mathbb{C}_Q = \mathbb{P}_C \amalg Q[1]$ and β (respectively, α) is a minimal left (respectively, right) add \mathbb{P}_U -approximation. This triangle is the direct sum of $n - \delta(U)$ triangles

$$(8) \quad \mathbb{P}_{B_i} \xrightarrow{\beta_i} (\mathbb{P}'_U)_i \xrightarrow{\alpha_i} \mathbb{X}_i \rightarrow,$$

where $B = \amalg_i B_i$ is a decomposition of B into a direct sum of indecomposable modules and the \mathbb{X}_i are the indecomposable direct summands of \mathbb{C}_Q .

Let $B = B' \amalg B''$, where B' is the direct sum of all indecomposable direct summands of B with the property that the minimal left add U -approximation $B_i \rightarrow U_i$ is not an epimorphism, and B'' is the complement of B' in B .

- (a) For each indecomposable direct summand B_i of B which is a summand of B' , there is an exact sequence

$$B_i \xrightarrow{\mu_i} U'_i \xrightarrow{\gamma_i} C_i \rightarrow 0,$$

where μ_i (respectively, γ_i) is a minimal left (respectively, right) add U -approximation and C_i is an indecomposable direct summand of C . This arises from part of the long exact sequence associated to (8):

$$H^0(\mathbb{P}_{B_i}) \rightarrow H^0((\mathbb{P}'_U)_i) \rightarrow H^0(\mathbb{X}_i) \rightarrow H^1(\mathbb{P}_{B_i}) = 0,$$

where $\mathbb{X}_i = \mathbb{P}_{C_i}$. The map $B_i \mapsto C_i$ is a correspondence between the indecomposable direct summands of B' and the indecomposable direct summands of C .

- (b) For each indecomposable direct summand B_i of B which is a summand of B'' , there is an exact sequence

$$Q_i \xrightarrow{\delta_i} B_i \xrightarrow{\mu_i} U'_i \rightarrow 0,$$

where Q_i is an indecomposable direct summand of Q , with $U'_i \in \text{add } U$ and with $\delta_i: Q_i \rightarrow B_i$ a minimal left $\mathcal{P}(\perp \tau U) = \text{add } B$ -approximation. This arises from part of the long exact sequence associated to (8):

$$H^{-1}(\mathbb{X}_i) \rightarrow H^0(\mathbb{P}_{B_i}) \rightarrow H^0((\mathbb{P}'_U)_i) \rightarrow H^0(\mathbb{X}_i) = 0,$$

where $\mathbb{X}_i = Q_i[1]$. The map $B_i \mapsto Q_i$ is a bijection between the indecomposable direct summands of B'' and the indecomposable direct summands of Q .

- (c) There is a bijection between the indecomposable direct summands of B and the indecomposable direct summands of the support τ -rigid object $C \amalg Q[1]$ in $C(\Lambda)$.

Proof. Parts (a) and (b) follow from Lemmas 3.3, 3.4, 3.6 and the above discussion. Part (c) is a direct consequence of parts (a) and (b). \square

We recall the following version of Wakamatsu's lemma from [1].

Lemma 3.8. [1, Lemma 2.6] *Let U be a τ -rigid module and let $\alpha: U' \rightarrow X$ be a right add U -approximation. Then $\ker \alpha$ lies in ${}^\perp(\tau U)$.*

Later we will need the following stronger version of Lemma 3.6(a), which is due to [1].

Lemma 3.9. [1, Lemma 2.20] *The map μ_i of Lemma 3.6 is a minimal left Gen U -approximation.*

Proof. The proof is essentially contained in the proof of Lemma 2.20 of [1], but we give the details for convenience. For an object V in $\text{Gen } U$ there is a short exact sequence

$$(9) \quad 0 \rightarrow Z \rightarrow U' \rightarrow V \rightarrow 0,$$

where $U' \rightarrow V$ is a right add U -approximation, and Z lies in ${}^{\perp}\tau U$ by Lemma 3.8. Since ${}^{\perp}\tau U \subseteq {}^{\perp}\tau B$, we have $\text{Hom}(Z, \tau B) = 0$, and hence $\text{Ext}^1(B, Z) = 0$, by the Auslander-Reiten formula. Applying $\text{Hom}(B_i, _)$ to the exact sequence (9) we get an exact sequence

$$\text{Hom}(B_i, U') \rightarrow \text{Hom}(B_i, V) \rightarrow \text{Ext}^1(B_i, Z).$$

Since the last term vanishes, the first map is an epimorphism.

Consider an arbitrary map $B_i \rightarrow V$. By the above, it factors $B_i \rightarrow U' \rightarrow V$. But the map $\mu_i: B_i \rightarrow U_i$ is a left add U -approximation by Lemma 3.6(a), and so $B_i \rightarrow U'$ factors $B_i \rightarrow U_i \rightarrow U'$, and hence $B_i \rightarrow V$ also factors through μ_i . This concludes the proof. \square

4. REDUCTION

We fix a τ -rigid Λ -module U throughout this section. Recall that a pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\text{mod } \Lambda$ is called a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. For a given torsion pair $(\mathcal{T}, \mathcal{F})$ and an arbitrary module X , there is a (unique up to isomorphism) exact sequence

$$0 \rightarrow t(X) \rightarrow X \rightarrow f(X) \rightarrow 0$$

with the property that $t(X)$ is in \mathcal{T} and $f(X)$ is in \mathcal{F} . It is known as the *canonical sequence* for X .

Lemma 4.1. [3, Theorem 5.10] *The pair $(\text{Gen } U, U^{\perp})$ is a torsion pair in $\text{mod } \Lambda$.*

From now on we only consider this torsion pair, and use the notation t and f relative to this pair.

We next recall some results and notions, mostly from [27]. Let $J(U) = U^{\perp} \cap {}^{\perp}\tau U$. The following summarizes some important facts about $J(U)$.

Proposition 4.2. *Let B be the Bongartz complement of U . Then we have:*

- (a) [27, Theorem 3.8] *The subcategory $J(U)$ is equivalent to $\text{mod } \text{End}(B \amalg U)/I$, where I is the ideal generated by all maps factoring through U .*
- (b) [7, Cor. 3.22], [15, Theorem 3.28] *The subcategory $J(U)$ is an exact abelian (wide) subcategory of $\text{mod } \Lambda$.*
- (c) [27, Theorem 3.8] *If U is indecomposable, then $J(U)$ has $n - 1$ simple modules up to isomorphism.*

The following result of Auslander and Smalø is very useful.

Lemma 4.3. [3, Proposition 5.8] *For X, Y in $\text{mod } \Lambda$ we have $\text{Hom}(X, \tau Y) = 0$ if and only if $\text{Ext}^1(Y, \text{Gen } X) = 0$*

Lemma 4.4. (a) *If X is in $\text{Gen } U$, then X is in $\mathcal{P}(\text{Gen } U)$ if and only if $X \amalg U$ is τ -rigid.*

- (b) *If $X \amalg U$ is τ -rigid, then $\text{Ext}^1(X, t(Z)) = 0$ for any module Z .*

Proof. We have (see Theorem 1.6) that $\mathcal{P}(\text{Gen } U) = \text{add}(C \amalg U)$, where $C \amalg U$ is τ -rigid. Hence, if X is in $\mathcal{P}(\text{Gen } U)$, then $X \amalg U$ is τ -rigid.

Conversely, assume $X \amalg U$ is τ -rigid. Then it follows from Lemma 4.3 that X lies in $\mathcal{P}(\text{Gen } U)$. This proves (a), and (b) is a direct consequence of (a). \square

Our next step towards the main result, is the following bijection.

Proposition 4.5. *For a τ -rigid module U , and with notation as before, the map f induces a bijection:*

$$\begin{array}{c} \text{Objects } X \text{ in } \text{ind mod } \Lambda \text{ such that } X \amalg U \text{ is } \tau\text{-rigid and } X \text{ is not in } \text{Gen } U \\ \updownarrow \\ \text{Objects in } \text{ind } J(U) \text{ which are } \tau\text{-rigid in } J(U) \end{array}$$

In order to prove Proposition 4.5, we will need the following lemmas.

Lemma 4.6. *If X is an indecomposable Λ -module such that $X \amalg U$ is τ -rigid, then either $f(X)$ is indecomposable or $f(X) = 0$. We have $f(X) = 0$ if and only if X is in $\text{Gen } U$.*

Proof. Note that $f(X) = 0$ if and only if X is in $\text{Gen } U$, by the definition of f . For the rest of the statement, it is sufficient to prove that there is a surjective ring map $\text{End}(X) \rightarrow \text{End}(f(X))$. The (well-known) functoriality of f gives a map from $\text{End}(X)$ to $\text{End}(f(X))$; we recall the construction now. Let ϕ be in $\text{End}(X)$ and consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & t(X) & \xrightarrow{\eta_X} & X & \xrightarrow{u_X} & f(X) \longrightarrow 0 \\ & & & & \downarrow \phi & & \\ 0 & \longrightarrow & t(X) & \xrightarrow{\eta_X} & X & \xrightarrow{u_X} & f(X) \longrightarrow 0 \end{array}$$

Since $\text{Hom}(t(X), f(X)) = 0$, there is a map $\theta: f(X) \rightarrow f(X)$ such that $u_X \phi = \theta u_X$. Since u_X is an epimorphism, there is a unique map θ with this property, so this gives a well defined map $\text{End}(X) \rightarrow \text{End}(f(X))$. It is then easy to check that this map is a ring map. We claim that it is surjective.

Consider part of the long exact sequence

$$\text{Hom}(X, X) \rightarrow \text{Hom}(X, f(X)) \rightarrow \text{Ext}^1(X, t(X))$$

obtained by applying $\text{Hom}(X, _)$ to the canonical sequence of X . We have that $\text{Ext}^1(X, t(X)) = 0$ by Lemma 4.4. Hence the map $\text{Hom}(X, X) \rightarrow \text{Hom}(X, f(X))$ is surjective. Furthermore, applying $\text{Hom}(_, f(X))$ to the canonical sequence gives that $\text{Hom}(f(X), f(X)) \simeq \text{Hom}(X, f(X))$, since $\text{Hom}(t(X), f(X)) = 0$. The claim follows. \square

Lemma 4.7. *Let X, Y be indecomposable modules not in $\text{Gen } U$, and such that both $U \amalg X$ and $U \amalg Y$ are τ -rigid. Then $f(X) \simeq f(Y)$ implies that $X \simeq Y$.*

Proof. Fix an isomorphism $\phi: f(X) \rightarrow f(Y)$, and consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & t(X) & \longrightarrow & X & \xrightarrow{u_X} & f(X) \longrightarrow 0 \\ & & & & & & \downarrow \phi \\ 0 & \longrightarrow & t(Y) & \longrightarrow & Y & \xrightarrow{u_Y} & f(Y) \longrightarrow 0 \end{array}$$

where the rows are the canonical sequences for X and Y . Consider part of the long exact sequence:

$$\mathrm{Hom}(X, Y) \xrightarrow{(X, u_Y)} \mathrm{Hom}(X, f(Y)) \rightarrow \mathrm{Ext}^1(X, t(Y)).$$

We have that $\mathrm{Ext}^1(X, t(Y)) = 0$ by Lemma 4.4, and hence the map

$$\mathrm{Hom}(X, Y) \xrightarrow{\mathrm{Hom}(X, u_Y)} \mathrm{Hom}(X, f(Y))$$

is surjective. Now choose a map $\psi: X \rightarrow Y$ satisfying $u_Y\psi = \phi u_X$. By a symmetric argument, we can also choose a map $\psi': Y \rightarrow X$ such that $u_X\psi' = \phi^{-1}u_Y$.

Now consider the composition $\psi'\psi$ in $\mathrm{End}(X)$. By Fitting's Lemma, any endomorphism of an indecomposable finite length module is either invertible or nilpotent. For any positive integer n we have $u_X(\psi'\psi)^n = (\phi^{-1}\phi)^n u_X = u_X \neq 0$. Note that u_X is non-zero, since X is not in $\mathrm{Gen} U$. Hence $\psi'\psi$ is not nilpotent and thus an automorphism. Therefore ψ is an isomorphism, and this finishes the proof. \square

Recall from Proposition 4.2 that $J(U)$ is equivalent to the module category of a finite dimensional algebra which we call Γ_U . We say that an object in $J(U)$ is τ -rigid in $J(U)$ if the corresponding Γ_U -module under this equivalence is τ -rigid. The Auslander-Reiten translate in Γ_U -mod induces a translate on $J(U)$, but in general this is not the same as the Auslander-Reiten translate in Λ -mod. In particular, a module which is τ -rigid in $J(U)$ in the above sense is not necessarily a τ -rigid Λ -module (see Example 6.3).

Lemma 4.8. *Assume $U \amalg X$ is τ -rigid in $\mathrm{mod} \Lambda$. Then either*

- (i) X is in $\mathcal{P}(\mathrm{Gen} U)$ and $f(X) = 0$, or
- (ii) $f(X)$ is τ -rigid in $J(U)$.

Proof. Clearly $f(X) = 0$ if and only if X is in $\mathrm{Gen} U$. It is also clear that $f(X)$ is in $J(U)$, since ${}^\perp\tau U$ contains X and is closed under factors. If X is in $\mathrm{Gen} U$ and $U \amalg X$ is τ -rigid, then X is in $\mathcal{P}(\mathrm{Gen} U)$ by Lemma 4.4.

So assume $f(X) \neq 0$. Consider the following exact sequence, obtained by applying $\mathrm{Hom}(\cdot, \mathrm{Gen} X \cap J(U))$ to the canonical sequence for X :

$$\mathrm{Hom}(t(X), \mathrm{Gen} X \cap J(U)) \rightarrow \mathrm{Ext}^1(f(X), \mathrm{Gen} X \cap J(U)) \rightarrow \mathrm{Ext}^1(X, \mathrm{Gen} X \cap J(U))$$

Since $\mathrm{Hom}(X, \tau X) = 0$ we have $\mathrm{Ext}^1(X, \mathrm{Gen} X) = 0$ by Lemma 4.3, so in particular $\mathrm{Ext}^1(X, \mathrm{Gen} X \cap J(U)) = 0$. We have $\mathrm{Hom}(t(X), \mathrm{Gen} X \cap J(U)) = 0$ since $t(X)$ is in $\mathrm{Gen} U$, and $\mathrm{Hom}(\mathrm{Gen} U, J(U)) = 0$, since $J(U) \subseteq U^\perp$. Therefore also $\mathrm{Ext}^1(f(X), \mathrm{Gen} X \cap J(U)) = 0$.

We have $\mathrm{Gen} f(X) \subseteq \mathrm{Gen} X$, and hence $\mathrm{Gen}_{J(U)} f(X) \subseteq \mathrm{Gen} X \cap J(U)$, so we have $\mathrm{Ext}^1(f(X), \mathrm{Gen}_{J(U)} f(X)) = 0$. Since $J(U)$ is an extension-closed subcategory of $\mathrm{mod} \Lambda$, it follows from Lemma 4.3 that $f(X)$ is τ -rigid in $J(U)$. \square

Note that this could also be seen by completing $U \amalg X$ to a τ -tilting module (using [1], and then applying [27, Theorem 3.15]).

We can now complete the proof of the Proposition.

Proof of Proposition 4.5. It follows from Lemma 4.8 that f sends objects X in $\mathrm{ind} \mathrm{mod} \Lambda$ such that $X \amalg U$ is τ -rigid and not in $\mathrm{Gen} U$ to objects in $\mathrm{ind} J(U)$ which are τ -rigid in

$J(U)$. By Lemma 4.6 f sends indecomposable modules to indecomposable modules and it follows from Lemma 4.7 that f induces an injective map.

It is a direct consequence of Theorem 3.15 in [27] that f induces a surjective map. \square

Lemma 4.9. *The map $X \mapsto f(X)$ induces a bijection between $\text{ind } \mathcal{P}({}^\perp \tau U) \setminus \text{ind } U$ and $\text{ind } \mathcal{P}(J(U))$.*

Proof. By Lemma 4.6, indecomposables are preserved by f . The result follows from this fact and a special case of [27, Prop. 3.14]. \square

Definition 4.10. *Define a map $\rho: \{\text{ind } \mathcal{P}(\text{Gen } U) \setminus \text{ind}(U)\} \cup \{\text{ind}(\mathcal{P}(\Lambda) \cap {}^\perp U)\} \rightarrow \text{ind } \mathcal{P}(J(U))$ as follows.*

- (i) *If X lies in $\{\text{ind } \mathcal{P}(\text{Gen } U) \setminus \text{ind}(U)\}$, let $\mathbb{P}_{U_X} \rightarrow \mathbb{P}_X$ be a minimal right add \mathbb{P}_U -approximation of \mathbb{P}_X in $D^b(\text{mod } \Lambda)$ (where U_X lies in $\text{add } U$), and complete this to a triangle $\mathbb{R}_X \rightarrow \mathbb{P}_{U_X} \rightarrow \mathbb{P}_X \rightarrow \cdot$. We set $\rho(X) = f(H^0(\mathbb{R}_X))$.*
- (ii) *If X lies in $\text{ind}(\mathcal{P}(\Lambda) \cap {}^\perp U)$, let $a_X: X \rightarrow B_X$ be a minimal left $\mathcal{P}({}^\perp \tau U)$ -approximation of X in $\text{mod } \Lambda$, so B_X is an indecomposable direct summand of the Bongartz complement of U . We set $\rho(X) = f(B_X)$.*

Remark 4.11. *Note that in case (i) we have that since X is a direct summand of the co-Bongartz complement of U , it follows from Lemmas 3.2 and 3.4 that $\mathbb{R}_X = \mathbb{P}_{B_X}$ for an indecomposable direct summand B_X of the Bongartz complement of U . By Proposition 3.7(a), taking $C_i = X$, there is an exact sequence*

$$B_X \xrightarrow{a_X} U_X \xrightarrow{b_X} X \rightarrow 0,$$

where $B_X = H^0(\mathbb{R}_X)$, the map a_X is a minimal left add U -approximation and b_X is a minimal right add U -approximation.

Note that in case (ii) we have that since X is a direct summand of Q , where $Q[1]$ is as in Theorem 1.6, it follows from Proposition 3.7(b), taking $Q_i = X$, that there is an exact sequence

$$X \xrightarrow{c_X} B_X \xrightarrow{d_X} U_X \rightarrow 0,$$

where d_X is a minimal left add U -approximation.

Now, combining Proposition 3.7 and Lemma 4.9, we obtain the following.

Proposition 4.12. *The map ρ defines a bijection*

$$\begin{array}{c} \{\text{ind } \mathcal{P}(\text{Gen } U) \setminus \text{ind}(U)\} \cup \{\text{ind}(\mathcal{P}(\Lambda) \cap {}^\perp U)\} \\ \updownarrow \\ \text{ind } \mathcal{P}(J(U)) \end{array}$$

We end this section with a lemma which will be useful for the proof of the main result.

Recall that a module Y_0 in a full subcategory \mathbb{Y} of $\text{mod } \Lambda$ is called *split projective* in \mathbb{Y} , if each epimorphism $Y \rightarrow Y_0$ with Y in \mathbb{Y} is split.

Lemma 4.13. *Let B be the Bongartz complement of U . Then each direct summand B_i in B is split projective in ${}^\perp \tau U$.*

Proof. Note that since B is the Bongartz complement of U we have that ${}^{\perp}\tau(B \amalg U) = {}^{\perp}\tau U$ and $\text{add}(B \amalg U) = \mathcal{P}({}^{\perp}\tau U)$, by Theorem 1.6.

Assume B_i is not split projective. Then there is an epimorphism $Q' \rightarrow B_i$, with Q' in $\text{add } B/B_i \amalg U$. This map factors through the right $\text{add } B/B_i \amalg U$ -approximation $Q \rightarrow B_i$, so in particular the map $Q \rightarrow B_i$ is also an epimorphism. Let B'_i be the kernel of $Q \rightarrow B_i$. Then, by Lemma 3.8, we have that B'_i is in ${}^{\perp}(\tau(B/B_i \amalg U)) = {}^{\perp}\tau U$. But then the sequence $0 \rightarrow B'_i \rightarrow Q \rightarrow B_i \rightarrow 0$ splits, since B_i is Ext-projective in ${}^{\perp}\tau U$. This is a contradiction. \square

5. MAIN THEOREM

In this section we complete the proof of Theorem 1.4, using the reduction technique of Section 4 and the bijection of Section 3.

Note that for a indecomposable projective module P , there is a primitive idempotent e in Λ , such that $P \simeq \Lambda e$, and we have that $J(P) = P^{\perp}$ is equivalent to $\text{mod}(\Lambda/\Lambda e\Lambda)$.

Lemma 5.1. *If $(\mathcal{U}_1, \dots, \mathcal{U}_{t-1}, \mathcal{U}_t)$ in $C(\Lambda)$ is a signed τ -exceptional sequence, then $t \leq n$.*

Proof. This is clear when $n = 1$. The statement follows by induction on n , since, by Proposition 4.2, we have that $\delta(\Lambda') = \delta(\Lambda) - 1$, when $\text{mod } \Lambda'$ is equivalent to $J(\mathcal{U}_t)$. \square

For an object \mathbb{X} in $C(\Lambda)$, we set

$$|\mathbb{X}| = \begin{cases} X, & \text{if } X = X \text{ is in } \text{mod } \Lambda; \\ Y, & \text{if } X = Y[1] \text{ is in } (\text{mod } \Lambda)[1]. \end{cases}$$

Warning: apart from $|\mathcal{U}_t|$, the modules $|\mathcal{U}_i|$ arising from a signed τ -exceptional sequence $(\mathcal{U}_1, \dots, \mathcal{U}_t)$ are not necessarily τ -rigid in $\text{mod } \Lambda$. However, we have the following.

Lemma 5.2. *Let U be a τ -rigid object in $\text{mod } \Lambda$ and suppose that Y is τ -rigid in $J(U)$. Then $\text{Ext}^1(Y, Y) = 0$.*

Proof. It follows from the Auslander-Reiten formula that $\text{Ext}_{J(U)}^1(Y, Y) = 0$. The result then follows from the fact that $J(U)$ is an extension closed subcategory of $\text{mod } \Lambda$. \square

Corollary 5.3. *Let $(\mathcal{U}_1, \dots, \mathcal{U}_t)$ in $C(\Lambda)$ be a signed τ -exceptional sequence. Then $\text{Ext}^1(|U_i|, |U_i|) = 0$ for all i .*

Proof. Firstly, $|\mathcal{U}_t|$ is τ -rigid in $\text{mod } \Lambda$, giving the result for $i = t$ using the Auslander-Reiten formula. The module $|\mathcal{U}_{t-1}|$ is τ -rigid in $J(\mathcal{U}_t)$, so the result for $i = t - 1$ follows from Lemma 5.2. The result for all i follows from an inductive argument. \square

We now restate our main result.

Theorem 5.4. *Let Λ be a finite dimensional algebra. For each $t \in \{1, \dots, n\}$ there is a bijection between the set of ordered support τ -rigid objects of length t in $C(\Lambda)$ and the set of signed τ -exceptional sequences of length t in $C(\Lambda)$.*

For $t = n$, we obtain.

Corollary 5.5. *Let Λ be a finite dimensional algebra. There is a bijection between the set of ordered support τ -tilting objects in $C(\Lambda)$ and the set of complete signed τ -exceptional sequences in $C(\Lambda)$.*

The remainder of this section is devoted to proving Theorem 5.4. The main idea of the proof is to work by induction on t , making use of Propositions 4.5 and 4.12, which we for convenience reformulate below.

Recall the definition of ρ from Definition 4.10 (see also Proposition 4.12). Also, note that by Lemma 4.4, we have that for U τ -rigid and $X \in \text{Gen } U$, then $X \amalg U$ is τ -rigid if and only if X is in $\mathcal{P}(\text{Gen } U)$. Hence, if $X \in \text{ind Gen } U \setminus \text{ind } U$ and $X \amalg U$ is τ -rigid, then ρ can be applied to X .

Proposition 5.6. *Let U be a τ -rigid module in $\text{mod } \Lambda$. Then there is a bijection \mathcal{E}_U between the sets*

$$\{X \in \text{ind mod } \Lambda \setminus \text{ind } U \mid X \amalg U \text{ is } \tau\text{-rigid}\} \cup \{\text{ind}(\mathcal{P}(\Lambda) \cap {}^\perp U)[1]\}$$

and

$$\{X \in \text{ind } J(U) \mid X \text{ } \tau\text{-rigid in } J(U)\} \cup \{\text{ind}(\mathcal{P}(J(U)))[1]\}$$

given by

$$\mathcal{E}_U(X) = \begin{cases} f(X), & X \in \{\text{ind mod } \Lambda \setminus \text{ind } U \mid X \amalg U \text{ is } \tau\text{-rigid and } X \notin \text{Gen } U\}; \\ \rho(X)[1], & X \in \{\text{ind mod } \Lambda \setminus \text{ind } U \mid X \amalg U \text{ is } \tau\text{-rigid and } X \in \text{Gen } U\}; \\ \rho(X[-1])[1], & X \in \text{ind}(\mathcal{P}(\Lambda) \cap {}^\perp U)[1]. \end{cases}$$

Remark 5.7. *The reduction technique of Proposition 5.6 can be seen as an extension of Jasso's reduction technique [27, Cor. 3.18]. Indeed, Jasso's technique forms an important part of the proof. However, it is important to note that modules generated by U (using the notation of Proposition 5.6) (which are killed by Jasso's map) are not sent to modules, but rather to 'signed' objects $P[1]$ in $C(J(U))$ (where P is a relative projective in $J(U)$). The cases of our proof dealing with such objects do not follow from Jasso's theorem.*

The bijection of Proposition 5.6 is needed for the proof of our main result, Theorem 5.4, even in the complete case.

We note that in recent work [5] (after this paper appeared on the Arxiv) Børve used the correspondence between support τ -tilting modules and 2-term silting objects, and results from [27, Section 4] to give an alternative proof of Proposition 5.6 using Iyama-Yoshino reduction [26, 4.2, 4.7].

We extend the domain of \mathcal{E}_U to

$$\text{add}(\{X \in \text{ind mod } \Lambda \setminus \text{ind } U \mid X \amalg U \text{ is } \tau\text{-rigid}\} \cup \{\text{ind}(\mathcal{P}(\Lambda) \cap {}^\perp U)[1]\})$$

by setting $\mathcal{E}_U(\amalg_{i=1}^s X_i) = \amalg_{i=1}^s \mathcal{E}_U(X_i)$ for objects X_1, X_2, \dots, X_s in

$$\{X \in \text{ind mod } \Lambda \setminus \text{ind } U \mid X \amalg U \text{ is } \tau\text{-rigid}\} \cup \{\text{ind}(\mathcal{P}(\Lambda) \cap {}^\perp U)[1]\}.$$

Proposition 5.8. *Let U be a τ -rigid module in $\text{mod } \Lambda$ with $\delta(U) = t'$. For any positive integer $t \leq n - t'$, the map \mathcal{E}_U induces a bijection between the set of support τ -rigid objects X in $C(\Lambda)$ such that $\delta(X) = t$, $X \amalg U$ is support τ -rigid and $\text{add } X \cap \text{add } U = 0$, and the set of support τ -rigid objects Y in $C(J(U))$ with $\delta(Y) = t$.*

Proof. We first need to prove that for any X in $C(\Lambda)$, if $X \amalg U$ is support τ -rigid with $\text{add } X \cap \text{add } U = 0$, then $\mathcal{E}_U(X) = \widetilde{X}$ is support τ -rigid in $C(J(U))$. For this, let X_i and X_j be two indecomposable direct summands in X , and consider the following cases.

Case I: Let X_i, X_j both be in $\text{mod } \Lambda$ and not in $\text{Gen } U$. Then by Lemma 4.8, we have that

$$\widetilde{X}_i \amalg \widetilde{X}_j = f(X_i) \amalg f(X_j) = f(X_i \amalg X_j)$$

is τ -rigid in $J(U)$.

Case II: Let X_i be in $\text{Gen } U$, and assume X_j is in $\text{mod } \Lambda$ but not in $\text{Gen } U$.

Then $\widetilde{X}_i[-1]$ is in $\mathcal{P}(J(U))$, and we need to prove that $\text{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = 0$. Note that

$$\widetilde{X}_i = \mathcal{E}_U(X_i) = \rho(X_i)[1] = f(H^0(\mathbb{R}_{X_i}))[1] = f(B_{X_i})[1]$$

(see Definition 4.10, Remark 4.11 and Proposition 5.6), where there is an exact sequence

$$B_{X_i} \xrightarrow{a_{X_i}} U_{X_i} \rightarrow X_i \rightarrow 0$$

with a_{X_i} a minimal left add U -approximation.

Moreover $\widetilde{X}_j = f(X_j)$, so we need to prove that $\text{Hom}(f(B_{X_i}), f(X_j)) = 0$. We have that B_{X_i} is in $\mathcal{P}({}^\perp \tau U)$ by Lemma 3.4, and $f(B_{X_i})$ is in $\mathcal{P}(J(U))$ by Lemma 4.9.

We have that $0 \rightarrow \text{Hom}(f(B_{X_i}), f(X_j)) \rightarrow \text{Hom}(B_{X_i}, f(X_j))$ is exact, so it suffices to show that $\text{Hom}(B_{X_i}, f(X_j)) = 0$. For this, apply $\text{Hom}(B_{X_i}, _)$ to the exact sequence

$$0 \rightarrow t(X_j) \rightarrow X_j \rightarrow f(X_j) \rightarrow 0$$

and consider the long exact sequence

$$0 \rightarrow \text{Hom}(B_{X_i}, t(X_j)) \rightarrow \text{Hom}(B_{X_i}, X_j) \rightarrow \text{Hom}(B_{X_i}, f(X_j)) \rightarrow \text{Ext}^1(B_{X_i}, t(X_j)).$$

We have that $t(X_j)$ is in $\text{Gen } U \subseteq {}^\perp \tau U$, and B_{X_i} is in $\mathcal{P}({}^\perp \tau U)$, so $\text{Ext}^1(B_{X_i}, t(X_j)) = 0$. Therefore it is sufficient to show that $\text{Hom}(B_{X_i}, t(X_j)) \rightarrow \text{Hom}(B_{X_i}, X_j)$ is an epimorphism.

So consider an arbitrary map $g: B_{X_i} \rightarrow X_j$. We first claim that this map factors through an object in $\text{add } U$. Recall from Definition 4.10 that there is a triangle

$$\mathbb{P}_{B_{X_i}} \rightarrow \mathbb{P}_{U_{X_i}} \rightarrow \mathbb{P}_{X_i} \rightarrow$$

with U_{X_i} in $\text{add } U$. Now apply $\text{Hom}_{\mathcal{K}}(_, \mathbb{P}_{X_j})$ to this triangle, and consider the exact sequence

$$\text{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}) \rightarrow \text{Hom}(\mathbb{P}_{B_{X_i}}, \mathbb{P}_{X_j}) \rightarrow \text{Hom}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1])$$

By assumption we have $\text{Hom}(X_j, \tau X_i) = 0$, and this implies that the last term $\text{Hom}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1])$ vanishes, by Lemma 2.3. Hence the map $\text{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}) \rightarrow \text{Hom}(\mathbb{P}_{B_{X_i}}, \mathbb{P}_{X_j})$ is an epimorphism, and so any map $\mathbb{P}_{B_{X_i}} \rightarrow \mathbb{P}_{X_j}$ factors through $\mathbb{P}_{U_{X_i}}$.

By Lemma 2.1, this means that the map g factors through U_{X_i} in $\text{add } U$. Assume $B_{X_i} \xrightarrow{g} X_j = B_{X_i} \xrightarrow{h} U_{X_i} \xrightarrow{p} X_j$. Then $U_{X_i} \xrightarrow{p} X_j$ factors through $t(X_j) \rightarrow X_j$. Hence g also factors through $t(X_j) \rightarrow X_j$, and we have proved the claim that $\text{Hom}(B_{X_i}, t(X_j)) \rightarrow \text{Hom}(B_{X_i}, X_j)$ is an epimorphism. It then follows that $\text{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = \text{Hom}(B_{X_i}, f(X_j)) = 0$.

Case III: Now assume $X_i = P[1]$ for P in $\mathcal{P}(\Lambda) \cap {}^\perp U$, and X_j lies in $\text{mod } \Lambda$ and not in $\text{Gen } U$. Then $\text{Hom}(P, X_j) = 0$. We need to prove that $\text{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = 0$. By Definition 4.10, there is an exact sequence:

$$(10) \quad P \xrightarrow{c_P} B_P \xrightarrow{d_P} U_P \rightarrow 0$$

By Definition 4.10, we have $\widetilde{X}_i = f(B_P)[1]$ and $\widetilde{X}_j = f(X_j)$.

Note that $\text{Hom}(P, X_j) = 0$ implies that $\text{Hom}(P, \widetilde{X}_j) = 0$, since P is projective. We also have $\text{Hom}(U, \widetilde{X}_j) = \text{Hom}(U, f(X_j)) = 0$, by definition of f .

Applying $\text{Hom}(\cdot, \widetilde{X}_j)$ to (10) gives an exact sequence

$$0 \rightarrow \text{Hom}(U_P, \widetilde{X}_j) \rightarrow \text{Hom}(B_P, \widetilde{X}_j) \rightarrow \text{Hom}(P, \widetilde{X}_j)$$

The end terms vanish, and hence we obtain that $\text{Hom}(B_P, \widetilde{X}_j) = 0$. Clearly then also $\text{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = \text{Hom}(f(B_P), \widetilde{X}_j) = 0$. This finishes the proof of Case III. Now combining the Cases I, II and III, the claim that $\mathcal{E}_U(X)$ is a support τ -rigid object in $J(U)$ follows.

Now, let $M = \coprod_{i=1}^t M_i$ be a support τ -rigid object in $\mathcal{C}(J(U))$ with $\delta(M) = t$, where each M_i is indecomposable. There are, by Proposition 5.6, unique indecomposable modules X_i such that $\mathcal{E}_U(X_i) = M_i$, the object $U \amalg X_i$ is support τ -rigid, and $X_i \notin \text{add } U$. We need to prove that $X = \coprod_{i=1}^t X_i$ is support τ -rigid as well. For this we consider two arbitrary summands X_i, X_j in X and the following cases.

Case I: Assume $X_i = P[1]$ with P in $\mathcal{P}(\Lambda) \cap {}^\perp U$, and assume that X_j is in $\text{mod } \Lambda$ is such that $X_j \amalg U$ is τ -rigid, X_j is not in $\text{Gen } U$, and $\text{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = 0$. We need to prove that $\text{Hom}(P, X_j) = 0$.

Note that by Definition 4.10, we have $\widetilde{X}_i[-1] = f(B_P)$, where $P \rightarrow B_P$ is a minimal $\mathcal{P}({}^\perp \tau U)$ -approximation of P . Moreover $\widetilde{X}_j = f(X_j)$.

We have that $\text{Hom}(f(B_P), f(X_j)) = \text{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = 0$. Apply $\text{Hom}(\cdot, f(X_j))$ to the exact sequence $0 \rightarrow t(B_P) \rightarrow B_P \rightarrow f(B_P) \rightarrow 0$ and consider the exact sequence

$$0 \rightarrow \text{Hom}(f(B_P), f(X_j)) \rightarrow \text{Hom}(B_P, f(X_j)) \rightarrow \text{Hom}(t(B_P), f(X_j))$$

The last term vanishes since $t(B_P)$ is in $\text{Gen } U$ and $f(X_j)$ is in U^\perp . We then obtain that also $\text{Hom}(B_P, f(X_j)) = 0$.

Since by assumption X is in ${}^\perp \tau U$, we also have that $f(X)$ is in ${}^\perp \tau U$. Hence any map $P \rightarrow f(X_j)$ factors through $B_P \rightarrow f(X_j)$, since $P \rightarrow B_P$ is a left ${}^\perp \tau U$ -approximation (see Lemma 3.5). Since $\text{Hom}(B_P, f(X_j)) = 0$, we have $\text{Hom}(P, f(X_j)) = 0$. We also have that $\text{Hom}(P, U) = 0$ implies that $\text{Hom}(P, \text{Gen } U) = 0$, so in particular $\text{Hom}(P, t(X_j)) = 0$.

Since $\text{Hom}(P, f(X_j)) = 0 = \text{Hom}(P, t(X_j))$, we indeed also have $\text{Hom}(P, X_j) = 0$, and this finishes Case I.

Case II: If X_i, X_j are both in $\text{mod } \Lambda$ and not in $\text{Gen } U$, then $\widetilde{X}_i = f(X_i), \widetilde{X}_j = f(X_j)$ is such that $\widetilde{X}_i \amalg \widetilde{X}_j = \widetilde{X_i \amalg X_j}$ is τ -rigid in $J(U)$, then $X_i \amalg X_j$ is τ -rigid according to [27, Cor. 3.18].

Case III: Now assume X_i is in $\mathcal{P}(\text{Gen } U)$, and hence \widetilde{X}_i in $\mathcal{P}(J(U))[1]$, while X_j is in $\text{mod } \Lambda$ but not in $\text{Gen } U$.

By assumption $\text{Hom}(\widetilde{X}_i[-1], \widetilde{X}_j) = 0$, and we need to prove that $\text{Hom}(X_i, \tau X_j) = 0 = \text{Hom}(X_j, \tau X_i)$

Since X_i is in $\text{Gen } U$, and $\text{Hom}(U, \tau X_j) = 0$, we have that also $\text{Hom}(X_i, \tau X_j) = 0$.

By Lemma 2.3 we have that in order to prove that $\text{Hom}(X_j, \tau X_i) = 0$, it is sufficient to prove that $\text{Hom}_{\mathcal{K}}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1]) = 0$.

We apply $\text{Hom}_{\mathcal{K}}(\cdot, \mathbb{P}_{X_j})$ to the triangle

$$\mathbb{P}_{B_{X_i}} \rightarrow \mathbb{P}_{U_{X_i}} \rightarrow \mathbb{P}_{X_i} \rightarrow$$

(see Definition 4.10) and consider the exact sequence

$$(11) \quad \text{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}) \rightarrow \text{Hom}(\mathbb{P}_{B_{X_i}}, \mathbb{P}_{X_j}) \rightarrow \text{Hom}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1]) \rightarrow \text{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}[1])$$

We first note that $\text{Hom}(X_j, \tau U) = 0$ implies that the last term $\text{Hom}_{\mathcal{K}}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}[1])$ vanishes. It is therefore sufficient to prove that the first map $\text{Hom}(\mathbb{P}_{U_{X_i}}, \mathbb{P}_{X_j}) \rightarrow \text{Hom}(\mathbb{P}_{B_{X_i}}, \mathbb{P}_{X_j})$ is an epimorphism, that is: we claim that any map $\mathbb{P}_{B_{X_i}} \rightarrow \mathbb{P}_{X_j}$ factors through $\mathbb{P}_{B_{X_i}} \rightarrow \mathbb{P}_{U_{X_i}}$. For this, it is sufficient that any map $B_{X_i} \rightarrow X_j$ factors through $B_{X_i} \rightarrow U_{X_i}$. Consider the exact sequence

$$(12) \quad \text{Hom}(B_{X_i}, t(X_j)) \rightarrow \text{Hom}(B_{X_i}, X_j) \rightarrow \text{Hom}(B_{X_i}, f(X_j))$$

obtained by applying $\text{Hom}(B_{X_i}, \cdot)$ to the canonical sequence for X_j . We claim that the last term vanishes. For this consider the exact sequence

$$(13) \quad \text{Hom}(f(B_{X_i}), f(X_j)) \rightarrow \text{Hom}(B_{X_i}, f(X_j)) \rightarrow \text{Hom}(t(B_{X_i}), f(X_j))$$

obtained by applying $\text{Hom}(\cdot, f(X_j))$ to the canonical sequence for B_{X_i} . In (13) the first term vanishes by assumption, and the last term vanishes since $t(B_{X_i})$ is in $\text{Gen } U$ and $f(X_j)$ is in U^\perp . Hence $\text{Hom}(B_{X_i}, f(X_j))$, which is the last term of sequence (12) also vanishes. This means that any map $B_{X_i} \rightarrow X_j$ factors $B_{X_i} \rightarrow t(X_j) \rightarrow X_j$.

By Lemma 3.9, the map $B_{X_i} \rightarrow U_{X_i}$ is a $\text{Gen } U$ -approximation. So the map $B_{X_i} \rightarrow t(X_j)$ factors $B_{X_i} \rightarrow U_{X_i} \rightarrow t(X_j)$, and hence our original map $B_{X_i} \rightarrow X_j$ factors $B_{X_i} \rightarrow U_{X_i} \rightarrow t(X_j) \rightarrow X_j$. We have now proved that any map $B_{X_i} \rightarrow X_j$ factors through $B_{X_i} \rightarrow U_{X_i}$, and hence any map $\mathbb{P}_{B_{X_i}} \rightarrow \mathbb{P}_{X_j}$ factors through $\mathbb{P}_{B_{X_i}} \rightarrow \mathbb{P}_{U_{X_i}}$. Therefore $\text{Hom}(\mathbb{P}_{X_i}, \mathbb{P}_{X_j}[1]) = 0$, which implies $\text{Hom}(X_j, \tau X_i) = 0$, and the claim is proved. This finishes the proof of case III.

Combining Cases I, II and III, and noting that the remaining cases can be checked trivially, proves the claim that $X = \coprod X_i$ is support τ -rigid, and this concludes the proof of the proposition. \square

We have the following special case of Proposition 5.8.

Corollary 5.9. *Let $t > 1$. Let U be an indecomposable τ -rigid module. The map \mathcal{E}_U induces a bijection between ordered support τ -rigid objects in $C(\Lambda)$ with last term U and length t , and ordered support τ -rigid objects of length $t - 1$ in $C(J(U))$.*

We need also to deal with the case where the last term in an ordered support τ -rigid objects in $C(\Lambda)$ is of the form $P[1]$. For this, we first observe the following.

Lemma 5.10. *Let P be a projective Λ -module, and consider $J(P[1]) = P^\perp$.*

- (a) *The τ -rigid modules in $J(P[1])$ are exactly the τ -rigid modules X in $\text{mod } \Lambda$ with $\text{Hom}(P, X) = 0$.*
- (b) *The map $Q \mapsto f_P(Q)$ gives a bijection between the indecomposables in $\mathcal{P}(\Lambda) \setminus \text{add } P$ and the indecomposables in $\mathcal{P}(P^\perp)$.*

Proof. Part (a) is a direct consequence of [1, Lemma 2.1]. Part (b) follows from Lemma 4.9. \square

For a projective module P in $\mathcal{P}(\Lambda)$, consider the map $\mathcal{E}_{P[1]}$ from

$$\text{add}\{X \in \text{ind mod } \Lambda \mid X \text{ } \tau\text{-rigid, and } \text{Hom}(P, X) = 0\} \cup (\text{ind } \mathcal{P}(\Lambda) \setminus \text{ind } P)[1]$$

to

$$\text{add}\{X \in \text{ind } J(P[1]) \mid X \text{ } \tau\text{-rigid}\} \cup \text{ind } \mathcal{P}(J(P[1]))[1]$$

defined as follows. For X indecomposable, we set

$$\mathcal{E}_{P[1]}(X) = \begin{cases} X & \text{if } X \text{ is } \tau\text{-rigid, and } \text{Hom}(P, X) = 0 \\ f_P(X)[1] & \text{if } X \in \mathcal{P}(\Lambda) \setminus \text{ind } P \end{cases}$$

For $X = \coprod_{i=1}^t X_i$, with each X_i in

$$\{X \in \text{ind mod } \Lambda \mid X \text{ } \tau\text{-rigid, and } \text{Hom}(P, X) = 0\} \cup (\text{ind } \mathcal{P}(\Lambda) \setminus \text{ind } P)[1],$$

we set $\mathcal{E}_{P[1]}(X) = \coprod_{i=1}^t \mathcal{E}_{P[1]}(X_i)$.

Proposition 5.11. *Let P be in $\mathcal{P}(\Lambda)$ with $\delta(P) = t'$.*

(a) *The map $\mathcal{E}_{P[1]}$ is a bijection between the sets*

$$\{X \in \text{ind mod } \Lambda \mid X \text{ is } \tau\text{-rigid and } \text{Hom}(P, X) = 0\} \cup \{\text{ind}(\mathcal{P}(\Lambda)) \setminus \text{ind } P\}[1]$$

and

$$\{X \in \text{ind } J(P[1]) \mid X \text{ is } \tau\text{-rigid}\} \cup \{\text{ind } \mathcal{P}(J(P[1]))[1]\}.$$

(b) *For any positive integer $t \leq n - t'$, the map $\mathcal{E}_{P[1]}$ induces a bijection between the set of support τ -rigid objects X in $C(\Lambda)$ such that $\delta(X) = t$, the object $X \amalg P[1]$ is support τ -rigid and $\text{add } X \cap \text{add } P[1] = 0$, and the set of support τ -rigid objects in $C(P^\perp)$ with t indecomposable direct summands.*

Proof. Part (a) follows directly from Lemma 5.10.

Part (b): Let $Q \notin \text{add } P$ be an indecomposable module in $\mathcal{P}(\Lambda)$, and let X be in P^\perp . Apply $\text{Hom}(_, X)$ to the canonical sequence

$$0 \rightarrow t_P(Q) \rightarrow Q \rightarrow f_P(Q) \rightarrow 0.$$

Since $\text{Hom}(P, X) = 0$, we have that $\text{Hom}(\text{Gen } P, X) = 0$ and thus that $\text{Hom}(t_P(Q), X) = 0$. It follows that $\text{Hom}(Q, X) \simeq \text{Hom}(f_P(Q), X)$. The claim follows from combining this with part (a). \square

Corollary 5.12. *Let $t > 1$. Let P be an indecomposable projective module. Then the map $\mathcal{E}_{P[1]}$ induces a bijection between ordered support τ -rigid objects in $C(\Lambda)$ with last term $P[1]$ and length t , and ordered support τ -rigid objects of length $t - 1$ in $C(J(P[1])) = C(P^\perp)$.*

We can now prove Theorem 5.4.

Proof of Theorem 5.4. We prove the result by induction on $\delta(\Lambda) = n$. The statement for $t = 1$ is clear. In particular, this deals with the case $n = 1$. So we assume the result to be true for algebras with a smaller number of indecomposable projective modules up to isomorphism. Let U be an indecomposable τ -rigid object in $\text{mod } \Lambda$. By Corollary 5.9, there is a bijection between the ordered support τ -rigid objects in $C(\Lambda)$ ending in U and the ordered support τ -rigid objects in $C(J(U))$. The ordered support τ -rigid objects in $C(J(U))$

are by the induction hypothesis in bijection with the signed τ -exceptional sequences in $C(J(U))$. And by definition a sequence (U_1, \dots, U_{t-1}) is a signed τ -exceptional sequence in $C(J(U))$ if and only if $(\mathcal{U}_1, \dots, \mathcal{U}_{t-1}, U)$ is a signed τ -exceptional sequence in $C(\Lambda)$.

Let P be an indecomposable module in $\mathcal{P}(\Lambda)$. In a similar way to the above, there is a bijection between ordered support τ -rigid objects in $C(\Lambda)$ ending in $P[1]$ and signed τ -exceptional sequences in $C(J(P[1]))$, using the induction hypothesis and Corollary 5.12. \square

Remark 5.13. *We now give a more explicit description of the bijection constructed in Theorem 5.4, and the inverse of this bijection. Let \mathbb{W} denote a wide subcategory of $\text{mod } \Lambda$ which is equivalent to a module category, and let $n_{\mathbb{W}}$ denote its rank. Let \mathcal{U} denote an indecomposable τ -rigid object in $C(\mathbb{W})$, that is either $\mathcal{U} = U$ for an indecomposable τ -rigid module in \mathbb{W} , or $\mathcal{U} = P[1]$, where P is indecomposable projective in \mathbb{W} . Recall, in particular, that by Proposition 4.2, we have that $J_{\mathbb{W}}(\mathcal{U})$ is a wide subcategory of \mathbb{W} , and hence of $\text{mod } \Lambda$, equivalent to a module category.*

Now, consider the bijections obtained by combining Propositions 5.6 and 5.11.

$$\begin{aligned} & \{\mathcal{X} \in \text{ind } C(\mathbb{W}) \setminus \mathcal{U} \mid \mathcal{X} \amalg \mathcal{U} \text{ is } \tau\text{-rigid}\} \\ & \quad \mathcal{E}_{\mathcal{U}}^{\mathbb{W}} \downarrow \quad \uparrow \mathcal{F}_{\mathcal{U}}^{\mathbb{W}} \\ & \{\mathcal{X} \in \text{ind } C(J(\mathcal{U})) \mid \mathcal{X} \text{ } \tau\text{-rigid in } J(\mathcal{U})\} \end{aligned}$$

Consider, for each $t = 1, \dots, n_{\mathbb{W}}$, the bijections

$$\begin{aligned} & \{\text{ordered } \tau\text{-rigid objects in } \mathbb{W} \text{ with } t \text{ indecomposable direct summands}\} \\ & \quad \Psi_t^{\mathbb{W}} \downarrow \quad \uparrow \Phi_t^{\mathbb{W}} \\ & \{\tau\text{-exceptional sequences in } \mathbb{W} \text{ of length } t\} \end{aligned}$$

where $\Psi_t^{\mathbb{W}}$ is the bijection constructed in Theorem 5.4, and $\Phi_t^{\mathbb{W}}$ is its inverse. Then we have

$$\Psi_t^{\mathbb{W}}(\mathcal{T}_1, \dots, \mathcal{T}_t) = (\Psi_{\mathcal{T}_{t-1}}^{J_{\mathbb{W}}(\mathcal{T}_t)}(\mathcal{E}_{\mathcal{T}_t}^{\mathbb{W}}(\mathcal{T}_1), \dots, \mathcal{E}_{\mathcal{T}_t}^{\mathbb{W}}(\mathcal{T}_{t-1})), \mathcal{T}_t)$$

Now let

$$\begin{array}{ll} \mathbb{W}_t = \mathbb{W} & \mathcal{U}_t = \mathcal{T}_t \\ \mathbb{W}_{t-1} = J_{\mathbb{W}_t}(\mathcal{U}_t) & \mathcal{U}_{t-1} = \mathcal{E}_{\mathcal{U}_t}^{\mathbb{W}_t}(\mathcal{T}_{t-1}) \\ \vdots & \vdots \\ \mathbb{W}_i = J_{\mathbb{W}_{i+1}}(\mathcal{U}_{i+1}) & \mathcal{U}_i = \mathcal{E}_{\mathcal{U}_{i+1}}^{\mathbb{W}_{i+1}} \dots \mathcal{E}_{\mathcal{U}_{t-1}}^{\mathbb{W}_{t-1}} \mathcal{E}_{\mathcal{U}_t}^{\mathbb{W}_t}(\mathcal{T}_i) \\ \vdots & \vdots \\ \mathbb{W}_1 = J_{\mathbb{W}_2}(\mathcal{U}_2) & \mathcal{U}_1 = \mathcal{E}_{\mathcal{U}_2}^{\mathbb{W}_2} \dots \mathcal{E}_{\mathcal{U}_{t-1}}^{\mathbb{W}_{t-1}} \mathcal{E}_{\mathcal{U}_t}^{\mathbb{W}_t}(\mathcal{T}_1) \end{array}$$

It is then straightforward to verify that

$$\Psi_t^{\mathbb{W}}(\mathcal{T}_1, \dots, \mathcal{T}_t) = (\mathcal{U}_1, \dots, \mathcal{U}_t).$$

and that the inverse bijection is given by

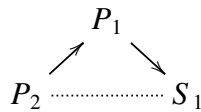
$$\Phi_t^{\mathbb{W}}(\mathcal{U}_1, \dots, \mathcal{U}_t) = (\mathcal{F}_{\mathcal{U}_t}^{\mathbb{W}_t} \dots \mathcal{F}_{\mathcal{U}_2}^{\mathbb{W}_2}(\mathcal{U}_1), \mathcal{F}_{\mathcal{U}_t}^{\mathbb{W}_t} \dots \mathcal{F}_{\mathcal{U}_3}^{\mathbb{W}_3}(\mathcal{U}_2), \dots, \mathcal{U}_t)$$

where $\mathbb{W}_t = \mathbb{W}$ and $\mathbb{W}_i = J_{\mathbb{W}_{i+1}}(\mathcal{U}_{i+1})$ for all i , as above.

6. EXAMPLES

Each example is given as the path algebra of a quiver modulo an admissible ideal of relations generated by paths. For each vertex i of the quiver, we denote by P_i, I_i, S_i the corresponding indecomposable projective (respectively, indecomposable injective, simple) module.

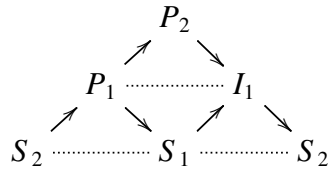
6.1. **Example 1:** Let Q be the quiver $1 \longrightarrow 2$, and let $\Lambda = kQ$. There are three indecomposable modules, $P_1, P_2 = S_2, S_1$, and the AR-quiver of $\text{mod } \Lambda$ is:



There are 5 support τ -tilting modules (= support tilting modules, since Λ is hereditary), and hence 10 ordered support τ -tilting modules. We list these in the table below, together with the corresponding complete signed τ -exceptional sequences.

Ordered support τ -tilting object	Signed τ -exc. sequence	Ordered support τ -tilting object	Signed τ -exc. sequence
(P_2, P_1)	(P_2, P_1)	$(P_2, P_1[1])$	$(P_2, P_1[1])$
(S_1, P_1)	$(P_2[1], P_1)$	$(P_2[1], P_1[1])$	$(P_2[1], P_1[1])$
(P_1, P_2)	(S_1, P_2)	$(S_1, P_2[1])$	$(S_1, P_2[1])$
$(P_1[1], P_2)$	$(S_1[1], P_2)$	$(P_1[1], P_2[1])$	$(S_1[1], P_2[1])$
(P_1, S_1)	(P_1, S_1)		
$(P_2[1], S_1)$	$(P_1[1], S_1)$		

6.2. **Example 2:** Let Q' be the quiver $1 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} 2$, and let $\Lambda' = kQ'/I$, where I is the ideal generated by the path $\beta\alpha$. There are 5 indecomposable modules, and the AR-quiver is:

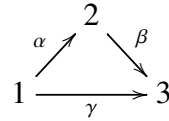


Note that the module I_1 is *not* τ -rigid in $\text{mod } \Lambda$, while the other four indecomposable modules are τ -rigid.

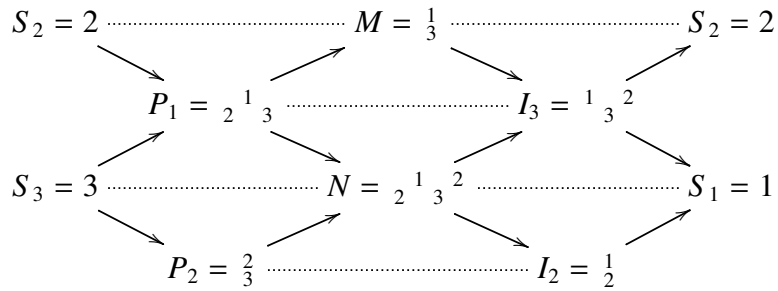
There are 6 support τ -tilting modules, and hence 12 ordered support τ -tilting modules. We list these in the table below, together with the corresponding complete signed τ -exceptional sequences.

Ordered support τ -tilting object	Signed τ -exc. sequence	Ordered support τ -tilting object	Signed τ -exc. sequence
(P_1, P_2)	(S_1, P_2)	$(P_2[1], S_1)$	$(P_1[1], S_1)$
(S_2, P_2)	$(S_1[1], P_2)$	(P_1, S_1)	(P_1, S_1)
(P_2, P_1)	(S_2, P_1)	$(S_2, P_1[1])$	$(S_2, P_1[1])$
(S_1, P_1)	$(S_2[1], P_1)$	$(P_2[1], P_1[1])$	$(S_2[1], P_1[1])$
(P_2, S_2)	(I_1, S_2)	$(S_1, P_2[1])$	$(S_1, P_2[1])$
$(P_1[1], S_2)$	$(I_1[1], S_2)$	$(P_1[1], P_2[1])$	$(S_1[1], P_2[1])$

6.3. **Example 3:** Let Q'' be the quiver



and let $\Lambda'' = kQ''/I$ where I is the ideal generated by the path $\beta\alpha$. There are 9 indecomposable modules, and the AR-quiver is



where the notation indicates which simple modules occur in the radical layers of the module, so $N = {}_2^1{}_3^2$ is a module of length 4, of radical length 2, and with $N/\text{rad } N \simeq S_1 \amalg S_2$.

The following table gives, for each τ -rigid indecomposable U , a list of the indecomposable modules in $J(U)$ and an algebra Γ_U such that $J(U) \simeq \text{mod } \Gamma$.

U	$J(U)$	Γ_U
$S_2 = 2$	$\{\frac{2}{3}, \frac{1}{3} \frac{2}{3}, 1\}$	$k(\bullet \longrightarrow \bullet)$
$S_3 = 3$	$\{2, \frac{1}{2}, 1\}$	$k(\bullet \longrightarrow \bullet)$
$P_1 = \frac{1}{2} \frac{1}{3}$	$\{3, \frac{2}{3}, 2\}$	$k(\bullet \longrightarrow \bullet)$
$P_2 = \frac{2}{3}$	$\{3, \frac{1}{3}, 1\}$	$k(\bullet \longrightarrow \bullet)$
$M = \frac{1}{3}$	$\{3, \frac{1}{2} \frac{1}{3}, \frac{1}{2}\}$	$k(\bullet \longrightarrow \bullet)$
$N = \frac{1}{2} \frac{1}{3} \frac{2}{3}$	$\{\frac{1}{3}, \frac{2}{3}\}$	$k(\bullet \quad \bullet)$
$I_2 = \frac{1}{2}$	$\{\frac{1}{2} \frac{1}{3}, \frac{1}{3}, \frac{1}{2} \frac{1}{3} \frac{2}{3}, \frac{1}{3} \frac{2}{3}, 2\}$	$k(\bullet \xrightleftharpoons[\beta]{\alpha} \bullet) / (\beta\alpha)$
S_1	$\{\frac{1}{2}, \frac{1}{3}\}$	$k(\bullet \quad \bullet)$

As remarked in the introduction, the module I_3 is *not* τ -rigid, but as an object in the module category $J(S_2)$, it is τ -rigid. Hence (P_2, I_3, S_2) is a τ -exceptional sequence. Since I_3 is projective in $J(S_2)$, we have that $(P_2, I_3[1], S_2)$ is also a τ -exceptional sequence.

We calculate the total number of signed τ -exceptional sequences as follows. For

$$U \in \{S_2, S_3 = P_3, P_2, P_1, M, P_3[1], P_2[1], P_1[1]\}$$

there are (by Example 1) 10 signed τ -exceptional sequences of the form $(-, -, U)$. For $U \in \{N, S_1\}$ there are 8 signed τ -exceptional sequences of the form $(-, -, U)$, while there are (by Example 2) 12 signed τ -exceptional sequences of the form $(-, -, I_2)$. Hence, in total there are 108 signed τ -exceptional sequences for this algebra.

We conclude with examples illustrating how we compute which signed τ -exceptional sequence is the image of a given support τ -tilting object under our bijection. Consider the ordered support τ -rigid object (M, I_2, P_1) . To compute $\mathcal{E}_{P_1}(M)$ we first note that M is in $\text{Gen } P_1$, so $\mathcal{E}_{P_1}(M) = \rho(M)$. Furthermore we have that \mathbb{P}_M is given by $P_2 \rightarrow P_1$ and so we have the triangle (8) in this case is:

$$P_2 \rightarrow P_1 \rightarrow \mathbb{P}_M$$

and hence $\rho(M) = (f_{P_1}(P_2))[1] = P_2[1]$. Similarly, we have that $\mathcal{E}_{P_1}(I_2) = \rho(I_2) = S_3[1]$. The ordered support τ -rigid object $(P_2[1], S_3[1])$ in $J(P_1)$ corresponds according to the table of Example 1 to the signed τ -exceptional sequence $(S_2[1], S_3[1])$. Hence our bijection maps the ordered support τ -rigid object (M, I_2, P_1) to the signed τ -exceptional sequence $(S_2[1], S_3[1], P_1)$.

Consider the ordered support τ -rigid object (M, P_1, I_2) . Note that $\text{Hom}(I_2, M) = 0 = \text{Hom}(I_2, P_1)$, so that $\mathcal{E}_{I_2}(M) = M$ and $\mathcal{E}_{I_2}(P_1) = P_1$. The ordered support τ -rigid object (M, P_1) in $J(I_2)$ corresponds, according to the table of Example 2, to the signed τ -exceptional sequence $(S_2[1], P_1)$ in $J(I_2)$ (note that S_2 is projective in $J_{J(I_2)}(P_1)$). Hence our bijection maps the ordered support τ -rigid object (M, P_1, I_2) to the signed τ -exceptional sequence $(S_2[1], P_1, I_2)$.

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