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MIRROR SYMMETRY FOR EXTENDED AFFINE WEYL GROUPS

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Abstract. We give a uniform, Lie-theoretic mirror symmetry construction for the Frobenius manifolds defined by Dubrovin–Zhang in [20] on the orbit spaces of extended affine Weyl groups, including exceptional Dynkin types. The B-model mirror is given by a one-dimensional Landau–Ginzburg superpotential constructed from a suitable degeneration of the family of spectral curves of the affine relativistic Toda chain for the corresponding affine Poisson–Lie group. As applications of our mirror theorem we give closed-form expressions for the flat coordinates of the Saito metric and the Frobenius prepotentials in all Dynkin types, compute the topological degree of the Lyashko–Looijenga mapping for certain higher genus Hurwitz space strata, and construct hydrodynamic bihamiltonian hierarchies (in both Lax–Sato and Hamiltonian form) that are root-theoretic generalisations of the long-wave limit of the extended Toda hierarchy.

Contents

1. Introduction

Frobenius manifolds, introduced by B. Dubrovin in [15] as a coordinate free formulation of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of 2D topological field theory, have sat for a good quarter of a century at a key point of confluence of algebraic geometry, singularity theory, quantum field theory, and the theory of integrable systems. In algebraic geometry, they serve as a model for the quantum co-homology (genus zero Gromov–Witten theory) of smooth projective varieties; in singularity theory, they encode the existence of pencils of flat pairings on the base of the mini-versal deformations of hypersurface singularities; in physics, they codify the associativity of the chiral ring of topologically twisted $\mathcal{N} = (2, 2)$ supersymmetric field theories in

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two dimensions; and in the theory of integrable hierarchies, they provide a loop-space formulation of hydrodynamic bihamiltonian integrable hierarchies in 1+1 dimensions.

On top of the physics-inspired examples of Frobenius manifolds coming from Witten's topological A- and B-twists of 2D-theories with four supercharges, an interesting source of Frobenius manifolds is well-known to arise in Lie theory. Let $\mathfrak{g}_{\mathcal{R}}$ be a simple complex Lie algebra associated to an irreducible root system $\mathcal{R}$, and write $\mathfrak{h}_{\mathcal{R}}$ and $\mathfrak{g}_{\mathcal{R}}^{(1)}$ for, respectively, its Cartan subalgebra and the associated untwisted affine Lie algebra. In a celebrated result [16], Dubrovin constructed a class of semi-simple polynomial Frobenius manifolds on the space of regular orbits of the reflection representation of $\mathrm{Weyl}(\mathfrak{g}_{\mathcal{R}})$ (and in fact, on the orbit spaces of the defining representation of any Coxeter group). A remarkable extension of this was provided by Dubrovin and Zhang [20], who defined a Frobenius manifold structure $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ on quotients of $\mathfrak{h}_{\mathcal{R}}\times\mathbb{C}$ by a suitable semi-direct product $\mathrm{Weyl}(\mathfrak{g}_{\mathcal{R}}^{(1)}) \rtimes \mathbb{Z}$. They furthermore provided a mirror symmetry construction for Dynkin type A, $\mathfrak{g}_{A_{N-1}} = \mathrm{sl}_N(\mathbb{C})$, in terms of Laurent-polynomial one-dimensional Landau–Ginzburg models, which was later generalised to classical Lie algebras in [19]. A question raised by [19, 20] was whether a similar uniform mirror symmetry construction for all Dynkin types could be established, including exceptional Lie algebras.

This paper gives a constructive Lie-theoretic answer to this question, which is furthermore entirely explicit, and provides closed-form expressions for the flat coordinates of the analogue of the Saito–Sekiguchi–Yano metric and for the Frobenius prepotential. Our mirror theorem has simultaneous implications for singularity theory, integrable systems, the Gromov–Witten theory of Fano orbicurves, and Seiberg–Witten theory, some of which are explored here.

1.1. Main results.

1.1.1. *Mirror symmetry for Dubrovin–Zhang Frobenius manifolds.* Our main result is the following general mirror theorem for Dubrovin–Zhang Frobenius manifolds (see Theorem 3.1 for the complete statement, and Table 1 for details of the notation employed). Let $\mathcal{H}_{g,\mathsf{m}}$ be the Hurwitz space of isomorphism classes $[\lambda : C_g \to \mathbb{P}^1]$ of covers of the complex line by a genus g curve C_g with ramification profile at infinity described by $\mathsf{m} \in \mathbb{N}_0^{\ell(\mathsf{m})}$, $\ell(\mathsf{m}) \geq 1$. Fixing a third kind differential ϕ on C_g with simple poles at $\lambda^{-1}([1:0])$ induces, as a particular case of a classical construction of Dubrovin [13, 15], a semi-simple Frobenius manifold structure $\mathcal{H}_{g,\mathsf{m}}^{\phi}$ on $\mathcal{H}_{g,\mathsf{m}}$.

Theorem 1.1 (=Theorem 3.1)**.** *For any simple Dynkin type $\mathcal{R}$ there exists a highest weight ω for the corresponding simple Lie algebra $\mathfrak{g}$, pairs of integers $(g_\omega, \mathsf{n}_\omega)$, an explicit embedding $\iota_\omega : \mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}} \hookrightarrow \mathcal{H}_{g_\omega,\mathsf{n}_\omega}^{\phi}$, and a choice of third kind differential on the fibres of the universal family $\pi : \mathcal{C}_{g_\omega} \to \mathcal{H}_{g_\omega,\mathsf{n}_\omega}$ such that ι_ω is a Frobenius manifold isomorphism onto its image $\mathcal{M}_{\omega}^{\mathrm{LG}} := \iota_\omega(\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}})$.*

In other words, ι_ω identifies the Frobenius manifold $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ with a distinguished stratum $\mathcal{M}_{\omega}^{\mathrm{LG}}$ of a Hurwitz space, which is an affine-linear subspace of $\mathcal{H}_{g_\omega,\mathsf{n}_\omega}^{\phi}$ in flat coordinates for the latter. The datum of the covering map and third kind differential on $\mathcal{H}_{g_\omega,\mathsf{n}_\omega}$ define a one-dimensional B-model superpotential for $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ in terms of a family of (trigonometric) meromorphic functions $\mathcal{M}_{\omega}^{\mathrm{LG}}$, whose Landau–Ginzburg residue formulas determine the Dubrovin–Zhang flat pencil of metrics and the

Frobenius product structure on $T\mathcal{M}_{\mathcal{R}}^{\rm DZ}$.

Theorem 1.1 is proved in two main steps. We first associate to $\mathcal{M}_{\mathcal{R}}^{\rm DZ}$ a family of spectral curves (a subvariety of a Hurwitz space) given by the characteristic equation for a pencil of group elements $\mathsf{g}(\lambda) \in \mathcal{G} := \exp \mathfrak{g}$ in the irreducible representation ρ_ω. The construction of the family hinges on determining all character relations of the form $\chi_{\wedge^k \rho_\omega} = \mathfrak{p}_k^\omega(\chi_{\rho_1}, \dots, \chi_{\rho_{l_{\mathcal{R}}}})$ in the Weyl character ring of $\mathcal{G}$, where ρ_i is the $i^{\rm th}$ fundamental representation of $\mathcal{G}$, and $l_{\mathcal{R}}$ is the rank of $\mathcal{R}$. Different choices of ω induce different families of spectral curves, and therefore different embeddings $\iota_\omega : \mathcal{M}_{\mathcal{R}}^{\rm DZ} \hookrightarrow \mathcal{H}_{g_\omega, \mathrm{m}_\omega}$ inside a parent Hurwitz space. Our construction is motivated by a conjectural relation of the almost-dual Frobenius manifold [18] for $\mathcal{R}$ of type ADE with the orbifold quantum co-homology of the associated simple surface singularity, as proposed in [6, 8], which is in turn described by a degeneration of a family of spectral curves for the relativistic Toda chain associated to (a co-extension of) the corresponding affine Poisson–Lie group of type ADE [23, 43]. The one-parameter family of group elements $g(\lambda)$ in our construction is given by the Lax operator for relativistic Toda, where λ is the spectral parameter, and the relation to the associated Dubrovin–Zhang Frobenius manifold of type ADE is suggested by analogous results for the simple Lie algebra case due to Lerche–Warner, Ito–Yang, and Dubrovin [18, 27, 32], and then generalising to all Dynkin types.

We prove Theorem 1.1 for all $\mathcal{R}$ with dominant weights ω in a minimal non-trivial Weyl orbit, but we also provide verifications that non-minimal choices of ω indeed give rise to isomorphic Frobenius manifolds. The target Hurwitz space $\mathcal{H}_{g_\omega, \mathsf{n}_\omega}$ is a space of rational functions ($g_\omega = 0$) for type $\mathcal{R} = A_l$, B_l, C_l, D_l and G_2, and it is a space of meromorphic functions on higher genus curves in the other exceptional types.

1.1.2. *Application I: Frobenius prepotentials.* The original Dubrovin–Zhang construction establishes the existence of a Frobenius manifold structure on $\mathcal{M}_{\mathcal{R}}^{\rm DZ}$ by abstractly constructing a flat pencil of metrics $\eta + \lambda\gamma$ on it, with γ being the extended Killing pairing on $\mathfrak{h}_{\mathcal{R}} \oplus \mathbb{C}$, without reference to an actual system of flat coordinates for η (the analogue of the Saito–Sekiguchi–Yano metric for finite reflection groups). From Theorem 1.1, the metric η and Frobenius product on the base of the family of spectral curves can then be computed using Landau–Ginzburg residue formulas for the superpotential: the associativity of the Frobenius product reduces the analysis of the pole structure of the Landau–Ginzburg residues to the sole poles of the superpotential, giving closed-form expressions for the flat coordinates of η and its prepotential. We then obtain the following

Theorem 1.2 (=Lemma 4.1 and Section 4.1)**.** *For all $\mathcal{R}$, we provide flat coordinates for the Saito metric of the Dubrovin–Zhang pencil and closed-form prepotentials for $\mathcal{M}_{\mathcal{R}}^{\rm DZ}$.*

Our expressions recover results of [19, 20] for classical Lie algebras; the statements for exceptional Dynkin types are new. Theorem 1.1 is key to the determination of the prepotential: the Landau–Ginzburg calculation reduces the computation of flat coordinates for η and a distinguished subset of structure constants to straightforward residue calculations on the spectral curves, from which the entire product structure on the Frobenius manifold can be recovered using WDVV equations.

1.1.3. *Application II: Lyaskho–Looijenga multiplicities of meromorphic functions.* The enumeration of isomorphism classes of covers of S^2 with prescribed ramification over a point is a classical problem in topology and enumerative combinatorics, going back to Hurwitz' formula for the case in which the covering surface is also a Riemann sphere. The result of the enumeration for a cover of arbitrary geometric genus g and branching profile $\mathsf{n} = (n_0, \dots, n_m)$ is the Hurwitz number $h_{g,\mathsf{n}}$, whose significance straddles several domains in enumerative combinatorics [25, 26], representation theory of the symmetric group [24], moduli of curves [22], and mathematical physics [5, 9, 12]. It was first noticed by Arnold [2] that when the branching profile has maximal degeneration (i.e., for polynomial maps $f : \mathbb{P}^1 \to \mathbb{P}^1$) this problem is intimately related to considering the topology of the complement of the discriminant for the base of the type A_l mini-versal deformation, and in particular to the degree of the Lyashko–Looijenga mapping [33, 34] $\mathrm{LL} : \mathbb{C}[\mu] \to \mathbb{C}[\mu]$, which assigns to a polynomial $\lambda(\mu)$ the unordered set of its critical values $\mathrm{LL}(\lambda)(\mu) = \prod_{\lambda'(\tilde{z})=0}(\mu - f(\tilde{z}))$. This is a finite polynomial map [2, 33], inducing a stratification of $\mathbb{C}[\mu]$ according to the degeneracy of the critical values of λ. The computation of the topological degree of this mapping on a given stratum, enumerating the number of polynomials sharing the same critical values counted with multiplicity, can usually be translated into a combinatorial problem enumerating some class of embedded graphs. This connection was used by Looijenga [33] to reprove Cayley's formula for the enumeration of marked trees (corresponding to the co-dimension zero stratum), and by Arnold [2] to encompass the case of Laurent polynomials (see also [30, 31] for generalisations to rational functions and discriminant strata). The extension of this combinatorial approach to arbitrary strata at higher genus, involving enumerations of suitable coloured oriented graphs (k-constellations), appears unwieldy [40]. However, when $\lambda(\mu)$ is the Landau–Ginzburg superpotential of a semi-simple, conformal Frobenius manifold, the graded structure of the latter can be used to determine the Lyashko–Looijenga multiplicity of $\lambda(\mu)$ by a direct application of the quasi-homogeneous Bezout theorem [3], with no combinatorics involved. In particular Theorem 1.1 has the following immediate consequence.

Theorem 1.3 (=Corollary 5.2 and Table 3)**.** *For all $\mathcal{R}$ we compute the Lyashko–Looijenga multiplicity of the stratum $\iota_\omega(\mathcal{M}^{\mathrm{DZ}}_{\mathcal{R}}) = \mathcal{M}^{\mathrm{LG}}_\omega \subset \mathcal{H}_{g_\omega, \mathsf{n}_\omega}$.*

This includes, in particular, the higher genus Hurwitz spaces appearing for types $\mathcal{R} = E_n$ and F_4 (see Table 3).

1.1.4. *Application III: the dispersionless extended type-$\mathcal{R}$ Toda hierarchy.* The datum of a semi-simple conformal Frobenius manifold is equivalent to the existence of a τ-symmetric quasi-linear integrable hierarchy, which is bihamiltonian with respect to a Dubrovin–Novikov hydrodynamic Poisson pencil. Having a description of the Frobenius manifold in terms of a closed-form prepotential allows to give an explicit presentation of the hierarchy in terms of an infinite set of commuting 1+1 PDEs in normal coordinates. The loop-space version of Theorem 1.2 is then the following

Theorem 1.4 (=Proposition 5.3)**.** *For all $\mathcal{R}$, we construct a bihamiltonian dispersionless hierarchy on the loop space $\mathcal{L}\mathcal{M}^{\mathrm{DZ}}$ in Hamiltonian form for the canonical Poisson pencil associated to $\mathcal{M}^{\mathrm{DZ}}$.*

For type A_n this integrable hierarchy is the zero-dispersion limit of Carlet's extended bigraded Toda hierarchy [10], and for type D_n it is the long-wave limit of the Cheng–Milanov extended D-type hierarchy [11]. For simply-laced $\mathcal{R}$, we expect that the principal hierarchies of Theorem 1.4 should coincide with the dispersionless limit of the Hirota integrable hierarchies constructed by Milanov–Shen–Tseng in [37]. The non-simply-laced cases are, to the best of our knowledge, new examples of hydrodynamic integrable hierarchies: our construction of the Landau–Ginzburg superpotential is highly suggestive that these should be obtained as symmetry reductions of the hierarchies in [37] by the usual folding procedure of the Dynkin diagram. Aside from laying the foundation for determining the prepotential of $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$, Theorem 1.1 also provides a dispersionless Lax formulation for the hierarchy as an explicit reduction of Krichever's genus g_ω, $\ell(\mathsf{n}_\omega)$-pointed universal Whitham hierarchy.

1.2. **Further applications.** We also highlight three further applications of Theorem 1.1, which are the subject of ongoing investigation and whose details will be provided in three separate publications.

1.2.1. *The orbifold Norbury–Scott conjecture.* When $\mathcal{R} = A_1$, the Frobenius manifold $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ famously coincides with the quantum cohomology $\mathrm{QH}(\mathbb{P}^1, \mathbb{C})$ of the complex projective line. In [39], the authors propose a higher genus version of this statmement and conjecture that the Chekhov–Eynard–Orantin topological recursion applied to the Landau–Ginzburg superpotential of $\mathbb{P}^1$ computes the n-point, genus-g Gromov–Witten invariants of $\mathbb{P}^1$ with descendant insertions of the Kähler class (the "stationary" invariants) in terms of explicit residues on the associated spectral curve (see [21] for a proof). It was shown in [41] that for type $\mathcal{R} = A_l$, D_l and E_l the Dubrovin–Zhang Frobenius manifolds $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ are isomorphic to the orbifold quantum cohomology of the Fano orbicurves $\mathscr{C}_{\mathcal{R}} = [\mathbb{C}^\star \backslash \mathbb{C}^2/\Gamma_{\mathcal{R}}]$, where $\Gamma_{\mathcal{R}} < SU(2)$, $|\Gamma_{\mathcal{R}}| < \infty$ is the McKay group of type $\mathcal{R}$. In particular,

$$\mathscr{C}_{\mathcal{R}} \simeq \begin{cases} \mathbb{P}(1,l), & \mathcal{R} = A_l, \\ \mathbb{P}_{2,2,l-2}, & \mathcal{R} = D_l, \\ \mathbb{P}_{2,3,l-3} & \mathcal{R} = E_l. \end{cases} \tag{1.1}$$

The construction of the LG superpotentials of Theorem 1.1 now associates a family of mirror spectral curves to the quantum cohomology of these orbifolds. As anticipated in [6], it is natural to conjecture that the Norbury–Scott theorem receives an orbifold generalisation through Theorem 1.1, whereby higher genus stationary Gromov–Witten invariants of $\mathscr{C}_{\mathcal{R}}$ can be computed by residue calculus on the respective type $\mathcal{R}$ spectral curve mirrors. The investigation of the correct phrasing for the topological recursion is ongoing.

1.2.2. *Seiberg–Witten theory.* For the case of polynomial Frobenius manifolds with $\mathcal{R} = A_l$, D_l or E_l, it was noted by a number of authors [18,27,32] that the *odd periods* of the Frobenius manifold (in the language of [18]) give the quantum periods of the Seiberg–Witten family of curves dual to $\mathcal{N} = 2$ pure super Yang–Mills theory on $\mathbb{R}^4$ with gauge group given by the compact real form of $\exp(\mathfrak{g})$. In [38], Nekrasov reformulated the Seiberg–Witten study of $\mathcal{N} = 2$, $d = 4$ gauge theories in the context of five-dimensional $\mathcal{N} = 1$ gauge theories compactified on a circle, by viewing the five-dimensional theory on $\mathbb{R}^4 \times S^1$ with gauge group G as, effectively, a four-dimensional theory with gauge group the extended loop group $\widehat{G}$. In this context the classical Coulomb vacua are parametrised by orbits of the associated extended affine Weyl group. It is only natural to conjecture that the construction of odd periods and Picard–Fuchs system for four dimensional Seiberg–Witten theory from the polynomial Frobenius manifolds can be lifted to provide solutions of five-dimensional Seiberg–Witten theory using their Dubrovin–Zhang, extended affine counterpart; this can indeed be explicitly checked for simply-laced cases. Considerations about folding in five dimensions also allow to treat non-simply laced Lie groups, which points to the existence of a new class of Frobenius manifolds having as monodromy group an extension of the *twisted* affine Weyl group.

1.2.3. *Saito determinants on discriminant strata.* In [1], the authors consider semi-simple Frobenius manifolds embedded as discriminant strata on the Dubrovin–Hertling polynomial Frobenius structures on the orbits of the reflection representation of Coxeter groups. In particular, they use the Landau–Ginzburg mirror superpotentials to establish structural results on the determinant of the restriction of the Saito metrics to arbitrary strata. A specific question asked by [1] is how much of that story can be lifted to the study of the Dubrovin–Zhang Frobenius manifolds on extended affine Weyl group orbits. The Landau–Ginzburg presentation of Theorem 1.1 unlocks the power to employ the same successful methodology in the affine setting as well.

1.3. **Organisation of the paper.** The paper is organised as follows. In Section 2 we recall the definition of the affine Lie theoretic Frobenius manifolds of Dubrovin–Zhang (DZ). In Section 3 we state how to construct Frobenius manifolds in terms of Landau–Ginzburg (LG) superpotentials defined on suitable strata of a Hurwitz space. We also recall the construction in [6], describing how to find LG-superpotentials for DZ-manifolds from the characteristic equation of a suitable degeneration of the Lax operator for the type $\mathcal{R}$ periodic relativistic Toda chain. This boils down to finding relations in the representation ring of $\mathcal{G}$, which we determine for all Dynkin types giving explicit algebraic expressions for the corresponding superpotentials. In Section 4 we prove the mirror theorem, and determine in turn closed-form prepotentials for the corresponding Dubrovin–Zhang Frobenius manifolds, including the hitherto unknown exceptional cases in type E_6, E_7 and F_4. Finally, in Section 5, we discuss the applications to the extended $\mathcal{R}$-type Toda hierarchies and the calculation of the multiplicities of the Lyashko–Looijenga map of $\mathcal{M}_\omega^{\rm LG}$. Our notation[1] is described in Table 1.

2. Dubrovin–Zhang Frobenius manifolds

2.1. **Generalities on Frobenius manifolds.** We start by recalling the basic definitions from the theory of Frobenius manifolds.

Definition 2.1. A (complex, holomorphic) Frobenius manifold is a tuple $\mathcal{M} = (M, \cdot, \eta, e, E)$, where M is a finite dimensional complex manifold such that at each point $p \in M$, the fibre T_pM of the holomorphic tangent bundle at p has the structure of a unital associative commutative algebra

[1]To declutter the polynomial expressions of the prepotentials of $\mathcal{M}_\mathcal{R}^{\rm DZ}$, and in a slight departure from the conventions in the Frobenius manifolds literature, components of a chart of $\mathcal{M}_\mathcal{R}^{\rm DZ}$ will consistently be written with lower indices; in particular the Einstein summation convention is *never* assumed.

$\mathcal{R}$	An irreducible root system
$l_{\mathcal{R}}$	The rank of $\mathcal{R}$
$\mathfrak{g}_{\mathcal{R}}$	The complex simple Lie algebra with root system $\mathcal{R}$
$\mathcal{G}_{\mathcal{R}}$	The simply connected complex simple Lie group $\exp(\mathfrak{g}_{\mathcal{R}})$
$\mathfrak{h}_{\mathcal{R}}$	The Cartan subalgebra of $\mathfrak{g}_{\mathcal{R}}$
$\mathcal{T}_{\mathcal{R}}$	The Cartan torus $\exp(\mathfrak{h}_{\mathcal{R}})$ of $\mathfrak{g}_{\mathcal{R}}$
g	A regular element of $\mathcal{G}_{\mathcal{R}}$
$\mathcal{W}_{\mathcal{R}}/\widehat{\mathcal{W}}_{\mathcal{R}}/\widetilde{\mathcal{W}}_{\mathcal{R}}$	The Weyl/affine Weyl/extended affine Weyl group of Dynkin type $\mathcal{R}$
h	A regular element of $\mathfrak{h}_{\mathcal{R}}$
$\{\alpha_1, \dots, \alpha_{l_{\mathcal{R}}}\}$	The set of simple roots of $\mathcal{R}$
$\{\omega_1, \dots, \omega_{l_{\mathcal{R}}}\}$	The set of fundamental weights of $\mathcal{R}$
$\Lambda_r(\mathcal{R})$ (resp. $\Lambda_r(\mathcal{R})^{\pm}$)	The lattice of roots of $\mathcal{R}$ (resp. the semi-group of positive/negative roots)
$\Lambda_w(\mathcal{R})$ (resp. $\Lambda_w(\mathcal{R})^{\pm}$)	The lattice of all weights of $\mathcal{R}$ (resp. the monoid of non-negative/non-positive weights)
ρ_ω	The irreducible representation of $\mathcal{G}_{\mathcal{R}}$ with highest weight ω
ρ_i	The i^{th} fundamental representation of $\mathcal{G}_{\mathcal{R}}$, $i = 1, \dots, l_{\mathcal{R}}$
$\Gamma(\rho)$	The weight system of the representation ρ
χ_ω	The formal character of ρ_ω
χ_i	The formal character of ρ_i
$[i_1 \dots i_{l_{\mathcal{R}}}]_{\mathcal{R}}$ (without commas)	Components of a weight in the ω-basis of $\mathcal{R}$
$(x_1, \dots, x_{l_{\mathcal{R}}})$	Linear coordinates on $\mathfrak{h}_{\mathcal{R}}$ w.r.t. the co-root basis $\{\alpha_1^*, \dots, \alpha_{l_{\mathcal{R}}}^*\}$
$(Q_1, \dots, Q_{l_{\mathcal{R}}})$	$(\exp(x_1), \dots, \exp(x_{l_{\mathcal{R}}}))$
$C_{\mathcal{R}}$	The Cartan matrix of $\mathcal{R}$
$\mathcal{K}_{\mathcal{R}}$	The symmetrised Cartan matrix of $\mathcal{R}$
$\mathsf{n} \vdash d$	A padded (i.e. parts are allowed to be zero) partition of $d \in \mathbb{N}$
$\lvert\mathsf{n}\rvert$ (resp. $\ell(\mathsf{n})$)	The length of (resp. the number of parts in) a partition n

TABLE 1. Notation employed throughout the text.

with multiplication $\cdot$ and identity element e, varying holomorphically. Additionally, η is a flat, holomorphic, non-degenerate symmetric $(0, 2)$-tensor such that the Frobenius property holds:

$$\eta(X \cdot Y, Z) = \eta(X, Y \cdot Z), \qquad \forall\, X, Y, Z \in \Gamma(M, TM). \tag{2.1}$$

Moreover, the following properties hold:

(1) the unit vector field is horizontal

$$\nabla e = 0, \tag{2.2}$$

w.r.t. the Levi-Civita connection ∇ associated to η;

(2) there exists a $(0,3)$-tensor $c \in \Gamma(M, \mathrm{Sym}^3 T^*M)$ such that

$$\nabla_W c(X, Y, Z), \tag{2.3}$$

is totally symmetric $\forall\, W, X, Y, Z \in \Gamma(M, TM)$;

(3) there exists $E \in \Gamma(M, TM)$ such that ∇E is covariantly constant, and the corresponding 1-parameter group of diffeomorphisms acts by conformal transformations of the metric, and by rescalings of the tangent algebras.

A complex Frobenius manifold is *semisimple* if the set $\mathrm{Discr}(M) = \{p \in M \mid \exists v \in T_pM \text{ with } v \cdot v = 0\}$ has positive complex co-dimension. Whenever E is in the group of units of $(T_pM, \cdot)$, one may define a second flat metric, $\gamma \in \Gamma(M, \mathrm{Sym}^2 T^*M)$, by

$$\gamma(E \cdot X, Y) = \eta(X, Y). \tag{2.4}$$

A key consequence of Definition 2.1 is that γ and η form a flat pencil of metrics, that is, $\gamma + \lambda\eta$ is a flat metric for any $\lambda \in \mathbb{C}$, and its Christoffel symbols satisfy $\Gamma(\gamma + \lambda\eta) = \Gamma(\gamma) + \lambda\Gamma(\eta)$.

Since the metric η is flat, a Frobenius manifold carries a canonical affine equivalence class of charts given by flat frames for η. Spelling out the Frobenius manifold axioms in this chart amounts to the following properties:

(i) the Gram matrix $\eta_{ij} \equiv \eta(\partial_i, \partial_j) = \dfrac{\partial^2 e(F)}{\partial t_i \partial t_j}$, where F is a holomorphic function called the prepotential, $\{t_i\}_{i=1,\cdots,\dim(M)}$ are flat coordinates for η, and ∂_i is shorthand for $\frac{\partial}{\partial t_i}$;

(ii) $c_{ijk} \equiv c(\partial_i, \partial_j, \partial_k) = \eta(\partial_i \cdot \partial_j, \partial_k) = \dfrac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}$;

(iii) $E = \sum_{d_i \neq 0} d_i t_i \partial_i + \sum_{d_i = 0} r_i \partial_i$, where $d_i = \deg(t_i)$;

(iv) $\gamma_{ij} = \sum_k E^k c_{kij}$;

(v) $\partial_i \cdot \partial_j = \sum_k c_{ij}^k \partial_k$, where $c_{ij}^k := \sum_m \eta^{km} c_{mij}$, and $\eta^{ij} := (\eta)^{-1}_{ij}$.

The prepotential, F, satisfies the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations;

$$\sum_{kl} \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \eta^{kl} \frac{\partial^3 F}{\partial t_l \partial t_m \partial t_n} = j \longleftrightarrow m. \tag{2.5}$$

The prepotential is thus defined up to scaling, allowed coordinate changes (linear combination of coordinates of coincident degrees), and up to terms at most quadratic in t's. Such transformations induce, by definition, an isomorphism of Frobenius manifolds. Hence, specifying a Frobenius manifold structure is tantamount to finding a weighted homogeneous solution to the WDVV-equations together with a choice of identity vector field.

2.2. **Frobenius manifolds from extended affine Weyl groups.** We will here give a condensed description of the Dubrovin–Zhang construction of semi-simple Frobenius manifold structures on the space of regular orbits of extended affine Weyl groups, which follows closely the account given in [6].

Let $\mathfrak{g}_{\mathcal{R}}$ be a complex simple Lie algebra associated to a root system $\mathcal{R}$, $\mathfrak{h}_{\mathcal{R}}$ the associated Cartan subalgebra, and $\mathcal{W}_{\mathcal{R}}$ the Weyl group. The construction of Dubrovin–Zhang Frobenius manifolds depends on a canonical choice[2] of a marked node in the Dynkin diagram of $\mathcal{R}$. This is the "attaching" node for the external node, that is, the one which if removed splits the Dynkin diagram into A-type pieces. Let this node be labelled $\bar{k}$, and let $\alpha_{\bar{k}}$ and $\omega_{\bar{k}}$ denote the corresponding simple root and fundamental weight, respectively. The action of $\mathcal{W}_{\mathcal{R}}$ on $\mathfrak{h}_{\mathcal{R}}$ may be lifted to an action of the affine Weyl group, $\widehat{\mathcal{W}}_{\mathcal{R}}$,

$$\widehat{\mathcal{W}}_{\mathcal{R}} \times \mathfrak{h}_{\mathcal{R}} \to \mathfrak{h}_{\mathcal{R}} \tag{2.6}$$

$$((w, \alpha^{\vee}), h) \mapsto w(h) + \alpha^{\vee}, \tag{2.7}$$

where $\widehat{\mathcal{W}}_{\mathcal{R}} \cong \mathcal{W}_{\mathcal{R}} \rtimes \Lambda_r^{\vee}(\mathcal{R})$, with $\Lambda_r^{\vee}(\mathcal{R})$ being the lattice of co-roots. Then the *extended* affine Weyl group $\widetilde{\mathcal{W}}_{\mathcal{R}}$ of type $\mathcal{R}$ is defined as $\widetilde{\mathcal{W}}_{\mathcal{R}} := \widehat{\mathcal{W}}_{\mathcal{R}} \rtimes \mathbb{Z}$ acting on $\mathfrak{h}_{\mathcal{R}} \oplus \mathbb{C}$ by

$$\widetilde{\mathcal{W}}_{\mathcal{R}} \times \mathfrak{h} \times \mathbb{C} \to \mathfrak{h}_{\mathcal{R}} \times \mathbb{C} \tag{2.8}$$

$$((w, \alpha^{\vee}, l_{\mathcal{R}}), (h, v)) \mapsto (w(h) + \alpha^{\vee} + l_{\mathcal{R}}\omega_{\bar{k}}, x_{l_{\mathcal{R}}+1} - l_{\mathcal{R}}), \tag{2.9}$$

where $l_{\mathcal{R}}$ is the rank of $\mathcal{R}$.

It is shown in [20] that the ring of invariants $\mathbb{C}[\mathfrak{h}_{\mathcal{R}} \times \mathbb{C}]^{\widetilde{\mathcal{W}}_{\mathcal{R}}}$ is isomorphic to a graded polynomial ring generated by a collection of $\widetilde{\mathcal{W}}_{\mathcal{R}}$-invariant Fourier polynomials $\{\tilde{y}_i\}_{i=1,\cdots,l_{\mathcal{R}}+1}$. It follows that we may define the regular orbit space of the extended affine Weyl group of $\mathcal{R}$ with marked node $\bar{k}$ as the GIT quotient

$$M_{\mathcal{R}}^{\mathrm{DZ}} := (\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}} \times \mathbb{C})//\widetilde{\mathcal{W}}_{\mathcal{R}} = \mathrm{Spec}(\mathcal{O}_{\mathfrak{h}_{\mathcal{R}}\times\mathbb{C}}(\mathfrak{h}^{\mathrm{reg}} \times \mathbb{C}))^{\widetilde{\mathcal{W}}_{\mathcal{R}}} \cong \mathcal{T}_{\mathcal{R}}^{\mathrm{reg}}/\mathcal{W}_{\mathcal{R}} \times \mathbb{C}^*, \tag{2.10}$$

where $\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}} := \mathfrak{h}_{\mathcal{R}} \backslash \Sigma$ is the set of regular elements in $\mathfrak{h}_{\mathcal{R}}$ (i.e. where $\mathcal{W}_{\mathcal{R}}$ acts freely), and $\mathcal{T}_{\mathcal{R}}^{\mathrm{reg}} = \exp(2\pi \mathrm{i} \mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}})$ is its image under the exponential map to the maximal torus $\mathcal{T}_{\mathcal{R}}$. By orthogonal extension of 4π times the Cartan–Killing form on $\mathfrak{h}_{\mathcal{R}}$, we obtain a non-degenerate pairing ξ on $\mathfrak{h}_{\mathcal{R}} \times \mathbb{C}$;

$$\xi(\partial_{x_i}, \partial_{x_j}) = \begin{cases} 4\pi^2\delta_{ij} & \text{if } i,j < l_{\mathcal{R}}+1 \\ -4\pi^2 & \text{if } i = j = l_{\mathcal{R}}+1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

Note that the construction of (2.10) realises $\tilde{y} : \mathcal{T}_{\mathcal{R}}^{\mathrm{reg}} \times \mathbb{C}^* \to M_{\mathcal{R}}^{\mathrm{DZ}}$ as a prinicpal $\widetilde{\mathcal{W}}_{\mathcal{R}}$-bundle, and local coordinates on $\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}}$ serve as local coordinates on the orbit space. A section of the principal bundle, $\tilde{\sigma}_i$, lifts an open $U \subset M_{\mathcal{R}}^{\mathrm{DZ}}$ to the i^{th} sheet of the cover $V_i \in \tilde{y}^{-1}(U) \equiv V_1 \sqcup \cdots \sqcup V_{l_{\mathcal{R}}+1}$. The following reconstruction theorem holds [20, Thm 2.1]:

Theorem 2.1. *There exists a unique semisimple Frobenius structure* $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}(M_{\mathcal{R}}^{\mathrm{DZ}}, e, E, \eta, \cdot)$ *such that:*

(1) The prepotential is polynomial in $t_1, \cdots, t_{l_{\mathcal{R}}}$, $e^{t_{l_{\mathcal{R}}+1}}$*;*

(2) $e = \partial_{\tilde{y}_{\bar{k}}} = \partial_{t_{\bar{k}}}$*;*

(3) $E = \dfrac{1}{2\pi i d_{\bar{k}}}\partial_{x_{l_{\mathcal{R}}+1}} = \sum_{j=1}^{l_{\mathcal{R}}} \dfrac{d_j}{d_{\bar{k}}} t_j \partial_j + \dfrac{1}{d_{\bar{k}}}\partial_{l_{\mathcal{R}}+1}$*;*

(4) The intersection form is given by $\gamma = \tilde{\sigma}_i^* \xi$.

[2]Non-canonical choices are considered in [19] for classical Lie algebras.

Here, $d_i := \langle \omega_i, \omega_{\bar{k}} \rangle$, i.e. the Coxeter exponents shifted by 1, is the degree of t_i, with $\{t_i\}_{i=1,\cdots,l_{\mathcal{R}}+1}$ being flat coordinates for the Saito metric $\eta = L_e g$. Such Frobenius manifolds will always be of charge one, or equivalently, the prepotential will be a degree 2 quasi-homogeneous function of its arguments.

3. Landau–Ginzburg superpotentials from Lie theory

3.1. **One-dimensional LG mirror symmetry.** Hurwitz spaces are moduli spaces parametrising ramified covers of the Riemann sphere. A point in such a space is a conjugacy class $[\lambda : C_g \mapsto \mathbb{P}^1]$, where C_g is a smooth genus g algebraic curve and λ is a morphism to the complex projective line realising C_g as a branched cover of $\mathbb{P}^1$; the equivalence relation is here given by automorphisms of the cover.

We consider Hurwitz spaces with fixed ramification over infinity. Let the preimage of ∞ consist of $m+1$ distinct points, denoted by $\infty_i \in C_g$ for $i = 0, \cdots, m$, with λ having degree $n_i + 1$ near ∞_i. We denote such a Hurwitz space by $\mathcal{H}_{g;\mathsf{n}}$, where $\mathsf{n} := (n_0, \cdots, n_m)$. This is an irreducible quasi-projective complex variety of dimension

$$d_{g;\mathsf{n}} := \dim(\mathcal{H}_{g;\mathsf{n}}) = 2g + 2m + \sum_{i=0}^{m} n_i. \tag{3.1}$$

We will write π, λ and Σ_i for, respectively, the universal family, the universal map, and the sections marking the ∞_i, as per the following commutative diagram:

$$\begin{array}{ccccc} C_g & \hookrightarrow & \mathcal{C} & \xrightarrow{\lambda} & \mathbb{P}^1 \\ P_i \uparrow \downarrow & & \Sigma_i \uparrow \downarrow \pi & & \\ [\lambda] & \overset{\mathrm{pt}}{\hookrightarrow} & \mathcal{H} & & \end{array} \tag{3.2}$$

We furthermore denote by $\mathrm{d} = \mathrm{d}_\pi$ the relative differential with respect to the universal family and $p_i^{\mathrm{cr}} \in C_g \simeq \pi^{-1}([\lambda])$ the critical points $\mathrm{d}\lambda = 0$ of the universal map. By the Riemann existence theorem, the critical values of λ, $\{u_i\}_{i=1,\cdots,d_{g;\mathsf{n}}}$, serve as local coordinates away from the closed subsets in $\mathcal{H}_{g;\mathsf{n}}$ in which $u_i = u_j$ for $i \neq j$, whose union is called the discriminant. Additionally, there is an action on a Hurwitz space given by the affine subgroup of the $\mathrm{PGL}_2(\mathbb{C})$-action on the target,

$$(C, \lambda) \mapsto (C, a\lambda + b), \qquad u_i \mapsto a u_i + b, \tag{3.3}$$

for $a, b \in \mathbb{C}$, and $\forall i = 1, \cdots, d_{g;\mathsf{n}}$. The Frobenius structure is constructed on a cover of $\mathcal{H}^{\phi}_{g;\mathsf{n}}$ which is defined by fixing a meromorphic differential[3] and a symplectic basis[4] of integral first homology

[3]The differential must be *admissible* for the cover λ, implying certain bounds on the order of its singularities and a set of vanishing conditions for the periods (see [15, Lecture 5]). The choices made in this paper will always result in admissible differentials.

[4]Changing this basis by an element of $\mathrm{Sp}(2g, \mathbb{Z})$ will give a Frobenius structure on the base which is related to the first by a type 1 transformation [15, App. B]. Note that this type of transformation in general changes the charge of a Frobenius manifold, except when $d = 1$, which is the case for Dubrovin–Zhang manifolds.

cycles on the fibres of the universal family. We first posit that the coordinate vector fields in this chart are idempotents of the algebra

$$\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij}\partial_{u_i}, \tag{3.4}$$

which provides the structure of a semi-simple commutative unital algebra with identity and Euler vector field

$$e = \sum_{i=1}^{d_{g;\mathsf{n}}} \partial_{u_i}, \quad E = \sum_{i=1}^{d_{g;\mathsf{n}}} u_i \partial_{u_i}, \tag{3.5}$$

arising as the generators of the affine action in (3.3). What remains to be defined is thus a flat, non-degenerate symmetric pairing playing the role of η. This is attained by choosing a suitable admissible quadratic differential [15], which for the purposes of this paper will always be the square of a third-kind meromorphic differential $\phi \in \Omega_C((\lambda))$ with at most simple poles at the poles of λ and vanishing periods. Then η is defined by the residue formula

$$\eta(X,Y) := \sum_i \operatorname*{Res}_{p_i^{\rm cr}} \frac{X(\lambda)Y(\lambda)}{\mathrm{d}\lambda}\phi^2, \tag{3.6}$$

for $X, Y \in \Gamma(\mathcal{H}_{g;\mathsf{n}}, T\mathcal{H}_{g;\mathsf{n}})$. Furthermore, combining (3.4) and (3.6) the 3-tensor $c(X,Y,Z)$ is defined by the LG formula

$$c(X,Y,Z) \equiv \eta(X, Y\cdot Z) = \sum_i \operatorname*{Res}_{p_i^{\rm cr}} \frac{X(\lambda)Y(\lambda)Z(\lambda)}{\mathrm{d}\lambda}\phi^2, \tag{3.7}$$

which clearly satisfies the Frobenius property. Moreover, the second flat pairing is obtained by replacing λ by $\log\lambda$ in (3.6),

$$\gamma(X,Y) = \sum_i \operatorname*{Res}_{p_i^{\rm cr}} \frac{X(\log\lambda)Y(\log\lambda)}{\mathrm{dlog}\ \lambda}\phi^2. \tag{3.8}$$

Since ϕ is admissible for λ in the language of [15], this construction satisfies the axioms of a Frobenius structure on $\mathcal{H}_{g;\mathsf{n}}$. We call the marked meromorphic function λ a *Landau-Ginzburg (LG) superpotential* for the Frobenius manifold, and ϕ its primary differential.

3.2. **Superpotentials for extended affine Weyl groups.** We give here a general method for the construction of spectral curves associated to affine relativistic Toda chains, as anticipated in [6], for arbitrary Dynkin types.

Let $\omega \in \Lambda_w^+(\mathcal{R})$ be the highest weight of a non-trivial irreducible representation $\rho_\omega \in \mathrm{Rep}(\mathcal{G}_\mathcal{R})$ of minimal dimension; in particular, ρ_ω is quasi-minuscule, and it is minuscule for all $\mathcal{R} \neq B_l$, F_4 and E_8. Consider the characteristic polynomial of $\mathsf{g} \in \mathcal{G}$ in the representation ρ_ω,

$$\mathcal{P}_\omega(\chi_1, \dots, \chi_{l_\mathcal{R}}; \mu) \ = \ \det_{\rho_\omega}(\mu\mathbf{1} - \mathsf{g}) = \sum_{k=0}^{\dim\rho_\omega} \mu^{\dim\rho_\omega - k}\chi_{\wedge^k\rho}(\mathsf{g}), \tag{3.9}$$

where the second equality is the co-factor expansion of the determinant. Recall that the representation ring of a simple Lie group is an integral polynomial ring generated by the fundamental representations,

$$\chi_{\wedge^k\rho}(\mathsf{g}) = \mathfrak{p}_k^\omega(\chi_1, \dots, \chi_{l_\mathcal{R}}) \in \mathbb{Z}[\chi_1, \dots, \chi_{l_\mathcal{R}}], \tag{3.10}$$

where $\chi_i \coloneqq \mathrm{Tr}_{\rho_i}(g)$ is the $i^{\rm th}$ fundamental character. Since ρ_ω is quasi-minuscule, $\mathcal{P}_\omega$ factorises as

$$\mathcal{P}_\omega = (\mu-1)^{\mathsf{z}_0}\mathcal{P}_\omega^{\rm red} = (\mu-1)^{|\mathsf{z}_0|} \prod_{0\neq\omega'\in\Gamma(\rho_\omega)} \left(\mu - \mathrm{e}^{\omega'\cdot h}\right), \tag{3.11}$$

where z_0 is the dimension of the zero weight space of ρ, and e^{h} with $[\mathrm{e}^{\mathsf{h}}]=[\mathsf{g}]$ is a choice of Cartan torus element conjugate to g: in particular, $\mathsf{z}_0=0$ and $\mathcal{P}_\omega^{\rm red}=\mathcal{P}_\omega$ for $\mathcal{R}\neq B_l$, F_4, or E_8. Consider, for $i=1,\dots,l_{\mathcal{R}}$ and as $w\coloneqq(w_1,\dots,w_{l_\mathcal{R}})\in\mathbb{C}^{l_\mathcal{R}}\times\mathbb{C}$ vary, the family of plane algebraic curves in $\mathrm{Spec}\mathbb{C}[\lambda,\mu]$ with fibre at w given by

$$C_w^{(\omega,i)} = \mathbb{V}\left(\mathcal{P}_\omega^{\rm red}\left(\chi_k = w_k - \delta_{ik}\frac{\lambda}{w_0}\right)\right). \tag{3.12}$$

By compactifying the fibres over w by taking the normalisation $\overline{C_w^{(\omega,i)}}$ of their Zariski closure in $\mathbb{P}^2$, and marking the λ-projection $(\lambda,\mu)\to\lambda\in\mathbb{P}^1$, we define a subvariety, isomorphic to the torus $\mathbb{C}^\star\times(\mathbb{C}^\star)^{l_\mathcal{R}}$ with coordinates $(w_0;w_1,\dots,w_{l_\mathcal{R}})$, of the Hurwitz space $\mathcal{H}_{g_{\omega,i},\mathsf{n}_{\omega,i}}$: here $g_\omega = h^{1,0}\left(\overline{C_w^{(\omega,i)}}\right)$, and $\mathsf{n}_{(\omega,i)}$ keeps track of the ramification of the poles of λ. We define a family of semi-simple, commutative, unital Frobenius algebras $\mathcal{M}_{\omega,i}^{\rm LG}$ on the tangent bundle of $\mathbb{C}^\star\times(\mathbb{C}^\star)^{l_\mathcal{R}}$ as follows

$$\eta(\partial_{w_i},\partial_{w_j}) = \sum_l \mathrm{Res}_{p_l^{\rm cr}} \frac{\partial_{w_i}\lambda\partial_{w_j}\lambda}{\mu^2\partial_\mu\lambda}\mathrm{d}\mu, \tag{3.13}$$

$$\eta(\partial_{w_i},\partial_{w_j}\cdot\partial_{w_k}) = \sum_l \mathrm{Res}_{p_l^{\rm cr}} \frac{\partial_{w_i}\lambda\partial_{w_j}\lambda\partial_{w_k}\lambda}{\mu^2\partial_\mu\lambda}\mathrm{d}\mu, \tag{3.14}$$

$$\gamma(\partial_{w_i},\partial_{w_j}) = \sum_l \mathrm{Res}_{p_l^{\rm cr}} \frac{\partial_{w_i}\lambda\partial_{w_j}\lambda\partial_{u_k}\lambda}{\mu^2\partial_\mu\lambda}\mathrm{d}\mu, \tag{3.15}$$

where $\{p_l^{\rm cr}\}_l$ are the ramification points of $\lambda:\overline{C_w^{(\omega,i)}}\to\mathbb{P}^1$. This doesn't yet give a Frobenius manifold, or indeed a Frobenius submanifold of $\mathcal{H}^\phi_{g_\omega,\mathsf{n}_\omega}$ with $\phi=\mathrm{d}\log\mu$, as η is not guaranteed to be either non-degenerate or flat at this stage. The following statement establishes that this is the case when $i=\bar{k}$, the label for the canonical node in the Dubrovin–Zhang construction. For this case, we write simply $\mathcal{M}_\omega^{\rm LG}\coloneqq\mathcal{M}_{\omega,\bar{k}}^{\rm LG}$.

Theorem 3.1 (Mirror symmetry for DZ Frobenius manifolds)**.** *The Landau–Ginzburg formulas* (3.13)–(3.15) *define a semi-simple, conformal Frobenius submanifold* $\iota_\omega:\mathcal{M}_\omega^{\rm LG}=(\mathbb{C}^\star\times(\mathbb{C}^\star)^{l_\mathcal{R}},\eta,e,E,\cdot)\hookrightarrow\mathcal{H}^\phi_{g_\omega,\mathsf{n}_\omega}$. *In particular,* (3.13) *and* (3.15) *give flat, non-degenerate metrics on* $T\mathcal{M}_\omega^{\rm LG}$*, and the identity and Euler vector fields read*

$$e = w_0^{-1}\partial_{u_s}, \qquad E = w_0\partial_{w_0}. \tag{3.16}$$

Furthermore,

$$\mathcal{M}_\omega^{\rm LG}\simeq\mathcal{M}_\mathcal{R}^{\rm DZ}. \tag{3.17}$$

The explicit embedding $\iota_\omega : \mathcal{M}_\omega^{\rm LG} \hookrightarrow \mathcal{H}^{\phi}_{g_\omega,\mathsf{n}_\omega}$ is described by the relations (3.11) in the character ring of $\mathcal{G}$, setting the coefficients associated to the interior of the Newton polytope of $\mathcal{P}^{\rm red}_\omega(\lambda,\mu)$ to be the polynomials $\mathfrak{p}^\omega_k(w_1,\ldots,w_r)$.

The proof of Theorem 3.1 requires two key steps. The first main stumbling block to overcome is the calculation of the exterior relations (3.11) in the Weyl character ring of $\mathcal{G}$, which was solved for simply-laced cases in [4, 7]. We here complete this to include all non-simply laced Lie groups as well. The second main step is to prove that the LG formulas (3.13)–(3.15), combined with the reconstruction theorem Theorem 2.1, establish the mirror statement of Theorem 3.1. In the remainder of this Section we perform the first step and construct explicitly the family of LG mirror duals to type-$\mathcal{R}$ Dubrovin–Zhang Frobenius manifolds.

3.3. **Superpotentials for classical Lie groups.** In the following we present the construction of the spectral curve for the classical root systems $\mathcal{R} = A_l$, B_l, C_l, D_l independently, and show how our construction for a weight ω corresponding to a minimal-dimensional representation ρ_ω recovers the mirror results of [19] for these cases. We will use the shorthand notation $\varepsilon_i \coloneqq \chi_{\wedge^i \rho}$ for the exterior characters of ρ_ω.

3.3.1. $\mathcal{R} = A_l$. The Dynkin diagram for affine A_l is shown in Figure 3.1. In this case we can choose any (non-affine) node to be the marked one, since the removal of any node from the corresponding finite Dynkin diagram results in two disconnected A-type pieces, with ranks adding up to $l-1$. In the figure, the canonical marked node is $\alpha_{\lfloor l/2 \rfloor}$, indicated by a $\times$ sign.

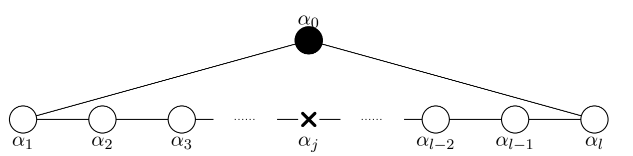

FIGURE 3.1. The affine Dynkin diagram of $\mathcal{R} = A_l$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

A choice of minimal, nontrivial, irreducible representation $\rho_\omega \coloneqq \rho_1 = (\mathbf{l+1})$ for $\mathrm{SL}_{\mathbb{C}}(l+1)$ is the defining $(l{+}1)$-dimensional representation, the other choice corresponding to its dual representation, $\rho_l = \wedge^l \Box$. We then have that $\varepsilon_i = \chi_i$ for $i = 1,\ldots,l$, and $\varepsilon_0 = \varepsilon_{l+1} = 1$. Then (3.9) becomes

$$\mathcal{P}_{[10\ldots0]_{A_l}} = \frac{(-1)^k \lambda \mu^k}{w_0} + 1 + (-1)^{l+1}\mu^{l+1} + \sum_{i=1}^{l} (-1)^i w_i \mu^i, \tag{3.18}$$

which defines a family of genus 0 curves. Setting (3.18) equal to zero and solving for λ gives

$$\lambda = \frac{(-1)^k w_0 (1 + (-1)^{l+1}\mu^{l+1} + \sum_{i=1}^{l} (-1)^i w_i \mu^i)}{\mu^k}, \tag{3.19}$$

which is, for every point in the moduli space, a meromorphic function of μ with poles at 0 and ∞ of orders $\bar{k}, l+1-\bar{k}$, respectively. We see that we have $l+1$ parameters $w_0, \cdots, w_l$, and so the resulting Frobenius manifold is $l+1$ dimensional[5]. In particular, it is an $l+1$ dimensional submanifold of the Hurwitz space $\mathcal{H}_{0;\mathsf{n}_\omega}$, with ramification profile $\mathsf{n}_\omega = (\bar{k}-1, l-\bar{k})$. This Hurwitz space, however, is of dimension $2+\bar{k}-1+l-\bar{k} = l+1$, and so the DZ-Frobenius manifold associated to A_l is isomorphic to (a full-dimensional ball inside) its associated Hurwitz space. This, as we will see, will not be the case for the other Dynkin types.

3.3.2. $\mathcal{R} = B_l$. For $l > 2$, the minimal, nontrivial, irreducible representation of $\mathrm{Spin}(2l+1)$ is the defining representation $\rho_1 = (\mathbf{2l+1})$ of the special orthogonal group in $(2l+1)$-dimensions[6], which is the irreducible representation with highest weight ω_1. In this case the marked node is $\bar{k} = l-1$, as depicted in Figure 3.2.

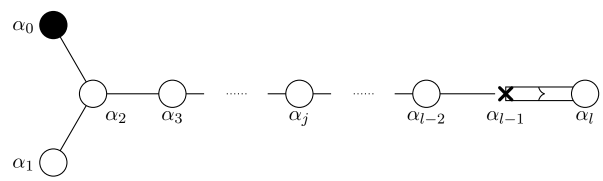

FIGURE 3.2. The affine Dynkin diagram of $\mathcal{R} = B_l$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

For $i < l$, the i^{th} fundamental representation ρ_i of B_l is the i^{th} exterior power of $(\mathbf{2l+1})$. For $i = l$, the decomposition of the tensor square of ρ_l leads to

$$\mathfrak{p}_i^{[10\ldots0]_{B_l}} = \begin{cases} \chi_i & \text{if } i < l \\ \chi_l^2 - \sum_{j=0}^{l-1} \chi_j & \text{if } i = l. \end{cases} \tag{3.20}$$

Together with the self-duality relation $\mathfrak{p}_i^{[10\ldots0]_{B_l}} = \mathfrak{p}_{2l+1-i}^{[10\ldots0]_{B_l}}$, we get that the curve is the zero locus of

$$\mathcal{P}_{[10\ldots0]_{B_l}} = \frac{(-1)^l(\mu-1)(\mu+1)^2\mu^{l-1}\lambda}{w_0} + \sum_{i=0}^{l-1}(-1)^i\mu^i(1-\mu^{2(l-i)+1})\varepsilon_i, \tag{3.21}$$

with $\varepsilon_i = w_i$ for $i < l$, and $\varepsilon_l = w_l^2 - \sum_{j=0}^{l-1} w_j$. Note that (3.21) has a factor of $(\mu-1)$, since $(\mathbf{2l+1})$ has a one-dimensional zero weight space. Setting to zero the reduced characteristic polynomial $\mathcal{P}^{\text{red}}_{[10\ldots0]_{B_l}} = \mathcal{P}_{[10\ldots0]_{B_l}}/(\mu-1)$ gives

$$\lambda = \frac{(-1)^l w_0}{\mu^{l-1}(\mu+1)^2} \sum_{j=0}^{2l} \mu^j \left(\sum_{i=0}^{\min(j,2l-j)} (-1)^i \varepsilon_i \right). \tag{3.22}$$

For each point in the moduli space, this is a rational function in μ with three poles at $0, -1$, and ∞ of orders $l-1, 2, l-1$, respectively. Hence, $\mathcal{M}^{\text{LG}}_{[10\ldots0]_{B_l}}$ is a sublocus in the $(2l+1)$-dimensional Hurwitz space $\mathcal{H}_{0;\mathsf{n}_\omega}$, with $\mathsf{n}_\omega = (l-2, 1, l-2)$. The latter carries an involution given by sending

[5] This is, in fact, the case for all DZ-manifolds associated to a simple Lie algebra of rank l.

[6] For $l = 2$, the 4-dimensional spin representation ρ_2 is both minimal and minuscule. It is also isomorphic to the vector representation ρ_1 of C_2, which is included in the discussion of the next section.

$\mu \to 1/\mu$, and $\mathcal{M}^{\mathrm{LG}}_{[10\ldots0]_{B_l}}$ is characterised as the $(l+1)$-dimensional stratum that is fixed by the involution.

3.3.3. $\mathcal{R} = C_l$. The minimal, nontrivial, irreducible representation for $\mathrm{Sp}(2l)$ is the defining representation $\rho_1 = (\mathbf{2l})$ of the rank $2l$ symplectic group. Again, this representation corresponds to the one in which ω_1 is highest weight. The canonical node is the l^{th} node, as shown in Figure 3.3.

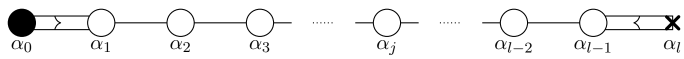

FIGURE 3.3. The affine Dynkin diagram of $\mathcal{R} = C_l$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

The exterior powers $\wedge^i \rho_1$ are reducible, with only fundamental representations appearing as direct summands in their decomposition, giving the character relations

$$\mathfrak{p}_i^{[10\ldots0]_{C_l}} = \begin{cases} \sum_{j=0}^{\frac{i}{2}} \chi_{2j} & i \text{ even}, \\ \sum_{j=0}^{\frac{i-1}{2}} \chi_{2j+1} & i \text{ odd}. \end{cases} \tag{3.23}$$

From this, and the fact that $\bar{k} = l$ for C_l, we see that the characteristic polynomial (3.9) is

$$\mathcal{P}_{[10\ldots0]_{C_l}} = \frac{(-1)^l \mu^l \lambda}{w_0} + \sum_{i=0}^{l-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(l-i)}) + (-1)^l \varepsilon_l \mu^l, \tag{3.24}$$

with $\varepsilon_{2i} = \sum_{j=0}^{i} \chi_{2j}$, $\varepsilon_{2i+1} = \sum_{j=0}^{i} \chi_{2j+1}$. Setting equal to zero and solving for λ gives

$$\lambda = \frac{(-1)^{l-1} w_0 \left(\sum_{i=0}^{l-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(l-i)}) + (-1)^l \varepsilon_l \mu^l \right)}{\mu^l}, \tag{3.25}$$

which is a rational function in μ with two poles at 0 and ∞ both of order l. Hence, the associated covering Hurwitz space is $\mathcal{H}_{0;\mathsf{n}_\omega}$, with $\mathsf{n}_\omega = (l-1, l-1)$, which has dimension $2l$. As before, there is an involution on this Hurwitz space sending $\mu \to 1/\mu$, with $\mathcal{M}^{\mathrm{LG}}_{[10\ldots0]_{C_l}}$ being the $(l+1)$-dimensional stratum that is fixed by it.

3.3.4. $\mathcal{R} = D_l$. For $l \geq 4$, the minimal, nontrivial, irreducible representation of $\mathrm{Spin}(2l)$ is the defining vector[7] representation $\rho_1 = (\mathbf{2l})_v$ of $\mathrm{SO}(2l)$, which corresponds to the irreducible representation with highest weight ω_1. The canonical node is the one with label $l-2$, as shown in Figure 3.4.

[7]For $l = 4$, this can be any of the irreducible 8-dimensional representations, related by triality.

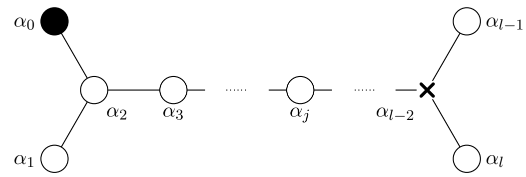

FIGURE 3.4. The affine Dynkin diagram of $\mathcal{R} = D_l$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

The character relations for D_l were found in [4] to be

$$\mathfrak{p}_i^{[10\ldots0]_{D_l}} = \chi_i, \quad i < l-2 \text{ or } i = l-1, \tag{3.26}$$

$$\mathfrak{p}_{l-2}^{[10\ldots0]_{D_l}} = \chi_{l-1}\chi_l - \begin{cases} \sum_{j=0}^{\frac{l}{2}-2} \chi_{2j+1} & \text{if } l \text{ is even,} \\ \sum_{j=0}^{\frac{l-3}{2}} \chi_{2j} & \text{if } l \text{ is odd,} \end{cases} \tag{3.27}$$

$$\mathfrak{p}_{l}^{[10\ldots0]_{D_l}} = \chi_{l-1}^2 + \chi_l^2 - 2\begin{cases} \sum_{j=0}^{\frac{l}{2}-1} \chi_{2j} & \text{if } l \text{ is even,} \\ \sum_{j=0}^{\frac{l-3}{2}} \chi_{2j+1} & \text{if } l \text{ is odd,} \end{cases} \tag{3.28}$$

and $\mathfrak{p}_i^{[10\ldots0]_{D_l}} = \mathfrak{p}_{2l-i}^{[10\ldots0]_{D_l}}$, so that

$$\mathcal{P}_{[10\ldots0]_{D_l}} = \frac{(-1)^l\mu^{l-2}(\mu^2-1)^2\lambda}{w_0} + \sum_{i=0}^{l-1}(-1)^i\varepsilon_i\mu^i(1+\mu^{2(l-i)}) + (-1)^l\mu^l\varepsilon_l, \tag{3.29}$$

where as before we denote $\varepsilon_i(w_1, \ldots, w_l) = \mathfrak{p}_i^{[10\ldots0]_{D_l}}(\chi_j = w_j)$. Setting (3.29) equal to zero and solving for λ gives

$$\lambda = (-1)^{l-1}\frac{w_0\left(\sum_{i=0}^{l-1}(-1)^i\varepsilon_i\mu^i(1+\mu^{2(l-i)}) + (-1)^l\mu^l\varepsilon_l\right)}{\mu^{l-2}(\mu^2-1)^2}, \tag{3.30}$$

which, for every point w, is a rational function in μ with four poles at 0, ∞, 1, -1 of orders $l-2$, $l-2$, 2, 2, respectively. Hence, the parent Hurwitz space is $\mathcal{H}_{0;\mathsf{n}_\omega}$, where $\mathsf{n}_\omega = (l-3, l-3, 1, 1)$, which has dimension $2l+2$. Once more this hosts an involution obtained by sending $\mu \to 1/\mu$, identifying $\mathcal{M}^{\mathrm{LG}}_{[10\ldots0]_{D_l}}$ as its fixed locus.

3.4. **Comparison with the Dubrovin–Strachan–Zhang–Zuo construction.** For the case of $\mathcal{R} = A_l$, an LG-superpotential was already found in the original paper [15];

$$\lambda = \sum_{j=0}^{k+m} a_j e^{i\phi(k-j)}, \tag{3.31}$$

with $a_0 a_{k+m} \neq 0$. In [19], the authors construct a three-integer parameter family of superpotentials

$$\lambda^{\mathrm{DSZZ}}(l,k,m) = \frac{4^m \mu^m \sum_{j=0}^{l} a_j 2^{-2(-j+k+m)} \left(\frac{\mu+1}{\sqrt{\mu}}\right)^{2(-j+k+m)}}{(\mu-1)^m}. \tag{3.32}$$

The key result of [19] is an identification of (3.32) with a superpotential for a Dubrovin–Zhang Frobenius manifold of type A_l, B_l, C_l, D_l, possibly with a non-canonical choice of marked node in the Dynkin diagram, for suitable choices of (l,k,m). In particular, the mirror theorem for the canonical label $\bar{k}$ is obtained by setting (l,k,m) equal to $(l,k,0)$, $(l,l-1,1)$, $(l,l,0)$, and $(l,l-2,1)$, respectively. We shall now show that the result of [19] coincides with our construction in the previous section.

3.4.1. $\mathcal{R} = A_l$. By using the fact that $k+m = l+1$, we get

$$\lambda^{\mathrm{DSZZ}}(l,k,0) = \frac{\sum_{i=0}^{l+1} a_i \mu^{l+1-i}}{\mu^k}, \tag{3.33}$$

which is the same as (3.19) by

$$a_i = (-1)^k w_0 \begin{cases} 1 & \text{if } i=0 \\ (-1)^{l+1} & \text{if } i = l+1 \\ w_i & \text{otherwise.} \end{cases} \tag{3.34}$$

3.4.2. $\mathcal{R} = B_l$. In the case of B_l we consider (3.32) with $k = l-1$, $m=1$, which is

$$\lambda^{\mathrm{DSZZ}}(l,l-1,1) = \frac{4\mu \sum_{j=0}^{l} a_j 2^{-2(l-j)} \left(\sqrt{\mu} + \frac{1}{\sqrt{\mu}}\right)^{2(l-j)}}{(\mu-1)^2}. \tag{3.35}$$

Simplifying (3.35) gives:

$$\frac{w_0}{(\mu-1)^2} \sum_{j=0}^{l} \frac{a_j 2^{-2(l-j-1)}(\mu+1)^{2(l-j)}}{w_0 \mu^{l-j-1}} = \frac{(-1)^l w_0}{(\mu+1)^2 \mu^{l-1}} \sum_{\beta=0}^{2l} (-1)^\beta C_\beta \mu^\beta, \tag{3.36}$$

where we have used the binomial theorem and let $\mu \mapsto -\mu$, with

$$C_\beta = \frac{(-1)^l}{w_0} \sum_{j,\alpha | j+\alpha=\beta} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{\alpha} = \frac{(-1)^l}{w_0} \sum_{j=0}^{\beta} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{\beta - j}. \tag{3.37}$$

On the other hand, the superpotential constructed from the spectral curve, (3.22), is given by

$$\lambda_{B_l} = \frac{(-1)^l w_0}{\mu^{l-1}(\mu+1)^2} \sum_{j=0}^{2l} \mu^j \left(\sum_{i=0}^{\min(j,2l-j)} (-1)^i \varepsilon_i \right), \tag{3.38}$$

which we can write as

$$\lambda_{B_l} = \frac{(-1)^l w_0}{\mu^{l-1}(\mu+1)^2} \sum_{j=0}^{2l} b_j \mu^j, \tag{3.39}$$

with $b_j = \sum_{i=0}^{\min(j,2l-j)}(-1)^i\varepsilon_i$; note that $b_j = b_{2l-j}$. This means that we want to match up $b_i = (-1)^i C_i$, hence

$$\sum_{j=0}^{\min(i,2l-i)} (-1)^j \varepsilon_j = \frac{(-1)^{l+i}}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{i-j}. \tag{3.40}$$

We claim that

$$\varepsilon_i = \frac{(-1)^l}{w_0} \sum_{j=0}^{l} a_j 2^{-2(l-i-1)} \binom{2l-2j+1}{i-j}. \tag{3.41}$$

Proof. The $i = 0$ case is clear giving $\varepsilon_0 = \dfrac{(-1)^l}{w_0} 2^{-2(l-1)} a_0$ obtained by taking $j = 0$.
So suppose $0 < i \leq l$. Then (3.40) becomes

$$\begin{aligned}
\sum_{j=0}^{i}(-1)^j\varepsilon_j &= \frac{(-1)^{l+i}}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{i-j} \\
\implies \varepsilon_i &= \frac{(-1)^l}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{i-j} + \frac{(-1)^l}{w_0} \sum_{j=0}^{i-1} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{i-1-j} \\
&= \frac{(-1)^l}{w_0} \left(a_i 2^{-2(l-i-1)} + \sum_{j=0}^{i-1} a_j 2^{-2(l-j-1)} \left(\binom{2(l-j)}{i-j} + \binom{2(l-j)}{i-1-j} \right) \right)
\end{aligned} \tag{3.42}$$

Furthermore,

$$\begin{aligned}
\binom{2(l-j)}{i-j} + \binom{2(l-j)}{i-1-j} &= \frac{(2(l-j))!}{(i-j)!(2l-i-j)!} + \frac{(2(l-j))!}{(i-j-1)!(2l-i-j+1)!} \\
&= \binom{2l-2j+1}{i-j},
\end{aligned}$$

which gives the result for $i \leq l$. Hence, since $\varepsilon_i = \varepsilon_{2l+1-i}$, we have (3.41) $\forall i$.

□

3.4.3. $\mathcal{R} = C_l$. In the case of C_l, we consider (3.32) with $k = l, m = 0$ which is

$$\lambda^{\mathrm{DSZZ}}(l,l,0) = \sum_{j=0}^{l} a_j 2^{-2(l-j)} \left(\sqrt{\mu} + \frac{1}{\sqrt{\mu}}\right)^{2(l-j)}. \tag{3.43}$$

Simplifying (3.43) gives:

$$\frac{(-1)^{l-1}w_0}{\mu^l} \sum_{j=0}^{l} \frac{(-1)^{l-1} a_j 2^{-2}(l-j)(\mu+1)^{2(l-j)}}{w_0 \mu^{-j}} = \frac{(-1)^{l-1}w_0}{\mu^l} \sum_{\beta=0}^{2l} C_\beta \mu^\beta, \tag{3.44}$$

where we again have used the binomial theorem, and with

$$C_\beta = \frac{(-1)^{l-1}}{w_0} \sum_{j=0}^{\beta} a_j 2^{-2(l-j)} \binom{2(l-j)}{\beta-j}. \tag{3.45}$$

Thus, the equivalence is obtained in the case of C_l by letting

$$\varepsilon_i \mapsto \frac{(-1)^{l+i-1}}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j)} \binom{2(l-j)}{i-j}. \tag{3.46}$$

3.4.4. $\mathcal{R} = D_l$. For D_l, we want to consider (3.32) with $k = l-2, m = 1$, which is of the form

$$\lambda^{\mathrm{DSZZ}}(l, l-2, 1) = \frac{4\mu \sum_{j=0}^{l} a_j 2^{-2(l-j-1)} \left(\sqrt{\mu} + \frac{1}{\sqrt{\mu}}\right)^{2(l-j-1)}}{(\mu-1)^2}. \tag{3.47}$$

This is equivalent to

$$\frac{(-1)^{l-1} w_0}{(\mu-1)^2 \mu^{l-2} (\mu+1)^2} \sum_{j=0}^{l} \frac{(-1)^{l-1} a_j 2^{-2(l-j-2)} (\mu+1)^{2(l-j)}}{w_0 \mu^{-j}} = \frac{(-1)^{l-1} w_0}{\mu^{l-2}(\mu^2-1)^2} \sum_{\beta=0}^{2l} C_\beta \mu^\beta, \tag{3.48}$$

with

$$C_\beta = \frac{(-1)^{l-1}}{w_0} \sum a_j 2^{-2(l-j-2)} \binom{2(l-j)}{\beta-j}, \tag{3.49}$$

where, again, the binomial theorem has been used. Hence, the map

$$\varepsilon_i \mapsto \frac{(-1)^{l+i-1}}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j-2)} \binom{2(l-j)}{\beta-j} \tag{3.50}$$

gives the equivalence.

3.5. **Superpotentials for exceptional Lie groups.** We present here the construction of the spectral curve for the exceptional types E_6, E_7, F_4, and G_2. The E_8 case was treated extensively in [6] and [7], and we only give a very brief presentation here.

As for the classical cases, the construction of the superpotential hinges on determining the character relations (3.10) for all k. Explicitly, for all dominant weights $\varpi \in \Lambda_w^+(\mathcal{R})$ we should determine $N_\omega^{(\omega,k)} \in \mathbb{Z}$ such that

$$\mathfrak{p}_k^\omega = \sum_{\varpi \in \Lambda_w^+(\mathcal{R})} N_\varpi^{(\omega,k)} \prod_{i=1}^{l_\mathcal{R}} \chi_i^{\varpi^{(i)}}. \tag{3.51}$$

Definition 3.1. A set of dominant weights $\Pi_\omega \subset \cup_k \Gamma(\wedge^k \rho_\omega)$ is called *pivotal* for ω if $\forall\ j = 1, \dots, \lfloor \frac{\dim \rho_\omega}{2} \rfloor$, $\varpi' \in \Gamma(\wedge^j \rho_\omega)$ $\exists \varpi \in \Pi_\omega$ such that $\varpi' \not\preceq \varpi$.

Here $\varpi' \preceq \varpi$ denotes the canonical partial ordering of weights, i.e. $\varpi' \preceq \varpi \Leftrightarrow \varpi - \varpi' = \sum_{i=1}^{l_\mathcal{R}} n_i \alpha_i$ with $n_i \geq 0$. As for the classical cases, we take ω to sit in a minimal non-trivial orbit of $\mathcal{W}_\mathcal{R}$, as described in Table 2.

$\mathcal{R}$	ω	ρ_ω	$\lvert\mathfrak{I}_\omega\rvert$
E_6	[100000] (resp. [000010])	$\mathbf{27}_{E_6}$ (resp. $\overline{\mathbf{27}}_{E_6}$)	111
E_7	[0000010]	$\mathbf{56}_{E_7}$	907
E_8	[00000010]	$\mathbf{248}_{E_8}$	950077
F_4	[0001]	$\mathbf{26}_{F_4}$	74
G_2	[10]	$\mathbf{7}_{G_2}$	5

TABLE 2. Highest weights of minimal representations for exceptional root systems.

Lemma 3.2. *The sets of dominant weights*

$$\Pi_\omega := \begin{cases} \{[010120],[120010],[200200],[110110],[001030], & \omega=[100000]_{E_6}, \\ [030000],[020020],[000041],[000050]\}, & \\ \{[0002022],[0001113],[0100132],[0101041],[1001051], & \\ [1001061],[0011031],[0010070],[0000204],[0110050], & \omega=[0000010]_{E_7}, \\ [0010070],[1100070],[1000090],[0003011],[0020040], & \\ [0004000],[0000105],[00000100],[0000006]\}, & \\ \{[22222222]\}, & \omega=[00000010]_{E_8}, \\ \{[0022]\}, & \omega=[0001]_{F_4}, \\ \{[20]\}, & \omega=[10]_{G_2}, \end{cases} \tag{3.52}$$

are pivotal and of minimal cardinality for ω.

For $\mathcal{R}=E_8$, F_4 and G_2 the weight system of ρ is the set of short roots of $\mathcal{R}$, and the single element of its minimal pivotal set is then the sum of the positive short roots. For $\mathcal{R}=E_6$, E_7 the pivotal sets of minimal cardinality in (3.52) can be constructed by direct inspection of the weight system.

Definition 3.2. We call the finite set

$$\mathfrak{I}_\omega := \left\{ \iota \in (\mathbb{Z}_+)^{l_\mathcal{R}} \middle| \exists\ \varpi\in\Pi_\omega \text{ s.t. } \sum_j (C_\mathcal{R})^{-1}_{jk}\iota_k \leq \sum_j (C_\mathcal{R})^{-1}_{jk}\varpi_k \right\} \tag{3.53}$$

the set of *admissible exponents* of the exterior algebra $\wedge\rho_\omega$.

In other words, ι is admissible if and only if $\sum_i \iota_i\omega_i \preceq \varpi$ for some pivotal weight ϖ. We will use the short-hand notation $\sum_i \iota_i\omega_i \npreceq \Pi_\omega$ when this happens. The terminology is justified by the following

Lemma 3.3. *We have that*

$$\iota\notin\mathfrak{I}_\omega \Rightarrow N_\iota^{(\omega,k)}=0 \quad \forall k. \tag{3.54}$$

Proof. Consider the representation space version of (3.51),

$$\wedge^k \rho_\omega = \bigoplus_\iota N_\iota^{(\omega,k)} \bigotimes_{i=1}^{l_\mathcal{R}} \otimes_{j=1}^{\iota_i} \rho_i, \tag{3.55}$$

and consider the subset of indices $\iota \in \mathbb{N}_0^{l_\mathcal{R}} \setminus \Pi_\omega$. The corresponding one-dimensional weight spaces must appear with zero coefficient on the l.h.s. of (3.55), and therefore their weighted occurences must sum up with total coefficient equal to zero on its r.h.s.. Write $\sigma_\iota := \prod_{j=1}^{l_\mathcal{R}} (\mathrm{dim}\rho_j)^{\iota_j}$ for the dimension of the tensor product representation $\otimes_j \rho_j$, and $\Omega_\mathcal{R} := \{p \mid \mathrm{dim}\rho_j\}_{j=1}^{l_\mathcal{R}}$. The monoid morphism

$$\begin{aligned} \mathfrak{f} : \Lambda_w(\mathcal{R})^+ \quad &\longrightarrow \quad \mathbb{N}^{|\Omega_\mathcal{R}|} \\ \{\iota_j\}_{j=1}^{l_\mathcal{R}} \quad &\longrightarrow \quad \{k_p\}_{p\in\Omega_\mathcal{R}} \end{aligned} \tag{3.56}$$

induced by the prime factor decomposition $\sigma_\iota = \prod_{p|\sigma_\iota} p^{k_p}$ is injective for all $\mathcal{R} \neq E_6$, as the reader can immediately verify, and for $\mathcal{R} = E_6$ it has fibres of order at most two under the complex conjugation $\omega_1 \leftrightarrow \omega_5$, $\omega_2 \leftrightarrow \omega_4$. This means that each summand $\bigotimes_i \rho_i^{\iota_i}$ in (3.55) is uniquely determined by its dimension and, for E_6, its reality type. We use this to endow $\Lambda_w(\mathcal{R})^+$ with a total order by stipulating that $\iota' < \iota$ if $\sigma_{\iota'} < \sigma_\iota$ or (for $\mathcal{R} = E_6$) $\sigma_{\iota'} = \sigma_\iota$ and $\iota_1 < \iota_5$. Let then $\iota_j^{\max}$ be the indices of the summand on the r.h.s. of (3.55) which is maximal under this total order. This summand contains the dominant weight $\sum_j \iota_j^{\max}\omega_j \not\preceq \Pi_\omega$ in its weight module, and by construction it is the unique direct summand in (3.55) containing it, so $N_\iota^{(\omega,k)} = 0$. Induction under the total order gives $N_\iota^{(\omega,k)} = 0$ for all $\sum_j \iota_j\omega_j \not\preceq \Pi_\omega$. The lemma then follows from writing down the condition $\sum_j \iota_j\omega_j \preceq \Pi_\omega$ in the root basis. □

By Lemma 3.3, the sum in the polynomial character decomposition (3.51) localises on the set of admissible exponents, whose cardinality $|\mathfrak{I}_\omega|$ is shown in Table 2. Evaluating (3.51) on a generic set of rational points $\{(Q_1^{(\kappa)}, \dots, Q_{l_\mathcal{R}}^{(\kappa)}) \in \mathbb{Q}^{l_\mathcal{R}}\}_{\kappa=1}^{|\mathfrak{I}_\omega|}$ in $\mathcal{T}_\mathcal{R}$ gives a rank-$|\mathfrak{I}_\omega|$ linear problem over the rationals, which can be solved exactly[8] for the integers $\{N_\iota^{(\omega,k)}\}_{\iota\in\mathfrak{I}_\omega}$. Using (3.12) and (3.51) we can then construct Landau–Ginzburg superpotentials for all $\mathcal{R}$, as we turn to describe.

3.5.1. $\mathcal{R} = E_6$. The Dynkin diagram for the affine E_6 root system is given in Figure 3.5, for which the canonical label is $\bar{k} = 3$.

In the case of E_6, there are two nontrivial minimal irreducible representations: the 27-dimensional fundamental representation $\rho_1 = (\mathbf{27})$ with highest weight ω_1 or its dual representation $\rho_5 = (\overline{\mathbf{27}})$ with highest weight ω_5, related by complex conjugation. The character relations (3.51) are

[8]This can be done using exact arithmetics over the integers, for example through the use of Dixon's algorithm. The fundamental characters on the r.h.s. of (3.51) are integral Laurent polynomials in $(Q_1, \dots, Q_{l_\mathcal{R}})$, whose form can be computed from the explicit knowledge of $\Gamma(\rho_i)$, $i = 1, \dots, l_\mathcal{R}$. The l.h.s. can similarly be computed from knowing $\Gamma(\rho_\omega)$ alone to evaluate the power sum virtual characters of ω, from which the exterior characters $\chi_{\wedge^k \rho_\omega}$ can be recovered using Newton's formulas. The resulting linear systems are easily solvable exactly on a computer for all $\mathcal{R} \neq E_8$, for which extra care is required in the choice of the sampling set $\{(Q_1^{(\kappa)}, \dots, Q_{l_\mathcal{R}}^{(\kappa)}) \in \mathbb{Q}^{l_\mathcal{R}}\}$. As explained more in detail in [7], special choices exist in this case reducing the determination of $N_\iota^{(\omega,k)}$ to a large number of smaller linear systems that can be solved using parallel computing.

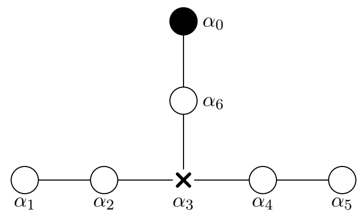

FIGURE 3.5. The affine Dynkin diagram of $\mathcal{R} = E_6$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

given explicitly for ρ_1 in Appendix A. The resulting family of spectral curves has fibres which are hyperelliptic curves of genus 5, with Newton polygon as shown in Figure 3.6, and ramification profile over ∞

$$\left(\overbrace{3,6}^{\mu=0}, \overbrace{6,3}^{\mu=\infty}, \overbrace{3}^{\mu=\varepsilon_3^j} \right), \tag{3.57}$$

where ε_3 is a primitive third root of unity. This realises $\mathcal{M}^{\rm LG}_{[100000]_{E_6}}$ as a 7-dimensional subvariety of the 42-dimensional Hurwitz space $\mathcal{H}_{g_\omega,\mathsf{n}_\omega}$ with $g_\omega = 5$ and $\mathsf{n}_\omega = (5,5,2,2,2,2,2)$, with the explicit embedding described by (A.1).

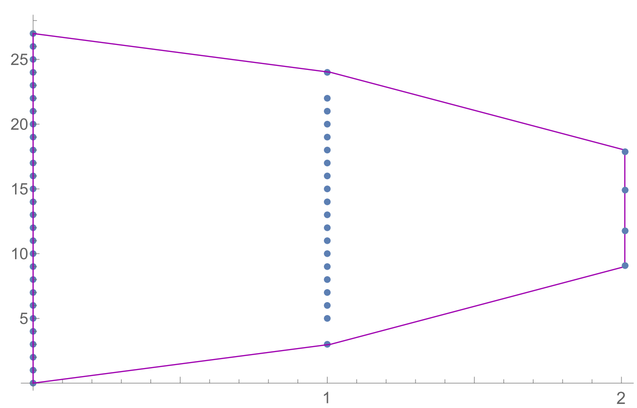

FIGURE 3.6. Newton polygon for the E_6-spectral curve.

3.5.2. $\mathcal{R} = E_7$. The Dynkin diagram for the affine E_7 root system is given in Figure 3.7, in which we see that the canonical label is $\bar{k} = 3$.

For this case there is a unique choice of minimal representation, corresponding to the 56-dimensional fundamental representation having highest weight ω_6. Choosing this representation gives character relations which we include in Appendix A for $k = 1, \ldots, 11$. The resulting family of spectral curves has fibres of genus 33, with a degree 3 morphism to $\mathbb{P}^1$ inducing a $3:1$ branched cover of the Riemann sphere with ramification profile over ∞

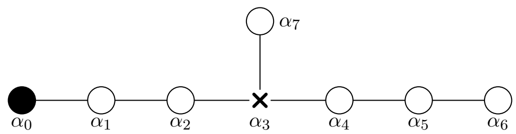

FIGURE 3.7. The affine Dynkin diagram of $\mathcal{R} = E_7$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

$$\Bigg(\overbrace{12,6,4}^{\mu=0},\ \overbrace{12,6,4}^{\mu=\infty},\ \overbrace{2}^{\mu=\pm 1}\ \overbrace{4}^{\mu=\pm i}\Bigg). \tag{3.58}$$

Hence, $\mathcal{M}_\omega^{\mathrm{LG}}$ is an 8-dimensional submanifold in the 130-dimensional Hurwitz space $\mathcal{H}_{g_\omega;\mathsf{n}_\omega}$, with $g_\omega = 33$ and $\mathsf{n}_\omega = (11,5,3,11,5,3,1,1,3,3)$. The associated Newton polygon is shown in Figure 3.8.

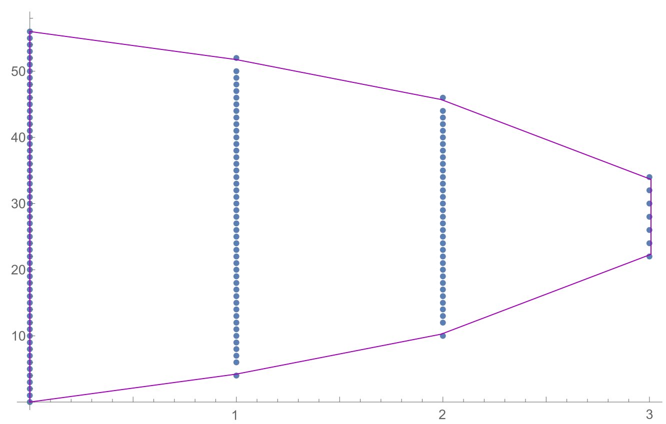

FIGURE 3.8. Newton polygon for the E_7-spectral curve.

3.5.3. $\mathcal{R} = E_8$. As mentioned, this case was thoroughly treated in [6], and we will provide only a brief presentation here. The Dynkin diagram for the affine E_8 root system is given in Figure 3.9, and the canonical label is, as for all the E-types, $\bar{k} = 3$.

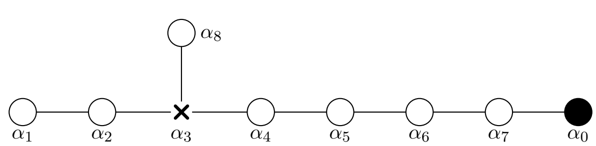

FIGURE 3.9. The affine Dynkin diagram of $\mathcal{R} = E_8$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

Here, the minimal, nontrivial, irreducible representation is the 248-dimensional adjoint representation. Explicit character relations were given in [7], where their derivation is explained in detail. It is shown in [6] that the resulting curve is of genus 128 and induces a cover of the Riemann sphere with ramification over ∞ at $\mu = 0, \infty$, in addition to second, third and fifth roots of unity, with ramification profile given in [6, Eq. (5.34)]. Explicit flat coordinates and prepotential can also be found in [6]. The resulting parent Hurwitz space is of dimension 518.

3.5.4. $\mathcal{R} = F_4$. The Dynkin diagram for the affine root system of type F_4 is shown in Figure 3.10. In this case the canonical node is the one corresponding to the fundamental weight ω_2.

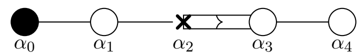

FIGURE 3.10. The affine Dynkin diagram of $\mathcal{R} = F_4$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

Here, $\omega = [0001]_{F_4}$ and ρ_ω will be the 26-dimensional fundamental representation, i.e. the irreducible representation of highest weight ω_4. The character relations are then given by

$$
\begin{aligned}
\mathfrak{p}_1^{[0001]_{F_4}} &= \chi_4, \\
\mathfrak{p}_2^{[0001]_{F_4}} &= \chi_1 + \chi_3, \\
\mathfrak{p}_3^{[0001]_{F_4}} &= \chi_2 + \chi_1\chi_4 - \chi_4, \\
\mathfrak{p}_4^{[0001]_{F_4}} &= \chi_1^2 + \chi_3\chi_1 - \chi_4^2 - \chi_2, \\
\mathfrak{p}_5^{[0001]_{F_4}} &= -\chi_4^3 + \chi_1^2\chi_4 - \chi_1\chi_4 - 2\chi_2\chi_4 + \chi_3\chi_4 + \chi_4 + \chi_3^2 - \chi_2 + \chi_3, \\
\mathfrak{p}_6^{[0001]_{F_4}} &= \chi_1^3 - \chi_1^2 - \chi_4^2\chi_1 - 3\chi_2\chi_1 + \chi_3\chi_4\chi_1 + \chi_4\chi_1 - \chi_1 - \chi_4^3 + \chi_4^2 - \chi_2 - \chi_2\chi_4 + \chi_3\chi_4 + \chi_4, \\
\mathfrak{p}_7^{[0001]_{F_4}} &= \chi_1\chi_4^3 - \chi_4^3 - \chi_1\chi_4^2 + \chi_2\chi_4^2 + 2\chi_4^2 - 2\chi_1\chi_4 - 2\chi_2\chi_4 - 3\chi_1\chi_3\chi_4 + 2\chi_3\chi_4 \\
&+ \chi_4 + \chi_1^2 - \chi_1 + \chi_1\chi_2 - \chi_2 + \chi_1^2\chi_3 - \chi_1\chi_3 - 2\chi_2\chi_3, \\
\mathfrak{p}_8^{[0001]_{F_4}} &= -\chi_1^3 + \chi_4^2\chi_1^2 - 2\chi_3\chi_1^2 + \chi_2\chi_1 - \chi_3\chi_1 + \chi_2\chi_4\chi_1 - \chi_3\chi_4\chi_1 - \chi_4\chi_1 \\
&+ \chi_3\chi_4^3 + \chi_4^3 - 2\chi_3^2 - \chi_4^2 - \chi_2 + 3\chi_2\chi_3 + \chi_3 - 3\chi_3^2\chi_4 + 2\chi_2\chi_4 - 2\chi_3\chi_4 + \chi_4, \\
\mathfrak{p}_9^{[0001]_{F_4}} &= \chi_4^5 - \chi_1\chi_4^3 - 4\chi_3\chi_4^3 - 2\chi_4^3 + 2\chi_1\chi_4^2 + 4\chi_2\chi_4^2 + \chi_1\chi_3\chi_4^2 + 2\chi_3^2\chi_4 - \chi_2 + \chi_1\chi_2\chi_4 + \chi_2^2 \\
&+ 2\chi_1\chi_3\chi_4 + 3\chi_3\chi_4 - 2\chi_1^2 + \chi_2\chi_4 - 2\chi_1\chi_3^2 - 2\chi_1\chi_2 + 2\chi_1\chi_4 - 2\chi_1^2\chi_3 - 2\chi_1\chi_3 - \chi_2\chi_3, \\
\mathfrak{p}_{10}^{[0001]_{F_4}} &= \chi_4^5 + \chi_1\chi_4^4 - \chi_1\chi_4^3 - 5\chi_3\chi_4^3 - 2\chi_4^3 - 2\chi_1^2\chi_4^2 + 3\chi_2\chi_4^2 - 3\chi_1\chi_3\chi_4^2 - \chi_3\chi_4^2 + 5\chi_3^2\chi_4 \\
&+ 3\chi_1\chi_4 + \chi_1\chi_2\chi_4 + \chi_2\chi_4 + 3\chi_1\chi_3\chi_4 + \chi_2\chi_3\chi_4 + 4\chi_3\chi_4 \\
&- \chi_4 + \chi_1^3 - \chi_2^2 + \chi_1\chi_3^2 + 3\chi_3^2 - \chi_1\chi_2 + \chi_2 + 2\chi_1^2\chi_3 + 3\chi_1\chi_3 - 3\chi_2\chi_3, \\
\mathfrak{p}_{11}^{[0001]_{F_4}} &= \chi_1\chi_4^4 + \chi_2\chi_4^3 - \chi_3\chi_4^3 + \chi_4^3 - \chi_1^2\chi_4^2 - 2\chi_1\chi_4^2 - 2\chi_2\chi_4^2 - 5\chi_1\chi_3\chi_4^2 - 2\chi_4^2 + 2\chi_3^2\chi_4 \\
&- \chi_1\chi_4\chi_2\chi_4 + \chi_1\chi_3\chi_4 - 3\chi_2\chi_3\chi_4 + \chi_3^3 + 3\chi_1^2 \\
&+ \chi_2^2 + 4\chi_1\chi_3^2 + 2\chi_3^2 + \chi_1 + 3\chi_1\chi_2 + 3\chi_1^2\chi_3 + 5\chi_1\chi_3 + 2\chi_2\chi_3 + \chi_3, \\
\mathfrak{p}_{12}^{[0001]_{F_4}} &= -\chi_4^5 - \chi_1\chi_4^4 + \chi_4^4 + 3\chi_1\chi_4^3 + 2\chi_3\chi_4^3 + 2\chi_1^2\chi_4^2 + \chi_3^2\chi_4^2 - \chi_1\chi_4^2 - \chi_2\chi_4^2 + \chi_1\chi_3\chi_4^2
\end{aligned}
$$

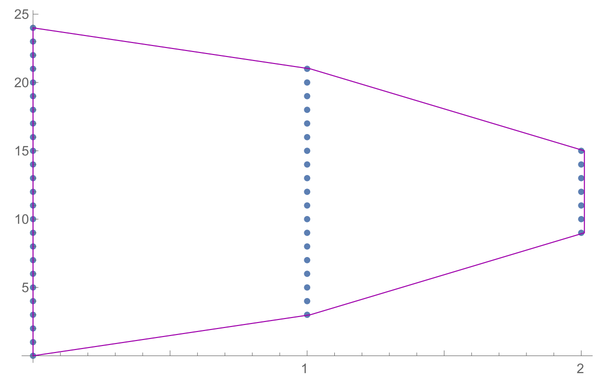

FIGURE 3.11. Newton polygon for the F_4-spectral curve.

$$
\begin{aligned}
&+ && \chi_3\chi_4^2 - 5\chi_1\chi_4 - 4\chi_2\chi_4 - 5\chi_1\chi_3\chi_4 - 3\chi_2\chi_3\chi_4 + \chi_3\chi_4 + \chi_4 - \chi_1^3 \\
&+ && 2\chi_2^2 - \chi_1\chi_3^2 + 3\chi_1\chi_2 - 3\chi_1\chi_3 - 2\chi_2\chi_3, \\
\mathfrak{p}_{13}^{[0001]_{F_4}} &= && -2\chi_4^5 - 2\chi_1\chi_4^4 + 2\chi_4^4 + 4\chi_1\chi_4^3 - 2\chi_2\chi_4^3 + 6\chi_3\chi_4^3 + 2\chi_4^3 + 4\chi_1^2\chi_4^2 + 2\chi_3^2\chi_4^2 \\
&- && 2\chi_1\chi_4^2 - 2\chi_2\chi_4^2 + 4\chi_1\chi_3\chi_4^2 - 2\chi_4^2 - 2\chi_1^2\chi_4 - 4\chi_3^2\chi_4 - 2\chi_1\chi_4 \\
&+ && 2\chi_1\chi_2\chi_4 - 4\chi_1\chi_3\chi_4 + 2\chi_2\chi_3\chi_4 - 4\chi_3\chi_4 - 2\chi_3^3 - 4\chi_1^2 \\
&- && 2\chi_2^2 - 4\chi_1\chi_3^2 - 4\chi_3^2 - 4\chi_1\chi_2 - 4\chi_1^2\chi_3 - 4\chi_1\chi_3 - 2\chi_2\chi_3 + 2,
\end{aligned}
\tag{3.59}
$$

with $\mathfrak{p}_{26-i}^{[0001]_{F_4}} = \mathfrak{p}_i^{[0001]_{F_4}}$. Note that the above relations in the character ring follow from those for $\mathcal{R} = E_6$ and $\rho = (\mathbf{27})$ or $\rho = (\overline{\mathbf{27}})$ by folding; in particular $\mathcal{M}^{\rm LG}_{F_4,[0001]_{F_4}}$ sits inside $\mathcal{M}^{\rm LG}_{[100000]_{E_6}}$ as the fixed locus of the involution $w_1 \leftrightarrow w_5$, $w_2 \leftrightarrow w_4$. The generic fibre $\overline{C_w^{(2)}}$ is a genus 4 hyperelliptic curve with ramification over ∞

$$
\left(\overbrace{3,6}^{\mu=0}, \overbrace{3,6}^{\mu=\infty}, \overbrace{3}^{\mu=\varepsilon_3}, \overbrace{3}^{\mu=\varepsilon_3^2} \right),
\tag{3.60}
$$

where ε_3 is a primitive third root of unity, and the associated Newton polygon is shown in Figure 3.11. The extended affine F_4-Frobenius manifold is thus realised as a 5-dimensional submanifold of the 36-dimensional Hurwitz space $\mathcal{H}_{g_\omega;\mathsf{n}_\omega}$, with $g_\omega = 4$ and $\mathsf{n}_\omega = (5, 5, 2, 2, 2, 2)$.

3.5.5. $\mathcal{R} = G_2$. The Dynkin diagram for the affine G_2 root system is given in Figure 3.12, and the canonical label is $\bar{k} = 2$.

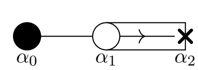

FIGURE 3.12. The affine Dynkin diagram of $\mathcal{R} = G_2$. The node corresponding to the affine root is marked in black, and the canonical marked node is indicated with a $\times$.

Here, $\rho_{\omega_1} = (\mathbf{7})$ is the 7-dimensional fundamental representation which is the irreducible representation with highest weight ω_1. We obtain the character relations

$$\mathfrak{p}_1^{[10]_{G_2}} = \chi_1, \quad \mathfrak{p}_2^{[10]_{G_2}} = \chi_1 + \chi_2, \quad \mathfrak{p}_3^{[10]_{G_2}} = \chi_1^2 - \chi_2, \tag{3.61}$$

and $\mathfrak{p}_{7-i}^{[10]_{G_2}} = \mathfrak{p}_i^{[10]_{G_2}}$, hence

$$\mathcal{P}_{[10]_{G_2}}(\lambda, \mu; w_0, w_1, w_2) \equiv \sum_{i=0}^{3} (-1)^i \mathfrak{p}_i^{[10]_{G_2}} \left(w_1, w_2 - \frac{\lambda}{w_0} \right) \mu^i (1 - \mu^{7-2i}),$$

and solving for λ

$$\lambda = \frac{w_0 \left(\mu^6 + (1-w_1)\mu^5 + (1+w_2)\mu^4 + (1 - w_1^2 + 2w_2)\mu^3 + (1+w_2)\mu^2 + (1-w_1)\mu + 1 \right)}{\mu^2(\mu+1)^2}. \tag{3.62}$$

As was the case for $\mathcal{R} = F_4$, the same superpotential could be obtained from the LG model of $\omega = [1000]_{D_4}$ by the order three folding of the D_4 Dynkin diagram, and $\mathcal{M}^{\rm LG}_{[10]_{G_2}}$ sits inside $\mathcal{M}^{\rm LG}_{[1000]_{D_4}}$ as the fixed locus of the triality action sending $(w_1, w_3, w_4) \to (w_{\epsilon(1)}, w_{\epsilon(3)}, w_{\epsilon(4)})$ with $\epsilon \in S_3$. The outcome is a family of rational functions in μ, with three poles at $\mu = 0, -1, \infty$, all of order two. This means that the resulting Frobenius manifold is a 3 dimensional sublocus in the 7 dimensional Hurwitz space $\mathcal{H}_{0;\mathsf{n}_\omega}$, with $\mathsf{n}_\omega = (1,1,1)$.

4. Mirror symmetry

Now that we have constructed the spectral curve for each Dynkin type, we move on to proving Theorem 3.1. For the classical Lie groups, a mirror symmetry statement is already present in [19], where LG-superpotentials are found by the way of a completely different path. In the exceptional cases the superpotentials are entirely new, and we give the flat coordinates and prepotentials arising from the spectral curves presented in the previous Section.

Lemma 4.1. *The following give flat coordinates for the metric η on $\mathcal{M}^{\rm LG}_\omega \subset \mathcal{H}^\phi_{g_\omega, \mathsf{n}_\omega}$ with $\phi = \mathrm{d}\log\mu$:*

$$\tau_{i;\alpha} = \operatorname*{Res}_{\infty_i} \kappa_i^{-\alpha} \log\mu \, d\lambda, \quad \alpha = 1, \cdots, n_i; \tag{4.1a}$$

$$\tau_j^{\rm ext} = \mathrm{p.v.} \int_{\infty_0}^{\infty_j} d\log\mu, \quad j = 1, \cdots, \ell(\mathsf{n}_\omega); \tag{4.1b}$$

$$\tau_k^{\rm res} = \operatorname*{Res}_{\infty_k} \lambda \, d\log\mu, \quad k = 1, \cdots, \ell(\mathsf{n}_\omega), \tag{4.1c}$$

where κ_i is defined near ∞_i by $\lambda(\kappa) = \kappa_i^{n_i+1} + \mathcal{O}(1)$, and p.v. *indicates subtraction of the divergent part in κ_i.*

For classical root systems this is a consequence of [15, Thm 5.1], where it is proved that these form part of a flat chart on a general Hurwitz space $\mathcal{H}^\phi_{g,\mathsf{n}}$, and the fact we proved in the previous Section that $\mathcal{M}^{\rm LG}_\omega$ for classical Lie algebras is the fixed locus inside a Hurwitz space $\mathcal{H}^\phi_{g_\omega,\mathsf{n}_\omega}$ of an involution, which is furthermore easily seen to act linearly in these coordinates. The general proof follows by a direct calculation for the exceptional root systems, as we shall show later, proving that $\mathcal{M}^{\rm LG}_\omega$ embeds as an affine subspace into $\mathcal{H}^{\mathrm{d}\log\mu}_{g_\omega,\mathsf{n}_\omega}$ in these cases as well. Note that the functions

(4.1a)–(4.1c) overparametrise $\mathcal{M}_\omega^{\mathrm{LG}}$ in general, and implicit in Lemma 4.1 is the statement that the vector space spanned by(4.1a)–(4.1c) always has dimension $\dim \mathcal{M}_\omega^{\mathrm{LG}} = l_{\mathcal{R}} + 1$, with some basis $\{t_0, \dots, t_{l_{\mathcal{R}}}\}$. In particular, it is immediate to see that we obtain a single coordinate from (4.1b) and any ∞_j;

$$t_{l_{\mathcal{R}}+1} := \frac{\log(w_0)}{d_{\bar{k}}} = q_j \tau_j^{\mathrm{ext}}, \quad q_j \in \mathbb{C} \tag{4.2}$$

Furthermore we obtain $l_{\mathcal{R}} - 1$ independent coordinates $\{t_1, \dots, t_{l_{\mathcal{R}}}\}$ from (4.1a),

$$\tau_{i;\alpha} = \sum_{j=1}^{l_{\mathcal{R}}} f_{i,\alpha}^{(j)} t_j, \quad f_{i,\alpha}^{(j)} \in \mathbb{C} \tag{4.3}$$

and a single independent coordinate from (4.1c):

$$t_{l_{\mathcal{R}}} = r_j \tau_k^{\mathrm{res}}, \quad r_j \in \mathbb{C}, \tag{4.4}$$

where $\partial_{l_{\mathcal{R}}} = e$. It is also straightforward to show from (4.1a)–(4.1c) that $t_i \in w_0^{\tilde{d}_i}\mathbb{Z}[w_1, \dots, w_{l_{\mathcal{R}}}]$ for some $\tilde{d}_i \in \mathbb{Q}$, and that the map $t(w)$ has a polynomial inverse $w_i \in \mathbb{Q}[t_1, \dots, t_{l_{\mathcal{R}}+1}, \mathrm{e}^{t_{l_{\mathcal{R}}+1}}]$.
We are now in position to prove Theorem 3.1.

Proof of Theorem 3.1. We identify the coordinates $\{t_1, \dots, t_{l_{\mathcal{R}}+1}\}$ constructed above with the flat coordinates of the metric η for the Dubrovin–Zhang Frobenius manifold $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ to define a complex manifold isomorphism from $\mathcal{M}_\omega^{\mathrm{LG}}$ to $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$. We shall now prove that this is in fact a Frobenius manifold isomorphism using the reconstruction theorem, Theorem 2.1.

Firstly, a direct calculation from Lemma 4.1 shows that

$$e = \frac{\partial_{w_k}}{w_0} = \frac{1}{w_0}\sum_{i=1}^{l_{\mathcal{R}+1}} \frac{\partial t_i}{\partial w_k}\partial_{t_i} = \partial_{t_{l_{\mathcal{R}}}} = \partial_{\tilde{y}_{\bar{k}}}, \tag{4.5}$$

$$E = w_0 \partial_{w_0} = w_0 \sum_{i=0}^{l_{\mathcal{R}}} \frac{\partial t_i}{\partial w_0}\partial_{t_i} = \frac{\partial_{t_{l_{\mathcal{R}}+1}}}{d_{\bar{k}}} + \sum_{i=1}^{l_{\mathcal{R}}} \frac{d_i t_i}{d_{\bar{k}}}\partial_{t_i}, \tag{4.6}$$

which matches the unit and the Euler vector fields of $\mathcal{M}_\omega^{\mathrm{LG}}$ with those of $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$.
We now turn to comparing the pairing γ from (3.15) with the Cartan–Killing pairing on $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$. Notice first that the argument of the residue in (3.15) has poles at $\partial_\lambda \mu = 0$, $\lambda, \mu = 0, \infty$, so by turning the contour around we can evaluate the sum of residues at the critical points by summing over residues at the poles and zeroes of λ and μ. For $j = 0$, only the poles of λ contribute, and we get

$$\begin{aligned} \gamma_{i0} &= \sum_{\lambda(p)=\infty} \mathrm{Res}_p \frac{\partial_{x_k}\lambda}{\mu^2 \partial_\mu \lambda} \mathrm{d}\mu = 0, \\ \gamma_{00} &= \sum_{\lambda(p)=\infty} \mathrm{Res}_p \frac{\lambda}{\mu^2 \partial_\mu \lambda} \mathrm{d}\mu = \sum_{\lambda(p)=\infty} \mathrm{ord}_p \lambda = c_\omega, \end{aligned} \tag{4.7}$$

for non-zero constants $c_\omega \in \mathbb{Q}$. For $i, j \neq 0$, only the zeroes of λ contribute. To see where these are located, we compute

$$\begin{aligned} \mathcal{P}^{\rm red}_\omega(0,\mu) &= \prod_{\omega' \in \Gamma(\rho_\omega)} (\mu - \mathrm{e}^{\omega'\cdot \mathsf{h}}), \\ &= \prod_{\omega' \in \Gamma(\rho_\omega)} (\mu - \mathrm{e}^{\sum_j \omega'_j x_j}), \end{aligned} \tag{4.8}$$

therefore λ has zeroes at $\mu = \mathrm{e}^{\sum_j \omega'_j x_j}$ for weights $\omega' \in \Gamma(\rho_\omega)$, and we get

$$\begin{aligned} & \mathrm{Res}_{\mu=\mathrm{e}^{\sum_l \omega'_l x_l}} \frac{\partial_{x_i}\mu\ \partial_{x_j}\mu}{\mu \partial_\lambda \mu} \frac{\mathrm{d}\lambda}{\lambda} = -\mathrm{Res}_{\mu=\mathrm{e}^{\sum_l \omega'_l x_l}} \frac{\partial_{x_i}\lambda \partial_{x_j}\lambda}{\lambda \partial_\mu \lambda} \frac{\mathrm{d}\mu}{\mu^2} \\ =& -\left.\frac{\partial_{x_i}\mathcal{P}^{\rm red}_\omega \partial_{x_j}\mathcal{P}^{\rm red}_\omega}{\mu^2(\partial_\mu \mathcal{P}^{\rm red}_\omega)^2}\right|_{\mu=\mathrm{e}^{\sum_l \omega'_l x_l}} \\ =& -\frac{\mathrm{e}^{2\sum_l \omega'_l x_l}\omega'_i\omega'_j \prod_{\omega'',\omega'''\neq\omega'}\left(\mathrm{e}^{\sum_l \omega'_l x_l} - \mathrm{e}^{\sum_l \omega''_l x_l}\right)\left(\mathrm{e}^{\sum_l \omega'_l x_l} - \mathrm{e}^{\sum_l \omega'''_l x_l}\right)}{\mathrm{e}^{2\sum_l \omega'_l x_l} \prod_{\omega'',\omega'''\neq\omega'}\left(\mathrm{e}^{\sum_l \omega'_l x_l} - \mathrm{e}^{\sum_l \omega''_l x_l}\right)\left(\mathrm{e}^{\sum_l \omega'_l x_l} - \mathrm{e}^{\sum_l \omega'''_l x_l}\right)} \\ =& -\omega'_i \omega'_j, \end{aligned} \tag{4.9}$$

where we have made use of the "thermodynamic identity" of [15, Lemma 4.6] to exchange $\mu \leftrightarrow \lambda$, the implicit function theorem, and (4.8). Summing over ω' gives

$$\gamma_{ij} = N_\omega \cdot (\mathcal{K}_\mathcal{R})_{ij}, \quad N_\omega \in \mathbb{Q}, \tag{4.10}$$

that is, up to normalisation, the Cartan–Killing pairing on $\mathfrak{h}_\mathcal{R} = \mathrm{span}_\mathbb{C}\{\partial_{x_1}, \dots, \partial_{x_{l_\mathcal{R}}}\}$. Together with (4.7) this shows that the two intersection forms coincide up to a linear change of variables.

The only missing piece to invoke the reconstruction theorem, Theorem 2.1, is to prove the first point in the list, namely the quasi-polynomiality of the prepotential of $\mathcal{M}^{\rm LG}_\omega$. This follows from an analysis of the residue formulas (3.14), as described in [6], in the w-chart for $\mathcal{M}^{\rm LG}_\omega$. Write $\eta = \sum_{ij} \widetilde{\eta}_{ij} \mathrm{d}w_i \mathrm{d}w_j$ and $c = \sum_{ijk} \widetilde{c}_{ijk} \mathrm{d}w_i \mathrm{d}w_j \mathrm{d}w_k$ for the components of the metric η and the c-tensor in this chart. As we did for the Gram matrix of the intersection pairing γ, we can try to compute these by turning the contour around and picking up residues of the argument of (3.14) in the complement of the region bounded by a closed loop encircling all critical points. These are located at the poles of λ, which can be easily read off from $\mathcal{P}^{\rm red}_\omega$, but also (for $g_\omega > 0$) at the ramification points of the λ-projection, $\mathrm{d}\mu = 0$, which are significantly harder to determine as functions of w. However, a simple count of the order of divergence shows that the latter all vanish when $i = 0$:

$$\tilde{c}_{0ij} = \sum_{\lambda(p)=\infty} \mathrm{Res}_p \frac{\partial_{w_0}\lambda \partial_{w_i}\lambda \partial_{w_j}\lambda}{\mu^2 \lambda'(\mu)} \mathrm{d}\mu, \tag{4.11}$$

and the remaining components can be computed using the WDVV equations in curved w-coordinates:

$$\sum_{kl} \left(\widetilde{c}_{ijk} \widetilde{\eta}^{kl} \widetilde{c}_{lmn} - \widetilde{c}_{imk} \widetilde{\eta}^{kl} \widetilde{c}_{ljn} \right) = 0. \tag{4.12}$$

Setting $n = 0$ and using (4.11) makes (4.12) into an inhomogeneous linear system of equations for the remaining unknown components of c, with coefficients being some polynomials in

$w_0, w_0^{-1}, w_1, w_2, \cdots, w_{l_\mathcal{R}}$. It can be verified directly that the resulting system has maximal rank and determines uniquely all components of c in terms of $\tilde{c}_{0ij}$. Thus it only remains to show that the structure constants $\tilde{c}_{0ij}$ are compatible with the quasi-polynomiality of the prepotential, and conclude by uniqueness. This follows by checking that the (heavily over-constrained) problem of finding $F_\mathcal{R} \in \mathbb{C}[t_1, \cdots, t_{l_{\mathcal{R}+1}}][\mathrm{e}^{t_{l_\mathcal{R}+1}}]$ satisfying $L_E F_\mathcal{R} = 2F_\mathcal{R}$, and such that the resulting c-tensor in w-coordinates returns (4.11), admits a unique solution up to quadratic terms, which can be verified by a direct analysis for all $\mathcal{R}$.

□

4.1. **Examples.** The construction of closed-form flat frames for η and prepotentials in all Dynkin types does not follow directly from the construction of the flat pencil in the original Dubrovin–Zhang construction[9]. One of the advantages of the mirror formulation in Theorem 3.1 is that these can now be easily computed using the Landau–Ginzburg formalism.

4.1.1. *Classical root systems.*

Example 4.1 ($\mathcal{R} = B_3$)**.** From (3.22) we have superpotential

$$\lambda_{B_3} = \frac{w_0(\mu^6 + 1 - (\mu^5 + \mu)(w_1 - 1) + (\mu^4 + \mu^2)(-w_1 + w_2 + 1) + \mu^3(-w_3^2 + 2w_2 + 2))}{\mu^2(\mu+1)^2}, \quad (4.13)$$

which gives flat coordinates

$$t_4 = \frac{\log w_0}{2}, \quad t_1 = w_0^{\frac{1}{2}}(w_1 + 1), \quad t_2 = w_0^{\frac{1}{2}} w_3, \quad w_0(w_1 + w_2 + 2). \quad (4.14)$$

We obtain the prepotential for B_3 to be

$$F_{B_3} = t_4 t_3^2 + \frac{1}{2} t_1^2 t_3 + t_2^2 t_3 - \frac{1}{48} t_1^4 - \frac{1}{24} t_2^4 + 2t_1 t_2^2 \mathrm{e}^{t_4} + t_1^2 e^{2t_4} + 2t_2^2 \mathrm{e}^{2t_4} + \frac{1}{2} \mathrm{e}^{4t_4}, \quad (4.15)$$

which is seen to be equivalent to the free energy in Example 2.7 in [20] by $F \mapsto \frac{F}{2}$ and $t_2 \leftrightarrow t_3$.

Example 4.2 ($\mathcal{R} = B_4$)**.** For B_4, (3.22) is

$$\begin{aligned} \lambda_{B_4} = \frac{w_0}{\mu^3(\mu+1)^2} \Big(& \mu^8 + 1 - (\mu^7 + \mu)(w_1 - 1) + (\mu^6 + \mu^2)(-w_1 + w_2 + 1) \\ & - (\mu^5 + \mu^3)(w_1 - w_2 + w_3 - 1) + \mu^4(w_4^2 - 2w_1 - 2w_3) \Big), \end{aligned} \quad (4.16)$$

which gives flat coordinates

$$t_5 = \frac{\log w_0}{3}, \quad t_1 = w_0^{\frac{1}{3}}(w_1 + 1), \quad t_2 = w_0^{\frac{1}{2}} w_4, \quad t_3 = w_0^{\frac{2}{3}}(4w_1 - w_1^2 + 6w_2 + 11),$$
$$t_4 = w_0(2w_1 + w_2 + w_3 + 2).$$

[9] A priori one could try to construct a quasi-homogeneous polynomial ansatz for the prepotential, impose the associativity of the Frobenius product, and solve for the coefficients. This typically results in a large system of non-linear equations, owing to the non-linearity of the WDVV equations, whose solution is already unviable e.g. for $\mathcal{R} = E_n$. The proof of Theorem 3.1 shows that the Landau–Ginzburg formulas reduce this to an explicit calculation of residues and a relatively small-rank, *linear* problem.

In this case, the resulting prepotential is given by

$$F_{B_4} = \frac{1}{1944}t_1^4t_3 - \frac{1}{9720}t_1^6 - \frac{1}{648}t_1^2t_3^2 + +\frac{1}{9}t_1t_3t_4 - \frac{1}{24}t_2^4 + t_2^2t_4 + \frac{1}{1944}t_3^3 + t_5t_4^2 + \frac{1}{3}t_1^2t_2^2\mathrm{e}^{t_5}$$
$$+ \frac{1}{3}t_2^2t_3\mathrm{e}^{t_5} + \frac{1}{36}t_1^4\mathrm{e}^{2t_5} + \frac{1}{18}t_1^2t_3\mathrm{e}^{2t_5} + 2t_1t_2^2\mathrm{e}^{2t_5} + \frac{1}{36}t_3^2\mathrm{e}^{2t_5} + 2t_2^2\mathrm{e}^{3t_5} + \frac{1}{2}t_1^2\mathrm{e}^{4t_5} + \frac{1}{3}\mathrm{e}^{6t_5}. \tag{4.18}$$

Example 4.3 ($\mathcal{R} = C_3$)**.** Here, the superpotential is given by

$$\lambda_{C_3} = \frac{w_0(\mu^6 + 1 - (\mu^5 + \mu)w_1 + (\mu^4 + \mu^2)(w_2 + 1) - \mu^3(w_1 + w_3)}{\mu^3}, \tag{4.19}$$

which leads to the following flat coordinates:

$$t_4 = \frac{\log w_0}{3}, \quad t_1 = w_0^{\frac{1}{3}}w_1, \quad t_2 = w_0^{\frac{2}{3}}(w_1^2 - 6(w_2 + 1)), \quad t_3 = w_0(w_1 + w_3). \tag{4.20}$$

In this case, the prepotential is

$$F_{C_3} = t_4t_3^2 - \frac{1}{9}t_1t_2t_3 - \frac{1}{9720}t_1^6 - \frac{1}{1944}t_1^4t_2 - \frac{1}{648}t_1^2t_2^2 - \frac{1}{1944}t_2^3 + \frac{1}{36}t_1^4\mathrm{e}^{2t_4} - \frac{1}{18}t_1^2t_2\mathrm{e}^{2t_4}$$
$$+ \frac{1}{36}t_2^2\mathrm{e}^{2t_4} + \frac{1}{2}t_1^2\mathrm{e}^{4t_4} + \frac{1}{3}\mathrm{e}^{6t_4}, \tag{4.21}$$

which is the same as the one found in Example 2.8 in [20] after letting $F \mapsto \frac{F}{2}$, and $t_2 \mapsto -6t_2$.

Example 4.4 ($\mathcal{R} = D_4$)**.** For $l = 4$, (3.30) becomes

$$\lambda_{D_4} = \frac{w_0(\mu^8 - w_1(\mu^7 + \mu) + w_2(\mu^6 + \mu^2) + (w_1 - w_3w_4)(\mu^5 + \mu^3) + 1)}{\mu^2(\mu^2 - 1)^2}, \tag{4.22}$$

which has poles at $\mu = 0, \infty, 1, -1$, all of order 2. The resulting flat coordinates are

$$t_5 = \frac{\log w_0}{2}, \quad t_1 = w_0^{\frac{1}{2}}w_1, \quad t_2 = w_0^{\frac{1}{2}}(w_3 + w_4), \quad t_3 = w_0^{\frac{1}{2}}(w_3 - w_4), \quad t_4 = w_0(w_2 + 2), \tag{4.23}$$

which leads the prepotential

$$F_{D_4} = t_5t_4^2 + \frac{1}{4}t_2^2t_4 + \frac{1}{4}t_3^2t_4 - \frac{1}{48}t_1^4 + \frac{1}{2}t_1^2t_4 - \frac{1}{384}t_2^4 - \frac{1}{64}t_2^2t_3^2 - \frac{1}{384}t_3^4 + \frac{1}{2}t_1t_2^2\mathrm{e}^{t_5} - \frac{1}{2}t_1t_3^2\mathrm{e}^{t_5} + t_1^2\mathrm{e}^{2t_5}$$
$$+ \frac{1}{2}t_2^2\mathrm{e}^{2t_5} + \frac{1}{2}t_3^2\mathrm{e}^{t_5} + \frac{1}{4}\mathrm{e}^{4t_5}. \tag{4.24}$$

Example 4.5 ($\mathcal{R} = D_5$)**.** For $l = 5$, (3.30) is given by

$$\lambda_{D_5} = \frac{w_0}{\mu^3(\mu^2 - 1)^2}\Big(\mu^{10} - w_1(\mu^9 + \mu) + w_2(\mu^8 + \mu^2) - w_3(\mu^7 + \mu^3) + (w_2 - w_4w_5 + 1)(\mu^6 + \mu^4)$$
$$- (w_4^2 + w_5^2 - 2w_1 - 2w_3)\mu^5 + 1\Big), \tag{4.25}$$

which has poles at $\mu = 0, \infty, 1, -1$, of orders $3, 3, 2, 2$, respectively. We can compute the flat coordinates

$$t_6 = \frac{\log w_0}{3}, \quad t_1 = w_0^{\frac{1}{3}}w_1, \quad t_2 = w_0^{\frac{1}{2}}(w_4 - w_5), \quad t_3 = w_0^{\frac{1}{2}}(w_4 + w_5),$$
$$t_4 = w_0^{\frac{2}{3}}(w_1^2 - 6(w_2 + 2)), \quad t_5 = w_0(2w_1 + w_3), \tag{4.26}$$

which give the prepotential

$$\begin{aligned}F_{D_5} =& t_6t_5^2 + \frac{1}{4}t_2^2t_5 + \frac{1}{4}t_3^2t_5 - \frac{1}{9720}t_1^6 - \frac{1}{1944}t_1^4t_4 + \frac{1}{648}t_1^2t_4^2 - \frac{1}{9}t_1t_4t_5 - \frac{1}{384}t_2^4 - \frac{1}{64}t_2^2t_3^2 - \frac{1}{384}t_3^4\\ &- \frac{1}{1944}t_4^3 - \frac{1}{12}t_1^2t_2^2e^{t_6} + \frac{1}{12}t_1^2t_3^2e^{t_6} + \frac{1}{12}t_2^2t_4e^{t_6} - \frac{1}{12}t_3^2t_4e^{t_6} + \frac{1}{36}t_1^4e^{2t_6} - \frac{1}{18}t_1^2t_4e^{2t_6}\\ &+ \frac{1}{2}t_1t_2^2e^{2t_6} + \frac{1}{2}t_1t_3^2e^{2t_6} + \frac{1}{36}t_4^2e^{2t_6} - \frac{1}{2}t_2^2e^{3t_6} + \frac{1}{2}t_3^2e^{3t_6} + \frac{1}{2}t_1^2e^{4t_6} + \frac{1}{3}e^{6t_6}.\end{aligned} \tag{4.27}$$

4.1.2. *Exceptional root systems.*

Example 4.6 ($\mathcal{R} = E_6$)**.** For the exceptional cases, we find expressions for λ near any ramification point by Puiseux expansions. By doing so, and using Lemma 4.1, we obtain the following flat coordinates for $\mathcal{R} = E_6$

$$\begin{aligned}t_7 = \frac{\log(w_0)}{6},\quad t_1 = w_0^{\frac{1}{3}}w_1,\quad t_2 = w_0^{\frac{1}{3}}w_5,\quad t_3 = w_0^{\frac{1}{2}}(w_6+2),\quad t_4 = w_0^{\frac{2}{3}}(-w_5^2 + 12w_1 + 6w_4),\\ t_5 = w_0^{\frac{2}{3}}(w_1^2 - 6w_2 - 12w_5),\quad t_6 = w_0(2w_1w_5 + w_3 + 3w_6 + 3).\end{aligned} \tag{4.28}$$

In this case, we find that the prepotential is given by

$$\begin{aligned}F_{E_6} =& -\frac{1}{19440}t_1^6 + \frac{1}{72}e^{2t_7}t_2t_1^4 - \frac{1}{3888}t_5t_1^4 + \frac{1}{6}e^{6t_7}t_1^3 + \frac{1}{6}e^{3t_7}t_3t_1^3 + \frac{5}{18}e^{4t_7}t_2^2t_1^2 - \frac{1}{1296}t_5^2t_1^2\\ &+ \frac{1}{36}e^{t_7}t_2^2t_3t_1^2 + \frac{1}{36}e^{4t_7}t_4t_1^2 + \frac{1}{36}e^{t_7}t_3t_4t_1^2 - \frac{1}{36}e^{2t_7}t_2t_5t_1^2 + \frac{1}{72}e^{2t_7}t_2^4t_1 + \frac{1}{2}e^{2t_7}t_2t_3^2t_1\\ &+ \frac{1}{72}e^{2t_7}t_4^2t_1 + \frac{1}{2}e^{8t_7}t_2t_1 + e^{5t_7}t_2t_3t_1 + \frac{1}{36}e^{2t_7}t_2^2t_4t_1 - \frac{1}{6}e^{3t_7}t_3t_5t_1 - \frac{1}{18}t_5t_6t_1 + \frac{1}{12}e^{12t_7}\\ &- \frac{1}{19440}t_2^6 - \frac{1}{96}t_3^4 + \frac{1}{6}e^{6t_7}t_2^3 + \frac{1}{3}e^{3t_7}t_3^3 + \frac{1}{3888}t_4^3 - \frac{1}{3888}t_5^3 + \frac{1}{2}e^{6t_7}t_3^2 - \frac{1}{1296}t_2^2t_4^2\\ &+ \frac{1}{72}e^{2t_7}t_2t_5^2 + \frac{1}{2}t_7t_6^2 + \frac{1}{6}e^{3t_7}t_2^3t_3 + \frac{1}{3888}t_2^4t_4 + \frac{1}{6}e^{3t_7}t_2t_3t_4 - \frac{1}{36}e^{4t_7}t_2^2t_5 - \frac{1}{36}e^{t_7}t_2^2t_3t_5\\ &- \frac{1}{36}e^{4t_7}t_4t_5 - \frac{1}{36}e^{t_7}t_3t_4t_5 + \frac{1}{4}t_3^2t_6 + \frac{1}{18}t_2t_4t_6.\end{aligned} \tag{4.29}$$

Example 4.7 ($\mathcal{R} = E_7$)**.** Similar to the E_6-case, by taking Puiseux expansions of the spectral curve, we obtain the following flat coordinates

$$\begin{aligned}t_8 = \frac{\log(w_0)}{12},\quad t_1 = w_0^{\frac{1}{4}}w_6,\quad t_2 = w_0^{\frac{1}{3}}(w_1+2),\quad t_3 = w_0^{\frac{1}{2}}(2w_6+w_7),\\ t_4 = w_0^{\frac{1}{2}}(-w_6^2 + 8w_1 + 4w_5 + 12),\quad t_5 = w_0^{\frac{2}{3}}(-w_1^2 + 26w_1 + 6w_2 + 12w_5 + 26),\\ t_6 = w_0^{\frac{3}{4}}(5w_6^3 + 24(6w_1 - w_5 + 21)w_6 + 96w_4 + 288w_7),\\ t_7 = w_0(3w_1^2 + 2w_1(w_5+8) + 3w_6^2 + 3w_6w_7 + 2w_2 + w_3 + 7w_5 + 14).\end{aligned} \tag{4.30}$$

The prepotential for E_7 takes the form

$$
\begin{aligned}
F_{E_7} = & -\frac{1}{4128768}t_1^8 + \frac{1}{18432}49\mathrm{e}^{6t_8}t_1^6 + \frac{1}{18432}\mathrm{e}^{2t_8}t_2t_1^6 + \frac{1}{294912}t_4t_1^6 + \frac{1}{384}\mathrm{e}^{3t_8}t_3t_1^5 - \frac{1}{2949120}t_6t_1^5 \\
& + \frac{19}{192}\mathrm{e}^{12t_8}t_1^4 + \frac{5}{288}\mathrm{e}^{4t_8}t_2^2t_1^4 - \frac{1}{49152}t_4^2t_1^4 + \frac{1}{8}\mathrm{e}^{8t_8}t_2t_1^4 + \frac{1}{1536}13\mathrm{e}^{6t_8}t_4t_1^4 + \frac{1}{1536}\mathrm{e}^{2t_8}t_2t_4t_1^4 \\
& + \frac{1}{576}\mathrm{e}^{4t_8}t_5t_1^4 + \frac{1}{4}\mathrm{e}^{9t_8}t_3t_1^3 + \frac{1}{576}\mathrm{e}^{t_8}t_2^2t_3t_1^3 + \frac{25}{96}\mathrm{e}^{5t_8}t_2t_3t_1^3 + \frac{7}{384}\mathrm{e}^{3t_8}t_3t_4t_1^3 + \frac{1}{576}\mathrm{e}^{t_8}t_3t_5t_1^3 \\
& + \frac{1}{9216}\mathrm{e}^{6t_8}t_6t_1^3 + \frac{1}{9216}\mathrm{e}^{2t_8}t_2t_6t_1^3 + \frac{1}{147456}t_4t_6t_1^3 + \frac{1}{6}\mathrm{e}^{18t_8}t_1^2 + \frac{1}{288}\mathrm{e}^{2t_8}t_2^4t_1^2 + \frac{5}{24}\mathrm{e}^{6t_8}t_2^3t_1^2 \\
& + \frac{1}{24576}t_4^3t_1^2 + \frac{5}{8}\mathrm{e}^{10t_8}t_2^2t_1^2 + \frac{5}{8}\mathrm{e}^{6t_8}t_3^2t_1^2 + \frac{1}{8}\mathrm{e}^{2t_8}t_2t_3^2t_1^2 + \frac{5}{512}\mathrm{e}^{6t_8}t_4^2t_1^2 + \frac{1}{512}\mathrm{e}^{2t_8}t_2t_4^2t_1^2 \\
& + \frac{1}{288}\mathrm{e}^{2t_8}t_5^2t_1^2 - \frac{1}{589824}t_6^2t_1^2 + \frac{1}{2}\mathrm{e}^{14t_8}t_2t_1^2 + \frac{1}{32}\mathrm{e}^{12t_8}t_4t_1^2 + \frac{1}{24}\mathrm{e}^{4t_8}t_2^2t_4t_1^2 + \frac{1}{8}\mathrm{e}^{8t_8}t_2t_4t_1^2 \\
& + \frac{1}{144}\mathrm{e}^{2t_8}t_2^2t_5t_1^2 + \frac{1}{24}\mathrm{e}^{6t_8}t_2t_5t_1^2 + \frac{1}{96}\mathrm{e}^{4t_8}t_4t_5t_1^2 + \frac{1}{384}\mathrm{e}^{3t_8}t_3t_6t_1^2 + \frac{1}{3}\mathrm{e}^{3t_8}t_3^3t_1 + \frac{1}{64}\mathrm{e}^{3t_8}t_3t_4^2t_1 \\
& + \frac{1}{6}\mathrm{e}^{3t_8}t_2^3t_3t_1 + \frac{7}{6}\mathrm{e}^{7t_8}t_2^2t_3t_1 + \mathrm{e}^{11t_8}t_2t_3t_1 + \frac{1}{4}\mathrm{e}^{9t_8}t_3t_4t_1 + \frac{1}{96}\mathrm{e}^{t_8}t_2^2t_3t_4t_1 + \frac{5}{16}\mathrm{e}^{5t_8}t_2t_3t_4t_1 \\
& + \frac{1}{6}\mathrm{e}^{7t_8}t_3t_5t_1 + \frac{1}{6}\mathrm{e}^{3t_8}t_2t_3t_5t_1 + \frac{1}{96}\mathrm{e}^{t_8}t_3t_4t_5t_1 + \frac{1}{576}\mathrm{e}^{4t_8}t_2^2t_6t_1 - \frac{1}{49152}t_4^2t_6t_1 + \frac{1}{1536}\mathrm{e}^{6t_8}t_4t_6t_1 \\
& + \frac{1}{1536}\mathrm{e}^{2t_8}t_2t_4t_6t_1 + \frac{1}{576}\mathrm{e}^{4t_8}t_5t_6t_1 + \frac{1}{384}t_6t_7t_1 + \frac{1}{24}\mathrm{e}^{24t_8} - \frac{1}{19440}t_2^6 + \frac{1}{72}\mathrm{e}^{4t_8}t_2^5 + \frac{5}{36}\mathrm{e}^{8t_8}t_2^4 \\
& - \frac{1}{96}t_3^4 - \frac{1}{49152}t_4^4 + \frac{1}{6}\mathrm{e}^{12t_8}t_2^3 + \frac{1}{384}\mathrm{e}^{6t_8}t_4^3 + \frac{1}{3888}t_5^3 + \frac{1}{4}\mathrm{e}^{16t_8}t_2^2 + \frac{1}{2}\mathrm{e}^{12t_8}t_3^2 + \frac{2}{3}\mathrm{e}^{4t_8}t_2^2t_3^2 \\
& + \mathrm{e}^{8t_8}t_2t_3^2 + \frac{1}{64}\mathrm{e}^{12t_8}t_4^2 + \frac{1}{64}\mathrm{e}^{4t_8}t_2^2t_4^2 + \frac{1}{72}\mathrm{e}^{8t_8}t_5^2 - \frac{1}{1296}t_2^2t_5^2 + \frac{1}{72}\mathrm{e}^{4t_8}t_2t_5^2 + \frac{1}{288}\mathrm{e}^{2t_8}t_4t_5^2 \\
& + \frac{1}{18432}\mathrm{e}^{6t_8}t_6^2 + \frac{1}{18432}\mathrm{e}^{2t_8}t_2t_6^2 + \frac{1}{294912}t_4t_6^2 + \frac{1}{2}t_8t_7^2 + \frac{1}{288}\mathrm{e}^{2t_8}t_2^4t_4 + \frac{1}{24}\mathrm{e}^{6t_8}t_2^3t_4 + \\
& \frac{1}{8}\mathrm{e}^{10t_8}t_2^2t_4 + \frac{1}{8}\mathrm{e}^{6t_8}t_3^2t_4 + \frac{1}{8}\mathrm{e}^{2t_8}t_2t_3^2t_4 + \frac{1}{3888}t_2^4t_5 + \frac{1}{36}\mathrm{e}^{4t_8}t_2^3t_5 + \frac{1}{36}\mathrm{e}^{8t_8}t_2^2t_5 + \frac{1}{6}\mathrm{e}^{4t_8}t_3^2t_5 \\
& + \frac{1}{144}\mathrm{e}^{2t_8}t_2^2t_4t_5 + \frac{1}{24}\mathrm{e}^{6t_8}t_2t_4t_5 + \frac{1}{576}\mathrm{e}^{t_8}t_2^2t_3t_6 + \frac{1}{96}\mathrm{e}^{5t_8}t_2t_3t_6 + \frac{1}{384}\mathrm{e}^{3t_8}t_3t_4t_6 + \frac{1}{576}\mathrm{e}^{t_8}t_3t_5t_6 \\
& + \frac{1}{4}t_3^2t_7 + \frac{1}{128}t_4^2t_7 + \frac{1}{18}t_2t_5t_7.
\end{aligned} \tag{4.31}
$$

Example 4.8 ($\mathcal{R} = F_4$)**.** In this case we obtain flat coordinates

$$
\begin{aligned}
& t_5 = \frac{1}{6}\log(w_0), \quad t_1 = w_0^{\frac{1}{3}}(w_4+1), \quad t_2 = w_0^{\frac{1}{2}}(w_4+w_1+2), \\
& t_3 = w_0^{\frac{2}{3}}(-w_4^2+16w_4+6w_3+6w_1+11), \quad t_4 = w_0(2w_4^2+6w_4+w_4w_1+w_3+w_2+4w_1+5).
\end{aligned} \tag{4.32}
$$

The resulting prepotential is

$$\begin{aligned}F_{F_4} =& -\frac{1}{9720}t_1^6 + \frac{1}{36}e^{2t_5}t_1^5 + \frac{5}{18}e^{4t_5}t_1^4 + \frac{1}{36}e^{t_5}t_2t_1^4 + \frac{1}{432}t_3t_1^4 + \frac{1}{3}e^{6t_5}t_1^3 + \frac{1}{3}e^{3t_5}t_2t_1^3 \\ &+ \frac{1}{4}e^{2t_5}t_3t_1^3 + \frac{1}{2}e^{8t_5}t_1^2 + \frac{1}{2}e^{2t_5}t_2^2t_1^2 - \frac{1}{32}t_3^2t_1^2 + e^{5t_5}t_2t_1^2 + \frac{1}{4}e^{4t_5}t_3t_1^2 + \frac{1}{4}e^{t_5}t_2t_3t_1^2 \\ &+ \frac{9}{16}e^{2t_5}t_3^2t_1 + \frac{3}{2}e^{3t_5}t_2t_3t_1 + \frac{1}{2}t_3t_4t_1 + \frac{1}{12}e^{12t_5} - \frac{1}{96}t_2^4 + \frac{1}{3}e^{3t_5}t_2^3 + \frac{3t_3^3}{64} + \frac{1}{2}e^{6t_5}t_2^2 \\ &+ \frac{9}{16}e^{4t_5}t_3^2 + \frac{9}{16}e^{t_5}t_2t_3^2 + \frac{1}{2}t_5t_4^2 + \frac{1}{4}t_2^2t_4.\end{aligned} \tag{4.33}$$

Example 4.9 ($\mathcal{R} = G_2$)**.** For G_2, we obtain flat coordinates

$$t_3 = \frac{\log(w_0)}{6}, \quad t_1 = w_0^{\frac{1}{2}}(w_1+1), \quad t_2 = w_0(2w_1+w_2+2). \tag{4.34}$$

In this case, the prepotential takes the form

$$F_{G_2} = \frac{1}{2}t_2^2t_3 + \frac{1}{4}t_2t_1^2 - \frac{1}{96}t_1^4 + \frac{1}{3}t_1^3e^{3t_3} + \frac{1}{2}t_1^2e^{6t_3} + \frac{1}{12}e^{12t_3}, \tag{4.35}$$

which matches exactly the expression found in [20] Example 2.4.

4.2. **Non-minimal irreducible representations.** It is argued in [6], based on the isomorphism of Toda flows on Prym–Tyurin varieties associated to different representations, [35, 36], that the Frobenius manifold obtained from the construction of Section 3.2 is independent of the choice of highest weight ω.

Let us verify this explicitly for $\mathcal{R} = G_2$. In this case picking $\omega = \omega_2$ gives the second smallest-dimensional non-trivial irreducible representation of G_2, which is the 14-dimensional adjoint representation $\rho_{\omega_2} = \mathfrak{g}_2$. By the same method of the previous section we obtain

$$\begin{aligned}\mathfrak{p}_0^{[01]_{G_2}} &= 1, \\ \mathfrak{p}_1^{[01]_{G_2}} &= \chi_2, \\ \mathfrak{p}_2^{[01]_{G_2}} &= \chi_1^3 - \chi_1^2 - \chi_1(2\chi_2+1), \\ \mathfrak{p}_3^{[01]_{G_2}} &= \chi_1^4 - \chi_1^3 - \chi_1^2(3\chi_2+1) + \chi_1 + 2\chi_2^2 + \chi_2, \\ \mathfrak{p}_4^{[01]_{G_2}} &= \chi_1^3(\chi_2-1) - \chi_1^2(\chi_2-1) - \chi_1(2\chi_2^2-\chi_2-1) - \chi_2^2 + \chi_2, \\ \mathfrak{p}_5^{[01]_{G_2}} &= \chi_1^5 - 2\chi_1^4 - 5\chi_1^3\chi_2 + \chi_1^2(3\chi_2+2) + \chi_1(6\chi_2^2+5\chi_2-1) + \chi_2^3 + 2\chi_2^2, \\ \mathfrak{p}_6^{[01]_{G_2}} &= \chi_1^4 - 3\chi_1^3\chi_2 + \chi_1^2(\chi_2^2-\chi_2-2) + \chi_1\chi_2(4\chi_2+3) + 2\chi_2^2 + \chi_2, \\ \mathfrak{p}_7^{[01]_{G_2}} &= +4\chi_1^4 + 2\chi_1^3(3\chi_2+1) + 2\chi_1^2(\chi_2^2-2\chi_2-3) \\ &- 2\chi_1\chi_2(4\chi_2+3) - 2\chi_2^3 - 6\chi_2^2 + 2 - 2\chi_1^5,\end{aligned} \tag{4.36}$$

and $\mathfrak{p}_i^{[01]_{G_2}} = \mathfrak{p}_{14-i}^{[01]_{G_2}}$ by reality of the adjoint representation. The characteristic polynomial (3.9) then factorises as

$$\mathcal{P}_{[01]_{G_2}}^{\mathrm{red}} = (\mu-1)^2\mathcal{P}_{G_2,\mathrm{short}}\mathcal{P}_{G_2,\mathrm{long}}, \tag{4.37}$$

with the three factors corresponding to the three irreducible Weyl orbits of the adjoint representation associated to the zero, short, and long roots of G_2:

$$\begin{aligned}
\mathcal{P}_{G_2,\mathrm{short}} &= \mathcal{P}^{\mathrm{red}}_{[10]_{G_2}}, \\
\mathcal{P}_{G_2,\mathrm{long}} &= \mu^6+\mu^5\left(w_1-w_2+1\right)+\mu^4\left(w_1^3-3w_2w_1-w_1-2w_2+1\right) \\
&+ \mu^3\left(2w_1^3-w_1^2-4w_2w_1-2w_1-w_2^2-4w_2+1\right) \\
&+ \mu^2\left(w_1^3-3w_2w_1-w_1-2w_2+1\right)+\mu\left(w_1-w_2+1\right)+1.
\end{aligned} \tag{4.38}$$

For any w, the curve $\overline{C}_w^{(1)} = \overline{\{\mathcal{P}_{G_2,\mathrm{long}}=0\}}$ is a $\mathbb{P}^1$, and the λ-projection has ramification profile

$$(\overbrace{1,2}^{\mu=0},\overbrace{1,2}^{\mu=\infty}). \tag{4.39}$$

The embedding $\iota_{[01]_{G_2}}:\mathcal{M}^{\mathrm{LG}}_{[01]_{G_2}}\hookrightarrow\mathcal{H}_{0;(0,1,0,1)}$ gives the same flat coordinates (4.34) as for the case $\omega=[10]_{G_2}$, and up to scaling the prepotential coincides with the prepotential (4.35), as expected.

4.3. **Non-canonical Dynkin marking.** In [6], it was proposed that the family of Frobenius algebras obtained in (3.12) through the shift of $w_j\to w_j+\delta_{ij}\frac{\lambda}{w_0}$ for any i should give the Frobenius structure corresponding to Dynkin node α_i. Theorem 3.1 shows that this is the case for $i=\bar{k}$, and this proposal is consistent with the analysis of the generalised type-A mirrors of [20]. However we now show that the conjecture is false in this form at the stated level of generality. Considering the case $\mathcal{R}=G_2$, we see that shifting w_1 instead of w_2 in (3.61) yields

$$\begin{aligned}
\mathcal{P}_{[10]_{G_2},i=1} = \left(\frac{\lambda}{w_0}\right)^2\mu^3+\frac{\lambda}{w_0}(-\mu^5-2w_1\mu^3-\mu-2)+\mu^6+(\mu^5+\mu)(1-w_1) \\
+(\mu^4+\mu^2)(w_2+1)-\mu^3(w_1^2-2w_2-1)+1.
\end{aligned} \tag{4.40}$$

By computing the metric $\eta=\sum_{ij}\tilde{\eta}_{ij}\mathrm{d}w^i\mathrm{d}w^j$ we get

$$\tilde{\eta}_{ij}=\begin{pmatrix}\frac{8w_1+1}{4w_0} & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{4.41}$$

which is clearly a singular matrix. Hence, (4.40) cannot define a Frobenius manifold structure, and the conjecture fails in this generality.

Remark 4.2. Opening for allowed changes in the primary differential, i.e. scalings and translations of $\mathrm{d}\log\mu$ which are type I (Legendre) transformations [15], under which the intersection form is invariant, does not resolve the nondegeneracy, and neither does attempting to change the pole structure manually akin to that of (3.32). Thus, the problem of finding Frobenius manifolds associated to non-canonical nodes for exceptional groups is still open, and indeed even the existence of non-canonical Frobenius manifold structures for DZ-manifolds of exceptional types is, at present, unknown.

5. Applications

5.1. **Topological degrees of Lyashko–Looijenga maps.** The semi-simple locus of a generically semi-simple, n-dimensional Frobenius manifold $\mathcal{M}$ is topologically a covering of finite multiplicity over a quotient by S_n of the complement of their discriminant, with covering map

$$\begin{aligned} \mathrm{LL}: \mathcal{M} &\longrightarrow (\mathbb{C}^n \setminus \mathrm{Discr}_{\mathcal{M}})/S_n \\ t &\longrightarrow \{e_1(u(t)), \dots, e_n(u(t))\} \end{aligned} \tag{5.1}$$

assigning to $t \in \mathcal{M}$ the unorderet set of its canonical coordinates in the form of their elementary symmetric polynomials $e_i(u_1, \dots, u_n)$. When $\mathcal{M}$ has a Landau–Ginzburg description as a holomorphic family of meromorphic functions, the application LL is the classical Lyashko–Looijenga (LL) mapping, sending a meromorphic function to the unordered set of its critical values.

As anticipated in Section 1.1.3, a direct corollary of Theorem 3.1 is the computation of the degree of the LL map of $\mathcal{M}_\omega^{\mathrm{LG}} \simeq \mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$. The calculation of the LL-degree can be tackled combinatorially through the enumeration of certain embedded graphs [29, Sec. 1.3.2], which unfortunately proves to be intractable for a general stratum of a Hurwitz space of arbitrary genus and ramification profile. In the case of $\deg \mathrm{LL}(\mathcal{M}_\omega^{\mathrm{LG}})$, however, its realisation as a conformal Frobenius manifold allows to bypass the problem altogether by the use of the quasi-homogeneous Bézout theorem. To this aim, note that

$$\det(z - (E(t)\cdot)) = \prod_{i=1}^{l_{\mathcal{R}}+1} (z - u_i(t)) = \sum_{j=0}^{l_{\mathcal{R}}+1} (-1)^j e_j(u_1(t), \dots, u_{l_{\mathcal{R}}+1}(t)) z^{l_{\mathcal{R}}+1-j}. \tag{5.2}$$

Setting $\mathcal{Q} := \mathrm{e}^{t_{l_{\mathcal{R}}+1}}$, it follows from Theorems 2.1 and 3.1 that the LL-map (5.1) is polynomial in $(t_1, \dots, t_{l_{\mathcal{R}}+1}, \mathcal{Q})$ since both the product and the Euler vector field are in (5.2). Moreover, it follows from the definition (3.4) of the quantum product in canonical coordinates that the canonical idempotents ∂_{u_i} have weight -1 under the Euler scaling, meaning that $e_i(u)$ is quasi-homogeneous of degree i. We can then avail ourselves of the graded generalisation of Bézout's theorem (see e.g. [29, Thm 3.3]).

Theorem 5.1. *Let $F : \mathbb{A}^n \to \mathbb{A}^n$ be a finite morphism induced by a quasi-homogeneous map $F_* : \mathbb{C}[y_1, \dots y_n] \to \mathbb{C}[x_1, \dots, x_n]$ with degrees p_i (resp. q_i) for y_i (resp. x_i) for $i = 1, \dots, n$. Then*

$$\deg F = \prod_{i=1}^{n} \frac{p_i}{q_i}. \tag{5.3}$$

An immediate consequence of Theorems 2.1, 3.1 and 5.1 is the following

Corollary 5.2. *The degree of the LL-map of the Hurwitz stratum $\mathcal{M}_\omega^{\mathrm{LG}}$ is*

$$\frac{(l_{\mathcal{R}}+1)!(\omega_{\bar{k}}, \omega_{\bar{k}})^{l_{\mathcal{R}}}}{\prod_{j=1}^{l_{\mathcal{R}}} (\omega_j, \omega_{\bar{k}})}. \tag{5.4}$$

$\mathcal{R}$	g_ω	n_ω	$d_{g_\omega;\mathsf{n}_\omega}$	$\iota_\omega(\mathcal{M}_{\mathcal{R}}^{\rm DZ})$	deg(LL)
A_l	0	$(\bar{k}-1, l-\bar{k})$	$l+1$	$\mathcal{H}_{g_\omega,\mathsf{n}_\omega}$	$\dfrac{(l-\bar{k})!(l+1)^l}{(l-\bar{k}+1)^{\bar{k}}\bar{k}^{l-\bar{k}}\prod_{j=\bar{k}+1}^{l}(l-j+1)}$
B_l	0	$(l-2, l-2, 1)$	$2l+1$	$\left(\mathcal{H}_{g_\omega,\mathsf{n}_\omega}\right)^{\mu_2}$	$2(l+1)l(l-1)^{l-1}$
C_l	0	$(l-1, l-1)$	$2l$	$\left(\mathcal{H}_{g_\omega,\mathsf{n}_\omega}\right)^{\mu_2}$	$(l+1)l^l$
D_l	0	$(l-3, l-3, 1, 1)$	$2l+2$	$\left(\mathcal{H}_{g_\omega,\mathsf{n}_\omega}\right)^{\mu_2}$	$4(l+1)l(l-1)(l-2)^{l-2}$
E_6	5	$(5, 5, 2, 2, 2, 2, 2)$	42	(A.1)	$2^3 \cdot 3^6 \cdot 5 \cdot 7$
E_7	33	$(11, 5, 3, 11, 5, 3, 1, 1, 3, 3)$	130	(A.2)	$2^{12} \cdot 3^3 \cdot 5 \cdot 7$
E_8	128	$(29, 29, 14, 14, 14, 14, 14, 14, 9, 9, 9, 9, 5, 5, 4, 4, 4, 4, 4, 4, 2, 2, 0, 0)$	518	[6, 7]	$2^4 \cdot 3^5 \cdot 5^5 \cdot 7$
F_4	4	$(5, 5, 2, 2, 2, 2)$	36	(3.59); $\left(\mathcal{M}^{\rm LG}_{[100000]_{E_6}}\right)^{\mu_2}$	$2^3 \cdot 3^3 \cdot 5$
G_2	0	$(1, 1, 1)$	7	(3.61); $\left(\mathcal{M}^{\rm LG}_{[1000]_{D_4}}\right)^{S_3}$	12

TABLE 3. Lyashko–Looijenga degrees for all Dynkin types.

We collect in Table 3 the calculation of the degrees for the minimal choices of weight ω. Our expectation that $\mathcal{M}_\omega^{\rm LG} \simeq \mathcal{M}_{\mathcal{R}}^{\rm DZ}$ for any of the infinitely many choices of dominant weight $\omega \in \Lambda_w^+(\mathcal{R})$ implies that the same formula (5.4) would hold for the Hurwitz strata associated to those non-minimal choices by the construction of Section 3.2. For type A_l, Corollary 5.2 recovers Arnold's formula for the LL-degree of the space of complex trigonometric polynomials, as was already shown in [20]. The fifth column indicates how $\iota_\omega(\mathcal{M}_{\mathcal{R}}^{\rm DZ})$ sits as a stratum inside the parent Hurwitz space $\mathcal{H}_{g_\omega,\mathsf{n}_\omega}$, either through explicit character relations in $\mathrm{Rep}(\mathcal{G}_{\mathcal{R}})$ or as a fixed locus of an automorphism of the Hurwitz space induced by the folding of the Dynkin diagram.

5.2. **The type-$\mathcal{R}$ extended Toda hierarchy.** Another notable consequence of the determination of the prepotential and the superpotential of $\mathcal{M}_{\mathcal{R}}^{\rm DZ}$ in Theorem 3.1 is the construction of a dispersionless bihamiltonian hierarchy of integrable PDEs on the loop space $\mathcal{L}\mathcal{M}_{\mathcal{R}}^{\rm DZ}$. This is amenable to a presentation both in normal and dispersionless Lax forms, and generalises the dispersionless limit of the extended bi-graded Toda hierarchy of [10, 20] (corresponding to $\mathcal{R} = A_l$) to general Dynkin types.

We first recall the general theory of principal hierarchies associated to Frobenius manifolds as formulated by Dubrovin [14]. Let $\mathcal{M}$ be an n-dimensional semi-simple complex Frobenius manifold, $\{t_\alpha : \mathcal{M} \to \mathbb{C}\}_{\alpha=1}^n$ a coordinate chart for $\mathcal{M}$ which is flat for the metric η, and $\mathcal{L}\mathcal{M} = \{\mathfrak{u} : S^1 \to \mathcal{M}\}$ the formal loop space of $\mathcal{M}$ – an element $\mathfrak{u} \in \mathcal{L}\mathcal{M}$ being an n-tuple $\mathfrak{u} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_n)$ with

$\mathfrak{u}_i \in \mathbb{C}[[X, X^{-1}]]$ a formal Laurent series in a periodic coordinate $X \in S^1$. The loop space $\mathcal{LM}$ can be endowed with a bihamiltonian pencil of hydrodynamic Poisson brackets

$$\{\mathfrak{u}_\alpha(X), \mathfrak{u}_\beta(Y)\}_{[\lambda]} = (\eta_{\alpha\beta} + \lambda\gamma_{\alpha\beta})\delta'(X-Y) + \lambda \sum_{\delta,\nu} \eta_{\alpha\nu}\Gamma^\nu_{\beta\delta}(\mathfrak{u})\partial_X \mathfrak{u}_\delta \delta(X-Y), \tag{5.5}$$

where Γ^i_{jk} denotes the Christoffel symbol of the Levi-Civita connection of γ in the flat coordinate chart t. Let ∇ denote the affine, torsion free connection on $T(\mathcal{M} \times \mathbb{C}^\star)$ defined by

$$\begin{aligned} \nabla_V W &= i_V \mathrm{d}_{\mathcal{M}} W + zV \cdot W, \\ \nabla_{\partial_z} W &= i_{\partial_z} \mathrm{d}_{\mathbb{C}^\star} W - E \cdot W - z^{-1}\mathcal{V}W, \end{aligned} \tag{5.6}$$

where $z \in \mathbb{C}^\star$, $V(z)$, $W(z) \in \Gamma(T\mathcal{M})$, and $\mathcal{V} \in \Gamma(\mathrm{End}(TM))$ is the grading operator defined in flat coordinates as $\mathcal{V}^\alpha_\beta = (1 - n/2)\delta^\alpha_\beta + \partial_\beta E^\alpha$. The Frobenius manifold axioms imply that ∇ is flat [15, Lecture 6], and a basis of horizontal sections $\sum_\alpha W^\alpha_\beta \partial_\alpha$ can be taken to have the form $W^\alpha_\beta = \sum_\nu \eta^{\alpha\nu}\partial_\beta h_\nu(\mathfrak{u}, z) z^{\mathcal{V}} z^R$ for some constant matrix R (determined by the monodromy data of the Frobenius manifold; see [17, Lecture 2]) and $h(\mathfrak{u}, z) \in \Gamma(\mathcal{O}_{\mathcal{M}})[[z]]$. Furthermore, the Taylor coefficients $h_{\alpha,m}(\mathfrak{u}) := [z^{m+1}]h_\alpha$ define Hamiltonian densities for which the corresponding local Hamiltonians,

$$H_{\alpha,m}[\mathfrak{u}] := \int_{S^1} h_{\alpha,m}((\mathfrak{u}(X))\mathrm{d}X \tag{5.7}$$

are in involution with respect to (5.5) for all λ,

$$\{H_{\alpha,m}, H_{\beta,n}\}_{[\lambda]} = 0. \tag{5.8}$$

The corresponding involutive Hamiltonian flows

$$\partial_{T_{\alpha,m}} \mathfrak{u}_\beta := \{\mathfrak{u}_\beta, H_{\alpha,m}\}_{[\lambda]} = \sum_{\delta\epsilon} \left[(\eta_{\beta\delta} + \lambda\gamma_{\beta\delta})\partial^2_{\mathfrak{u}^\delta \mathfrak{u}^\epsilon} h_{\alpha,m}(\mathfrak{u})\mathfrak{u}^\epsilon_X + \lambda \sum_\nu \eta_{\beta\nu}\Gamma^\nu_{\epsilon\delta}\partial_{\mathfrak{u}^\delta} h_{\alpha,m}(\mathfrak{u})\partial_X \mathfrak{u}_\epsilon \right] \tag{5.9}$$

for $\alpha = 1, \dots, n$ and $m = 0, \dots, \infty$ define an integrable hierarchy of quasi-linear PDEs on $\mathcal{LM}$, called the *principal hierarchy* of $\mathcal{M}$; the dependent variables $\mathfrak{u}_i = \mathfrak{u}_i(X, T)$ are called the normal coordinates of the hierarchy. This hierarchy moreover satisfies the τ-symmetry condition

$$\partial_{T_{\mu,m}} h_{\nu,n}((\mathfrak{u}(X,T)) = \partial_{T_{\nu,n}} h_{\mu,m}(\mathfrak{u}(X,T)) = \frac{\partial^3 \log \tau(X,T)}{\partial X \partial T_{\mu,m} \partial T_{\nu,n}}, \tag{5.10}$$

for some function $\tau(X; (T_{\mu,m})_{\mu,m})$. In particular, $\mathfrak{u}_\alpha(X,T) = \partial^2_{X,T_{\alpha,0}} \log \tau$.

5.2.1. *The dispersionless extended $\mathcal{R}$-type Toda hierarchy: Hamiltonian and Lax–Sato form.* In the case of the Dubrovin–Zhang Frobenius manifolds of type $\mathcal{R}$, Theorem 3.1 can be used effectively to solve the problem of finding the fundamental solutions of (5.6) and obtain the presentation of the principal hierarchy in normal coordinates (5.9). An immediate adaptation of [15, Proposition 6.3] gives the following

Proposition 5.3. *With conventions as in Lemma 4.1, let* $\widetilde{h}_{i,\alpha}(\tau,z)$*,* $\widetilde{h}^{\rm ext}_j(\tau,z)$*,* $\widetilde{h}^{\rm res}_k(\tau,z)$ *be the flat coordinates for the deformed connection* (5.6) *on* $\mathcal{M}_{\mathcal{R}}^{\rm DZ}\times\mathbb{C}^\star$ *normalised such that* $\widetilde{h}_{i,\alpha}=\tau_{i,\alpha}+\mathcal{O}(z)$*,* $\widetilde{h}^{\rm ext}_j=\tau^{\rm ext}_j+\mathcal{O}(z)$*,* $\widetilde{h}^{\rm res}_k=\tau^{\rm res}_k+\mathcal{O}(z)$*. Then,*

$$\widetilde{h}_{i,\alpha}(\tau,z) = -\frac{n_i+1}{\alpha}\operatorname*{Res}_{\infty_i}\kappa_i^\alpha\,{}_1F_1\big(1,1+\frac{\alpha}{n_i+1};z\lambda(\mu)\big)\frac{\mathrm{d}\mu}{\mu}, \tag{5.11}$$

$$\widetilde{h}^{\rm ext}_j(\tau,z) = \mathrm{p.v.}\int_{\infty_0}^{\infty_j}\mathrm{e}^{z\lambda}\frac{\mathrm{d}\mu}{\mu}, \tag{5.12}$$

$$\widetilde{h}^{\rm res}_k(\tau,z) = \operatorname*{Res}_{\infty_i}\frac{\mathrm{e}^{z\lambda}-1}{z}\frac{\mathrm{d}\mu}{\mu}, \tag{5.13}$$

where ${}_1F_1(a,b;x)=\sum_{n=0}^\infty\frac{(a)_nx^n}{(b)_nn!}$ *is Kummer's confluent hypergeometric function, and* $(a)_n:=\Gamma(a+n)/\Gamma(a)$*.*

We call the bihamiltonian integrable hierarchy defined by (5.9) and (5.11)–(5.13) the *dispersionless extended $\mathcal{R}$-type Toda hierarchy.* For $\mathcal{R}=A_n$, this coincides with the dispersionless limit of the bi-graded Toda hierarchy of [10]. The adjective "extended" refers to the Hamiltonian flows generated by $H^{\rm ext}_j[\mathfrak{u}]$, which are higher order versions of the space translation: when $\mathcal{R}$ is simply-laced these encode the non-stationary Gromov–Witten invariants of the type-$\mathcal{R}$ $\mathbb{P}^1$ orbifold given by insertions of descendents of the identity.

Theorem 3.1 also provides a dispersionless Lax–Sato description of (5.9) and (5.11)–(5.13) as a specific reduction of the universal genus-g_ω Whitham hierarchy with $\ell(\mathsf{n}_\omega)$ punctures [14, 28]. Let $\widetilde{C}_w$ be the universal covering of $\overline{C_w^{\omega,\bar{k}}}\setminus\{\infty_i\}_i$, the fibre at w of the Landau–Ginzburg family of Theorem 3.1, viewed as an analytic variety. Following [15], we consider second and third kind differentials $\Omega_{i,\alpha}$, $\Omega^{\rm ext}_j$, $\Omega^{\rm res}_k$ defined on $\widetilde{C}_w$ such that

$$\begin{aligned}
\Omega_{i,\alpha;m} &= -\frac{1}{n_i+1}\left[\left(\frac{\alpha}{n_i+1}\right)_{m+1}\right]^{-1}\mathrm{d}\lambda^{\alpha/(n_i+1)+m}+\text{regular},\\
\Omega^{\rm ext}_{j;m} &= \begin{cases}-\frac{\mathrm{d}\psi_m(\lambda)}{n_j+1}+\text{regular near }\infty_i,\\ \frac{\mathrm{d}\psi_m(\lambda)}{n_0+1}+\text{regular near }\infty_0,\end{cases}\\
\Omega^{\rm res}_{i;m} &= -\mathrm{d}\left(\frac{\lambda^{m+1}}{(m+1)!}\right)+\text{regular},
\end{aligned} \tag{5.14}$$

where $\psi_m(\lambda):=\lambda^m/m!(\log\lambda-H_m)$, and H_m is the $m^{\rm th}$ harmonic number. Then, (5.9) and (5.11)–(5.13) are equivalent to the dispersionless Lax system

$$\partial_{T_{(i,\alpha);m}}\lambda=\{\lambda,q_{i,\alpha;m}\}_{\rm LS},\quad \partial_{T^{\rm ext}_{i;m}}\lambda=\{\lambda,q^{\rm ext}_{j;m}\}_{\rm LS},\quad \partial_{T^{\rm res}_{i;m}}\lambda=\{\lambda,q^{\rm res}_{i;m}\}_{\rm LS}, \tag{5.15}$$

where $q_{i,\alpha;m}(\mu):=\int^\mu\Omega_{i,\alpha;m}$, $q^{\rm ext}_{j;m}(\mu):=\int^\mu\Omega^{\rm ext}_{j;m}$ and $q^{\rm res}_{j;m}(\mu):=\int^\mu\Omega^{\rm res}_{j;m}$, and

$$\{f(\mu,X),g(\mu,X)\}_{\rm LS}:=\mu\left(\partial_\mu f\partial_X g-\partial_X f\partial_\mu g\right). \tag{5.16}$$

Having a closed-form superpotential for $\mathcal{M}_{\mathcal{R}}^{\rm DZ}$ from Theorem 3.1 in particular provides explicit expressions for the Lax–Sato and Hamiltonian densities.

Example 5.1 ($\mathcal{R} = G_2$)**.** Let us consider for example $\mathcal{R} = G_2$. We construct explicitly the whole tower of Hamiltonian densities for the stationary flows of the type-G_2 dispersionless Toda hierarchy. In principle, these can be computed (up to a triangular linear transformation in the flow variables $T^{\alpha,m}$) by imposing the recursion relation, coming from the first line of (5.6),

$$\partial^2_{t_\alpha t_\beta} h_{\gamma,m} = \sum_\delta c^\delta_{\alpha\beta} \partial_{t_\delta} h_{\gamma,m-1} \tag{5.17}$$

with $\gamma = 1,2,3,\ m \geq 0,\ h_{\gamma,0} = \sum_\delta \eta_{\gamma,\delta} t_\delta$, and $c^\gamma_{\alpha\beta}$ are the structure constants of the quantum product determined by the prepotential (4.35). While the recursion (5.17) is ostensibly very hard to solve directly, the combination of Theorem 3.1 and Proposition 5.3 allows to give closed forms for the stationary Hamiltonians $H_{\gamma,m}$, $\gamma = 1,3$, parametrically in m.

This is most easily achieved in flat coordinates (x_0, x_1, x_2) for the second metric γ, and in Hamiltonian form for the corresponding Poisson bracket. From (3.62) we have

$$\lambda(\mu) = \frac{w_0}{\mu^2(\mu+1)^2} \prod_{i=1}^6 (\mu - a_i(x)), \tag{5.18}$$

with

$$a_1 = \mathrm{e}^{x_1}, \quad a_2 = \mathrm{e}^{-x_1+x_2}, \quad a_3 = \mathrm{e}^{2x_1-x_2}, \quad a_4 = \mathrm{e}^{-x_1}, \quad a_5 = \mathrm{e}^{x_1-x_2}, \quad a_6 = \mathrm{e}^{x_2-2x_1}. \tag{5.19}$$

Labelling the punctures at $\mu = 0$, -1 and ∞ as ∞_0, ∞_1 and ∞_2 respectively we have:

$$\begin{aligned} \widetilde{h}^{\mathrm{res}}_{0;m} &= -\widetilde{h}^{\mathrm{res}}_{2;m} = h_{3,m}, & \widetilde{h}^{\mathrm{res}}_{1;m} &= 0, \\ \widetilde{h}_{0,1/2;m} &= \widetilde{h}_{2,1/2;m} = h_{1,m}, & \widetilde{h}_{1,1/2;m} &= -2\widetilde{h}_{0,1/2;m}. \end{aligned} \tag{5.20}$$

From (5.11) and (5.13) we then get that

$$\begin{aligned} h_{1,m} &= \sum_{j=0}^{2m} \sum_{\substack{k_1,\dots,k_6=0,\dots,m \\ \sum_i k_i = 2m-j}} \frac{(2m)_j \mathrm{e}^{m x_0}}{j! m!} \prod_{i=1}^6 \frac{a_i(x)^{m-k_i} (m-k_i+1)_{k_i}}{k_i!}, \\ h_{3,m} &= \sum_{j=0}^{2m+1} \sum_{\substack{k_1,\dots,k_6=0,\dots,2m+1 \\ \sum_i k_i = 2m+1-j}} \frac{(2m+1)_j \mathrm{e}^{(m+1/2)x_0}}{j! (\frac{3}{2})_m} \prod_{i=1}^6 \frac{a_i(x)^{m+1/2-k_i} (m-k_i+3/2)_{k_i}}{k_i!}, \end{aligned} \tag{5.21}$$

where $(a)_m = \Gamma(a+m)/\Gamma(m)$ is the usual notation for the Pochhammer symbol. In these coordinates, the Gram matrix of the second metric γ and its inverse read

$$(\gamma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 12 & -6 \\ 0 & -6 & 4 \end{pmatrix}, \qquad (\gamma^{-1}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}. \tag{5.22}$$

Let $x_i = f_i(t_1, t_2, t_3)$, $i = 0,1,2$ be the change-of-variables expressing the x-coordinates in the flat coordinate chart (t_1, t_2, t_3) for the first metric η, and define accordingly $\mathfrak{w}_i = f_i(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3)$ for the corresponding dependent variables for the principal hierarchy. In these coordinates, the second Poisson bracket takes the form

$$\{\mathfrak{w}_i(X), \mathfrak{w}_j(Y)\}_\infty = \gamma_{ij}\delta'(X-Y) \tag{5.23}$$

and the stationary flows are given by

$$\frac{\partial \mathfrak{w}_i}{\partial T_{j,m}} = \{\mathfrak{w}_i, H_{j,m}\}_\infty = \sum_{k=0,1,2} \gamma_{ik}\partial_{\mathfrak{w}_k} h_{j,m}(\mathfrak{w})\partial_X \mathfrak{w}_k, \quad j = 1, 3. \tag{5.24}$$

Remark 5.4. In [42], a deformation scheme for the genus zero universal Whitham hierarchy is introduced in terms of a Moyal-type quantisation of the dispersionless Lax–Sato formalism. It would be intriguing to apply these ideas to the cases when $\mathcal{M}_\omega^{\rm LG}$ embeds in a genus $g_\omega = 0$ Hurwitz space, and verify that the resulting dispersionful deformation of the Principal Hierarchy is compatible with the hierarchy obtained by the quantisation of the underlying semi-simple CohFT.

Appendix A. Character relations for E_6 and E_7

A.1. $\mathcal{R} = E_6$. The character relations for $\rho = \rho_{\omega_1}$ and $k = 1, \dots, 13$ are:

$$\begin{aligned}
\mathfrak{p}_0^{[100000]_{E_6}} &= 1, \\
\mathfrak{p}_1^{[100000]_{E_6}} &= \chi_1, \\
\mathfrak{p}_2^{[100000]_{E_6}} &= \chi_2, \\
\mathfrak{p}_3^{[100000]_{E_6}} &= \chi_3, \\
\mathfrak{p}_4^{[100000]_{E_6}} &= -\chi_5^2 - \chi_2\chi_5 + \chi_1 + \chi_4 + \chi_4\chi_6, \\
\mathfrak{p}_5^{[100000]_{E_6}} &= \chi_1^2 - 2\chi_5^2\chi_1 + 2\chi_4\chi_1 + \chi_4^2 + \chi_5\chi_6^2 + \chi_2 - 2\chi_3\chi_5 + \chi_5 - \chi_2\chi_6 - \chi_5\chi_6, \\
\mathfrak{p}_6^{[100000]_{E_6}} &= \chi_1\chi_5 - \chi_5^3 - \chi_2\chi_5^2 + 2\chi_4\chi_5 - 2\chi_1\chi_6\chi_5 + \chi_4\chi_6\chi_5 + \chi_6^3 \\
&+ 2\chi_1\chi_2 - 2\chi_3 + \chi_2\chi_4 - 3\chi_3\chi_6, \\
\mathfrak{p}_7^{[100000]_{E_6}} &= 2\chi_2^2 + \chi_5\chi_2 - 2\chi_5\chi_6\chi_2 + \chi_3\chi_5^2 + \chi_1\chi_6^2 + \chi_4\chi_6^2 - 3\chi_1\chi_3 \\
&- 2\chi_3\chi_4 + \chi_4 - \chi_5^2\chi_6 - \chi_1\chi_6 + \chi_4\chi_6, \\
\mathfrak{p}_8^{[100000]_{E_6}} &= \chi_2\chi_5^3 - \chi_1\chi_6\chi_5^2 + \chi_6^2\chi_5 + \chi_1\chi_2\chi_5 - 2\chi_3\chi_5 - 3\chi_2\chi_4\chi_5 + \chi_3\chi_6\chi_5 \\
&- 2\chi_6\chi_5 + \chi_5 - \chi_1^2 - \chi_2\chi_6^2 + \chi_2\chi_3 + \chi_1\chi_4 + \chi_1^2\chi_6 - \chi_2\chi_6 + 2\chi_1\chi_4\chi_6, \\
\mathfrak{p}_9^{[100000]_{E_6}} &= \chi_1\chi_5^4 - \chi_6\chi_5^3 + \chi_2\chi_5^2 - 4\chi_1\chi_4\chi_5^2 + \chi_2\chi_6\chi_5^2 - \chi_2^2\chi_5 - \chi_1\chi_6^2\chi_5 - 4\chi_1\chi_5 \\
&+ \chi_4\chi_5 + 3\chi_4\chi_6\chi_5 + \chi_6^3 + \chi_3^2 + 2\chi_1\chi_4^2 + \chi_1\chi_2 - 6\chi_3 + 4\chi_1^2\chi_4 - 4\chi_2\chi_4 \\
&+ 4\chi_1\chi_3\chi_5 - 2\chi_2\chi_4\chi_6 - 3\chi_3\chi_6 + 3 \\
\mathfrak{p}_{10}^{[100000]_{E_6}} &= \chi_5^5 - 5\chi_4\chi_5^3 + \chi_1\chi_6\chi_5^3 - \chi_6^2\chi_5^2 - \chi_1\chi_2\chi_5^2 + 5\chi_3\chi_5^2 - \chi_5^2 - 2\chi_1^2\chi_5 + 5\chi_4^2\chi_5 + \chi_2\chi_5 \\
&+ \chi_2\chi_3\chi_5 + 4\chi_1\chi_4\chi_5 + \chi_1^2\chi_6\chi_5 - 2\chi_2\chi_6\chi_5 - 3\chi_1\chi_4\chi_6\chi_5 + \chi_1\chi_6^2 + 2\chi_4\chi_6^2 + 2\chi_1 \\
&+ \chi_1^2\chi_2 - 5\chi_1\chi_3 + 2\chi_1\chi_2\chi_4 - 5\chi_3\chi_4 - \chi_2^2\chi_6 - 2\chi_1\chi_6 + \chi_1\chi_3\chi_6 - \chi_4\chi_6, \\
\mathfrak{p}_{11}^{[100000]_{E_6}} &= \chi_6\chi_5^4 - \chi_2\chi_5^3 + \chi_1\chi_3\chi_5^2 - \chi_4\chi_5^2 + 2\chi_1\chi_6\chi_5^2 - 4\chi_4\chi_6\chi_5^2 - 2\chi_6^2\chi_5 - \chi_1\chi_2\chi_5 + \chi_2\chi_6
\end{aligned}$$

$$\begin{aligned}
&+ \quad 3\chi_3\chi_5 + 3\chi_2\chi_4\chi_5 - 2\chi_1\chi_2\chi_6\chi_5 + 3\chi_3\chi_6\chi_5 - \chi_6\chi_5 + \chi_1\chi_2^2 + 2\chi_4^2 + 2\chi_4^2\chi_6 \\
&+ \quad 2\chi_1^2\chi_6^2 - 2\chi_2\chi_6^2 + 2\chi_2 - 3\chi_1^2\chi_3 + \chi_2\chi_3 + \chi_2^2\chi_4 + \chi_1\chi_4 - 2\chi_1\chi_3\chi_4 - 2\chi_1^2\chi_6, \\
\mathfrak{p}_{12}^{[100000]_{E_6}} &= \quad 2\chi_6\chi_1^3 + \chi_5^2\chi_1^2 - \chi_4\chi_1^2 - 2\chi_2\chi_5\chi_1^2 - \chi_5^2\chi_6\chi_1^2 + \chi_4\chi_6\chi_1^2 + 3\chi_5\chi_6^2\chi_1 + 2\chi_2\chi_1 + 6\chi_3\chi_6 \\
&- \quad 3\chi_2\chi_3\chi_1 + \chi_2\chi_4\chi_5\chi_1 + \chi_5\chi_1 - 5\chi_2\chi_6\chi_1 - 2\chi_5\chi_6\chi_1 + \chi_2^3 + \chi_3\chi_5^3 - 3\chi_4\chi_5\chi_6 - \chi_1^3 \\
&- \quad \chi_5^3 - 2\chi_6^3 + 3\chi_3^2 - \chi_3 + \chi_2\chi_4 + 3\chi_2^2\chi_5 - 3\chi_3\chi_4\chi_5 + 2\chi_4\chi_5 + \chi_5^3\chi_6 - \chi_2\chi_5^2\chi_6, \\
\mathfrak{p}_{13}^{[100000]_{E_6}} &= \quad \chi_1^4 - 2\chi_5^2\chi_1^3 + 2\chi_4\chi_1^3 + \chi_4^2\chi_1^2 - 3\chi_2\chi_1^2 - 2\chi_3\chi_5\chi_1^2 - \chi_5\chi_1^2 - \chi_2\chi_6\chi_1^2 + 4\chi_5\chi_6\chi_1^2 \\
&+ \quad 2\chi_5^3\chi_1 + 2\chi_2\chi_5^2\chi_1 - 2\chi_6^2\chi_1 + \chi_3\chi_1 - 4\chi_2\chi_4\chi_1 + \chi_2^2\chi_5\chi_1 - 4\chi_4\chi_5\chi_1 - \chi_5^3\chi_6\chi_1 \\
&+ \quad 3\chi_3\chi_6\chi_1 + \chi_4\chi_5\chi_6\chi_1 - \chi_6\chi_1 + 2\chi_1 + \chi_2^2 - 2\chi_2\chi_4^2 + \chi_3\chi_5^2 + \chi_2\chi_4\chi_5^2 \\
&+ \quad \chi_5^2\chi_6^2 - 3\chi_3\chi_4 + 2\chi_4 + \chi_2\chi_5 - \chi_2\chi_3\chi_5 + \chi_2^2\chi_6 - 2\chi_5^2\chi_6 - 2\chi_2\chi_5\chi_6. \qquad (A.1)
\end{aligned}$$

A.2. $\mathcal{R} = E_7$**.** We include here the character relations for $\rho = \rho_{\omega_6}$ and $k = 1, \ldots, 11$; note that by reality, we have $\mathfrak{p}^{\rho}_{56-k} = \mathfrak{p}^{\rho}_k$. The full set of character relations for k up to 28 is available upon request.

$$\begin{aligned}
\mathfrak{p}_0^{[0000010]_{E_7}} &= \quad 1, \\
\mathfrak{p}_1^{[0000010]_{E_7}} &= \quad \chi_6, \\
\mathfrak{p}_2^{[0000010]_{E_7}} &= \quad \chi_5 + 1, \\
\mathfrak{p}_3^{[0000010]_{E_7}} &= \quad \chi_4 + \chi_6, \\
\mathfrak{p}_4^{[0000010]_{E_7}} &= \quad \chi_3 + \chi_5 + 1, \\
\mathfrak{p}_5^{[0000010]_{E_7}} &= \quad -(\chi_1 - 1)\chi_4 + \left(-\chi_1^2 + \chi_1 + \chi_2 + \chi_5 + 1\right)\chi_6 + \chi_2\chi_7, \\
\mathfrak{p}_6^{[0000010]_{E_7}} &= \quad -2\chi_1^3 + (1 - 2\chi_5)\chi_1^2 + \left(\chi_6^2 - \chi_7\chi_6 + \chi_7^2 + 4\chi_2 - 2\chi_3 + 2\chi_5 + 2\right)\chi_1 + \chi_2^2 + \chi_5^2 \\
&- \quad \chi_3 + 2\chi_5 + 2\chi_2(\chi_5 + 1) + \chi_4\chi_6 - \chi_4\chi_7 - \chi_6\chi_7 + 1, \\
\mathfrak{p}_7^{[0000010]_{E_7}} &= \quad \chi_4\left(-\chi_1^2 + \chi_1 + \chi_2 + 2\chi_5 + 2\right) + (-\chi_1^3 + (2\chi_2 + \chi_5 + 3)\chi_1 + 2\chi_2 - 2\chi_3 + \chi_5 \\
&+ \quad 1)\chi_6 + \chi_7\left(-2\chi_1^2 + (\chi_2 - 2\chi_5 + 1)\chi_1 + \chi_7^2 + 3\chi_2 - 3\chi_3\right), \\
\mathfrak{p}_8^{[0000010]_{E_7}} &= \quad (2\chi_6^2 + \chi_4\chi_6 - 2\chi_7\chi_6 + \chi_7^2 + 4\chi_2 - 2\chi_1^3 - 4\chi_3 + 2\chi_5 - 2\chi_4\chi_7)\chi_1 \\
&+ \quad \chi_2^2 + 2\chi_4^2 + \chi_6^2 + \chi_5\chi_7^2 + \chi_7^2 - 2\chi_3 - 3\chi_3\chi_5 + 3\chi_4\chi_6 + \chi_4\chi_7 - \chi_5\chi_6\chi_7 + \chi_6\chi_7 \\
&+ \quad \chi_2(\chi_6^2 + \chi_7\chi_6 + \chi_7^2 - 2\chi_3 + 2\chi_5) + (\chi_3 - 2\chi_5 - \chi_6\chi_7 + 2)\,\chi_1^2, \\
\mathfrak{p}_9^{[0000010]_{E_7}} &= \quad (\chi_1 + 2)\chi_6^3 - 2\chi_1\chi_7\chi_6^2 + (-\chi_1^3 + \chi_1^2 + \left(\chi_7^2 + 2\chi_2 - 2\chi_3 - 2\chi_5\right)\chi_1 - \chi_5^2 \\
&- \quad 3\chi_3 - 2\chi_5 + \chi_2(\chi_5 + 1) - 1)\chi_6 + (-(\chi_5 + 2)\chi_1^2 + (\chi_2 + \chi_3 + \chi_5 \\
&+ \quad 2)\chi_1 + \chi_5^2 - \chi_3 + 3\chi_5 + 2\chi_2(\chi_5 + 1) + 2)\chi_7 + \chi_4(\chi_1^3 - \chi_1^2 + (-3\chi_2 + \chi_5 \\
&+ \quad 2)\chi_1 - \chi_7^2 - 2\chi_2 + \chi_3 + 3\chi_5 - \chi_6\chi_7 + 4), \\
\mathfrak{p}_{10}^{[0000010]_{E_7}} &= \quad \chi_5\chi_1^4 - \chi_6\chi_7\chi_1^3 + \left(-\chi_6^2 - 2\chi_7\chi_6 + \chi_7^2 - (4\chi_2 + 1)\chi_5 + \chi_4(\chi_6 + \chi_7)\right)\chi_1^2
\end{aligned}$$

$$
\begin{aligned}
&- && (\chi_4^2+3\chi_7\chi_4-4\chi_5^2-\chi_2\chi_6^2-2\chi_6^2+\chi_7^2-3\chi_2\chi_6\chi_7-3\chi_6\chi_7+\chi_5\left(4\chi_6^2+\chi_7^2-2\right))\chi_1 \\
&+ && \chi_6\chi_7^3+\chi_3^2+4\chi_2\chi_5^2+2\chi_5^2+\chi_2\chi_6^2-5\chi_5\chi_6^2-5\chi_6^2-\chi_2\chi_7^2-\chi_5\chi_7^2+3\chi_6^4 \\
&+ && 5\chi_5-4\chi_2\chi_4\chi_6+3\chi_4\chi_6+\chi_4\chi_5\chi_6-2\chi_2\chi_4\chi_7+\chi_4\chi_7+2\chi_2\chi_6\chi_7+2\chi_5\chi_6\chi_7 \\
&+ && 2\chi_2\chi_5+\chi_6\chi_7+\chi_3\left(-6\chi_6^2-3\chi_7\chi_6+(4\chi_1+5)\chi_5+4\right)-\chi_7^2+2\chi_2^2\chi_5+3, \\
\mathfrak{p}_{11}^{[0000010]_{E_7}} &= && \left(-\chi_1^2+4\chi_1+2\chi_5\right)\chi_6^3+(\chi_1-\chi_2-2(\chi_5+1))\chi_7\chi_6^2+(\chi_1^5-5(\chi_2+1)\chi_1^3 \\
&+ && \left(-\chi_7^2+\chi_2+5\chi_3-4\chi_5+3\right)\chi_1^2+(5\chi_2^2+4(\chi_5+2)\chi_2-2\chi_5^2 \\
&- && 8\chi_3-1)\chi_1+3\chi_2^2-\chi_5^2+\chi_5\chi_7^2+\chi_7^2-2\chi_3-5\chi_3\chi_5 \\
&- && \chi_5+\chi_2\left(2\chi_7^2-5\chi_3+7\chi_5+2\right)+1)\chi_6-\chi_4^2\chi_7+\chi_4(2\chi_1^3-(\chi_5-1)\chi_1^2 \\
&+ && \left(\chi_6^2-2\chi_7\chi_6-6\chi_2+\chi_3+4\chi_5-3\right)\chi_1+\chi_5^2-\chi_6^2-\chi_7^2+5\chi_3 \\
&+ && 3\chi_5+\chi_2(2\chi_5-7)+2\chi_6\chi_7+3)+\chi_7(-\chi_1^4+\chi_5\chi_1^3+(4\chi_2-2\chi_5+3)\chi_1^2+(\chi_5^2+\chi_5 \\
&+ && \chi_7^2-4\chi_3-\chi_2(3\chi_5+1)-3)\chi_1-2\chi_2^2+\chi_5^2+3\chi_3+\chi_3\chi_5+3\chi_5-\chi_2(\chi_5+2)+2).
\end{aligned} \tag{A.2}
$$

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