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Expressing Discrete Spatial Relations under Granularity

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Abstract

The logic **UBiSKt** is a bi-intuitionistic modal logic with universal modalities having a semantics in which formulae are interpreted as subgraphs. We use the logic to formalize the idea of looking at subgraphs at different levels of detail. We show how the logic can be used to define various spatial relations between subgraphs. These relations are considered with respect to change in level of detail. We use an existing formulation of generalized partitions on graphs and hypergraphs in terms of relations to give a novel generalization of the classical modal logic **S5** to a bi-intuitionistic modal logic we call **UBiSKtS5**.

1. Introduction

The logic **UBiSKt** is a bi-intuitionistic modal logic with universal modalities first introduced in [28] and studied also in [26]. It is an extension of the logic **BiSKt** [30, 24]. Propositional bi-intuitionistic logic was first studied by Rauszer [20, 21, 22], who refers to it as *H-B* (Heyting-Brower) logic. The reader is referred to [12] for a recent study of Rauszer's works on bi-intuitonistic logic. The semantics of **UBiSKt** is given by a set U with a preorder $H \subseteq U \times U$. Under certain additional conditions on H, the structure (U, H) can be seen as a graph, or more generally as a hypergraph. While there are many possible notions of what a relation on a hypergraph should be, we follow the approach introduced in [29], where relations on hypergraphs satisfy the so called *stability* condition on H, namely $H \,;\, R \,;\, H \subseteq R$ where ; is the operation of relational composition. This choice is justified in [29] by the fact that there is a bijective correspondence between stable relations and union preserving functions on the lattice of subgraphs. Modal operators in **UBiSKt** are interpreted with respect to a stable relation $R \subseteq U \times U$.

Qualitative spatial relations [6] are used in artificial intelligence to model commonsense notions such as regions of space overlapping, touching only at their boundaries, or being separate. Various spatial calculi have been developed,

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such as the Region-Connection Calculus (RCC) [19]. The RCC models various topological spatial relations in dense spaces, and defines them by using first-order logic. In [28] **UBiSKt** is used to model qualitative relations in discrete space. This works because the logic has a semantics in which formulae are interpreted as subgraphs of a hypergraph. In this paper we will show that our logic can also deal with subgraphs and spatial relations between subgraphs at different levels of detail. The idea of zooming out, or viewing a situation in a less detailed way, is commonplace. For instance, zooming out on an image (a set of pixels) we expect narrow cracks to fuse and narrow spikes to become invisible. This idea is formalized in the discipline of mathematical morphology [16]: instead of being able to see individual pixels, only groups of pixels of a particular pattern (copies of a structuring element) can be seen. This idea is known to extend to graphs (not just sets of pixels) [30, 7].

Classically, the modal logic **S4** is based on a relation that is reflexive and transitive, so it can be seen as the logic of preorders. Similarly, the classical modal logic **S5** can be seen as the logic of equivalence relations. These systems are discussed in the context of rough set theory [18] by Yao and Lin [32].

In [25] the correspondence between equivalence relations and partitions on sets is generalised to the case of hypergraphs. The results there provide a foundation for a theory of rough hypergraphs. It is shown that stable relations having suitable properties correspond to partitions of a certain type on hypergraphs (cf. [25, Theorem 7]). In this paper we show, for the first time, that these properties of R can be expressed by formulae in **UBiSKt**. This allows the definition of a novel bi-intuitionistic version of **S5**, denoted H**UBiSKtS5** below. This provides a logic for partitions of hypergraphs, or hypergraphs at different levels of detail in the sense of [25].

The paper is structured as follows: in Section 2 we review the notions of graphs and hypergraphs, we introduce the syntax and the semantics of **UBiSKt** and the connections between the logic and the structures of hypergraphs and graphs. We also show that there are formulae in **UBiSKt** that capture the notion of hypergraph, and the special case of a hypergraph with no edges (i.e. a set). In Section 3 a Hilbert style axiomatization for **UBiSKt** is introduced. Soundness, strong completeness and decidability are proved, thus extending the work done in [26]. In Section 4 we review the work done in [28, 26], encoding topological spatial relations between subgraphs. We formalize the idea of changing the level of detail on a graph, and we present the spatial relations in a zoomed-out fashion. In Section 5 we review the approach of [25]. We first introduce a modal extension of **UBiSKt** where R is both reflexive and transitive, and then we analyse the additional properties required on R in order to induce a partition on hypergraphs. This gives a bi-intuitionistic generalization of **S5**. We also look at how the spatial relations are encoded in this modal system. Section 6 provides conclusions and makes suggestions for further work.

2. UBiSKt as a Logic of Graphs and Hypergraphs

2.1. Hypergraphs and Graphs as Posets

In this paper we work with undirected graphs, and with the related and more general idea of a hypergraph. One definition of hypergraph is as follows, in which $\mathcal{P}(N)$ denotes the power-set of N.

Definition 1. An *edge-node hypergraph* is a triple (E, N, i) where E and N are disjoint sets, E is called the set of *edges*, N is called the set of *nodes*, and $i : E \to \mathcal{P}(N)$ is a function satisfying $i(e) \neq \varnothing$ for each $e \in E$.

An alternative definition has a single set comprising all edges and nodes together with an incidence relation.

Definition 2. A *hypergraph* (U, H) consists of a set U and a reflexive relation $H \subseteq U \times U$ such that for all $u, v, w \in U$, uHv and vHw implies $v = u$ or $w = v$. Given $u \in U$, u is an *edge* if there is some $v \in U$ such that uHv and $u \neq v$. An element $u \in U$ that is not an edge is called a *node*.

It is straightforward to check that the relation H is a partial order. The two definitions are equivalent as shown by Proposition 1 in [25, p79].

Proposition 3. There is a bijective correspondence between edge-node hypergraphs (Definition 1) and hypergraphs in the sense of Definition 2.

Proposition 3 states that any edge-node hypergraph uniquely corresponds to a poset (U, H) where U is the set of edges and nodes together and H describes the edge-node incidence. Let us briefly explain how we can go from one construction to the other one, and vice-versa. Let $K = (E, N, i)$ be an edge-node hypergraph. We construct a hypergraph (U_K, H_K) as follows: $U_K = E \cup N$ and given two elements $u, v \in U_K$, the relation $u \; H_K \; v$ holds iff $u = v$ or $u \in E$ and $v \in N$ and $v \in i(u)$. We need to check that $u \; H_K \; v$ and $v \; H_K \; w$ implies that $u = v$ or $v = w$, so that (U_K, H_K) satisfies Definition 2. Suppose that (u, v) and (v, w) are both in H_K and that $u \neq v$. Then $u \in E$ and $v \in N$ and $v \in i(u)$ by our definition of H_K. Now if also $v \neq w$ holds, we have that $v \in E$, but this is impossible as E and N are disjoint sets. Hence we conclude that $v = w$. In the opposite direction, let $G = (U, H)$ be a hypergraph. We can construct an edge-node hypergraph (E_G, N_G, i_G) as follows. E_G is the set of all edges of G, so $E_G = \{ u \in U \mid \exists v (u \; H \; v \text{ and } u \neq v) \}$, and similarly N_G is the set of all nodes of G. Then we can define the function $i_G : E_G \to \mathcal{P}(N_G)$ by: $i_G(u) = \{ v \in U \mid u \; H \; v \text{ and } u \neq v \}$. We can easily check that no node in G is also an edge, by Definition 2, thus the sets E_G and N_G are disjoint, and $i_G(u)$ is always non-empty for any edge u. Thus (E_G, N_G, i_G) satisfies Definition 1, and it is an edge-node hypergraph. Finally it is possible to check that the constructions are inverse of each other, namely that if from $G = (U, H)$ we construct $K = (E_G, N_G, i_G)$, then $(U_K, H_K) = G$, and if from $K = (E, N, i)$ we construct $G = (U_K, H_K)$, then $(E_G, N_G, i_G) = K$. For a detailed proof of this fact, we refer the reader to [25] (p. 79-80).

A graph is a special case of a hypergraph in the following sense.

Definition 4. A graph is a hypergraph (U, H) where for all $u \in U$, the set $\{v \in U \mid uHv \text{ and } v \neq u\}$ has cardinality at most 2.

By Proposition 3 the above definition is equivalent to the following, more common in the literature and using separate sets of edges and of nodes.

Definition 5. An edge-node graph (E, N, i) is an edge-node hypergraph where for all $e \in E$, $1 \leq |i(e)| \leq 2$.

From now on we adopt the terminology 'hypergraph' and 'graph' in the sense of Definitions 2 and 4. A graph in this sense is undirected and might have multiple edges between the same pair nodes as well as loops on nodes. These structures are also known as multi-graphs, or pseudo-graphs [13], but we will refer to them simply as graphs. Figure 1 shows an edge-node graph, its representation as edge-node hypergraph, and the associated poset (U, H).

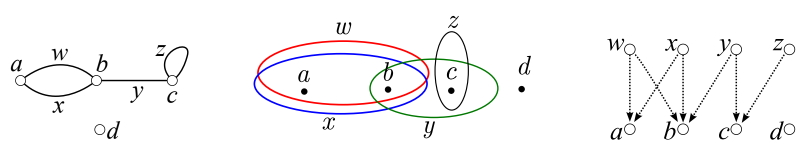


Figure 1: The edge-node graph on the left has four nodes, a, b, c, d, and four edges w, x, y, z. It can also be represented as an edge-node hypergraph, shown in the middle. The corresponding poset for this graph is the reflexive closure of the relation on the set $U = \{a, b, c, d, w, x, y, z\}$ shown on the right hand side.

A subgraph of a hypergraph as in Definition 2, and thus of a graph, can be seen as a subset of U such that if an edge is present, then all its end-points are present as well. This idea can be formalised by the following definition.

Definition 6. Given a hypergraph (U, H) and a set $K \subseteq U$, K is a *subgraph* if $u \in K$ and uHv jointly imply $v \in K$.

2.2. Kripke Semantics for **UBiSKt**

Let Prop be a countable set of propositional variables. Our syntax $\mathcal{L}$ for bi-intuitionistic stable tense logic with universal modalities consists of all logical connectives of bi-intuitionistic logic, i.e., two constant symbols $\bot$ and $\top$, disjunction $\vee$, conjunction $\wedge$, implication $\rightarrow$, coimplication $\prec$, and the set $\{\blacklozenge, \Box, \mathsf{A}, \mathsf{E}\}$ of modal operators. The set $\mathsf{Form}_{\mathcal{L}}$ of all formulae in $\mathcal{L}$ is defined inductively as follows:

$$\varphi ::= p \mid \top \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \rightarrow \varphi) \mid (\varphi \prec \varphi) \mid \blacklozenge \varphi \mid \Box \varphi \mid \mathsf{E}\,\varphi \mid \mathsf{A}\,\varphi \quad (p \in \mathsf{Prop}).$$

We follow the standard rules for omission of the parentheses. We define the following abbreviations:

$$\neg\varphi := \varphi \rightarrow \bot, \quad ⫟\varphi := \top \prec \varphi, \quad \varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi),$$
$$\Diamond\,\varphi := ⫟\Box\neg\varphi, \quad \blacksquare\,\varphi := \neg\blacklozenge⫟\varphi.$$

Definition 7. Let $R, S \subseteq U \times U$. The *converse* of R is $\breve{R} = \{(x, y) \in U \times U \mid (y, x) \in R\}$. The *relational complement* of R is $\overline{R} = \{(x, y) \in U \times U \mid (x, y) \notin R\}$. The *relational composition* of R followed by S is $R;S = \{(x, y) \in U \times U \mid (x, z) \in R$ and $(z, y) \in S$ for some $z\}$. We use R^2 to mean $R ; R$. The *identity relation* $I \subseteq U \times U$ is the set $\{ (u, u) \mid u \in U \}$.

We also remind the reader that the following holds: $R \subseteq S$ implies $\breve{R} \subseteq \breve{S}$ and $\breve{\overline{R}} = \overline{\breve{R}}$ (cf. [15]).

Definition 8. Let H be a preorder on a set U. We say that $X \subseteq U$ is an *upset* if X is closed under H-successors, i.e., uHv and $u \in X$ jointly imply $v \in X$ for all elements u, $v \in U$. Given a preorder (U, H), a binary relation $R \subseteq U \times U$ is *stable* if it satisfies $H ; R ; H \subseteq R$.

It is easy to see that a relation R on U is stable if and only if $R ; H \subseteq R$ and $H ; R \subseteq R$.

Even if R is a stable relation, its converse $\breve{R}$ may not be stable. A simple example is given by the construction (U, H, R) where $U = \{u, v\}$, $H = \{ (u, u), (v, v), (u, v) \}$ and $R = \{ (u, v) \}$. Here we have that $H ; R ; H \subseteq R$, but $(v, v) \in \breve{R} ; H$ and $(v, v) \notin \breve{R}$, hence $\breve{R}$ is not stable. Stable relations are thus not closed under converse.

However, the set of all stable relations on a fixed preorder (U, H) supports a weaker form of converse. Instead of a single involutory operation there is a pair of operations forming an adjunction as shown in [29, Corollary 12, p447]. These two operations are know as the left and right converse as they are left and right adjoints to each other. In this paper we only need the left converse of a stable relation R, which can be characterized as the smallest stable relation containing $\breve{R}$. It can be defined as follows.

Definition 9. The *left converse* $\curvearrowleft R$ of a stable relation R is $H ; \breve{R} ; H$.

Definition 10. We say that $F = (U, H, R)$ is a *frame* if U is a nonempty set, H is a preorder on U, and R is a *stable* binary relation on U. A *valuation* on a frame $F = (U, H, R)$ is a mapping V from Prop to the set of all upsets of (U, H). A pair $M = (F, V)$ is a *model* if $F = (U, H, R)$ is a frame and V is a valuation. Given a model $M = (U, H, R, V)$, a state $u \in U$ and a formula φ, the satisfaction relation $M, u \models \varphi$ is defined inductively as follows:

$$
\begin{array}{lll}
M, u \models p & \text{iff} & u \in V(p), \\
M, u \models \top, & & \\
M, u \not\models \bot, & & \\
M, u \models \varphi \vee \psi & \text{iff} & M, u \models \varphi \text{ or } M, u \models \psi, \\
M, u \models \varphi \wedge \psi & \text{iff} & M, u \models \varphi \text{ and } M, u \models \psi, \\
M, u \models \varphi \to \psi & \text{iff} & \text{For all } v \in U \ ((uHv \text{ and } M, v \models \varphi) \text{ imply } M, v \models \psi), \\
M, u \models \varphi \prec \psi & \text{iff} & \text{For some } v \in U \ (vHu \text{ and } M, v \models \varphi \text{ and } M, v \not\models \psi), \\
M, u \models \blacklozenge \varphi & \text{iff} & \text{For some } v \in U \ (vRu \text{ and } M, v \models \varphi), \\
M, u \models \Box \varphi & \text{iff} & \text{For all } v \in U \ (uRv \text{ implies } M, v \models \varphi), \\
M, u \models \mathsf{E} \varphi & \text{iff} & \text{For some } v \in U \ (M, v \models \varphi), \\
M, u \models \mathsf{A} \varphi & \text{iff} & \text{For all } v \in U \ (M, v \models \varphi).
\end{array}
$$

The *truth set* $[\![\varphi]\!]_M$ of a formula φ in a model M is defined by $[\![\varphi]\!]_M := \{\, u \in U \mid M, u \models \varphi \,\}$. If the underlying model M in $[\![\varphi]\!]_M$ is clear from the context, we drop the subscript and simply write $[\![\varphi]\!]$. We write $M \models \varphi$ (read: 'φ is valid in M') to mean that $[\![\varphi]\!]_M = U$ or $M, u \models \varphi$ for all states $u \in U$. For a set Γ of formulae, $M \models \Gamma$ means that $M \models \gamma$ for all $\gamma \in \Gamma$. Given any frame $F = (U, H, R)$, we say that a formula φ is *valid* in F (written: $F \models \varphi$) if $(F, V) \models \varphi$ for any valuation V and any state $u \in U$, i.e., $[\![\varphi]\!]_{(F,V)} = U$.

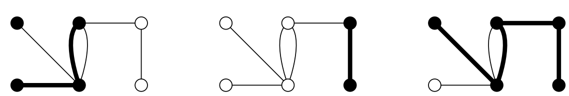
Proposition 11. Given any model M, the truth set $[\![\varphi]\!]_M$ is an upset.

Proof. By induction on φ. When φ is of the form $\mathsf{E}\,\psi$ or $\mathsf{A}\,\psi$, we remark that $[\![\varphi]\!]_M = U$ or $[\![\varphi]\!]_M = \varnothing$, which are both trivially upsets. ■

As for the abbreviated symbols, we may derive the following satisfaction conditions with the help of Proposition 11:

$M, u \models \neg\varphi$	iff	For all $v \in U$ (uHv implies $M, v \not\models \varphi$),
$M, u \models \lrcorner\,\varphi$	iff	For some $v \in U$ (vHu and $M, v \not\models \varphi$),
$M, u \models \Diamond\,\varphi$	iff	For some $v \in U$ ($(v, u) \in \curvearrowright R$ and $M, v \models \varphi$),
$M, u \models \blacksquare\,\varphi$	iff	For all $v \in U$ ($(u, v) \in \curvearrowright R$ implies $M, v \models \varphi$).

The semantics of **UBiSKt** is built on a set with a preorder. We have seen how a hypergraph gives rise to a poset, that is also a preorder. So, a special case of a model as in Definition 10 is when (U, H) is a hypergraph.



K $\qquad\qquad$ $\neg K$ $\qquad\qquad$ $\lrcorner\,K$

Figure 2: A graph with a subgraph K and the two kinds of complement of K. We remind the reader that the graph is represented as the set $U = N \cup E$ where N is the set of nodes and E is the set of edges, and H is the reflexive closure of the incidence relation between edges and nodes.

By Definition 6, the subgraphs of a hypergraph (U, H) are exactly the upsets as in Definition 8. Since any formula φ in the logic is interpreted as the upset $[\![\varphi]\!]_M$, formulae in the logic can be regarded as names for subgraphs of an underlying hypergraph (U, H). Similarly, operations in the logic provide operations on subgraphs, following the semantics defined earlier (Definition 10). Fig. 2 shows a graph with a subgraph and the two operations of complement $\neg$ and $\lrcorner$, where the leftmost is a graph $G = (U, H)$ with subgraph K and the remaining graphs are the subgraphs obtained by the operation $\neg$ and $\lrcorner$. We note, in Fig. 2, that $\neg K$ is the largest subgraph disjoint from K and $\lrcorner\,K$ is the smallest subgraph whose union with K gives all of the underlying graph G.

Definition 12. Given a set $\Gamma \cup \{\, \varphi \,\}$ of formulae and a class $\mathbb{F}$ of frames, φ is a *semantic consequence* of Γ in $\mathbb{F}$ (notation: $\Gamma \models_{\mathbb{F}} \varphi$) if, whenever $M, u \models \gamma$ for all $\gamma \in \Gamma$, it holds that $M, u \models \varphi$ for all models $M = (U, H, R, V)$ such that $(U, H, R) \in \mathbb{F}$ and all states $u \in U$. When Γ is a singleton $\{\psi\}$, we simply write $\psi \models_{\mathbb{F}} \varphi$ instead of $\{\psi\} \models \varphi$. When Γ is empty, we also simply write $\models_{\mathbb{F}} \varphi$ instead of $\varnothing \models_{\mathbb{F}} \varphi$. If $\mathbb{F}$ is the class of all frames, we drop the subscript to write $\Gamma \models \varphi$ provided no confusion arises.

Definition 13. We say that a set Γ of formulae *defines* a class $\mathbb{F}$ of frames if whenever F is a frame, $F \in \mathbb{F}$ iff $F \models \varphi$ for all formulae $\varphi \in \Gamma$. When Γ is a singleton $\{\varphi\}$, we simply say that φ defines a class $\mathbb{F}$.

The following frame definability result appears in [30, Theorem 10].

Proposition 14 ([30])**.** Let $F = (U, H, R)$ be a frame. Let $S_i \in \{R, \smile R\}$ for $i = 1, \ldots, m$ and for each i let

$$\mathsf{B}_i = \begin{cases} \square & \text{if } S_i = R, \\ \blacksquare & \text{if } S_i = \smile R, \end{cases} \quad \text{and let } \mathsf{D}_i = \begin{cases} \blacklozenge & \text{if } S_i = R, \\ \lozenge & \text{if } S_i = \smile R. \end{cases}$$

The following are equivalent for any k where $0 \leqslant k \leqslant m$. (1) $S_1; \cdots; S_k \subseteq S_{k+1}; \cdots; S_m$; (2) $F \models \mathsf{D}_k \cdots \mathsf{D}_1 p \to \mathsf{D}_m \cdots \mathsf{D}_{k+1} p$; (3) $F \models \mathsf{B}_{k+1} \cdots \mathsf{B}_m p \to \mathsf{B}_1 \cdots \mathsf{B}_k p$. In the case that $k = 0$, it should be understood that $S_1; \cdots; S_k = H$ and that $\mathsf{D}_k \cdots \mathsf{D}_1 p$ means p as does $\mathsf{B}_1 \cdots \mathsf{B}_k p$. The special case of $k = m$ is treated similarly.

2.3. Defining the Notion of Hypergraph with **UBiSKt**

Hypergraphs can be regarded as posets (U, H) where only two kinds of elements occur: 0-dimensional elements or nodes, H-incident only with themselves, and 1-dimensional elements or edges, H-incident with nodes and themselves. Given a poset (U, H), this property[1] can be expressed as a constraint on H: $(H \cap \breve{\overline{H}})^2 = \varnothing$. For a generic preorder H, imposing this constraint amounts to requiring that it should not have two consecutive asymmetric relational steps: for any x, y and $z \in U$, if $xHyHz$ holds then at least one of the following must hold: yHx or zHy. When the preorder H is additionally anti-symmetric, so when the symmetric steps are just identity loops, it is clear the constraint limits U to two layers, one of edges and one of nodes.

In what follows we are going to show that there is a formula in **UBiSKt** that corresponds to this essential property of hypergraphs, so that when we restrict our attention to frames where H is also anti-symmetric, the formula is valid in the class of hypergraphs-frames, and only there. The following results are valid for any frame $F = (U, H, R)$ but are independent from the relation R.

Theorem 15. Let $F = (U, H, R)$ be a frame. Then $F \models q \vee (q \to (p \vee \neg p))$ iff $(H \cap \breve{\overline{H}})^2 = \varnothing$.

[1]Later on, with Definition 70, we are going to call this property of a preorder *two-tierness*, following terminology introduced in [25].

Proof. For the right-to-left direction: let us assume that the condition on the frame $F = (U, H, R)$ holds. We prove that the formula is valid. So, let us take any valuation V and any state $u \in U$, and put $M = (F, V)$. Suppose that $M, u \nvDash q$. Our goal is to show that $M, u \models q \to (p \lor \neg p)$. So fix any $v \in U$ such that uHv and $M, v \models q$ and $M, v \nvDash p$. It suffices to show that $M, v \models \neg p$. Furthermore, let us fix any $j \in U$ such that vHj. Our goal is to show that $M, j \nvDash p$. By the condition $(H \cap \overline{\breve{H}})^2 = \varnothing$, we have that i) vHu holds, or ii) jHv holds. In the first case $M, v \models q$ and vHu imply that $M, u \models q$, that contradicts $M, u \nvDash q$. So jHv must be the case. But $M, v \not\models p$ and jHv jointly imply $M, j \not\models p$, as desired. Therefore we have shown that $(H \cap \overline{\breve{H}})^2 = \varnothing$ implies $(F, V), u \models q \lor (q \to (p \lor \neg p))$ for any V and $u \in U$.

For the left-to-right direction, we assume that $F \models q \lor (q \to (p \lor \neg p))$. To prove that $(H \cap \overline{\breve{H}})^2 = \varnothing$, we show that xHy and yHz jointly imply yHx or zHy, for any x, y, $z \in U$. Fix any x, y, $z \in U$ such that xHy and yHz. Moreover, we assume that yHx fails. Then our goal is to show that zHy. Let us define the following valuation V such that $V(p) = \{ u \in U \mid zHu \}$ and $V(q) = \{ v \in U \mid yHv \}$. It is clear that $V(p)$ and $V(q)$ are upsets. Write $M = (F, V)$. By our assumption, we obtain $M, x \models q \lor (q \to (p \lor \neg p))$. Since yHx fails, we have $M, x \not\models q$. Thus $M, x \models q \to (p \lor \neg p)$. By xHy and $M, y \models q$, we get $M, y \models p \lor \neg p$. Since yHz and $z \in V(p)$, we obtain $M, y \not\models \neg p$. It follows from $M, y \models p \lor \neg p$ that $M, y \models p$ hence zHy, as required. ■

Corollary 16. Let $F = (U, H, R)$ be a frame such that H is anti-symmetric, i.e. H is a partial order. Then $F \models q \lor (q \to (p \lor \neg p))$ iff F is a hypergraph.

Proof. The condition $(H \cap \overline{\breve{H}})^2 = \varnothing$ says that for all $x, y, z \in U$, xHy and yHz imply yHx or zHy, so that for all $x, y, z \in U$, if xHy and yHz then $x(H \cap \breve{H})y$ or $y(H \cap \breve{H})z$. When H is anti-symmetric $H \cap \breve{H} = I$, so we have that for all $x, y, z \in U$, if xHy and yHz then $y = x$ or $z = y$, thus (U, H) is a hypergraph by Definition 2. ■

Remark 17. We note that the frame correspondence of the formulae $q \lor (q \to (p \lor \neg p))$ and $p \lor \neg p$ and their generalizations are studied also in [4, 23] in the syntax of intuitionistic logic. It is remarked that a Kripke frame for intuitionistic logic in [4] is a *partial ordering* not a preordering, while a Kripke frame in [23] is a preordering.[2] The novelty of our contribution here consists in interpreting these frame correspondence results in terms of hypergraphs.

[2]It is well known that the intuitionistic logic is sound and complete for both the class of partial orderings and the class of preorderings. This is because we can define a partial ordering (U', H') from a preordering (U, H) as follows: first we define the equivalence relation xH^*y iff xHy and yHx. Then, taking the quotient structure of (U, H) in terms of H^*, we get a partial ordering (U', H'). As for the frame definability, however, there is still a difference. For example, $p \lor \neg p$ defines the symmetry of H within the class of pre-orderings but the same formula defines within the class of partial orderings that the partial order H is simply the identity relation I, as shown in Corollary 19.

Furthermore, as is well known (cf. [23]), the law of excluded middle $p \vee \neg p$ corresponds to the property of H being symmetric. When H is anti-symmetric, the formula corresponds to the property of $H = I$, in the sense of a hypergraph having no edges.

Theorem 18. Let $F = (U, H, R)$ be a frame. Then $F \models p \vee \neg p$ iff $H \subseteq \breve{H}$.

Corollary 19. Let $F = (U, H, R)$ be a frame such that H is anti-symmetric. Then $F \models p \vee \neg p$ iff $H = I$.

Proof. The condition $H \subseteq \breve{H}$ expresses symmetry of H and it is equivalent to $H = H \cap \breve{H}$. When H is a partial order H is also anti-symmetric and we get $H \cap \breve{H} = I$. Therefore for $F = (U, H)$, with H being a partial order, $F \models p \vee \neg p$ iff $H = H \cap \breve{H} = I$. ■

The results show that, by adding the extra assumption of anti-symmetry, by the formula $q \vee (q \to (p \vee \neg p))$ or by the formula $p \vee \neg p$, we can restrict to the class of frames that are hypergraphs or the class consisting of sets respectively.

3. Axiomatizing the Logic UBiSKt

3.1. *Hilbert System of Bi-intuitionistic Stable Tense Logic with Universal Modalities*

Table 1 provides the Hilbert system H**UBiSKt**. Roughly speaking, it is a bi-intuitionistic tense analogue of a Hilbert system for the ordinary modal logic with the universal modalities [11] (see also [2, p.417]). It consists of all the axioms and inference rules of the bi-intuitionistic stable tense logic [24] as well as new axioms and inference rules for the universal modalities A and E. Let us comment on the new axioms and rules. Axioms (`A14`) and (`A15`) and rules (`MonA`) and (`MonE`) are required to capture the adjunction $\mathsf{E} \dashv \mathsf{A}$, i.e., $\vdash \mathsf{E}\,\varphi \to \psi$ iff $\vdash \varphi \to \mathsf{A}\,\psi$ for all formulas φ and ψ (this is established in Proposition 24(15)). Axioms (`A16`) and (`A17`) are the T-axiom and the 4-axiom for A respectively. Axioms (`A18`) and (`A19`) describe how A interacts with E. Axioms (`A20`), (`A21`) and (`A22`) explain how A interacts with $\square$, $\blacklozenge$ and $\prec$, respectively.

Remark 20. It is noted that our axiomatization for the bi-intuitionistic part is much simpler than Rauszer's axiomatization [21]. A recent study [12] on bi-intuitionistic logic revealed that Rauszer's proofs for the semantic completeness of her axiomatization were faulty. The study also noted [12, p.280] that [24] provided the first correct proof of the strong completeness of bi-intuitionistic logic.

Definition 21. We say that a set Λ of formulae is a *bi-intuitionistic stable tense logic with universal modalities* (for short, *ubist*-logic) if Λ contains all the axioms of Table 1 and is closed under all the rules of Table 1. Given a *ubist*-logic Λ and a set $\Gamma \cup \{\varphi\}$ of formulae, we say that φ is Λ-*provable* from Γ (notation: $\Gamma \vdash_\Lambda \varphi$) if there is a finite set $\Gamma' \subseteq \Gamma$ such that $\bigwedge \Gamma' \to \varphi \in \Lambda$,

Table 1: Hilbert System **HUBiSKt**

Axioms and Rules for Intuitionistic Logic			
(A0)	$p \to (q \to p)$		
(A1)	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$		
(A2)	$p \to (p \vee q)$	(A3)	$q \to (p \vee q)$
(A4)	$(p \to r) \to ((q \to r) \to (p \vee q \to r))$	(A5)	$(p \wedge q) \to p$
(A6)	$(p \wedge q) \to q$	(A7)	$(p \to (q \to p \wedge q))$
(A8)	$\bot \to p$	(A9)	$p \to \top$
(MP)	From φ and $\varphi \to \psi$, infer ψ		
(US)	From φ, infer a substitution instance φ' of φ		
Additional Axioms and Rules for Bi-intuitionistic Logic			
(A10)	$p \to (q \vee (p \prec q))$	(A11)	$((q \vee r) \prec q) \to r$
(Mon-$\prec$)	From $\delta_1 \to \delta_2$, infer $(\delta_1 \prec \psi) \to (\delta_2 \prec \psi)$		
Additional Axioms and Rules for Tense Operators			
(A12)	$p \to \Box\blacklozenge p$	(A13)	$\blacklozenge\Box p \to p$
(Mon$\Box$)	From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$	(Mon$\blacklozenge$)	From $\varphi \to \psi$, infer $\blacklozenge\varphi \to \blacklozenge\psi$
Additional Axioms and Rules for Universal Modalities			
(A14)	$p \to \mathsf{A}\,\mathsf{E}\,p$	(A15)	$\mathsf{E}\,\mathsf{A}\,p \to p$
(A16)	$\mathsf{A}\,p \to p$	(A17)	$\mathsf{A}\,p \to \mathsf{A}\,\mathsf{A}\,p$
(A18)	$\mathsf{A}\,\neg p \leftrightarrow \neg\,\mathsf{E}\,p$	(A19)	$(\mathsf{A}\,p \wedge \mathsf{E}\,q) \to \mathsf{E}(p \wedge q)$
(A20)	$\mathsf{A}\,p \to \Box p$	(A21)	$(\mathsf{A}\,p \wedge \blacklozenge q) \to \blacklozenge(p \wedge q)$
(A22)	$(\mathsf{A}\,p \wedge (q \prec r)) \to ((p \wedge q) \prec r)$		
(Mon A)	From $\varphi \to \psi$, infer $\mathsf{A}\,\varphi \to \mathsf{A}\,\psi$	(Mon E)	From $\varphi \to \psi$, infer $\mathsf{E}\,\varphi \to \mathsf{E}\,\psi$

where $\bigwedge \Delta$ is the conjunction of all elements of Δ and $\bigwedge \Delta := \top$ when Δ is empty. Moreover, when $\Gamma = \varnothing$ we simply write $\vdash_\Lambda \varphi$ instead of $\varnothing \vdash_\Lambda \varphi$. Note that $\vdash_\Lambda \varphi$ is equivalent to $\varphi \in \Lambda$. It is remarked that the intersection $\bigcap_{i \in I} \Lambda_i$ of a family $\{\, \Lambda_i \,\}_{i \in I}$ of *ubist*-logics is also a *ubist*-logic. We define **UBiSKt** as the smallest *ubist*-logic $\bigcap \{\, \Lambda \mid \Lambda$ is a *ubist*-logic $\}$. Given a set Σ of formulae, the smallest *ubist*-logic **UBiSKt**Σ containing Σ is defined by: **UBiSKt**$\Sigma :=$ $\bigcap \{\, \Lambda \mid \Lambda$ is a *ubist*-logic and $\Sigma \subseteq \Lambda \,\}$.

In what follows in this paper, we assume that the reader is familiar with theorems and derived inference rules in intuitionistic logic (even if the reader is not familiar with a Hilbert-style axiomatization of intuitionistic logic, it is possible to verify theorems and derived rules in a natural deduction calculus where the rule of reductio ad absurdum cannot be used, i.e. if we assume $\neg\varphi$ to get $\bot$, then we can conclude φ). For the reader's convenience, we list some classical validities that are *non-theorems* of intuitionistic logic: $\varphi \vee \neg\varphi$, $\neg\neg\varphi \to \varphi$, $(\varphi \to \psi) \to (\neg\varphi \vee \psi)$, $\neg(\varphi \wedge \psi) \to (\neg\varphi \vee \neg\psi)$, $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$, etc. The reader cannot rely on these formulas to reason within **UBiSKt**. Finally, let us demonstrate that $\top$ is a theorem in terms of axioms and rules of intuitionistic logic. By (A8) and (US), we obtain $\bot \to \bot$ is a theorem of **UBiSKt**. It follows from axiom (A9) and (US) that $(\bot \to \bot) \to \top$ is a theorem of **UBiSKt**. Finally, we use (MP) to conclude $\top$ is a theorem of **UBiSKt**.

Theorem 22 (Soundness)**.** Given any formula φ, $\vdash_{\mathbf{UBiSKt}} \varphi$ implies $\models \varphi$.

Proof. Since **UBiSKt** *without* the universal modalities is already known to be sound [24], we focus on some of the new axioms and rules. Let $M = (U, H, R, V)$ be a model. The validity of axioms (A19), (A20) and (A22) is shown by the fact that $M, x \models p$ for every $x \in U$ implies $\llbracket \mathsf{A}\, p \rrbracket_M = U$. Let us check the validity of (A18) in detail. To show $\models \mathsf{A}\, \neg p \leftrightarrow \neg\, \mathsf{E}\, p$, it suffices to show $\mathsf{A}\, \neg p \models \neg\, \mathsf{E}\, p$ and $\neg\, \mathsf{E}\, p \models \mathsf{A}\, \neg p$. Firstly, fix any $x \in U$ such that $M, x \models \mathsf{A}\, \neg p$, which implies $\llbracket \neg p \rrbracket_M = U$. To show $M, x \models \neg\, \mathsf{E}\, p$, fix any $y \in U$ such that xHy. Our goal is to show $M, y \not\models \mathsf{E}\, p$, i.e., $V(p) = \varnothing$. But this is an easy consequence from $\llbracket \neg p \rrbracket_M = U$. Secondly, assume that $M, x \models \neg\, \mathsf{E}\, p$. Then $M, x \not\models \mathsf{E}\, p$ by xHx. This implies $V(p) = \varnothing$. To show $M, x \models \mathsf{A}\, \neg p$, fix any $y \in U$. Our goal is to establish $M, y \models \neg p$. But this is easy from $V(p) = \varnothing$. ■

By axioms (A12) and (A13) and rules (Mon$\Box$) and (Mon$\blacklozenge$), we can establish the following equivalence, i.e., an adjunction "$\blacklozenge \dashv \Box$" (see [24, Proposition 6]).

Proposition 23 ([24])**.** For any *ubist*-logic Λ, $\vdash \blacklozenge\, \varphi \to \psi$ iff $\vdash \varphi \to \Box \psi$.

Our proof of the following proposition can be found in Appendix A.

Proposition 24. All the following hold for any *ubist*-logic.

1. $\vdash \neg \bigvee_{j \in J} \varphi_j \leftrightarrow \bigwedge_{j \in J} \neg \varphi_j$.
2. $\vdash (\psi \prec \gamma) \to \rho$ iff $\vdash \psi \to (\gamma \vee \rho)$.
3. If $\vdash \varphi \leftrightarrow \psi$ then $\vdash (\gamma \prec \varphi) \leftrightarrow (\gamma \prec \psi)$.
4. $\vdash \neg(\varphi \prec \varphi)$.
5. $\vdash \varphi \vee \lrcorner\, \varphi$.
6. $\vdash \lrcorner\, \lrcorner\, \varphi \to \varphi$.
7. $\vdash \neg \varphi \to \lrcorner\, \varphi$.
8. $\vdash \varphi \to \neg \psi$ iff $\vdash \psi \to \neg \varphi$.
9. $\vdash \lrcorner\, \varphi \to \psi$ iff $\vdash \lrcorner\, \psi \to \varphi$.
10. $\vdash \lrcorner\, \neg \varphi \to \psi$ iff $\vdash \varphi \to \neg\, \lrcorner\, \psi$.
11. $\vdash \varphi \to \lrcorner\, \neg \varphi$.
12. $\vdash \neg\, \lrcorner\, \varphi \to \varphi$.
13. If $\vdash \varphi \to \psi$ then $\vdash \lrcorner\, \psi \to \lrcorner\, \varphi$.
14. $\vdash \lrcorner(\varphi \wedge \lrcorner\, \varphi)$.
15. $\vdash \mathsf{E}\, \varphi \to \psi$ iff $\vdash \varphi \to \mathsf{A}\, \psi$.
16. $\vdash \varphi \to \mathsf{E}\, \varphi$.
17. $\vdash \mathsf{E}\, \mathsf{E}\, \varphi \to \mathsf{E}\, \varphi$.
18. $\vdash \mathsf{A}\, \mathsf{E}\, \varphi \leftrightarrow \mathsf{E}\, \varphi$.
19. $\vdash \mathsf{A}\, \neg\, \mathsf{A}\, \varphi \leftrightarrow \neg\, \mathsf{A}\, \varphi$.
20. $\vdash (\mathsf{A}\, \varphi \wedge \Box \psi) \to \Box(\varphi \wedge \psi)$.
21. $\vdash \mathsf{A}\, \varphi \to \Box\, \mathsf{A}\, \varphi$.
22. $\vdash \blacklozenge\, \mathsf{A}\, \varphi \to \mathsf{A}\, \varphi$.
23. $\vdash \neg\, \mathsf{A}\, \varphi \leftrightarrow \lrcorner\, \mathsf{A}\, \varphi$.
24. $\vdash \mathsf{A}\, \varphi \vee \neg\, \mathsf{A}\, \varphi$.
25. $\vdash \neg\, \mathsf{E}\, \varphi \leftrightarrow \lrcorner\, \mathsf{E}\, \varphi$.
26. $\vdash \mathsf{E}\, \varphi \vee \neg\, \mathsf{E}\, \varphi$.
27. $\vdash \mathsf{E}\, \varphi \leftrightarrow \neg\, \mathsf{A}\, \neg \varphi$.
28. $\vdash \mathsf{A}(\lrcorner\, \varphi \to \psi) \leftrightarrow \mathsf{A}(\lrcorner\, \psi \to \varphi)$.
29. $\vdash \mathsf{E}(\lrcorner\, \neg \varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \lrcorner\, \neg \psi)$.

We note the following, generally known as a "replacement theorem".

Proposition 25. Let Λ be any *ubist*-logic, φ and ψ be formulas and let χ be a formula containing φ as a subformula. Let χ' be the result of replacing some occurrences of φ in χ by ψ. Suppose that $\vdash_\Lambda \varphi \leftrightarrow \psi$. Then $\vdash_\Lambda \chi \leftrightarrow \chi'$.

Proof. The replacement theorem is well known as a standard result in intuitionistic logic and in normal modal logics. It is proved by induction on the structure of χ. For a *ubist*-logic Λ we additionally need to justify that $\vdash_\Lambda \delta_1 \leftrightarrow \delta_2$ implies both (1) $\vdash_\Lambda (\delta_1 \prec \rho) \leftrightarrow (\delta_2 \prec \rho)$ and also (2) $\vdash_\Lambda (\rho \prec \delta_1) \leftrightarrow (\rho \prec \delta_2)$. The first of these follows from the rule ($\mathtt{Mon}\text{-}\prec$). The second is item 3 of Proposition 24. The monotonicity rules for the universal modalities are then all that is needed to complete the proof. ■

3.2. Strong Completeness of **UBiSKt** *via a Canonical Model*

Given a finite set Δ of sets of formulae, $\bigvee \Delta$ is defined as the disjunction of all formulae in Δ, where $\bigvee \varnothing$ is understood as $\bot$.

Definition 26. Let Λ be a *ubist*-logic. A pair (Γ, Δ) of formulae is Λ-*provable* if $\Gamma \vdash_\Lambda \bigvee \Delta'$ for some finite $\Delta' \subseteq \Delta$. We say that a pair (Γ, Δ) of formulae is Λ-*unprovable* if it is not Λ-provable. A pair (Γ, Δ) is *complete* if $\Gamma \cup \Delta = \mathsf{Form}_{\mathcal{L}}$, i.e., $\varphi \in \Gamma$ or $\varphi \in \Delta$ for all formulae φ.

We remark that a pair (Γ, Δ) is Λ-unprovable iff $\nvdash_\Lambda \bigwedge \Gamma' \to \bigvee \Delta'$ for all finite $\Gamma' \subseteq \Gamma$ and all finite $\Delta' \subseteq \Delta$. The following two lemmas hold because Λ 'contains' intuitionistic logic.

Lemma 27. Let Λ be a *ubist*-logic and (Γ, Δ) a complete and Λ-unprovable pair. Then,

1. ($\Gamma \vdash_\Lambda \varphi$ implies $\varphi \in \Gamma$) for all formulae φ.
2. $\Lambda \subseteq \Gamma$.
3. If $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$.
4. $\bot \notin \Gamma$.
5. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$.
6. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Lemma 28. Let Λ be a *ubist*-logic. Given a Λ-unprovable pair (Γ, Δ), there exists a complete and Λ-unprovable pair (Γ^+, Δ^+) such that $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^+$.

Definition 29. Let Λ be a *ubist*-logic and (Γ, Δ) be a pair which is complete and Λ-unprovable. The Λ-canonical model $M^\Lambda_{(\Gamma,\Delta)} = (U^\Lambda, H^\Lambda, R^\Lambda, V^\Lambda)$ is defined as:

- $U^\Lambda := \{ (\Sigma, \Theta) \mid (\Sigma, \Theta) \text{ is complete, } \Lambda\text{-unprovable, and } (\Gamma, \Delta) S^\Lambda (\Sigma, \Theta) \}$ where the relation S^Λ is defined as:

$$(\Gamma, \Delta) S^\Lambda (\Sigma, \Theta) \quad \text{iff} \quad (\mathsf{A}\,\varphi \in \Gamma \text{ iff } \mathsf{A}\,\varphi \in \Sigma) \text{ for all formulae } \varphi.$$

- $(\Sigma_1, \Theta_1) H^\Lambda (\Sigma_2, \Theta_2)$ iff $\Sigma_1 \subseteq \Sigma_2$.
- $(\Sigma_1, \Theta_1) R^\Lambda (\Sigma_2, \Theta_2)$ iff ($\Box\varphi \in \Sigma_1$ implies $\varphi \in \Sigma_2$) for all formulae φ.
- $(\Sigma, \Theta) \in V^\Lambda(p)$ iff $p \in \Sigma$.

Let $F^\Lambda_{(\Gamma,\Delta)} = (U^\Lambda, H^\Lambda, R^\Lambda)$ be the Λ-canonical frame.

It is clear that H^Λ is not merely a pre-order but also a partial order. Moreover, $(\Sigma_1, \Theta_1) H^\Lambda (\Sigma_2, \Theta_2)$ implies $\Theta_2 \subseteq \Theta_1$ by completeness. We also note that the relation S^Λ in the definition of U^Λ is symmetric and transitive and is shown to be reflexive by definition and Lemma 27. Therefore, S^Λ is an equivalence relation. In this sense, U^Λ can be regarded as an S^Λ-equivalence class of the pair (Γ, Δ). The domain restriction to the S^Λ-equivalence class of (Γ, Δ) makes the universal modalities A and E behave as "real" universal modalities in our canonical model.

Lemma 30. Let $(\Sigma_i, \Theta_i) \in U^\Lambda$ ($i = 1$ or 2). The following are all equivalent:

1. $(\Sigma_1, \Theta_1) R^\Lambda (\Sigma_2, \Theta_2)$.
2. ($\varphi \in \Theta_2$ implies $\Box\varphi \in \Theta_1$) for all formulae φ.
3. ($\varphi \in \Sigma_1$ implies $\blacklozenge\varphi \in \Sigma_2$) for all formulae φ.
4. ($\blacklozenge\varphi \in \Theta_2$ implies $\varphi \in \Theta_1$) for all formulae φ.

Lemma 31. R^Λ is stable in the Λ-canonical model $M^\Lambda_{(\Gamma,\Delta)}$.

Proof. It is easy to see that V^Λ is a valuation. So, we prove that R^Λ is stable. Suppose that $(\Sigma_1, \Theta_1) H^\Lambda (\Sigma_2, \Theta_2) R^\Lambda (\Sigma_3, \Theta_3) H^\Lambda (\Sigma_4, \Theta_4)$. Our goal is to establish $(\Sigma_1, \Theta_1) R^\Lambda (\Sigma_4, \Theta_4)$. Let us fix any formula φ such that $\Box\varphi \in \Sigma_1$. We establish that $\varphi \in \Sigma_4$. By $(\Sigma_1, \Theta_1) H^\Lambda (\Sigma_2, \Theta_2)$, we get $\Box\varphi \in \Sigma_2$. It follows from $(\Sigma_2, \Theta_2) R^\Lambda (\Sigma_3, \Theta_3)$ that $\varphi \in \Sigma_3$. Finally, we conclude from $(\Sigma_3, \Theta_3) H^\Lambda (\Sigma_4, \Theta_4)$ that $\varphi \in \Sigma_4$, as desired. ■

Lemma 32. Let (Γ, Δ) and (Σ, Θ) be complete Λ-unprovable pairs such that $(\Gamma, \Delta) S^\Lambda (\Sigma, \Theta)$.

1. If $\psi \to \rho \notin \Sigma$, then $(\{\psi\} \cup \Sigma \cup \{\mathsf{A}\gamma \mid \mathsf{A}\gamma \in \Gamma\}, \{\rho\} \cup \{\mathsf{A}\delta \mid \mathsf{A}\delta \in \Delta\})$ is Λ-unprovable.
2. If $\psi \prec \rho \in \Sigma$, then $(\{\psi\} \cup \{\mathsf{A}\gamma \mid \mathsf{A}\gamma \in \Gamma\}, \{\rho\} \cup \Theta \cup \{\mathsf{A}\delta \mid \mathsf{A}\delta \in \Delta\})$ is Λ-unprovable.
3. If $\Box\psi \notin \Sigma$, then $(\{\sigma \mid \Box\sigma \in \Sigma\} \cup \{\mathsf{A}\gamma \mid \mathsf{A}\gamma \in \Gamma\}, \{\psi\} \cup \{\mathsf{A}\delta \mid \mathsf{A}\delta \in \Delta\})$ is Λ-unprovable.
4. If $\blacklozenge\psi \in \Sigma$, then $(\{\psi\} \cup \{\mathsf{A}\gamma \mid \mathsf{A}\gamma \in \Gamma\}, \{\theta \mid \blacklozenge\theta \in \Theta\} \cup \{\mathsf{A}\delta \mid \mathsf{A}\delta \in \Delta\})$ is Λ-unprovable.

5. If $\mathsf{A}\,\psi \notin \Sigma$, then $(\{\,\mathsf{A}\,\gamma \,|\, \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\psi\,\} \cup \{\,\mathsf{A}\,\delta \,|\, \mathsf{A}\,\delta \in \Delta\,\})$ is Λ-unprovable.

6. If $\mathsf{E}\,\psi \in \Sigma$, then $(\{\,\psi\,\} \cup \{\,\mathsf{A}\,\gamma \,|\, \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\mathsf{A}\,\delta \,|\, \mathsf{A}\,\delta \in \Delta\,\})$ is Λ-unprovable.

The proof of Lemma 32 can be found in Appendix B.

Lemma 33 (Truth Lemma)**.** Let Λ be a *ubist*-logic and (Γ, Δ) be a complete and Λ-unprovable pair. Then, for any formula φ and any complete Λ-unprovable pair (Σ, Θ), in the model generated by (Γ, Δ), the following equivalence holds:

$$\varphi \in \Sigma \text{ iff } M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma, \Theta) \models \varphi.$$

Proof. By induction on φ. When φ is a propositional variable from Prop, it is immediate from the definition of V^{Λ}. When φ is of the form $\psi \wedge \rho$ or $\psi \vee \rho$, we can establish the equivalence by Lemma 27 and the induction hypothesis. We now deal with the remaining cases, where φ is of the form $\psi \to \rho$, $\psi \prec \rho$, $\blacklozenge\,\psi$, $\Box\psi$, $\mathsf{E}\,\psi$ or $\mathsf{A}\,\psi$.

- Let φ be of the form $\psi \to \rho$. Fix any $(\Sigma_1, \Theta_1) \in U^{\Lambda}$. For the right-to-left direction, we prove the contrapositive implication. Assume that $\psi \to \rho \notin \Sigma_1$. By Lemma 32(1), we know that the pair

  $$(\{\,\psi\,\} \cup \Sigma_1 \cup \{\,\mathsf{A}\,\gamma \,|\, \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\rho\,\} \cup \{\,\mathsf{A}\,\delta \,|\, \mathsf{A}\,\delta \in \Delta\,\})$$

  is Λ-unprovable. It follows from Lemma 28 that the above pair can be extended to a Λ-unprovable and complete pair (Σ_2, Θ_2). We can be sure that $(\Sigma_2, \Theta_2) \in U^{\Lambda}$ by construction. Since $\psi \in \Sigma_2$, $\rho \in \Theta_2$ and $(\Sigma_1, \Theta_1) H^{\Lambda} (\Sigma_2, \Theta_2)$ hold, we get $M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma_1, \Theta_1) \not\models \psi \to \rho$ from the induction hypothesis.

  Next, we prove the left-to-right direction. Suppose that $\psi \to \rho \in \Sigma_1$. To show that $M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma_1, \Theta_1) \models \psi \to \rho$, fix any $(\Sigma_2, \Theta_2) \in U^{\Lambda}$ such that $\Sigma_1 \subseteq \Sigma_2$ and $M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma_2, \Theta_2) \models \psi$, which implies $\psi \in \Sigma_2$ by the induction hypothesis. Our goal is to show that $M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma_2, \Theta_2) \models \rho$. By induction hypothesis, it suffices to show $\rho \in \Sigma_2$. Since $\psi \to \rho \in \Sigma_1 \subseteq \Sigma_2$, we have $\psi \to \rho \in \Sigma_2$. Since $\psi \in \Sigma_2$, we can derive $\rho \in \Sigma_2$, as desired.

- Let φ be of the form $\psi \prec \rho$. Fix any $(\Sigma_1, \Theta_1) \in U^{\Lambda}$. The left-to-right direction is established by Lemma 32(2) and Lemma 28 with the help of the induction hypothesis. So, we focus on the right-to-left direction. Suppose that $M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma_1, \Theta_1) \models \psi \prec \rho$, i.e., we can find a pair $(\Sigma_2, \Theta_2) \in U^{\Lambda}$ such that $\Theta_1 \subseteq \Theta_2$ and $M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma_2, \Theta_2) \models \psi$ and $M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma_2, \Theta_2) \not\models \rho$. By induction hypothesis, we obtain $\psi \in \Sigma_2$ and $\rho \notin \Sigma_2$, i.e., $\rho \in \Theta_2$. Suppose for contradiction that $\psi \prec \rho \notin \Sigma_1$, i.e., $\psi \prec \rho \in \Theta_1$. Since $\Theta_1 \subseteq \Theta_2$, we get $\psi \prec \rho \in \Theta_2$. Since $\psi \vdash_{\Lambda} \rho, \psi \prec \rho$ holds, it implies that (Σ_2, Θ_2) is Λ-provable, which is the desired contradiction with Λ-*un*provability of (Σ_2, Θ_2).

- Let φ be of the form $\blacklozenge\psi$. Fix any $(\Sigma_1, \Theta_1) \in U^\Lambda$. The left-to-right direction is established by Lemma 32(4), Lemma 30 and Lemma 28 with the help of the induction hypothesis. So, we focus on the right-to-left direction. Suppose that $M^\Lambda_{(\Gamma,\Delta)}, (\Sigma_1, \Theta_1) \models \blacklozenge\psi$, i.e., we can find a pair $(\Sigma_2, \Theta_2) \in U^\Lambda$ such that $(\Sigma_2, \Theta_2) R^\Lambda (\Sigma_1, \Theta_1)$ and $M^\Lambda_{(\Gamma,\Delta)}, (\Sigma_2, \Theta_2) \models \psi$. By the induction hypothesis, $\psi \in \Sigma_2$. By $(\Sigma_2, \Theta_2) R^\Lambda (\Sigma_1, \Theta_1)$ and Lemma 30, we obtain $\blacklozenge\psi \in \Sigma_1$, as desired.

- Let φ be of the form $\Box\psi$. Fix any $(\Sigma_1, \Theta_1) \in U^\Lambda$. The right-to-left direction is established by Lemma 32(3) and Lemma 28 with the help of the induction hypothesis. So, we prove the left-to-right direction. Suppose that $\Box\psi \in \Sigma_1$. To show $M^\Lambda_{(\Gamma,\Delta)}, (\Sigma_1, \Theta_1) \models \Box\psi$, let us fix any pair $(\Sigma_2, \Theta_2) \in U^\Lambda$ such that $(\Sigma_1, \Theta_1) R^\Lambda (\Sigma_2, \Theta_2)$. We show that $M^\Lambda_{(\Gamma,\Delta)}, (\Sigma_2, \Theta_2) \models \psi$, i.e., $\psi \in \Sigma_2$ by the induction hypothesis. By definition of R^Λ and $\Box\psi \in \Sigma_1$, we obtain $\psi \in \Sigma_1$, as desired.

- Let φ be of the form $\mathsf{A}\psi$. Fix any $(\Sigma_1, \Theta_1) \in U^\Lambda$. The right-to-left direction is established by Lemma 32(5) and Lemma 28 with the help of the induction hypothesis. So, we prove the left-to-right direction. Suppose that $\mathsf{A}\psi \in \Sigma_1$. To show $M^\Lambda_{(\Gamma,\Delta)}, (\Sigma_1, \Theta_1) \models \mathsf{A}\psi$, let us fix any pair $(\Sigma_2, \Theta_2) \in U^\Lambda$. By induction hypothesis, it suffices to show $\psi \in \Sigma_2$. By $(\Sigma_1, \Theta_1), (\Sigma_2, \Theta_2) \in U^\Lambda$, we have $(\Gamma, \Delta) S^\Lambda (\Sigma_1, \Theta_1)$ and $(\Gamma, \Delta) S^\Lambda (\Sigma_2, \Theta_2)$. It follows from $\mathsf{A}\psi \in \Sigma_1$ that $\mathsf{A}\psi \in \Sigma_2$, which implies $\psi \in \Sigma_2$ by the axiom (`A16`) (T-axiom for A).

- Let φ be of the form $\mathsf{E}\psi$. Fix any $(\Sigma_1, \Theta_1) \in U^\Lambda$. The left-to-right direction is established by Lemma 32(6) and Lemma 28 with the help of the induction hypothesis. So, we focus on the right-to-left direction. Suppose that $M^\Lambda_{(\Gamma,\Delta)}, (\Sigma_1, \Theta_1) \models \mathsf{E}\psi$, i.e., we can find a pair $(\Sigma_2, \Theta_2) \in U^\Lambda$ such that $M^\Lambda_{(\Gamma,\Delta)}, (\Sigma_2, \Theta_2) \models \psi$. By induction hypothesis, $\psi \in \Sigma_2$. By the axiom (`A14`) and $\psi \in \Sigma_2$, $\mathsf{A}\mathsf{E}\psi \in \Sigma_2$. By $(\Sigma_1, \Theta_1), (\Sigma_2, \Theta_2) \in U^\Lambda$, we have $(\Gamma, \Delta) S^\Lambda (\Sigma_1, \Theta_1)$ and $(\Gamma, \Delta) S^\Lambda (\Sigma_2, \Theta_2)$. It follows from $\mathsf{A}\mathsf{E}\psi \in \Sigma_2$ that $\mathsf{A}\mathsf{E}\psi \in \Sigma_1$, which implies $\mathsf{E}\psi \in \Sigma_1$ by the axiom (`A16`) (T-axiom for A). ■

Theorem 34 (Strong Completeness of **UBiSKt**)**.** If $\Gamma \models \varphi$ then $\Gamma \vdash_{\mathbf{UBiSKt}} \varphi$, for every set $\Gamma \cup \{\varphi\}$ of formulae.

Proof. Put $\Lambda := \mathbf{UBiSKt}$. Fix any set $\Gamma \cup \{\varphi\}$ of formulae. We prove the contrapositive implication and so assume that $\Gamma \nvdash_\Lambda \varphi$. It follows that $(\Gamma, \{\varphi\})$ is Λ-unprovable. By Lemma 28, we can find a complete and Λ-unprovable pair $(\Sigma, \Theta) \in U^\Lambda$ such that $\Gamma \subseteq \Sigma$ and $\varphi \in \Theta$. By Lemma 33 (Truth Lemma), $M^\Lambda_{(\Sigma,\Theta)}, (\Sigma, \Theta) \models \gamma$ for all $\gamma \in \Gamma$ and $M^\Lambda_{(\Sigma,\Theta)}, (\Sigma, \Theta) \not\models \varphi$. Since M^Λ is a model by Lemma 31, we can conclude $\Gamma \not\models \varphi$, as desired. ■

Definition 35. We define $\mathbb{HG}$ as the class of all frames (U, H, R) such that (U, H) is a hypergraph, and let $\mathbf{bd}_2$ be the formula $q \vee (q \to (p \vee \neg p))$.

Lemma 36. Given a *ubist*-logic Λ such that $\mathbf{bd}_2 \in \Lambda$, the Λ-canonical frame $F^{\Lambda}_{(\Gamma,\Delta)} = (U^{\Lambda}, H^{\Lambda}, R^{\Lambda})$ satisfies $(H^{\Lambda} \cap \overline{\breve{H}^{\Lambda}})^2 = \varnothing$, i.e., $F^{\Lambda}_{(\Gamma,\Delta)} \in \mathbb{HG}$.

Proof. We prove $(\Gamma_1, \Delta_1)H^{\Lambda}(\Gamma_2, \Delta_2)H^{\Lambda}(\Gamma_3, \Delta_3)$ implies $(\Gamma_2, \Delta_2)H^{\Lambda}(\Gamma_1, \Delta_1)$ or $(\Gamma_3, \Delta_3)H^{\Lambda}(\Gamma_2, \Delta_2)$. Suppose that $(\Gamma_1, \Delta_1)H^{\Lambda}(\Gamma_2, \Delta_2)H^{\Lambda}(\Gamma_3, \Delta_3)$ and assume that $(\Gamma_2, \Delta_2)H^{\Lambda}(\Gamma_1, \Delta_1)$ fails, i.e., $\Gamma_2 \not\subseteq \Gamma_1$. To show that $(\Gamma_3, \Delta_3)H^{\Lambda}(\Gamma_2, \Delta_2)$, let us suppose that $\varphi \in \Gamma_3$. Our goal is to establish $\varphi \in \Gamma_2$. Since $\Gamma_2 \not\subseteq \Gamma_1$, there exists a formula ψ such that $\psi \in \Gamma_2$ but $\psi \notin \Gamma_1$. By $\mathbf{bd}_2 \in \Lambda$, $\psi \vee (\psi \to (\varphi \vee \neg\varphi)) \in \Gamma_1$ hence $\psi \to (\varphi \vee \neg\varphi) \in \Gamma_1$ by $\psi \notin \Gamma_1$. It follows from $\psi \in \Gamma_2$ that $\varphi \vee \neg\varphi \in \Gamma_2$. Since $\varphi \in \Gamma_3$ and $(\Gamma_2, \Delta_2)H^{\Lambda}(\Gamma_3, \Delta_3)$, we obtain $\neg\varphi \notin \Gamma_2$. Therefore, we deduce from $\varphi \vee \neg\varphi \in \Gamma_2$ that $\varphi \in \Gamma_2$, as desired. Because H^{Λ} is antisymmetric, we conclude that $F^{\Lambda}_{(\Gamma,\Delta)} \in \mathbb{HG}$. ■

By Lemmas 33 and 36 and Theorem 15, we can establish the following.

Theorem 37. If $\Gamma \models_{\mathbb{HG}} \varphi$ then $\Gamma \vdash_{\mathbf{UBiSKtbd_2}} \varphi$, for every set $\Gamma \cup \{\varphi\}$ of formulae.

3.3. Finite Model Property and Decidability of **UBiSKt**

This section shows that a technique employed in [24] for **BiSKt** also works for **UBiSKt** to establish the finite model property and decidability of **UBiSKt**.

Let $M = (U, H, R, V)$ be a model and Δ a subformula closed set of formulae. We define an equivalence relation $\sim_\Delta$ by

$$x \sim_\Delta y \text{ iff } (M, x \models \varphi \text{ iff } M, y \models \varphi) \text{ for all } \varphi \in \Delta.$$

When $x \sim_\Delta y$ holds, we say that x and y are Δ-equivalent. We use $[x]$ to mean the equivalence class $\{\, y \in U \mid x \sim_\Delta y \,\}$ of $x \in U$.

Definition 38 (Filtration [24])**.** We say that a model $M_\Delta = (U_\Delta, H_\Delta, R_\Delta, V_\Delta)$ is a *filtration* of a model $M = (U, H, R, V)$ through a subformula closed set Δ of formulae if the following seven conditions are satisfied.

1. $U_\Delta = \{\, [x] \mid x \in U \,\}$.
2. For all $x, y \in U$, if xHy then $[x]H_\Delta[y]$.
3. For all $x, y \in U$ and $\varphi \in \Delta$, if $[x]H_\Delta[y]$ and $M, x \models \varphi$ then $M, y \models \varphi$.
4. For all $x, y \in U$, if xRy then $[x]R_\Delta[y]$.
5. For all $x, y \in U$ and $\Box\varphi \in \Delta$, if $[x]R_\Delta[y]$ and $M, x \models \Box\varphi$ then $M, y \models \varphi$.
6. For all $x, y \in U$ and $\blacklozenge\varphi \in \Delta$, if $[x]R_\Delta[y]$ and $M, x \models \varphi$ then $M, y \models \blacklozenge\varphi$.
7. $V_\Delta(p) = \{\, [x] \mid x \in V(p) \,\}$ for all $p \in \Delta$.

When Δ is finite, we note that U_Δ is also finite.

Proposition 39. Let $M_\Delta = (U_\Delta, H_\Delta, R_\Delta, V_\Delta)$ be a filtration of a model $M = (U, H, R, V)$ through a subformula closed set Δ of formulae. Then for every $x \in U$ and every $\varphi \in \Delta$, the following equivalence holds: $M, x \models \varphi$ iff $M_\Delta, [x] \models \varphi$.

Proof. We only show the case where φ is of the form $\mathsf{A}\,\psi$, since an argument for the case where φ is of the form $\mathsf{E}\,\psi$ is similar and arguments for the remaining cases are similarly done for **BiSKt** [24]. Note that $\psi \in \Delta$. We proceed as follows: $M, x \models \mathsf{A}\,\psi$ iff $M, y \models \psi$ for all $y \in U$ iff $M_\Delta, [y] \models \psi$ for all $y \in U$ by $\psi \in \Delta$ and induction hypothesis. The last statement is also equivalent with $M_\Delta, [x] \models \mathsf{A}\,\psi$, as desired. ■

Filtrations always exist as demonstrated by the following definition and proposition from [24] which provide a specific example of a filtration for any frame. We remark that the filtration in Definition 40 is called the *finest filtration* and is shown to be the smallest filtration in [14].

Definition 40 ([24])**.** Given a frame $F = (U, H, R)$ and a subformula closed set Δ, $\underline{H}_\Delta$ and $\underline{R}_\Delta$ are defined by:

$$\begin{array}{lll} [x]\underline{H}_\Delta[y] & \text{iff} & x'Hy' \text{ for some } x' \in [x] \text{ and some } y' \in [y], \\ [x]\underline{R}_\Delta[y] & \text{iff} & x'Ry' \text{ for some } x' \in [x] \text{ and some } y' \in [y]. \end{array}$$

Put $\underline{R}^s_\Delta := \underline{H}^+_\Delta \,;\, \underline{R}_\Delta \,;\, \underline{H}^+_\Delta$ where X^+ is the transitive closure of the binary relation X.

Proposition 41 ([24])**.** Let $M = (U, H, R, V)$ be a model and Δ a subformula closed set. Then $M^s_\Delta := (U_\Delta, \underline{H}^+_\Delta, \underline{R}^s_\Delta, V_\Delta)$ is a model and a filtration of M through Δ.

Theorem 42 (Decidability of **HUBiSKt**)**.** For every formula φ which is a non-theorem of **HUBiSKt**, there is a finite frame F such that $F \not\models \varphi$. Therefore, **HUBiSKt** is decidable.

Proof. It suffices to show the first part, since decidability follows from the first part and the finite axiomatization of **HUBiSKt**. Suppose that φ is not a theorem of **HUBiSKt**. By Theorem 34, there is a model $M = (U, H, R, V)$ such that $M \not\models \varphi$. Put Δ as the set of all subformulae of φ and note that Δ is finite. By Proposition 39, we obtain $M^s_\Delta \not\models \varphi$ hence $F^s_\Delta \not\models \varphi$, where F^s_Δ is the frame part of M^s_Δ. ■

4. Spatial Relations and Granularity

Logic-based methods have been widely used in Artificial Intelligence to describe in a qualitative way spatial relations between objects. One of the most widely cited accounts is known as the RCC (Region-Connection-Calculus) [19], a first-order theory with a primitive predicate of Connection, C, between regions of the space. From Connection a notion of Parthood is defined by $P(X, Y)$ iff $\forall Z(C(X, Z) \to C(Y, Z))$. Using Connection and Parthood, a set of eight

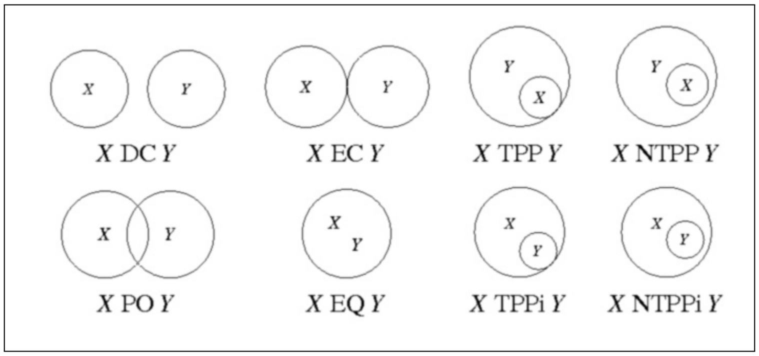


Figure 3: The set of RCC-8 spatial relations

Jointly Exhaustive and Pairwise Disjoint Spatial Relations between regions is obtained. This is known as the RCC-8 in Fig. 3. The RCC-8 can distinguish Non-tangential Proper Part (NTPP) from Tangential Proper Part (TPP). RCC-8 can express the relation of sharing only a part, or Partial Overlapping (PO), as well as connection on the boundaries, or External Connection (EC) and disjointedness, or Disconnection (DC). Equality (EQ), and the inverses of TPP and NTPP are also included.

It is well known that the RCC does not model discrete space, by which we mean a space where regions cannot be sub-divided infinitely often into strictly smaller sub-regions. Once atomic regions are allowed, that is non-empty regions of the space without any proper parts, the RCC theory becomes contradictory [19]. The ability to represent and reason about discrete space is important, as for example networks (maps, transport networks, social networks) are naturally represented by discrete structures such as graphs. We have seen that **UBiSKt** can describe incidence structures as hypergraphs, and graphs are a type of hypergraph. This fact has been used in previous work in the context of qualitative spatial reasoning. In [28, 26], **UBiSKt** is used to model RCC-8 style spatial relations between discrete regions, namely subgraphs of a graph-universe as in Definition 6. Formulae are names of subgraphs, and **UBiSKt** can encode the notion of Connection and other spatial relations between subgraphs. Among related works on discrete qualitative spatial reasoning we mention Galton's discrete mereotopology [9, 10] (for the recent development of the framework, the reader is referred to [5, 1]). Here the RCC relations are defined on a type of discrete space called an Adjacency space: a nonempty set W with a relation of adjacency $\alpha \subseteq W \times W$. As Galton explains [10], adjacency spaces can be regarded as graphs, the elements of W being the nodes, and the adjacency relation holding between two nodes meaning an edge exists between them. However, as also Galton underlines [10], an important difference emerges between adjacency spaces and graphs when we consider *substructures*. A subgraph of a graph is specified by a subset of nodes *and* a subset of edges, with the proviso that every time an edge e belongs to a subgraph K, all the nodes e is incident with belong to K. Thus there may be different subgraphs sharing the same set of nodes, but having different edges. However, a substructure of an adjacency space is defined by giving only its nodes, and the adjacency (or otherwise) of two nodes is automatically inherited. Moreover multiple edges may occur between the same pair of nodes in the graphs we consider here, unlike adjacency spaces. Cousty et al. [7] argue that edges need to play a more central role, and make the key observation that sets of nodes which differ only in their edges need to be regarded as distinct.

In the next section we are going to show that the propositional (i.e., bi-intuitionistic) part of **UBiSKt** can be used to encode spatial relations between subgraphs. In the following section we are going to show that the modal part of **UBiSKt** can be used to model granularity of subgraphs, thought of as varying the level of detail at which they are presented. We will also explore how spatial relations at different levels of detail can be encoded. To do so, we consider notions taken from the discipline of Mathematical Morphology. At this stage, no assumptions are made on R other than stability.

4.1. Discrete Spatial Relations in **UBiSKt**: *a Summary of Previous Work*

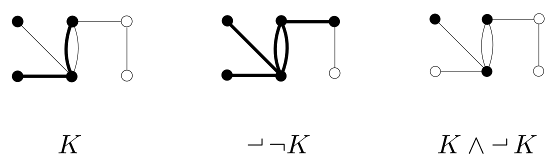


Figure 4: A subgraph K with its one-edge expansion and its nodes-boundary.

The **UBiSKt** operator $⌐\neg$, the effect of which on a subgraph is shown in Fig. 4 can be seen as a one-edge expansion operator: it includes K and it takes all the nodes that are one edge away from the boundary of K, this latter being encoded by the operation $K \wedge ⌐K$ as in Fig 4. As shown in [28], we may regard $⌐\neg$ as $\Diamond$ arising from the left converse $\curvearrowleft H$ of H. Indeed, when we restrict our attention to the class of models $M = (U, H, R, V)$ satisfying $R = H$, we note that the modal operator $\Diamond$ arising from the left converse $\curvearrowleft R$ of R is equivalent to $⌐\neg$, while the modal operator $\blacksquare$ is equivalent to $\neg⌐$. This is shown by the following lemma.

Lemma 43. Let $M = (U, H, R, V)$ be a model where $R = H$, and let φ be any formula. Then $[\![\Diamond\varphi]\!]_M = [\![⌐\neg\varphi]\!]_M$ and $[\![\blacksquare\varphi]\!]_M = [\![\neg⌐\varphi]\!]_M$.

Proof. We sketch the proof. First we notice that it follows from Theorem 4 in [30] that given a frame $F = (U, H, R)$ and an upset $X \subseteq U$, $\Diamond X = ⌐\Box\neg X$ and $\blacksquare X = \neg\blacklozenge⌐X$, where $\Diamond$ and $\blacksquare$ are interpreted with respect to $\curvearrowleft R$ and $\blacklozenge$ and $\Box$ are interpreted with respect to R as in Definition 10. Let $M = (U, H, R, V)$ be a model where $R = H$, so $\Diamond$ and $\blacksquare$ are interpreted with respect to $\curvearrowleft H$ and $\blacklozenge$ and $\Box$ are interpreted with respect to H. In this model, for any $[\![\varphi]\!]_M \in \{ [\![\psi]\!]_M \mid \psi \in \mathsf{Form}_{\mathcal{L}} \}$, we have that $[\![\blacklozenge \varphi]\!]_M = \{ u \in U \mid \exists v(vHu \text{ and } v \in [\![\varphi]\!]_M) \} = [\![\varphi]\!]_M$, from Definition 10 and as $[\![\varphi]\!]_M$ is an upset and by reflexivity of H. Similar reasoning holds for $[\![\Box\varphi]\!]_M = \{ u \in U \mid \forall v(uHv \text{ implies } v \in [\![\varphi]\!]_M) \} = [\![\varphi]\!]_M$. Thus $[\![\Diamond\varphi]\!]_M = [\![\neg\!\!\!\!\!\!- \Box\neg\varphi]\!]_M = [\![\neg\!\!\!\!\!\!- \neg\varphi]\!]_M$ when $R = H$, and similarly $[\![\blacksquare \varphi]\!]_M = [\![\neg \blacklozenge \neg\!\!\!\!\!\!- \varphi]\!]_M = [\![\neg \neg\!\!\!\!\!\!- \varphi]\!]_M$. ■

We also remark that given a model M, the truth of the formula $\mathsf{A}\,\varphi$ means that $[\![\varphi]\!]_M = U$ and truth of $\mathsf{E}\,\varphi$ means that $[\![\varphi]\!] \neq \varnothing$.

Using the closure operator and the universal modalities the spatial relation of connection between subgraphs $[\![\varphi]\!]$ and $[\![\psi]\!]$ can be expressed by an appropriate formula in **UBiSKt**:

$$C(\varphi, \psi) := \mathsf{E}(\neg\!\!\!\!\!\!- \neg\varphi \wedge \psi).$$

The formula states that $[\![\neg\!\!\!\!\!\!- \neg\varphi]\!]_M \cap [\![\psi]\!]_M \neq \varnothing$. The subgraphs are connected in a model M if the one-edge expansion of the first subgraph intersects the latter subgraph. The two subgraphs are one edge apart, in the limit case.

Beside connection, the following spatial relations can be defined in **UBiSKt**: Part, non-Part, Overlapping, Proper Part, Non-tangential Proper Part, Tangential Proper Part, External Connection, Disconnection, Partial overlapping, Equality, and the Inverse of Non-tangential Proper Part and Tangential Proper Part respectively. We list each relation with its corresponding formula in Table 2. The reasoning about spatial relations in **UBiSKt** can be implemented using the Hilbert system H**UBiSKt** or the alternative tableau-based proof system Tab**UBiSKt**, that has been automatized using the theorem-prover generator MeTtel [31] (see [27] to access the automated theorem-prover for Tab**UBiSKt** and for spatial reasoning examples using H**UBiSKt** and Tab**UBiSKt**).

Table 2: Spatial Relations and the corresponding formulae

Spatial Relation	Formula
$P(\varphi, \psi)$	$\mathsf{A}(\varphi \to \psi)$
$non\text{-}P(\varphi, \psi)$	$\mathsf{E}(\varphi \prec \psi)$
$O(\varphi, \psi)$	$\mathsf{E}(\varphi \wedge \psi)$
$PP(\varphi, \psi)$	$P(\varphi, \psi) \wedge non\text{-}P(\psi, \varphi)$
$NTPP(\varphi, \psi)$	$PP(\varphi, \psi) \wedge P(\neg\!\!\!\!\!\!- \neg\varphi, \psi)$
$TPP(\varphi, \psi)$	$PP(\varphi, \psi) \wedge non\text{-}P(\neg\!\!\!\!\!\!- \neg\varphi, \psi)$
$EC(\varphi, \psi)$	$C(\varphi, \psi) \wedge \neg O(\varphi, \psi)$
$DC(\varphi, \psi)$	$\neg C(\varphi, \psi)$
$PO(\varphi, \psi)$	$O(\varphi, \psi) \wedge non\text{-}P(\varphi, \psi)$ $\wedge non\text{-}P(\psi, \varphi)$
$EQ(\varphi, \psi)$	$P(\varphi, \psi) \wedge P(\psi, \varphi)$
$NTPP^i(\varphi, \psi)$	$NTPP(\psi, \varphi)$
$TPP^i(\varphi, \psi)$	$TPP(\psi, \varphi)$

4.2. Granularity and Spatial Relations under Granularity

The idea of zooming out, or viewing a situation in a less detailed way, is commonplace. Intuitively, zooming out on an image (a set of pixels) we expect

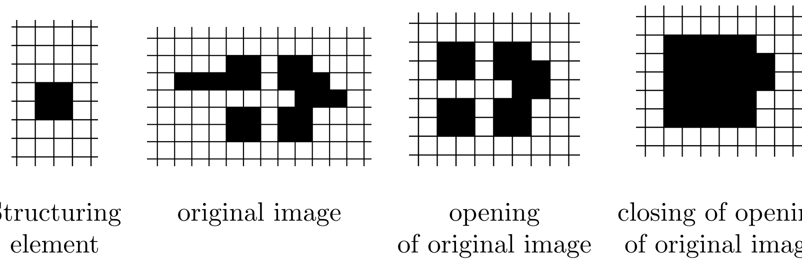


Figure 5: Approximation of a subset of $\mathbb{Z}^2$ by a 2×2 structuring element.

narrow cracks to fuse and narrow spikes to become invisible. This intuitive expectation can be formalized in mathematical morphology. The idea here is that instead of being able to see individual pixels, only groups of pixels can be seen. This is illustrated in Fig. 5 using the operations of opening and closing by a structuring element. For details of mathematical morphology see [16], but here it is sufficient to know that the opening consists of the image formed by (overlapping) copies of the structuring element within the original, and that closing consists of the complement of the (overlapping) copies of the structuring element but rotated by half a turn, that can be placed wholly outside the original.

As explained in [16] the operations of mathematical morphology are not restricted to approximating subsets of a grid of pixels by a structuring element, but apply in the context of any subset of a set with an arbitrary binary relation on the set instead of a structuring element. As [30] shows, we can extend this to a preorder (U, H) and approximate upsets in this structure by means of a stable relation R. Given $X \subseteq U$, we use $X \oplus R$ (dilation of X) to denote the set $\{u \in U \mid \exists v(vRu \wedge v \in X)\}$, and use $R \ominus X$ (erosion of X) to denote $\{u \in U \mid \forall v(uRv \Rightarrow v \in X)\}$. It is well known that for R fixed the operations $\_ \oplus R$ and $R \ominus \_$ form an adjunction on the lattice $\mathcal{P}(U)$, with $\_ \oplus R$ left adjoint to $R \ominus \_$, in the following sense.

Definition 44. Let $(V, \leq_V)$ and $(W, \leq_W)$ be partially ordered sets. An *adjunction* between V and W is a pair of functions $f : V \to W$ and $g : W \to V$ such that $f(v) \leq_W w$ iff $v \leq_V g(w)$, for all $v \in V$ and $w \in W$. The function f is called the *left adjoint* and g is the *right adjoint*.

Thus the following property of dilation and erosion follows.

Lemma 45. Given a set U, two subsets $A \subseteq U$ and $B \subseteq U$ and a relation $R \subseteq U \times U$, we have that $A \oplus R \subseteq B$ is equivalent to $A \subseteq R \ominus B$.

The opening of X by R is denoted $X \circ R$ and defined as $(R \ominus X) \oplus R$ and the closing is $X \bullet R = R \ominus (X \oplus R)$. The connection between mathematical morphology and modal logic has been studied in [3] in the set based case, and extended to the graph based case in [30]. Here, the modalities $\Diamond$, $\blacklozenge$, $\Box$ and $\blacksquare$ function as semantic operators taking upsets to upsets, with $\Diamond$ associated to $X \mapsto X \oplus \smile R$, $\blacklozenge$ associated to $X \mapsto X \oplus R$, $\Box$ associated to $R \ominus X$ and $\blacksquare$ associated to $\smile R \ominus X$. So, given a propositional variable p representing an upset, opening and closing of the upset are expressible in the logic by the formulae $\blacklozenge\Box p$ and $\Box\blacklozenge p$ respectively. This extends to **UBiSKt**, as it is an extension of the logic studied in [30]. In this setting, the idea of opening as fitting copies of a structuring element inside an image remains meaningful. Copies of the structuring element correspond to R-dilates in the following sense.

Definition 46. A subset $X \subseteq U$ is an R-dilate if $X = \{u\} \oplus R$ for some $u \in U$.

Notice that $\{ u \} \oplus R$ is the set of all R-successor of u, sometimes indicated as $R(\{ u \})$. Stability implies that R-dilates are always upsets.

Opening can be expressed in terms of R-dilates as follows.

Lemma 47. $X \circ R = \bigcup\{ \{ x \} \oplus R \mid \{ x \} \oplus R \subseteq X \}$

Proof.

$$\begin{aligned}
\bigcup\{ \{ x \} \oplus R \mid \{ x \} \oplus R \subseteq X \} &= \{ u \in U \mid \exists x(xRu \text{ and } \{ x \} \oplus R \subseteq X) \} \\
&= \{ u \in U \mid \exists x(xRu \text{ and } \forall y(xRy \Rightarrow y \in X)) \} \\
&= \{ u \in U \mid \exists x(xRu \text{ and } x \in R \ominus X \} \\
&= (R \ominus X) \oplus R.
\end{aligned}$$

■

So the opening of X is the union of all R-dilates belonging to X. It is straightforward to check that closing can also be expressed in terms of dilates:

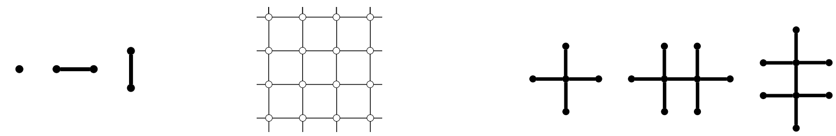
$$X \bullet R = \{u \in U \mid \{u\} \oplus R \subseteq \bigcup\{\{x\} \oplus R \mid x \in X\}\}.$$

To give concrete examples of R-dilates, let (U, H) be the graph with $\mathbb{Z}^2$ for nodes and two nodes are connected by an edge if exactly one of their coordinates differs by 1, and the other coordinates are equal to each other. We refer to this as the graph $\mathbb{Z}^2$, visualized as in Fig. 6. The dilates by H and by $\smile H$ of a node, a horizontal edge, and a vertical edge respectively are shown in the figure.

We can think of $(X \circ R) \bullet R$ as a granular version of X in which we cannot 'see' arbitrary upsets, but only ones that can be described in terms of the R-dilates. As we have seen, opening and closing correspond to specific sequences of modalities in the logic. So, given a representable upset, we can capture its granular version by a formula in the logic.

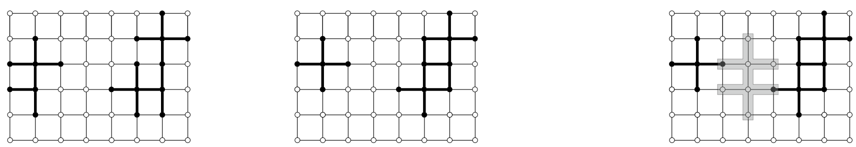
Definition 48. The formula 'coarsely φ' is defined by $\mathsf{G}\varphi := \Box\blacklozenge\blacklozenge\Box\varphi$.

We notice that the closing of the opening of a region is known in mathematical morphology as an alternating filter. This gives a way of zooming-out



Dilates of H (U, H) is infinite grid, $\mathbb{Z}^2$ Dilates of $\circlearrowright H$

Figure 6: Consider the graph (U, H) in the center where we remind the reader that $U = N \cup E$ being N the set of nodes and E the set of edges, and H is the reflexive closure of the incidence relation from edges to nodes. On the left we show shapes of H-dilates, $\{ u \} \oplus H$, when u is a node, a horizontal edge, and a vertical edge respectively. On the right the shape of $\circlearrowright H$-dilates are shown.



Two upsets | Their approximations by opening then closing by $\circlearrowright H$ | The approximations can be joined by a dilate

Figure 7: Granular View by Relation $\circlearrowright H$

for a region, but how should we define connection between coarse regions? The issue is that the space underlying the regions should become coarser – regions disconnected may become connected for example. In the same way that coarse regions are described in terms of dilates, a coarse version of connection can be formulated using dilates. To motivate this consider Fig. 7 which shows the idea that coarse regions are coarsely connected if there is a dilate intersecting both. Visually and informally the idea is that the gap between can be bridged by a dilate. Requiring an R-dilate joining two regions seems a suitable notion of coarse connection, as it extends the intuition of connection at the detailed level given in Section 4. Indeed two upsets X and Y are connected at the detailed level if the gap between them can be bridged by an H-dilate, or in other words if they are an edge apart. Going to the granular level, single H-dilates are no longer “visible”, and the space has coarser atomic parts: copies of the structuring element, i.e. R-dilates.

Definition 49. An R-dilate, D, *joins* upsets X and Y if $X \cap D \neq \varnothing$ and $Y \cap D \neq \varnothing$.

It is easy to see that requiring there be an R-dilate that joins X and Y amounts to requiring that, given the R-dilates intersecting X, at least one of them intersects Y.

It is well known that dilation is a monoid action in which the monoid of relations on a fixed set under composition acts on the set of subsets of the set. In particular, the following holds; we include a proof for completeness.

Lemma 50. If R and S are relations on a set U and $X \subseteq U$ then $X \oplus (R\,;S) = (X \oplus R) \oplus S$.

Proof.

$$
\begin{aligned}
(X \oplus R) \oplus S &= \{\, u \in U \mid \exists y(ySu \text{ and } y \in X \oplus R) \,\} \\
&= \{\, u \in U \mid \exists y(ySu \text{ and } \exists z(zRy \text{ and } z \in X)) \,\} \\
&= \{\, u \in U \mid \exists z \exists y((zRy \text{ and } ySu) \text{ and } z \in X) \,\} \\
&= \{\, u \in U \mid \exists z((z,u) \in R\,;S \text{ and } z \in X) \,\} \\
&= X \oplus (R\,;S).
\end{aligned}
$$
■

Lemma 51. Let X be an upset and R a stable relation. The union of the R-dilates intersecting X is $X \oplus (\circlearrowleft R\,;R)$.

Proof. First we show that the union of the R-dilates intersecting X is $(X \oplus \breve{R}) \oplus R$. If $\{u\} \oplus R$ intersects X, for some $u \in U$, then there is an $x \in X$ such that $\{u\} \subseteq \{x\} \oplus \breve{R}$. Hence $\{u\} \oplus R \subseteq (\{x\} \oplus \breve{R}) \oplus R \subseteq (X \oplus \breve{R}) \oplus R$. In the other direction, if $y \in (X \oplus \breve{R}) \oplus R$, then there is some $u \in U$ and $x \in X$ such that uRy and uRx, so that $y \in \{u\} \oplus R$ with $\{u\} \oplus R$ intersecting X. Now, since $\breve{R} \subseteq \circlearrowleft R$ (see Definition 9), $(X \oplus \breve{R}) \oplus R \subseteq (X \oplus \circlearrowleft R) \oplus R = X \oplus (\circlearrowleft R; R)$. Also $X \oplus (\circlearrowleft R; R) = X \oplus (H\,; \breve{R}; H\,; R) = X \oplus (\breve{R}; H\,; R) \subseteq X \oplus (\breve{R}; R) = (X \oplus \breve{R}) \oplus R$ because X is an upset and R is stable. So $(X \oplus \breve{R}) \oplus R = X \oplus (\circlearrowleft R\,; R)$. ■

Proposition 52. There is an R-dilate joining upsets X and Y iff $(X \oplus (\circlearrowleft R\,; R)) \cap Y \neq \varnothing$.

Proof. The union of R-dilates intersecting X is $X \oplus (\circlearrowleft R\,; R)$ from Lemma 51. This intersects Y iff $(X \oplus (\circlearrowleft R\,; R)) \cap Y \neq \varnothing$. ■

The above discussion provides a semantic justification for the following definition.

Definition 53 (Coarse connection)**.** $C_G(\varphi, \psi) := \mathsf{E}(\blacklozenge \Diamond \mathsf{G}\varphi \wedge \mathsf{G}\psi)$.

Note that in a model M where $R = H$, we have that $[\![\mathsf{G}\varphi]\!]_M = [\![\varphi]\!]_M$. First we notice that $[\![\blacklozenge\, \varphi]\!]_M = [\![\varphi]\!]_M \oplus H$ and $[\![\Box\varphi]\!]_M = H \ominus [\![\varphi]\!]_M$ in such a model. Then we have that $[\![\varphi]\!]_M \oplus H = [\![\varphi]\!]_M = H \ominus [\![\varphi]\!]_M$, as $[\![\varphi]\!]_M$ is an upset and by reflexivity of H. Moreover, as noticed in Section 4.1 Lemma 43, $\urcorner\neg$ can be regarded as $\Diamond$ arising from $\circlearrowleft H$, the left converse of H. Hence, in a model where $R = H$, the relation of coarse connection $C_G(\varphi, \psi)$ is equivalent to the standard notion of edge-connection $C(\varphi, \psi)$, presented in Section 4.1, as $\blacklozenge \Diamond \mathsf{G}\varphi$ can be reduced to $\urcorner\neg\varphi$, and $\mathsf{G}\psi$ can be reduced to ψ.

As we would expect, our notion of coarse connection is symmetric as follows.

Proposition 54. $\vdash_{\mathbf{HUBiSKt}} \mathsf{E}(\blacklozenge \Diamond\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \blacklozenge \Diamond\psi)$.

Proof. We have the following derivation in H**UBiSKt**. $\vdash \neg(\blacklozenge \Diamond\varphi \wedge \psi) \leftrightarrow (\blacklozenge \Diamond\varphi \to \neg\psi)$ because $\neg(\alpha \wedge \beta) \leftrightarrow (\alpha \to \neg\beta)$ is a theorem in intuitionistic logic. Thus by ($\mathtt{Mon}\,\mathsf{A}$) we have that $\vdash \mathsf{A}\neg(\blacklozenge \Diamond \varphi \wedge \psi) \leftrightarrow \mathsf{A}(\blacklozenge \Diamond \varphi \to \neg\psi)$.

Now $\vdash \mathsf{A}(\blacklozenge \lozenge \varphi \to \neg\psi) \leftrightarrow \mathsf{A}(\lozenge \varphi \to \Box\neg\psi)$ and $\vdash \mathsf{A}(\lozenge \varphi \to \Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \blacksquare\Box\neg\psi)$ by adjunction between $\blacklozenge$ and $\Box$, and between $\lozenge$ and $\blacksquare$.
Then $\vdash \mathsf{A}(\varphi \to \blacksquare\Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \neg\blacklozenge\neg\Box\neg\psi)$ and $\mathsf{A}(\varphi \to \neg\blacklozenge\neg\Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \neg\blacklozenge\lozenge\psi)$ because $\blacksquare\alpha \leftrightarrow \neg\blacklozenge\neg\alpha$ and $\lozenge\alpha \leftrightarrow \neg\Box\neg\alpha$ are both abbreviations in the syntax. By $\vdash (\varphi \to \neg\blacklozenge\lozenge\psi) \leftrightarrow \neg(\varphi \wedge \blacklozenge\lozenge\psi)$ and thus by $(\mathtt{Mon}\,\mathsf{A})$ we have that $\vdash \mathsf{A}(\varphi \to \neg\blacklozenge\lozenge\psi) \leftrightarrow \mathsf{A}\neg(\varphi \wedge \blacklozenge\lozenge\psi)$, and then $\vdash \mathsf{A}\neg(\blacklozenge\lozenge\varphi \wedge \psi) \leftrightarrow \mathsf{A}\neg(\varphi \wedge \blacklozenge\lozenge\psi)$ by sequence of equivalences. Therefore $\vdash \neg\mathsf{A}\neg(\blacklozenge\lozenge\varphi \wedge \psi) \leftrightarrow \neg\mathsf{A}\neg(\varphi \wedge \blacklozenge\lozenge\psi)$ that is $\vdash \mathsf{E}(\blacklozenge\lozenge\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \blacklozenge\lozenge\psi)$ by item 27 of Proposition 24. ■

Similarly to connection, we can define a notion of coarse parthood in terms of R-dilates. The standard notion of parthood at the detailed level (Table 2) says that, given upsets X and Y, X is part of Y if and only if all the H-dilates in X lie in Y. A suitable notion of coarse parthood will require that X is a coarse part of Y if and only if all the R-dilates in X lie also in Y.

Proposition 55. Let X and Y be upsets, and R a stable relation. The following are equivalent: 1) all the R-dilates included in X are included in Y and 2) $(R \ominus X) \subseteq (R \ominus Y)$.

Proof. The union of all the R-dilates in X is the opening of X: $X \circ R = (R \ominus X) \oplus R$. Hence, requiring that all the R-dilates in X lie in Y amounts to require that $((R \ominus X) \oplus R) \subseteq Y$. By properties of adjunctions this is equivalent to $(R \ominus X) \subseteq (R \ominus Y)$. ■

Lemma 56 ([30])**.** Let M be a model. Then $[\![\varphi]\!]_M \subseteq [\![\psi]\!]_M$ iff $M \models \mathsf{A}(\varphi \to \psi)$.

The above reasoning together with Lemma 56 provides a semantic justification for the following definition of coarse parthood between coarse regions.

Definition 57 (Coarse parthood)**.** $P_G(\varphi, \psi) := \mathsf{A}(\Box\mathsf{G}\varphi \to \Box\mathsf{G}\psi)$.

The negation of the notion of coarse parthood will give a notion of coarse non-parthood: this requires that there is at least an R-dilate included in X such that it is not included in Y. From Proposition 55, we know that this is equivalent to $R \ominus X \not\subseteq R \ominus Y$.

Lemma 58. Let M be a model. Then $[\![\varphi]\!]_M \not\subseteq [\![\psi]\!]_M$ iff $M \models \mathsf{E}(\varphi \prec \psi)$.

Proof. Let φ and ψ be formulae and $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M$ the associated upsets in a given model M. In what follows we omit the subscript M. Suppose $[\![\varphi]\!] \not\subseteq [\![\psi]\!]$; therefore for some $u \in U$, $u \in [\![\varphi]\!]$ and $u \notin [\![\psi]\!]$. Since H is reflexive, uHu holds, hence there is a $v \in U$ such that vHu and $v \in [\![\varphi]\!]$ and $v \notin [\![\psi]\!]$. By Definition 10 this means that $M, u \models \varphi \prec \psi$, hence $M \models \mathsf{E}(\varphi \prec \psi)$. For the other direction, $M \models \mathsf{E}(\varphi \prec \psi)$ iff for some $u \in U$ $M, u \models \varphi \prec \psi$. Hence there is a $v \in U$ such that vHu and $M, v \models \varphi$ and $M, v \not\models \psi$, that means that $v \in [\![\varphi]\!]$ and $v \notin [\![\psi]\!]$, for some $v \in U$. Therefore $[\![\varphi]\!] \not\subseteq [\![\psi]\!]$. ■

Because of Lemma 58 we propose the following definition.

Definition 59 (Coarse non-parthood)**.** non-$P_G(\varphi, \psi) := \mathsf{E}(\Box\mathsf{G}\varphi \prec \Box\mathsf{G}\psi)$.

We now analyze how to extend the spatial relation of overlapping to the granular level. Two upsets X and Y overlap at the detailed level if and only if there is an H-dilate that lies both in X and Y. Following this idea, a suitable notion of coarse overlapping requires a non-empty R-dilate that lies both in X and Y.

Proposition 60. Let X and Y be upsets and R a stable relation. The following are equivalent: 1) there is a non-empty R-dilate that lies both in X and in Y and 2) $(X \cap Y) \circ R \neq \varnothing$.

Proof. $(X \cap Y) \circ R$ is the opening of $X \cap Y$, so union of all R-dilates both in X and in Y. Hence requiring that there is a non-empty R-dilate that lies both in X and in Y amounts to require that the opening of $X \cap Y$ is non-empty: $(X \cap Y) \circ R \neq \varnothing$. ■

Thus we define coarse overlapping between coarse regions as follows.

Definition 61 (Coarse overlapping)**.** $O_G(\varphi, \psi) := \mathsf{E}(\blacklozenge\Box(\mathsf{G}\varphi \wedge \mathsf{G}\psi))$.

As an example, in Fig. 8 on the left we show two upsets (red and black) that intersect, but an R-dilate will not fit inside the region of intersection ($R = \smile H$). Therefore the spatial relation O_G does not hold. If the region of intersection is at least as big as an R-dilate, as happens on the right of the figure, then the relation O_G does hold.

An upset X is a non-tangential part of an upset Y at the detailed level if X is part of Y and $\neg\neg X$, the closure of X, is still part of Y. This means that all the H-dilates that intersect X lie in Y. Hence, a suitable notion of coarse non-tangential part between upsets X and Y is obtained by requiring that X is coarse part of Y and all the R-dilates intersecting X lie in Y.

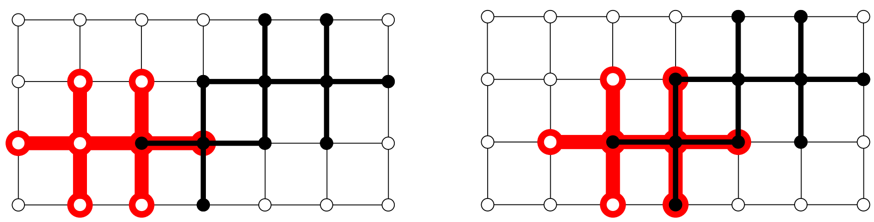
Proposition 62. Let X and Y be upsets and R a stable relation. The following are equivalent: 1) all the R-dilates overlapping X lie in Y, and 2) $X \oplus \smile R \subseteq R \ominus Y$.

Proof. The union of the R-dilates overlapping X is included in Y if $(X \oplus \smile R) \oplus R \subseteq Y$ by Lemma 51. This is equivalent to $X \oplus \smile R \subseteq R \ominus Y$ by properties of adjunctions. ■

The above reasoning provides a semantic justification for the following definition.

Definition 63 (Coarse non-tangential part)**.** $NTP_G(\varphi, \psi) := \mathsf{A}(\Box\mathsf{G}\varphi \rightarrow \Box\mathsf{G}\psi) \wedge \mathsf{A}(\Diamond\mathsf{G}\varphi \rightarrow \Box\mathsf{G}\psi)$.

Finally, we analyze the notion of coarse tangential part. At the detailed level, an upset X is a tangential part of Y if X is part of Y and there is at least one H-dilate intersecting X that does not lie in Y. This is obtained by requiring that the closure of X is not part of Y. Hence, at the granular level we



Two upsets not sharing a whole R-dilate   Two upsets sharing a whole R-dilate

Figure 8: Cases of coarse non-overlapping and of coarse overlapping, where R is $\curvearrowleft H$.

will require that the union of all R-dilates intersecting X does not lie in Y. This means that we have to negate the requirement for NTP_G: by Proposition 62 this is $X \oplus \curvearrowleft R \not\subseteq R \ominus Y$. Because of this and Lemma 58 we propose the following.

Definition 64 (Coarse tangential part)**.** $TP_G(\varphi, \psi) := \mathsf{A}(\Box\mathsf{G}\varphi \rightarrow \Box\mathsf{G}\psi) \wedge \mathsf{E}(\Diamond\mathsf{G}\varphi \prec \Box\mathsf{G}\psi)$.

Using the predicates C_G, P_G, non-P_G, O_G, NTP_G and TP_G, a set of RCC-8 style coarse spatial relations between coarse subgraphs can be obtained in the obvious way from Table 2. For example the coarse spatial relation of external connection $EC_G(\varphi, \psi)$ will be defined as $C_G(\varphi, \psi) \wedge \neg O_G(\varphi, \psi)$.

5. A Modal Logic for Graph and Hypergraph Partitions

In the previous section we have explored the idea of looking at subgraphs at a different level of detail. The intuition is that instead of being able to see all the minimal upsets of a graph – individual nodes and individual edges together with their incident nodes – we can only see things that can be described by a structuring element, which serves as a probe through which we look at the graph. The notion of structuring element is captured by a relation in modal logic, and by a stable relation when the underlying domain is a graph.

However the information about how certain elements can coalesce together does not need to come from a well defined shape, as it is with a morphological structuring element. Elements can be grouped together if they share certain attributes. In rough set theory [18], attributes defined on a set provide an equivalence relation on the set, and thus a partition of the set. Indistinguishable elements coalesce into granules, blocks of the partition, and this gives a coarser view of the original set. Then, given any subset X of the given set, two kinds of approximation can be considered: $\underline{X}$ or the lower approximation, and $\overline{X}$ or the upper approximation. It is well known that rough set theory has connections with the modal logic system **S5**, where indeed R is an equivalence relation, with **S5**-$\Box$ associated to $X \mapsto \underline{X}$ and **S5**-$\blacklozenge$ associated to $X \mapsto \overline{X}$ [32].

In this section we propose a generalization of **S5** to the bi-intuitionisitic setting. In the classical case, the axioms of **S5** correspond to the requirement that the accessibility relation be an equivalence relation. For a hypergraph (U, H) requiring that a stable relation R be an equivalence relation on U is too restrictive. In particular, [25, Theorem 6] shows that, when (U, H) has a single connected component, the only stable equivalence relation is the universal relation $U \times U$. We note that [25] uses the terminology 'graphical relation' instead of 'stable relation'. In seeking a suitable notion for a partition of a hypergraph and for corresponding conditions on R that are analogous to being an equivalence relation [25] identifies two conditions in addition to reflexivity and transitivity. We recall the details of these conditions in Section 5.2 below. The following subsection, 5.3, shows that these two conditions correspond to formulae of **UBiSKt**. The final subsection here, 5.4, explores granular spatial relations in the specific case of **S5** granularity. First, however, in 5.1, we examine **S4** for **UBiSKt**.

5.1. System **S4** *for* **UBiSKt**

Reflexivity of an arbitrary relation R can be expressed as the relational inclusion $I \subseteq R$. Transitivity can be expressed as $R \,;\, R \subseteq R$. Given a set and a preorder (U, H), and a stable relation $R \subseteq U \times U$, we have that $I \subseteq R$ iff $H \subseteq R$. This follows from the fact that $I \subseteq H$ as H is a preorder and therefore it is reflexive, and by stability of R. If $H \subseteq R$ (reflexivity of a stable relation) and R is transitive then we have that $H \,;\, R \,;\, H \subseteq R \,;\, R \,;\, R \subseteq R$, that means that R is stable (see Definition 8).

From the correspondence theorem in [30] we have that the properties of reflexivity and transitivity of a stable relation R can be expressed as formulae in the logic **BiSKt**, and therefore in **UBiSKt**, as this latter is an extension of the former. We will refer to the formula $p \to \blacklozenge p$ (or to its equivalent box-form $\Box p \to p$) as the *reflexivity axiom*, since imposing it is equivalent to the constraint $H \subseteq R$. We will refer to the formula $\blacklozenge \blacklozenge p \to \blacklozenge p$ (or to its equivalent box-form $\Box p \to \Box\Box p$) as the *transitivity axiom*, since imposing it is equivalent to the constraint $R \,;\, R \subseteq R$.

Definition 65. We define **S4** to be the set $\{p \to \blacklozenge p, \blacklozenge \blacklozenge p \to \blacklozenge p\}$ and $\mathbb{S}4$ to be the class of frames (U, H, R) such that R is reflexive and transitive.

Theorem 66. Let Λ be a *ubist*-logic such that $\mathbf{S4} \subseteq \Lambda$. Then all the following hold in Λ.

1. $\blacklozenge \Box p \leftrightarrow \Box p$
2. $\Box \blacklozenge p \leftrightarrow \blacklozenge p$
3. $\Box \blacklozenge \blacklozenge \Box p \leftrightarrow \Box p$
4. $\blacklozenge \Box\Box \blacklozenge \leftrightarrow \blacklozenge p$

Proof. 1. For the left-to-right direction, we proceed as follows: $\vdash \Box p \to \Box\Box p$ is the transitivity axiom. By the adjunction $\blacklozenge \dashv \Box$ (Proposition 23), this is equivalent to $\vdash \blacklozenge \Box p \to \Box p$. For the right-to-left direction, $\vdash \Box p \to \blacklozenge \Box p$ holds by an instantiation of the reflexivity axiom.

2. For the right-to-left direction, we proceed as follows: $\vdash \blacklozenge\blacklozenge p \to \blacklozenge p$ is the transitivity axiom. By the adjunction $\blacklozenge \dashv \Box$ (Proposition 23), this is equivalent to $\vdash \blacklozenge p \to \Box\blacklozenge p$. For the left-to-right direction, $\vdash \Box\blacklozenge p \to \blacklozenge p$ holds by an instantiation of the reflexivity axiom.

3. $\vdash \Box\blacklozenge\blacklozenge\Box p \leftrightarrow \blacklozenge\blacklozenge\Box p$ by item 2, $\vdash \blacklozenge\blacklozenge\Box p \leftrightarrow \blacklozenge\Box p$ by reflexivity and transitivity axioms, and $\vdash \blacklozenge\Box p \leftrightarrow \Box p$ by item 1. Hence $\vdash \Box\blacklozenge\blacklozenge\Box p \leftrightarrow \Box p$.

4. $\vdash \blacklozenge\Box\Box\blacklozenge p \leftrightarrow \Box\Box\blacklozenge p$ by item 1, $\vdash \Box\Box\blacklozenge p \leftrightarrow \Box\blacklozenge p$ by reflexivity and transitivity axioms and $\vdash \Box\blacklozenge p \leftrightarrow \blacklozenge p$ by and item 2. Hence $\vdash \blacklozenge\Box\Box\blacklozenge p \leftrightarrow \blacklozenge p$. ■

Theorem 66 shows that when we assume the **S4**-axioms, the notion of granulation presented in section 4.2 corresponds to the lower approximation as in Rough Set Theory, and the other order of composition of closing and opening corresponds to upper approximation.

Lemma 67. Given a *ubist*-logic Λ such that $\mathbf{S4} \subseteq \Lambda$, the Λ-canonical frame $F^{\Lambda}_{(\Gamma,\Delta)} = (U^{\Lambda}, H^{\Lambda}, R^{\Lambda})$ satisfies both $H^{\Lambda} \subseteq R^{\Lambda}$ and $R^{\Lambda} ; R^{\Lambda} \subseteq R^{\Lambda}$, i.e., $F^{\Lambda}_{(\Gamma,\Delta)} \in \mathbb{S}4$.

Proof. First we establish $H^{\Lambda} \subseteq R^{\Lambda}$. Suppose that $(\Sigma_1, \Theta_1)H^{\Lambda}(\Sigma_2, \Theta_2)$, i.e., $\Sigma_1 \subseteq \Sigma_2$. To show $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_2, \Theta_2)$, we use Lemma 30(3) to assume that $\varphi \in \Sigma_1$. Our goal is to show $\blacklozenge\varphi \in \Sigma_2$. But this is clear from $\varphi \to \blacklozenge\varphi \in \Sigma_1$ (by $\mathbf{S4} \subseteq \Lambda$) and $\Sigma_1 \subseteq \Sigma_2$. Second, we prove $R^{\Lambda} ; R^{\Lambda} \subseteq R^{\Lambda}$. Assume that $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_2, \Theta_2)$ and $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_3, \Theta_3)$. To show that $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_3, \Theta_3)$, suppose that $\varphi \in \Sigma_1$. We show that $\blacklozenge\varphi \in \Sigma_3$. By assumption, we have $\blacklozenge\blacklozenge\varphi \in \Sigma_3$, which implies $\blacklozenge\varphi \in \Sigma_3$ by $\blacklozenge\blacklozenge\varphi \to \blacklozenge\varphi \in \Sigma_3$ (by $\mathbf{S4} \subseteq \Lambda$). ■

We can establish the following strong completeness results.

Theorem 68. 1. **UBiSKt** extended with **S4** is sound and strongly complete for the class $\mathbb{S}4$, i.e., $\Gamma \models_{\mathbb{S}4} \varphi$ iff $\Gamma \vdash_{\mathbf{UBiSKtS4}} \varphi$ for every set $\Gamma \cup \{\varphi\}$ of formulae.

2. **UBiSKt** extended with **S4** and $\mathbf{bd}_2$ is sound and strongly complete for the class of frames $(U, H, R) \in \mathbb{S}4$ where (U, H) is a hypergraph, i.e., $\Gamma \models_{\mathbb{HG}\cap\mathbb{S}4} \varphi$ iff $\Gamma \vdash_{\mathbf{UBiSKtS4bd_2}} \varphi$ for every set $\Gamma \cup \{\varphi\}$ of formulae.

Proof. We establish item 2 alone. Let $\Lambda := \mathbf{UBiSKtS4bd_2}$. The soundness follows from Proposition 14 and Theorem 15. For the strong completeness, it suffices to show that the canonical frame $F^{\Lambda}_{(\Gamma,\Delta)}$ belongs to $\mathbb{HG} \cap \mathbb{S}4$. This is established by Lemmas 36 and 67. ■

5.2. Hypergraph Partitions and Associated Relations

In this section conditions on a stable relation R, analogous to the three properties of an equivalence relation are stated. These conditions have appeared already in [25], but without the connection to logical formulae which is provided in section 5.3. The conditions are thus reviewed here to prepare for their logical counterparts, which can be seen as providing one possible bi-intuitionistic generalization of the classical **S5**.

Suppose that a hypergraph (U, H) additionally supports a reflexive and transitive relation $R \subseteq U \times U$, so that $H \subseteq R$ and $R \,;\, R \subseteq R$. The symmetric part of R, that is $R \cap \breve{R}$, is then an equivalence relation on the set U. Furthermore, the set U' of equivalence classes is naturally partially ordered by the relation H', where for $X, Y \in U'$ it holds that $X \; H' \; Y$ iff there are $x \in X$ and $y \in Y$ such that xRy.

This provides one way to obtain a granular version of the poset (U, H) with the less detailed version being (U', H'). An example of this is given in Fig. 9. There are two reasons why a granular description of hypergraphs is more complex than simply a reflexive and transitive relation. One immediate objection is that (U', H') may not be a hypergraph. For example, taking $U = \{a, b, c\}$, and taking H to be the identity relation on U, the relation R can be the reflexive, transitive closure of $\{(a, b), (b, c)\}$. But this example reveals the second problem. In seeking a granular view of hypergraphs it seems natural to look for a way that reduces to the ordinary notion of a partition on a set in the special case that the hypergraph has nodes but no edges. The example starts with a set, in the guise of a discrete partial order, but results in the same underlying set with a partial order that is not discrete.

This discussion motivates the problem of finding additional constraints on a reflexive transitive relation R on (U, H) such that both of these problems are avoided, without being unnecessarily restrictive on the granular views of hypergraphs that can be obtained.

An answer has been provided in [25] which establishes a bijective correspondence between certain partitions of a hypergraph (U, H) and relations R on U which are reflexive, transitive, and satisfy two additional constraints. The blocks or cells of these partitions can contain both edges and nodes of (U, H), and the blocks constitute either edges or nodes in the quotient structure (U', H'). The constraints on R are as follows.

Definition 69. The relation R is *symmetrically generated* if $R \subseteq \overleftrightarrow{R} \,;\, H \,;\, \overleftrightarrow{R}$, where $\overleftrightarrow{R} := R \cap \breve{R}$.

If R is symmetrically generated and additionally transitive, then $R = \overleftrightarrow{R} ; H ; \overleftrightarrow{R}$, by transitivity and stability of R. Notice that, when $H = I$, so when (U, H) is a set, R being symmetrically generated corresponds to R being symmetric.

Definition 70. The relation R is *two-tier* if $(R \cap \overline{\breve{R}})^2 = \varnothing$.

5.3. Two-tier and Symmetric Generation Axioms: an **S5** *System for* **UBiSKt**

In this section we show that there are two formulae corresponding to the properties of a frame of being two-tier and symmetrically generated.

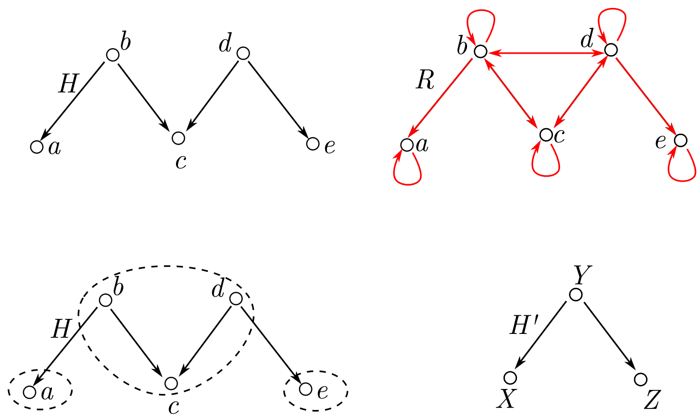


Figure 9: Top-left: a hypergraph (U, H), (reflexive loops of H are left implicit). Top-right: the relation $R \subseteq U \times U$ is the transitive closure of the relation shown. The symmetric part of R, $R \cap \breve{R} = \{(b,c),(c,b),(b,d),(d,b),(d,c),(c,d)\} \cup I$ generates a partition of U (bottom-left). Bottom-right: the quotient structure (U', H'). The elements of U' are $X = \{a\}$, $Y = \{b, c, d\}$ and $Z = \{e\}$. H', that is the reflexive closure of the relation shown on the set U'.

Theorem 71. Let $F = (U, H, R)$ be a frame where R is reflexive and transitive. Then $F \models \Box q \vee \Box(\Box q \to (\Box p \vee \Box\neg\Box p))$ iff $(R \cap \breve{\overline{R}})^2 = \varnothing$.

Recall that the following equivalence holds: $(R \cap \breve{\overline{R}})^2 = \varnothing$ iff xRy and yRz jointly imply yRx or zRy for all x, y, $z \in U$.

Proof. **Right-to-left direction**: Assume that $(R \cap \breve{\overline{R}})^2 = \varnothing$. Fix any valuation V and any $w \in U$. Put $M = (F, V)$ and assume that $M, w \not\models \Box q$. To show that $M, w \models \Box(\Box q \to (\Box p \vee \Box\neg\Box p))$, fix any v such that wRv and $M, v \models \Box q$. We show that $M, v \models \Box p \vee \Box\neg\Box p$. So suppose that $M, v \not\models \Box p$ and let us show that $M, v \models \Box\neg\Box p$. Let us fix any u such that vRu.To show that $M, u \models \neg\Box p$, let us fix any i such that uHi. Our goal is to show that $M, i \not\models \Box p$. By $v(R\,;H)i$, we have vRi. Since wRv and vRi, two-tierness implies vRw or iRv. If vRw, we should have $M, w \models \Box q$ by transitivity of R and $M, v \models \Box q$. But this is a contradiction with $M, w \not\models \Box q$. So we have iRv. By $M, v \not\models \Box p$, we can find a state $x \in U$ such that vRx and $M, x \not\models p$. By transitivity of R and iRv, we have $M, i \not\models \Box p$.
Left-to-right direction: Suppose that $F \models \Box q \vee \Box(\Box q \to (\Box p \vee \Box\neg\Box p))$. To show the two-tierness of R, let us suppose that wRv and vRu and that vRw fails. Our goal is to show that uRv. Define $V(p) = R(u) := \{\, a \in U \mid uRa \,\}$ and $V(q) = R(v)$, where we note that both sets are upsets by stability of R. Let us write $M = (F, V)$. We have $M, w \not\models q$ and wRw hence $M, w \not\models \Box q$. By the initial supposition (the validity of the formula), we obtain $M, w \models \Box(\Box q \to (\Box p \vee \Box\neg\Box p))$. Because wRv and $M, v \models \Box q$ by our definition of V, we obtain $M, v \models \Box p \vee \Box\neg\Box p$. Let us establish $M, v \not\models \Box\neg\Box p$. It suffices to show

$M, u \not\models \neg\Box p$. This is clear from uHu and $M, u \models \Box p$ by our definition of V. It follows from $M, v \models \Box p \vee \Box\neg\Box p$ that $M, v \models \Box p$, which implies $R(v) \subseteq R(u)$. Since vRv, we conclude uRv, as required. ■

Definition 72. We use $\mathbf{t}_2$ to mean the formula $\Box q \vee \Box(\Box q \to (\Box p \vee \Box\neg\Box p))$. Let $\mathbb{T}_2$ be the class of all frames $F = (U, H, R)$ such that (U, R) is a preorder, i.e., $(U, H, R) \in \mathbb{S}4$, and $(R \cap \overline{\breve{R}})^2 = \varnothing$.

Lemma 73. Given a *ubist*-logic Λ such that $\mathbf{S4} \cup \{\, \mathbf{t}_2 \,\} \subseteq \Lambda$, the Λ-canonical frame $F^{\Lambda}_{(\Gamma,\Delta)} = (U^{\Lambda}, H^{\Lambda}, R^{\Lambda})$ is a preorder and it also satisfies two-tierness, i.e., $((R^{\Lambda} \cap \overline{\breve{R^{\Lambda}}}))^2 = \varnothing$.

Proof. It suffices to show the two-tierness of R^{Λ}. Suppose $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_2, \Theta_2)$ and $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_3, \Theta_3)$. We need to prove that either $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_1, \Theta_1)$ or $(\Sigma_3, \Theta_3)R^{\Lambda}(\Sigma_2, \Theta_2)$ holds. Suppose that $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_1, \Theta_1)$ fails, i.e., we can find a formula ψ such that $\Box\psi \in \Sigma_2$ and $\psi \notin \Sigma_1$. To show $(\Sigma_3, \Theta_3)R^{\Lambda}(\Sigma_2, \Theta_2)$, fix any $\Box\varphi \in \Sigma_3$. Our goal is to establish $\varphi \in \Sigma_2$. Since $\Box\psi \to \psi \in \Sigma_1$ ($\because$ $\mathbf{S4} \subseteq \Lambda$) and $\psi \notin \Sigma_1$, we get $\Box\psi \notin \Sigma_1$. Because any substitution instance of $\mathbf{t}_2$ is in Σ_2, we deduce from $\Box\psi \notin \Sigma_1$ that $\Box(\Box\psi \to (\Box\varphi \vee \Box\neg\Box\varphi)) \in \Sigma_1$. By $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_2, \Theta_2)$, we obtain $\Box\psi \to (\Box\varphi \vee \Box\neg\Box\varphi) \in \Sigma_2$. Since $\Box\psi \in \Sigma_2$, we have $\Box\varphi \vee \Box\neg\Box\varphi \in \Sigma_2$. Recall that our goal is to show $\varphi \in \Sigma_2$. Because $\Box\varphi \to \varphi \in \Sigma_2$ (by $\mathbf{S4} \subseteq \Lambda$), our goal is reduced to establish that $\Box\varphi \in \Sigma_2$. By our assumption of $\Box\varphi \in \Sigma_3$ and $(\Sigma_3, \Theta_3)H^{\Lambda}(\Sigma_3, \Theta_3)$, we get $\neg\Box\varphi \notin \Sigma_3$. Since $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_3, \Theta_3)$, we get $\Box\neg\Box\varphi \notin \Sigma_2$. It follows from $\Box\varphi \vee \Box\neg\Box\varphi \in \Sigma_2$ that $\Box\varphi \in \Sigma_2$, as desired. ■

Theorem 74. The logic **UBiSKt** extended with **S4** and $\mathbf{t}_2$ is sound and strongly complete for the class $\mathbb{T}_2$, i.e., $\Gamma \models_{\mathbb{T}_2} \varphi$ iff $\Gamma \vdash_{\mathbf{UBiSKtS4t_2}} \varphi$ for every set $\Gamma \cup \{\varphi\}$ of formulae.

Proof. Let $\Lambda := \mathbf{UBiSKtS4t}_2$. The soundness follows from Proposition 14 and Theorem 71. For the strong completeness, it suffices to show that the canonical frame $F^{\Lambda}_{(\Gamma,\Delta)}$ belongs to $\mathbb{T}_2$. This is established by Lemmas 73 and 67. ■

Theorem 75. Let $F = (U, H, R)$ be a frame where R is reflexive and transitive. Then the following equivalence holds:

$$F \models (\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r)) \text{ iff } R \subseteq \overleftrightarrow{R} \,;\, H \,;\, \overleftrightarrow{R}.$$

Proof. **Right-to-left direction**: Let us fix any valuation V. Let $M = (F, V)$. Let us fix $u \in U$. We need to show that $M, u \models (\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r))$. Let us take any $v \in U$ such that uHv and $M, v \models \Box(\Box p \vee (\Box q \to \Box r))$. We need to show that $M, v \models \Box p \vee \Box(\Box q \to \Box r)$. Let us assume that $M, v \not\models \Box p$, i.e., we can find a $w \in U$ such that vRw and $M, w \not\models p$. Now we need to show that $M, v \models \Box(\Box q \to \Box r)$. Let us fix any $x \in U$ such that vRx. We show that $M, x \models \Box q \to \Box r$. Fix any $a \in U$ such that xHa and $M, a \models \Box q$. To show that $M, a \models \Box r$, fix any $b \in U$ such that aRb. Our goal is to establish $M, b \models r$.

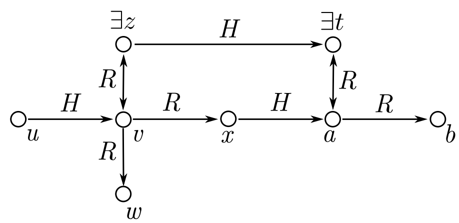


Figure 10: Model constructed for proving the right-to-left direction of Theorem 75

By R being stable, we know that vRa. By R being symmetrically generated, we also know that there are $z, t \in U$ such that $v\overleftrightarrow{R}zHt\overleftrightarrow{R}a$. See the model in Fig. 10. Since vRz and $M, v \models \Box(\Box p \vee (\Box q \to \Box r))$, we have that $M, z \models \Box p \vee (\Box q \to \Box r)$. Since $M, v \not\models \Box p$ and zRv, we know that $M, z \not\models \Box\Box p$. Then, by transitivity of R, $M, z \not\models \Box p$. Thus it must be the case that $M, z \models \Box q \to \Box r$. Then, as zHt, we have the implication from $M, t \models \Box q$ to $M, t \models \Box r$. But since aRt and $M, a \models \Box q$ by assumption, we deduce from transitivity of R that $M, t \models \Box q$. Therefore $M, t \models \Box r$. By transitivity tRb holds from tRa and aRb, and then $M, b \models r$, as wanted.

Left-to-right direction: Assume the validity of the formula in $F = (U, H, R)$. Suppose that xRy. We need to show that $x\overleftrightarrow{R} \,;\, H \,;\, \overleftrightarrow{R}y$. Let us use $\downarrow u$ to mean the set $\{\, v \in U \mid vHu \,\}$. Consider the following valuation V such that $V(p) = U \setminus \downarrow x$, $V(q) = R(y)$ (the set of all R-successors of y), $V(r) := U \setminus \downarrow y$. It is immediate to see that all these sets are upsets. Let us write $M = (F, V)$. By our assumption, we get $M, x \models (\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r))$.

By $x \notin V(p)$ and xRx, we have $M, x \not\models \Box p$. Moreover we can prove that $M, x \not\models \Box(\Box q \to \Box r)$ as follows: By xRy and yHy, it suffices to show that $M, y \models \Box q$ and $M, y \not\models \Box r$. By our definition of V, the former is easy and the latter holds by yRy and $M, y \not\models r$. This finishes showing $M, x \not\models \Box(\Box q \to \Box r)$. Therefore, $M, x \not\models \Box p \vee \Box(\Box q \to \Box r)$ hence $M, x \not\models \Box(\Box p \vee (\Box q \to \Box r))$. So we can find a state $z \in U$ such that xRz and $M, z \not\models \Box p$ and $M, z \not\models \Box q \to \Box r$. It follows from $M, z \not\models \Box p$ that $z(R \,;\, H)x$ and so zRx by stability of R. Thus we have $x\overleftrightarrow{R}z$. On the other hand, from $M, z \not\models \Box q \to \Box r$ we can find a state w such that zHw and $M, w \models \Box q$ and $M, w \not\models \Box r$. By $M, w \not\models \Box r$, we know that $w(R \,;\, H)y$ and so wRy. But it follows from $M, w \models \Box q$ that $R(w) \subseteq R(y)$, which implies yRw hence $w\overleftrightarrow{R}y$. Therefore, we have $x\overleftrightarrow{R}z$, zHw and $w\overleftrightarrow{R}y$, i.e., $x\overleftrightarrow{R} \,;\, H \,;\, \overleftrightarrow{R}y$, as desired. ■

Remark 76. Ono [17] studied several intuitionistic modal logics which correspond to modal logic **S5** based on classical propositional logic. His notion of Kripke frame is equivalent with a frame $F = (U, H, R)$ such that R is reflexive (i.e., $H \subseteq R$) and transitive. In particular, he observed that the formula $(\Box p \to \Box q) \to \Box(\Box p \to \Box q)$ defines the property $R \subseteq H \,;\, (R \cap \breve{R})$ and that $\Box(\Box q \vee q) \to (\Box p \vee \Box q)$ defines $R \subseteq (R \cap \breve{R}) \,;\, H$. He also proved that extensions of the intuitionistic version of classical modal logic **S4** with these formulae are sound and semantically complete (cf. [17, Theorem 3.2]). But the formula $(\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r))$ given in Theorem 75 is not considered in [17].

Definition 77. We use **sg** to mean the formula $(\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r))$. We also use **S5** to mean the set $\mathbf{S4} \cup \{\, \mathbf{t}_2, \mathbf{sg} \,\}$. Let $\mathbb{SG}$ be the class of all frames $F = (U, H, R)$ such that (U, R) is a preorder, i.e., $(U, H, R) \in \mathbb{S}4$, and F is symmetrically generated. We also use $\mathbb{HGS}5$ to mean $\mathbb{HG} \cap \mathbb{T}_2 \cap \mathbb{SG}$, where we recall that $\mathbb{T}_2 \subseteq \mathbb{S}4$.

By Proposition 14, Corollary 16 and Theorems 71 and 75, we obtain the following.

Theorem 78. The logic **UBiSKt** extended with **S5** is sound for the class $\mathbb{HGS}5$, i.e., if $\Gamma \vdash_{\mathbf{UBiSKtS5}} \varphi$ then $\Gamma \models_{\mathbb{HGS}5} \varphi$ for every set $\Gamma \cup \{\varphi\}$ of formulae.

Thus, extending **HUBiSKt** with **S4**-axioms, $\mathbf{t}_2$ and **sg**, we can obtain a logic in which we can represent and reason about hypergraph partitions and related approximations. This extends the connection between rough set theory and the classical modal logic **S5**, to the theory of rough hypergraphs and this intuitionistic modal system, that we call **HUBiSKtS5**. Finally we note that the (strong) completeness result for **HUBiSKtS5** is still open and we conjecture that it holds.

Conjecture 79. The logic **UBiSKt** extended with **S5** is strongly complete for the class $\mathbb{HGS}5$.

5.4. **S5** *Granularity for Spatial Relations*

In Section 4.2 we have seen how connection and other spatial relations can be expressed when we look at regions of a graph at another level of detail, induced by a relation R. We now consider the special case that R has the properties discussed in Section 5.2. We have already seen in Section 5.1 that when R is reflexive and transitive the notion of granulation as was presented in Section 4.2 is equivalent to the lower approximation, or erosion, of the subgraph at issue (by Theorem 66 item 3). That is, in **HUBiSKtS5** we have $\vdash \mathsf{G}\varphi \leftrightarrow \Box\varphi$. We use Proposition 25 to show that the following definitions of coarse spatial relations for an **S5** granularity are the correct special cases of the ones given earlier. We use the subscript $G\mathbf{S5}$ to denote this form of granularity, so that $C_{G\mathbf{S5}}$ is used for coarse connection in this setting. In some of the later definitions the justification requires appeal to additional facts to express formulae in a simple way. In these cases we combine definition and a proof.

Definition 80 (**S5** Coarse Connection)**.** $C_{G\mathbf{S5}}(\varphi, \psi) := \mathsf{E}(\blacklozenge \lozenge \Box\varphi \wedge \Box\psi)$.

This formulation of the predicate of coarse connection in **HUBiSKtS5** supports the view that the classical notion of equivalence relation in the context of hypergraphs, i.e. a stable relation that is additionally reflexive, transitive and symmetric, is too restrictive, as argued in [25]. Indeed symmetry of R in a frame corresponds to the property $R = \backsim R$, that implies the validity of the formula $\blacklozenge p \leftrightarrow \lozenge p$ in all symmetric frames (see correspondence results from [30]). Therefore, in the extension of H**UBiSKtS4** with the symmetry axiom, we can derive the following: $\mathsf{E}(\blacklozenge \lozenge \Box\varphi \wedge \Box\psi) \leftrightarrow \mathsf{E}(\blacklozenge \blacklozenge \Box\varphi \wedge \Box\psi)$ by the symmetry axiom, $\mathsf{E}(\blacklozenge \blacklozenge \Box\varphi \wedge \Box\psi) \leftrightarrow \mathsf{E}(\blacklozenge \Box\varphi \wedge \Box\psi)$ by the reflexivity and transitivity axioms and $\mathsf{E}(\blacklozenge \Box\varphi \wedge \Box\psi) \leftrightarrow \mathsf{E}(\Box\varphi \wedge \Box\psi)$ by Theorem 66 item 1, thus $\mathsf{E}(\blacklozenge \lozenge \Box\varphi \wedge \Box\psi) \leftrightarrow \mathsf{E}(\Box\varphi \wedge \Box\psi)$ holds in this system. Hence the notion of coarse connection collapses to a notion of overlapping, the formula $\mathsf{E}(\Box\varphi \wedge \Box\psi)$ meaning that the granulation of φ overlaps with the granulation of ψ. Cases of external-connection, i.e. edge-connection, will not occur in such a setting.

To give an example of **S5**-connection, let us look at the graph-partition in Fig. 9. R is transitive, reflexive and symmetrically generated, but not symmetric, and (U, H, R) is a frame, let us call it F (reflexivity and transitivity imply stability of R, as already noticed); let us impose a valuation V such that $V(p) = \{a\}$ and $V(q) = \{e\}$ for propositional variables p, q in the language. We remark that these sets are upsets. Let $M = (F, V)$. Then the granulation of the subgraph $\{a\}$ is $[\![\Box p]\!]_M = R \ominus [\![p]\!]_M = R \ominus \{a\} = \{a\}$ and the granulation of subgraph $\{e\}$ is $[\![\Box q]\!]_M = R \ominus [\![p]\!]_M = R \ominus \{e\} = \{e\}$. Also $[\![\blacklozenge \lozenge \Box p]\!]_M = U$, (notice that here we see that the formula $\blacklozenge \lozenge \Box p \leftrightarrow \Box p$ is not a theorem in H**UBiSKtS5**, therefore connection can be distinguished from overlapping) and $M \models \mathsf{E}(\blacklozenge \lozenge \Box p \wedge \Box q)$. Indeed the granulations of $\{a\}$ and $\{e\}$ are connected by the edge $Y = \{b, c, d\}$ in the quotient structure.
Notice that, when $H = I$, so when (U, H) is a set, the symmetric generation property of R is equivalent to symmetry of R, as $R \subseteq \overleftrightarrow{R} \,;\, H \,;\, \overleftrightarrow{R}$ iff $R \subseteq \breve{R}$ iff $R = \breve{R}$ iff $R = \backsim R$ since $\breve{R} = \backsim R$ when $H = I$. Indeed in this case it is correct to assume that granular connection collapses to a form of overlapping, as no edges are present in a set, so the only possible form of connection between two sets is when they overlap.

Proposition and Definition 81 (**S5** Coarse Parthood)**.**
In H**UBiSKtS5** it is a theorem that $P_G(\varphi, \psi) \leftrightarrow \mathsf{A}(\Box\varphi \to \Box\psi)$. We thus define $P_{G\mathbf{S5}}(\varphi, \psi) := \mathsf{A}(\Box\varphi \to \Box\psi)$.

Proof. The spatial relation of coarse parthood $P_G(\varphi, \psi)$ was defined in Section 4.2 by the formula $\mathsf{A}(\Box\mathsf{G}\varphi \to \Box\mathsf{G}\psi)$. By Theorem 66 item 3, we deduce that the following holds in **S5**: $\vdash \mathsf{A}(\Box\mathsf{G}\varphi \to \Box\mathsf{G}\psi) \leftrightarrow \mathsf{A}(\Box\Box\varphi \to \Box\Box\psi)$. Also the following holds by reflexivity and transitivity axioms: $\vdash \mathsf{A}(\Box\Box\varphi \to \Box\Box\psi) \leftrightarrow \mathsf{A}(\Box\varphi \to \Box\psi)$. Thus $\vdash \mathsf{A}(\Box\mathsf{G}\varphi \to \Box\mathsf{G}\psi) \leftrightarrow \mathsf{A}(\Box\varphi \to \Box\psi)$. ■

Proposition and Definition 82 (**S5** Coarse Overlapping)**.** In H**UBiSKtS5** it is a theorem that $O_G(\varphi, \psi) \leftrightarrow \mathsf{E}(\Box\varphi \wedge \Box\psi)$. Thus we define $O_{G\mathbf{S5}}(\varphi, \psi) := \mathsf{E}(\Box\varphi \wedge \Box\psi)$.

Proof. The spatial relation of coarse overlapping $O_G(\varphi, \psi)$ was defined in Section 4.2 by the formula $\mathsf{E}(\blacklozenge \Box(\mathsf{G}\varphi \wedge \mathsf{G}\psi))$. By Theorem 66 item 3, in **S5** we have $\vdash \mathsf{E}(\blacklozenge \Box(\mathsf{G}\varphi \wedge \mathsf{G}\psi)) \leftrightarrow \mathsf{E}(\blacklozenge \Box(\Box\varphi \wedge \Box\psi))$. Moreover we have that in this system $\vdash \mathsf{E}(\blacklozenge\Box(\Box\varphi\wedge\Box\psi)\leftrightarrow\mathsf{E}(\Box(\Box\varphi\wedge\Box\psi)$ by Theorem 66 item 1, and $\vdash \mathsf{E}(\Box(\Box\varphi\wedge\Box\psi)\leftrightarrow\mathsf{E}(\Box\Box\varphi\wedge\Box\Box\psi)$ by distributivity of $\Box$ over $\wedge$. Finally, by reflexivity and transititivity axioms, $\vdash \mathsf{E}(\Box\Box\varphi\wedge\Box\Box\psi)\leftrightarrow\mathsf{E}(\Box\varphi\wedge\Box\psi)$. ■

Definition 83 (**S5** Coarse Non-tangential part)**.**
$NTP_{G\mathbf{S5}}(\varphi,\psi) := \mathsf{A}(\Box\varphi\to\Box\psi)\wedge\mathsf{A}(\Diamond\Box\varphi\to\Box\psi)$.

Also here a remark is important, as for coarse connection. In the extension of **HUBiSKtS4** with the symmetry axiom $\blacklozenge p\leftrightarrow\Diamond p$ we have $\vdash \mathsf{A}(\Diamond\Box\varphi\to\Box\psi)\leftrightarrow\mathsf{A}(\blacklozenge\Box\varphi\to\Box\psi)$, and $\vdash \mathsf{A}(\blacklozenge\Box\varphi\to\psi)\leftrightarrow\mathsf{A}(\Box\varphi\to\Box\psi)$ by item 1 of Theorem 66. In this extension the predicate of Non-tangential parthood collapses with the predicate of parthood. When $H=I$, so when (U,H) is a set and thus the symmetrically generated property of R is equivalent to the symmetry of R, it is correct to say that Non-tangential parthood is just parthood: a set has no edges, thus it has no boundary-nodes, therefore any of its subsets is a Non-tangential part.

Finally, let us look at the predicate of tangential-part in **HUBiSKtS5**.

Proposition and Definition 84 (**S5** Coarse Tangential part)**.** The biconditional $TP_G(\varphi,\psi)\leftrightarrow\mathsf{A}(\Box\varphi\to\Box\psi)\wedge\mathsf{E}(\Diamond\Box\varphi\prec\Box\psi)$ is a theorem in **HUBiSKtS5**. Thus we define $TP_{G\mathbf{S5}}(\varphi,\psi) := \mathsf{A}(\Box\varphi\to\Box\psi)\wedge\mathsf{E}(\Diamond\Box\varphi\prec\Box\psi)$.

Proof. The first conjunct is the spatial relation of coarse parthood in **S5**, that has already been analysed above. Then we have that $\vdash \mathsf{E}(\Diamond\mathsf{G}\varphi\prec\Box\mathsf{G}\psi)\leftrightarrow\mathsf{E}(\Diamond\Box\varphi\prec\Box\Box\psi)$ by item 3 of Theorem 66, and $\vdash \mathsf{E}(\Diamond\Box\varphi\prec\Box\Box\psi)\leftrightarrow\mathsf{E}(\Diamond\Box\varphi\prec\Box\psi)$ by transitivity and reflexivity axioms. ■

When a stable relation R is reflexive, transitive and symmetric, i.e. it is an equivalence relation, the predicate of coarse tangential part from Section 4.2 can be reduced to $\mathsf{A}(\Box\varphi\to\Box\psi)\wedge\mathsf{E}(\Box\varphi\prec\Box\psi)$. This leads to contradiction, as $\Box\varphi$ would be a part and a non-part of $\Box\psi$ at the same time. The spatial relation of coarse tangential parts could not occur in such a system, because its formula is equivalent to contradiction. When (U,H) is a set, i.e. when $H=I$, it is correct to assume that the spatial relation of coarse tangential part between two regions never occurs, because, as we have seen earlier, only Non-tangential parts of a set exist, since a set is a hypergraph with an empty set of edges.

6. Conclusions and Further Work

In this paper we have introduced a sound, complete and decidable axiomatization for the logic **UBiSKt**. We have explored the role of the logic **UBiSKt** in representing spatial knowledge at different levels of detail. We have shown how to extend the well known connection between the classical modal system **S5** and equivalence relations to the setting of graphs and hypergraphs, introducing a new bi-intuitionistic modal system, that we called **HUBiSKtS5**. We have seen how to encode various spatial relations within this system.

There are many directions for further work. A similarity relation on a set is usually obtained from an equivalence relation by dropping the transitivity constraint. We might ask whether similarity relations on hypergraphs correspond to reflexive, symmetrically generated and two-tier relations.

We have considered graphs and hypergraphs; these are one-dimensional discrete structures. Mathematical morphology can also be applied to higher dimensional generalisations of hypergraphs, as shown in [8], where morphological operators for simplicial complexes are considered. The description of hypergraphs as posets suggests that it is possible to consider these higher dimensional structures, and therefore to obtain a more general logic for partitions.

In this paper we focused on the Hilbert-style axiomatization for the logic. However, a tableau calculus for **UBiSKtS5** could be obtained, by extending the tableau calculus for **UBiSKt**, Tab**UBiSKt**, presented in [26]. It could be automated using the theorem-prover generator MeTtel [31], as it has been done for Tab**UBiSKt** [26].

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Appendix A. Proof of Proposition 24

Proof. We provide proofs of a selection of these.

Item 2 is the adjunction property connecting $\prec$ and $\vee$. Given $\vdash (\psi \prec \gamma) \to \rho$ we obtain $\vdash (\gamma \vee (\psi \prec \gamma)) \to (\gamma \vee \rho)$ by monotonicity of $\vee$ in the second component: $\vdash \delta_1 \to \delta_2$ implies $\vdash (\varphi \vee \delta_1) \to (\varphi \vee \delta_2)$. Then use $\vdash \psi \to (\gamma \vee (\psi \prec \gamma))$ (A10). For the converse, given $\vdash \psi \to (\gamma \vee \rho)$, we get $\vdash (\psi \prec \gamma) \to ((\gamma \vee \rho) \prec \gamma)$ by (Mon-$\prec$). Then use $\vdash ((\gamma \vee \rho) \prec \gamma) \to \rho$ (A11) to complete the proof.

For item 3 it suffices to show $\vdash \varphi \to \psi$ implies $\vdash (\gamma \prec \psi) \to (\gamma \prec \varphi)$. From $\vdash \varphi \to \psi$ use monotonicity of $\vee$ in first component to obtain $\vdash (\varphi \vee (\gamma \prec \varphi)) \to (\psi \vee (\gamma \prec \varphi))$. Now, $\vdash \gamma \to (\varphi \vee (\gamma \prec \varphi))$ (A10), so that $\gamma \to (\psi \vee (\gamma \prec \varphi))$, and the result follows by item 2.

Item 4 is shown as follows. Since $\vdash \varphi \to (\varphi \vee \bot)$ is a theorem of intuitionistic logic, we obtain $\vdash (\varphi \prec \varphi) \to \bot$ by item 2. For item 5, we show $\vdash \varphi \vee (\top \prec \varphi)$, which is equivalent to $\vdash \top \to (\varphi \vee (\top \prec \varphi))$. We can derive this from $\vdash (\top \prec \varphi) \to (\top \prec \varphi)$ by item 2.

Item 6 is shown by items 5 and 2.

Item 7 can be derived with the help of item 5.

Items 8 and 9 are easy. Item 10 is an easy consequence of items 8 and 9. Item 11 is shown as follows. Since $\vdash \varphi \to \neg\neg\varphi$ and $\vdash \neg\neg\varphi \to \lrcorner\neg\varphi$ by item 7, we obtain $\vdash \varphi \to \lrcorner\neg\varphi$, as desired. Item 12 is similarly shown to item 11. Item 13 is shown as: Suppose that $\vdash \varphi \to \psi$. By item 6, we obtain $\vdash \lrcorner\lrcorner\varphi \to \psi$. By item 9, we conclude $\vdash \lrcorner\psi \to \lrcorner\varphi$, as desired. Let us move to item 14. We proceed as follows: By intuitionistic logic we have $\vdash (\varphi \wedge \lrcorner\varphi) \to \varphi$. Then we deduce from item 13 that $\vdash \lrcorner\varphi \to \lrcorner(\varphi \wedge \lrcorner\varphi)$, which implies $\vdash (\varphi \wedge \lrcorner\varphi) \to \lrcorner(\varphi \wedge \lrcorner\varphi)$ by intuitionistic logic. By item 6, we get $\vdash \lrcorner\lrcorner(\varphi \wedge \lrcorner\varphi) \to \lrcorner(\varphi \wedge \lrcorner\varphi)$. With the help of item 2 and the definition of $\lrcorner$, this is equivalent with $\vdash \lrcorner(\varphi \wedge \lrcorner\varphi)$, as required.

Items 16 and 17 are established by (A16) $\mathsf{A}\,p \to p$ and (A17) $\mathsf{A}\,p \to \mathsf{A}\,\mathsf{A}\,p$ by item 15 respectively.

For item 18, it suffices to show the right-to-left implication since the other half is just an instance of (A16). The right-to-left direction is obtained from item 17 by item 15.

For item 19, it suffices to prove the left-to-right implication by (A16). By item 15, it suffices to prove $\vdash \mathsf{E}\,\neg\,\mathsf{A}\,\varphi \to \neg\,\mathsf{A}\,\varphi$, i.e., $\vdash (\mathsf{E}\,\neg\,\mathsf{A}\,\varphi \wedge \mathsf{A}\,\varphi) \to \bot$. This can be established by $\vdash \mathsf{E}\,\bot \leftrightarrow \bot$ and (A19).

Item 20 is established by (A20) and the commutativity of $\Box$ over finite conjunctions.

Item 21 is shown by (A20) and the equivalence of $\mathsf{A}\,\mathsf{A}\,\varphi$ and $\mathsf{A}\,\varphi$ (due to (A16) and (A17)).

Item 22 is established from item 21 and the adjunction "$\blacklozenge \dashv \Box$" (Proposition 23).

For item 23, it suffices to prove $\vdash \lrcorner\,\mathsf{A}\,\varphi \to \neg\,\mathsf{A}\,\varphi$ since the other half is shown by item 7. To show our goal, it suffices to show $\vdash ((\top \prec \mathsf{A}\,\varphi) \wedge \mathsf{A}\,\varphi) \to \bot$. It follows from (A17) and monotonicity of $\wedge$ that $\vdash ((\top \prec \mathsf{A}\,\varphi) \wedge \mathsf{A}\,\varphi) \to ((\top \prec \mathsf{A}\,\varphi) \wedge \mathsf{A}\,\mathsf{A}\,\varphi)$. By (A22), we obtain $\vdash ((\top \prec \mathsf{A}\,\varphi) \wedge \mathsf{A}\,\mathsf{A}\,\varphi) \to ((\mathsf{A}\,\varphi \wedge \top) \prec \mathsf{A}\,\varphi)$. So, we get

$\vdash ((\top \prec \mathsf{A}\,\varphi) \wedge \mathsf{A}\,\varphi) \to ((\mathsf{A}\,\varphi \wedge \top) \prec \mathsf{A}\,\varphi)$, by A16 and A17. By noting that $((\mathsf{A}\,\varphi \wedge \top) \prec \mathsf{A}\,\varphi) \to \bot$ by items 4 and 3, we can obtain our goal.

Item 24 is established from items 5 and 23.

For item 25, the left-to-right implication is just item 7 and the other half is shown as follows. This follows from $\vdash (\neg\!\!\!\!\!\!\!\!\; {}^{\llcorner}\,\mathsf{E}\,\varphi \wedge \mathsf{E}\,\varphi) \to \bot$. By item 18, we show $\vdash ((\top \prec \mathsf{E}\,\varphi) \wedge \mathsf{A}\,\mathsf{E}\,\varphi) \to \bot$. This follows from $\vdash ((\top \prec \mathsf{E}\,\varphi) \wedge \mathsf{A}\,\mathsf{E}\,\varphi) \to ((\top \wedge \mathsf{E}\,\varphi) \prec \mathsf{E}\,\varphi)$, established by (A22) and from $\vdash ((\top \wedge \mathsf{E}\,\varphi) \prec \mathsf{E}\,\varphi) \to \bot$, established by items 3 and 4. Item 26 is established similarly to item 24.

For item 27, we proceed as follows. For the left-to-right direction, it suffices to show that $\vdash (\mathsf{E}\,\varphi \wedge \mathsf{A}\,\neg\varphi) \to \bot$. But this is easy from the axiom (A19) and $\vdash \mathsf{E}\,\bot \leftrightarrow \bot$. For the right-to-left direction, it suffices to prove $\vdash (\neg\,\mathsf{A}\,\neg\varphi \wedge \neg\,\mathsf{E}\,\varphi) \to \bot$ by item 26. By the axiom (A18), we show $\vdash (\neg\,\mathsf{A}\,\neg\varphi \wedge \mathsf{A}\,\neg\varphi) \to \bot$, which is trivial.

Let us move to item 28. It suffices to show the left-to-right implication. To show it, it suffices to prove $\vdash \mathsf{A}(\daleth\,\varphi \to \psi) \to (\daleth\,\psi \to \varphi)$ by the rule (MonA) and the axiom (A17). By the axiom (A22), we get $\vdash (\mathsf{A}(\daleth\,\varphi \to \psi) \wedge \daleth\,\psi) \to ((\daleth\,\varphi \to \psi) \prec \psi)$ (It is noted that $(\mathsf{A}(\daleth\,\varphi \to \psi) \wedge (\top \prec \psi)) \to (((\daleth\,\varphi \to \psi) \wedge \top) \to \psi)$ is an instance of (A22) and it is easy to see that $\vdash ((\daleth\,\varphi \to \psi) \wedge \top) \leftrightarrow (\daleth\,\varphi \to \psi)$). We note that $\vdash (\daleth\,\varphi \to \psi) \to (\psi \vee \varphi)$, which is derivable with the help of $\vdash \varphi \vee \daleth\,\varphi$ (due to item 5). It follows from item 2 that $\vdash ((\daleth\,\varphi \to \psi) \prec \psi) \to \varphi$. Thus, we get $\vdash (\mathsf{A}(\daleth\,\varphi \to \psi) \wedge \daleth\,\psi) \to \varphi$, which is equivalent with $\vdash (\mathsf{A}(\daleth\,\varphi \to \psi) \to (\daleth\,\psi \to \varphi)$. This finishes to prove item 28.

Finally, let us show item 29. By item 28, we obtain $\vdash \mathsf{A}(\daleth\,\neg\varphi \to \neg\psi) \leftrightarrow \mathsf{A}(\daleth\,\neg\psi \to \neg\varphi)$, which is equivalent with $\vdash \mathsf{A}\,\neg(\daleth\,\neg\varphi \wedge \psi) \leftrightarrow \mathsf{A}\,\neg(\daleth\,\neg\psi \wedge \varphi)$. This implies $\vdash \neg\,\mathsf{A}\,\neg(\daleth\,\neg\varphi \wedge \psi) \leftrightarrow \neg\,\mathsf{A}\,\neg(\daleth\,\neg\psi \wedge \varphi)$. By item 27, we obtain our desired goal. ∎

Appendix B. Proof of Lemma 32

First we need the following proposition.

Proposition 85. The set of *disjunctive universal formulae* is defined inductively as:

$$\alpha ::= \bot \mid \mathsf{A}\,\varphi \mid \alpha \vee \alpha,$$

where $\varphi \in \mathsf{Form}_{\mathcal{L}}$. Let Λ be a *ubist*-logic and α be a disjunctive universal formula. Then we have:

1. $\vdash_\Lambda (\alpha \leftrightarrow \top) \vee (\alpha \leftrightarrow \bot)$.
2. $\vdash_\Lambda \alpha \vee \neg\alpha$.
3. $\vdash_\Lambda (\psi \to (\rho \vee \alpha)) \to ((\psi \to \rho) \vee \alpha)$ for every $\psi, \rho \in \mathsf{Form}_{\mathcal{L}}$.
4. $\vdash_\Lambda \Box(\psi \vee \alpha) \to ((\Box\psi) \vee \alpha)$ for every $\psi \in \mathsf{Form}_{\mathcal{L}}$.
5. $\vdash_\Lambda \mathsf{A}(\psi \vee \alpha) \to ((\mathsf{A}\,\psi) \vee \alpha)$ for every $\psi \in \mathsf{Form}_{\mathcal{L}}$.

Proof. 1. By induction on the complexity of α. When α is of the form $\mathsf{A}\,\varphi$, our goal follows from Proposition 24(24). When α is of the form $\alpha \vee \beta$, it is trivial.

2. This is an easy consequence of item 1.

3. This is also easily established by item1.

4. Let us put α as $\bigvee_{j\in J} \mathsf{A}\,\varphi_j$. By item 2, it suffices to show

$$\vdash_\Lambda \left(\neg \bigvee_{j\in J} \mathsf{A}\,\varphi_j \wedge \Box\left(\psi \vee \bigvee_{j\in J} \mathsf{A}\,\varphi_j\right)\right) \to \Box\psi.$$

By Proposition 24(1), it suffices to show

$$\vdash_\Lambda \left(\bigwedge_{j\in J} \neg\,\mathsf{A}\,\varphi_j \wedge \Box\left(\psi \vee \bigvee_{j\in J} \mathsf{A}\,\varphi_j\right)\right) \to \Box\psi.$$

Moreover, by Proposition 24(19), it suffices to show

$$\vdash_\Lambda \left(\bigwedge_{j\in J} \mathsf{A}\,\neg\,\mathsf{A}\,\varphi_j \wedge \Box\left(\psi \vee \bigvee_{j\in J} \mathsf{A}\,\varphi_j\right)\right) \to \Box\psi.$$

Since A commutes over finite conjunction, we show that

$$\vdash_\Lambda \left(\mathsf{A}\left(\bigwedge_{j\in J} \neg\,\mathsf{A}\,\varphi_j\right) \wedge \Box\left(\psi \vee \bigvee_{j\in J} \mathsf{A}\,\varphi_j\right)\right) \to \Box\psi,$$

which is derived from Proposition 24(20).

5. Let us put α as $\bigvee_{j\in J} \mathsf{A}\,\varphi_j$ as above. By item 2, it suffices to show

$$\vdash_\Lambda \left(\neg \bigvee_{j\in J} \mathsf{A}\,\varphi_j \wedge \mathsf{A}\left(\psi \vee \bigvee_{j\in J} \mathsf{A}\,\varphi_j\right)\right) \to \mathsf{A}\,\psi.$$

By Proposition 24(1), it suffices to show

$$\vdash_\Lambda \left(\bigwedge_{j\in J} \neg\,\mathsf{A}\,\varphi_j \wedge \mathsf{A}\left(\psi \vee \bigvee_{j\in J} \mathsf{A}\,\varphi_j\right)\right) \to \mathsf{A}\,\psi.$$

Moreover, by Proposition 24(19), it suffices to show

$$\vdash_\Lambda \left(\bigwedge_{j\in J} \mathsf{A}\,\neg\,\mathsf{A}\,\varphi_j \wedge \mathsf{A}\left(\psi \vee \bigvee_{j\in J} \mathsf{A}\,\varphi_j\right)\right) \to \mathsf{A}\,\psi.$$

Since A commutes over finite conjunctions, we show that

$$\vdash_\Lambda \left(\mathsf{A}\left(\bigwedge_{j\in J} \neg\,\mathsf{A}\,\varphi_j\right) \wedge \mathsf{A}\left(\psi \vee \bigvee_{j\in J} \mathsf{A}\,\varphi_j\right)\right) \to \mathsf{A}\,\psi,$$

which is derived from the distributivity of A over finite conjunctions. ■

Now we can give the proof of Lemma 32 as follows.

Proof. 1. Let $\psi \to \rho \notin \Sigma$. Suppose for contradiction that

$$(\{\,\psi\,\} \cup \Sigma \cup \{\,\mathsf{A}\,\gamma \mid \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\rho\,\} \cup \{\,\mathsf{A}\,\delta \mid \mathsf{A}\,\delta \in \Delta\,\})$$

is Λ-provable. Then there exist $\sigma_1, \ldots, \sigma_k \in \Sigma$ and $\mathsf{A}\,\gamma_1, \ldots, \mathsf{A}\,\gamma_n \in \Gamma$ and $\mathsf{A}\,\delta_1, \ldots, \mathsf{A}\,\delta_m \in \Delta$ such that

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \bigwedge_{1\leqslant l\leqslant k} \sigma_l \wedge \psi\right) \to \left(\rho \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right).$$

It follows by Proposition 24(2) that

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \bigwedge_{1\leqslant l\leqslant k} \sigma_l\right) \to \left(\psi \to \left(\rho \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)\right).$$

Since $\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j$ is a disjunctive universal formula, we have

$$\vdash_\Lambda \left(\psi \to \left(\rho \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)\right) \to \left((\psi \to \rho) \vee \left(\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)\right)$$

by Proposition 85(3). Therefore, we obtain:

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \bigwedge_{1\leqslant l\leqslant k} \sigma_l\right) \to \left((\psi \to \rho) \vee \left(\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)\right).$$

By $(\Gamma, \Delta) S^\Lambda (\Sigma, \Theta)$ and $\mathsf{A}\,\gamma_i \in \Gamma$ and $\mathsf{A}\,\delta_j \in \Delta$, we obtain $\mathsf{A}\,\gamma_i \in \Sigma$ and $\mathsf{A}\,\delta_j \in \Theta$ hence $\mathsf{A}\,\delta_j \notin \Sigma$. Together with $\sigma_l \in \Sigma$, these imply $\psi \to \rho \in \Sigma$. A contradiction with our initial supposition.

2. Let $\psi \prec \rho \in \Sigma$. Suppose for contradiction that

$$(\{\,\psi\,\} \cup \{\,\mathsf{A}\,\gamma \mid \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\rho\,\} \cup \Theta \cup \{\,\mathsf{A}\,\delta \mid \mathsf{A}\,\delta \in \Delta\,\})$$

is Λ-provable. Then there exist $\mathsf{A}\,\gamma_1, \ldots, \mathsf{A}\,\gamma_n \in \Gamma$ and $\mathsf{A}\,\delta_1, \ldots, \mathsf{A}\,\delta_m \in \Delta$ and $\theta_1, \ldots, \theta_k \in \Theta$ such that

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \psi\right) \to \left(\rho \vee \bigvee_{1\leqslant l\leqslant k} \theta_l \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right).$$

It follows that

$$\vdash_\Lambda \left(\left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \psi\right) \prec \rho\right) \to \left(\bigvee_{1\leqslant l\leqslant k} \theta_l \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right).$$

By the distributivity of A over finite conjunctions, (`A17`) and (`A22`), we have

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i\right) \wedge (\psi \prec \rho) \to \left(\left(\left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i\right) \wedge \psi\right) \prec \rho\right).$$

Therefore, we get

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i\right) \wedge (\psi \prec \rho) \to \left(\bigvee_{1\leqslant l\leqslant k} \theta_l \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right).$$

By $(\Gamma,\Delta)S^{\Lambda}(\Sigma,\Theta)$ and $\mathsf{A}\,\gamma_i \in \Gamma$ and $\mathsf{A}\,\delta_j \in \Delta$, we obtain $\mathsf{A}\,\gamma_i \in \Sigma$ and $\mathsf{A}\,\delta_j \in \Theta$. Together with $\psi \prec \rho \in \Sigma$ and $\theta_l \in \Theta$, we obtain the Λ-provability of (Σ,Θ). A contradiction.

3. Let $\Box\psi \notin \Sigma$. Suppose to the contrary that

$$(\{\,\sigma \mid \Box\sigma \in \Sigma\,\} \cup \{\,\mathsf{A}\,\gamma \mid \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\psi\,\} \cup \{\,\mathsf{A}\,\delta \mid \mathsf{A}\,\delta \in \Delta\,\})$$

is Λ-provable. Then there exist $\sigma_1,\ldots,\sigma_k$ and $\mathsf{A}\,\gamma_1,\ldots,\mathsf{A}\,\gamma_n \in \Gamma$ and $\mathsf{A}\,\delta_1,\ldots,\mathsf{A}\,\delta_m \in \Delta$ such that

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \bigwedge_{1\leqslant l\leqslant k} \sigma_l\right) \to \left(\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)$$

and $\Box\sigma_l \in \Sigma$ $(1 \leqslant l \leqslant k)$. By ($\mathtt{Mon}\Box$),

$$\vdash_\Lambda \Box\left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \bigwedge_{1\leqslant l\leqslant k} \sigma_l\right) \to \Box\left(\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right).$$

Since $\Box$ distributes over finite conjunctions and $\vdash_\Lambda \mathsf{A}\,\varphi \to \Box\mathsf{A}\,\varphi$ (by Proposition 24(21)), we obtain

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \bigwedge_{1\leqslant l\leqslant k} \Box\sigma_l\right) \to \Box\left(\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right).$$

Since $\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j$ is a disjunctive universal formula, we have

$$\vdash_\Lambda \Box\left(\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right) \to \left(\Box\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)$$

by Proposition 85(4). Thus we obtain

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \bigwedge_{1\leqslant l\leqslant k} \Box\sigma_l\right) \to \left(\Box\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right).$$

By $(\Gamma,\Delta)S^{\Lambda}(\Sigma,\Theta)$ and $\mathsf{A}\,\gamma_i \in \Gamma$ and $\mathsf{A}\,\delta_j \in \Delta$, we obtain $\mathsf{A}\,\gamma_i \in \Sigma$ and $\mathsf{A}\,\delta_j \in \Theta$ hence $\mathsf{A}\,\delta_j \notin \Sigma$. Together with $\Box\sigma_l \in \Sigma$, we obtain $\Box\psi \in \Sigma$ but this is a contradiction.

4. Let $\blacklozenge\,\psi \in \Sigma$. Suppose for contradiction that

$$(\{\,\psi\,\} \cup \{\,\mathsf{A}\,\gamma \mid \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\theta \mid \blacklozenge\,\theta \in \Theta\,\} \cup \{\,\mathsf{A}\,\delta \mid \mathsf{A}\,\delta \in \Delta\,\})$$

is Λ-provable. Then there exist $\mathsf{A}\,\gamma_1,\ldots,\mathsf{A}\,\gamma_n \in \Gamma$ and $\mathsf{A}\,\delta_1,\ldots,\mathsf{A}\,\delta_m \in \Delta$ and $\theta_1,\ldots,\theta_k$ such that

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \psi\right) \to \left(\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j \vee \bigvee_{1\leqslant l\leqslant k} \theta_l\right)$$

and $\blacklozenge\,\theta_l \in \Theta$ $(1 \leqslant l \leqslant k)$. By ($\mathtt{Mon}\blacklozenge$), we have

$$\vdash_\Lambda \blacklozenge\left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \psi\right) \to \blacklozenge\left(\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j \vee \bigvee_{1\leqslant l\leqslant k} \theta_l\right).$$

Since $\blacklozenge$ distributes over finite disjunctions and $\vdash_\Lambda \blacklozenge \mathsf{A}\,\varphi \to \mathsf{A}\,\varphi$ (due to Proposition 24(22), we have

$$\vdash_\Lambda \blacklozenge\left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \psi\right) \to \left(\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j \vee \bigvee_{1\leqslant l\leqslant k} \blacklozenge\,\theta_l\right).$$

Again by $\vdash_\Lambda \blacklozenge \mathsf{A}\,\varphi \to \mathsf{A}\,\varphi$, (A17), (A21) and distributivity of A over finite conjunctions, we have

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \blacklozenge\,\psi\right) \to \blacklozenge\left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \psi\right).$$

Thus, we have

$$\vdash_\Lambda \left(\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \wedge \blacklozenge\,\psi\right) \to \left(\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j \vee \bigvee_{1\leqslant l\leqslant k} \blacklozenge\,\theta_l\right).$$

By $(\Gamma,\Delta)S^\Lambda(\Sigma,\Theta)$ and $\mathsf{A}\,\gamma_i \in \Gamma$ and $\mathsf{A}\,\delta_j \in \Delta$, we obtain $\mathsf{A}\,\gamma_i \in \Sigma$ and $\mathsf{A}\,\delta_j \in \Theta$ hence $\mathsf{A}\,\delta_j \notin \Sigma$. Together with $\blacklozenge\,\psi \in \Sigma$, we get a contradiction with $\blacklozenge\,\theta_l \in \Theta$ for all $1 \leqslant l \leqslant k$.

5. Suppose that $\mathsf{A}\,\psi \notin \Sigma$. Assume for contradiction that

$$(\{\,\mathsf{A}\,\gamma \mid \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\psi\,\} \cup \{\,\mathsf{A}\,\delta \mid \mathsf{A}\,\delta \in \Delta\,\})$$

is Λ-provable. Then there exist $\mathsf{A}\,\gamma_1,\ldots,\mathsf{A}\,\gamma_n \in \Gamma$ and $\mathsf{A}\,\delta_1,\ldots,\mathsf{A}\,\delta_m \in \Delta$ such that $\vdash_\Lambda \bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \to \left(\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)$. By (Mon A), we obtain $\vdash_\Lambda \mathsf{A}\bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \to \mathsf{A}\left(\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)$. By the distributivity of A over finite conjunctions and $\vdash_\Lambda \mathsf{A}\,\varphi \leftrightarrow \mathsf{A}\,\mathsf{A}\,\varphi$ due to (A16) and (A17)), $\vdash_\Lambda \bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \to \mathsf{A}\left(\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)$. Since $\bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j$ is a disjunctive universal formula, we have

$$\vdash_\Lambda \mathsf{A}\left(\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right) \to \left(\mathsf{A}\,\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right)$$

by Proposition 85(5). Thus, now we get

$$\vdash_\Lambda \bigwedge_{1\leqslant i\leqslant n} \mathsf{A}\,\gamma_i \to \left(\mathsf{A}\,\psi \vee \bigvee_{1\leqslant j\leqslant m} \mathsf{A}\,\delta_j\right).$$

By $(\Gamma,\Delta)S^\Lambda(\Sigma,\Theta)$ and $\mathsf{A}\,\gamma_i \in \Gamma$ and $\mathsf{A}\,\delta_j \in \Delta$, we obtain $\mathsf{A}\,\gamma_i \in \Sigma$ and $\mathsf{A}\,\delta_j \in \Theta$ hence $\mathsf{A}\,\delta_j \notin \Sigma$. These implies $\mathsf{A}\,\psi \in \Sigma$, a contradiction.

6. Suppose that $\mathsf{E}\,\psi \in \Sigma$. Assume for contradiction that

$$(\{\,\psi\,\} \cup \{\,\mathsf{A}\,\gamma \mid \mathsf{A}\,\gamma \in \Gamma\,\}, \{\,\mathsf{A}\,\delta \mid \mathsf{A}\,\delta \in \Delta\,\})$$

is Λ-provable. Then there exist $\mathsf{A}\,\gamma_1, \ldots, \mathsf{A}\,\gamma_n \in \Gamma$ and $\mathsf{A}\,\delta_1, \ldots, \mathsf{A}\,\delta_m \in \Delta$ such that $\vdash_\Lambda \psi \wedge \bigwedge_{1 \leqslant i \leqslant n} \mathsf{A}\,\gamma_i \to \bigvee_{1 \leqslant j \leqslant m} \mathsf{A}\,\delta_j$. By ($\mathtt{Mon}\,\mathsf{E}$), we get $\vdash_\Lambda \mathsf{E}(\psi \wedge \bigwedge_{1 \leqslant i \leqslant n} \mathsf{A}\,\gamma_i) \to \mathsf{E}\left(\bigvee_{1 \leqslant j \leqslant m} \mathsf{A}\,\delta_j\right)$. Since E distributes over finite disjunctions and $\vdash_\Lambda \mathsf{E}\,\mathsf{A}\,\varphi \leftrightarrow \mathsf{A}\,\varphi$ (by Proposition 24(16) and ($\mathtt{A17}$) and "$\mathsf{E} \dashv \mathsf{A}$", i.e., Proposition 24(15)), the following holds: $\vdash_\Lambda \mathsf{E}(\psi \wedge \bigwedge_{1 \leqslant i \leqslant n} \mathsf{A}\,\gamma_i) \to \left(\bigvee_{1 \leqslant j \leqslant m} \mathsf{A}\,\delta_j\right)$. By the distributivity of A over finite conjunctions and the axioms ($\mathtt{A17}$) and ($\mathtt{A19}$), $\vdash_\Lambda (\mathsf{E}\,\psi \wedge \bigwedge_{1 \leqslant i \leqslant n} \mathsf{A}\,\gamma_i) \to \mathsf{E}(\psi \wedge \bigwedge_{1 \leqslant i \leqslant n} \mathsf{A}\,\gamma_i)$. Thus, we get

$$\vdash_\Lambda (\mathsf{E}\,\psi \wedge \bigwedge_{1 \leqslant i \leqslant n} \mathsf{A}\,\gamma_i) \to \left(\bigvee_{1 \leqslant j \leqslant m} \mathsf{A}\,\delta_j\right).$$

By $(\Gamma, \Delta) S^\Lambda (\Sigma, \Theta)$ and $\mathsf{A}\,\gamma_i \in \Gamma$ and $\mathsf{A}\,\delta_j \in \Delta$, we obtain $\mathsf{A}\,\gamma_i \in \Sigma$ and $\mathsf{A}\,\delta_j \in \Theta$ hence $\mathsf{A}\,\delta_j \notin \Sigma$. Together with our initial assumption $\mathsf{E}\,\psi \in \Sigma$, we can obtain a desired contradiction. ■