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The sympathetic sceptic's guide to semigroup representations

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Abstract. This is an elementary, examples driven, introduction to the representation theory of finite semigroups. We illustrate the Clifford-Munn correspondence between the representations of a semigroup and the representations of its maximal subgroups. The emphasis throughout is on naturally occurring examples.

Introduction

This is an elementary introduction to the representation theory of finite semigroups. As the title suggests, it is not necessarily intended for semigroup theorists.

We start with a quick primer on the semigroups that will interest us – the inverse and regular monoids – and spend a certain amount of energy selling these objects to the general mathematical public. Section 2 introduces from scratch linear actions of semigroups on vector spaces, where the emphasis is on those aspects of the theory that are in common with group representations. By this point the symmetric group S_n will have appeared a number of times, so we divert to describe its "atomic" representations. There are two fundamental constructions, reduction and induction, that connect group theory and semigroup theory, at least when it comes to representations. These are described in Sections 3-4. Section 5 contains, what is, from the point of view of these notes, the fundamental theorem of semigroup representation theory: the Clifford-Munn correspondence. It gives a mechanism for producing the atomic representations of semigroups using only knowledge from group theory. The last section is essentially a gratuitous excuse to draw pictures of our favourite polytope, the permutohedron, dressed up as a worked example of the representations of an interesting Renner monoid.

Throughout, three running examples, S_n (a group), I_n (an inverse monoid) and T_n (a regular monoid) are used as illustration. By the end of Section 4 the emphasis will have completely moved to inverse monoids. We also start with actions on vector spaces over an arbitrary field k, but in later sections we retreat to the relative safety of representations over $\mathbb{C}$. We borrow heavily from a number of sources – full attributions are given in the Notes and References section at the end.

1. Semigroups

A semigroup is a set equipped with an associative multiplication. This leaves us with quite a bit of scope! In this section we feel our way towards a manageable class of semigroups to study. Our guiding principle will be the role of inverses in semigroup theory.

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We start with three finite examples that are the most typical of their type. Throughout, we write $[n]$ for the set $\{1, 2, \ldots, n\}$.

- *The symmetric group* S_n: consisting of all bijections $g : [n] \to [n]$ with multiplication the usual composition of maps.
- *The symmetric inverse monoid* I_n: consisting of all *partial* bijections $s : [n] \to [n]$, i.e. bijections $s : X \to Y$ where $X, Y \subseteq [n]$. The multiplication is composition of partial maps as shown in Figure 1. All our functions, actions, etc, will be on the left, so the partial map st

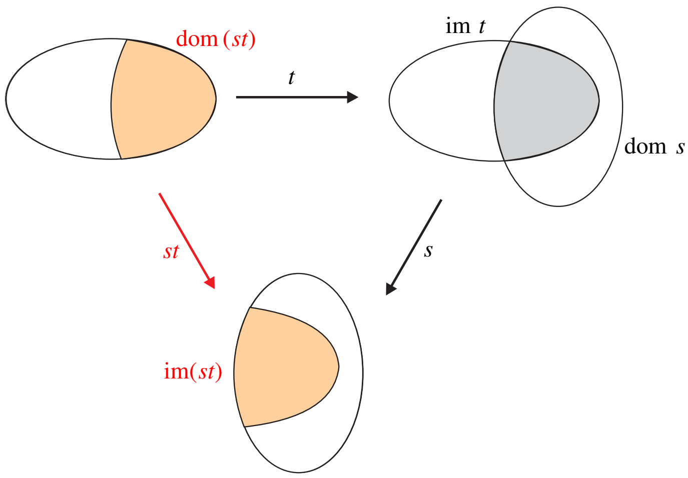


Fig. 1. composition of partial maps of $[n]$.

  has domain the t-preimage of $\operatorname{im} t \cap \operatorname{dom} s$ and image the s-image of $\operatorname{im} t \cap \operatorname{dom} s$, and is the usual composition of t followed by s between these two sets. If $\operatorname{im} t \cap \operatorname{dom} s$ is empty, then st is the unique bijection $\varnothing \to \varnothing$, which we will call the zero map 0.
- *The full transformation monoid* T_n: consisting of all mappings $s : [n] \to [n]$ with multiplication the usual composition of maps.

Inverses in semigroup theory. Naively, a semigroup is a group, except without inverses. But to completely rule out inverses in a semigroup is unnecessarily defeatist. The elements of S_n are "global" symmetries of the set $[n]$ – with global inverses – while the elements of I_n are "local" symmetries of $[n]$, with local inverses to match. Our three running examples motivate three ways in which inverses arise:

(i). The symmetric group S_n is a group, obviously, so for every $g \in S_n$ there is a unique $h \in S_n$ with $gh = \mathrm{id} = hg$. Write $h = g^{-1}$ as usual.

(ii). If $s : X \to Y$ is an element of I_n then there is a unique $X \leftarrow Y : t$, that is the inverse of s, but only defined on the set Y. Indeed, $st = \mathrm{id}_Y$ and $ts = \mathrm{id}_X$, where $\mathrm{id}_X : X \to X$ and $\mathrm{id}_Y : Y \to Y$ are partial identities, and in particular, idempotents (i.e: $\mathrm{id}_X\mathrm{id}_X = \mathrm{id}_X$ and $\mathrm{id}_Y\mathrm{id}_Y = \mathrm{id}_Y$).

As a working definition of the local inverse of s, we could take it to be an element t such that st and ts are idempotents, but not necessarily the global idempotent id. It turns out that this isn't quite satisfactory, as any map defined on some subset of the image of s, and equal to the inverse of s on this subset, also has this property.

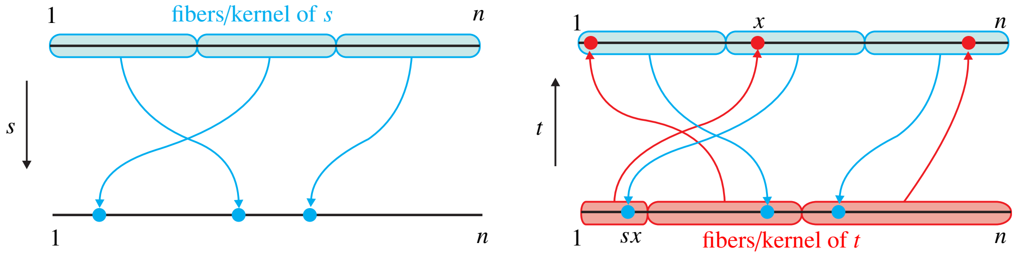


Fig. 2. inverses in T_n.

Instead, we have $s\,\mathrm{id}_X = s = \mathrm{id}_Y s$. Together with $st = \mathrm{id}_Y$ and $ts = \mathrm{id}_X$ we get that t satisfies $sts = s$; similarly $tst = t$.

A semigroup with the property that for every s there is a *unique* s^* satisfying

$$ss^*s = s \text{ and } s^*ss^* = s^* \tag{1}$$

is called an inverse semigroup; a semigroup with an identity id is a monoid, and an inverse semigroup with an identity id is an *inverse monoid*. I_n is thus the most inverse monoid-like of the inverse monoids. We will sometimes call s^* an inverse "in the sense of semigroup theory" and reserve the notation s^{-1} for inverses in a group.

(iii). Definition (1) of inverses opens up new possibilities. The element $s \in T_n$ shown on the left of Figure 2 has *kernel* the partition of $[n]$ whose blocks are the *fibers* of s: the s-preimages of a point in the image of s. Now construct an element $t \in T_n$ in the following way:

- partition the image copy of $[n]$ so that each block of the partition contains exactly one element of the image of s; this partition will be the kernel of t.
- For each block in this new kernel choose an x in the domain copy of $[n]$ such that the block contains the point sx; then define t so that it maps this block to x; see the right of Figure 2.

The t just constructed satisfies $sts = s$ and $tst = t$; conversely, any t satisfying these relations must come about in this way. But this t is clearly not unique – there is choice in the partition of the image $[n]$ and for each block of this partition, choice in the x so that the block is labelled by sx.

A monoid with the property that for every s there is *some* t satisfying $sts = s$ and $tst = t$ is called a *regular monoid*.

From now on: S will be a finite regular monoid.

The structure of semigroups: Green's relations. These allow us to draw strategic pictures of semigroups. Define an equivalence relation $\mathcal{L}$ on S by $s\mathcal{L}t$ when $Ss = St$, where $Ss = \{rs : r \in S\}$ is a left ideal (hence the "$\mathcal{L}$"). Dually, define $s\mathcal{R}t$ when $sS = tS$.

In S_n, and indeed any group, these relations are trivial: any two elements are $\mathcal{L}$ and $\mathcal{R}$-related. In I_n and T_n they take a particularly simple form:

- $s\mathcal{L}t$ when the fibers of s are equal to the fibers of t (or s and t have the same kernel). In I_n this is equivalent to $\operatorname{dom} s = \operatorname{dom} t$.
- $s\mathcal{R}t$ when $\operatorname{im} s = \operatorname{im} t$.

If we consider the equivalence relation $\langle \mathcal{L}, \mathcal{R} \rangle$ generated by $\mathcal{L}$ and $\mathcal{R}$, then something very nice happens. The $\mathcal{L}$-class of any element t that is $\mathcal{R}$-related to s intersects the $\mathcal{R}$-class of any element r that is $\mathcal{L}$-related to s. It is particularly easy to see for I_n as on the left of Figure 3.

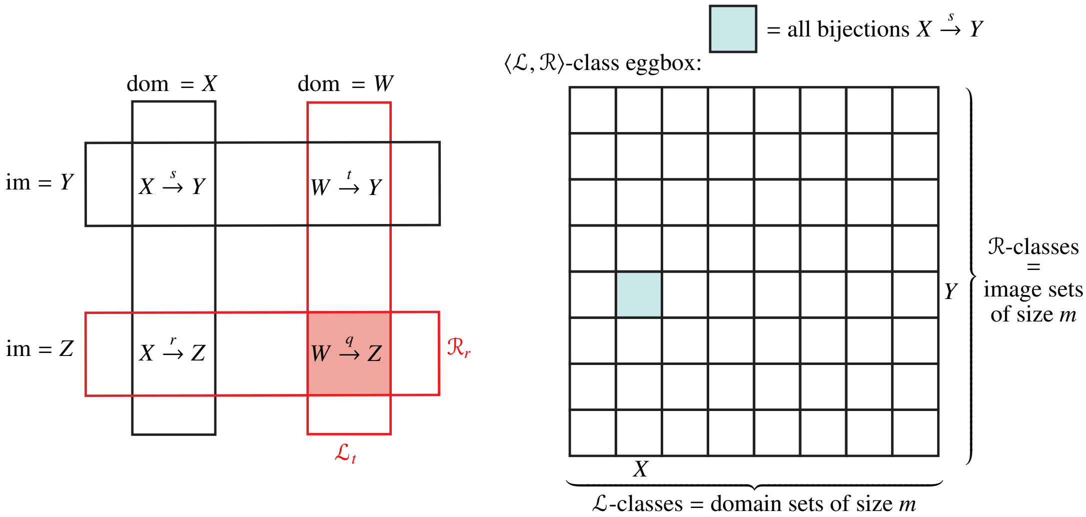


Fig. 3. In the symmetric inverse monoid I_n, the $\mathcal{L}$-class $\mathcal{L}_t$ of any element t that is $\mathcal{R}$-related to s intersects the $\mathcal{R}$-class $\mathcal{R}_r$ of any element r that is $\mathcal{L}$-related to s *(left)* and an eggbox grid of a $\langle\mathcal{L},\mathcal{R}\rangle$-class *(right)* partitioned into mutually intersecting $\mathcal{L}$ and $\mathcal{R}$-classes.

The $\langle\mathcal{L},\mathcal{R}\rangle$-classes are thus partitioned into $\mathcal{L}$-classes and partitioned into $\mathcal{R}$-classes, with any $\mathcal{L}$-class intersecting any $\mathcal{R}$-class and vice-versa. Semigroup theorists call this grid an "eggbox" – see the right of Figure 3.

Moreover, pursuing the ideal theme, define a relation $\mathcal{J}$ on S by $s\mathcal{J}t$ when $SsS = StS$. Again, this has a simple form in I_n and T_n, with $s\mathcal{J}t$ when $\operatorname{im} s$ and $\operatorname{im} t$ are sets of the same size. But the $\langle\mathcal{L},\mathcal{R}\rangle$-class of I_n in Figure 3 consists precisely of those partial bijections whose image has the fixed size $|\operatorname{im} s| = |Y|$. Thus $\mathcal{J} = \langle\mathcal{L},\mathcal{R}\rangle$ in I_n, and in general for any finite S.

Our final relation is $\mathcal{H} = \mathcal{L}\cap\mathcal{R}$, so that $s\mathcal{H}t$ when they are both $\mathcal{L}$ and $\mathcal{R}$-related. In I_n and T_n a pair $s\mathcal{H}t$ means that s and t have the same fibers (or domains in I_n) and the same images. The $\mathcal{H}$-classes are thus the small boxes in the eggbox grid with one marked on the right of Figure 3. Write $\mathcal{L}_s, \mathcal{R}_s, \mathcal{J}_s$ and $\mathcal{H}_s$ for the equivalence class of $s \in S$ under these relations.

The $\mathcal{J}$-classes are not just floating around in the ether in a disembodied fashion. They can be compared to each other; in other words, they form a poset. This is what we mean by "strategic picture".

Again we can see this quite naturally by looking at I_n and T_n, where the $\mathcal{J}$-classes are parametrised by the possible sizes of the images: by $\{0,\dots,n\}$ in I_n and by $\{1,\dots,n\}$ in T_n. Indeed, SsS consists of the maps having image size $\leq |\operatorname{im} s|$, so that $SsS \subseteq StS$ exactly when $|\operatorname{im} s| \leq |\operatorname{im} t|$. We will write $\mathcal{J}_m$ for the $\mathcal{J}$-class consisting of those maps with image size m.

In general, define a partial order on the $\mathcal{J}$-classes of a semigroup S by $\mathcal{J}_s \leq \mathcal{J}_t$ whenever $SsS \subseteq StS$.

Idempotents. An idempotent is an element e with the property that $e^2 = e$. In a group there is precisely one: the identity id. But in I_n there are others, and in T_n even more again.

The idempotents in I_n are the maps $\mathrm{id}_X : X \to X$ that are the identity on some $X \subseteq [n]$; they are the *partial identities*. In the eggbox on the right of Figure 3, id_X lives on the diagonal in the row and column labelled by X. Moreover, id_X is the only idempotent in its row and column.

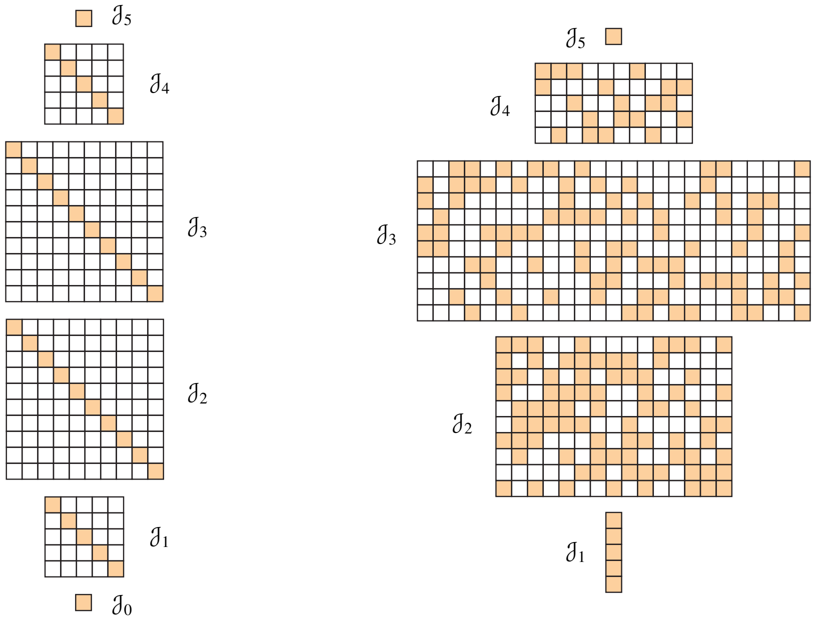


Fig. 4. The $\mathcal{H}$-classes containing idempotents in I_5 *(left)* and T_5 *(right)*.

In fact, this is true for any inverse semigroup: each $\mathcal{R}$-class and each $\mathcal{L}$-class contains a unique idempotent.

To find idempotents in T_n, fix any partition of $[n]$; this will be the fibers/kernel of e. In each fiber fix a point, and then define e to map each fiber to the point chosen in it – see Figure 5. (A slicker way to say it is that e restricts to the identity on its image.) If the fibers – and hence the $\mathcal{L}$-class – are fixed, there is still wiggle-room in the choice of point in each one. So a given $\mathcal{L}$-class may contain several idempotents. Dually, fixing some image points (and hence the $\mathcal{R}$-class) there are many partitions of $[n]$ with a unique image point in each block of the partition, and so several idempotents in a given $\mathcal{R}$-class. This behaviour is typical of regular, non-inverse semigroups. Figure 4 compares the $\mathcal{H}$-classes containing idempotents in I_5 and T_5.

In any case, in both I_n and T_n – and in a regular monoid in general – each $\mathcal{H}$-class contains at most one idempotent.

Subgroups. In any monoid the *units* are the elements that have inverses in the sense of group theory, and these form a subgroup. In our three examples S_n, I_n and T_n, these are the bijections

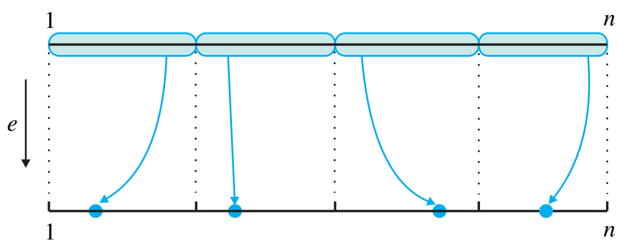


Fig. 5. An idempotent in T_n.

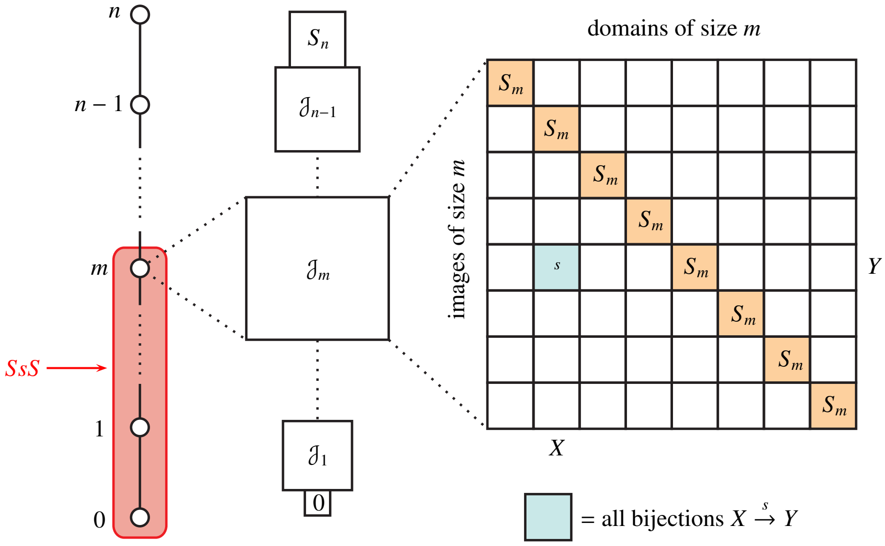


Fig. 6. Strategic picture of I_n. The $\mathcal{J}$-class poset *(left)*, the stacked eggboxes *(middle)* and the eggbox picture of the $\mathcal{J}$-class $\mathcal{J}_m$ *(right)*; there are $\binom{n}{m}$ rows and columns with the maximal subgroups $\cong S_m$ down the diagonal.

$[n] \to [n]$, so the group of units is S_n with identity id : $[n] \to [n]$. In S_n this is the whole story, but in I_n there are other subgroups, *disjoint from the units*, and in T_n even more again.

For $X \subseteq [n]$ fixed, the bijections $X \to X$ form a subgroup of I_n isomorphic to S_m, where $m = |X|$. This subgroup is precisely the diagonal $\mathcal{H}$-class containing the idempotent $\mathrm{id}_X : X \to X$.

In general, if $\mathcal{H}_e$ is the $\mathcal{H}$-class containing the idempotent e, then this is a subgroup of S with identity e. Moreover, any subgroup of S is a subgroup of an $\mathcal{H}_e$ for some e, hence these are *maximal subgroups* of S. We write G_e for $\mathcal{H}_e$ from now on, to stress its group structure.

T_n has many more $\mathcal{H}$-classes containing idempotents, hence many more maximal subgroups. The $\mathcal{H}$-class containing the idempotent e on the left of Figure 7 consists of the maps with fibers $X_1, \ldots, X_m$ and image points $y_1, \ldots, y_m$, and where these maps give a bijection $\{X_1, \ldots, X_m\} \to$

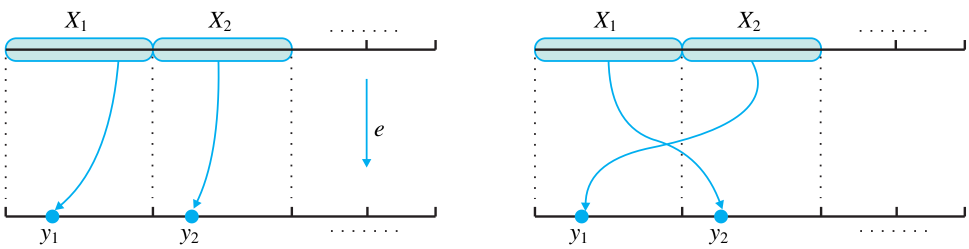


Fig. 7. Maximal subgroup of T_n with identity the idempotent e *(left)* and a typical element *(right)*.

$\{y_1, \ldots, y_m\}$. The maximal subgroups of T_n are thus symmetric groups S_m as well, but in a slightly different way to I_n. Figure 4 therefore also shows these S_m subgroups (shaded), for $0 \leq m \leq 5$, in I_5 and T_5.

The strategic picture for I_n: is given in Figure 6. The $\mathcal{J}$-class poset is on the left – the image sizes $\{0, \ldots, n\}$ with their usual total order – and the stacked eggboxes are in the middle. The

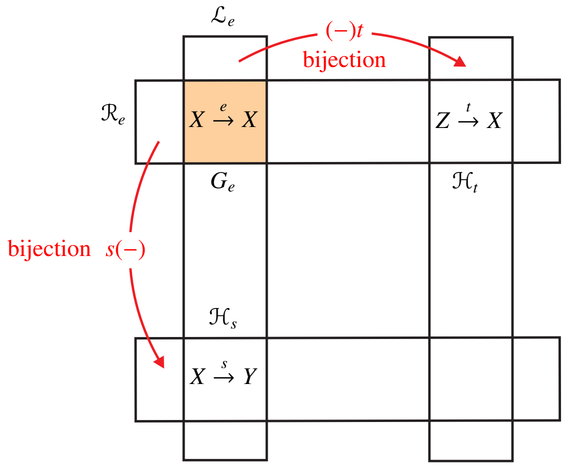


Fig. 8. Green's lemma.

maximal $\mathcal{J}$-class consists of all the bijections with image size n, so is the symmetric group S_n, and the minimal $\mathcal{J}$-class has single element the zero map $0 : \varnothing \to \varnothing$. The class $\mathcal{J}_m$ has rows and columns indexed by the $\binom{n}{m}$ subsets of size m, with the blue box in Figure 6 containing the bijections $s : X \to Y$. The idempotents $\mathrm{id}_X : X \to X$ lie down the diagonal, with the maximal subgroups consisting of all the bijections $X \to X$ for fixed $|X| = m$, and thus $\cong S_m$.

Green's lemma. The maximal subgroups can be used to parametrise the $\mathcal{L}$ and $\mathcal{R}$ classes containing them – indeed, the $\mathcal{H}$-classes in $\mathcal{R}_e$ are like right cosets of the subgroup G_e and the $\mathcal{H}$-classes in $\mathcal{L}_e$ are like left cosets.

It is easy to see in I_n: let e be the idempotent $\mathrm{id}_X : X \to X$, contained in the maximal subgroup G_e of all bijections $X \to X$ (see Figure 8). An $s \in \mathcal{L}_e$ is a bijection $s : X \to Y$. For any $g \in G_e$, the composition

$$X \xrightarrow{g} X \xrightarrow{s} Y$$

is a bijection $X \to Y$, and all such bijections arise in this way via some g. Put another way, left multiplication by s is a bijection $s(-) : G_e \to \mathcal{H}_s$. Thus:

$$\text{every element of the } \mathcal{H}\text{-class } \mathcal{H}_s \text{ can be uniquely expressed as } sg \text{ for some } g \in G_e \qquad (2)$$

(so $\mathcal{H}_s = sG_e$ is the left coset in $\mathcal{L}_e$ of the maximal subgroup G_e). Dually, if $t \in \mathcal{R}_e$ is some bijection $t : Z \to X$ then every element of $\mathcal{H}_t$ has a unique expression as gt for some $g \in G_e$, and so $\mathcal{H}_t$ is the right coset $G_e t$ of G_e in $\mathcal{R}_e$. These observations are called Green's lemma.

An important consequence is that the maximal subgroups in a fixed $\mathcal{J}$-class are isomorphic. Again we see it in I_n; let $e = \mathrm{id}_X$ and $f = \mathrm{id}_Y$ be idempotents in the $\mathcal{J}$-class $\mathcal{J}_e = \mathcal{J}_f$ and let G_e, G_f be the corresponding maximal subgroups – see Figure 9. Somewhat incidentally, $G_e \cong S_m \cong G_f$ with $|X| = m = |Y|$, but this isomorphism also arises naturally as follows. "Complete the square" of $\mathcal{H}$-classes that has the maximal subgroups at its diagonal corners, and fix a representative $s : X \to Y$ of the $\mathcal{H}$-class lying the the same column as G_e and row as G_f. The inverse $s^* : Y \to X$ then lies in the diagonally opposite $\mathcal{H}$-class.

Any $h : Y \to Y$ in the group G_f can now be decomposed as:

$$Y \xrightarrow{h} Y = Y \xrightarrow{s^*} X \xrightarrow{g} X \xrightarrow{s} Y$$

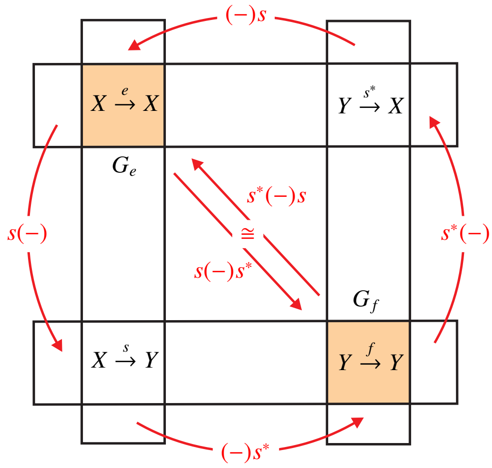


Fig. 9. Maximal subgroups are isomorphic.

for some $g \in G_e$, and the map $g \mapsto sgs^*$ is a homomorphism $G_e \to G_f$ with inverse the map $h \mapsto s^*hs$.

The inverse monoids $S(G, L)$. Mathematics contains many examples of a group acting on a poset or a lattice – we now describe an inverse monoid that wraps up the group, the lattice and the action into a single object. It turns out that many naturally occurring inverse monoids arise this way. It is also a particularly useful format for understanding their representations – we will find it essential for the examples of §6.

Let G be a finite group and L a finite lattice – a poset in which every pair of elements a, b has a greatest lower bound, or *meet* $a \wedge b$, and a least upper bound, or *join* $a \vee b$. Suppose that G acts on L: each $g \in G$ gives rise to a poset map $a \mapsto g \cdot a$, so that if $a \leq b$ in L then $g \cdot a \leq g \cdot b$. As this must also be true for g^{-1}, we have $a \leq b$ iff $g \cdot a \leq g \cdot b$.

We form a semigroup $S(G, L)$ out of this input data: the elements have *expressions* of the form g_a where $g \in G$ and $a \in L$. Two different expressions can represent the same element:

$$g_a = h_b \text{ in } S(G, L) \text{ iff } a = b \text{ and } g^{-1}h \cdot c = c \text{ for all } c \leq a \tag{3}$$

Finally, the product is given by

$$g_a h_b = (gh)_{h^{-1} \cdot a \wedge b} \tag{4}$$

where gh is the product of g and h in G. If it seems a little mysterious, you can think of g_a as the element of the symmetric inverse monoid on the set L and having domain the interval $L_{\leq a} = \{c \in L : c \leq a\}$ with effect the restriction of g to this interval. Then (4) is just the composition of partial bijections for I_L and (3) warns us that different elements of G can restrict to the same partial bijection in $S(G, L)$.

As L is a finite lattice it has a maximum $\mathbf{1} = \bigvee_{a \in L} a$ and a minimum $\mathbf{0} = \bigwedge_{a \in L} a$, hence $S(G, L)$ has an identity $\mathrm{id}_{\mathbf{1}}$, where id is the identity in G, and a zero $g_{\mathbf{0}}$, for any $g \in G$ (as $g \cdot \mathbf{0} = \mathbf{0} = h \cdot \mathbf{0}$ for any other h, we have by (3) that $g_{\mathbf{0}} = h_{\mathbf{0}}$). More significantly, g_a has the semigroup inverse $g_a^* = g_{g \cdot a}^{-1}$ so that $S(G, L)$ is an inverse monoid.

The Green's relation structure follows the dictates of the symmetric inverse monoid on L: we have $g_a\mathcal{L}h_b$ exactly when $a = b$ and $g_a\mathcal{R}h_b$ when $g \cdot a = h \cdot b$. In particular the $\mathcal{L}$-class of g_a consists of all the h_a as $h \in G$ varies, and the $\mathcal{R}$-class of all the $h_{h^{-1}\cdot a}$.

The $\mathcal{H}$-class of g_a consists of the h_a for those $h \in G$ such that $h \cdot a = g \cdot a$. The $\mathcal{J}$-classes correspond to the orbits of the G-action on L; if $\{a_1, \ldots, a_m\}$ is an orbit, then the eggbox decomposition of the corresponding $\mathcal{J}$-class has rows and columns indexed by the a_i and the $\mathcal{J}$-class consists of all the g_a where $g \in G$ and a is one of the a_i.

To complete the strategic picture, let $\mathcal{J}_1$ and $\mathcal{J}_2$ be $\mathcal{J}$-classes corresponding to the G-orbits $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_\ell\}$. Then $\mathcal{J}_1 \leq \mathcal{J}_2$ exactly when $a_i \leq b_j$ in L for some a_i and some b_j (or equivalently $a_i \leq b_j$ for any a_i and some b_j, or, any b_j and some a_i).

The idempotents of $S(G, L)$ are the id_a for id the identity of G, and the units are the $g_{\mathbf{1}}$ with $\mathbf{1}$ the maximum of L. By (3)-(4) the units form an isomorphic copy of G in $S(G, L)$.

The maximal subgroup G_a containing the idempotent id_a consists of the g_a with $g \cdot a = a$, subject to our running ambiguity (3). It turns out that the ambiguity can be easily ironed out: let G^a be those elements of G with $g \cdot a = a$ and let $G^{\leq a}$ be those elements of G with $g \cdot c = c$ for all $c \leq a$. Then $G^{\leq a}$ is a normal subgroup of G^a and there is an isomorphism

$$G_a \cong G^a/G^{\leq a} \tag{5}$$

from the maximal subgroup G_a to this sub-quotient of the group of units G.

Let $s = g_a$ be an element in the $\mathcal{L}$-class of the maximal subgroup G_a and $s(-) : G_a \to \mathcal{H}_s$ the bijection promised by Green's lemma. We can make the decomposition (2) quite explicit: if $t = h_a$ is another element of $\mathcal{H}_s$ then

$$t = s \cdot g_b^{-1} h_a \tag{6}$$

where $b = g \cdot a$ and $g_b^{-1} h_a \in G_a$.

Example 1. We can shoehorn I_n into this setting: $G = S_n$ and L is the lattice of subsets of $[n]$ ordered by inclusion with S_n acting on L in the obvious way. We leave the reader to show that $S(S_n, L) \cong I_n$ via the map that sends g_a to the partial permutation obtained by restricting g to the subset a.

Exercise 1. If $[G : H]$ is the index of the subgroup H in G, show that $|S(G, L)| = \sum_{a \in L} [G : G^{\leq a}]$. Hence $|I_n| = \sum_{X \subseteq [n]} [S_n : S_X]$, where S_X is the symmetric group on the set X.

Example 2. An important lattice in combinatorics is the *partition lattice $\Pi(n)$*, having elements the partitions $\Lambda = \{\Lambda_1, \ldots, \Lambda_p\}$ of $[n]$ ordered by $\{\Lambda_1, \ldots, \Lambda_p\} \leq \{\Delta_1, \ldots, \Delta_q\}$ iff each Λ_i is a subset of some Δ_j. The symmetric group S_n acts on $\Pi(n)$ via $g \cdot \{\Lambda_1, \ldots, \Lambda_p\} = \{g\Lambda_1, \ldots, g\Lambda_p\}$. The resulting $S(G, L)$ is called the monoid of uniform block permutations and the strategic picture, when $n = 4$, is in Figure 10. The $\mathcal{J}$-class poset is the poset of partitions $\lambda = \{\lambda_1, \ldots, \lambda_p\}$ of the integer n (see the beginning of the Interlude), and the corresponding maximal subgroup is isomorphic to the Young subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_p}$. In particular, the order of the monoid of uniform block permutations is $\sum_{\Lambda \in \Pi(n)} [S_n : S_{\lambda_1} \times \cdots \times S_{\lambda_p}]$.

2. Representations

This section contains the basics of representation theory that are common to all finite regular monoids. The theme is the extent to which representations can be decomposed into "atomic"

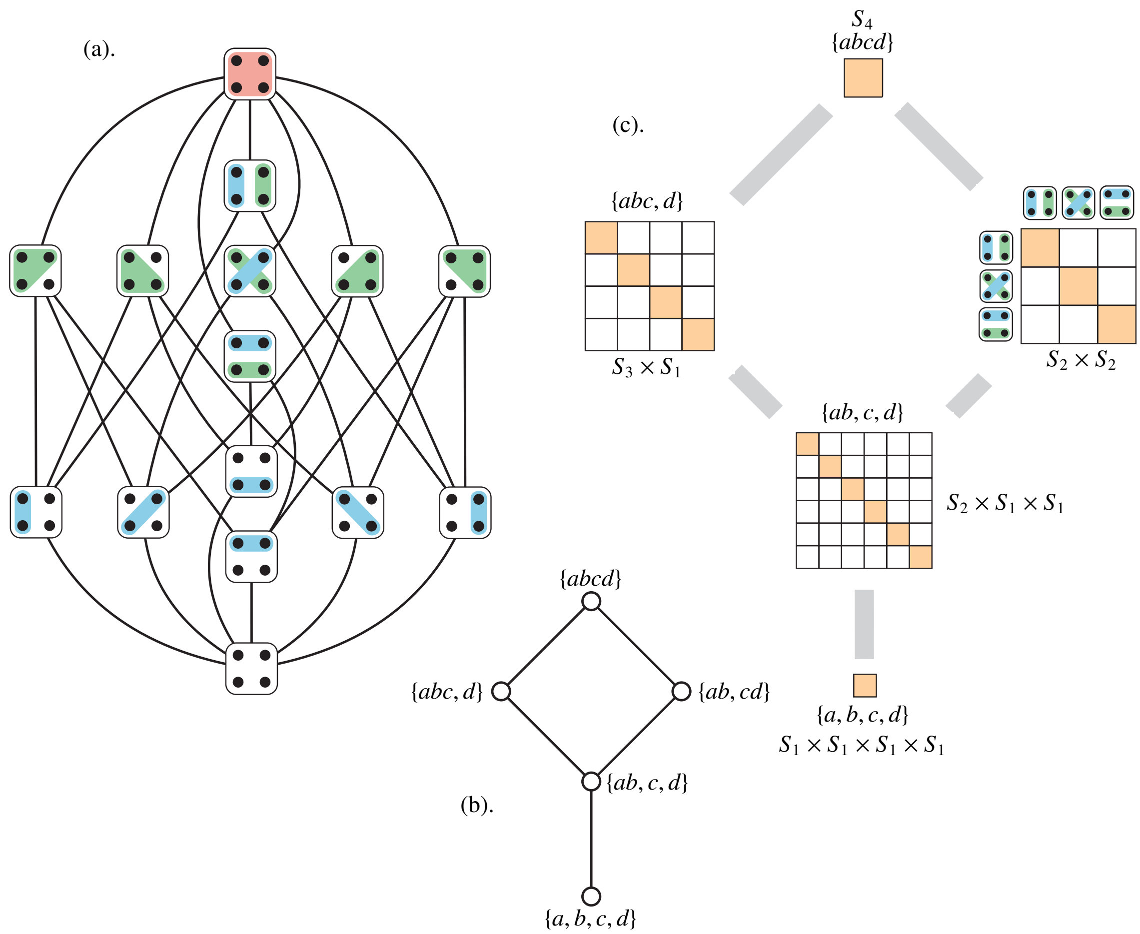


Fig. 10. Strategic picture of $S(G, L)$ when $G = S_4$ and $L = \Pi(4)$ from Example 2: (a). the partition lattice $\Pi(4)$; (b). the poset of $\mathcal{J}$-classes labelled by the type of partitions; (c). the strategic picture.

pieces. These can then be reassembled to get a handle on the sociology of the representations of a semigroup. It turns out that this is almost always possible for groups and inverse monoids, but less so for regular, non-inverse monoids.

Throughout, k is a field and V a finite dimensional vector space over k. Let $End(V)$ be the monoid, under composition, of all vector space homomorphisms (or linear maps) $V \to V$.

An *S-action on V* or *linear representation of S* is a monoid homomorphism

$$\varphi : S \to End(V).$$

We adopt the convention that all monoid homomorphisms send 1's to 1's, so that $\varphi(1_S)$ is the identity homomorpism id : $V \to V$. In particular im $\varphi \neq \{0\}$, and so our representations are not *null*. If S is a group then necessarily im $\varphi \subset GL(V)$, the group of vector space isomorphisms (or invertible linear maps) $V \to V$. The notion of a semigroup representation is thus a straight generalisation of that of a group representation.

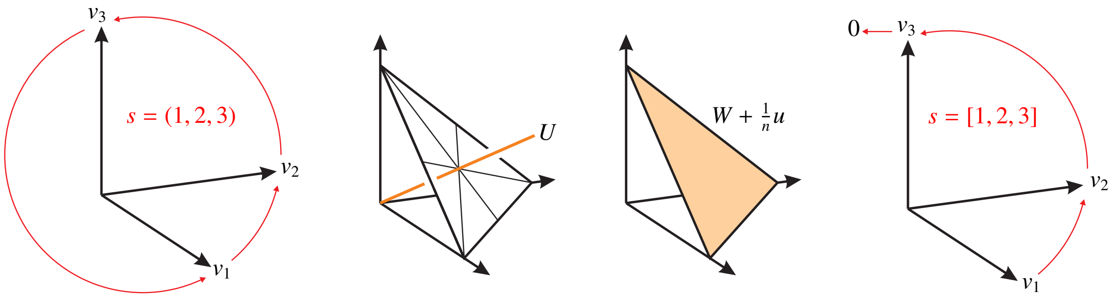


Fig. 11. *From left to right*: Permutation action of S_n; the line U which is the k-span of $u = v_1 + \cdots + v_n$; the (affine) hyperplane $W + \frac{1}{n}u$ coming from the reflectional representation; the partial reflection action of I_n. The pictures are for $n = 3$ and the notation for partial permutations is described in the Notes and References section.

We will identify $s \in S$ and $\varphi(s) \in End(V)$, so that if $v \in V$, we just write $s \cdot v$, or even sv, for the effect of the linear map $\varphi(s)$ on the vector v. Mostly we will just write V for an S-representation without explicit reference to the action.

The following representation of our three running examples S_n, I_n and T_n will turn out to display the full range of possible behaviours:

Example 3 (mapping representations). Fix a basis $\{v_1, \ldots, v_n\}$ for the space V and for $s \in S_n, I_n$ or T_n define

$$s \cdot v_i = v_{s(i)} \quad (s \in S_n, T_n) \qquad \text{or} \qquad s \cdot v_i = \begin{cases} v_{s(i)}, & \text{if } i \in \text{dom } s \\ 0, & \text{else.} \end{cases} \qquad (s \in I_n) \tag{7}$$

and then extend linearly. To analyse the structure of the mapping representations, we need to know how to decompose representations in general.

Sub-representations and reducibility. These allow us to understand representations in the large. If V is a representation and U is a subspace left invariant by the S-action, i.e. $SU = U$, then we call U an *(S-)subrepresentation* of V. The quotient space V/U then carries an S-action via $s \cdot (v + U) = sv + U$, well-defined, as $sU = U$. There is then a 1-1 correspondence between the subrepresentations of V/U and the subrepresentations W of V such that $U \subseteq W \subseteq V$.

If V has a proper, non-zero subrepresentation U, then call V *reducible*; V is *irreducible* if the only subrepresentations are $\{0\}$ and V.

A subrepresentation U of V is *maximal* when $U \neq V$ but for any subrepresentation W of V with $U \subseteq W \subseteq V$ we have either $W = U$ or $W = V$. Because of the 1-1 correspondence mentioned above, U is maximal exactly when V/U is irreducible.

If U, W are subrepresentations of V such that $V = U \oplus W$ as vector spaces, then the representation V is the *(internal) direct sum* of U and W. Externally, if U and W are arbitrary S-representations then the vector space direct sum $U \oplus W$ carries an S-action via $s \cdot (u + w) = su + sw$, and $U \oplus W$ is the *(external) direct sum* of U and W.

Example 4 (the permuting coordinates and reflectional representations of S_n). The S_n-action in (7) is by "permuting coordinates" (or more accurately, permuting basis vectors). In particular, for $(i, j) \in S_n$ the resulting isomorphism $V \to V$ is the reflection in the hyperplane with equation $x_i - x_j = 0$. As S_n is generated by the transpositions, its image in $GL(V)$ is generated by reflections, i.e: is a *reflection group*.

The vector $u = v_1 + \cdots + v_n$ is fixed by any permutation in S_n, so that if U is the k-span of u then $S_nU = U$, a subrepresentation. As each vector in U is fixed by every element of S_n, this is the *trivial representation* of S_n (see Figure 11).

Thus if $n > 1$ then the permuting coordinates representation V is reducible. Moreover, as U is 1-dimensional it has only the two subspaces, S_n-invariant or otherwise, namely $\{0\}$ and U. Hence U is irreducible. When $n = 1$ we have $V = U$ is irreducible.

Now let W be the hyperplane with equation $x_1 + x_2 + \cdots + x_n = 0$, that is, the set of points whose coordinates with respect to the v_i sum to 0. Permuting the coordinates of such a vector doesn't change the fact that they add to 0, hence W is also a subrepresentation of V. Figure 11 has the plane W when $n = 3$, shifted off the origin to make it easier to see. For reasons that are maybe a little obscure at the moment, W is called the *reflectional representation* of S_n.

Moreover, if the characteristic char(k) of the field does not divide $\dim V = n$, then W is irreducible. For suppose that $X \neq \{0\}$ is a S_n-invariant subspace of W and let $v \in X$ with $v \neq 0$. If all the coordinates of v are equal to some $\lambda \in k$, then these sum to 0 to give $n\lambda = 0$, hence – by the restriction on the characteristic – we must have $\lambda = 0$, hence $v = 0$, a contradiction. The vector v must therefore have two coordinates that are different. For each $1 \leq i < n$ we can engineer a $g_i \in S_n$ such that in the vector $g_i \cdot v$ it is the i-th and $(i + 1)$-st coordinates that are different. Then $(i, i + 1)g_i \cdot v - g_i \cdot v$ is a non-zero multiple of $v_i - v_{i+1}$. As X is S_n-invariant we conclude that for each $1 \leq i < n$ the vector $v_i - v_{i+1}$ is an element of X. But these vectors form a basis for W, so $X = W$, and W is irreducible as claimed.

The permuting coordinates representation of S_n can thus be decomposed $V = U \oplus W$ into the trivial and reflectional representations, with both of these irreducible.

Example 5 (the partial reflectional representation of I_n). The I_n-action (7) is by partial permutations of coordinates and the image of I_n in $End(V)$ is a *reflection monoid*.

Assume that $n > 1$. The line U spanned by $u = v_1 + \cdots + v_n$ is no longer I_n-invariant: if $s \in I_n$ is the partial identity with domain $\{1\}$ then $s \cdot u = v_1 \notin U$. Similarly W is not I_n-invariant.

In fact, V itself is irreducible: for suppose U is an I_n-invariant subspace containing $0 \neq v \in U$ with $v = \sum \lambda_i v_i$ and $\lambda_j \neq 0$ for some j. For each $1 \leq i \leq n$ let $s_i \in I_n$ be the partial permutation shown in Figure 12. Then $s_i \cdot v = \lambda_j v_i$, hence $v_i \in U$ for all i, and so $U = V$.

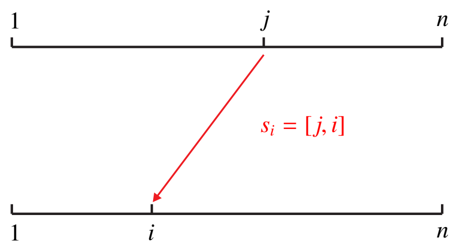


Fig. 12. $s_i \in I_n$ for $1 \leq i \leq n$; the notation $[j, i]$ is explained in the Notes and References section.

Example 6 (the mapping representation of T_n). Again suppose that $n > 1$. The hyperplane W goes back to being a subrepresentation of V: if $w \in W$ with $w = \sum \lambda_i v_i$ where $\sum \lambda_i = 0$, then sw for $s \in T_n$ is shown in Figure 13. In particular the non-zero coordinates of sw are sums of the coordinates of w, and so still sum to 0: i.e. $sw = \sum \mu_i v_i$ where $\mu_i = \sum \lambda_{ij}$, the sum over the j in the fiber of i. As $\sum \mu_i = \sum \lambda_i = 0$, we get $sw \in W$.

If S is any monoid, V an S-representation and T a submonoid of S, then it is easy to see that restricting the S-action to T makes V into a T-representation. This observation gives that

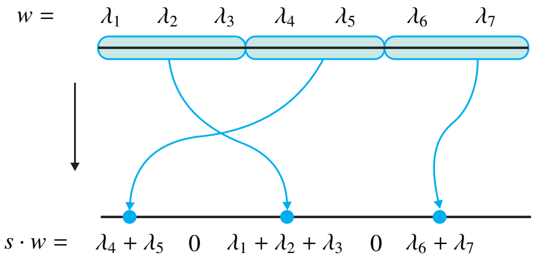


Fig. 13. The hyperplane W is a T_n-subrepresentation.

W is irreducible: if X is a subrepresentation of W then it is an S_n-subrepresentation of the S_n-representation W. The irreducibility of this – when char(k) does not divide n – then gives $X = \{0\}$ or W.

Just as for I_n, the line U spanned by $v_1 + \cdots + v_n$ is not a subrepresentation: for example when s is the constant map in T_n that sends all of $[n]$ to 1, then $su = nv_1 \notin U$.

In fact, when $n > 2$ we claim that there are *no* 1-dimensional subrepresentations of the T_n-mapping representation V. For, suppose that $v \neq 0$ so that $v = \sum \lambda_i v_i$ with $\lambda_j \neq 0$ for some j. If $s_1, s_2 \in T_n$ are given by Figure 14, where everything apart from j is sent to n, with s_i sending j to i, then

$$s_1(v) = \lambda_j v_1 + \Big(\sum_{i \neq j} \lambda_i\Big) v_n \quad \text{and} \quad s_2(v) = \lambda_j v_2 + \Big(\sum_{i \neq j} \lambda_i\Big) v_n.$$

As these are independent, any non-trivial T_n-invariant subspace must be at least 2-dimensional, and the claim follows.

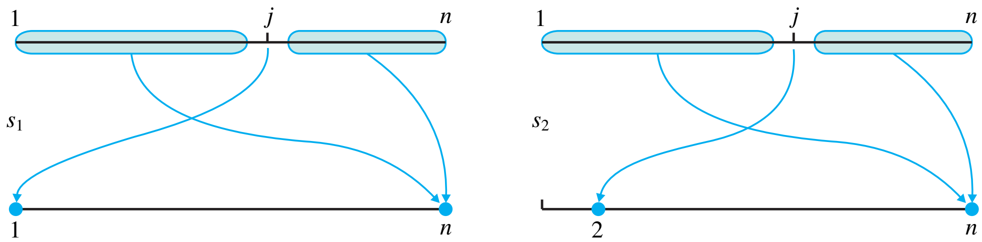


Fig. 14. The s_1 and s_2 in T_n for Example 6.

Semisimplicity. If S is a finite regular monoid and k a field, then the pair (S, k) is *semisimple* when every S-representation V over k can be decomposed

$$V = \bigoplus V_i$$

with the V_i irreducible subrepresentations of V.

Such a decomposition is then unique in the following sense: a *morphism* $q : V \to U$ of S-representations is a linear map that commutes with the S-actions on V and U, i.e. for all $s \in S$

the diagram

$$\begin{tikzcd} V \arrow[r, "s\cdot(-)"] \arrow[d, "q"'] & V \arrow[d, "q"] \\ U \arrow[r, "s\cdot(-)"] & U \end{tikzcd}$$

commutes, where the top $s \cdot (-)$ is the S-action on V and the bottom is the S-action on U. Call q an *isomorphism* if it is a bijective morphism. Uniqueness then means:

Theorem 1 (Jordan-Hölder). *Let V be an S-representation and $V = \bigoplus V_i$ with the subrepresentations V_i irreducible. If W is an irreducible subrepresentation of V then W is isomorphic to one of the V_i.*

We saw at the end of Example 4 that the reflectional representation of S_n can be so decomposed, and indeed:

Theorem 2 (Mashke). *If S is a finite group then (S, k) is semisimple if and only if the characteristic char(k) does not divide the order of S.*

In particular, (S_n, k) is semisimple exactly when char(k) doesn't divide $n!$, so that characteristic 0 representations can always be decomposed. The situation for inverse monoids is almost as good:

Theorem 3 (Munn-Oganesyan). *If S is a finite inverse monoid then (S, k) is semisimple if and only if char(k) does not divide the order of any subgroup of S.*

As any subgroup of S is in turn a subgroup of a maximal subgroup G_e for some idempotent e, it suffices that the characteristic does not divide the order of any G_e.

For our model inverse monoid I_n, we have already seen that the maximal subgroups are isomorphic to S_m for $1 \leq m \leq n$. The pair (I_n, k) is thus semisimple when the characteristic of k does not divide $m!$ for any $m \leq n$, i.e. when it does not divide $n!$ We therefore get the same condition for the semisimplicity of I_n and S_n representations.

For T_n things are not so good. If the mapping representation V of T_n is decomposable $V = \bigoplus V_i$ with the V_i irreducible, then by Theorem 1, one of the V_i is isomorphic to the representation on the hyperplane W with equation $x_1 + \cdots x_n = 0$ that we saw above. The decomposition of V must then be $V \cong W \oplus W'$ with W' a 1-dimensional subrepresentation. But we have seen for $n > 2$ that there are no 1-dimensional subrepresentations of V, and so no such decomposition of V can exist when $n > 2$.

Thus the pair (T_n, k) is not semisimple, when $n > 2$, for *any* k whose characteristic does not divide n. In particular, not even for k of characteristic 0.

Here then is what we have learned from the three examples: in characteristic 0 the partial permuting coordinates (or reflectional) representation of I_n is "atomic"; the permuting coordinates representation of S_n is not atomic but can be decomposed into pieces that are; the mapping representation of T_n is not atomic and cannot even be decomposed into atomic pieces.

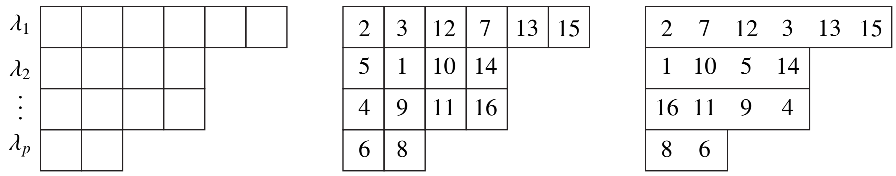


Fig. 15. A Young diagram *(left)*, a tableau T *(middle)* and the resulting tabloid $\{T\}$ *(right)* corresponding to a partition $\lambda = \{\lambda_1, \ldots, \lambda_p\} \vdash n$.

Interlude: the symmetric group

The moral of §§3-5 is that the representations of a (finite regular) monoid S are largely driven by the representations of its maximal subgroups. In every example that we have seen so far these maximal subgroups have been symmetric groups, or products of symmetric groups. It seems reasonable then to understand better the representations of the symmetric group, at least when $k = \mathbb{C}$.

We do this in a completely self-contained-tailored-to-S_n way, without any reference to the general theory of representations of finite groups. This will make it seem a little like pulling a rabbit out of a hat; the reader who is interested in the broader context of these facts should consult the Notes and References at the end.

By Theorem 2, any S_n-representation over $\mathbb{C}$ is a direct sum of irreducible representations; we will thus content ourselves with describing just these. Despite the comments in the previous paragraph, we allow ourselves one general fact: the irreducible representations over $\mathbb{C}$ of a finite group are in 1-1 correspondence with the conjugacy classes of the group. For S_n, these in turn are in 1-1 correspondence with the possible cycle structures of permutations of degree n and *these in turn* with the partitions of the integer n: the integer sequences $\lambda_1 \geq \ldots \geq \lambda_p > 0$ with $\sum \lambda_i = n$. Write $\lambda = \{\lambda_1, \ldots, \lambda_p\} \vdash n$.

Fix $\lambda = \{\lambda_1, \ldots, \lambda_p\} \vdash n$ a partition of n. A Young diagram of shape λ illustrates the structure of λ, as on the left of Figure 15, and a *tableau* T is a Young diagram filled with entries from $[n]$ with no repeats allowed – as in the middle of Figure 15. The tableau is standard, or T is a *standard tableau*, when the entries increase along the rows and down the columns. Finally, a tableau T yields a *tabloid* $\{T\}$, which is just a tableau where we no longer care about the ordering in the rows – see the right of Figure 15.

The symmetric group S_n acts on the set of tableau of shape λ, via $g : T \mapsto gT$ for $g \in S_n$, where gT is the tableau that has $g(i)$ in the box in which T has i. This action extends to the set of tabloids of shape λ via $g \cdot \{T\} = \{gT\}$. For a tableau T, the column group c_T is defined

$$c_T = \{g \in S_n : g \text{ preserves each column of } T\} \subseteq S_n$$

Let M^λ be the $\mathbb{C}$-vector space with basis the tabloids $\{T\}$ of the fixed shape λ. Via the action above, S_n acts on M^λ by permuting the basis vectors. For the partition $\lambda = \{n-1, 1\}$ we will see below that M^λ is the permuting coordinates representation of §2. Now we have other representations of S_n.

In any case, the M^λ are in general reducible – much like the permuting coordinates representation – and we will pass to a particular subrepresentation. If T is a tableau then let $v_T \in M^\lambda$ be the vector

$$v_T = \sum_{h \in c_T} \operatorname{sign}(h)\, h \cdot \{T\} \tag{8}$$

where $\mathrm{sign}(h) = 1$ or -1 depending on whether h is an even or odd permutation. We will see below that in general the vectors v_T, as T ranges over the tableau of shape λ, are *not* independent. (The v_T for T standard *are* an independent subset, although we won't need this fact here.) In any case, let

$$S^{\lambda} := \text{ the subspace of } M^{\lambda} \text{ spanned by the } v_T. \tag{9}$$

It turns out that S^{λ} is an irreducible subrepresentation of M^{λ}, and as λ varies over the partitions of n, the S^{λ} – called *Specht* representations – give a complete and non-redundant list of the irreducible S_n-representations over $\mathbb{C}$.

Example 7. If $\lambda = \begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \cdots & \; \\ \hline\end{array}$ then there is a single tabloid:

$$T = \begin{array}{|c|}\hline 1 \quad 2 \quad 3 \quad \cdots\cdots \quad n \\ \hline\end{array}$$

and the column group c_T is trivial. There is thus just one vector $v_T = \{T\}$ in the 1-dimensional space M^{λ}, with $g \cdot \{T\} = \{T\}$ for all $g \in S_n$, so that $M^{\lambda} = S^{\lambda}$ is the trivial 1-dimensional representation of S_n.

Example 8. At the other extreme we have:

$$\lambda = \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \vdots \\ \hline \; \\ \hline\end{array}$$

For any tableau of this shape the column group c_T is the full symmetric group S_n, and upto sign, there is just one of the vectors

$$v_T = \sum_{h \in S_n} \mathrm{sign}(h)\, h \cdot \{T\} = A - B,$$

where A is the sum of those terms with h even and B the sum involving those with h odd. An even permutation $g \in S_n$ preserves both summands and an odd one swaps them over, so that

$$g \cdot v_T = \begin{cases} A - B = v_T, & g \text{ even}, \\ B - A = -v_T, & g \text{ odd}. \end{cases} = \mathrm{sign}(g) \cdot v_T$$

The resulting Specht representation S^{λ} is thus 1-dimensional (hence irreducible) but not the trivial representation; it is called the *sign representation* of S_n.

Example 9. If now $\lambda = \begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \cdots & \; \\ \hline \; \\ \cline{1-1}\end{array}$ then M^{λ} is an n-dimensional space with basis the tabloids:

$$v_1 = \begin{array}{|l|}\hline 2 \quad 3 \quad \cdots\cdots \qquad n \\ \hline \multicolumn{1}{|l|}{1} \\ \cline{1-1}\end{array}, \quad \cdots, v_n = \begin{array}{|l|}\hline 2 \quad 3 \quad \cdots\cdots \quad n-1 \\ \hline \multicolumn{1}{|l|}{n} \\ \cline{1-1}\end{array}$$

$S^{\ydiagram{3}}$	$S^{\ydiagram{1,1,1}}$	$S^{\ydiagram{2,1}}$
1-dimensional trivial representation	1-dimensional sign representation	2-dimensional reflectional representation

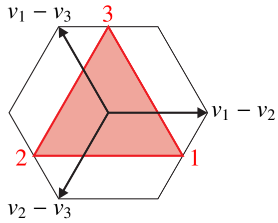


Fig. 16. The three irreducible S_3 representations over $\mathbb{C}$: the trivial representation, the sign representation and the reflectional representation. The last corresponds to the symmetries of an equilateral triangle.

and the S_n-action is $g \cdot v_i = v_{g \cdot i}$; it is thus the permuting coordinates representation of Example 3. If

$$T = \begin{array}{|c|c|c|c|c|}\hline i & & & \cdots\cdots & \\ \hline j \\ \cline{1-1}\end{array} \quad \text{then} \quad v_T = \begin{array}{|c|c|}\hline j & \cdots\cdots \\ \hline i \\ \cline{1-1}\end{array} - \begin{array}{|c|c|}\hline i & \cdots\cdots \\ \hline j \\ \cline{1-1}\end{array}$$

i.e. $v_T = v_i - v_j \in M^\lambda$, and the v_T, as T ranges over the tableau of shape λ, give the vectors $\{v_i - v_j\}_{1 \leq i \neq j \leq n}$, which span the hyperplane in M^λ having equation $\sum x_i = 0$ (and as promised, the v_T, as T ranges over all tableau, form a dependent set). The restriction of the S_n-action on M^λ to this hyperplane then gives that S^λ the reflectional representation of Example 4.

When $n = 3$ the possible λ are $\ydiagram{3}$, $\ydiagram{1,1,1}$ and $\ydiagram{2,1}$ and the S^λ are the three examples in Figure 16.

Exercise 2. Show that the exterior power $\bigwedge^p S^{\ydiagram{n-1,1}} \cong S^{\ydiagram{n-p,1,\ldots,1}}$, where there are $n - p$ boxes in the first row of the Young diagram of the second Specht representation. (The exterior powers of the reflectional representation are thus also irreducible).

3. Reduction

This section and the next give two constructions for shuttling back and forth between representations of a finite regular monoid S and representations of the maximal subgroups G_e of S. The first of these – reduction – squashes S-representations down to G_e-representations; the second, induction, blows up G_e-representations into S-representations. In Section 5 we will see that with a little care in the choice of e, these constructions turn out to be inverses of each other. Throughout, the underlying field $k = \mathbb{C}$.

We start with two examples that illustrate all the key features:

Example 10. Let $S = I_n$, the symmetric invere monoid, and let V be the partial reflectional representation of Example 5; we saw in that example that V is irreducible with basis $\{v_1, \ldots, v_n\}$.

Now let e be an idempotent in the $\mathcal{J}$-class $\mathcal{J}_m$ in the strategic picture for I_n of Figure 6. This idempotent is the identity map $\mathrm{id}_X : X \to X$ on some subset $X \subseteq [n]$ of size m and is the identity of the maximal subgroup G_e consisting of all bijections $X \to X$, which in turn is a copy of the symmetric group S_m.

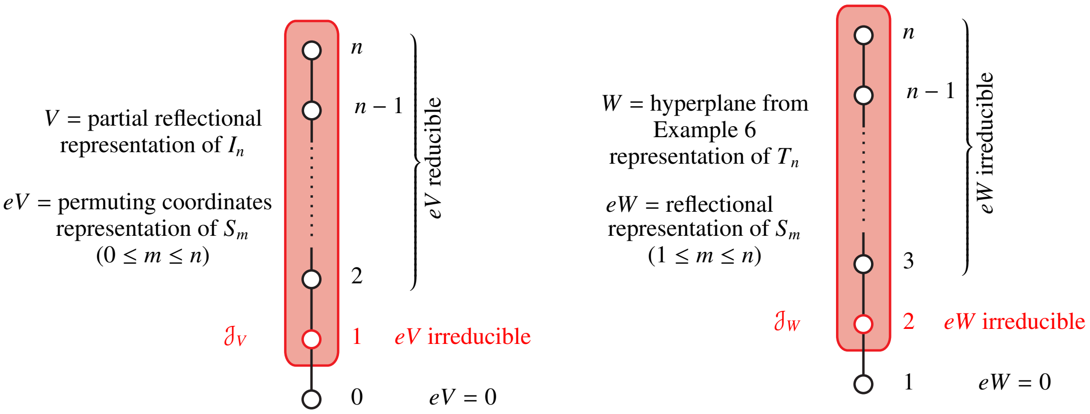


Fig. 17. Reducing irreducible representations of I_n *(left)* and T_n *(right)*. The apexes $\mathcal{J}_V$, in red, are at the bottom of the red intervals.

To squash V down to a representation of $G_e \cong S_m$, we take its image under e: let $eV := e \cdot V = \{ev : v \in V\}$. Then, as $e \cdot v_i \neq 0$ exactly when $i \in \text{dom}(e) = X$, in which case $e \cdot v_i = v_i$, the space eV has basis the v_i for $i \in X$. Define an action of G_e on eV by:

$$g \cdot (ev) = (ge) \cdot v, \qquad (g \in G_e) \tag{10}$$

observing, as e is an identity for G_e, that $(ge) \cdot v = (eg) \cdot v = e \cdot (gv) \in eV$. Indeed, eV with this action is just the permuting coordinates representation of S_m given in Example 4. We saw there that this representation is reducible when $m \geq 2$, irreducible when $m = 1$, and if $e \in \mathcal{J}_0$ is the zero map then $eV = 0$.

Upto isomorphism of representations, eV doesn't depend on the choice of the idempotent e in $\mathcal{J}_m$. For suppose that $f = \text{id}_Y : Y \to Y$ is another idempotent in $\mathcal{J}_m$, with Y a subset of size m, and f the identity of the maximal subgroup G_f. We know from Figure 9 that $G_f \cong S_Y \cong S_X \cong G_e$ via the map $h \mapsto s^* hs$, where s is some bijection $X \to Y$ and s^* is its semigroup inverse. Defining a G_f-action on fV as in (10) gives a representation of G_f.

The spaces fV and eV are incidentally isomorphic as they both have dimension m; but the map $f \cdot v \mapsto (s^* f) \cdot v$ naturally gives an isomorphism $fV \to eV$ that respects the actions of G_f and G_e: firstly $s^* f = es^*$, so that $(s^* f) \cdot v = (es^*) \cdot v = e \cdot (s^* v) \in eV$. Then

$$\begin{tikzcd}
fV \arrow[r, "h(-)"] \arrow[d, "\cong"'] & fV \arrow[d, "\cong"] \\
eV \arrow[r, "s^*hs(-)"] & eV
\end{tikzcd}$$

commutes, and so the representations fV and eV are indeed isomorphic as claimed. The results of the example are summarised on the left of Figure 17.

Example 11. The calculations, if not necessarily the results, are similar for the mapping representation of T_n from Example 6. Now V is reducible, so we start instead with the hyperplane W consisting of the $w = \sum \lambda_i v_i$ with $\sum \lambda_i = 0$; this is an irreducible representation of T_n.

Let e be the idempotent in $\mathcal{J}_m$ (the maps $[n] \to [n]$ having image size m) given in Figure 18. Then the maximal subgroup G_e consists of all bijections from the fibres of e to the image of

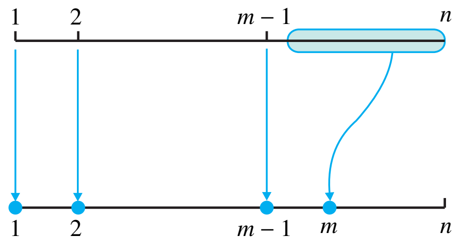


Fig. 18. $e \in \mathcal{J}_m \subset T_n$ for Example 11.

e – again, isomorphic to the symmetric group S_m. The space eV has basis $\{v_1, \ldots, v_m\}$ and the subspace eW is the hyperplane in eV whose coordinates add to 0 with respect to this basis. Via the action (10) the space eV is again the permuting coordinates representation of S_m and eW is now the reflectional representation – see Figure 17 (right).

The general picture is as follows: let S be a finite regular monoid and V an *irreducible* representation of S. Choose an idempotent in each $\mathcal{J}$-class of S; the spaces eV, equipped with the action (10), are then representations of the maximal subgroups G_e for the various choices of e. It doesn't matter, upto isomorphism of the resulting representations, which idempotent in a given $\mathcal{J}$-class is chosen.

In particular, whether $eV = 0$, or not, is a property of the $\mathcal{J}$-class containing e. The $\mathcal{J}$-classes for which $eV \neq 0$ form an *interval* in the poset of $\mathcal{J}$-classes: there is a $\mathcal{J}$-class $\mathcal{J}_V$ such that

$$eV \neq 0 \text{ exactly when } e \in \mathcal{J}_s \text{ with } \mathcal{J}_s \geq \mathcal{J}_V.$$

The $\mathcal{J}$-class $\mathcal{J}_V$ is called the *apex* of the representation V, although "trough" would probably be a better name. Figure 19 shows the idea for the $\mathcal{J}$-class poset of Figure 10 (left and middle) and generically (right).

In Example 10, the partial reflectional representation has apex the $\mathcal{J}$-class $\mathcal{J}_1$ consisting of the partial bijections on sets of size 1; the interval of $\mathcal{J}$-classes $\geq \mathcal{J}_1$ is marked on the left in Figure 17. For the irreducible representation W of T_n in Example 11, the apex is the $\mathcal{J}$-class $\mathcal{J}_2$ of maps with image size 2 – see the right of Figure 17.

In both examples, starting with an irreducible S-representation V, we get an irreducible G_e-representation *when e is in the apex $\mathcal{J}$-class $\mathcal{J}_V$*. If f is an idempotent lying in a $\mathcal{J}$-class strictly greater than $\mathcal{J}_V$, then the resulting G_f-representation may or may not be irreducible.

Definition. (reduced representations) *Let V be an irreducible representation of S with apex $\mathcal{J}_V$ and $e \in \mathcal{J}_V$ an idempotent. Then the reduced G_e-representation is given by*

$$V \downarrow G_e := eV \tag{11}$$

together with the G_e-action (10).

Exercise 3. We can verify the general picture for I_n and then for inverse monoids of the form $S = S(G, L)$. The reader might want to leave this exercise until after they have read §5.

1. Let $S = I_n$ and let V be a representation of S. Let I be the set of $\mathcal{J}$-classes $\mathcal{J}_e$ such that $eV \neq 0$. Show that $I \neq \varnothing$ and if $\mathcal{J}_e \in I$ and $\mathcal{J}_e \leq \mathcal{J}_f$ in the $\mathcal{J}$-class poset, then $\mathcal{J}_f \in I$. Hence I forms a (closed) interval in the $\mathcal{J}$-class poset. Denote the minimum element by $\mathcal{J}_V$.

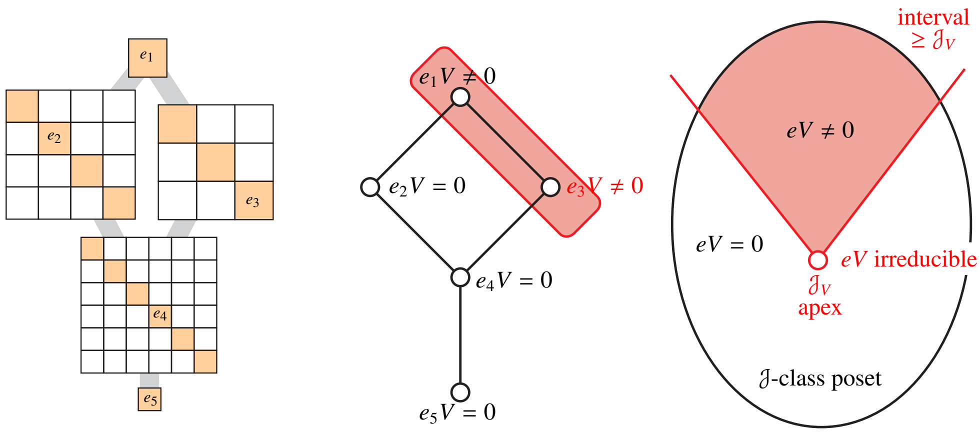


Fig. 19. Schematic of reduction: the example of Figure 10 *(left and middle)* and the generic set-up *(right)*.

2. Let e be an idempotent in this minimal $\mathcal{J}$-class $\mathcal{J}_V$ and let $T = \{s_i\}$ be a collection of representatives for the $\mathcal{H}$-classes in the $\mathcal{L}$-class $\mathcal{L}_e$. Suppose that we have $0 \neq eW \neq eV$ with eW a $G_e \cong S_m$-subrepresentation of eV, and consider the subspace
$$U = \sum_{s_i \in T} s_i eW \tag{12}$$
of V. Show that U is an I_n-subrepresentation of V.
3. If now V is an irreducible I_n-representation, then use the arguments of §5 (about the composition $\mathrm{Irr}_m(I_n) \to \mathrm{Irr}(S_m) \to \mathrm{Irr}_m(I_n)$) to show that $V = \bigoplus_{s_i \in T} s_i eV$. Deduce that $0 \neq U \neq V$ for the U of (12) and hence that eV is irreducible as a S_m-representation.
4. Repeat the whole thing for an inverse monoid of the form $S = S(G, L)$.

4. Induction

Induction is the opposite of reduction: it takes as input a representation of a maximal subgroup and spits out a representation of the whole semigroup.

We start with the general construction when $k = \mathbb{C}$. Let e be an idempotent in the semigroup S, with G_e the maximal subgroup having identity e, and let V be a representation of the group G_e.

The induction of V to an S-representation is controlled by the $\mathcal{H}$-classes that are in the $\mathcal{L}$-class $\mathcal{L}_e$ containing G_e. Choose a transversal for these $\mathcal{H}$-classes, i.e. a set $T = \{s_i\}$ with exactly one s_i in each $\mathcal{H}$-class of $\mathcal{L}_e$; choose e itself as the representative in the $\mathcal{H}$-class G_e. The transversal is just scaffolding for the construction – the resulting S-representation is independent of the choice of T.

For each $s_i \in T$, let V_i be an isomorphic copy of the space V having the elements
$$V_i = \{s_i \otimes v : v \in V\}$$
and vector space operations given by $\lambda(s_i \otimes v) + \mu(s_i \otimes u) = s_i \otimes (\lambda v + \mu u)$. The "$s_i \otimes$" notation is just a device to tell us which particular copy of V we are working in; other than that it serves no purpose and just comes along for the ride in the vector space operation on V_i (although see the comments in the Note and References section)

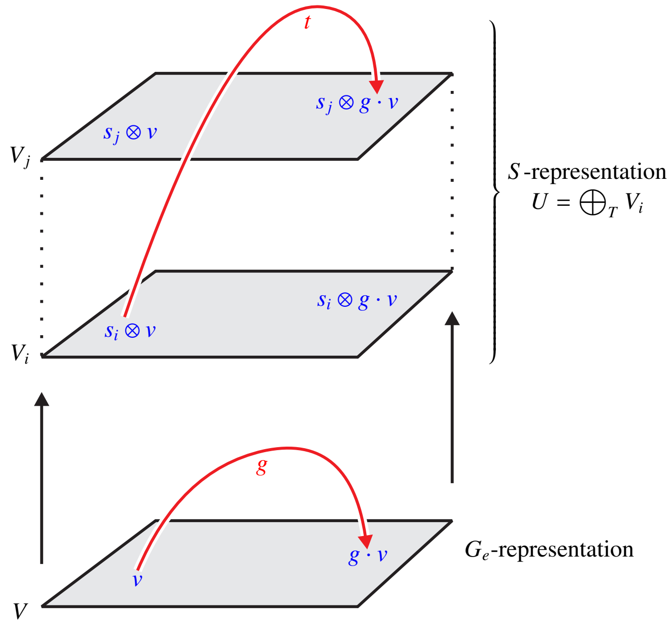


Fig. 20. Schematic of the first step of induction.

Let U be the space

$$U = \bigoplus_{s_i \in T} V_i \tag{13}$$

and define an S-action on U by

$$t \cdot (s_i \otimes v) = \begin{cases} s_j \otimes g \cdot v, & \text{if } ts_i \in \mathcal{L}_e, \text{ hence } ts_i = s_j g \\ 0, & \text{if } ts_i \notin \mathcal{L}_e, \end{cases} \tag{14}$$

for $t \in S$. The action of t thus kills the vector $s_i \otimes v$, unless ts_i is also in the $\mathcal{L}$-class $\mathcal{L}_e$, in which case by (2) there are unique $s_j \in T$ and $g \in G_e$ with $ts_i = s_j g$. The vector v is then moved in V by the action of g, with the resulting image transferred to the corresponding element of V_j – see Figure 20.

Example 12 (trivial S_1 to trivial I_n). Let $S = I_n$ and e be the zero map $0 : \varnothing \to \varnothing$. The subgroup G_e is the trivial group (or S_1!) with the single element 0. If V is the trivial representation of S_1 then (slightly confusingly) $0 \cdot v = v$ for all $v \in V$, and the transversal T consists of the one element $\{0\}$, so $U = V_0$ is a single copy of V. Finally, for any $t \in I_n$ we have $t \cdot (0 \otimes v) = 0 \otimes 0 \cdot v = 0 \otimes v$, and so U is the trivial 1-dimensional representation of I_n.

Example 13. At the opposite end of the strategic picture for I_n we have the idempotent $e = \text{id} : [n] \to [n]$, the identity of the group of units S_n. If V is any representation of S_n, then again we have a transversal T containing the single element $\{e\}$ and so $U = V_e$. Moreover $t \cdot U = 0$ unless $t \in S_n$ is also a unit, in which case it has effect that of the representation V.

Every S_n-representation is thus also an I_n-representation, just by making that part of I_n not in S_n (i.e. the stacked $\mathcal{J}$-classes $0, \mathcal{J}_1, \dots, \mathcal{J}_{n-1}$ in the middle of Figure 6) act as the zero map. It is easy to check that in general any G-representation is also an S-representation in this way, when G is the group of units of S.

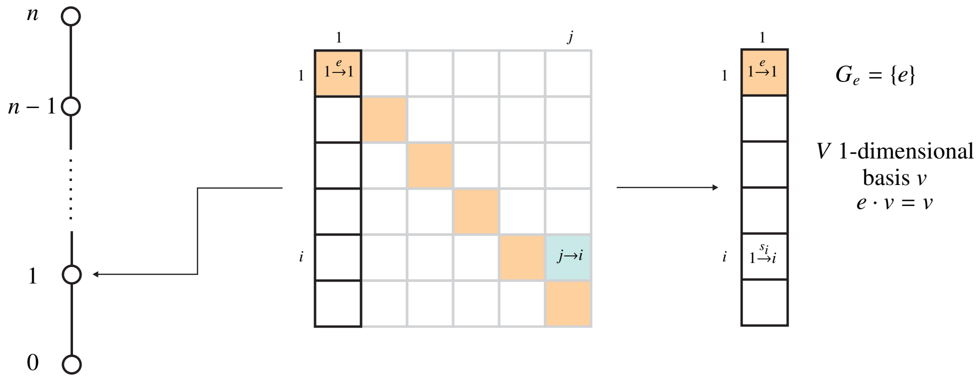


Fig. 21. The trivial representation of the subgroup G_e, where $e : 1 \mapsto 1$, induces up to the partial reflectional representation of I_n.

Example 14 (trivial S_1 to partial reflectional I_n). Moving up one rung from the bottom in the strategic picture for I_n in Figure 6, let e be the partial identity $1 \mapsto 1$ with domain and image $\{1\}$, so that G_e is again the trivial group $\{e\}$. Also again, let V be the trivial 1-dimensional representation of G_e, with basis vector v, and action $e \cdot v = v$.

The $\mathcal{J}$-class containing the subgroup G_e consists of all the bijections with domain and image of size 1 (Figure 21) and $\mathcal{L}_e$ is the column of all the maps with domain 1. There is no choice for the representatives T: they are the partial bijections $s_i : 1 \mapsto i$. The copy V_i of V has basis the vector $s_i \otimes v$, and so the space U of (13) is n-dimensional with basis $\{s_i \otimes v\}_{1 \leq i \leq n}$.

For the I_n-action, we have $ts_i \in \mathcal{L}_e$ when it has domain 1, and this is exactly when i lies in the domain of t, in which case $ts_i = s_{t(i)} = s_{t(i)}e$. Thus

$$t \cdot (s_i \otimes v) = \begin{cases} s_{t(i)} \otimes v, & \text{if } i \in \text{dom}\,(t), \\ 0, & \text{else.} \end{cases}$$

Replacing $s_i \otimes v$ by v_i we get the formula (7), and so U is the partial reflectional representation of I_n.

In Examples 12-14 an irreducible G_e-representation V becomes an irreducible S-representation U. We are not always so lucky:

Example 15 (trivial S_1 to mapping T_n). This is very similar to Example 14. Let S be the full transformation monoid T_n, with strategic picture on the left of Figure 22.

The $\mathcal{J}$-class at the bottom consists of the maps with image size 1 – the constant maps. There is a single $\mathcal{L}$-class and n $\mathcal{R}$-classes, each containing the single constant map $e_i : [n] \mapsto i$ for $1 \leq i \leq n$. These are all idempotents, so that every $\mathcal{H}$-class in this $\mathcal{J}$-class is a maximal subgroup. (You can see this in Figure 4, where every box of $\mathcal{J}_1$ is a maximal subgroup). In anycase, there is no choice once again for T, which must be the $\{e_i\}_{1 \leq i \leq n}$.

Let e_1 be the constant map $e : [n] \mapsto 1$ and V be the trivial G_{e_1}-representation with basis the vector v. For each i, the space V_i has basis the vector $e_i \otimes v$ and the U of (13) is n-dimensional with basis $\{e_i \otimes v\}_{1 \leq i \leq n}$.

For any $t \in T_n$ we have $t \cdot e_i = e_{t(i)} = e_{t(i)}e_1$, so that (14) becomes $t \cdot (e_i \otimes v) = e_{t(i)} \otimes v$, and once again we have the formula (7), after replacing $e_i \otimes v$ with v_i. The space U thus carries the

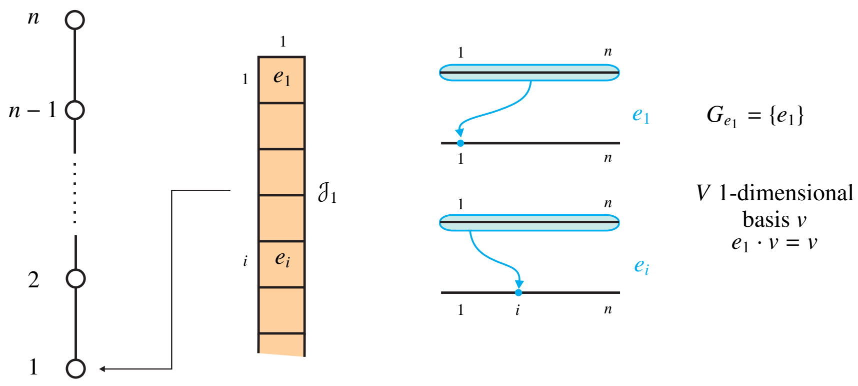


Fig. 22. An irreducible V doesn't necessarily give an irreducible U: the trivial representation of the subgroup G_{e_1}, where e_1 is the constant map $[n] \mapsto 1$, induces up to the mapping representation of T_n.

mapping representation of T_n, which is reducible, even though the seeding representation V of the subgroup $G_{e_1} \cong S_1$ is irreducible.

Before leaving the example we observe something for later on: the $\mathcal{R}$-class $\mathcal{R}_e$ contains just the single element $\{e_1\}$ with $e_1 \cdot (e_i \otimes v) = e_1 \otimes v$ for all i. Suppose that $u \in U$ is a vector that is annihilated by e_1, i.e.

$$u = \sum \lambda_i(e_i \otimes v) \text{ with } e_1 \cdot u = 0.$$

Then

$$e_1 \cdot u = 0 \Leftrightarrow \Big(\sum \lambda_i\Big)(e_1 \otimes v) = 0 \Leftrightarrow \sum \lambda_i = 0,$$

so that, after replacing $e_i \otimes v$ with v_i, the set of such annihilated vectors is the hyperplane W consisting of the $w = \sum \lambda_i v_i$ where $\sum \lambda_i = 0$.

These annihilated vectors thus form a subrepresentation of U. Even more is true: U is the n-dimensional mapping representation and $W \subset U$ is $(n-1)$-dimensional, so the quotient representation U/W is 1-dimensional, hence irreducible. In particular, W is a maximal subrepresentation of U.

Returning to generalities, let V be a representation of the maximal subgroup G_e and U the space given in (13). As in the example just done, consider the vectors in U that are annihilated by the elements of the $\mathcal{R}$-class $\mathcal{R}_e$:

$$\mathrm{Ann}_e(U) := \{u \in U : s \cdot u = 0 \text{ for all } s \in \mathcal{R}_e\}. \tag{15}$$

Definition. (induced representations) *Let V be a representation of the maximal subgroup G_e of S and U be the S-representation given by (13) and (14). Then the S-representation induced by V is the quotient*

$$V \uparrow S := U/Ann_e(U) \tag{16}$$

As in Example 15, if V is irreducible, then $\mathrm{Ann}_e(U)$ is a maximal subrepresentation of U, and $V \uparrow S$ is irreducible. The construction depends only on the $\mathcal{J}$-class of e: if e and f lie in the same $\mathcal{J}$-class then the resulting induced representations are isomorphic.

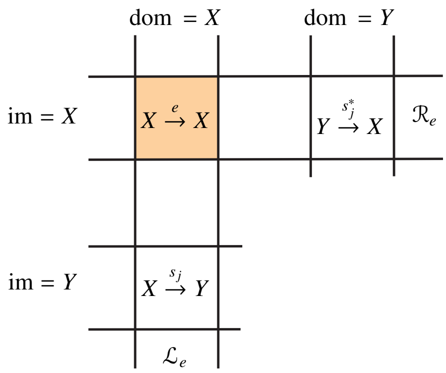


Fig. 23. $\text{Ann}_e(U) = 0$ in I_n.

Example 16. We can verify some of these general claims in the setting of I_n. In the Exercise following, we do this for an inverse monoid of the form $S(G, L)$.

Suppose that $|X| = m$ and V is a representation of the maximal subgroup $G_e \cong S_m$, with e the partial identity $X \to X$, and let U be the I_n-representation described in (13)-(14). We show first that the annihilator $\text{Ann}_e(U)$ is trivial.

To see this, let $T = \{s_i\}$ be the transversal used for the induction and $u = \sum_i s_i \otimes v_i \in \text{Ann}_e(U)$. Fix an $s_j : X \to Y \in T$ with $s_j^* : Y \to X$ the semigroup inverse of s_j – see Figure 23. Then for any i we have:

$$s_j^* s_i \in \mathcal{L}_e \Leftrightarrow \text{dom}\,(s_j^* s_i) = X \Leftrightarrow \text{im}(s_i) = \text{dom}\,(s_j^*) \Leftrightarrow \text{im}(s_i) = Y \Leftrightarrow s_i = s_j.$$

Thus, on the one hand, by (14):

$$s_j^* \cdot u = s_j^* \cdot (s_j \otimes v_j) = e \otimes v_j,$$

while on the other, $s_j^* \in \mathcal{R}_e$ and $u \in \text{Ann}_e(U)$ gives $s_j^* \cdot u = 0$. The conclusion is that $v_j = 0$ and hence $s_j \otimes v_j = 0$. Letting j vary we see that $u = 0$ and hence $\text{Ann}_e(U) = 0$ as claimed.

The annihilator is then certainly an I_n-subrepresentation of U, albeit for trivial reasons! Suppose now that V is an *irreducible* S_m-representation. We claim that the annihilator is a maximal subrepresentation, or equivalently, that U is an irreducible I_n-representation. Let u be a non-zero vector in $U = \bigoplus_T s_i \otimes V$ with $u = \sum_i s_i \otimes v_i$ and $s_j \otimes v_j \neq 0$ for some j (so that in particular, $v_j \neq 0$). Let $W = I_n \cdot u \subset U$ be the set (hence subspace – Exercise) of all images of u under the elements of I_n.

We claim that $W = U$. We have $s_j^* \cdot u \in W$, where as above

$$s_j^* \cdot u = e \otimes v_j$$

with $0 \neq e \otimes v_j \in V$. On the one hand, we have $S_m \cdot (e \otimes v_j) \subseteq W$ (as $S_m \subset I_n$), while on the other $S_m \cdot (e \otimes v_j)$ is a non-zero S_m-subrepresentation of the irreducible S_m-representation V. The conclusion is that $S_m \cdot (e \otimes v_j) = V$, and hence $V \subseteq W$. For any $s_i \in T$ we have by the definition of the action in (14) that

$$s_i \cdot (e \otimes v) = s_i \otimes v$$

and so $s_i \cdot V = s_i \otimes V$. Thus, as soon as we have $V \subseteq W$ then we have $s_i \otimes V \subseteq W$ for each s_i, and thus $U \subseteq W$. This proves the claim, and so the I_n-representation U is irreducible.

To summarise: if V is an irreducible S_m-representation then the induced representation $V \uparrow I_n$ is the space U, and this in turn is an irreducible I_n-representation.

Exercise 4. Let S be an inverse monoid of the form $S = S(G, L)$ and V an irreducible representation of the maximal subgroup G_e. Let U be the S-representation given by (13)-(14).

1. Show that $\mathrm{Ann}_e(U) = 0$ (*hint :* prove the following fact first: if the finite group G acts on the lattice L, and if a, b lie in the same G-orbit with $a \leq b$, then $a = b$; in other words, distinct elements of L in the same G-orbit are not comparable.)
2. Mimic the argument above for I_n to show that U is irreducible.

Here is how induction works for an inverse monoid of the form $S = S(G, L)$. Let $e = \mathrm{id}_a$ be an idempotent and V a representation of the maximal subgroup G_a given in (5).

The $\mathcal{H}$-classes in $\mathcal{L}_e$ are parametrised by the G-orbit of a, say $G \cdot a = \{a, b, \ldots\}$; let $\alpha = \mathrm{id}, \beta, \ldots$ be elements of G such that

$$\alpha : a \mapsto a, \beta : a \mapsto b, \ldots$$

We then take our transversal T to be $\alpha_a, \beta_a, \ldots$ In light of Exercise 4, the induced representation is carried by the space

$$V \uparrow S = \bigoplus_{\beta_a \in T} b \otimes V$$

where (after simplifying notation a little) $b \otimes V = \{b \otimes v : v \in V\}$ and, as usual, the vector space operations happen in the "v" coordinate, with the "$b \otimes$" just a label for the copy of V.

Suppose that $s = g_c$ is some element of S. To understand the action in (14) we need to compute products like $s\beta_a$: as $s\beta_a = g_c\beta_a = (g\beta)_{\beta^{-1} \cdot c \wedge a}$, we have that $s\beta_a$ lies in $\mathcal{L}_e$ exactly when

$$\beta^{-1} \cdot c \wedge a = a \Leftrightarrow a \leq \beta^{-1} \cdot c \Leftrightarrow b \leq c$$

(recall that $\beta \in G$ sends a to b). Moreover, if $b \leq c$, then $s\beta_a$ lies in the $\mathcal{H}$-class $\mathcal{H}_d$, where $d = g \cdot b$. The description (6) gives the element $\delta_d^{-1} s\beta_a = (\delta^{-1} g\beta)_a$ of G_a and

$$s\beta_a = \delta_a \cdot (\delta^{-1} g\beta)_a$$

Thus, for $b \otimes v$ an element of $V \uparrow S$ and for g_c an element of $S = S(G, L)$ we have the action:

$$g_c \cdot (b \otimes v) = \begin{cases} d \otimes h \cdot v, & \text{if } b \leq c \\ 0, & \text{if } b \nleq c, \end{cases} \tag{17}$$

where $d = g \cdot b$ and $h = (\delta^{-1} g\beta)_a$ with $\delta : a \mapsto d$ one of the elements of G chosen above.

We can say more. If s lies in the $\mathcal{J}$-class $\mathcal{J}_c$ with $\mathcal{J}_a \nleq \mathcal{J}_c$, then for any b in the G-orbit of a we have $b \nleq c$ in L, hence $s \cdot (b \otimes v) = 0$, and so $s \cdot V \uparrow S = 0$. On the other hand, if $s = \mathrm{id}_c \in \mathcal{J}_c$ with $a \leq c$ and if $a \otimes v \neq 0$ in $V \uparrow S$, then $s \cdot (a \otimes v) = a \otimes v$, and so $s \cdot V \uparrow S \neq 0$. The conclusion is that $s \cdot V \uparrow S \neq 0$ precisely for those s lying in the $\mathcal{J}$-classes that are $\geq \mathcal{J}_a$ in the $\mathcal{J}$-class poset. In particular, the apex of $V \uparrow S$, for V a representation of G_a, is $\mathcal{J}_a$.

Example 17. We return to T_n and Example 15 where $e_1 : [n] \mapsto 1$ is our idempotent, V is the trivial representation of $G_{e_1} \cong S_1$ and U the mapping representation of T_n. We saw at the end of Example 15 that $\mathrm{Ann}_e(U)$ is the hyperplane W in U consisting of the $w = \sum \lambda_i v_i$ with $\sum \lambda_i = 0$. The induced representation $V \uparrow T_n = U/W$ is thus 1-dimensional, and as $v_i - v_j \in W$ we have $v_i + W = v_j + W$ for all i and j. Taking $v_1 + W$ to be the basis vector for $V \uparrow T_n$, we have for any $t \in T_n$ that:

$$t \cdot (v_1 + W) = t(v_1) + W = v_{t(1)} + W = v_1 + W,$$

so that $V \uparrow T_n$ is the trivial representation.

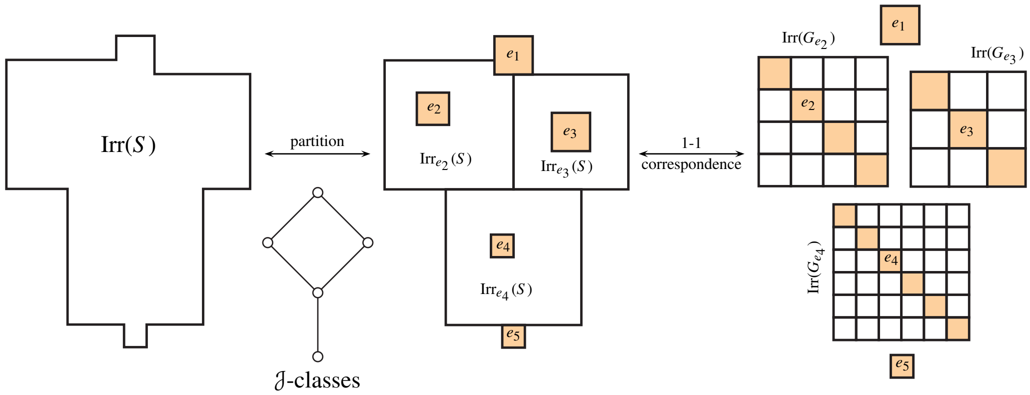


Fig. 24. Schematic of the Clifford-Munn correspondence: the irreducibles of S *(left)* partitioned into their various apexes *(middle)* which in turn are in 1-1 correspondence with the irreducibles of the corresponding maximal subgroups *(right)*.

5. The Clifford-Munn correspondence

Induction creates irreducible representations of a (finite regular) monoid out of irreducible representations of its maximal subgroups. With a little care in the accounting, this process gives a 1-1 correspondence between the irreducible S-representations and the irreducibles of a certain collection of maximal subgroups. This bijection is called the *Clifford-Munn correspondence*.

The bijection comes about by showing that reduction is the inverse of induction; for us, this is the principal purpose of reduction. The apex of an S-representation V tells us the "right" maximal subgroup to reduce to.

Figure 24 illustrates the correspondence, where as usual, the strategic picture of S drives the whole process. Let $\text{Irr}(S)$ be the set of isomorphism classes of irreducible S-representations and $E = \{e_i\}$ be a set of idempotents in 1-1 correspondence with the $\mathcal{J}$-classes of S. For $e \in E$ let

$$\text{Irr}_e(S) = \{V \in \text{Irr}(S) : \mathcal{J}_V = \mathcal{J}_e\},$$

be the irreducible S-representations V whose apex $\mathcal{J}_V$ is the $\mathcal{J}$-class $\mathcal{J}_e$ containing e. Every irreducible V has a uniquely determined apex – the set of S-irreducibles $\text{Irr}(S)$ is thus partitioned into the $\text{Irr}_e(S)$ as e ranges over E. Finally, let $\text{Irr}(G_e)$ be the irreducible representations of the maximal subgroup G_e.

Theorem 4 (Clifford-Munn correspondence). *For a fixed $e \in E$, the maps:*

$$Irr_e(S) \underset{V\uparrow S \leftarrow V}{\overset{V\to V\downarrow G_e}{\rightleftarrows}} Irr(G_e)$$

are mutual inverses, inducing a bijection $Irr(S) \rightleftarrows \bigcup_{e\in T} Irr(G_e)$.

We will prove the correspondence in the context of the symmetric inverse monoid I_n when $k = \mathbb{C}$. Exercise 5 at the end of the section asks for a proof for an inverse monoid of the form $S = S(G, L)$.

Fix then, in I_n, the $\mathcal{J}$-class $\mathcal{J}_m$ for some $0 \leq m \leq n$ and the idempotent $e = \text{id} : [m] \to [m]$. The maximal subgroup G_e is isomorphic to S_m and consists of all partial bijections $[m] \to [m]$. As we have a nice total order on the $\mathcal{J}$-classes, we write $\text{Irr}_m(I_n)$ for $\text{Irr}_e(I_n)$.

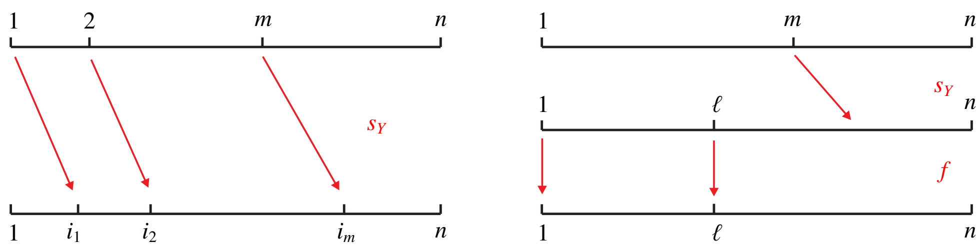


Fig. 25. The $s_Y \in \mathcal{L}_e$ in a transversal T *(left)* and $f s_Y$ *(right)*.

The map $Irr_m(I_n) \to Irr(S_m)$ *given by* $V \mapsto V \downarrow S_m$*:* as $V \in \mathrm{Irr}_m(I_n)$, it is irreducible with apex $\mathcal{J}_m$, and hence $V \downarrow S_m = eV$ is an irreducible S_m-representation by §3. Thus $V \downarrow S_m \in \mathrm{Irr}(S_m)$.

The map $Irr(S_m) \to Irr_m(I_n)$ *given by* $V \mapsto V \uparrow I_n$*:* first, we show that this is indeed a map. For $V \in \mathrm{Irr}(S_m)$, we saw in Example 16 that $\mathrm{Ann}_e(U) = 0$ where U is the I_n-representation given in (13)-(14) and that $V \uparrow I_n = U$ is irreducible. Thus $V \uparrow I_n \in \mathrm{Irr}(I_n)$; we need it to be in $\mathrm{Irr}_m(I_n)$, i.e. to have apex the $\mathcal{J}$-class $\mathcal{J}_m$. The following essentially repeats the more general arguments immediately preceding Example 17, but in a concrete setting.

The $\mathcal{L}$-class $\mathcal{L}_e$ consists of all the partial bijections with domain $[m]$. If $Y = \{i_1, \dots, i_m\}$ is some subset of size m, then let $s_Y : [m] \to Y$ be the map $s_Y : j \mapsto i_j$ given on the left of Figure 25. We take $T = \{s_Y\}$ to be the transversal used in the induction process, for Y ranging over all m-subsets of $[n]$. Thus

$$V \uparrow I_n = U = \bigoplus_{s_Y} V_Y$$

where V_Y is the vector space consisting of the vectors $s_Y \otimes v$ for $v \in V$. The I_n-action on U is given by (14).

We claim the following: if f is an idempotent, then $f(V \uparrow I_n) \neq 0$ exactly when f lies in a $\mathcal{J}$-class $\mathcal{J}_\ell$ with $\mathcal{J}_m \leq \mathcal{J}_\ell$. Moreover, $e(V \uparrow I_n)$ is itself isomorphic, as an S_m-representation, to V.

We choose f conveniently in its $\mathcal{J}$-class $\mathcal{J}_\ell$: $f = \mathrm{id} : [\ell] \to [\ell]$. We have $\mathcal{J}_\ell < \mathcal{J}_m$ exactly when $\ell < m$, in which case the right part of Figure 25 shows that $m \notin \mathrm{dom}\,(f s_Y)$ for any m-subset Y. Hence $f s_Y \notin \mathcal{L}_e$ (the partial bijections with domain $[m]$) for any Y, and so by (14)

$$f \cdot (s_Y \otimes v) = 0$$

for all Y and all v. Thus $f(V \uparrow I_n) = 0$ when $\mathcal{J}_\ell < \mathcal{J}_m$. If now $f = e$ then we have

$$e s_Y \in \mathcal{L}_e \Leftrightarrow \mathrm{dom}\,(e s_Y) = [m] \Leftrightarrow Y = [m] \Leftrightarrow s_Y = e$$

in which case

$$e \cdot (s_Y \otimes v) \neq 0 \Leftrightarrow s_Y \otimes v = e \otimes v.$$

The map $e \otimes v \mapsto v$ is then an isomorphism of vector spaces $e(V \uparrow I_n) \to V$, and for any $g \in S_m$ the diagram

$$\begin{tikzcd}
e \otimes v \arrow[r, "g(-)"] \arrow[d] & e \otimes gv \arrow[d] \\
v \arrow[r, "g(-)"] & gv
\end{tikzcd}$$

commutes. Thus, the S_m-representations $e(V \uparrow I_n)$ and V are isomorphic as claimed. Finally, Exercise 3 part 1 gives that $f(V \uparrow I_n) \neq 0$ when $\mathcal{J}_\ell > \mathcal{J}_m$. This establishes all our claims.

In particular, $\mathcal{J}_m$ is the apex of the I_n-representation $V \uparrow I_n$, and so $V \uparrow I_n$ is indeed in $\mathrm{Irr}_m(I_n)$.

The composition $Irr(S_m) \to Irr_m(I_n) \to Irr(S_m)$: We have just seen, for V an irreducible S_m-representation, that $e(V \uparrow I_n) \cong V$. Thus $(V \uparrow I_n) \downarrow S_m \cong V$, and the composition is the identity.

The composition $Irr_m(I_n) \to Irr(S_m) \to Irr_m(I_n)$: we now show that $(V \downarrow S_m) \uparrow I_n \cong V$ when V is an irreducible I_n-representation with apex $\mathcal{J}_V = \mathcal{J}_m$. The strategy is to reconstruct the representation $(V \downarrow S_m) \uparrow I_n$ inside V.

We have already the idempotent $e = \mathrm{id} : [m] \to [m]$ and the transversal $T = \{s_Y\}$ in Figure 25 for the m-sized subsets Y of $[n]$.

Consider now the subspaces $(s_Y e)V$ of V for the various Y. Then:

- Each vector space $(s_Y e)V$ is isomorphic to eV: the linear map $eV \to (s_Y e)V$ given by $ev \mapsto (s_Y e)v$ has inverse the map $(s_Y e)v \mapsto s_Y^*(s_Y e)v = e^2 v = ev$, and so is an isomorphism.
- The sum $\sum_Y (s_Y e)V$ of these spaces is direct: for which we need to show that for a fixed subset Y, the intersection
$$s_Y eV \cap \sum_{Z \neq Y} s_Z eV \tag{18}$$
is the zero space. We have just seen that s_Y^* gives an isomorphism $s_Y eV \to eV$, hence maps the subspace $s_Y eV \cap \sum_{Z\neq Y} s_Z eV$ of $s_Y eV$ isomorphically onto its image in eV. But
$$s_Y^*\Big(s_Y eV \cap \sum_{Z \neq Y} s_Z eV\Big) \subseteq s_Y^* s_Y eV \cap s_Y^*\Big(\sum_{Z\neq Y} s_Z eV\Big) = eV \cap \sum_{Z \neq Y} s_Y^* s_Z eV \tag{19}$$
where $Z \neq Y$ gives that the domain of $s_Y^* s_Z e$ has size strictly less than m, and so $s_Y^* s_Z e$ lies in a $\mathcal{J}$-class lower down the strategic picture than $\mathcal{J}_m$ does. As $\mathcal{J}_m$ is the apex of V we have $s_Y^* s_Z eV = 0$ for all Z, so that the right hand side of (19) is 0, and hence (18) is too.
- Restricting the S-action on V to the subspace $\bigoplus_Y s_Y eV$: if $t \in I_n$ then there are two possibilities for the product ts_Y. Either:
  (i). $ts_Y \in \mathcal{L}_e$, in which case by (2), there is a $g \in G_e$ and an m-subset Z such that $ts_Y = s_Z g$; or
  (ii). $ts_Y \notin \mathcal{L}_e$, and since this $\mathcal{L}$-class consists of all partial bijections with domain $[m]$, and $\mathrm{dom}\,(ts_Y) \subseteq [m]$, we have that $\mathrm{dom}\,(ts_Y)$ is a proper subset of $[m]$. In particular ts_Y lies in a $\mathcal{J}$-class lower down the strategic picture than $\mathcal{J}_m$.
  The S-action on $\bigoplus_Y s_Y eV$ is therefore given by
$$t \cdot (s_Y e) \cdot v = \begin{cases} (s_Z e) \cdot (g \cdot v), & \text{if } ts_Y \in \mathcal{L}_e, \text{ or} \\ 0, & \text{else.} \end{cases}$$

We conclude, first of all, that the subspace $\bigoplus_Y s_Y eV$ is in fact a subrepresentation of V; moreover $\bigoplus_Y s_Y eV$ contains, by taking $Y = [m]$, the subspace $eV \neq 0$. Thus $\bigoplus_Y s_Y eV$ is a non-zero subrepresentation of the irreducible representation V, hence
$$\bigoplus_Y s_Y eV = V.$$

Finally, if $ts_Y \in \mathcal{L}_e$ then the diagram

$$\begin{tikzcd}
(s_Y e) \cdot v \arrow[r, "t(-)"] \arrow[d] & (s_Z e) \cdot (g \cdot v) \arrow[d] \\
s_Y \otimes e \cdot v \arrow[r, "t(-)"] & s_Z \otimes (eg) \cdot v
\end{tikzcd}$$

commutes (it trivially commutes if $ts_Y \notin \mathcal{L}_e$). Thus $(V \downarrow S_m) \uparrow I_n \cong V$ as I_n-representations, and the composition $\text{Irr}_m(I_n) \to \text{Irr}(S_m) \to \text{Irr}_m(I_n)$ is the identity map.

This completes the proof of the Clifford-Munn correspondence when $S = I_n$.

Exercise 5. Mimic the proof above for an inverse monoid S of the form $S = S(G, L)$ (*hint:* much of the proof can be found scattered among what we have already said).

Example 18 (The irreducibles of I_n). We are finally in a position to describe the irreducible representations over $\mathbb{C}$ of the symmetric inverse monoid I_n. By Theorem 3, every I_n-representation over $\mathbb{C}$ is a direct sum of these. As we will be doing things this way in §6 – and this is sort of a dry run at it – we will use the $S(S_n, L)$ description of I_n that we saw at the end of §1, where L is the lattice of subsets of $[n]$. This allows us to follow the recipe for induction given at the end of §4.

Fix an m in the range $0 \leq m \leq n$, hence a $\mathcal{J}$-class corresponding to the S_n-orbit on L consisting of the subsets of $[n]$ having size m. Let $a = \{1, 2, \dots, m\}$ and G_a the maximal subgroup containing the idempotent id_a. The elements of G_a are the g_a where $g \in S_n$ is such that $g \cdot a = a$ (rather than being the bijections $a \to a$ as they would be in the "usual" way of describing I_n). Finally, let λ be a partition of m and S^λ be the Specht representation spanned by the v_T in (8) as T ranges over the tableau of shape λ.

We will describe the representation $S^\lambda \uparrow I_n$. The Clifford-Munn correspondence tells us that the $S^\lambda \uparrow I_n$, as both λ and m vary in $\lambda \vdash m$, form a complete and non-redundant list of the I_n-irreducibles over $\mathbb{C}$.

If $b = \{i_1, \dots, i_m\}$ is a subset of $[n]$ of size m, then let β be an element of S_n that sends $j \in a$ to $i_j \in b$. We then take the transversal T needed for induction to be the resulting β_a as b ranges over the subsets of size m.

If T is a tableau of shape λ filled with entries from a, then $\beta \cdot T$ is a tableau of shape λ filled with entries from b. Let $S^{\lambda,b}$ be a copy of S^λ, spanned by the

$$b \otimes v_T = \sum_{h \in c_{\beta \cdot T}} \text{sign}(h)\, h \cdot \{\beta \cdot T\},$$

as T varies over the tableau (on a), and where $c_{\beta \cdot T}$ are those elements of the symmetric group on the set b preserving the columns of T. The vector $b \otimes v_T$ is just the vector v_T, but with every occurence of $j \in a$ in a tabloid replaced by $i_j \in b$, and $S^{\lambda,b}$ is the space spanned by the $b \otimes v_T$.

The representation $S^\lambda \uparrow I_n$ acts on the space

$$S^\lambda \uparrow I_n = \bigoplus_{|b|=m} S^{\lambda,b}$$

To see how, fix an $s = g_c \in I_n$. We saw at the end of §4 that the apex of $S^\lambda \uparrow I_n$ is the $\mathcal{J}$-class containing the maximal subgroup G_a that we started with, so if $|c| < m$ we get $s \cdot S^\lambda \uparrow I_n = 0$.

On the other hand, by (17), if c has size at least m, then it will *not* kill those summands $S^{\lambda,b}$ for which $b \subseteq c$. In this case $s\beta_a$ lies in the $\mathcal{H}$-class labelled by the subset $d = g \cdot b$, so that for $b \otimes v_T$ spanning $S^{\lambda,b}$ we get

$$s \cdot (b \otimes v_T) = d \otimes h \cdot v_T$$

where $h = (\delta^{-1} g\beta)_a \in G_a$.

6. A sexy example

For the purposes of these notes, "sexy" will mean a certain family of Renner monoids. These encode much of the structure of algebraic monoids, and are ubiquitous in nature.

We first set the examples up in the form $S(G, L)$ from Section 1. As usual G is the symmetric group S_n, but the lattice is one we haven't seen before. Let L_0 consist of the *ordered* partitions of $[n]$, i.e. the *tuples* $\Lambda = (\Lambda_1, \dots, \Lambda_p)$ with $\{\Lambda_1, \dots, \Lambda_p\}$ a partition of $[n]$. Partially order the ordered partitions via $(\Lambda_1, \dots, \Lambda_p) \leq (\Delta_1, \dots, \Delta_q)$ if and only if

- each $\Lambda_i \subseteq$ some Δ_j, and
- if $\Lambda_i \subseteq \Delta_j$ and for $i < k$ we have $\Lambda_k \subseteq \Delta_\ell$, then $j \leq \ell$.

L_0 then has maximum element the ordered partition $([n])$ with a single block and *minimal* elements the ordered partitions where every block has size one; these minima are in 1-1 correspondence with the permutations of $[n]$.

Formally adjoin a minimum $\mathbf{0}$ to L_0 to get the lattice L. The S_n-action on L is the usual $g \cdot (\Lambda_1, \dots, \Lambda_p) = (g \cdot \Lambda_1, \dots, g \cdot \Lambda_p)$ together with $g \cdot \mathbf{0} = \mathbf{0}$.

A short diversion on where the example comes from. A linear algebraic group $\mathbb{G}$, over an algebraically closed field k, is an affine algebraic variety over k, together with a morphism $\varphi : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ of varieties, such that the product $gh := \varphi(g, h)$ gives $\mathbb{G}$ the structure of a group. Generalising this idea, a *linear algebraic monoid* $\mathbb{M}$ arises when $\varphi : \mathbb{M} \times \mathbb{M} \to \mathbb{M}$ gives $\mathbb{M}$ the structure of a monoid.

The canonical examples are $\mathbb{G} = \mathrm{GL}_n k$, the group of invertible matrices over k, and $\mathbb{M} = \mathrm{M}_n k$, the monoid of all $n \times n$ matrices over k (both under multiplication). In fact $\mathrm{GL}_n k$ is the group of units of the monoid $\mathrm{M}_n k$, and indeed for sensible $\mathbb{M}$ the group of units $\mathbb{G}$ is an algebraic group with Zariski closure $\overline{\mathbb{G}} = \mathbb{M}$.

There is a standard construction of algebraic monoids that starts with a sensible algebraic group $\mathbb{G}_0$ and a sensible representation $f : \mathbb{G}_0 \to GL(V)$. The resulting algebraic monoid is then $\mathbb{M} = \overline{k^\times f(\mathbb{G}_0)} \subset \mathrm{M}_m k$ with group of units $\mathbb{G} = k^\times f(\mathbb{G}_0)$. For example, if $\mathbb{G}_0 = \mathrm{SL}_n, \mathrm{SO}_n$ and Sp_n and f is the natural representation of $\mathbb{G}_0$, then the resulting $\mathbb{M}$ are the *classical monoids*: the general linear monoids $\mathrm{M}_n = \overline{k^\times \mathrm{SL}_n}$, the orthogonal monoids $\mathrm{MSO}_n = \overline{k^\times \mathrm{SO}_n}$ and the symplectic monoids $\mathrm{MSp}_n = \overline{k^\times \mathrm{Sp}_n}$.

Associated to a (reductive) algebraic group $\mathbb{G}$ is a finite group – called the Weyl group – that encodes much of the structure of $\mathbb{G}$; for a (reductive) algebraic monoid $\mathbb{M}$ there is a finite inverse monoid R – called the *Renner monoid* – that plays an analogous role. For example, the Weyl group of $\mathrm{GL}_n k$ is the symmetric group S_n and the Renner monoid of $\mathrm{M}_n k$ is the symmetric inverse monoid I_n. In general the group of units of the Renner monoid R of $\mathbb{M}$ is the Weyl group W of the algebraic group of units $\mathbb{G}$ of $\mathbb{M}$.

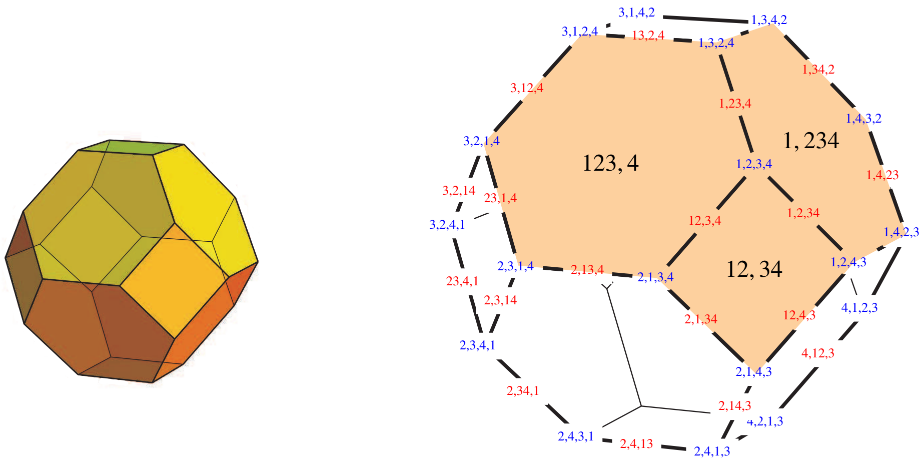


Fig. 26. The 3-permutohedron, after identifying the hyperplane $x_1 + x_2 + x_2 + x_4 = 10 \subset \mathbb{R}^4$ with $\mathbb{R}^3$ *(left)* and part of the lattice of ordered partitions overlaid on the faces *(right)*, with the blocks of the partitions separated by commas.

If R is the Renner monoid of the algebraic monoid $\mathbb{M}$ having group of units the algebraic group $\mathbb{G}$, then R is an inverse monoid of the form $S(W, L)$: the group W is the Weyl group of $\mathbb{G}$ and the lattice L turns out to be the face lattice of a convex polytope.

To see what this means, a polytope P in $\mathbb{R}^m$ is the convex hull of a finite set of points. It has r-dimensional faces, for $-1 \leq r \leq m$, with the 0-dimensional faces being the vertices, 1-dimensional faces the edges, and so on, with P itself the unique m-dimensional face; for formal reasons (mainly so that we get a lattice below) we take the empty set $\varnothing$ to be the unique face of dimension -1. The *face lattice* of P consists of the faces ordered by inclusion; it is a lattice with meet $\sigma \wedge \tau$ the intersection and join $\sigma \vee \tau$ the smallest face containing both σ and τ.

The Renner monoid of $\mathrm{M}_n k$ has the form $S(W, L)$ where W is the symmetric group S_n and L is the face lattice of an $(n-1)$-dimensional simplex. If $[n] = \{1, \ldots, n\}$ are the labels of the vertices of the simplex, then L is the lattice of subsets of $[n]$ ordered by inclusion, and the S_n-action on L is the usual one. This is the description of I_n we gave at the end of §1.

Now to the example we are interested in: let $\mathbb{G}_0 = \mathrm{SL}_n$ and V_0 be the natural module for $\mathbb{G}_0$. Let $\bigwedge^p V_0$ be the p-th exterior power of V_0 and finally

$$V = \bigotimes_{p=1}^{n-1} \bigwedge^{p} V_0, \text{ with } \dim V := m = \prod_{p=1}^{n-1} \binom{n}{p}.$$

If $f : \mathbb{G}_0 \to GL(V)$ is the corresponding representation then let $\mathbb{M} = \overline{k^\times f(\mathbb{G}_0)}$ and let R be the Renner monoid of $\mathbb{M}$. Then $R \cong S(W, L)$ with W the symmetric group S_n and L the face lattice of the $(n-1)$-dimensional permutohedron. This is the polytope in $\mathbb{R}^n$ obtained by taking the convex hull of the $n!$ points arising from all permutations of the coordinates of the point $(1, 2, \ldots, n)$. As all these points lie in the hyperplane with equation $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$, the polytope is actually $(n-1)$-dimensional.

The face lattice of the permutohedron is isomorphic to the lattice L of ordered partitions, with $\mathbf{0}$ adjoined, described at the beginning of the section. Figure 26 shows the $n = 4$ case.

To describe the irreducible representations over $\mathbb{C}$ of our Renner monoid, we use the $S(G,L)$ description from the beginning of the section, and start by drilling down a little more into the structure of the monoid, following §1.

First, we have our usual ambiguity with the elements of $S(G,L)$, where $g_a = h_b$ when $a = b$ and $g^{-1}h \cdot c = c$ for all $c \leq a$. In this case it turns out to disappear. If a is the ordered partition $(\Lambda_1, \dots, \Lambda_p)$ and c is a minimal element $\neq \mathbf{0}$ with the property that $c \leq a$, then c has the form

$$c = (\{x_{11}, \dots, x_{1q_1}\}, \dots, \{x_{p1}, \dots, x_{pq_p}\})$$

where $\Lambda_1 = \{x_{11}, \dots, x_{1q_1}\}, \dots, \Lambda_p = \{x_{p1}, \dots, x_{pq_p}\}$. If k is an element of S_n with $k \cdot c = c$ then $k = \text{id}$. Thus $g_a = h_b$ iff $a = b$ and $g = h$.

If $(\Lambda_1, \dots, \Lambda_p)$ is an ordered partition with $\lambda_i = |\Lambda_i|$, then the ordered tuple $(\lambda_1, \dots, \lambda_p)$ is called a *composition* of n: namely, the λ_i are totally ordered with $\sum \lambda_i = n$. Call the composition the *type* of the ordered partition. Two ordered partitions are then in the same S_n-orbit when they have the same type, and the $\mathcal{J}$-class poset has elements the compositions ordered by $(\lambda_1, \dots, \lambda_p) \leq (\mu_1, \dots, \mu_q)$ whenever

$$(\lambda_1, \dots, \lambda_p) = (\lambda_{11}, \dots, \lambda_{1m_1}, \dots, \lambda_{p1}, \dots, \lambda_{p,m_p})$$

with $\mu_i = \lambda_{i1} + \ldots + \lambda_{i,p_i}$. Figure 27 shows this poset when $n = 4$.

Fix a composition $(\lambda_1, \dots, \lambda_p)$ and consider the $\mathcal{J}$-class of ordered partitions of this type. If a is one of them, then the maximal subgroup is $G_a = \{g_a : g \cdot a = a\}$, and this is isomorphic to the Young subgroup $S_{\lambda_1} \times \cdots \times S_{\lambda_p}$ of S_n. We saw above that our usual ambiguity in expressing elements vanishes in R; this is why there is no need to form a quotient when describing G_a.

Let $a = (\Lambda_1, \dots, \Lambda_p)$ be the ordered partition of type $(\lambda_1, \dots, \lambda_p)$ given by:

$$\Lambda_1 = \{1, \dots, \lambda_1\}, \dots, \Lambda_p = \{n - \lambda_p + 1, \dots, n\}, \tag{20}$$

We now describe the irreducible representations of R that arise by inducing up those of the maximal subgroup $G_a \cong S_{\lambda_1} \times \cdots \times S_{\lambda_p}$. Varying the composition produces a complete list of the irreducibles over $\mathbb{C}$ of the Renner monoid R. We borrow one more fact from the representation theory of finite groups: if $\{V_i\}_{i \in I}$ and $\{U_j\}_{j \in J}$ are the irreducibles of the groups G and H, then the $\{V_i \otimes U_j\}_{I \times J}$ are the irreducibles of $G \times H$, with $V_i \otimes U_j$ a $(G \times H)$-representation via the action $(g,h) \cdot v \otimes u = g \cdot v \otimes h \cdot u$.

Now fix partitions $\mu_1 \vdash \lambda_1, \dots, \mu_p \vdash \lambda_p$ and consider the irreducible $(S_{\lambda_1} \times \cdots \times S_{\lambda_p})$-representation

$$S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \tag{21}$$

where S^{μ_i} is the Specht representation of S_{λ_i} corresponding to the partition $\mu_i \vdash \lambda_i$. The representation (21) is spanned by the vectors

$$v_{T_1} \otimes \cdots \otimes v_{T_p}$$

defined in (8) and as the T_i range over the tableau of shape μ_i filled with the numbers Λ_i in (20). To describe $S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \uparrow R$, let $b = (\Delta_1, \dots, \Delta_p)$ be another ordered partition of type $(\lambda_1, \dots, \lambda_p)$ and let $\beta \in S_n$ be such that $\beta : \Lambda_i \mapsto \Delta_i$ in an order preserving way, i.e. if $x < y \in \Lambda_i$ then $\beta(x) < \beta(y) \in \Delta_i$. Let $S^{\mu_1,\beta} \otimes \cdots \otimes S^{\mu_p,\beta}$ be a copy of (21) with spanning vectors of the form:

$$\beta \otimes v_{T_1} \otimes \cdots \otimes v_{T_p},$$

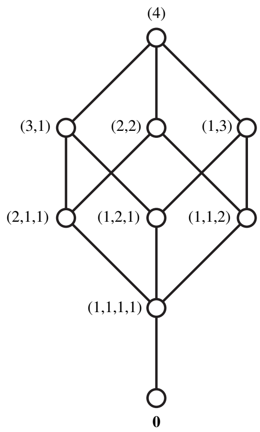


Fig. 27. The $\mathcal{J}$-class poset of our Renner monoid R when $n = 4$, corresponding to the poset of compositions of 4.

defined to be $v_{T_1} \otimes \cdots \otimes v_{T_p}$, but with every occurence of j in a tabloid replaced by $\beta(j)$. (Warning: this vector is linear in the v_{T_i} coordinates only; the "$\beta\otimes$", as usual, is just notation that comes along for the ride).

The representation $S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \uparrow R$ is carried by the space

$$S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \uparrow R = \bigoplus_b S^{\mu_1,\beta} \otimes \cdots \otimes S^{\mu_p,\beta}$$

with the direct sum over the ordered partitions of type $(\lambda_1, \ldots, \lambda_p)$. Let $s = g_c \in R$ with $g \in S_n$ and c the ordered partition $(\Omega_1, \ldots, \Omega_q)$ of type $(\omega_1, \ldots, \omega_q)$. If $(\lambda_1, \ldots, \lambda_p) \nleq (\omega_1, \ldots, \omega_q)$ then

$$s \cdot (S^{\mu_1} \otimes \cdots \otimes S^{\mu_p} \uparrow R) = 0.$$

Otherwise, when $(\lambda_1, \ldots, \lambda_p) \leq (\omega_1, \ldots, \omega_q)$ we have $s \cdot (S^{\mu_1,\beta} \otimes \cdots \otimes S^{\mu_p,\beta}) \neq 0$ when $b = (\Delta_1, \ldots, \Delta_p) \leq (\Omega_1, \ldots, \Omega_q)$ and in this case

$$s \cdot (\beta \otimes v_{T_1} \otimes \cdots \otimes v_{T_p}) = \delta \otimes h \cdot (v_{T_1} \otimes \cdots \otimes v_{T_p})$$

where $d = g \cdot b$ and $h = (\delta^{-1} g \beta)_a \in G_a \cong S_{\lambda_1} \times \cdots \times S_{\lambda_p}$.

Notes and References

There are numerous books that deal with semigroup representations, starting with the classic [CP61, Chapter 5]; more modern sources are [GM09, Ste16]. The reader who has got this far will see large overlap with [GM09], making [Ste16] a good next step. The original papers of Clifford [Cli42] and Munn [Mun55]-[Mun64] are still very readable, as is the later reworking by Rhodes [RZ91].

Semigroups. The standard reference on semigroups is [How95], where we have followed Chapters 1, 2 and 5; see also [CP61, Law98, GM09]. The three running examples are very much in the style of [GM09]. For the reader who is wondering about the "full" transformation semigroup, there is a partial version PT_n, which is a sort-of-amalgam of I_n and T_n; see [GM09, Chapter 2].

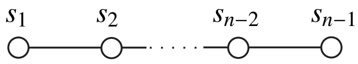
The restriction to finite regular monoids is purely to make things cleaner. An expert (who shouldn't be reading this anyway) can make the appropriate adjustments, especially in Section 5. One convenience that results is that the relation $\langle\mathcal{L},\mathcal{R}\rangle$, usually called $\mathcal{D}$ by semigroup theorists, coincides with $\mathcal{J}$. So all mention of $\mathcal{D}$ (which gives the eggbox pictures) has been merged with $\mathcal{J}$ (which gives the partial order on the eggboxes). Figure 4 is adapted from a picture by James East.

The inverse monoids $S(G, L)$ appear in [EF10, Section 9.2] as monoids of partial permutations, although they are implicit in the literature. Their purpose in [EF10] is to shed light on the *factorisable* inverse monoids: these are monoids S with the property that $S = EG = GE$, where G is the group of units of S and E the idempotents – see [CH74, Fit10]. Exercise 1 can be done by counting the g_a, but bearing in mind the ambiguity; another way, more natural in this context, is to count up the entries in the boxes in the strategic picture. The monoid of uniform block permutations of Example 2 first appears in [Fit03]. The picture of the Hasse diagram for the partition lattice $\Pi(4)$ in Figure 10 is based on one by Tilman Piesk [JB].

Representations. There are many books on group representation theory; we have followed the notation and style of [FH91, Part I]. In particular the approach is elementary, *aka* "module-free". In this section the representations are over an arbitrary field k; one moral to be extracted at the end is that in dealing with semigroup representations in characteristic $p > 0$, one needs to be just as careful, if not more careful, than one does in group representation theory. The emphasis thus moves to $k = \mathbb{C}$ in later sections. Another omission is the theory of semigroup characters, which is well developed for the running examples.

The restriction to monoids (rather than semigroups) and monoid homomorphisms removes null representations from consideration – this makes a number of statements less cluttered.

The standard reference on reflection groups is [Hum90]. A finite reflection group (acting on a real vector space) can be boiled down to a very concise piece of combinatorial data called a Coxeter symbol. Starting from a Coxeter symbol one can construct a representation of the reflection group, called the reflectional representation; a fundamental result in the theory of reflection groups is that the reflectional representation is irreducible. Starting from the type A Coxeter symbol:



this process gives the reflectional representation of Example 4. The elementary argument showing that this is irreducible was supplied by Michael Bate.

Munn [Mun57b] extends the cycle notation for permutations in S_n to elements $s \in I_n$ in the following neat way: for $x \in [n]$, repeated application of s either results in a *cycle*: $x, s(x), s^2(x), \ldots, s^{k+1}(x) = x$, in which case we write $(x, s(x), \ldots, s^k(x))$ as usual; or, $s^k(x)$ is the first iteration of S that does not lie in the domain of s, so that no more applications of s can be made. In this case we have a *link* $[x, s(x), \ldots, s^k(x)]$. Any $s \in I_n$ can then be written uniquely as a juxtaposition of disjoint cycles and links; the element $[1, 2, 3] \in I_3$ on the right of Figure 11 is an example. Reflection monoids appear in [EF10], where I_n is a Boolean monoid of type A.

The formulation of semisimplicity suffers a little from the module-free approach, where it is cleaner to talk in terms of the semisimplicity of the semigroup algebra kS. We have also avoided the notion of decomposability: the mapping representation of T_n is thus indecomposable but not irreducible, *even in characteristic* 0. One imagines that this is the aspect of the whole thing that group theorists find most distressing. Theorem 1 is standard – we have followed [Wei03, Theorem 6.1.15]; Theorem 2 similarly (see e.g. [Wei03, Theorem 3.1.14]); Theorem 3 is less well known, except to the cognoscenti; see [CP61, Ste16]. For I_n and T_n see also [GM09, Section 11.5].

Interlude: the symmetric group. A standard introductory text to all aspects of the representations of S_n is [Sag01]; for the Young tableau of this section we have followed [Ful97, Section 7.2]; see also [FH91, Chapter 4]. The irreducibles of S_n are more commonly called Specht *modules* rather than representations; as we are not mentioning modules, we hope the change of nomenclature is not too discombobulating. The representation $S^{\ydiagram{2,1}}$ in Figure 16 is S_3 as the symmetries – obtained by permuting its three vertices – of the equilateral triangle. In general $S^{\begin{ytableau} \, & \, & \, & \cdots & \, \\ \, \end{ytableau}}$ is the representation of S_n acting as the symmetries of the regular $(n-1)$-simplex; it is another incarnation of the reflectional representation of S_n mentioned in Example 4 and in the notes to the previous section. The number of irreducible representations of S_n over $\mathbb{C}$ is equal to the number $p(n)$ of partitions $\lambda \vdash n$; there is no known closed formula for $p(n)$, but many weird and wonderful properties are known. To choose just one, there is the generating function

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \left(\frac{1}{1-x^k} \right)$$

Exercise 2 is [FH91, Exercise 4.6].

Reduction. We have generally followed [GMS09]. The philosophy of the Clifford-Munn correspondence described in Section 5 is that knowledge of group representations yields knowledge of semigroup representations. The passage from groups to semigroups is the induction construction of Section 4. The current section is thus a little more perfunctory, as reduction – for us – is merely the inverse construction, its principal purpose being to establish the Clifford-Munn bijection. The usual terminology is "restriction" in much of the literature, but we have gone for reduction on two counts: it is first of all a double restriction – in that an action of S on V is being restricted to both a subgroup of S and a subspace of V – and secondly, *re*duction seems a more satisfying counterpoint to *in*duction.

Induction. The section is based on [GM09, Chapter 11]. Like there we adopt an elementary approach; for example, in module-theoretic terms the representation U is $kS \otimes_{kG_e} V$; the notation $s_i \otimes v$ for the elements of the copy V_i of V is a nod to this. That the construction is independent of the transversal T is [GM09, Theorem 11.3.1(ii)]. The general picture is from [GMS09, Theorem 7]. The justification in Example 16 that U is an irreducible I_n-representation closely follows [GM09, Theorem 11.3.1].

The Clifford-Munn correspondence. Again we have followed [GM09, GMS09] for the general picture. The irreducibles of the symmetric inverse monoid in Example 18 are a venerable topic. Munn [Mun57b] took a character-theoretic approach while Grood [Gro02] constructed the "Specht" representations for I_n from scratch, and seemingly without reference to the Clifford-Munn correspondence. Our approach follows [Alb], where this and representations of other Boolean reflection monoids are described.

A sexy example. For algebraic groups and Weyl groups see [Hum75] and for algebraic monoids and Renner monoids, the books of Putcha and Renner [Put88, Ren05]. A beautiful expository article is [Sol95]; the example in this section is taken from [Sol95, Example 5.7].

The meaning of "sensible", when talking about algebraic groups and monoids, depends on the context. If $\mathbb{M}$ is irreducible, meaning its underlying variety is irreducible, then the units $\mathbb{G}$ are a connected algebraic group with $\overline{\mathbb{G}} = \mathbb{M}$. If $\mathbb{G}_0$ is connected semisimple and the representation $f : \mathbb{G}_0 \to GL(V)$ is rational with finite kernel, then we have the construction for $\mathbb{M} = \overline{k^\times f(\mathbb{G}_0)}$ described.

The Renner monoid of $\mathrm{M}_n k$ is isomorphic to the symmetric inverse monoid; in this incarnation, I_n is called the *Rook monoid* and consists of the $n \times n$ matrices, with $0, 1$-entries, such that each row and column contains *at most* one 1. The name comes about as the matrices can be identified with $n \times n$ chessboards, with rooks in the positions occupied by 1's, and with the property that no two rooks are attacking each other. Warning: the Renner monoid is not in general a submonoid of $\mathbb{M}$, much as the Weyl group is not in general a subgroup of $\mathbb{G}$; both $\mathrm{GL}_n k$ and $\mathrm{M}_n k$ are a little special in this way.

Good references for polytopes are [Grü03, Zie95] where one can also find the combinatorial description of the face polytope of a permutohedron in terms of ordered partitions.

[Alb] Majed Albaity, *Representations of reflection monoids*. (in preparation).

[JB] John Baez, *Lattice of partitions*, American Mathematical Society Blogs, available at `blogs.ams.org`.

[CH74] S. Y. Chen and S. C. Hsieh, *Factorizable inverse semigroups*, Semigroup Forum **8** (1974), no. 4, 283–297.

[Cli42] A. H. Clifford, *Matrix representations of completely simple semigroups*, Amer. J. Math. **64** (1942), 327–342.

[CP61] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups. Vol. I*, Mathematical Surveys, No. 7, American Mathematical Society, Providence, R.I., 1961.

[CP67] ———, *The algebraic theory of semigroups. Vol. II*, Mathematical Surveys, No. 7, American Mathematical Society, Providence, R.I., 1967. MR0218472

[EF13] Brent Everitt and John Fountain, *Partial mirror symmetry, lattice presentations and algebraic monoids*, Proc. London Math. Soc. **107** (2013), no. 3, 414–450.

[EF10] ———, *Partial symmetry, reflection monoids and Coxeter groups*, Adv. Math. **223** (2010), no. 5, 1782–1814.

[Fit10] D. G. FitzGerald, *Factorizable inverse monoids*, Semigroup Forum **80** (2010), no. 3, 484–509.

[Fit03] ———, *A presentation for the monoid of uniform block permutations*, Bull. Austral. Math. Soc. **68** (2003), no. 2, 317–324.

[Fou17] John Fountain, *Sandy Green: his impact on semigroup theory*, Semigroup Forum **94** (2017), no. 1, 1–5.

[Fou10] ———, *The work of Douglas Munn and its legacy*, Semigroup Forum **81** (2010), no. 1, 2–25.

[Ful97] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[FH91] William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991. A first course; Readings in Mathematics.

[GM09] Olexandr Ganyushkin and Volodymyr Mazorchuk, *Classical finite transformation semigroups*, Algebra and Applications, vol. 9, Springer-Verlag London, Ltd., London, 2009. An introduction.

[GMS09] Olexandr Ganyushkin, Volodymyr Mazorchuk, and Benjamin Steinberg, *On the irreducible representations of a finite semigroup*, Proc. Amer. Math. Soc. **137** (2009), no. 11, 3585–3592.

[Gro02] Cheryl Grood, *A Specht module analog for the rook monoid*, Electron. J. Combin. **9** (2002), no. 1, Research Paper 2, 10.

[Grü03] Branko Grünbaum, *Convex polytopes*, 2nd ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[How95] John M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs. New Series, vol. 12, The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[Hum75] ———, *Linear algebraic groups*, Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[Law98] Mark V. Lawson, *Inverse semigroups*, World Scientific Publishing Co. Inc., River Edge, NJ, 1998. The theory of partial symmetries.

[Mun64] W. D. Munn, *Matrix representations of inverse semigroups*, Proc. London Math. Soc. (3) **14** (1964), 165–181.

[Mun61] ______, *A class of irreducible matrix representations of an arbitrary inverse semigroup*, Proc. Glasgow Math. Assoc. **5** (1961), 41–48.

[Mun60] ______, *Irreducible matrix representations of semigroups*, Quart. J. Math. Oxford Ser. (2) **11** (1960), 295–309.

[Mun57a] ______, *Matrix representations of semigroups*, Proc. Cambrdige Philos. Soc. **53** (1957), 5–12.

[Mun57b] ______, *The characters of the symmetric inverse semigroup*, Proc. Cambridge Philos. Soc. **53** (1957), 13–18.

[Mun55] ______, *On semigroup algebras*, Proc. Cambridge Philos. Soc. **51** (1955), 1–15.

[Oga55] V. A. Oganesyan, *On the semisimplicity of a system algebra*, Akad. Nauk Armyan. SSR. Dokl. **21** (1955), 145–147 (Russian, with Armenian summary).

[Pon56] J. S. Ponizovskii, *On matrix representations of semigroups (Russian)*, Mat. Sb. **38** (1956), 241–260.

[Put96] Mohan S. Putcha, *Complex representations of finite monoids*, Proc. London Math. Soc. (3) **73** (1996), no. 3, 623–641.

[Put88] ______, *Linear algebraic monoids*, London Mathematical Society Lecture Note Series, vol. 133, Cambridge University Press, Cambridge, 1988.

[Ren05] Lex E. Renner, *Linear algebraic monoids*, Encyclopaedia of Mathematical Sciences, vol. 134, Springer-Verlag, Berlin, 2005. ; Invariant Theory and Algebraic Transformation Groups, V.

[RZ91] John Rhodes and Yechezkel Zalcstein, *Elementary representation and character theory of finite semigroups and its application*, Monoids and semigroups with applications (Berkeley, CA, 1989), World Sci. Publ., River Edge, NJ, 1991, pp. 334–367.

[Sag01] Bruce E. Sagan, *The symmetric group*, 2nd ed., Graduate Texts in Mathematics, vol. 203, Springer-Verlag, New York, 2001. Representations, combinatorial algorithms, and symmetric functions.

[Sol95] Louis Solomon, *An introduction to reductive monoids*, Semigroups, formal languages and groups (York, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 466, Kluwer Acad. Publ., Dordrecht, 1995, pp. 295–352.

[Ste16] Benjamin Steinberg, *Representation theory of finite monoids*, Universitext, Springer, Cham, 2016.

[Wei03] Steven H. Weintraub, *Representation theory of finite groups: algebra and arithmetic*, Graduate Studies in Mathematics, vol. 59, American Mathematical Society, Providence, RI, 2003.

[Zal71] Yechezkel Zalcstein, *Studies in the representation theory of finite semigroups*, Trans. Amer. Math. Soc. **161** (1971), 71–87.

[Zie95] Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR1311028