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LAFFORGUE PSEUDOCHARACTERS AND PARITIES OF LIMITS OF GALOIS REPRESENTATIONS

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Abstract. Let F be a CM field with totally real subfield F^+ and let π be a C-algebraic cuspidal automorphic representation of the unitary group $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$, whose archimedean components are discrete series or non-degenerate limit of discrete series representations. We attach to π a Galois representation $R_\pi\colon \mathrm{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(a,b)(\overline{\mathbf{Q}}_\ell)$ such that, for any complex conjugation element c, $R_\pi(c)$ is as predicted by the Buzzard–Gee conjecture [BG14]. As a corollary, we deduce that the Galois representations attached to certain irregular, C-algebraic essentially conjugate self-dual cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ are *odd* in the sense of Bellaïche–Chenevier [BC11].

1. Introduction

If $\rho\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_\ell)$ is the ℓ-adic Galois representation attached to a classical modular eigenform, then ρ is *odd*: for any choice of complex conjugation $c \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, we have $\det(\rho(c)) = -1$. For Galois representations attached to automorphic representations of general reductive groups, this notion of oddness has been generalised by Buzzard–Gee [BG14], where it is interpreted as local-global compatibility at the archimedean place of $\mathbf{Q}$.

In this paper, we study the image of complex conjugation for Galois representations attached to certain irregular automorphic representations of unitary groups. Let F be a CM field with totally real subfield F^+. Let $\mathrm{U}(a,b)$ be the unitary group defined over F^+ by the Hermitian matrix $\begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}$ and let ${}^C\mathrm{U} = (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \mathrm{Gal}(\overline{F}/F^+)$ be its C-group in the sense of [BG14, Section 5] (see Section 2.3.3). Our main result is the following theorem:

Theorem 1.1 (Theorem 3.16)**.** *Let F be a CM field with totally real subfield F^+. Let π be a C-algebraic cuspidal automorphic representation of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$ such that, for each archimedean place v of F^+, π_v is either a discrete series or a non-degenerate limit of discrete series representation. Let ℓ be a prime at which π is unramified. Then there exists a Galois representation*

$$R_\pi\colon \mathrm{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(a,b)(\overline{\mathbf{Q}}_\ell)$$

attached to π that satisfies local-global compatibility at unramified primes and such that, for any complex conjugation element $c \in \mathrm{Gal}(\overline{F}/F^+)$, $R_\pi(c) = (g_c, 1)c$, where $g_c^t = g_c$, as predicted by the Buzzard–Gee conjecture [BG14, Conj. 5.3.4].

If π is an automorphic representation of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$ such that π_v is a non-degenerate limit of discrete series representation, and if ℓ is a prime at which π is unramified, then Goldring–Koskivirta [GK19, Theorem 10.5.3] have recently attached an ℓ-adic Galois representation

$$\rho_\pi\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell) \hookrightarrow {}^C\mathrm{U}(a,b)(\overline{\mathbf{Q}}_\ell)$$

to π. However, since π is an automorphic representation over F^+, its associated Galois representation should be a representation of $\mathrm{Gal}(\overline{F}/F^+)$. Our contribution is to extend ρ_π to a representation of $\mathrm{Gal}(\overline{F}/F^+)$ with the correct image on complex conjugation elements.

1.1. **The sign of an essentially conjugate self-dual representation**

Fix a complex conjugation element $c \in \mathrm{Gal}(\overline{F}/F^+)$. For any representation $\sigma\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_m(\overline{\mathbf{Q}}_\ell)$ we put $\sigma^c(g) := \sigma(cgc)$ and write $\sigma^\vee$ for its dual representation.

Definition 1.2. If $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ is a Galois representation and $\chi\colon \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbf{Q}}_\ell^\times$ is a character such that $\chi = \chi^c$, then we say that (ρ, χ) is *essentially conjugate self-dual* if $\rho^c \simeq \rho^\vee \otimes \chi$.

If (ρ, χ) is an essentially conjugate self-dual representation and if ρ is irreducible, then Bellaïche–Chenevier [BC11] have introduced the following notion of the sign of (ρ, χ): since ρ is irreducible, by Schur's Lemma, there is a matrix $A \in \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$, unique up to scalar multiplication, such that $\rho^c = A\rho^\vee A^{-1}\chi$. Applying this relation twice, we see that AA^{-t} commutes with ρ and hence, by Schur's Lemma again, that $A^t = \lambda A$, where $\lambda = \pm 1$. We call λ the *Bellaïche–Chenevier sign* of (ρ, χ) and call (ρ, χ) *odd* if $\lambda = 1$.

The representation ρ_π constructed by Goldring–Koskivirta is essentially conjugate self-dual with respect to $\chi = \varepsilon^{1-n}$, where ε is the ℓ-adic cyclotomic character. We will see in Proposition 2.20 that $(\rho_\pi, \varepsilon^{1-n})$ being odd is equivalent to ρ lifting to a representation $\mathrm{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(a,b)(\overline{\mathbf{Q}}_\ell)$ that satisfies the Buzzard–Gee conjecture on the image of complex conjugation. Applying Theorem 1.1, we deduce the following theorem for automorphic representations over $\mathrm{GL}_n(\mathbf{A}_F)$, which generalises the main result of [BC11]:

Theorem 1.3 (Theorem 2.24)**.** *Let (π, μ) be a C-algebraic essentially conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Here, $\mu\colon \mathbf{A}_F^\times \to \mathbf{C}^\times$ denotes a character such that $\pi^c \cong \pi^\vee \otimes \mu$. Assume that, for each archimedean place v of F, π_v descends to a C-algebraic discrete series or non-degenerate limit of discrete series representation of $\mathrm{U}(a,b)$. Let ℓ be a prime at which π is unramified. Then there exists a Galois representation*

$$\rho_\pi\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$$

attached to π such that $\rho_\pi^c \simeq \rho_\pi^\vee \otimes \rho_\mu\varepsilon^{1-n}$, where ρ_μ is the ℓ-adic Galois character associated to μ. Moreover, there exists a totally odd polarisation of $(\rho_\pi, \rho_\mu\varepsilon^{1-n})$ (in the sense of [BLGGT14, Section 2.1]*). In particular, if r is an irreducible subrepresentation of ρ_π that satisfies $r^c \simeq r^\vee \otimes \rho_\mu\varepsilon^{1-n}$ and appears with multiplicity one in the decomposition of ρ_π into irreducible subrepresentations, then $(r, \rho_\mu\varepsilon^{1-n})$ is odd.*

When π is regular, Theorem 1.3 is the main result of [BC11]. For an application of the oddness of these Galois representations see [Ber18], in particular Remark 2.7.

Since π is cuspidal, the representation ρ_π is conjectured to be irreducible and, hence, the condition on the multiplicity of r should be vacuously true. When π is regular, the uniqueness of r in the decomposition of ρ_π is automatic since ρ_π has distinct Hodge–Tate weights. However, when π is not regular, this multiplicity freeness is an open problem in general.

We note that our method of establishing oddness is different to that of [BC11], who use unitary eigenvarieties to deform (a form related to) π into a p-adic family of automorphic forms with generically irreducible Galois representations. A key advantage of our argument is that it does not require any multiplicity freeness considerations for proving Theorem 1.1, which is important in the irregular setting.

1.2. **Our method**

When π is irregular, [GK19] construct ρ_π, via its corresponding pseudocharacter, as a limit of Galois representations attached to regular automorphic representations. Although the Galois representations attached to regular automorphic representations are known to be odd [BC11], it is not clear that this property should be preserved after taking a limit: oddness is not encoded in the trace of ρ_π. Moreover, we do not know that ρ_π is irreducible and, hence, a priori, it need not have a sign at all, nor any lift to a representation valued in ${}^C\mathrm{U}(a,b)(\overline{\mathbf{Q}}_\ell)$.

Our solution to these problems is to work with Lafforgue's pseudocharacters [Laf18] in place of Taylor's classical pseudocharacters [Tay91]. By the work of Goldring–Koskivirta, for each $n \in \mathbf{N}$, the system of Hecke eigenvalues of π is congruent modulo ℓ^n to the system of Hecke eigenvalues of a mod ℓ^n cohomological eigenform π_n. By [BC11, Theorem 1.2] and Proposition 2.20, the Galois representation ρ_n attached to π_n lifts to a representation R_n that is valued in ${}^C\mathrm{U}(a,b)$ with the correct sign. Finally, a computation in invariant theory shows that the limit of a sequence of ${}^C\mathrm{U}(a,b)$-valued representations is valued in ${}^C\mathrm{U}(a,b)$ and that the sign is preserved in the limit.

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2. Preliminaries

2.1. **Notation**

For each prime ℓ, we fix once and for all an isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$. For a number field L and a Hecke character $\mu\colon \mathbf{A}_L^\times \to \mathbf{C}^\times$ we write $\rho_\mu\colon \mathrm{Gal}(\overline{L}/L) \to \overline{\mathbf{Q}}_\ell^\times$ for the corresponding Galois character. We let ε denote the ℓ-adic cyclotomic character.

2.2. **Unitary groups**

Let F be a CM field with totally real subfield F^+. For $x \in F$, let $\overline{x}$ denote the image of x under the non-trivial element of $\mathrm{Gal}(F/F^+)$. Fix an integer $n \in \mathbf{N}$ and a matrix $J \in \mathrm{GL}_n(F)$ with $\overline{J} = J^t$.

Definition 2.1. The *unitary group* $\mathrm{U}(J)$ is the algebraic group over F^+ whose R-points are

$$\mathrm{U}(J)(R) = \left\{ g \in \mathrm{GL}_n(R \otimes_{F^+} F) : gJ\overline{g}^t = J \right\}$$

for any F^+-algebra R.

Definition 2.2. The *general unitary group* $\mathrm{GU}(J)$ is the algebraic group over F^+ whose R-points are

$$\mathrm{GU}(J)(R) = \left\{ g \in \mathrm{GL}_n(R \otimes_{F^+} F) : gJ\overline{g}^t = \lambda J,\ \lambda \in R^\times \right\}$$

for any F^+-algebra R.

Typically, we take $J = J_{a,b} = \begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}$ and write $\mathrm{U}(a,b)$ for the corresponding unitary group. We call (a,b) the signature of $\mathrm{U}(a,b)$.

For any F-algebra R, the canonical isomorphism

$$\begin{aligned} R \otimes_{F^+} F &\xrightarrow{\sim} R \oplus R \\ (r \otimes x) &\mapsto (rx, r\overline{x}). \end{aligned}$$

allows us to identify $\mathrm{U}(J)_{/F}$ with GL_n and $\mathrm{GU}(J)_{/F}$ with $\mathrm{GL}_n \times \mathrm{GL}_1$.

2.2.1. **Root data and the L-group.** If B is the upper-triangular Borel of $\mathrm{U}(a,b)_{/\overline{F}}$ and if T is the diagonal torus, then the based root datum of $\mathrm{U}(a,b)_{/\overline{F}} \cong \mathrm{GL}_n$ is given by $\Psi(B,T) = (X^*, \Delta^*, X_*, \Delta_*)$ with

- $X^* = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{a_1} \cdots t_n^{a_n} : a_i \in \mathbf{Z} \right\}$;
- $\Delta^* = \{E_i - E_{i+1} : i = 1, \ldots, n-1\}$, where E_i denotes the character $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i$;
- $X_* = \left\{ t \mapsto \begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n} \end{pmatrix} : a_i \in \mathbf{Z} \right\}$;
- $\Delta_* = \left\{ E'_i - E'_{i+1} \right\}$, where E'_i denotes the cocharacter $t \mapsto \mathrm{diag}(1, \ldots, 1, t, 1, \ldots, 1)$ with the non-trivial part in the i^{th} position.

The identification of $\mathrm{U}(a,b)_{/\overline{F}}$ with GL_n identifies the dual group $\widehat{\mathrm{U}(a,b)}$ with GL_n, however the action of Galois is different.

Definition 2.3. The L-group of $\mathrm{U}(a,b)$ is

$${}^L\mathrm{U} = {}^L\mathrm{U}(a,b) = \mathrm{GL}_n \rtimes \mathrm{Gal}(\overline{F}/F^+),$$

where $\mathrm{Gal}(\overline{F}/F^+)$ acts, via its quotient $\mathrm{Gal}(F/F^+)$, by

$$c \cdot g = \Phi_n g^{-t} \Phi_n^{-1},$$

where $g \in \mathrm{GL}_n$, c is the non-trivial element of $\mathrm{Gal}(F/F^+)$ and Φ_n is the matrix whose ij^{th} entry is $(-1)^{i+1}\delta_{i,n-j+1}$.

Definition 2.4. The L-group of $\mathrm{GU}(a,b)$ is

$${}^L\mathrm{GU} = {}^L\mathrm{GU}(a,b) = (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \mathrm{Gal}(\overline{F}/F^+),$$

where $\mathrm{Gal}(\overline{F}/F^+)$ acts, via its quotient $\mathrm{Gal}(F/F^+)$, by

$$c \cdot (g, \lambda) = (\Phi_n g^{-t} \Phi_n^{-1}, \det(g)\lambda),$$

where $(g, \lambda) \in \mathrm{GL}_n \times \mathrm{GL}_1$ and c is the non-trivial element of $\mathrm{Gal}(F/F^+)$.

We note that the L-group of $\mathrm{U}(a,b)$ does not depend on the signature (a,b).

2.3. **Algebraic automorphic representations and the C-group**

The Langlands conjectures predict a relationship between *algebraic* automorphic representations and Galois representations. There are two natural notions of what it means to be algebraic, which Buzzard–Gee [BG14] call L-algebraic and C-algebraic. When $G = \mathrm{GL}_n$, there is a simple method using a *twisting element* to go from a C-algebraic automorphic representation of G to an L-algebraic one: if π is C-algebraic, then $\pi \otimes |\cdot|^{(n-1)/2}$ is L-algebraic. However, in general, and in particular when $G = \mathrm{U}(a,b)$, these notions are indeed distinct. As a result, the Galois representations attached to C-algebraic automorphic representations of $\mathrm{U}(a,b)$ will not be valued in the L-group of $\mathrm{U}(a,b)$ but in its C-group, defined in [BG14]. In this subsection, we recall the notions of L-algebraic and C-algebraic representations and define the C-group of $\mathrm{U}(a,b)$.

2.3.1. **Algebraic automorphic representations.** Let k be either $\mathbf{R}$ or $\mathbf{C}$. Let G be a reductive group over a number field F, with fixed maximal torus T, Borel B and based root datum $\Psi(B,T) = (X^*, \Delta^*, X_*, \Delta_*)$. To an irreducible admissible complex representation of $G(k)$, Langlands [Lan89] associates a $\widehat{G}(\mathbf{C})$-conjugacy class (called *L-parameter*) of admissible homomorphisms

$$r = r_\pi \colon W_k \to {}^L G(\mathbf{C}),$$

where

$$W_k = \begin{cases} \mathbf{C}^\times & k = \mathbf{C} \\ \mathbf{C}^\times \sqcup j\mathbf{C}^\times & k = \mathbf{R} \end{cases}$$

is the Weil group of k; if $k = \mathbf{R}$, then $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbf{C}^\times$.

Fix a maximal torus $\widehat{T}$ in $\widehat{G}_{\mathbf{C}}$ equipped with an identification $X_*(\widehat{T}) = X^*(T)$, and conjugate r so that $r(\mathbf{C}^\times) \subset \widehat{T}(\mathbf{C})$. We find that, for $z \in \mathbf{C}^\times$,

$$r(z) = \lambda(z)\lambda_c(\overline{z}),$$

where $\lambda, \lambda_c \in X_*(\widehat{T}) \otimes \mathbf{C}$ and $\lambda - \lambda_c \in X_*(\widehat{T})$. Let $\delta \in X_*(\widehat{T}) \otimes \mathbf{C}$ denote half the sum of the positive roots.

Definition 2.5. We say that π is *L-algebraic* if $\lambda, \lambda_c \in X_*(\widehat{T})$. We say that π is *C-algebraic* if $\lambda, \lambda_c \in \delta + X_*(\widehat{T})$.

If π is an automorphic representation of $G(\mathbf{A}_F)$, we say that π is L-algebraic (resp. C-algebraic) if π_v is L-algebraic (resp. C-algebraic) for every archimedean place v.

2.3.2. **Twisting elements.** If half the sum of the positive roots δ is itself a root, then the notions of L-algebraic and C-algebraic coincide. More generally, if $X^*(T)$ contains a twisting element θ, then Buzzard–Gee [BG14, §5.2] give a recipe to go between L-algebraic and C-algebraic representations, which we now recall.

Definition 2.6 ([BG14, Definition 5.2.1])**.** An element $\theta \in X^*(T)$ is a *twisting element* if θ is $\mathrm{Gal}(\overline{F}/F)$-stable and $\langle \theta, \alpha^\vee \rangle = 1 \in \mathbf{Z}$ for all simple coroots $\alpha^\vee$.

Let S' denote the maximal split torus quotient of G. If θ is a twisting element, then $\theta - \delta \in \frac{1}{2}X^*(S')$, and we can define a character $|\cdot|^{\theta-\delta}$ of $G(F)\backslash G(\mathbf{A}_F)$ as the composite

$$G(\mathbf{A}_F) \to S'(\mathbf{A}_F) \xrightarrow{2(\theta-\delta)} \mathbf{A}_F^\times \xrightarrow{|\cdot|} \mathbf{R}_{>0} \xrightarrow{\sqrt{\cdot}} \mathbf{R}_{>0}.$$

Using this character $|\cdot|^{\theta-\delta}$, we can go between L-algebraic and C-algebraic representations:

Proposition 2.7 ([BG14, Proposition 5.2.2])**.** *If θ is a twisting element, then an automorphic representation π is C-algebraic if and only if $\pi \otimes |\cdot|^{\theta-\delta}$ is L-algebraic.*

Example 2.8. If $G = \mathrm{GL}_n$ and n is even, then δ is not a root. However, the element $\theta = (n-1, n-2, \ldots, 1, 0) \in \mathbf{Z}^n \cong X^*(T)$ is a twisting element, and the character $|\cdot|^{\theta-\delta}$ is equal to $|\cdot|^{(n-1)/2}$.

2.3.3. **The C-group.**

Proposition 2.9. *Let $G = \mathrm{U}(a,b)$. If $n = a+b$ is odd, then $\delta \in X^*(T)$. If n is even, then $X^*(T)$ does not contain a twisting element.*

Proof. Identify $X^*(T)$ with $\mathbf{Z}^n$ in the obvious way. Then $\delta = \frac{1}{2}(n-1, n-3, n-5, \ldots, -n+3, -n+1) \in X^*(T)$ if and only if n is odd.

Suppose that n is even and that $\theta = (a_1, \ldots, a_n)$ is a twisting element. Then, for each each $i = 1, \ldots, n-1$, since $\langle \theta, E_i' - E_{i+1}' \rangle = 1$, we have $a_i = a_{i+1} + 1$. Hence, $\theta = (a_1, a_1 - 1, \ldots, a_1 - n + 1)$. It is clear that no element of this form can be stable under the action of Galois: we have

$$c \cdot \theta = (n-1-a_1, n-2-a_1, \ldots, -a_1),$$

so if $c \cdot \theta = \theta$, then $a_1 = \frac{n-1}{2}$, which is a root only if n is odd. $\square$

Hence, in general, we cannot go between L-algebraic and C-algebraic automorphic representations of $\mathrm{U}(a,b)$. To solve this problem, Buzzard–Gee construct an extension $\widetilde{\mathrm{U}(a,b)}$, such that

$$1 \to \mathbf{G}_m \to \widetilde{\mathrm{U}(a,b)} \to \mathrm{U}(a,b) \to 1$$

is exact and $\widetilde{\mathrm{U}(a,b)}$ contains a twisting element. The C-group of $\mathrm{U}(a,b)$ is then defined to be the L-group of $\widetilde{\mathrm{U}(a,b)}$.

Lemma 2.10. *The group ${}^C\mathrm{U} = {}^C\mathrm{U}(a,b)$ is isomorphic to*

$${}^C\mathrm{U} \cong \widehat{\widetilde{\mathrm{U}(a,b)}} \rtimes \mathrm{Gal}(\overline{F}/F^+),$$

where

$$\widehat{\widetilde{U(a,b)}} \cong \mathrm{GL}_n \times \mathrm{GL}_1$$

and $\mathrm{Gal}(\overline{F}/F^+)$ *acts via the quotient* $\mathrm{Gal}(F/F^+)$*: if c is the non-trivial element of* $\mathrm{Gal}(F/F^+)$ *and $(g,\mu) \in \mathrm{GL}_n \times \mathrm{GL}_1$, then*

$$c \cdot (g, \mu) = (g^{-t}\mu^{1-n}, \mu).$$

Proof. Following [BG14, Prop 5.3.3], we find that

$${}^C\mathrm{U} \cong \widehat{\widetilde{\mathrm{U}(a,b)}} \rtimes \mathrm{Gal}(\overline{F}/F^+),$$

where

$$\widehat{\widetilde{U(a,b)}} \cong \frac{\mathrm{GL}_n \times \mathrm{GL}_1}{\langle ((-I_n)^{n-1}, -1) \rangle},$$

and $\mathrm{Gal}(\overline{F}/F^+)$ acts via the quotient $\mathrm{Gal}(F/F^+)$: if c is the non-trivial element of $\mathrm{Gal}(F/F^+)$ and $(g,\mu) \in \widehat{\widetilde{\mathrm{U}(a,b)}}$, then

$$c \cdot (g, \mu) = (\Phi_n g^{-t} \Phi_n^{-1}, \mu).$$

The map

$${}^C\mathrm{U} \cong \frac{\mathrm{GL}_n \times \mathrm{GL}_1}{\langle ((-I_n)^{n-1}, -1) \rangle} \rtimes \mathrm{Gal}(\overline{F}/F^+) \to (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \mathrm{Gal}(\overline{F}/F^+)$$

given by

$$(g,\mu)\mapsto(g\mu^{1-n},\mu^2)$$

and

$$c\mapsto(\Phi_n,-1)c$$

defines an isomorphism from the group defined in [BG14, Prop 5.3.3] to the group we have defined. ☐

Remark 2.11. Via the above lemma, we see that the group ${}^C\mathrm{U}$ is closely related to the group $\mathscr{G}_n$ defined in [CHT08, Section 1]. Indeed, $\mathscr{G}_n=(\mathrm{GL}_n\times\mathrm{GL}_1)\rtimes\mathrm{Gal}(\overline{F}/F^+)$, but with $c\in\mathrm{Gal}(F/F^+)$ acting as $c\cdot(g,\mu)=(g^{-t}\mu,\mu)$. We note that the map $(g,\mu)\mapsto(g,\mu^{1-n})$ defines an isogeny ${}^C\mathrm{U}\to\mathscr{G}_n$. We refer the reader to [BG14, Section 8.3] for further details.

The map $\mathbf{G}_m\to\widetilde{\mathrm{U}(a,b)}$ induces a map $d\colon{}^C\mathrm{U}(a,b)\to\mathbf{G}_m$, given by $(g,\mu)\mapsto\mu$ and $c\mapsto-1$.

Let $\widetilde{T}\subseteq\widetilde{\mathrm{U}(a,b)}$ be the pullback of the torus $T\subseteq\mathrm{U}(a,b)$. Let $\hat{\xi}\in X_*(\hat{\widetilde{T}})$ be given by

$$z\mapsto(1,z).$$

Let $\theta=\delta+\frac{1}{2}\hat{\xi}\in X_*(\hat{\widetilde{T}})$; explicitly,

$$\theta\colon z\mapsto\left(\begin{pmatrix}1&&&\\&z^{-1}&&\\&&\ddots&\\&&&z^{1-n}\end{pmatrix},z\right).$$

Then θ is a twisting element.

2.3.4. **Algebraic automorphic representations of** $\mathrm{U}(a,b)$**.** In the specific case of an automorphic representation π of $\mathrm{U}(a,b)$, we now make explicit what it means for π to be C- or L-algebraic. For $z=re^{i\theta}\in\mathbf{C}$ with $r\in\mathbf{R}_{>0}$ and $a\in\mathbf{Z}$ we write $(z/\overline{z})^{a/2}$ for $e^{ia\theta}$.

Lemma 2.12. *Consider $r_\pi\colon W_{\mathbf{R}}\to{}^L\mathrm{U}(a,b)(\mathbf{C})$ such that*

$$z\in\mathbf{C}^\times\mapsto\begin{pmatrix}(z/\overline{z})^{a_1}&&&\\&(z/\overline{z})^{a_2}&&\\&&\ddots&\\&&&(z/\overline{z})^{a_n}\end{pmatrix}$$

with $a_i\in\frac{1}{2}\mathbf{Z}$.

(1) π is C-algebraic (i.e. $a_i\in\mathbf{Z}+\frac{n-1}{2}$) if and only if $r(j)=(A\Phi_n^{-1})c$ with $A=A^t$.

(2) π is L-algebraic (i.e. $a_i\in\mathbf{Z}$) if and only if $r(j)=(A\Phi_n^{-1})c$ with $A=(-1)^{n-1}A^t$.

(3) Assume that there exists $i\in\{1,\dots,n\}$ such that $a_i\neq a_j$ for all $j\neq i$. Then π is C-algebraic.

Proof. Writing $r(j)=(A\Phi_n^{-1})c$, the semi-direct product relation implies that $r(j^2)=(AA^{-t}(-1)^{n-1})$.

Observe that $r(-1)=(-1)^{n-1}$ if and only if π is C-algebraic and that $r(-1)=1$ if and only if π is L-algebraic (if n is odd, then π is C-algebraic if and only if it is L-algebraic).

Since $j^2=-1$, it follows that $A=A^t$ if and only if π is C-algebraic and that $A=(-1)^{n-1}A^t$ if and only if π is L-algebraic.

For (3) we note that the relationship $jzj^{-1}=\overline{z}$ implies

$$Ar(z)=r(z)A.$$

Assuming, without loss of generality, that $a_1 \neq a_2$, it follows that A is of the form

$$A = \begin{pmatrix} A_1 & 0 & * \\ 0 & A_2 & * \\ * & * & * \end{pmatrix}.$$

In particular, A cannot satisfy $A^t = -A$. □

2.4. Galois representations attached to automorphic representations of $\mathrm{U}(a,b)$

Let F be a CM field—i.e. a totally imaginary quadratic extension of a totally real subfield F^+—and let π be a cuspidal automorphic representation of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$. In this subsection, we recall results associating $\mathrm{Gal}(\overline{F}/F)$-representations to π.

Let $H = \mathrm{Res}_{F/F^+}(\mathrm{GL}_n)$. For an automorphic representation π, let χ_π denote its central character. For a place v of F^+ at which π is unramified and a place w of F above v, define the base change of π_v to H, denoted $\mathrm{BC}(\pi_v)$, and its w-part $\mathrm{BC}(\pi_v)_w$ as in [HLTT16, Section 1.3]. Write rec_{F_w} for the (unramified) local Langlands correspondence, normalized as in [HT01].

The following theorem is the work of many people; for a reference see, for example, [HLTT16, Corollary 1.3] or [Ski12] (but we state a version over general CM fields for the regular discrete series case covered by [Shi11], which only requires Labesses restricted base change):

Theorem 2.13 (Clozel, Harris, Taylor, Labessse, Morel, Shin)**.** *Let π be a cuspidal automorphic representation of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$. Let S be the set of primes of F lying above rational primes at which F and π are ramified. Suppose that, for each archimedean place v of F^+, π_v is a regular discrete series representation. Then there exists a compatible system of ℓ-adic Galois representations*

$$\rho_\pi \colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$$

such that

$$\rho_\pi^c \simeq \rho_\pi^\vee \otimes \varepsilon^{1-n}$$

and such that

$$(\rho_\pi|_{W_{F_w}})^{\mathrm{ss}} \cong \mathrm{rec}_{F_w}(\mathrm{BC}(\pi_v)_w \otimes |\cdot|_w^{\frac{1-n}{2}})$$

for $w \notin S$ and $w \mid v$. These representations are de Rham for primes above ℓ.

For imaginary CM fields (i.e. those containing an imaginary quadratic field), stronger local-global compatibility statements can be proved [Sha19, Theorem 2.2].

Using Theorem 2.13, Goldring–Koskivirta [GK19] prove the following result (a similar result is proved by Pilloni-Stroh [PS16]) for certain irregular automorphic representations. We refer the reader to [GK19, 10.1.2] for the definition of non-degenerate limit of discrete series representations.

Theorem 2.14 ([GK19, Theorem 10.5.3])**.** *Let π be a C-algebraic cuspidal automorphic representation of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$. Let S be the set of primes of F lying above rational primes at which F and π are ramified. Suppose that, for each archimedean place v of F^+, π_v is a discrete series or a non-degenerate limit of discrete series representation. Then, for each prime ℓ at which π is unramified, there exists an ℓ-adic Galois representation*

$$\rho_\pi \colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$$

such that

$$\rho_\pi^c \simeq \rho_\pi^\vee \otimes \varepsilon^{1-n}$$

and such that

$$(\rho_\pi|_{W_{F_w}})^{\mathrm{ss}} \cong \mathrm{rec}_{F_w}(\mathrm{BC}(\pi_v)_w \otimes |\cdot|_w^{\frac{1-n}{2}})$$

for $w \notin S$.

Remark 2.15. Note that the condition that π is C-algebraic is often satisfied automatically. Indeed, by [GK19, Section 10.5.3], non-degenerate limit of discrete series representations correspond to Langlands parameters as in Lemma 2.12, with parameters a_i of multiplicity at most two (whereas discrete series representations have all a_i, $i = 1, \dots n$ distinct). By Lemma 2.12, these representations are automatically C-algebraic unless each a_i has multiplicity exactly two.

2.5. Polarised Galois representations and the Bellaïche–Chenevier sign

In the previous subsection, we recalled the existence of Galois representations

$$\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$$

attached to automorphic representations π of $\mathrm{U}(a,b)$. In this subsection, we show how to lift these representations to representations

$$R\colon \mathrm{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell)$$

and we relate the image of complex conjugation elements under R to the Bellaïche–Chenevier sign of ρ.

2.5.1. Polarised Galois representations.

We begin by recalling the notion of a polarised Galois representation of $\mathrm{Gal}(\overline{F}/F)$. For a more detailed discussion, we refer the reader to [BLGGT14, Section 2.1].

Definition 2.16. A *polarised* ℓ-adic Galois representation of $\mathrm{Gal}(\overline{F}/F)$ is a triple $(\rho, \chi, \langle\cdot,\cdot\rangle)$, where:

- $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ is a Galois representation;
- $\chi\colon \mathrm{Gal}(\overline{F}/F^+) \to \overline{\mathbf{Q}}_\ell^\times$ is a Galois character such that $\chi(c)$ is independent of the choice of complex conjugation c in $\mathrm{Gal}(\overline{F}/F^+)$.
- $\langle\cdot,\cdot\rangle$ is a pairing on $\overline{\mathbf{Q}}_\ell^n$

such that, for all $x, y \in \overline{\mathbf{Q}}_\ell^n$:

- $\langle x, y\rangle = -\chi(c)\langle y, x\rangle$.
- $\langle \rho(g)x, \rho^c(g)y\rangle = \chi(g)\langle x, y\rangle$ for all $g \in \mathrm{Gal}(\overline{F}/F)$.

If $(\rho, \chi, \langle\cdot,\cdot\rangle)$ is a polarised Galois representation, then there is a matrix $A \in \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$, such that, for all $x, y \in \overline{\mathbf{Q}}_\ell^n$,

$$\langle x, y\rangle = x^t A^{-1} y.$$

Since $\langle \rho(g)x, \rho^c(g)y\rangle = \chi(g)\langle x, y\rangle$ for all $g \in \mathrm{Gal}(\overline{F}/F)$, we see that

$$\rho^c = A\rho^\vee A^{-1}\chi,$$

so that (ρ, χ) is essentially conjugate self-dual. Moreover, the condition that $\langle x, y\rangle = -\chi(c)\langle y, x\rangle$, where c is the non-trivial element of $\mathrm{Gal}(F/F^+)$, ensures that $x^t A^{-1} y = -\chi(c) x^t A^{-t} y$. Since $x, y \in \overline{\mathbf{Q}}_\ell^n$ were arbitrary, we see that $A = -\chi(c)A^t$. We call $-\chi(c)$ the *sign* of $(\rho, \chi, \langle\cdot,\cdot\rangle)$. Note that, under our assumption on χ, the sign is independent of the choice of c. If ρ is irreducible, then the sign is exactly the Bellaïche–Chenevier sign of (ρ, χ), as recalled in Section 1.1. In our terminology the result of [BC11] is equivalent to saying that $(\rho_\pi, \varepsilon^{1-n}\eta^n_{F/F^+}, \langle\cdot,\cdot\rangle)$ is polarised, where η_{F/F^+} denotes the quadratic Galois character corresponding to F/F^+.

If (ρ, χ) is essentially conjugate self-dual and if $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ is irreducible, then there is a natural way to extend (ρ, χ) to a polarised Galois representation. Indeed, there is a matrix A, unique up to scalar multiplication, such that $\rho^c = A\rho^\vee A^{-1}\chi$ and such that $A = \lambda A^t$ where $\lambda = \pm 1$. Since $\chi = \chi^c$, χ extends to a character of $\mathrm{Gal}(\overline{F}/F^+)$, and we choose this extension so that $\chi(c) = -\lambda$. If we define a pairing $\langle\cdot,\cdot\rangle$ on $\overline{\mathbf{Q}}_\ell$ using A^{-1}, then $(\rho, \chi, \langle\cdot,\cdot\rangle)$ is a polarised Galois representation.

More generally, suppose that (ρ, χ) is essentially conjugate self-dual, that ρ is semisimple and that every irreducible subrepresentation r of ρ for which $r^c \simeq r \otimes \chi$ has sign λ. Then there is still a choice of polarisation for (ρ, χ). Indeed, we can write

$$\rho = \left(\bigoplus_i r_i\right) \oplus \left(\bigoplus_j s_j \oplus (s_j^c)^\vee \chi\right),$$

where the r_i are essentially conjugate self-dual with sign λ. We can define a polarisation on each r_i as before: for each j, if $\dim(s_j) = n_j$, then the matrix

$$\begin{pmatrix} & I_{n_j} \\ \lambda I_{n_j} & \end{pmatrix}$$

defines an invariant pairing on the essentially conjugate self-dual representation $s_j \oplus (s_j^c)^\vee \chi$. Taking the direct sum of these polarised Galois representations gives a polarisation of ρ with the correct sign.

Remark 2.17. In general, the converse of this construction fails. Given a polarised Galois representation $(\rho, \chi, \langle\cdot,\cdot\rangle)$ with sign λ, it is not true in general that every essentially conjugate self-dual subrepresentation of ρ has Bellaïche–Chenevier sign λ. For example, if (r, χ) is an essentially conjugate self-dual Galois representation with sign -1, then we can define two polarisations on $\rho = r \oplus r$ with different signs. Indeed, if $r^c = BrB^{-1}\chi$ with $B = -B^t$, then let

$$A_1 = \begin{pmatrix} B & \\ & B \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} & -B \\ B & \end{pmatrix}.$$

Then (ρ, χ) has sign -1 with respect to the pairing $\langle x, y\rangle = x^t A_1^{-1} y$, while it has sign 1 with respect to the pairing $\langle x, y\rangle = x^t A_2^{-1} y$.

Nevertheless, given a polarised Galois representation $(\rho, \chi, \langle\cdot,\cdot\rangle$ with sign λ, any subrepresentation r of ρ that is essentially conjugate self-dual with respect to χ and appears with multiplicity one in the decomposition of ρ will have sign λ. We record this fact in the following lemma:

Lemma 2.18. *Let $(\rho, \chi, \langle\cdot,\cdot\rangle)$ be a polarised Galois representation with sign λ. Suppose that r is an irreducible subrepresentation of ρ that appears with multiplicity one in the decomposition of ρ and is such that $r^c \simeq r^\vee \otimes \chi$. Then (r, χ) has sign λ.*

Proof. Write $\rho = (\bigoplus_i r_i) \oplus \left(\bigoplus_j s_j \oplus (s_j^c)^\vee \chi\right)$, where, for each i, (r_i, χ) is essentially conjugate self-dual. Write A for the matrix such that $\langle x, y\rangle = x^t A^{-1} y$. Then

$$\rho^c = A\rho^\vee A^{-1}\chi.$$

In particular, A permutes the r_i's and since r has multiplicity one in the decomposition of ρ, there must be a submatrix A_r of the block diagonal of A such that $r^c = A_r r^\vee A_r^{-1}\chi$. In particular, $A_r = \lambda A_r^t$, so (r, χ) has sign λ. $\square$

2.5.2. **Galois representations valued in ${}^C\mathrm{U}(a, b)$.** Recall that, by Lemma 2.10,

$$ {}^C\mathrm{U} \cong (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \mathrm{Gal}(\overline{F}/F^+), $$

where the action of Galois is given by

$$ c \cdot (g, \mu) = (g\mu^{1-n}, \mu). $$

Moreover, recall from Section 2.3.3 that there is a map $d\colon {}^C\mathrm{U} \to \mathrm{GL}_1$ given by

$$ (g, \mu) \in \mathrm{GL}_n \times \mathrm{GL}_1 \mapsto \mu, \qquad c \mapsto -1. $$

If π is a cuspidal automorphic representation of the form considered in Theorem 2.13 and if π is C-algebraic (often satisfied by Remark 2.15, e.g. if π_v is a discrete series representation for all $v \mid \infty$), then its associated Galois representation should be valued in ${}^C\mathrm{U} = {}^C\mathrm{U}(a, b)$. In this subsection, we prove the following theorem, which proves [BG14, Conjecture 5.3.4] for such π and primes ℓ at which π is unramified:

Theorem 2.19. *Let π be a (necessarily C-algebraic) cuspidal automorphic representation of $\mathrm{U}(a, b)(\mathbf{A}_{F^+})$ such that, for each archimedean place v, π_v is a discrete series representation. Then for each prime ℓ there exists a continuous Galois representation*

$$ R_\pi\colon\ \mathrm{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell) $$

such that:

(1) The composition of R_π with the projection ${}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell) \to \mathrm{Gal}(\overline{F}/F^+)$ is the identity.

(2) The composition of R_π with the map $d\colon {}^C\mathrm{U} \to \mathbf{G}_m$ is the cyclotomic character ε.

(3) R_π satisfies local-global compatibility at unramified primes: for each place v of F^+ lying over a rational prime $p \neq \ell$ at which both F and π are unramified, the local representation $(R_\pi|_{W_{F_v^+}})^{\mathrm{ss}}$ is $\mathrm{GL}_n \times \mathrm{GL}_1$-conjugate to the representation sending $w \in W_{F_v^+}$ to $r_{\pi_v}(w)\hat{\xi}(|w|^{1/2})$. Here r_{π_v} is the local Langlands correspondence normalised as in [BG14, Section 2.2] *and $\hat{\xi}$ is the map $\mathbf{C}^\times \to (\mathrm{GL}_n \times \mathrm{GL}_1)(\mathbf{C})$ defined in Section 2.3.3.*

(4) If v is a place diving ℓ, then R_π is de Rham, i.e. for any faithful representation ${}^C\mathrm{U} \to \mathrm{GL}_N$ the resulting N-dimensional representation is de Rham.

(5) For any complex conjugation c, the image $R_\pi(c)$ is $(\mathrm{GL}_n \times \mathrm{GL}_1)(\overline{\mathbf{Q}}_\ell)$-conjugate to $(I_n, 1)c$.[1]

(6) The representation ρ_π obtained in Theorem 2.13 is the projection onto GL_n of the restriction $R_\pi|_{\mathrm{Gal}(\overline{F}/F)}$.

The theorem follows from Theorem 2.13 along with the following proposition, which is essentially a combination of [CHT08, Lemma 2.1.1] and [BG14, Section 8.3].

Proposition 2.20. *Let $\chi\colon\ \mathrm{Gal}(\overline{F}/F^+) \to \overline{\mathbf{Q}}_\ell^\times$ be a character with $\chi(c)$ independent of the choice of c. Let η_{F/F^+} be the quadratic character of $\mathrm{Gal}(\overline{F}/F^+)$ with kernel $\mathrm{Gal}(\overline{F}/F)$. There is a bijection between:*

[1]Since $R(c)$ has order 2, it is equal to $(g, \mu)c$, where $\mu = \pm 1$ and $g^t = \mu g$. The content of this statement is that $\mu = 1$, after which, by further conjugation, we can ensure that $g = I_n$.

(1) Isomorphism classes of representations

$$R\colon \mathrm{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell)$$

taken up to $\mathrm{GL}_n \times \mathrm{GL}_1$*-conjugacy, such that the composite of* R *and the projection onto* $\mathrm{Gal}(\overline{F}/F^+)$ *is the identity and such that* $d \circ R = \chi$*.*

(2) Isomorphism classes of polarised Galois representations $(\rho, \chi^{1-n}\eta^n_{F/F^+}, \langle\cdot,\cdot\rangle)$ *as in Definition 2.16 for*

- $\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$*,*
- $\chi\colon \mathrm{Gal}(\overline{F}/F^+) \to \overline{\mathbf{Q}}_\ell^\times$*,*
- $\langle\cdot,\cdot\rangle$ *a pairing on* $\overline{\mathbf{Q}}_\ell^n$*.*

In this bijection, $R(c)$ *has a representative of the form* $(A, -\chi(c))c$*, where* $A \in \mathrm{GL}_n(k)$ *defines the pairing* $\langle\cdot,\cdot\rangle$*.*

Remark 2.21. The fact that $(\rho, \chi^{1-n}\eta_{F/F^+}, \langle\cdot,\cdot\rangle)$ is polarised is crucial to this proposition. For example, an essentially conjugate self-dual representation ρ such that $\rho \simeq \rho_1 \oplus \rho_2$, where ρ_1 is even and ρ_2 is odd, would not lift to a representation valued in ${}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell)$.

Proof. For $\sigma \in \mathrm{Gal}(\overline{F}/F)$, define:

(1) $\rho(\sigma) = \mathrm{pr}(R(\sigma))$, where pr is the projection map $\mathrm{GL}_n \times \mathrm{GL}_1 \to \mathrm{GL}_n$;

(2) $\langle x, y\rangle = x^t A^{-1} y$, where $R(c) = (A, \mu_c)c$.

Note that, since $R(c)^2 = (AA^{-t}(\mu_c)^{1-n}, \mu_c^2) = (I_n, 1)$, we have $\mu_c = \pm 1$ and $AA^{-t} = \mu_c I_n$, so that $(\rho, \chi^{1-n}\eta^n_{F/F^+}, \langle\cdot,\cdot\rangle)$ has sign μ_c. Finally, since $d \circ R = \chi$, we see that $\chi(c) = -\mu_c$, so $-\chi(c)^{1-n}\eta^n_{F/F^+}(c) = -(-\mu_c)^{1-n}(-1)^n = \mu_c^{1-n} = \mu_c$ (since $\mu_c = 1$ whenever n is odd). Hence, $(\rho, \chi^{1-n}\eta^n_{F/F^+}, \langle\cdot,\cdot\rangle)$ is polarised, as required.

Conversely, given a polarised representation $(\rho, \chi^{1-n}\eta^n_{F/F^+}, \langle\cdot,\cdot\rangle)$, for $\sigma \in \mathrm{Gal}(\overline{F}/F)$, define

$$R(\sigma) = (\rho(\sigma), \chi(\sigma))\sigma$$

and

$$R(c) = (A, -\chi(c))c.$$

Note that $R(c)^2 = (AA^{-t}(-\chi(c))^{1-n}, \chi(c)^2) = (I_n, 1)$, because, by definition, $(\rho, \chi^{1-n}\eta^n_{F/F^+}, \langle\cdot,\cdot\rangle)$ has sign $-\chi(c)^{1-n}\eta^n_{F/F^+}(c) = -\chi(c)^{1-n}(-1)^n = (-\chi(c))^{1-n}$.

∎

We deduce Theorem 2.19:

Proof of Theorem 2.19. By Theorem 2.13 and [BC11], there is a polarised Galois representation $(\rho_\pi, \varepsilon^{1-n}\eta^n_{F/F^+}, \langle\cdot,\cdot\rangle)$ attached to π. By Proposition 2.20, this representation lifts uniquely to a representation

$$R_\pi\colon \mathrm{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell)$$

such that $d \circ R_\pi = \varepsilon$, the projection to $\mathrm{Gal}(\overline{F}/F^+)$ is the identity and $R(c) = (A, 1)c$ with A symmetric and non-singular. If A and B are non-singular symmetric matrices, then by Sylvester's law of inertia, A and B are congruent over $\overline{\mathbf{Q}}_\ell$: there exists $h \in \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ such that $hAh^t = B$. In particular, since A is symmetric, $hAh^t = I_n$ for some h, and conjugating $R_\pi(c)$ by $(h, 1)$, we find that

$$(I_n, 1)c = (h, 1)(A, 1)c(h^{-1}, 1).$$

Hence, R satisfies conditions (1), (2), (5) and (6). Since de Rham representations are potentially semistable (and vice versa) property (4) follows directly from Theorem 2.13.

It remains to check local-global compatibility at unramified places. Denote by $\Phi(G)$ the set of L-parameters of a reductive group G over a local non-archimedean field k (either F_v^+ or F_w in the following), given by conjugacy classes of admissible homomorphisms $W_k \times \mathrm{SU}(2) \to {}^L G(\overline{k}) = \widehat{G}(\overline{k}) \rtimes W_k$. Note that the bijection of Proposition 2.20 is induced from a sequence of L-homomorphisms (i.e. homomorphisms such that that the induced maps on the dual group are complex analytic and the induced maps on W_k are trivial): ${}^C\mathrm{U}(a,b) \to {}^L\mathrm{U}(a,b) \to {}^L(\mathrm{Res}_{F/F^+}(\mathrm{GL}_n))$. For places v inert in F/F^+, $(\rho_\pi \otimes \varepsilon^{(n-1)/2})|_{G_{F_v^+}}$ is conjugate self-dual of parity 1 in the sense of [Mok15, (2.2.4) and (2.2.5)], as the sign of $(\rho_\pi, \varepsilon^{1-n}\eta^n_{F/F^+}, \langle \cdot, \cdot \rangle)$ is $-\varepsilon(c) = +1$. Therefore, for places w of F both above inert and split primes, it suffices to compare the L-parameters under the injection $\Phi(\mathrm{U}(a,b)) \hookrightarrow \Phi(\mathrm{Res}_{F/F^+}(\mathrm{GL}_n)) = \Phi(\mathrm{GL}_{n/F})$ arising from base change (as opposed to twisted base change; see [Mok15, Lemma 2.2.1]). By the proof of Proposition 2.20, $R(\sigma) = (\rho_\pi(\sigma), \varepsilon(\sigma))\sigma$ for $\sigma \in \mathrm{Gal}(\overline{F}/F)$. Hence, this compatibility follows from that of Theorem 2.13, which asserts that the corresponding L-parameter $W_{F_w} \to \mathrm{GL}_n(\mathbf{C})$ matches the one associated to $\mathrm{BC}(\pi_v)_w \otimes |\cdot|_w^{\frac{1-n}{2}}$. $\square$

Remark 2.22. The arguments of this section break down if we try to work with automorphic representations of $\mathrm{GU}(a,b)$ instead of automorphic representations of $\mathrm{U}(a,b)$. On the automorphic side, the base change map from automorphic representations of $\mathrm{GU}(a,b)$ to automorphic representations of $\mathrm{GL}_n \times \mathrm{GL}_1$ is two-to-one (whereas the base change map from $\mathrm{U}(a,b)$ to GL_n is injective). In particular, there should be two distinct lifts of the GL_n valued representation ρ_π to a ${}^C\mathrm{GU}$-valued representation. An analogous version of Proposition 2.20 indeed gives a two-to-one map from suitable ${}^C\mathrm{GU}$-valued representations to suitable polarised Galois representations which preserves the sign at infinity. However, we are unable to show that either of the two lifts of ρ_π match up to π at all unramified primes. In Section 3.6, we will see that the invariant theory of $\mathrm{GU}(a,b)$ suggests that this question is genuinely difficult.

2.6. Galois representations attached to polarised automorphic representations of GL_n

Let F be a CM field and let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Assume that π is essentially conjugate self-dual, i.e. that there exists a Hecke character $\mu\colon \mathbf{A}_F^\times \to \mathbf{C}^\times$ such that $\pi^c = \pi^\vee \otimes \mu$. Furthermore, assume that there exists a character $\mu_0\colon \mathbf{A}_{F^+}^\times/(F^+)^\times \to \mathbf{C}^\times$ such that $\mu = \mu_0 \circ \mathrm{Nm}_{F/F^+}$ and $\mu_{0,v}(-1)$ is independent of $v \mid \infty$.

Following [FP19], we say that π is *weakly regular* if all its infinitesimal characters for F_v with $v \mid \infty$ are of the form $a_v = (a_{1,v}, \dots, a_{n,v})$, where the $a_{i,v}$ have multiplicity at most two. Moreover, we say that π is *odd* if the Asai L-function $L(s, \pi, \mathrm{Asai}^{(-1)^{n-1}\varepsilon(\mu_0)} \otimes \mu_0^{-1})$ has a pole at $s = 1$. For precise definitions of this Langlands L-function, the representations $\mathrm{Asai}^\pm\colon {}^L\mathrm{Res}_{F/F^+}(\mathrm{GL}_n) \to \mathrm{GL}(\mathbf{C}^n \otimes \mathbf{C}^n)$ and the sign $\varepsilon(\mu_0)$, we refer to [FP19, Section 9.1]. We note here that, since π is cuspidal, the Rankin–Selberg L-function

$$L(\pi \otimes \pi^\vee, s) = L(s, \pi, \mathrm{Asai}^+ \otimes \mu_0^{-1}) L(s, \pi, \mathrm{Asai}^- \otimes \mu_0^{-1})$$

has a simple pole since at $s = 1$, and neither of the Asai L-values vanishes at $s = 1$.

In [FP19], Fakhruddin–Pilloni combine the results of [GK19] and [PS16] (see Theorem 2.14) with Mok's proof in [Mok15] of the Arthur classification for quasi-split unitary groups to obtain:

Theorem 2.23 ([FP19, Theorem 9.10])**.** *Let F be a CM field and π be a weakly regular C-algebraic odd cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, such that $\pi^c = \pi^\vee \otimes \mu$ for*

$\mu\colon \mathbf{A}_F^\times \to \mathbf{C}^\times$ as above. Then there exists a Galois representation

$$\rho_\pi\colon \operatorname{Gal}(\overline{F}/F) \to \operatorname{GL}_n(\overline{\mathbf{Q}}_\ell)$$

that is unramified at all finite places $w \nmid \ell$ at which π is unramified, satisfies local-global compatibility up to semisimplification—i.e. $(\rho_\pi|_{W_{F_w}})^{\mathrm{ss}} \cong \operatorname{rec}_{F_w}(\pi_w \otimes |\cdot|_w^{\frac{1-n}{2}})$—at unramified places w, and is such that $\rho_\pi^c \simeq \rho_\pi^\vee \otimes \rho_\mu \varepsilon^{1-n}$.

[FP19, Theorem 9.11] also proves a result towards local-global compatibility at places dividing ℓ.

We now deduce Theorem 1.3 from Theorem 1.1.

Theorem 2.24 (Theorem 1.3)**.** *Keep the notation and assumptions of Theorem 2.23. If r is an irreducible subrepresentation of ρ_π that satisfies $r^c \simeq r^\vee \otimes \rho_\mu \varepsilon^{1-n}$ and appears with multiplicity one in the decomposition of ρ_π into irreducible subrepresentations, then $(r, \rho_\mu \varepsilon^{1-n})$ is odd.*

Proof. As explained in [FP19, Section 9.1.2, Theorem 9.6], there is an algebraic Hecke character $\psi\colon \mathbf{A}_F \to \mathbf{C}^\times$ such that $\pi_0 = \pi \otimes \psi$ is conjugate self-dual (i.e. $\pi_0^c \cong \pi_0^\vee$), and the results of [Mok15] imply that a C-algebraic odd representation π_0 of $\operatorname{GL}_n(\mathbf{A}_F)$ that is conjugate self-dual descends to a C-algebraic representation $\tilde{\pi}_0$ of the quasi-split unitary group $\mathrm{U}(n)/F^+$ (which equals $\mathrm{U}(n/2, n/2)$ for n even and $\mathrm{U}(\frac{n+1}{2}, \frac{n-1}{2})$ for n odd). The definition of weakly regular is chosen exactly to ensure that this descent is a non-degenerate limit of discrete series (see Remark 2.15).

The result for π_0 follows therefore from Theorem 1.1, together with Proposition 2.20 and Lemma 2.18. The result for π is then immediate from [BC11, Lemma 2.1]. $\square$

3. Lafforgue pseudocharacters and invariant theory

In this section, we prove that a Galois representation constructed as a limit of pseudocharacters of odd representations is odd, from which we deduce Theorem 1.1. Our method is to reconstruct the Galois representations using Lafforgue pseudocharacters in place of Taylor's pseudocharacters. This method was previously applied in a simpler case by the second author in [Wei18] to prove that the Galois representations attached to low weight Siegel modular forms are valued in GSp_4.

Lafforgue pseudocharacters were introduced by Vincent Lafforgue as part of his proof of the automorphic-to-Galois direction of the geometric Langlands correspondence for general reductive groups [Laf18]. Rather than following Lafforgue's original approach [Laf18, Section 11], we use a categorical approach due to Weidner [Wei20]. Our exposition follows that of the second author in [Wei19].

3.1. FFS-algebras

Let **FFS** be the category of free, finitely-generated semigroups and let **FFG** be the category of free, finitely-generated groups. If I is a finite set, let $\mathrm{FS}(I)$ denote the free semigroup generated by I and let $\mathrm{FG}(I)$ denote the free group generated by I.

If $I \to J$ is a morphism of sets, then there is a corresponding group homomorphism $\mathrm{FS}(I) \to \mathrm{FS}(J)$. However, not all morphisms in **FFS** and **FFG** are of this form.

Lemma 3.1 ([Wei20, Lemma 3])**.** *Any morphism in **FFS** is a composition of morphisms of the following types:*

- *morphisms* $\mathrm{FS}(I) \to \mathrm{FS}(J)$ *that send generators to generators, i.e. those induced by morphisms* $I \to J$ *of finite sets;*
- *morphisms*

$$\mathrm{FS}(\{x_1, \dots, x_n\}) \to \mathrm{FS}(\{y_1, \dots, y_{n+1}\})$$
$$x_i \mapsto \begin{cases} y_i & i < n \\ y_n y_{n+1} & i = n. \end{cases}$$

Any morphism in **FFG** *is a composition of morphisms of the above two types (with* FS *replaced by* FG*) and morphisms*

$$\mathrm{FG}(\{x_1, \dots, x_n\}) \to \mathrm{FG}(\{y_1, \dots, y_n\})$$
$$x_i \mapsto \begin{cases} y_i & i < n \\ y_n^{-1} & i = n. \end{cases}$$

Definition 3.2. Let R be a topological ring. An **FFS**-*algebra* (resp. **FFG**-algebra) is a covariant functor from **FFS** (resp. **FFG**) to the category R-**alg** of topological R-algebras. Morphisms of **FFS**-algebras and **FFS**-algebras are natural transformations of functors.

We will be interested in the following two examples:

Examples 3.3 ([Wei20, Examples 1,2])**.**

(1) Let Γ be a topological group and let A be a topological R-algebra. For a finite set I, let $\Gamma^I = \mathrm{Hom}_{\mathbf{Set}}(I, \Gamma)$. We define a covariant functor

$$\mathrm{C}(\Gamma^\bullet, A) \colon \mathbf{FFG} \to R\text{-}\mathbf{alg}$$

as follows. For each finite set I, let $\mathrm{C}(\Gamma^I, A)$ denote the R-algebra of continuous set maps $\Gamma^I \to A$. Then there is a natural isomorphism

$$\Gamma^I = \mathrm{Hom}_{\mathbf{Set}}(I, \Gamma) \cong \mathrm{Hom}_{\mathbf{FFG}}(\mathrm{FG}(I), \Gamma)$$

and, hence, the association

$$\mathrm{FG}(I) \mapsto \mathrm{C}(\mathrm{Hom}_{\mathbf{FFG}}(\mathrm{FG}(I), \Gamma), A) \cong \mathrm{C}(\Gamma^I, A)$$

is well-defined. Moreover, a morphism $\phi\colon \mathrm{FG}(I) \to \mathrm{FG}(J)$ in **FFG** induces a morphism of sets

$$\Gamma^J \cong \mathrm{Hom}_{\mathbf{FFG}}(\mathrm{FG}(J), \Gamma) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathbf{FFG}}(\mathrm{FG}(I), \Gamma) \cong \Gamma^I,$$

and therefore a morphism of R-algebras

$$\mathrm{C}(\Gamma^I, A) \to \mathrm{C}(\Gamma^J, A).$$

Hence, the functor

$$\mathrm{C}(\Gamma^\bullet, A)\colon\ \mathrm{FG}(I) \mapsto \mathrm{C}(\Gamma^I, A)$$

is an **FFG**-algebra.

(2) Let G, X be affine group schemes over R, and let G act on X compatibly with the group structure on X. For any finite set I, G acts diagonally on X^I, and, hence, G acts on the coordinate ring $R[X^I]$ of X^I.

For each finite set I, let $R[X^I]^G$ be the R-algebra of fixed points of $R[X^I]$ under the action of G. A morphism $\phi\colon\ \mathrm{FG}(I) \to \mathrm{FG}(J)$ in **FFG** induces a morphism of R-schemes $X^J \to X^I$, and thus a R-algebra morphism $R[X^I]^G \to R[X^J]^G$. The corresponding covariant functor

$$R[X^\bullet]^G\colon\ \mathrm{FS}(I) \mapsto R[X^I]^G$$

is an **FFG**-algebra.

Note that any **FFG**-algebra is naturally an **FFS**-algebra. Hence, we may also consider both $\mathrm{C}(\Gamma^\bullet, A)$ and $R[X^\bullet]^G$ as **FFS**-algebras.

3.2. **Lafforgue pseudocharacters**

Let R be a topological ring and let G be a reductive group over R. Let G° denote the identity connected component of G, which we assume is split. Then G° acts on G by conjugation, and we can form the **FFS**-algebra $R[G^\bullet]^{G^\circ}$.

Definition 3.4. Let Γ be a topological group and let A be a topological R-algebra. A continuous *G-pseudocharacter* of Γ over A is an **FFS**-algebra morphism

$$\Theta^\bullet\colon R[G^\bullet]^{G^\circ} \to \mathrm{C}(\Gamma^\bullet, A).$$

Remarks 3.5.

(1) Unwinding this definition recovers Lafforgue's original definition [Laf18, Definition 11.3]. Indeed, Lafforgue defines a continuous pseudocharacter as a collection $(\Theta_n)_{n\geq 1}$ of algebra maps

$$\Theta_n\colon R[G^n]^{G^\circ} \to \mathrm{C}(\Gamma^n, A)$$

that are compatible in the following sense:

(a) If $n, m \geq 1$ are integers and $\zeta\colon \{1, \dots, m\} \to \{1, \dots, n\}$, then for every $f \in R[G^m]^{G^\circ}$ and $\gamma_1, \dots \gamma_n \in \Gamma$, we have

$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}),$$

where $f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$.

(b) For every integer $n \geq 1$, $f \in R[G^n]^{G^\circ}$ and $\gamma_1, \dots \gamma_{n+1} \in \Gamma$, we have

$$\Theta_{n+1}(\hat{f})(\gamma_1, \dots, \gamma_{n+1}) = \Theta_n(f)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n\gamma_{n+1}),$$

where $\hat{f}(g_1, \dots, g_{n+1}) = f(g_1, \dots, g_{n-1}, g_n g_{n+1})$.

By definition, an **FFS**-algebra morphism $R[G^\bullet]^{G^\circ} \to \mathrm{C}(\Gamma^\bullet, A)$ consists of a collection of R-algebra morphisms $\Theta^I\colon \mathbf{R}[G^I]^{G^\circ} \to \mathrm{C}(\Gamma^I, A)$ such that, for any semigroup homomorphism $\phi\colon \mathrm{FS}(I) \to \mathrm{FS}(J)$, the following diagram commutes:

$$\begin{array}{ccc} R[G^I]^{G^\circ} & \xrightarrow{\Theta^I} & \mathrm{C}(\Gamma^I, A) \\ \downarrow & & \downarrow \\ R[G^J]^{G^\circ} & \xrightarrow{\Theta^J} & \mathrm{C}(\Gamma^J, A) \end{array}$$

Here, the vertical arrows are those induced by ϕ. By Lemma 3.1, checking that this diagram commutes for all morphisms ϕ is equivalent to verifying conditions (a) and (b) above.

(2) Suppose that G is a connected linear algebraic group with a fixed embedding $G \hookrightarrow \mathrm{GL}_r$ for some r. Let χ denote the composition of this embedding with the usual trace function. Then $\chi \in \mathbf{Z}[G]^G$ and, if $\Theta^\bullet$ is a G-pseudocharacter of Γ over A, then

$$\Theta^1(\chi) \in \mathrm{C}(\Gamma, A)$$

is a classical pseudocharacter. In fact, we will see in Section 3.4 that when $G = \mathrm{GL}_n$, $\Theta^\bullet$ is completely determined by $\Theta^1(\chi)$ [Laf18, Remark 11.8]. In particular, the notion of a G-pseudocharacter is a generalisation of the notion of a classical pseudocharacter.

3.3. **Lafforgue pseudocharacters and G-valued representations**

The key motivation for introducing Lafforgue pseudocharacters is their connection to G-valued representations. From now on, assume that $R = \mathbf{Z}$, so that G is a reductive group over $\mathbf{Z}$ with G° split, and A is a topological ring.

Lemma 3.6 ([BHKT19, Lemma 4.3])**.** *Let $\rho\colon \Gamma \to G(A)$ be a continuous representation of Γ. Define*

$$(\mathrm{Tr}\,\rho)^\bullet\colon \mathbf{Z}[G^\bullet]^{G^\circ} \to \mathrm{C}(\Gamma^\bullet, A)$$

by

$$(\mathrm{Tr}\,\rho)^I(f)\big((\gamma_i)_{i\in I}\big) = f\big((\rho(\gamma_i))_{i\in I}\big)$$

for each finite set I and for each $f \in \mathbf{Z}[G^I]^{G^\circ}$.

Then $(\mathrm{Tr}\,\rho)^\bullet$ is a continuous G-pseudocharacter of Γ over A. Moreover, $(\mathrm{Tr}\,\rho)^\bullet$ depends only on the $G^\circ(A)$-conjugacy class of ρ.

In fact, in many cases, the converse of Lemma 3.6 is also true. Let k be an algebraically closed field and let $\rho\colon \Gamma \to G(k)$ be a representation of Γ. If $G = \mathrm{GL}_n$ and ρ is semisimple, then Taylor [Tay91] (for $\mathrm{char}(k) = 0$) and Rouquier (for $(\mathrm{char}(k), n!) = 1$, see [Rou96] and [Che14]) proved that ρ can be recovered from its classical pseudocharacter. To state the generalisation of this fact for G-pseudocharacters, we first define what it means for ρ to be semisimple in general.

Definition 3.7 ([BHKT19, Definitions 3.3, 3.5])**.** Let H denote the Zariski closure of $\rho(\Gamma)$.

(1) We say that ρ is *G-irreducible* if there is no proper parabolic subgroup of G containing H.

(2) We say that ρ is *semisimple* or *G-completely reducible* if, for any parabolic subgroup $P \subseteq G$ containing H, there exists a Levi subgroup of P containing H.

Theorem 3.8 ([Laf18, Proposition 11.7], [BHKT19, Theorem 4.5])**.** *Let k be an algebraically closed field. The assignment $\rho \mapsto (\mathrm{Tr}\,\rho)^\bullet$ defines a bijection between the following two sets:*

(1) The set of $G^\circ(k)$-conjugacy classes of G-completely reducible continuous homomorphisms $\rho\colon \Gamma \to G(k)$;

(2) The set of continuous G-pseudocharacters $\Theta^\bullet\colon \mathbf{Z}[G^\bullet]^{G^\circ} \to \mathrm{C}(\Gamma^\bullet, k)$ of Γ over k.

Remark 3.9. Note that, by [Wei20, Theorem 5], Theorem 3.8 holds with **FFS**-algebra morphisms (i.e. natural transformations of functors **FFS** $\to$ R-**alg**) replaced by **FFG**-algebra morphisms (natural transformations of functors **FFG** $\to$ R-**alg**). In particular, we will often work with G-pseudocharacters $\Theta^\bullet$ that are, moreover, **FFG**-algebra morphisms $\Theta^\bullet\colon \mathbf{Z}[G^\bullet]^{G^\circ} \to \mathrm{C}(\Gamma^\bullet, k)$, rather than just **FFS**-algebra morphisms.

We finish this subsection by recording a generalisation of [Tay91, Lemma 1], which notes that we are free to change the R-algebra A. This lemma is valid whether we work with **FFS**-algebras or **FFG**-algebras. Part (i) is part of [BHKT19, Lemma 4.4].

Lemma 3.10. *Let A be a topological R-algebra, and let Γ be a topological group.*

(1) Let $h\colon A \to A'$ be a continuous morphism of R-algebras, and let $\Theta^\bullet$ be a continuous G-pseudocharacter of Γ over A. Then $h_*(\Theta) = h \circ \Theta^\bullet$ is a continuous G-pseudocharacter of Γ over A'.

(2) Let $h\colon A \hookrightarrow A'$ be a continuous injective morphism of R-algebras. Define a collection of maps $\Theta^\bullet$, where, for each finite set I, $\Theta^I\colon R[G^I]^{G^\circ} \to \mathrm{C}(\Gamma^I, A)$ is a map of sets.

Suppose that $h\circ\Theta^\bullet$ is a continuous G-pseudocharacter of Γ over A'. Then $\Theta^\bullet$ is a continuous G-pseudocharacter over A.

3.4. ${}^C\mathrm{U}$-pseudocharacters

Definition 3.11 ([Wei20, Definition 3])**.** Let $A^\bullet$ be an **FFS**-algebra (resp. **FFG**-algebra). Given a subset $\Sigma \subseteq \bigsqcup_I A^I$, define the **FFS**- (resp. **FFG**-) *span* of Σ in $A^\bullet$ to be the smallest **FFS**-subalgebra (resp. **FFG**-subalgebra) $B^\bullet$ of $A^\bullet$, such that $\Sigma \subseteq \bigsqcup_I B^I$. We say that Σ generates $A^\bullet$ if the span of Σ in $A^\bullet$ is the whole of $A^\bullet$.

Example 3.12. Suppose that k is a field of characteristic 0. By results of Procesi [Pro76], the **FFS**-algebra $k[\mathrm{GL}_n^\bullet]^{\mathrm{GL}_n}$ is spanned by the elements $\mathrm{Tr}, \det^{-1} \in k[\mathrm{GL}_n]^{\mathrm{GL}_n}$. Similarly, as an **FFG**-algebra, $k[\mathrm{GL}_n^\bullet]^{\mathrm{GL}_n}$ is spanned by Tr.[2] If $\Theta^\bullet$ is any GL_n-pseudocharacter of a group Γ over k, then, since k is $\mathbf{Z}$-flat, by [BHKT19, Remark 4.2], the data of a GL_n-pseudocharacter is equivalent to the data of a **FFS**-algebra homomorphism $k[\mathrm{GL}_n^\bullet]^{\mathrm{GL}_n} \to \mathrm{C}(\Gamma^\bullet, k)$. Hence, by [Wei20, Theorem 5], $\Theta^\bullet$ is completely determined by its classical pseudocharacter $\Theta^1(\mathrm{Tr}) \in \mathrm{C}(\Gamma, k)$.

More generally, if R is any ring, then $R[\mathrm{GL}_n^\bullet]^{\mathrm{GL}_n}$ is generated as an **FFS**-algebra by $s_i, \det^{-1} \in R[\mathrm{GL}_n]^{\mathrm{GL}_n}$, where s_i is the i^{th} coefficient of the characteristic polynomial [DCP17, Theorem 1.10]. In particular, if $\Theta^\bullet$ is any GL_n-pseudocharacter, $\Theta^\bullet$ is completely determined by the maps $\Theta^1(s_i), \Theta^1(\det^{-1}) \in \mathrm{C}(\Gamma, k)$.

Recall from Lemma 2.10 that ${}^C\mathrm{U} \cong \widehat{\widehat{U}} \rtimes \mathrm{Gal}(\overline{F}/F^+)$, where $\widehat{\widehat{U}} \cong \mathrm{GL}_n \times \mathrm{GL}_1$ and $\mathrm{Gal}(\overline{F}/F^+)$ acts via the quotient $\mathrm{Gal}(F/F^+)$: if $c \in \mathrm{Gal}(F/F^+)$ is the non-trivial element, then $c \cdot (g, \mu) = (g^{-t}\mu^{1-n}, \mu)$. For the remainder of this section, we work with the quotient

$$(\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \mathrm{Gal}(F/F^+),$$

which, abusing notation, we will continue to call ${}^C\mathrm{U}$. Note that if $\rho\colon \mathrm{Gal}(\overline{F}/F^+) \to (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \mathrm{Gal}(F/F^+)$ is any representation such that $\rho(c)$ projects to $c \in \mathrm{Gal}(F/F^+)$, then there is a unique lift of ρ to a representation valued in ${}^C\mathrm{U}$ such that $\rho(\sigma)$ projects to $\sigma \in \mathrm{Gal}(\overline{F}/F^+)$ for all σ. Hence, we lose nothing in replacing ${}^C\mathrm{U}$ by its quotient.

In the remainder of this subsection, we prove the main technical result of this paper, in which we compute a generating set for $\mathbf{Z}[{}^C\mathrm{U}^\bullet]^{\widehat{\widehat{U}}}$ as an **FFG**-algebra.

Theorem 3.13. *As an* **FFG***-algebra,* $\mathbf{Z}[{}^C\mathrm{U}^\bullet]^{\widehat{\widehat{U}}}$ *is spanned by the elements*

- $(g, \mu) \mapsto s_m(g) \qquad (g, \mu)c \mapsto 0,$
- $(g, \mu) \mapsto \mu \qquad (g, \mu)c \mapsto 0,$
- $(g, \mu) \mapsto 0 \qquad (g, \mu)c \mapsto \mu,$

of $\mathbf{Z}[{}^C\mathrm{U}]^{\widehat{\widehat{U}}}$*. Here,* s_m*,* $m = 1, \ldots, n$*, is the* m^{th} *coefficient of the characteristic polynomial.*

Proof. We begin by making some reductions.

Lemma 3.14. *Let* $H = \mathrm{GL}_n \times \mathrm{GL}_1$ *and let* $r \in \mathbf{N}$*. Then*

$$\mathbf{Z}[{}^C\mathrm{U}^r]^{\widehat{\widehat{U}}} \cong \prod_{x \in \mathrm{Gal}(F/F^+)^r} \mathbf{Z}[(H^r)x]^H.$$

Proof. For an affine scheme $X_{/\mathbf{Z}}$, write $\mathbf{Z}[X]$ for the $\mathbf{Z}$-algebra such that $X = \mathrm{Spec}(\mathbf{Z}[X])$. We have $H = \mathrm{GL}_n \times \mathrm{GL}_1 = \widehat{\widehat{U}}$. Let

$${}^C\mathrm{U}^r /\!\!/ H := \mathrm{Spec}(\mathbf{Z}[{}^C\mathrm{U}^r]^H),$$

[2] If $X \in \mathrm{GL}_n$, then $\det(X)$ can be expressed as a polynomial in $\mathrm{Tr}(X^i)$. Hence, as an **FFG**-algebra, $\det^{-1}$ is in the **FFG**-subalgebra generated by Tr.

where H acts by diagonal conjugation, and let

$$\pi \colon {}^{C}\mathrm{U}^r \to {}^{C}\mathrm{U}^r /\!\!/ H$$

be the quotient map. As a $\mathbf{Z}$-scheme,

$${}^{C}\mathrm{U}^r = \bigsqcup_{x \in \mathrm{Gal}(F/F^+)^r} (H^r)x,$$

where the subsets $(H^r)x \subseteq {}^{C}\mathrm{U}^r$ are closed and pairwise disjoint. Hence,

$$\mathbf{Z}[{}^{C}\mathrm{U}^r] \cong \prod_{x \in \mathrm{Gal}(F/F^+)^r} \mathbf{Z}[(H^r)x].$$

Moreover, the subsets $(H^r)x$ are stable under the conjugation action of H. Hence, by [Ses77, Theorem 3], the subsets $\pi((Hx)^r)$ are closed, disjoint subsets of ${}^{C}\mathrm{U}^r /\!\!/ H$ and, since π is surjective, we see that

$${}^{C}\mathrm{U}^r /\!\!/ H = \bigsqcup_{x \in \mathrm{Gal}(F/F^+)^r} \pi((H^r)x).$$

It follows that

$$\mathbf{Z}[{}^{C}\mathrm{U}^r]^H \cong \prod_{x \in \mathrm{Gal}(F/F^+)^r} \mathbf{Z}[(H^r)x]^H.$$

□

Consider a component $(H^r)x$, where $x = (\underbrace{c, \ldots, c}_{r_1 \text{ times}}, \underbrace{1, \ldots, 1}_{r_2 \text{ times}})$. Recall that $(\gamma, \nu) \in \mathrm{GL}_n \times \mathrm{GL}_1$ acts by conjugation on H and as

$$(g, \mu)c \mapsto (\gamma g \gamma^t \nu^{n-1}, \mu)$$

on Hc. Since $\mathrm{GL}_n \times \mathrm{GL}_1$ acts trivially on the GL_1 component, we have

$$\mathbf{Z}[Hc]^H = \mathbf{Z}[\mathrm{GL}_1] \otimes \mathbf{Z}[\mathrm{GL}_n]^H,$$

where the action of H on GL_n is given by $(\gamma, \nu) \cdot g = \gamma g \gamma^t \nu^{n-1}$. In particular, we see that

$$\mathbf{Z}[(H^r)x]^H \subseteq \mathbf{Z}[\mathrm{GL}_1^r] \otimes \mathbf{Z}[\mathrm{GL}_n^{r_1} \times \mathrm{GL}_n^{r_2}]^{\mathrm{GL}_n}, \tag{3.1}$$

where the action of GL_n on the first r_1 copies of GL_n is by $\gamma \cdot g = \gamma g \gamma^t$ and the action on the second r_2 copies is by conjugation.

In what follows, we compute a generating set for $\mathbf{Z}[\mathrm{GL}_1^r] \otimes \mathbf{Z}[\mathrm{GL}_n^{r_1} \times \mathrm{GL}_n^{r_2}]^{\mathrm{GL}_n}$. It will turn out that these generators are all elements of $\mathbf{Z}[(H^r)x]^H$, from which it follows that the inclusion in (3.1) is an equality.

Now, the map $\mathrm{GL}_n \hookrightarrow \mathrm{M}_n \times \mathrm{M}_n$ that sends $g \mapsto (g, g^{-1})$ is a closed embedding. Hence, by [Ses77, Theorem 3], the restriction map

$$\mathbf{Z}[\mathrm{M}_n^{r_1} \times \mathrm{M}_n^{r_1} \times \mathrm{M}_n^{r_2} \times \mathrm{M}_n^{r_2}]^{\mathrm{GL}_n} \to \mathbf{Z}[\mathrm{GL}_n^{r_1} \times \mathrm{GL}_n^{r_2}]^{\mathrm{GL}_n}, \tag{3.2}$$

is a surjection.

Consider $\mathbf{Z}[\mathrm{M}_n^{r_1} \times \mathrm{M}_n^{r_1} \times \mathrm{M}_n^{r_2} \times \mathrm{M}_n^{r_2}]^{\mathrm{GL}_n}$. Here, $\gamma \in \mathrm{GL}_n$ acts on an element

$$(A_1, \ldots, A_{r_1}, B_1, \ldots, B_{r_1}, C_1, \ldots, C_{2r_2}) \in \mathrm{M}_n^{2r_1 + 2r_2}$$

by taking it to

$$(\gamma A_1 \gamma^t, \ldots, \gamma A_{r_1} \gamma^t, \gamma^{-t} B_1 \gamma^{-1}, \ldots, \gamma^{-t} B_{r_1} \gamma^{-1}, \gamma C_1 \gamma^{-1}, \ldots, \gamma C_{2r_2} \gamma^{-1}).$$

For the remainder of the proof, we will denote matrices in M_n by A, B or C depending on how GL_n acts as above.

Lemma 3.15. *The ring* $\mathbf{Z}[\mathrm{M}_n^{r_1} \times \mathrm{M}_n^{r_1} \times \mathrm{M}_n^{2r_2}]^{\mathrm{GL}_n}$ *is spanned by*

$$\{(A_1, \ldots, A_{r_1}, B_1, \ldots, B_{r_1}, C_1, \ldots, C_{2r_2}) \mapsto s_m(M) : m = 1, \ldots, n\},$$

where s_m *is the* m^{th} *coefficient of the characteristic polynomial and* M *varies over the free semigroup generated by*

$$\left\{C_k, A_i N^t B_j, A_i^t N^t B_j, A_i N^t B_j^t, A_i^t N^t B_j^t : 1 \le i, j \le r_1, 1 \le k \le 2r_2\right\}$$

where N *is in the free semigroup generated by* $\{C_i : 1 \le i \le 2r_2\}$.

Proof. If k is an arbitrary infinite field, then by [Zub99, Theorem 2.1], the ring $k[\mathrm{M}_n^{r_1} \times \mathrm{M}_n^{r_1} \times \mathrm{M}_n^{2r_2}]^{\mathrm{GL}_n}$ is generated by the maps

$$\{(A_1, \ldots, A_{r_1}, B_1, \ldots, B_{r_1}, C_1, \ldots, C_{2r_2}) \mapsto s_m(M) : m = 1, \ldots, n\},$$

where s_m is the m^{th} coefficient of the characteristic polynomial and M varies over the free semigroup generated by

$$\left\{C_k, A_i N^t B_j, A_i^t N^t B_j, A_i N^t B_j^t, A_i^t N^t B_j^t : 1 \le i, j \le r_1, 1 \le k \le 2r_2\right\},$$

where N is in the free semigroup generated by $\{C_i : 1 \le i \le 2r_2\}$.

Note that, although Zubkov's result is stated only in the case that $r_1 = 1$, the proof works in general.

To deduce the result over $\mathbf{Z}$, we use an argument similar to that in [DCP17, 15.2.1]. To ease notation, for any ring R, we will write $R[\mathrm{M}_n^r]$ instead of $R[\mathrm{M}_n^{r_1} \times \mathrm{M}_n^{r_1} \times \mathrm{M}_n^{2r_2}]$ and we will write $R[s_m(M)]$ for the subring of $R[\mathrm{M}_n^r]$ generated by the elements in the statement of the Lemma.

Suppose for contradiction that $\mathbf{Z}[s_m(M)] \subsetneq \mathbf{Z}[\mathrm{M}_n^r]^{\mathrm{GL}_n}$. For each prime ℓ, let $\mathbf{Z}_{(\ell)}$ denote the localisation of $\mathbf{Z}$ at ℓ. By [Ses77, Lemma 2], for every ℓ,

$$\mathbf{Z}[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{Z})} \otimes_{\mathbf{Z}} \mathbf{Z}_{(\ell)} = \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{Z}_{(\ell)})}.$$

Since localisation is exact and $\mathbf{Z}[s_m(M)] \subsetneq \mathbf{Z}[\mathrm{M}_n^r]^{\mathrm{GL}_n}$, there exists some ℓ such that $\mathbf{Z}_{(\ell)}[s_m(M)] \subsetneq \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n}$.

Note that $\mathrm{GL}_n(\mathbf{Z}_{(\ell)})$ is Zariski dense in $\mathrm{GL}_n(\mathbf{Q})$. Hence,

$$\mathbf{Q}[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{Q})} = \mathbf{Q}[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{Z}_{(\ell)})}.$$

Moreover, we have

$$\mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{Z}_{(\ell)})} = \mathbf{Q}[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{Q})} \cap \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r].$$

For each prime ℓ, from the exact sequence

$$0 \to \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r] \xrightarrow{\times \ell} \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r] \to \mathbf{F}_\ell[\mathrm{M}_n^r]$$

we obtain an exact sequence

$$0 \to \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{Z}_{(\ell)})} \xrightarrow{\times \ell} \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{Z}_{(\ell)})} \to \mathbf{F}_\ell[\mathrm{M}_n^r]^{\mathrm{GL}_n(\mathbf{F}_\ell)}.$$

Here, we are using the fact that $\mathrm{GL}_n(\mathbf{Z}_{(\ell)}) \to \mathrm{GL}_n(\mathbf{F}_\ell)$ is a surjection. Hence, we have an injective map

$$\mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n} \otimes \mathbf{F}_\ell \hookrightarrow \mathbf{F}_\ell[\mathrm{M}_n^r]^{\mathrm{GL}_n}.$$

Now, $\mathbf{Z}_{(\ell)}[s_m(M)]$ is a subring of $\mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n}$ and the two rings coincide over $\mathbf{Q}$. Suppose for contradiction that

$$\mathbf{Z}_{(\ell)}[s_m(M)] \subsetneq \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n}.$$

Then there exists some degree $d = (d_1, \ldots, d_r)$ such that there is a strict inclusion

$$\mathbf{Z}_{(\ell)}[s_m(M)]_d \subsetneq \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]^{\mathrm{GL}_n}_d$$

of multihomogeneous components of degree d. Note that both $\mathbf{Z}_{(\ell)}[s_m(M)]_d$ and $\mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]_d^{\mathrm{GL}_n}$ are finitely generated $\mathbf{Z}_{(\ell)}$-modules. Hence, by Nakayama's lemma, the map

$$\mathbf{Z}_{(\ell)}[s_m(M)]_d \otimes \mathbf{F}_\ell \to \mathbf{Z}_{(\ell)}[\mathrm{M}_n^r]_d^{\mathrm{GL}_n} \otimes \mathbf{F}_\ell \hookrightarrow \mathbf{F}_\ell[\mathrm{M}_n^r]_d^{\mathrm{GL}_n}$$

is not surjective. Since, by definition, this map factors through $\mathbf{F}_\ell[s_m(M)]_d$, we find that $\mathbf{F}_\ell[s_m(M)]_d \subsetneq \mathbf{F}_\ell[\mathrm{M}_n^r]_d^{\mathrm{GL}_n}$. It follows that, $\mathbf{F}_\ell[s_m(M)] \subsetneq \mathbf{F}_\ell[\mathrm{M}_n^r]^{\mathrm{GL}_n}$.

Now, if k is any infinite field of characteristic ℓ, then, by [Ses77, Lemma 2], $\mathbf{F}_\ell[\mathrm{M}_n^r]^{\mathrm{GL}_n} \otimes_{\mathbf{F}_\ell} k = k[\mathrm{M}_n^r]^{\mathrm{GL}_n}$. Since field extensions are faithfully flat, we obtain a strict inclusion

$$k[s_m(M)] \subsetneq k[\mathrm{M}_n^r]^{\mathrm{GL}_n},$$

contradicting Zubkov's above result. The result follows. □

Unwinding Lemma 3.15 via (3.2), we see that the ring of invariants $\mathbf{Z}[\mathrm{GL}_n^{r_1} \times \mathrm{GL}_n^{r_2}]^{\mathrm{GL}_n}$ is spanned by

$$\{(g_1, \dots, g_{r_1}, h_{r_1+1}, \dots, h_{r_1+r_2}) \mapsto s_m(M) : m = 1, \dots, n\},$$

where M is in the free group generated by

$$\left\{h_k, g_i N^t g_j^{-1}, g_i N^t g_j^{-t} : 1 \le i, j \le r_1, r_1 + 1 \le k \le r_1 + r_2\right\},$$

where N is in the free group generated by $\{h_i : r_1 + 1 \le i \le r_1 + r_2\}$.

Note that all these generators are further invariant under the action of $\mathrm{GL}_n \times \mathrm{GL}_1$ on GL_n by $(\gamma, \nu) \cdot g = \gamma g \gamma^t \nu^{n-1}$. In particular, the inclusion of (3.1) is an equality.

Now, $\mathbf{Z}[\mathrm{GL}_1^r]$ is generated by functions mapping $(\mu_1, \dots, \mu_r)$ to an element of the free semigroup generated by $\{\mu_i, \mu_i^{-1} : 1 \le i \le r\}$. Hence, using (3.1), we see that, for $x = (\underbrace{c, \dots, c}_{r_1 \text{ times}}, \underbrace{1, \dots, 1}_{r_2 \text{ times}})$, the ring of invariants $\mathbf{Z}[H^r x]^H$ is generated by maps

$$\{((g_1, \mu_1)c, \dots, (g_{r_1}, \mu_{r_1})c, (h_{r_1+1}, \mu_{r_1+1}), \dots, (h_{r_1+r_2}, \mu_{r_1+r_2})) \mapsto \lambda : i = 1, \dots, n\},$$

where λ falls into one of the following two cases:

- $\lambda = s_m(M)$ for some $m = 1, \dots, n$ and M is in the free group generated by

  $$\left\{h_k, g_i N^t g_j^{-1}, g_i N^t g_j^{-t} : 1 \le i, j \le r_1, r_1 + 1 \le k \le r_1 + r_2\right\},$$

  where N is in the free group generated by $\{h_i : r_1 + 1 \le i \le r_1 + r_2\}$.
- λ is in the free group generated by $\{\mu_i : 1 \le i \le r_1 + r_2\}$.

Finally, observe that if $(g_i, \mu_i)c, (g_j, \mu_j)c \in Hc$ and $(N, \mu) \in H$, then

$$(g_i, \mu_i)c \cdot (N, \mu)^{-1} \cdot (g_j, \mu_j)c = (g_i N^t g_j^{-t} (\mu_j \mu^{-1})^{1-n}, \mu_i \mu_j \mu^{-1}) \in H$$

and

$$(g_i, \mu_i)c \cdot (N, \mu)^{-1} \cdot ((g_j, \mu_j)c)^{-1} = (g_i N^t g_j^{-1} \mu^{n-1}, \mu_i \mu_j^{-1} \mu^{-1}) \in H.$$

We see, for example, that the invariant

$$f : \big((g_i, \mu_i)c, (g_j, \mu_j)c\big) \mapsto \mathrm{Tr}(g_i g_j^{-t})$$

in $\mathbf{Z}[{}^C\mathrm{U}^2]^H$ is equal to the product of the map $(g_j, \mu_j)c \mapsto \mu_j^{n-1} \in \mathbf{Z}[{}^C\mathrm{U}]^H$ with the map

$$(g, \mu) \mapsto \mathrm{Tr}(g)$$

applied to the product of $(g_i, \mu_i)c$ and $(g_j, \mu_j)c$. Thus, f is in the **FFG**-algebra span of the two elements $(g, \mu) \mapsto \mathrm{Tr}(g)$ and $(g, \mu)c \mapsto \mu$ of $\mathbf{Z}[{}^C\mathrm{U}]^H$.

Similarly, by repeatedly applying the above relations along with Lemma 3.1, we see that the elements $s_m(M)$, where M is in the free group generated by

$$\left\{h_k, g_i N^t g_j^{-1}, g_i N^t g_j^{-t} : 1 \le i, j \le r_1, r_1 + 1 \le k \le r_1 + r_2\right\},$$

are already in the **FFG**-algebra span of the elements

- $(g,\mu) \mapsto s_m(g) \qquad (g,\mu)c \mapsto 0, \qquad m = 1,\dots,n$
- $(g,\mu) \mapsto \mu \qquad (g,\mu)c \mapsto 0,$
- $(g,\mu) \mapsto 0 \qquad (g,\mu)c \mapsto \mu,$

from which Theorem 3.13 follows. □

3.5. **Oddness in low weight**

We can now prove Theorem 1.1. The precise statement is as follows:

Theorem 3.16 (Theorem 1.1)**.** *Let F be a CM field with totally real subfield F^+. Let π be a C-algebraic cuspidal automorphic representation of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$ such that, for each archimedean place v of F^+, π_v is a discrete series or a non-degenerate limit of discrete series representation. Then, for each prime ℓ at which π is unramified, there exists a continuous, semisimple (in the sense of Definition 3.7) Galois representation*

$$R_\pi \colon \mathrm{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell)$$

such that:

(1) The composition of R_π with the projection ${}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell) \to \mathrm{Gal}(\overline{F}/F^+)$ is the identity.

(2) The composition of R_π with the map $d\colon {}^C\mathrm{U} \to \mathbf{G}_m$ is the cyclotomic character ε.

(3) R_π satisfies local-global compatibility at unramified primes: for each place v of F^+ lying over a rational prime $p \neq \ell$ at which both F and π are unramified, the local representation $(R_\pi|_{W_{F_v^+}})^{\mathrm{ss}}$ is $\mathrm{GL}_n \times \mathrm{GL}_1$-conjugate to the representation sending $w \in W_{F_v^+}$ to $r_{\pi_v}(w)\hat{\xi}(|w|^{1/2})$, where r_{π_v} is the local Langlands correspondence normalised as in [BG14, Section 2.2] *and $\hat{\xi}$ is the map $\mathbf{C}^\times \to (\mathrm{GL}_n \times \mathrm{GL}_1)(\mathbf{C})$, defined in Section 2.3.3.*

(4) For any complex conjugation c the image $R(c)$ is $(\mathrm{GL}_n \times \mathrm{GL}_1)(\overline{\mathbf{Q}}_\ell)$-conjugate to $(I_n, 1)c$.

Proof of Theorem 3.16. The result follows by replacing Taylor's pseudocharacters with Lafforgue's pseudocharacters in the proof of [GK19, Theorem 10.5.3]. Let $\mathbf{T}$ be the abstract Hecke algebra generated by the Hecke operators of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$ away from the conductor of π, the discriminant of $F/\mathbf{Q}$ and ℓ. Let E be the finite extension of $\mathbf{Q}_\ell$ generated by the Hecke parameters of π, and let $\theta\colon \mathbf{T} \to \mathcal{O}_E$ be the Hecke map associated to π. Then [GK19] consider the reductions of θ modulo ℓ^m for $m \ge 1$, which correspond to eigenclasses in coherent cohomology of the reduction of the Shimura variety corresponding to $\mathrm{U}(a,b)$ modulo ℓ^m.

In [GK19, Theorem 10.4.1], they associate Galois representations to such torsion classes by proving that these Hecke maps factor through a Hecke algebra $\mathbf{T}_m$ acting on cuspidal automorphic representations of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$ with regular discrete series. In particular, they produce (see [GK19, (10.6.2)]) a sequence of Galois representations

$$\rho_m \colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbf{T}_m \otimes \overline{\mathbf{Q}}_\ell)$$

such that:

- $\mathbf{T}_m$ is the Hecke algebra (which [GK19] denote by $\mathcal{H}^{0,+}(\nu + ak\eta_\omega)$, with a, k depending on m) parametrising automorphic representations of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$ of a certain regular weight depending on m. In particular, $\mathbf{T}_m \otimes \mathbf{Z}_\ell$ is reduced and flat as a $\mathbf{Z}_\ell$-algebra (which follows e.g. from [HLTT16, Lemma 5.11]).

- For each m, the map $\theta\colon \mathbf{T} \to \mathcal{O}_E \to \mathcal{O}_E/\ell^m$ factors through a map $r_m\colon \mathbf{T}_m \to \mathcal{O}_E/\ell^m$. In other words, the Hecke eigenvalues of π are congruent modulo ℓ^m to the eigenvalues of a regular form π_m of $\mathrm{U}(a,b)(\mathbf{A}_{F^+})$.

By Theorem 2.19 the Galois representation ρ_m lifts to a representation

$$R_m\colon \operatorname{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(\mathbf{T}_m \otimes \overline{\mathbf{Q}}_\ell)$$

such that $R_m(c)$ is conjugate to $(I_n, 1)c$. Let $\Theta_m^\bullet$ be the ${}^C\mathrm{U}$-pseudocharacter of $\operatorname{Gal}(\overline{F}/F^+)$ over $\mathbf{T}_m \otimes \overline{\mathbf{Q}}_\ell$ attached to R_m by Theorem 3.8. By Remark 3.9, we may consider $\Theta_m^\bullet$ as an **FFG**-algebra morphism. Hence, by Theorem 3.13, $\Theta_m^\bullet$ is completely determined by

$$\Theta_m(f)\colon \operatorname{Gal}(\overline{F}/F^+) \to \mathbf{T}_m \otimes \overline{\mathbf{Q}}_\ell,$$

where f varies over the elements

- $(g,\mu) \mapsto s_i(g) \qquad (g,\mu)c \mapsto 0, \qquad i = 1,\dots,n$
- $(g,\mu) \mapsto \mu \qquad (g,\mu)c \mapsto 0,$
- $(g,\mu) \mapsto 0 \qquad (g,\mu)c \mapsto \mu,$

of $\mathbf{Z}[{}^C\mathrm{U}]^{\widehat{\widehat{U}}}$. Note that, since $\mathbf{T}_m \otimes \mathbf{Z}_\ell$ is flat, $\mathbf{T}_m \otimes \overline{\mathbf{Z}}_\ell \hookrightarrow \mathbf{T}_m \otimes \overline{\mathbf{Q}}_\ell$. Since the characteristic polynomial of ρ_m has coefficients in $\mathbf{T}_m \otimes \overline{\mathbf{Z}}_\ell$ and since $d \circ R_m = \varepsilon$, it follows that, for each such f, $\Theta_m(f)$ factors through $\mathbf{T}_m \otimes \overline{\mathbf{Z}}_\ell$. Thus, by Lemma 3.10, $\Theta_m^\bullet$ is a ${}^C\mathrm{U}$-pseudocharacter of $\operatorname{Gal}(\overline{F}/F^+)$ over $\mathbf{T}_m \otimes \overline{\mathbf{Z}}_\ell$. In particular, this argument promotes the Taylor pseudocharacter of [GK19, (10.6.3)] to a Lafforgue pseudocharacter, thereby strengthening [GK19, Theorem 10.4.1]. Hence, composing $\Theta_m^\bullet$ with the map r_m, we obtain a ${}^C\mathrm{U}$-pseudorepresentation of $\operatorname{Gal}(\overline{F}/F^+)$ over $\mathcal{O}_E/\ell^m$. Moreover, if $m' > m$, then

$$(r_m \circ \Theta_m)^\bullet = (r_{m'} \circ \Theta_{m'})^\bullet \pmod{\ell^m}.$$

Hence, we can form a ${}^C\mathrm{U}$-pseudocharacter

$$\Theta^\bullet = \varprojlim_m (r_m \circ \Theta_m)^\bullet$$

of $\operatorname{Gal}(\overline{F}/F^+)$ over $\mathcal{O}_E$. Viewing $\mathcal{O}_E$ as a subalgebra of $\overline{\mathbf{Q}}_\ell$ and applying Theorem 3.8, we obtain the Galois representation

$$R_\pi\colon \operatorname{Gal}(\overline{F}/F^+) \to {}^C\mathrm{U}(\overline{\mathbf{Q}}_\ell).$$

That $R_\pi(c)$ is conjugate to $(I_n, 1)c$ follows from the fact that $\Theta(f)$ is the limit of $\Theta_m(f)$ when f is the map

$$(g,\mu) \mapsto 0 \quad (g,\mu)c \mapsto \mu.$$

The fact that R_π satisfies local-global compatibility at unramified primes follows from the fact that $\Theta(f)$ is the limit of $\Theta_m(f)$, where f is one of the elements

- $(g,\mu) \mapsto s_i(g) \qquad (g,\mu)c \mapsto 0,$
- $(g,\mu) \mapsto \mu \qquad (g,\mu)c \mapsto 0.$

□

3.6. $\mathrm{GU}(a,b)$-representations

We conclude by highlighting a constraint to our approach and why it cannot be used to prove an analogous result for automorphic representations of $\mathrm{GU}(a,b)$.

The key input to the above proof is Theorem 3.13, which shows that $\mathbf{Z}[{}^C\mathrm{U}^\bullet]^{\widehat{\widehat{U}}}$ is spanned by elements of $\mathbf{Z}[{}^C\mathrm{U}]^{\widehat{\widehat{\mathrm{U}}}}$ and, therefore, that any ${}^C\mathrm{U}$-pseudocharacter $\Theta^\bullet$ is completely determined by its action on elements of $\mathbf{Z}[{}^C\mathrm{U}]^{\widehat{\widehat{U}}}$. This input is crucial in proving that the pseudocharacters $\Theta^\bullet_m$, which were a priori defined over $\mathbf{T}\otimes\overline{\mathbf{Q}}_\ell$, are actually defined over $\mathbf{T}\otimes\overline{\mathbf{Z}}_\ell$. Indeed, it is only the elements $f\in\mathbf{Z}[{}^C\mathrm{U}]^{\widehat{\widehat{U}}}$ for which the map $\Theta_m(f)\colon \mathrm{Gal}(\overline{F}/F^+)\to\mathbf{T}\otimes\overline{\mathbf{Q}}_\ell$ can be related to automorphic data by viewing its action on Frobenius elements.

On the other hand, this key input does not hold for the C-group of $\mathrm{GU}(a,b)$ when $n=a+b$ is even. By [BG14, Prop 5.3.3], we have

$$ {}^C\mathrm{GU}\cong\frac{\mathrm{GL}_n\times\mathrm{GL}_1\times\mathrm{GL}_1}{\langle((-I_n)^{n-1},1,-1)\rangle}\rtimes\mathrm{Gal}(\overline{F}/F^+) $$

where $c\in\mathrm{Gal}(F/F^+)$ acts on $(g,\lambda,\mu)\in\frac{\mathrm{GL}_n\times\mathrm{GL}_1\times\mathrm{GL}_1}{\langle((-I_n)^{n-1},1,-1)\rangle}$ by

$$ c\cdot(g,\lambda,\mu)=(\Phi_n g^{-t}\Phi_n^{-1},\det(g)\lambda,\mu). $$

Similarly to the case of SO_{2n} (c.f. [Wei20, Lemma 18]), the full polarisation pl of the Pfaffian function $(g,\lambda,\mu)c\mapsto\lambda\cdot\mathrm{pf}(g\Phi_n\mu-(g\Phi_n\mu)^t)$ is an element of $\mathbf{Z}[{}^C\mathrm{GU}^n]^{\widehat{\widehat{\mathrm{GU}}}}$ that cannot be generated by elements of $\mathbf{Z}[{}^C\mathrm{GU}]^{\widehat{\widehat{\mathrm{GU}}}}$. We see that, when $a+b$ is even, the **FFG** algebra $\mathbf{Z}[{}^C\mathrm{GU}^\bullet]^{\widehat{\widehat{\mathrm{GU}}}}$ is *not* generated by elements of $\mathbf{Z}[{}^C\mathrm{GU}]^{\widehat{\widehat{\mathrm{GU}}}}$.

The failure of Theorem 3.13 for ${}^C\mathrm{GU}$ is closely related to the fact that ${}^C\mathrm{GU}$ is not an acceptable group (c.f. [Lar94] and [Wei20, Theorem 19]): there exist ${}^C\mathrm{GU}$-valued representations that are everywhere locally conjugate but not globally conjugate.[3] In particular, if $\Theta^\bullet$ is pseudocharacter attached to a GU-valued representation, then the actions of $\Theta(f)$ on Frobenius elements for $f\in\mathbf{Z}[{}^C\mathrm{GU}]^{\widehat{\widehat{\mathrm{GU}}}}$ are not enough to uniquely determine $\Theta^\bullet$.

For example, when $n=a+b$ is a multiple of 4, the two representations

$$ R_1,R_2\colon(\mathbf{Z}/4\mathbf{Z})^2\to{}^C\mathrm{GU}(\overline{\mathbf{Q}}_\ell) $$

defined by

$$ R_1\colon(0,1)\mapsto\left(\begin{pmatrix}I_m & \\ & \Phi_m\end{pmatrix}\Phi_n,1,1\right)c;\quad(1,0)\mapsto\left(\begin{pmatrix}\Phi_m & \\ & I_m\end{pmatrix}\Phi_n,1,1\right)c $$

and

$$ R_2\colon(0,1)\mapsto\left(\begin{pmatrix}\zeta_n I_m & \\ & \zeta_n\Phi_m\end{pmatrix}\Phi_n,1,1\right)c;\quad(1,0)\mapsto\left(\begin{pmatrix}\zeta_n\Phi_m & \\ & \zeta_n I_m\end{pmatrix}\Phi_n,1,1\right)c, $$

where ζ_n is a primitive n^{th} root of unity, are everywhere locally conjugate, but are not globally conjugate.

[3] Two representations $\rho_1,\rho_2\colon\Gamma\to G(k)$ are everywhere locally conjugate if, for every $\gamma\in\Gamma$, there exists $g\in G(k)$ such that $\rho_1(\gamma)=g\rho_2(\gamma)g^{-1}$. They are globally conjugate if g can be chosen independently of γ.

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