Article:

Amiri-Aref, M., Shiripour, S. and Ruiz-Hernández, D. (2021) Exact and approximate heuristics for the rectilinear Weber location problem with a line barrier. Computers & Operations Research, 132. 105293. ISSN 0305-0548

https://doi.org/10.1016/j.cor.2021.105293

Exact and approximate heuristics for the rectilinear Weber location problem with a line barrier

Mehdi Amiri-Aref, Saber Shiripour, Diego Ruiz-Hernández

Abstract

In this article, we propose an extension of the multi-Weber facility location problem with rectilinear-distance in the presence of passages over a non-horizontal line barrier. For the single-facility case, we develop an exact heuristic based on a divide-and-conquer approach that outperforms alternative heuristics available in literature. The multiple facilities case is solved by means of the application of an alternate-location-allocation heuristic heuristic, characterized by embedded exact and approximate procedures. For large instances, we propose a heuristic (with polynomial time complexity) which provides near-optimal solutions in a short computational time and a negligible gap. Finally, for testing purposes, we use a benchmark based on the transformation of the main problem into an equivalent p-median problem. Experimental results evidence the efficiency and validity of the proposed heuristics, which are capable of obtaining high quality solutions within acceptable computation times.

Keywords

Facility Location; Multi-facility Weber problem; Line barrier; Heuristics; P-median

1 Introduction

In this paper, we extend the available literature on the multi-facility Weber location problem with line barrier by allowing the barrier to take any slope with respect to the horizontal axis. For the special case of a single-facility, we propose a divide-and-conquer strategy that terminates at an optimal solution. For the solution of the multi-facility case, we propose an alternate location/allocation procedure that decomposes the main problem into a set of single-facility location sub-problems with passages and a number of set-partitioning sub-problems for allocation.

An early definition of a barrier region in planar location theory, introduced by Katz and Cooper (1981), is that of a region in which neither placing a facility in nor travelling through is allowed. Aneja and Parlar (1994) used Dijkstra's algorithm and simulated-annealing to find an approximate solution. Butt and Cavalier (1996) studied the Euclidian distance single Weber location problem in the presence of barrier regions; while McGarvey and Cavalier (2003) developed a branch and bound algorithm, called big square small square (BSSS). Klamroth (2001) introduced a class of facility location problems with barrier in which a line barrier expands across the plane dividing it into two separate sub-planes. Traversing across such barriers is allowable only through a finite set of passages and ignoring their presence may, in practice, cause an inappropriate location of facilities with increased travelling distances. The author showed that this type of problems is generally non-convex and NP-hard, and proposed an algorithm for solving it. Later, Klamroth and Wiecek (2002) extended this problem to a multi-objective median problem and presented a reduction-based algorithm for bi-criteria problems. Sarkar et al. (2007) and Kelachankuttu et al. (2007) also studied the Weber problem with barrier and showed how the barrier distance function can affect location of facilities. Canbolat and Wesolowsky (2010) introduced a single Weber location problem with a probabilistic line barrier. Shiripour et al. (2012) and Akyüz (2017) extended that problem to a multi-Weber problem and applied heuristic and meta-heuristic methods to solve large-size instances. Canbolat and Wesolowsky (2012) studied the Weber location problem in the presence of line and polyhedral barriers using the Varignon frame and proposed an approximate local optimal solution. Mahmud (2013), Gharravi (2013), Javadian et al. (2014), and Oğuz et al. (2018) also studied the presence of a line barrier in a rectilinear-distance single-facility location problem. Regarding the study of barrier problems with non-horizontal line barrier, Canbolat and Wesolowsky (2010) suggested that, by means of a rotation of the axes, the problem can be transformed into a horizontal barrier one. However, in such cases, although the distance between any two points will remain the same as under the rectilinear distance, the underlying distance function will be different.

This article extends the available work on multi-facility location problems with non-horizontal line barrier by proposing an heuristic approach that preserves the rectilinear distance function. For the special case of a single-facility, we propose a divide-and-conquer strategy that terminates at an optimal solution. The proposed approach has polynomial time-complexity with respect to the number of passages, however, despite the considerably rapid convergence rate to optimality of our heuristic, its time complexity grows exponentially as the number of existing facilities increases. For this reason, a fast and efficient approximation procedure, embedded in the divide-and-conquer approach, is applied, terminating the heuristic at an optimal/near-optimal solution in polynomial computational time. Numerical results show the superiority of this heuristic in terms of CPU time with respect to the exact one.

For the solution of the multi-facility case, we propose an alternate-location-allocation procedure. This procedure decomposes the main problem into a set of single-facility location sub-problems with passages and a number of set-partitioning sub-problems for allocation. Bischoff et al. (2009) utilized this procedure to solve a multi-Weber location problem where travelling was possible through the barrier's extreme points. They applied a continuous relaxation to the location sub-problems and executing a genetic-algorithm-based heuristic, developed by Bischoff and Klamroth (2007), to solve single-facility location sub-problems. In contrast, for the location procedure, we maintain the continuity in the location sub-problems and execute an exact algorithm for solving each sub-problem. This enables us to solve real-life sized problems. Finally, we also consider the discretisation-based solution method proposed by Larson and Sadiq (1983) as a benchmark to the solutions found by our proposed methods. The discretisation approach leads to the p-median problem. Proposing a method for solving this problem is beyond the scope of this paper and, therefore, we rely in general methods as those implemented in general-purpose solvers.

The rest of this work is organised as follows. In Section 2, we introduce the problem and its mathematical formulation. In Section 3, we describe the exact and approximate heuristics proposed for the single and the multi-facility problems. Section 4 presents the computational results and a discussion on the performance of the proposed heuristics. Conclusions, key findings, and directions for future study are provided in Section 5.

2 Problem description and formulation

In this section, we address the problem of locating a set of new facilities under the presence of a straight-line barrier with general slope and a limited number of passages.

Let us start by introducing some basic notation. The problem is defined in the closed unit plane $[0,1]^2$. The line barrier is defined as $L = \{(x,y) : y = mx + b, x \in [0,1], y \in [0,1]\}$, where m is a real value representing the slope of the barrier and $b \in [0,1]$ is the intercept. There is a number K of passages located along the line barrier. The coordinates of each of these passages are denoted by $P_k = (r_{k1}, r_{k2}), k = 1, \ldots, K$. Set $\mathbb{K} = \{P_1, \ldots, P_K\}$ represents the collection of all passages in barrier L. The line barrier with passages is represented by $L_{\mathcal{B}} = L \setminus \mathbb{K}$.

For a given line barrier with passages, $L_{\mathcal{B}}$, we define the "admissible region", where locating facilities and travel is allowed, as $\mathcal{F} = [0,1]^2 \setminus L_{\mathcal{B}}$. The line barrier divides the admissible region into two sub-planes, each denoted by $\mathcal{F}^m$, $m = 1, 2$.

Let $d_1(X,Y)$ be the regular 1-norm distance between any two points X and Y in $\mathcal{F}$; and use $d_1^{L_{\mathcal{B}}}(X,Y)$ for the 1-norm barrier distance between X and Y given the existence of barrier $L_{\mathcal{B}}$. The 1-norm barrier distance can be defined as the 1-norm shortest path between points $X \in \mathcal{F}^m$

and $Y \in \mathcal{F}^{m'}$, where $m, m' = 1, 2$, in the presence of a line barrier with passages, $L_{\mathcal{B}}$, i.e.

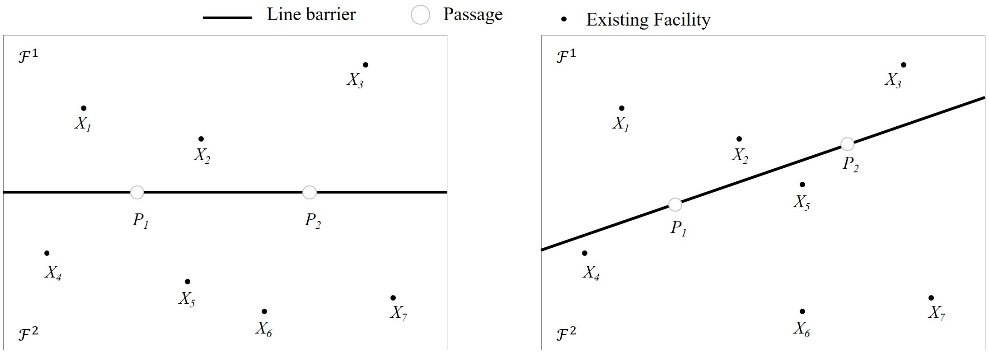
$$d_1^{L_{\mathcal{B}}}(X,Y) = \begin{cases} d_1(X,Y) & m = m' \\ \min_{k \in K} \{d_1(X,P_k) + d_1(P_k,Y)\} & m \neq m' \end{cases} \tag{1}$$

Notice that the lower part of (1) can be explicitly expressed as:

$$\min_{k=1,\ldots K} \{|x_1 - r_{k1}| + |x_2 - r_{k2}| + |y_1 - r_{k1}| + |y_2 - r_{k2}|\} \tag{2}$$

Points $X, Y \in \mathcal{F}$ are called 1-*visible* if $d_1(X,Y) = d_1^{L_{\mathcal{B}}}(X,Y)$; otherwise, if $d_1(X,Y) < d_1^{L_{\mathcal{B}}}(X,Y)$, X and Y are called 1-*shadow* points. The set of *visible* points to Y is given by $V_1(Y) = \left\{X \in \mathcal{F} : d_1^{L_{\mathcal{B}}}(X,Y) = d_1(X,Y)\right\}$; correspondingly, the set of *shadow* points to Y, is represented by $S_1(Y) = \left\{X \in \mathcal{F} : d_1^{L_{\mathcal{B}}}(X,Y) > d_1(X,Y)\right\}$.

Let us assume that there exist I facilities located in the admissible region. Each of these facilities has coordinates $X_i = (x_{i1}, x_{i2}) \in \mathcal{F}, i = 1, \ldots, I$, and weight $w_i \in \mathbb{R}^+$ representing the demand of the facility. This is illustrated in Figure 1. Also assume that we want to optimally locate a number J of new facilities. Each new facility j is characterised by its coordinates in the admissible region, i.e. $Y_j = (y_{j1}, y_{j2}) \in \mathcal{F}, j = 1, \ldots, J$. The 1-norm barrier distance for any pair of facilities (X_i, Y_j) can be obtained by means of equation (1).



(a) Horizontal line barrier

(b) Line barrier with general slope

Figure 1: Admissible region with existing facilities barrier line with two passages.

In the following sections, we present the formulation of the single facility location problem with a line barrier with general slope and K passages, and its extension to the multi-facility case.

2.1 Single-facility Weber Problem with K Passages on a Line Barrier (SFWP$_K$)

Consider the problem of locating a single facility $Y = (y_1, y_2)$ in the admissible region $\mathcal{F}$, so the total weighted distance between the new and all existing facilities is minimised. This problem can be expressed as

$$SFWP_K : \qquad \min_{Y \in \mathcal{F}} \sum_{i \in \mathcal{I}} w_i \cdot d_1^{L_\mathcal{B}}(X_i, Y) \tag{3}$$

where $\mathcal{I} = \{i : i = 1, \ldots, I\}$.

It is well known that in a Weber problem with rectilinear distances, the new facility's x and y-coordinates are independent and can be optimized separately. Canbolat and Wesolowsky (2012) used this separability property and presented a problem with a horizontal line barrier where the new facility's y-coordinate was independent of its x-coordinate. In such case, the new facility's y-coordinate determines the optimal sub-plane and the optimal x-coordinate can be found by solving the following optimisation problem (provided that the optimal sub-plane of the new facility is $\mathcal{F}^m$):

$$\min_{Y \in \mathcal{F}^m} \sum_{i=1}^{I} w_i \cdot d_1^{L_\mathcal{B}}(X_i, Y) = \sum_{i \in \mathcal{I}^1} w_i \left(|y_1 - x_{i1}| + |y_2 - x_{i2}|\right)$$
$$+ \sum_{i \in \mathcal{I}^2} w_i \cdot \min_{k=1,\ldots,K} \{|x_{i1} - r_{k1}| + |x_{i2} - r_{k2}| + |y_1 - r_{k1}| + |y_2 - r_{k2}|\} \tag{4}$$

where $\mathcal{I}^m = \{i : X_i \in \mathcal{F}^m\}$, $m, m' = 1, 2, m \neq m'$.

The first part of (4) minimizes the total regular rectilinear distance between the new facility and all the facilities located in the same sub-plane, whereas the second part minimizes the shortest path (through the passages) between the new facility and the facilities located in the opposite sub-plane. Whilst Canbolat and Wesolowsky's modelling approach is efficient when the line barrier is horizontally positioned on the plane, in the problem discussed in this paper the new facility's x and y coordinates are not separable due to the fact that the line barrier is non-horizontal. In this case, both coordinates of the new facility are determined by the position of the passages on the non-horizontal barrier. Therefore, we need a solution heuristic for problem SFWP$_K$ that simultaneously determines the new facility's optimal coordinates. This problem can be viewed as an extended version of the assignment problem, whose computational complexity had been extensively discussed in the literature. With this motivation, an exact algorithm characterised by polynomial growth with respect to the number of passages is developed in Section 3. This algorithm outperforms the one proposed by Klamroth (2001).

2.2 Multi-facility Weber Problem with K Passages on a Line Barrier (MFWP$_K$)

In this section, we introduce a multi-facility Weber problem with K passages on a line barrier with general non-zero slope. We refer to this as the MFWP$_K$ problem. The main assumption is that a subset of the existing facilities is uniquely allocated to and served by each new facility. The allocation is exhaustive. The objective is to find the location of J new facilities in the admissible region $\mathcal{F}$, so the total weighted travelled distance from the existing facilities to their assigned new facility is minimised. Before presenting the model formulation, we need to introduce the allocation variable U_{ij}, which takes value 1 when facility $i \in \mathcal{I}$ is allocated to facility $j \in \mathcal{J}$, and zero otherwise. The problem is given by equations (5) to (8).

$$MFWP_K: \qquad \min \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} w_i \, U_{ij} \, d_1^{L_\mathcal{B}} \, (X_i, Y_j) \tag{5}$$

$$s.t. \qquad \sum_{j \in \mathcal{J}} U_{ij} = 1 \qquad i \in \mathcal{I} \tag{6}$$

$$Y_j \in \mathcal{F} \qquad j \in \mathcal{J} \tag{7}$$

$$U_{ij} \in \{0, 1\} \qquad i \in \mathcal{I}, j \in \mathcal{J} \tag{8}$$

Constraints (6) guarantee that each facility is uniquely allocated to a facility in $\mathcal{J}$; constraints (7) impose the facilities to be located in the admissible region, $\mathcal{F}$; and (8) are standard binary constraints.

Bischoff et al. (2009) applied an alternate-location-allocation heuristic method based on the multi-start version of Cooper's method (Brimberg and Salhi, 2005), and iteratively solved the mixed-integer formulation of the location and allocation sub-problems to reach local optimum solutions. They showed that the enumeration of all feasible location-allocation sub-problems is useful only for very small size instances, suggesting that heuristics should be developed for larger sizes. However, to the best of our knowledge, no efficient heuristic has yet been proposed for the MFWP$_K$ problem, hence the need of efficient solution methods as those presented in Section 3.

3 Solution Methods

In this section, we present a collection of heuristics for solving the SFWP$_K$ and the MFWP$_K$ problems. Two of these heuristics, an exact and an approximate ones, have been developed for solving the single facility problem. These heuristics are based on a divide-and-conquer strategy. For solving the MFWP$_K$, we propose an alternate-location-allocation heuristic that calls the aforementioned heuristics for finding a solution to the decomposed location sub-problems. Before introducing the heuristics, we briefly explain how the non-convex solution space of the original problem can be divided into a set of convex sub-spaces; and formulate an appropriate barrier distance function.

3.1 Convex sub-spaces, distance functions and pre-processing

Two preliminary steps are required for our proposed heuristics. The first one consists of decomposing the solution space of the original problem into a set of convex sub-spaces; the second one computes the distances between any two points in the constructed convex sub-spaces.

The line barrier, $L_{\mathcal{B}}$, divides the plane into two sub-planes. Let us assume that $L_{\mathcal{B}}$ has K passages, which are labelled increasingly according to their x-coordinates, namely $r_{1,1} \leq r_{2,1} \leq \ldots \leq r_{K,1}$. We can then construct a collection of convex sub-spaces by drawing K vertical lines $x = r_{k,1}$, $k = 1, \ldots, K$. Each sub-plane spawns $K+1$ convex sub-spaces, each of them labelled H^{ml}, with $m = 1, 2$, and $l = 1, \ldots, K+1$, where

$$H^{1l} = \left\{(x, y)\,|r_{l-1,1} < x \leq r_{l,1}, y > ax + b\right\}, l = 2, \ldots, K \tag{9}$$

$$H^{2l} = \left\{(x, y)\,|r_{l-1,1} < x \leq r_{l,1}, y \leq ax + b\right\}, , l = 2, \ldots, K \tag{10}$$

with the special cases

$$H^{11} = \left\{(x, y)\,|0 \leq x \leq r_{1,1}, y > ax + b\right\} \tag{11}$$

$$H^{21} = \left\{(x, y)\,|0 \leq x \leq r_{1,1}, y \leq ax + b\right\} \tag{12}$$

and

$$H^{1,K+1} = \left\{(x, y)\,|r_{K,1} < x \leq 1, y > ax + b\right\} \tag{13}$$

$$H^{2,K+1} = \{(x,y)\,|r_{K,1} < x \leq 1, y \leq ax + b\} \tag{14}$$

The resulting convex sub-spaces are pairwise disjoint, i.e. $\bigcap_{1\leq l\leq K+1} H^{ml} = \emptyset$, satisfying $\bigcup_{1\leq l\leq K+1} H^{ml} = \mathcal{F}^m$.

The next step is to define the distance functions. Consider a new facility Y located in sub-plane $H^{\mu\ell}$. For the particular case of a horizontal line barrier, see Figure 2, the distance from Y to any existing facility X located in the same sub-plane is simply the standard rectilinear distance. Likewise, the distance to any facility X located in the opposite sub-plane and different sub-space is also given by the rectilinear distance. Finally, the distance from Y to any existing facility $X \in H^{m\ell}$, where $m \neq \mu$, is a barrier distance. In general, the distance between a new facility $Y \in H^{\mu\ell}$ and any existing facility $X \in H^{ml}$ in the presence of a horizontal line barrier is given by

$$d_1^{L_\mathcal{B}}(X,Y) = \begin{cases} d_1(X,Y), & (m=\mu) \vee (m \neq \mu \wedge l \neq \ell) \\ \min_{h=l-1,l}\{d_1(X,P_h) + d_1(P_h,Y)\}, & m \neq \mu \wedge l = \ell \end{cases} \tag{15}$$

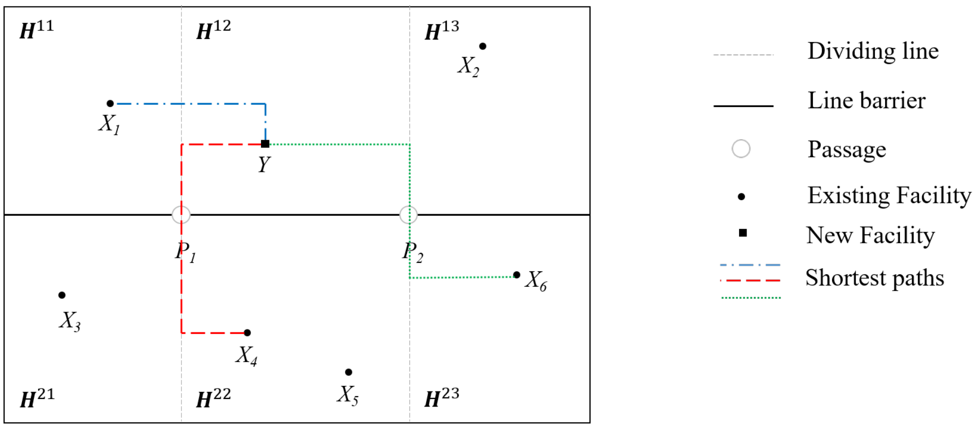


Figure 2: Existing facilities with 2 passages on a horizontal line barrier.

The more general case of a non-horizontal line barrier (without any loss of generality, let us assume that the line barrier has positive slope, i.e. $a > 0$ in $L_\mathcal{B}$) is illustrated in Figure 3. Consider a candidate location $Y \in H^{\mu\ell}$ for a new facility and any existing facility $X \in H^{ml}$. It is easy to see that if the existing facility X is in the same sub-plane as Y $(m = \mu)$, i.e. X and Y are 1-*visible*, the distance between them is the regular rectilinear distance. The same holds for the cases where X and Y are located in different sub-planes and either $(m > \mu \wedge l < \ell)$ or $(m < \mu \wedge l > \ell)$. Otherwise, the barrier distance holds, as facilities X and Y are 1-*shadow*. Equation (16) summarises this:

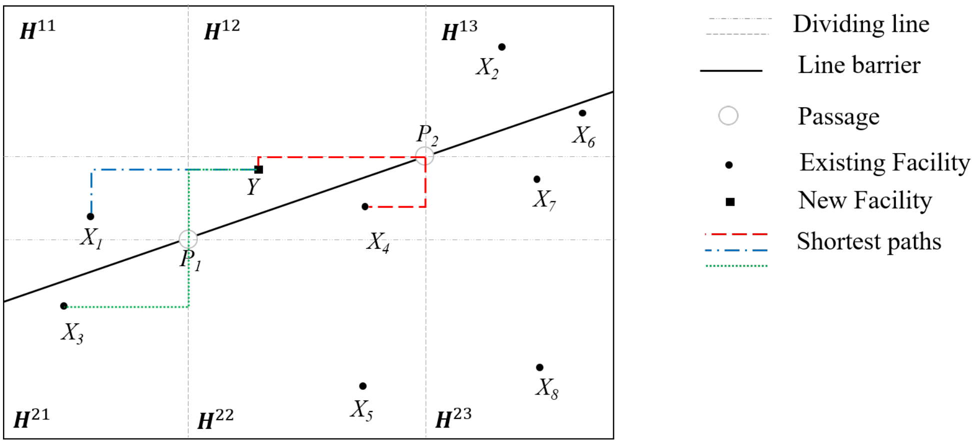


Figure 3: Existing facilities with 2 passages on a non-horizontal line barrier.

$$d_1^{L_{\mathcal{B}}}(X,Y) = \begin{cases} d_1(X,Y), & (m=\mu) \vee (m>\mu \wedge l<\ell) \\ & \vee (m<\mu \wedge l>\ell) \\ \min_{1\le h\le K}\left\{d_1(X,P_h)+d_1(P_h,Y)\right\}, & \text{otherwise} \end{cases} \tag{16}$$

Before presenting our heuristic approach, it is convenient to introduce some additional notation and pre-processing. Denote the subset of existing facilities located in sub-space H^{ml} by $\mathbb{I}^{ml} = \left\{X_i | X_i \in H^{ml}, i = 1,\ldots,I\right\}$ for $l = 1,\ldots,K$ and $m = 1,2$. Obtain the power set corresponding to each $\mathbb{I}^{ml}$. This set is represented by $\wp\left(\mathbb{I}^{ml}\right) = \left\{\mathcal{A}_q^{ml}, q = 1,\ldots,2^{\left|\mathbb{I}^{ml}\right|}\right\}$, where $\mathcal{A}_q^{ml}$ represents each of the q subsets in the power set of $\mathbb{I}^{ml}$.

In the following sections, we develop an exact approach for solving the Weber problem, with one and multiple facilities, in the presence of a line barrier with general slope and K passages.

3.2 Exact divide-and-conquer-based strategy (EDC) for the $\boldsymbol{SFWP}_K$

The general solution for the planar location problem with passages on a horizontal line barrier is based on the solution of an alternative (unconstrained) Weber problem in which the passages are interpreted as artificial facilities bearing the weight of the facilities located on the opposite sub-plane (Klamroth, 2001). In that approach, assigning the weight of existing facilities to the artificial ones was a combinatorial problem with solution time growing exponentially in the number of passages. Instead, the solution approach we are proposing, referred to as the Exact Divide-and-Conquer-based strategy (EDC), is based on dividing the solution plane into several sub-planes, defined by the passage points, and identifying the best solution in every sub-plane. This strategy has the virtue of allowing us to address the problems where the line barrier has a general non-zero slope. Our exact heuristic, based on the divide-and-conquer strategy, has a polynomial time complexity with respect to the number of passages, dominating the time complexity of other algorithms proposed in literature.

The exact solution approach proposed here starts with a preliminary pre-processing of input data described in Section 3.1. The main idea of this solution approach is to identify all possible assignments of passages to all existing facilities located in the opposite sub-plane, and to solve unconstrained Weber problems for each sub-space. The best among these solutions solves the Weber problem in the presence of a line barrier with general slope.

Heuristic approach

In this section, we describe the procedure for finding the optimal location of one new facility under the presence of a line barrier with general slope and K passages. A verbal summary of this procedure is presented in the Appendix of this article.

For each candidate sub-space $H^{\mu\ell}$, we define a set $\mathbb{A}^{\mu\ell}$ that includes the subset of existing facilities that are 1-visible to all points in $H^{\mu\ell}$ and all relevant passages defining the sub-spaces that contain shadow facilities, i.e.

$$\mathbb{A}^{\mu\ell} = \begin{cases} \{X_i \notin \mathbb{I}^{2j}, j = \ell \ldots K+1\} \cup \{P_h, h = \ell - 1 \ldots, K\}, & \mu = 1 \\ \{X_i \notin \mathbb{I}^{1j}, j = 1 \ldots \ell\} \cup \{P_h, h = 1, \ldots, \ell\}, & \mu = 2 \end{cases} \tag{17}$$

Please notice that P_0 and P_{K+1} correspond to the extremes of the barrier and, therefore, are not passages. Consequently, they are not included in $\mathbb{A}^{\mu\ell}$. Also notice that when the line barrier is horizontal, the involved passages are only P_{l-1} and P_l.

In our ic approach we interpret the passages as artificial facilities bearing the weight of the facilities located on a given sub-plane. This allocation of weights is conducted in the following way. For all subspaces H^{mj} (with $j = 1, \ldots, \ell$, for $m = 1$ and $j = \ell, \ldots, K+1$ for $m = 2$) the demand of the facilities in each $\mathcal{A}_q^{mj} \in \wp\left(\mathbb{I}^{mj}\right)$ is assigned to passage P_{j-1}, while the demand of those in its complement, $\left(\mathcal{A}_q^{mj}\right)^c = \mathbb{I}^{mj} \backslash \mathcal{A}_q^{mj}$, goes to passage P_j.

Let us, using $\oplus$ to represent the Cartesian product, introduce the following two sets. For $\mu = 1$ we have

$$\begin{aligned} \mathbb{D}^{1,\ell} &= \wp\left(\mathbb{I}^{2,\ell}\right) \oplus \ldots \oplus \wp\left(\mathbb{I}^{2,K+1}\right) \\ &= \left\{\left\{\mathcal{A}_1^{2,\ell}, \ldots, \mathcal{A}_1^{2,K}, \mathcal{A}_1^{2,K+1}\right\}, \left\{\mathcal{A}_1^{2,\ell}, \ldots, \mathcal{A}_1^{2,K}, \mathcal{A}_2^{2,K+1}\right\}, \right. \\ &\quad \left. \ldots, \left\{\mathcal{A}_1^{2,\ell}, \ldots, \mathcal{A}_1^{2,K}, \mathcal{A}_{|\wp(\mathbb{I}^{2,K+1})|}^{2,K+1}\right\}, \ldots, \left\{\mathcal{A}_{\left|\wp\left(\mathbb{I}^{2,\ell}\right)\right|}^{2,\ell}, \ldots, \mathcal{A}_{|\wp(\mathbb{I}^{2,K})|}^{2,K}, \mathcal{A}_{|\wp(\mathbb{I}^{2,K+1})|}^{2,K+1}\right\}\right\} \end{aligned} \tag{18}$$

which has $\delta^{1\ell} = \Pi_{h=\ell}^{K+l} \left|\wp\left(\mathbb{I}^{2,h}\right)\right|$ elements. Correspondingly, for $\mu = 2$ we define

$$
\begin{aligned}
\mathbb{D}^{2,\ell} = \wp\left(\mathbb{I}^{1,1}\right) \oplus \ldots \oplus \wp\left(\mathbb{I}^{1,\ell}\right)& \\
= \Big\{\left\{\mathcal{A}_1^{1,1},\ldots,\mathcal{A}_1^{1,\ell-1},\mathcal{A}_1^{1,\ell}\right\},\left\{\mathcal{A}_1^{1,1},\ldots,\mathcal{A}_1^{1,\ell-1},\mathcal{A}_2^{1,\ell}\right\},& \\
\ldots,\left\{\mathcal{A}_1^{1,1},\ldots,\mathcal{A}_1^{1,\ell-1},\mathcal{A}^{1,\ell}_{\left|\wp\left(\mathbb{I}^{1,\ell}\right)\right|}\right\},\ldots,\left\{\mathcal{A}^{1,1}_{\left|\wp\left(\mathbb{I}^{1,1}\right)\right|},\ldots,\mathcal{A}^{1,\ell-1}_{\left|\wp\left(\mathbb{I}^{1,\ell-1}\right)\right|},\mathcal{A}^{1,\ell}_{\left|\wp\left(\mathbb{I}^{1,\ell}\right)\right|}\right\}\Big\}& \quad (19)
\end{aligned}
$$

with cardinality $\delta^{2\ell} = \Pi_{h=1}^{\ell}\left|\wp\left(\mathbb{I}^{1,h}\right)\right|$.

Let us define $\xi_{\mathbf{q}}^{\mu\ell}$ as the $\mathbf{q}^{th}$ element in $\mathbb{D}^{\mu\ell}$, and $\xi_{\mathbf{q},(h)}^{\mu\ell}$ as the h^{th} element of $\xi_{\mathbf{q}}^{\mu\ell}$. For example, in (18), $\xi_1^{1,\ell} = \left\{\mathcal{A}_1^{2,\ell},\ldots,\mathcal{A}_1^{2,K},\mathcal{A}_1^{2,K+1}\right\}$, and $\xi_{1,(1)}^{1\ell} = \mathcal{A}_1^{2\ell}$. With these sets, we can obtain our weights as follows:

$$
g_{\mathbf{q}}^{1,\ell+h-2} = \begin{cases} \sum_{i:X_i\in\xi^{1,\ell}_{\mathbf{q},(h)}} w_i, & h=1 \\ \sum_{i:X_i\in\left(\xi^{1,\ell}_{\mathbf{q},(h-1)}\right)^c} w_i + \sum_{i:X_i\in\xi^{1,\ell}_{\mathbf{q},(h)}} w_i, & h=2,\ldots,K-\ell+2 \end{cases} \tag{20}
$$

for $\mathbf{q} = 1,\ldots\delta^{1\ell}$, and

$$
g_{\mathbf{q}}^{2,h} = \begin{cases} \sum_{i:X_i\in\xi^{2,\ell}_{\mathbf{q},(h)}} w_i + \sum_{i:X_i\in\left(\xi^{2,\ell}_{\mathbf{q},(h+1)}\right)^c} w_i, & h=1\ldots,\ell-1 \\ \sum_{i:X_i\in\xi^{2,\ell}_{\mathbf{q},(h)}} w_i, & h=\ell \end{cases} \tag{21}
$$

for $\mathbf{q} = 1,\ldots\delta^{2\ell}$.

Finally, we denote $\Omega_{\mathbf{q}}^{\mu\ell}$ as the set of weights associated to each element in $\mathbb{A}^{\mu\ell}$ (please see equation (17) and the discussion around) corresponding to the $\mathbf{q}^{th}$ possible allocation of weights, $\xi_{\mathbf{q}}^{\mu\ell} \in \mathbb{D}^{\mu\ell}$, namely

$$
\Omega_{\mathbf{q}}^{\mu\ell} = \left\{\left\{w_h : X_h \in \mathbb{A}^{\mu\ell}\right\},\left\{g_{\mathbf{q}}^{\mu h} : P_h \in \mathbb{A}^{\mu\ell}\right\}\right\}, \quad \mu = 1,2 \tag{22}
$$

Each possible pair $\left\{\mathbb{A}^{\mu\ell},\Omega_{\mathbf{q}}^{\mu\ell}\right\}$ defines a Weber problem with rectilinear distance. For each candidate region $H^{\mu\ell}$, this results in a family of $\delta^{\mu\ell}$ Weber problems, each of them with solution $Y_{\mathbf{q}}^{\mu\ell}$ and objective function value

$$
f_{\mathbf{q}}^{\mu\ell}\left(Y_{\mathbf{q}}^{\mu\ell}\right) = \sum_{j=1}^{\left|\mathbb{A}^{\mu\ell}\right|} \Omega_{\mathbf{q},(j)}^{\mu\ell}\, d_1\left(\mathbb{A}_j^{\mu\ell}, Y_{\mathbf{q}}^{\mu\ell}\right) \tag{23}
$$

with $\mathbf{q} = 1,\ldots,\delta^{\mu\ell}$, and $\Omega_{\mathbf{q},(j)}^{\mu\ell}$ representing the j^{th} element of set $\Omega_{\mathbf{q}}^{\mu\ell}$.

We take, for candidate region $H^{\mu\ell}$, the solution that returns the minimal value of the objective function, i.e. $\overline{Y}^{\mu\ell} = \operatorname{argmin}_{\mathbf{q}}\left\{f_{\mathbf{q}}^{\mu\ell}\left(Y_{\mathbf{q}}^{\mu\ell}\right)\right\}$. The corresponding value of the objective function is represented by $\overline{f}^{\mu\ell}$. The optimal value of the main problem, denoted by f^*, is defined as the infimum of $\overline{f}^{\mu\ell}$ over all convex sub-spaces $H^{\mu\ell}$. The corresponding solution, Y^*, is the optimal location for the new facility and the solution to the Weber problem with line barrier, $SFWP_K$.

Several efficient algorithms have been proposed for solving the unconstrained Weber problem (see for example Hamacher (1995) and references in Klamroth (2006)). For the numerical analysis in this manuscript, we used the algorithm proposed by Francis et al. (1992).

Unfortunately, notwithstanding the prompt convergence rate to optimality of the procedure outlined above, the problem's size grows exponentially with the number of existing facilities located in each sub-space. Let $O\left(2^{\left|\mathbb{I}^{ml}\right|}\right)$ be the computational complexity of enumerating all possible assignments of existing facilities in H^{ml} to the involved passages, and $O(\left|\mathbb{A}^{ml}\right| \log \left|\mathbb{A}^{ml}\right|)$ the complexity of sorting data for the median method. Notice that the median method is performed for each combination of assignments, therefore, the algorithm complexity is $O\left(\left|\mathbb{A}^{ml}\right| \log \left|\mathbb{A}^{ml}\right| 2^{\left|\mathbb{I}^{ml}\right|}\right)$ for each subspace. As this procedure is executed for each convex sub-space, if we take $M = \max_{m=1,2;l=1,\ldots,K+1}\left\{\left|A^{ml}\right|\right\}$, careful analysis of equation (17) reveals that its overall complexity is bounded by $O\left(KM \log\left(M\right) 2^{KI}\right)$. This suggests that, for the particular case of a horizontal line barrier, the exact divide-and-conquer approach outlined in this section has polynomial growth (this is so because only two passages are involved in the computation of the barrier distance, i.e. $O\left(M \log\left(M\right) 2^{I}\right)$), whereas the one proposed by Klamroth (2001) presented exponential growth. Unfortunately, the exact divide-and-conquer procedure's computational time grows exponentially on number of existing facilities in each convex space, i.e. $\left|\mathbb{I}^{ml}\right|$. This exponential growth is due to the number of combinations of possible assignments of existing facilities to the involved passages. On the other hand, when the barrier is non-horizontal, the computational time of the proposed EDC procedure has exponential growth with respect to both, the number of passages and the number of existing facilities. To overcome this, we propose an approximation procedure, which improves the algorithm's time complexity considerably. This is presented in the following section.

3.3 Approximate divide-and-conquer-based strategy (ADC) for the $SFWP_K$

In this section, we propose an approximate heuristic for solving the Weber problem with line barrier with a general slope by applying the strategy used in the exact approach, together with a simplification procedure for reducing the number of assignments. We refer to this as the Approximate divide-and-conquer-based strategy (ADC).

After decomposing the problem space into several convex sub-spaces as explained in Section 3.1, for each sub-space $H^{ml}, m = 1, 2,\ l = 1, \ldots, K + 1$, which contains shadow points to space $H^{\mu\ell}$, we can introduce a *dummy* point $\widehat{X}^{ml} = \left(\hat{x}_1^{ml}, \hat{x}_2^{ml}\right)$, that substitutes all $X_i \in H^{ml}$, where $\hat{x}_h^{ml} = \text{med}_{i:X_i \in H^{ml}}\left\{x_{ih}\right\},\ h = 1, 2$. It is a well known result that the median point carries the weight $\hat{w}^{ml} = \sum_{i:X_i \in H^{ml}} w_i$. Therefore, instead of considering all combinations of assignments of facilities $X_i \in H^{ml}$ to all involved passages, we replace them by our dummy point $\widehat{X}^{ml}$ with associated weight $\hat{w}^{ml}$. This simple procedure reduces the computational complexity significantly, as enumerating all combinations of assignments is now reduced to choosing between two possible passages (P_{l-1} and P_l) for the point $\widehat{X}^{ml}$ in each H^{ml}. The shortest distance between $\widehat{X}^{ml} \in H^{ml}$ and a new facility $Y \in H^{\mu\ell}$ is given by:

$$\min_{h=l-1,l} \left\{ \hat{w}^{ml} \left(d_1 \left(\widehat{X}^{ml}, P_h \right) + d_1 \left(P_h, Y \right) \right) \right\} \tag{24}$$

This substitution must be conducted for all the sub-spaces defined by the passages involved in each particular sub-problem. Notice that for the particular case when the barrier is horizontal, only two passages are involved and, therefore, only one dummy point is generated. However, when the barrier has a general slope, several passages are involved, resulting in the need of as many dummy points, one for each involved sub-space (please see discussion around equation (17)). In such cases, the corresponding power set is simply $\wp \left(\widehat{X}^{ml} \right) = \left\{ \emptyset, \widehat{X}^{ml} \right\}$. This reduces the computational complexity of the procedure with a minimal cost in precision.

The approximate procedure starts with a pre-processing step, involving the convex sub-space construction procedure, power set generation, and obtaining the dummy point associated to each sub-space. It then proceeds with the divide-and-conquer strategy for solving unconstrained Weber problems for each sub-space using the median method. Details of these are given below. As before, a verbal summary of our heuristic is presented in the Appendix of this article.

Heuristic Approach

Similar to the EDC strategy, we start by creating $2(K+1)$ convex sub-spaces. For each convex sub-space H^{ml}, we follow the procedure described above around equation (17) for building the associated subset $\mathbb{A}^{ml}$, which includes all existing facilities which are 1-visible to all points in H^{ml}, together with all involved passages. The dummy point $\widehat{X}^{ml}$ is then obtained and its weights sequentially assigned black to each of the two involved passages. To obtain the sets of weights we introduce the following elements:

$$\mathbb{D}^{1,\ell} = \left\{ \emptyset, \widehat{X}^{2,\ell} \right\} \oplus \ldots \oplus \left\{ \emptyset, \widehat{X}^{2,K} \right\} \oplus \left\{ \widehat{X}^{2,K+1} \right\} \tag{25}$$

which has $\delta^{1\ell} = 2^{K-\ell+1}$ elements. Correspondingly, for $\mu = 2$ we define

$$\mathbb{D}^{2,\ell} = \left\{ \widehat{X}^{1,1} \right\} \oplus \left\{ \emptyset, \widehat{X}^{1,2} \right\} \oplus \ldots \oplus \left\{ \emptyset, \widehat{X}^{1,\ell} \right\} \tag{26}$$

with cardinal $\delta^{2\ell} = 2^{\ell-1}$.

Using the same notational convention as before, and using $1[\xi]$ to represent the indicator function of set ξ, we have that

$$\hat{g}_{\mathbf{q}}^{1,\ell+h-2} = \begin{cases} \hat{w}^{2,\ell+h-1} \cdot 1\left[\xi_{\mathbf{q},(h)}^{1,\ell} \neq \emptyset\right], & h = 1 \\ \hat{w}^{2,\ell+h-2} \cdot 1\left[\xi_{\mathbf{q},(h-1)}^{1,\ell} = \emptyset\right] + \hat{w}^{2,\ell+h-1} \cdot 1\left[\xi_{\mathbf{q},(h)}^{1,\ell} \neq \emptyset\right] & h = 2, \ldots, K - \ell + 2 \end{cases} \tag{27}$$

for $\mathbf{q} = 1, \ldots \delta^{1,\ell}$, and

$$\hat{g}_{\mathbf{q}}^{2,h} = \begin{cases} \hat{w}^{1,h+1} \cdot 1\left[\xi_{\mathbf{q},(h+1)}^{2,\ell} = \emptyset\right] + \hat{w}^{1,h} \cdot 1\left[\xi_{\mathbf{q},(h)}^{2,\ell} \neq \emptyset\right], & h = 1, \ldots, \ell - 1 \\ \hat{w}^{1,h} \cdot 1\left[\xi_{\mathbf{q},(h)}^{2,\ell} \neq \emptyset\right], & h = \ell \end{cases} \tag{28}$$

for $\mathbf{q} = 1, \ldots \delta^{2\ell}$.

Finally, the sets of weights associated to each possible allocation, $\mathbf{q}$, are given by:

$$\widehat{\Omega}_{\mathbf{q}}^{\mu\ell} = \left\{\left\{w_h : X_h \in \mathbb{A}^{\mu\ell}\right\}, \left\{\hat{g}_{\mathbf{q}}^{\mu h} : P_h \in \mathbb{A}^{\mu\ell}\right\}\right\}, \quad \mu = 1, 2 \tag{29}$$

As with the case of the exact heuristic, this results in a family of $\delta^{\mu\ell}$ Weber problems for each candidate region $H^{\mu\ell}$, each of them with solution $\widehat{Y}_{\mathbf{q}}^{\mu\ell}$ and objective function value

$$\hat{f}_{\mathbf{q}}^{\mu\ell}\left(\widehat{Y}_{\mathbf{q}}^{\mu\ell}\right) = \sum_{j=1}^{\left|\mathbb{A}^{\mu\ell}\right|} \widehat{\Omega}_{\mathbf{q},(j)}^{\mu\ell}\, d_1\left(\mathbb{A}_j^{\mu\ell}, \widehat{Y}_{\mathbf{q}}^{\mu\ell}\right) \tag{30}$$

with $\mathbf{q} = 1, \ldots, \delta^{\mu\ell}$.

For region $H^{\mu\ell}$, we take the solution that minimises the objective function over all possible allocations $\mathbf{q}$, i.e. $\widehat{Y}^{\mu\ell} = \text{argmin}_{\mathbf{q}}\left\{\hat{f}_{\mathbf{q}}^{\mu\ell}\left(\widehat{Y}_{\mathbf{q}}^{\mu\ell}\right)\right\}$. This value is represented by $\hat{f}^{\mu\ell}$. The optimal value of the $SFWP_K$ problem, denoted by $\hat{f}^*$, is defined as the infimum of $\hat{f}^{\mu\ell}$ over all convex sub-spaces $H^{\mu\ell}$. The corresponding solution, $\widehat{Y}^*$, is the optimal location for the new facility.

The complexity of the approximate heuristic for any given sub-space H^{ml} is given by $O\left(\left|\mathbb{A}^{ml}\right| \log \left|\mathbb{A}^{ml}\right|\, 2^{K^{ml}}\right)$, where black

$$K^{ml} = \begin{cases} K - l + 2, & m = 1 \\ l, & m = 2 \end{cases}.$$

Using M as defined at the end of Section 3.2, we have that the complexity of our heuristic is bounded by $O\left(KM \log(M)\, 2^K\right)$. Therefore, the approximate heuristic has a polynomial time complexity in the number of existing facilities and an exponential time complexity in the number of passages.

3.4 An alternate-location-allocation heuristic MFWP$_K$

The procedures described in the previous sections were aimed at solving the single-facility Weber problem in the presence of a line barrier with general slope. In this Section, we address the multi-facility variant of this problem.

The alternate-location-allocation (ALA) heuristic, first formulated by Cooper (1963) and later developed by Brimberg and Salhi (2005), is a standard method to solve location-allocation problems stemming from the fact that location and allocation sub-problems are easier to solve individually. In other words, when the location of the new facilities is given, the multi-Weber problem reduces to a set-partitioning problem, where each customer is assigned to its closest facility. On the other hand, given the assignment of demand points to facilities, the multi-Weber problem reduces to a number of independent single-facility location problems. Bischoff et al. (2009) studied the properties of these heuristics and showed that they remain valid in the presence of barriers. Next, we present the location and allocation sub-problems within the context of our problem.

black

3.4.1 Set-partitioning sub-problem

Let us recall the MFWP$_K$, i.e. expressions (5) to (8), where the location and allocation decision variables are represented by Y_j and U_{ij}, respectively. When the location of new facilities is given, the allocation segment of the MFWP$_K$ becomes a set-partitioning (sub-)problem, referred to as SPSP. The SPSP, minimising objective function (31) subject to constraints (6) and (8), assigns each existing facility to the nearest new facility location (notice that the $\widetilde{Y}$s in the objective function are given locations for new facilities).

$$\sum_{i=1}^{I}\sum_{j=1}^{J} w_i U_{ij} black d_1^{L_{\mathcal{B}}}\left(X_i, \widetilde{Y}_j\right) \tag{31}$$

black

3.4.2 Single-facility Weber sub-problems

Assume that the allocation variables U_{ij} in MFWP$_K$ are given, with their corresponding values represented by $\widetilde{U}_{ij}$. This implies that the new facility to which each existing facility is allocated is known. Let us denote set of existing facilities that are assigned to the j^{th} new facility by $\mathcal{I}_j, j = 1, \ldots, J$. In this framework, the new facility locations are independent from each other and, consequently, the MFWP$_K$ can be decomposed into J single-facility Weber sub-problems. Each of these sub-problems can be represented as:

$$\min \quad \sum_{i \in \mathcal{I}_j} w_i \widetilde{U}_{ij} black d_1^{L_\mathcal{B}} \left(X_i, Y_j\right) \tag{32}$$

$$s.t. \quad Y_j \in \mathcal{F} \tag{33}$$

where,black $d_1^{L_\mathcal{B}} \left(X_i, Y_j\right)$ is given by (2) , and $j = 1, \ldots, J$. The objective function, (32), minimizes the total travelled distance from each location to the associated demand points, given a pre-determined allocation. This problem can be solved by the divide-and-conquer-based approaches presented in Sections 3.2 and 3.3.

black

3.4.3 Alternate-location-allocation heuristic

In this section, we propose an alternate-location-allocation (ALA) heuristic blackconsisting of iteratively recalling SPSPs and SFWP$_K$s. A verbal summary of this procedure is presented in the Appendix of this article.

For solving the MFWP$_K$, we first run the SPSPs by providing random locations for the new facilities. The solution of SPSP is a set of clusters in which each of the existing facilities is assigned to the nearest new facility. Given this set of independent clusters, the MFWP$_K$ can be decomposed into a number of SFWP$_K$s that can be solved by either the EDC or the ADC heuristics. The output of each SFWP$_K$ is the location of a new facility for each cluster. These new locations are then fed to the SPSPs, and the new solution is used for updating the clusters. This procedure is repeated until certain termination criterion is satisfied. In the understanding that initial solutions may have a great impact on the final solution's quality, it is advisable to repeat this procedure several times with different initial locations, this will avoid getting trapped in local optimum solution, although attaining a global solution cannot be guaranteed.

blackRemark

While the ALA heuristic with embedded EDC converges to optimal/near optimal solutions, it is less efficient in large-scale instances, due to its inherent exponential complexity. In comparison, thanks to the polynomial time complexity of the ADC, the ALA heuristic with an embedded ADC is capable to get high quality (near optimal) solutions in short computational time. In order to highlight the efficiency and the quality of the proposed heuristics, in the next section we present a benchmark solution method .

black

3.4.4 Benchmark solution method

In order to obtain a benchmark for our proposed heuristics, we implemented the method proposed by Larson and Sadiq (1983) for reformulating the rectilinear-distance Weber problem with line barrier to an equivalent discrete location problem.

Larson and Sadiq suggested that the Weber problem with line barrier can be transformed into a discrete p-median problem by means of expanding the set of candidate nodes. This is attained by considering the nodes defined by the intersection of the horizontal and vertical lines passing through all existing facilities and barrier extreme points. The rectilinear distance is then used for obtaining the pairwise distance matrix between all, existing and generated, nodes. A p-median problem is then solved considering the existing facilities as demand nodes and the intersection points as candidate nodes. However, this method has the problem of generating a large solution set. For a given Weber problem with barrier characterized by I existing facilities and K passage points, the number of intersection points is equal to $(I+K)^2$. In other words, this method constructs a p-median problem with $(I+K)^2$ candidate points, significantly increasing the solution set. The number of alternative solutions to be considered to find the best solution is bounded by

$$\binom{(I+K)^2}{J} = \frac{(I+K)^2!}{J!\left((I+K)^2 - J\right)!}$$

By considering the passage points on the line barrier as the barrier extreme points, the problem described above can be solved for obtaining a benchmark for the procedure proposed in this section. The resulting p-median problem can be solved by any of the multiple methods available in literature (see, for example, Daskin, 2011).

In the following Section, the performance of the proposed heuristics is assessed by means of number of numerical experiments.

4 Experimental Results

For the numerical assessment of our heuristics, we generated a wide range of problem instances that vary in terms of number of existing facilities, number of passages, number of facilities to locate, and slopes for the line barrier. The coordinates of existing facilities were generated as extractions from a two-dimensional uniform distribution in the 100×100 square. The number of existing facilities used in the different experiments was taken from the set $a = \{10, 20, 50, 100, 150, 200, 250, 500, 1000\}$. In order to capture the different weights of the existing facilities, w_i, we generated random continuous numbers in the interval $[0, 1]$. For the instances corresponding to each number of existing facilities, we generated passage points on a horizontal line barrier inside the unit square. The number of passage points generated for each instance is given by the set $b = \{2, 5, 10, 15, 20, 25, 50\}$. For the multi-facility location problem, we solved all combinations of instances produced by the product $a \times b$ with the number of new facilities taken from the set $c = \{1, 2, 3, 5, 10\}$. The combination of the set of number of existing facilities, a, the set of number of passage points, b, and the set of number of new facilities, c, yields a total of $|a| \times |b| \times |c|$=315 instances for problems with horizontal barrier. We also evaluated our heuristics for instances where the line barrier has a positive slope. In such cases, the number of passage points was taken from the set $b' = \{5, 15, 25, 50\}$. We consider two different slopes for the line barrier: $\pi/10$ and $\pi/5$ radians, respectively. In this case, the number of existing facilities is taken from the set black $a' = \{10, 20, 100, 200\}$, while the number of new facilities is taken from $c' = \{1, 2, 5, 10\}$. In total, $2 \times |a'| \times |b'| \times |c'| = 128$ instances were analysed for the barrier with positive slope case. Overall, 443 instances were tested. The single-facility problems were solved using the EDC and ADC heuristics; the multi-facility problems were solved by means of the ALA heuristic using either EDC (ALA-EDC) or ADC (ALA-ADC); black finally each problem was also solved using a general-purpose solver (Gurobi 8.1.) as a benchmark. It is worth mentioning that since the ALA heuristic is started with a random initial solution, blackthe reported values -and associated computation times- are the over 10 alternative replications. When the solver was used, the relative MIP optimality gap was also set to 10^{-4}. All the experiments were ran on a 64-bit operating system server with a 2.7 GHz CPU Intel(R) processor and 72 GB of RAM. The proposed heuristics were coded in Matlab 7.10. We set a time limit of 3600 seconds (1 hour) for the Matlab code.

The rest of the Section is organised as follows. We first discuss the computational time of the proposed heuristics and their efficiency. Later, we evaluate the heuristics' accuracy, providing the percentage relative error (PRE) functions for each instance under the corresponding heuristic. Finally, we illustrate the impact of the number of passage points on the objective function and compare it with that of problem without barrier.

4.1 Heuristic Validity and Accuracy Assessment

We assessed the validity and accuracy of the proposed heuristics by computing the associated PRE of each heuristic, and comparing it against the optimal solution obtained for instances for which a general-purpose solver can reach the relative MIP optimality gap within the runtime limit. The PRE of instances with I existing facilities, K passages, and J new facilities are denoted by $PRE_E(I, K, J)$ and $PRE_A(I, K, J)$, and given by:

$$PRE_E(I, K, J) = \frac{\mathcal{Z}_E\,(I, K, J) - \mathcal{Z}_{pM}\,(I, K, J)}{\mathcal{Z}_{pM}\,(I, K, J)} \times 100\% \tag{34}$$

$$PRE_A(I, K, J) = \frac{\mathcal{Z}_A\,(I, K, J) - \mathcal{Z}_{pM}\,(I, K, J)}{\mathcal{Z}_{pM}\,(I, K, J)} \times 100\% \tag{35}$$

Notice that $PRE_E(I,K,J)$ represents the percentage relative error of the EDC heuristic for the special case where $J = 1$, and the ALA-EDC heuristic for the cases where $J > 1$; whilst $PRE_A(I,K,J)$ represents the percentage relative error of the ADC heuristic for $J = 1$, and the ALA-ADC heuristic for the cases where $J > 1$. The value $\mathcal{Z}_E(I,K,J)$ represents the best objective function value of instances with I existing facilities, K passages, and J new facilities, obtained with the EDC heuristic for $J = 1$, and with the ALA-EDC heuristics for cases with $J > 1$. Likewise, $\mathcal{Z}_A(I,K,J)$ represents the best objective function value for instances with I existing facilities, K passages, and J new facilities obtained using the ADC heuristic for cases where $J = 1$ and the ALA-ADC for $J > 1$. Finally, $\mathcal{Z}_{pM}(I,K,J)$ denotes the optimal objective function value, obtained by direct solution of the p-Median formulation, for instances with I existing facilities, K passages, and J new facilities.

Tables 1 and 2 show the results obtained for instances in the presence of horizontal and non-horizontal line barriers, respectively. Each cell provides $PRE_E(I,K,J)$ and $PRE_A(I,K,J)$ values corresponding to different combinations of the number of new and existing facilities (rows) and the number of passages (columns). In order to provide a general view over different instance sizes, we compute the following average values for $PRE_E(I,K,J)$ and $PRE_A(I,K,J)$:

$$
\begin{aligned}
\overline{PRE}_E(I,J) &= \frac{1}{|b|}\sum_{K\in b} PRE_E(I,K,J)\\
\overline{PRE}_A(I,J) &= \frac{1}{|b|}\sum_{K\in b} PRE_A(I,K,J)\\
\overline{\overline{PRE}}_E(J) &= \frac{1}{|a|\,|b|}\sum_{I\in a}\sum_{K\in b} PRE_E(I,K,J)\\
\overline{\overline{PRE}}_A(J) &= \frac{1}{|a|\,|b|}\sum_{I\in a}\sum_{K\in b} PRE_A(I,K,J)\\
\overline{\overline{PRE}}_E(K) &= \frac{1}{|a|\,|c|}\sum_{I\in a}\sum_{J\in c} PRE_E(I,K,J)\\
\overline{\overline{PRE}}_A(K) &= \frac{1}{|a|\,|c|}\sum_{I\in a}\sum_{J\in c} PRE_A(I,K,J)
\end{aligned}
$$

For the case of a horizontal barrier, the following general conclusions can be drawn from Table 1:

i) Direct solution of the p-Median formulation is practical only for small-size instances;

ii) EDC reaches optimality within the runtime limit for the single-facility instances ($\overline{\overline{PRE}}_E(J) = 0.00\%$);

iii) ALA(EDC) unnoticeably deviates from optimality as number of new facilities increases, due mainly to the randomness in the initial solutions;

iv) ADC approaches optimal or near-optimal solutions within the runtime limit for the single-facility instances with negligible error ($\overline{\overline{PRE}}_A\,(J) = 0.05\%$);

v) ALA(ADC) converges to near optimal solution as number of new facilities increases with acceptable relative error ($0.07\% < \overline{\overline{PRE}}_A\,(J) < 0.93\%$).

Correspondingly, when the barrier is not horizontal, we can observe in Table 2 that:

i) Direct solution of the p-Median formulation is useful only for small-size instances;

ii) EDC reaches optimality within the runtime limit for the single-facility instances ($\overline{\overline{PRE}}_E\,(J) = 0.00\%$);

iii) ADC approaches optimal or near-optimal solutions within the runtime limit for the single-facility instances with negligible error $\overline{\overline{PRE}}_A\,(J) = 0.73\%$ and ($\overline{\overline{PRE}}_A\,(J) = 0.84\%$ when the slope of the barrier is $\pi/10$ and $\pi/5$, respectively).

Tables 1 and 2 also illustrate that in spite of the fluctuation of $\overline{\overline{PRE}}_E\,(K)$ and $\overline{\overline{PRE}}_A\,(K)$ over the number of passages, their value generally decreases as the number of passages increase. Results in Table 1 show that the overall PRE average of EDC and ALA(EDC) is only 0.16%; and the one of the ADC and ALA(ADC) is 0.35%. This highlights the better performance of the proposed heuristics with respect to the p-Median formulation. It is worth mentioning that the reason for which the ALA(EDC) heuristic, in spite of executing an optimal procedure, does not reach 0.00% is the presence of the random initial solution at the first iteration. Results in Table 2 also reveal that the bigger slope of a line barrier, the bigger overall PRE average.

For those instances where the direct solution of the p-Median is not practical, we consider $\mathcal{Z}_E(I,K,J)$ as a benchmark for, mainly, multi-facility (i.e. $J > 1$) instances and denote $PRE'_A\,(I,K,J)$ as the percentage relative error (PRE) of the ALA(ADC) heuristic, i.e.

$$PRE'_A(I,K,J) = \frac{\mathcal{Z}_A\,(I,K,J) - \mathcal{Z}_E\,(I,K,J)}{\mathcal{Z}_E\,(I,K,J)} \times 100\% \tag{36}$$

Tables 2 and 3 show the following average of $PRE'_A(I,K,J)$, for horizontal and non-horizontal line barriers:

$$\overline{PRE'}_A\,(I,J) = \frac{1}{|b|}\sum_{K\in b} PRE'_A(I,K,J)$$
$$\overline{\overline{PRE'}}_A\,(J) = \frac{1}{|a|\,|b|}\sum_{I\in a}\sum_{K\in b} PRE'_A(I,K,J)$$
$$\overline{\overline{PRE'}}_A\,(K) = \frac{1}{|a|\,|c|}\sum_{I\in a}\sum_{J\in c} PRE'_A(I,K,J)$$

The numerical results show that, in general, the ALA(ADC) heuristic produces solutions of similar quality to those obtained by the ALA(EDC) heuristic, with an overall PRE average of about 0.38%, 0.61%, and 0.91%, for solving Weber problems with zero, $\pi/10$, and $\pi/5$ radians slopes, respectively. Considering the efficiency of the ALA(ADC) as an approximate heuristic to solve the large-size instances, the results suggest

i) The ALA(ADC) can be implemented for very large-size instances (for example, $I = 500, 1000$) providing high-quality solutions;

ii) $\overline{\overline{PRE'}}_A(J)$ increases as number of new facilities increase;

iii) $\overline{\overline{PRE'}}_A(J)$ increases as the line barrier takes larger slopes.

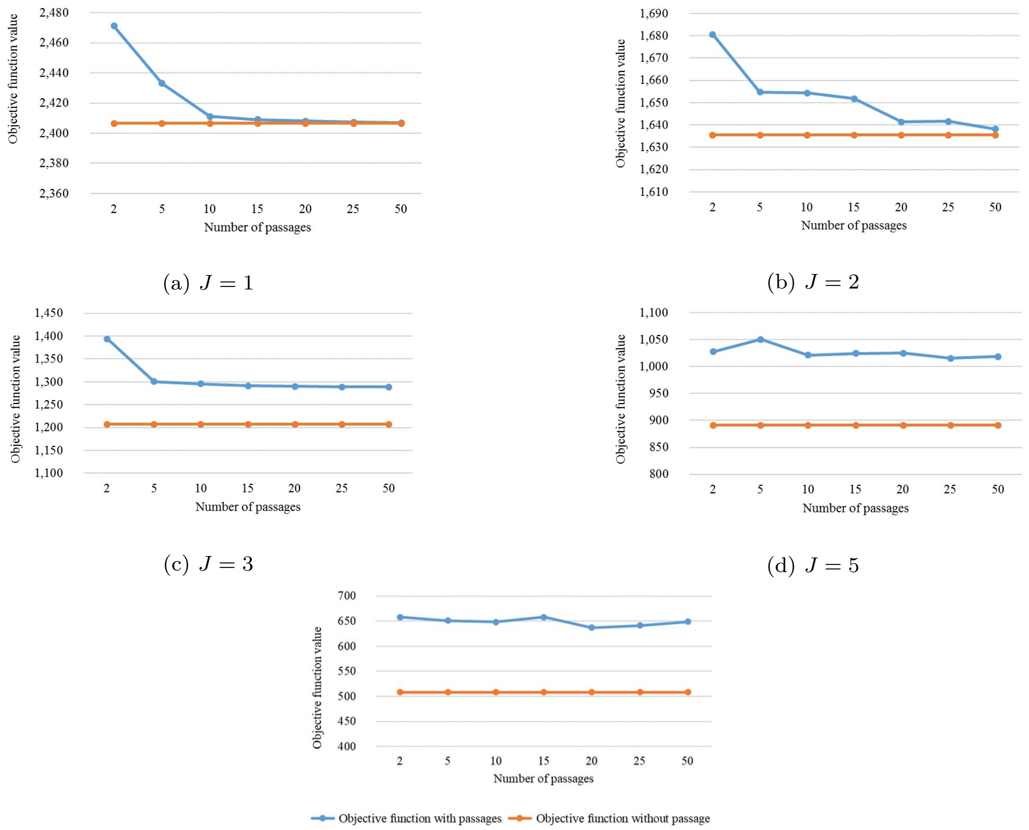
4.2 Impact of Passages

We choose the instances with 100 existing facilities for illustrating the impact of passage points on the total travel distance. Figure 4 shows five displays corresponding to different numbers of new facilities to locate ($J = 1, 2, 3, 5$ and 10). It can be seen that, for the cases where $J = 1, 2, 3$ the objective function tends to flatten-up as the number of passages on the line barrier increases, converging to the one of a problem without barrier. In general, the results suggest that considering the passage points in the mathematical modelling becomes more relevant as the number of new facilities to locate increases, and that ignoring the presence of barriers may cause blacksub-optimal decisions.

4.3 Assessment of Computational Time

In the following lines, we compare the performance of the heuristics introduced in Section 3 in terms of their average CPU computational time, illustrated in logarithmic scale in Figure 5. Each of the panels in the figure represents a different numbers of new facilities, the horizontal axis in each plot represents different numbers of existing facilities.

Results indicate that there is a significant difference between CPU time of the single-facility problem (Figure 5a) and the multi-facility problems (Figure 5b - 5e). black Figure 5a, illustrates the performance of the EDC and ADC heuristics, together with the solution to the p-Median formulation obtained with Gurobi, pM. It can be observed that by direct implementation of the p-Median formulation, it was possible to solve all instances of the single facility problem within the runtime limit, while the EDC heuristic experiences a rapid increase in CPU time as instance-size increases. Additionally, results show that the ADC heuristic solves the instances in a considerably shorter computational time than pM. On the other hand, Figures 5b - 5e depict



(a) $J = 1$

(b) $J = 2$

(c) $J = 3$

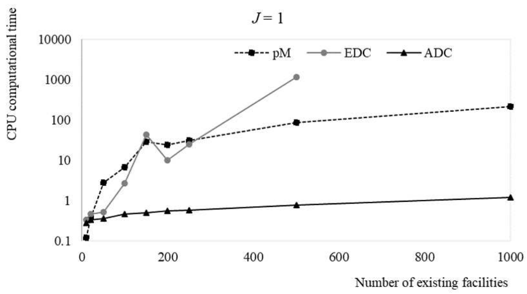
(d) $J = 5$

(e) $J = 10$

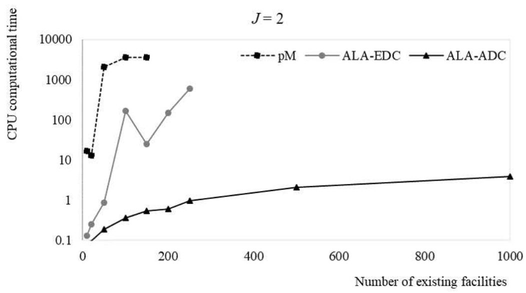
Figure 4: Objective function convergence presentation.

CPU times of the ALA-EDC and ALA-ADC heuristics, and the associated p-Median formulation, pM, for different sizes of the multi-facility location problems. The results suggest that in most instances, the CPU time of the pM solver rapidly hits the runtime limit with no convergence result, which can be attributed to the high complexity of the p-median problem. These figures also show that the CPU times of ALA-EDC, has a remarkable increasing trend as instance's size increases, while the that of ALA-ADC grows gently with increasing the instance size. The ALA-ADC performs at noticeably faster rate as the instance size increases, showing its fitness for solving very large instances within the runtime limit.

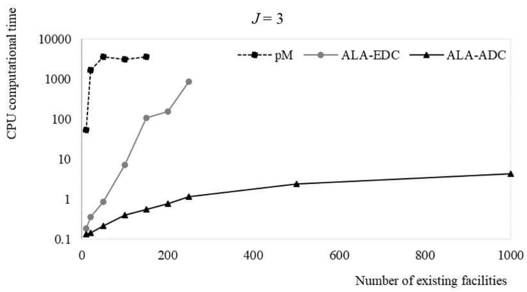
Overall, the following conclusions can be obtained: a) while direct solution of the p-Median problem is a useful method for solving small-size instances of the single-facility problem, it can be more expensive in terms of computational time as the instance size grows; b) the ALA-EDC heuristic can be applied to many instances, however, as the number of new facilities increases it becomes less efficient due to the exponential complexity inherent in EDC heuristic; and c) the ALA-ADC heuristic outperforms the other alternatives discussed above for almost all instances. Moreover, it seems fairly safe to affirm that it can be of use even for very large sized instances. For the sake of clarity, Table 4 shows the average computation times for different instances. The



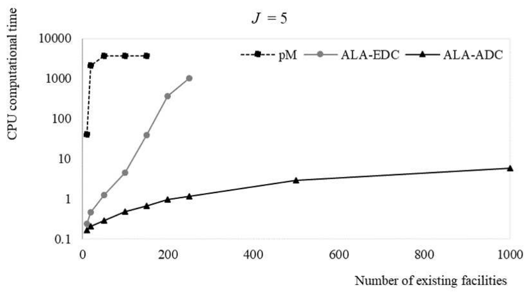
(a) $J = 1$



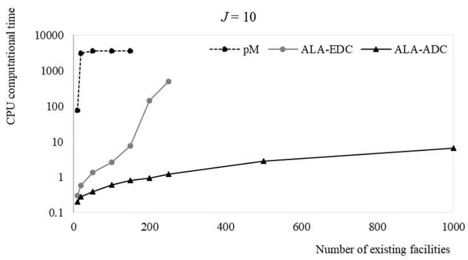
(b) $J = 2$



(c) $J = 3$



(d) $J = 5$



(e) $J = 10$

Figure 5: Computational time.

reported time refers to the average execution time for instances with the indicated numbers of existing facilities (I) and new facilities (J), irrespective of the number of passages. Please also notice that whenever convergence was not reached within 1 hour of execution, the execution time reported in the table is 3600 sec.

5 Conclusions

In this paper, we address single and multi-facility Weber location problems with rectilinear distance in the presence of line barrier with general slope and a fixed number of passages. For solving this problem, we first proposed an exact heuristic (EDC) that outperforms other available alternatives. However, it still shows exponential growth on the number of facilities and is unfit for solving very large-size instances. To overcome this, we proposed an approximation heuristic (ADC) that shows polynomial time complexity.

For the single facility case, the proposed heuristics (based on a divide-and-conquer strategy) reduce the time complexity of other available heuristics to a polynomial computational time. Additionally, experimental results suggest that the approximate heuristic is remarkably efficient in finding a near-optimal solution within polynomial time. It is also shown that it performs very well in large-sized instances, with negligible average optimality gap, with a short computational time.

We also propose an alternate-location-allocation heuristic for solving the multi-facility case, which is also based on a divide-and-conquer strategy. Results show that the performance of this heuristic is as good as the one that uses an exact divide-and-conquer procedure. We compared the results of the proposed heuristics with the equivalent p-median problem as a benchmark method. Numerical experiments show that the proposed heuristic strongly outperforms the benchmark.

To conclude, the methodologies presented in this manuscript constitute a considerable improvement in the computational time and solution quality of the Weber problem with passages on a barrier. Moreover, they extend the available literature by addressing the problem under the presence of a non-horizontal line barrier. However, some questions remain open, pointing at future research directions. For example, this problem can be revisited by considering a Weber problem's formulation with Euclidean distances, where the nonlinearity inherent to such approach will demand more sophisticated solution methodologies. It will also be worth to investigate the case where the line barrier is not monotonic (e.g. sinusoidal barriers).

Table 1: The performance of the exact and the approximate algorithms (Weber problem with a horizontal line barrier).

	$\|K\| = 2$		$\|K\| = 5$		$\|K\| = 10$		$\|K\| = 15$		$\|K\| = 20$		$\|K\| = 25$		$\|K\| = 50$					
	PRE_E	PRE_A	PRE_E	PRE_A	PRE_E	PRE_A	PRE_E	PRE_A	PRE_E	PRE_A	PRE_E	PRE_A	PRE_E	PRE_A	$\overline{PRE}_E(I,J)$	$\overline{PRE}_A(I,J)$	$\overline{\overline{PRE}}_E(J)$	$\overline{\overline{PRE}}_A(J)$
$\|J\| = 1$																		
$\|I\| = 10$	0.00%	0.00%	0.00%	0.46%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.07%		
$\|I\| = 20$	0.00%	0.15%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.02%		
$\|I\| = 50$	0.00%	0.00%	0.00%	0.03%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%		
$\|I\| = 100$	0.00%	0.13%	0.00%	0.05%	0.02%	0.00%	0.00%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.03%		
$\|I\| = 150$	0.00%	0.53%	0.00%	0.08%	0.00%	0.03%	0.00%	0.21%	0.00%	0.07%	0.00%	0.00%	0.00%	0.09%	0.00%	0.14%	**0.00%**	**0.05%**
$\|J\| = 2$																		
$\|I\| = 10$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.29%	0.00%	0.04%		
$\|I\| = 20$	0.00%	0.20%	0.00%	0.16%	0.00%	0.14%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.18%	0.00%	0.10%		
$\|I\| = 50$	0.00%	0.00%	0.00%	0.41%	0.00%	0.00%	0.56%	0.56%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.08%	0.14%		
$\|I\| = 100$	0.02%	0.02%	0.00%	0.18%	0.00%	0.00%	0.03%	0.04%	0.00%	0.01%	0.00%	0.00%	0.00%	0.02%	0.01%	0.04%		
$\|I\| = 150$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	0.00%	0.09%	N.A.	N.A.	0.00%	0.00%	N.A.	N.A.	0.00%	0.05%	**0.02%**	**0.07%**
$\|J\| = 3$																		
$\|I\| = 10$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%		
$\|I\| = 20$	0.00%	0.00%	0.00%	0.21%	0.00%	0.19%	0.00%	0.00%	0.00%	0.66%	0.00%	0.69%	0.00%	0.24%	0.00%	0.28%		
$\|I\| = 50$	0.00%	0.00%	0.07%	0.75%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%	0.11%		
$\|I\| = 100$	0.00%	0.00%	0.00%	0.14%	0.00%	0.00%	0.00%	0.04%	0.01%	0.01%	0.00%	0.00%	0.00%	0.00%	0.01%	0.02%		
$\|I\| = 150$	0.48%	0.48%	0.14%	0.14%	0.24%	0.24%	0.00%	0.00%	0.13%	0.13%	0.04%	0.04%	N.A.	N.A.	0.17%	0.17%	**0.04%**	**0.12%**
$\|J\| = 5$																		
$\|I\| = 10$	0.00%	0.25%	0.58%	2.11%	0.00%	0.28%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.08%	0.38%		
$\|I\| = 20$	0.00%	0.00%	0.00%	1.39%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.75%	0.00%	0.31%		
$\|I\| = 50$	0.00%	0.00%	0.19%	0.29%	1.38%	2.10%	0.00%	1.02%	1.09%	1.21%	0.00%	1.47%	0.00%	0.00%	0.38%	0.87%		
$\|I\| = 100$	0.00%	0.02%	0.00%	0.00%	0.06%	0.06%	0.58%	0.58%	0.00%	0.00%	0.00%	1.02%	0.29%	0.32%	0.14%	0.29%		
$\|I\| = 150$	0.03%	0.03%	N.A.	N.A.	0.01%	0.01%	0.07%	0.12%	0.01%	0.01%	N.A.	N.A.	N.A.	N.A.	0.03%	0.04%	**0.13%**	**0.38%**
$\|J\| = 10$																		
$\|I\| = 10$	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%		
$\|I\| = 20$	0.00%	3.26%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.47%		
$\|I\| = 50$	0.65%	1.25%	1.55%	2.65%	3.35%	3.35%	0.22%	0.22%	4.79%	5.77%	0.22%	0.23%	0.23%	0.76%	1.57%	2.03%		
$\|I\| = 100$	0.13%	0.16%	0.18%	0.46%	0.70%	1.95%	0.35%	0.47%	0.46%	0.62%	0.74%	0.67%	0.41%	0.82%	0.42%	0.73%		
$\|I\| = 150$	0.13%	0.13%	N.A.	N.A.	N.A.	N.A.	0.34%	0.56%	0.88%	3.52%	N.A.	N.A.	N.A.	N.A.	0.45%	1.40%	**0.49%**	**0.93%**
$\overline{\overline{\overline{PRE}}}_E(K)$	**0.07%**		**0.16%**		**0.34%**		**0.10%**		**0.36%**		**0.07%**		**0.06%**		*0.16%*			
$\overline{\overline{\overline{PRE}}}_A(K)$		**0.31%**		**0.51%**		**0.48%**		**0.17%**		**0.58%**		**0.21%**		**0.20%**		*0.35%*		

N.A.: No Answer.

Table 2: The performance of the exact and the approximate algorithms (Weber problem with a non-horizontal line barrier).

Slope = $\pi/10$										
			$\lvert K\rvert = 5$	$\lvert K\rvert = 15$	$\lvert K\rvert = 25$	$\lvert K\rvert = 50$	$\overline{PRE}_E(I,J)$	$\overline{PRE}_A(I,J)$	$\overline{\overline{PRE}}_E(J)$	$\overline{\overline{PRE}}_A(J)$
$\lvert J\rvert = 1$	$\lvert I\rvert = 10$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.09%	0.12%	0.11%	0.11%		**0.11%**		
	$\lvert I\rvert = 20$	PRE_E	0.64%	0.44%	0.17%	0.03%	**0.32%**			
		PRE_A	1.02%	0.80%	0.52%	0.45%		**0.70%**		
	$\lvert I\rvert = 100$	PRE_E	0.62%	0.23%	0.08%	0.02%	**0.24%**		**0.19%**	
		PRE_A	1.77%	1.65%	0.92%	1.22%		**1.39%**		**0.73%**
	$\lvert I\rvert = 200$	$PRE_A^{'}$	1.39%	1.43%	1.33%	N.A.	**1.38%**			
$\lvert J\rvert = 2$	$\lvert I\rvert = 10$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.11%	0.09%	0.12%	0.10%		**0.11%**		
	$\lvert I\rvert = 20$	PRE_E	0.91%	0.01%	0.26%	0.38%	**0.39%**			
		PRE_A	1.43%	0.59%	0.84%	0.80%		**0.92%**		
	$\lvert I\rvert = 100$	PRE_E	1.55%	0.39%	0.40%	0.02%	**0.59%**		**0.33%**	
		PRE_A	2.30%	1.77%	1.60%	1.51%		**1.79%**		**0.94%**
	$\lvert I\rvert = 200$	$PRE_A^{'}$	1.31%	N.A.	N.A.	N.A.	**1.31%**			
$\lvert J\rvert = 5$	$\lvert I\rvert = 10$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.10%	0.09%	0.09%	0.11%		**0.10%**		
	$\lvert I\rvert = 20$	PRE_E	0.17%	0.55%	0.55%	0.00%	**0.32%**			
		PRE_A	0.96%	0.63%	0.47%	0.34%		**0.60%**		
	$\lvert I\rvert = 100$	PRE_E	1.35%	0.87%	0.40%	0.03%	**0.65%**		**0.32%**	
		PRE_A	2.51%	3.11%	1.40%	1.05%		**2.02%**		**0.90%**
	$\lvert I\rvert = 200$	$PRE_A^{'}$	N.A.	N.A.	N.A.	N.A.	**N.A.**			
$\lvert J\rvert = 10$	$\lvert I\rvert = 10$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.00%	0.00%	0.00%	0.00%		**0.00%**		
	$\lvert I\rvert = 20$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.66%	0.41%	0.52%	0.34%		**0.48%**		
	$\lvert I\rvert = 100$	PRE_E	2.35%	0.40%	0.61%	0.22%	**0.89%**		**0.300%**	
		PRE_A	3.52%	1.22%	1.40%	1.21%		**0.77%**		**0.00%**
	$\lvert I\rvert = 200$	$PRE_A^{'}$	1.48%	N.A.	N.A.	N.A.	**1.48%**			
		$\overline{\overline{PRE}}_E(K)$	**0.63%**	**0.24%**	**0.21%**	**0.06%**				

Table 2: *Cont.* The performance of the exact and the approximate algorithms (Weber problem with a non-horizontal line barrier).

Slope = $\pi/5$										
			$\|K\| = 5$	$\|K\| = 15$	$\|K\| = 25$	$\|K\| = 50$	$\overline{PRE}_E(I,J)$	$\overline{PRE}_A(I,J)$	$\overline{\overline{PRE}}_E(J)$	$\overline{\overline{PRE}}_A(J)$
$\|J\| = 1$	$\|I\| = 10$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.08%	0.08%	0.12%	0.10%		**0.10%**		
	$\|I\| = 20$	PRE_E	0.80%	0.44%	0.17%	0.03%	**0.36%**			
		PRE_A	1.17%	0.88%	0.56%	0.34%		**0.74%**		
	$\|I\| = 100$	PRE_E	2.67%	0.49%	0.08%	0.02%	**0.82%**		**0.39%**	
		PRE_A	3.61%	1.13%	0.74%	1.32%		**1.70%**		**0.84%**
	$\|I\| = 200$	PRE_A'	1.16%	1.36%	1.12%	N.A.	**1.21%**			
$\|J\| = 2$	$\|I\| = 10$	PRE_E	0.22%	0.75%	0.00%	0.22%	**0.30%**			
		PRE_A	0.30%	0.84%	0.12%	0.30%		**0.39%**		
	$\|I\| = 20$	PRE_E	1.10%	0.34%	0.75%	0.48%	**0.67%**			
		PRE_A	1.75%	0.99%	1.18%	1.07%		**1.25%**		
	$\|I\| = 100$	PRE_E	0.16%	1.13%	0.88%	0.02%	**0.55%**		**0.50%**	
		PRE_A	1.44%	2.36%	2.13%	1.05%		**1.74%**		**1.13%**
	$\|I\| = 200$	PRE_A'	N.A.	0.73%	1.38%	N.A.	**1.06%**			
$\|J\| = 5$	$\|I\| = 10$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.08%	0.09%	0.10%	0.08%		**0.09%**		
	$\|I\| = 20$	PRE_E	0.16%	1.34%	1.08%	0.08%	**0.66%**			
		PRE_A	0.60%	1.85%	1.53%	0.76%		**1.19%**		
	$\|I\| = 100$	PRE_E	0.00%	2.56%	0.00%	0.27%	**0.71%**		**0.46%**	
		PRE_A	0.81%	3.96%	0.94%	1.70%		**1.85%**		**1.04%**
	$\|I\| = 200$	PRE_A'	N.A.	1.05%	1.25%	N.A.	**1.15%**			
$\|J\| = 10$	$\|I\| = 10$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.00%	0.00%	0.00%	0.00%		**0.00%**		
	$\|I\| = 20$	PRE_E	0.00%	0.00%	0.00%	0.00%	**0.00%**			
		PRE_A	0.42%	0.54%	0.55%	0.68%		**0.55%**		
	$\|I\| = 100$	PRE_E	0.00%	0.23%	0.12%	0.11%	**0.12%**		**0.04%**	
		PRE_A	1.12%	1.28%	1.24%	1.04%		**1.17%**		**0.57%**
	$\|I\| = 200$	PRE_A'	N.A.	1.30%	1.07%	N.A.	**1.19%**			
		$\overline{\overline{PRE}}_E(K)$	**0.43%**	**0.61%**	**0.26%**	**0.10%**				

Table 3: The performance of the approximate algorithms (Weber problem with a horizontal line barrier).

		$\lvert K\rvert = 2$	$\lvert K\rvert = 5$	$\lvert K\rvert = 10$	$\lvert K\rvert = 15$	$\lvert K\rvert = 20$	$\lvert K\rvert = 25$	$\lvert K\rvert = 50$	$\overline{PRE}'_A(I,J)$	$\overline{\overline{PRE}}'_A(J)$
$\lvert J\rvert = 1$	$\lvert I\rvert = 200$	black0.19%	black0.10%	black0.00%	black0.00%	black0.00%	black0.01%	black0.00%	black0.04%	
	$\lvert I\rvert = 250$	black0.16%	black0.09%	black0.09%	black0.00%	black0.00%	black0.00%	black0.00%	black0.05%	
	$\lvert I\rvert = 500$	black0.25%	black0.19%	black0.10%	black0.00%	black0.03%	black0.00%	black0.00%	black0.08%	
	$\lvert I\rvert = 1000$	black0.30%	black0.15%	black0.01%	black0.01%	N.A.	N.A.	N.A.	black0.12%	black**0.07%**
$\lvert J\rvert = 2$	$\lvert I\rvert = 200$	N.A.	N.A.	0.00%	0.12%	0.11%	0.13%	0.00%	0.07%	
	$\lvert I\rvert = 250$	N.A.	N.A.	0.00%	0.00%	0.05%	N.A.	N.A.	0.02%	
	$\lvert I\rvert = 500$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	
	$\lvert I\rvert = 1000$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	**0.04%**
$\lvert J\rvert = 3$	$\lvert I\rvert = 200$	N.A.	0.02%	0.00%	0.01%	0.02%	0.00%	0.00%	0.01%	
	$\lvert I\rvert = 250$	N.A.	0.00%	0.00%	0.00%	0.00%	N.A.	N.A.	0.00%	
	$\lvert I\rvert = 500$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	
	$\lvert I\rvert = 1000$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	**0.01%**
$\lvert J\rvert = 5$	$\lvert I\rvert = 200$	N.A.	0.00%	0.00%	0.02%	0.00%	0.00%	0.01%	0.00%	
	$\lvert I\rvert = 250$	N.A.	0.00%	N.A.	0.00%	0.00%	N.A.	N.A.	0.00%	
	$\lvert I\rvert = 500$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	
	$\lvert I\rvert = 1000$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	**0.00%**
$\lvert J\rvert = 10$	$\lvert I\rvert = 200$	N.A.	0.00%	0.00%	0.00%	0.00%	0.74%	0.00%	0.11%	
	$\lvert I\rvert = 250$	N.A.	0.00%	0.00%	0.00%	0.00%	N.A.	N.A.	0.00%	
	$\lvert I\rvert = 500$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	
	$\lvert I\rvert = 1000$	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.	**0.05%**
	$\overline{PRE}'_A(K)$	black0.22%	black0.05%	black0.01%	black0.01%	black0.02%	black0.13%	black0.00%	black0.05%	black***0.03%***

N.A.: No Answer.

Table 4: Average computation time (Weber problem with a horizontal line barrier).

		$\lvert J\rvert = 1$	$\lvert J\rvert = 2$	$\lvert J\rvert = 3$	$\lvert J\rvert = 5$	$\lvert J\rvert = 10$	Average
$\lvert I\rvert = 10$	p-median	black9.0	0.1	0.0	0.0	0.0	black1.8
	EDC-ALA	0.3	2.6	3.7	4.7	6.0	3.5
	ADC-ALA	0.3	1.8	2.7	3.3	4.1	2.4
$\lvert I\rvert = 20$	p-median	black9.1	0.5	0.4	0.4	0.2	black 2.1
	EDC-ALA	0.5	5.0	7.3	9.1	11.9	6.8
	ADC-ALA	0.3	2.0	2.9	4.1	5.7	3.0
$\lvert I\rvert = 50$	p-median	black16.6	44.9	37.1	29.7	21.9	black30.0
	EDC-ALA	0.5	17.3	17.1	24.7	27.7	17.5
	ADC-ALA	0.4	3.7	4.4	5.8	7.8	4.4
$\lvert I\rvert = 100$	p-median	black62.7	279.1	232.6	724.3	365.4	black332.8
	EDC-ALA	516.6	618.2	145.3	88.4	52	284.1
	ADC-ALA	0.5	7.3	7.9	9.6	12.3	7.5
$\lvert I\rvert = 200$	p-median	black344	3600	3600	3600	3600	black2949
	EDC-ALA	1548.6	2100.8	1857.2	1633.9	1196.9	1667.5
	ADC-ALA	0.6	12.1	15.4	19.4	19.1	13.3
$\lvert I\rvert = 250$	p-median	black573	3600	3600	3600	3600	black2995
	EDC-ALA	1557	3457.2	3600	3053.9	2569.5	2847.5
	ADC-ALA	0.6	19	23.2	22.9	24.5	18.0
$\lvert I\rvert = 500$	p-median	black1356	3600	3600	3600	3600	black3151
	EDC-ALA	2897.1	3600	3600	3600	3600	3459.4
	ADC-ALA	0.8	41.5	47.5	57	56.9	40.7
$\lvert I\rvert = 1000$	p-median	3600	3600	3600	3600	3600	3600
	EDC-ALA	3600	3600	3600	3600	3600	3600
	ADC-ALA	1.2	76.7	85.3	117.7	134.1	83.0

Appendix

Table 1: Summary of EDC for $SFWP_K$.

Step 1. Decompose the solution space into convex sub-spaces, following equations (9) – (14).

Step 2. Calculate distances, using equations (15) – (16).

Step 3. For each constructed sub-space,

- Step 3.1. Identify the relevant *1-visible* points using equation (17).
- Step 3.2. Enumerate all possible combinations of relevant *1-shadow* points and the associated passages.
- Step 3.3. For each possible combination,
  - Step 3.3.1. Assign the weight of corresponding *1-shadow* points to the associated passages according to equations (20) – (21).
  - Step 3.3.2. Solve an unconstrained single-facility Weber problem (23) with the set of *1-visible* points and the associated passage points using the weights allocated in step 3.3.1.
- Step 3.4. Save the solution that returns the minimum objective function value over all possible combinations.

Step 4 Report the solution that returns the minimum value of the objective function over all constructed sub-spaces.

Table 2: Summary of ADC for $SFWP_K$.

Step 1. Decompose the solution space into convex sub-spaces, following equations (9) – (14).

Step 2. Calculate distances, using equations (15) – (16).

Step 3. For each constructed sub-space,

Step 3.1. Identify the relevant *1-visible* points using equation (17).

Step 3.2. Replace the relevant *1-shadow* points by their corresponding *median* point. Such points are assigned weights given by $\hat{w}^{ml} = \sum_{i:X_i \in H^{ml}} w_i$.

Step 3.3. Enumerate possible combinations between the generated *median* point and the associated passages.

Step 3.4. For each possible combination,

Step 3.4.1. Assign the weight of the *median* point to the associated passages according to the equations (27) – (28).

Step 3.4.2. Solve an unconstrained single-facility Weber problem (30) with *1-visible* points and the associated passage points using the weights allocated in 3.4.1.

Step 3.5. Save the solution that returns the minimum objective function value over all possible combinations.

Step 4 Report the solution that returns the minimum value of the objective function over all constructed sub-spaces.

Table 3: Summary of ALA for $MFWP_K$.

Step 1. Provide initial random locations for the new facilities to the $MFWP_K$ problem (5) - (8).

Step 2. Given the location of new facilities, solve the SPSP (31) subject to (6) and (8)

Step 3. With the given location of new facilities and their corresponding allocated existing facilities, decompose the $MFWP_K$ to a number of $SFWP_K$ problems, (32)–(33).

Step 4. For each new facility,

Step 3.1. Solve the generated $SFWP_K$s by either the EDC or ADC algorithms.

Step 4 If the termination criterion is met, Stop; otherwise, go to Step 2.

References

Akyüz, M. H. (2017). The capacitated multi-facility weber problem with polyhedral barriers: Efficient heuristic methods. *Computers & Industrial Engineering*, 113:221–240.

Aneja, Y. P. and Parlar, M. (1994). Algorithms for weber facility location in the presence of forbidden regions and/or barriers to travel. *Transportation Science*, 28(1):70–76.

Bischoff, M., Fleischmann, T., and Klamroth, K. (2009). The multi-facility location–allocation problem with polyhedral barriers. *Computers & Operations Research*, 36(5):1376–1392.

Bischoff, M. and Klamroth, K. (2007). An efficient solution method for weber problems with barriers based on genetic algorithms. *European Journal of Operational Research*, 177(1):22–41.

Brimberg, J. and Salhi, S. (2005). A continuous location-allocation problem with zone-dependent fixed cost. *Annals of Operations Research*, 136(1):99–115.

Butt, S. E. and Cavalier, T. M. (1996). An efficient algorithm for facility location in the presence of forbidden regions. *European Journal of Operational Research*, 90(1):56–70.

Canbolat, M. S. and Wesolowsky, G. O. (2010). The rectilinear distance weber problem in the presence of a probabilistic line barrier. *European Journal of Operational Research*, 202(1):114–121.

Canbolat, M. S. and Wesolowsky, G. O. (2012). On the use of the varignon frame for single facility weber problems in the presence of convex barriers. *European Journal of Operational Research*, 217(2):241–247.

Cooper, L. (1963). Location-allocation problems. *Operations research*, 11(3):331–343.

Daskin, M. S. (2011). *Network and discrete location: models, algorithms, and applications*. John Wiley & Sons.

Francis, R. L., White, J. A., and McGinnis, L. F. (1992). *Facility layout and location: an analytical approach*. Prentice-Hall Englewood Cliffs, NJ.

Gharravi, H. G. (2013). Rectilinear interdiction problem by locating a line barrier. Master's thesis, Middle East Technical University.

Hamacher, H. W. (1995). *Mathematische Lösungsverfahren für planare Standortprobleme*, volume 246. Springer.

Javadian, N., Tavakkoli-Moghaddam, R., Amiri-Aref, M., and Shiripour, S. (2014). Two metaheuristics for a multi-period minisum location–relocation problem with line restriction. *The International Journal of Advanced Manufacturing Technology*, 71(5-8):1033–1048.

Katz, I. N. and Cooper, L. (1981). Facility location in the presence of forbidden regions, i: Formulation and the case of euclidean distance with one forbidden circle. *European Journal of Operational Research*, 6(2):166–173.

Kelachankuttu, H., Batta, R., and Nagi, R. (2007). Contour line construction for a new rectangular facility in an existing layout with rectangular departments. *European Journal of Operational Research*, 180(1):149–162.

Klamroth, K. (2001). Planar weber location problems with line barriers. *Optimization*, 49(5-6):517–527.

Klamroth, K. (2006). *Single-facility location problems with barriers*. Springer Science & Business Media.

Klamroth, K. and Wiecek, M. M. (2002). A bi-objective median location problem with a line barrier. *Operations Research*, 50(4):670–679.

Larson, R. C. and Sadiq, G. (1983). Facility locations with the manhattan metric in the presence of barriers to travel. *Operations Research*, 31(4):652–669.

Mahmud, T. M. T. (2013). *Simulated Annealing Approach in Solving the Minimax Problem with Fixed Line Barrier*. PhD thesis, Universiti Teknologi Malaysia.

McGarvey, R. G. and Cavalier, T. M. (2003). A global optimal approach to facility location in the presence of forbidden regions. *Computers & industrial engineering*, 45(1):1–15.

Oğuz, M., Bektaş, T., and Bennell, J. A. (2018). Multicommodity flows and benders decomposition for restricted continuous location problems. *European Journal of Operational Research*, 266(3):851–863.

Sarkar, A., Batta, R., and Nagi, R. (2007). Placing a finite size facility with a center objective on a rectangular plane with barriers. *European Journal of Operational Research*, 179(3):1160–1176.

Shiripour, S., Mahdavi, I., Amiri-Aref, M., Mohammadnia-Otaghsara, M., and Mahdavi-Amiri, N. (2012). Multi-facility location problems in the presence of a probabilistic line barrier: a mixed integer quadratic programming model. *International Journal of Production Research*, 50(15):3988–4008.