G. Kokarev (2021) "Berger's Inequality in the Presence of Upper Sectional Curvature Bound,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–42
https://doi.org/10.1093/imrn/rnab073

Berger's Inequality in the Presence of Upper Sectional Curvature Bound

Gerasim Kokarev*

School of Mathematics, The University of Leeds, Leeds LS2 9JT, UK

**Correspondence to be sent to: e-mail: G.Kokarev@leeds.ac.uk*

We obtain inequalities for all Laplace eigenvalues of Riemannian manifolds with an upper sectional curvature bound, whose rudiment version for the 1st Laplace eigenvalue was discovered by Berger in 1979. We show that our inequalities continue to hold for conformal metrics, and moreover, extend naturally to minimal submanifolds. In addition, we obtain explicit upper bounds for Laplace eigenvalues of minimal submanifolds in terms of geometric quantities of the ambient space.

1 Statement and Discussion of Results

1.1 Introduction

Let (M, g) be a closed Riemannian manifold of dimension m and $\mathrm{inj}(g)$ its injectivity radius. A classical result by Berger [5] in 1979 says that for every $0 < r < \mathrm{inj}(g)$, there exists a point $p \in M$ such that the 1st nonzero Dirichlet eigenvalue of a geodesic ball $B(p, r)$ in M satisfies the inequality

$$\lambda_0(B(p,r)) \leqslant C_1(m)\frac{Vol_g(M)}{r^{m+2}},$$

where $C_1(m)$ is a positive constant that depends on the dimension m only. He uses this inequality to obtain the following upper bound for the 1st nonzero Laplace eigenvalue of M.

Received March 08, 2020; Revised March 03, 2021; Accepted March 04, 2021
Communicated by Prof. Toby Colding

Berger's inequality. Let (M,g) be a closed Riemannian manifold that admits an involutive isometry without fixed points. Then, its 1st nonzero Laplace eigenvalue satisfies the inequality

$$\lambda_1(g) \leqslant C_2(m)\frac{Vol_g(M)}{\mathrm{inj}(g)^{m+2}}, \tag{1.1}$$

where $C_2(m)$ is a constant that depends on the dimension m only.

Berger [5] asks under what other geometric hypotheses on M inequality (1.1) may hold. The 1st answers are given by Bérard and Besson [4], who show that this inequality holds for homogeneous Riemannian manifolds and locally harmonic spaces. In a seminal paper, Croke [17] proves, among other results, a version of inequality (1.1) that uses the convexity radius $\mathrm{conv}(g)$ instead of $\mathrm{inj}(g)$ and holds for *arbitrary* closed Riemannian manifolds. More precisely, he shows that

$$\lambda_1(g) \leqslant C_3(m)\frac{Vol_g(M)^2}{\mathrm{conv}(g)^{2m+2}}. \tag{1.2}$$

The argument in [17] uses a slightly different (to the one above) estimate for the 1st nonzero Dirichlet eigenvalue of geodesic balls, which actually yields inequalities for all Laplace eigenvalues

$$\lambda_k(g) \leqslant C_3(m)\frac{Vol_g(M)^2}{\mathrm{conv}(g)^{2m+2}}k^{2m}, \tag{1.3}$$

where $k \geqslant 1$ is an arbitrary integer. These inequalities for higher eigenvalues do not seem to appear in the literature, and we refer to Appendix A for related details.

The purpose of this paper is to prove a version of the Berger inequality (1.1) for all Laplace eigenvalues of Riemannian manifolds with an upper sectional curvature bound and their minimal submanifolds. For example, we show that inequality (1.1), as well as its neat version for higher Laplace eigenvalues, holds for manifolds of nonpositive sectional curvature. More importantly, we show that these eigenvalue inequalities are *conformal* in nature, that is, the ratio $Vol_g(M)/\mathrm{inj}(g)^m$ controls Laplace eigenvalues of all metrics conformal to g. We also discover another interesting feature of these inequalities—they are naturally *inherited my minimal submanifolds* in M. Below, we discuss the results in detail.

1.2 Conformal nature of the Berger inequality

Let (M,g) be a closed m-dimensional Riemannian manifold whose sectional curvatures are not greater than $\delta \geqslant 0$. By $\mathrm{rad}(g)$, we denote the quantity $\min\{\mathrm{inj}(g), \pi/(2\sqrt{\delta})\}$. When $\delta = 0$, we always assume that $\pi/(2\sqrt{\delta})$ equals $+\infty$, and hence, $\mathrm{rad}(g)$ coincides with the injectivity radius $\mathrm{inj}(g)$. Further, we denote by

$$0 = \lambda_0(g) < \lambda_1(g) \leqslant \lambda_2(g) \leqslant \ldots \leqslant \lambda_k(g) \leqslant \ldots$$

the Laplace eigenvalues of a metric g on M repeated with respect to multiplicity. Our 1st result gives the following conformal bounds for all Laplace eigenvalues.

Theorem 1.1. Let (M,g) be a closed Riemannian manifold whose sectional curvatures are not greater than $\delta \geqslant 0$. Then, for any Riemannian metric $\tilde{g}$ conformal to g its Laplace eigenvalues satisfy the inequalities

$$\lambda_k(\tilde{g}) Vol_{\tilde{g}}(M)^{2/m} \leqslant C_4(m) \left(\frac{Vol_g(M)}{\mathrm{rad}(g)^m}\right)^{1+2/m} k^{2/m}$$

for any $k \geqslant 1$, where $\mathrm{rad}(g)$ equals $\min\{\mathrm{inj}(g), \pi/(2\sqrt{\delta})\}$ and $C_3(m)$ is a positive constant that depends on the dimension m of M only. In particular, the Laplace eigenvalues of the metric g satisfy the inequalities

$$\lambda_k(g) \leqslant C_4(m) \frac{Vol_g(M)}{\mathrm{rad}(g)^{m+2}} k^{2/m} \tag{1.4}$$

for any $k \geqslant 1$.

Note that under the hypotheses of Theorem 1.1, even the inequality for the 1st nonzero Laplace eigenvalue in (1.4) seems to be absent in the literature. When the sectional curvatures of M are nonpositive, inequalities (1.4) give a neat generalisation of the Berger inequality and improve Croke's inequalities (1.3). To our knowledge, it is unknown whether in the absence of a curvature hypothesis the power k^{2m} in inequalities (1.3) can be replaced by the asymptotically sharp power $k^{2/m}$.

Recall that a celebrated result by Korevaar [23] says that for any closed m-dimensional Riemannian manifold $(M,\tilde{g})$ its Laplace eigenvalues satisfy the inequalities

$$\lambda_k(\tilde{g})Vol_{\tilde{g}}(M)^{2/m} \leqslant Ck^{2/m},$$

where C is the constant that depends on the conformal class of a metric $\tilde{g}$ in a rather implicit way. Thus, Theorem 1.1 can be viewed as an explicit version of Korevaar's result that describes the dependance of the constant on the geometry of a background metric g in a given conformal class. Upper bounds for Laplace eigenvalues in terms of other conformal invariants can be also found in [21, 25]. Using the Weyl law

$$\lambda_k(g)Vol_g(M)^{2/m} \sim \frac{4\pi^2}{\omega_m^{2/m}}k^{2/m} \qquad \text{as } k \to +\infty,$$

where ω_m is the volume of a unit ball in the m-dimensional Euclidean space, we may pass to the limit as $k \to +\infty$ in the inequalities in Theorem 1.1 to obtain that $Vol_g(M) \geqslant C_4(m)\mathrm{rad}(g)^m$. This inequality is well known: it is a consequence of standard volume comparison theorems and is reminiscent to Berger's isoembolic inequality [6]; see the discussion in Section 2. Thus, the collection of inequalities (1.4) can be viewed as a quantised version of the classical geometric inequality.

The proof of Theorem 1.1 builds on the results from [16, 19] and [21]. The key ingredient is a construction of disjoint sets whose measure is carefully controlled by our geometric hypotheses. Though similar ideas, originating in the work by Buser [8] and Korevaar [23], have been used in a few papers recently, see for example [21, 22, 24, 25], our hypotheses are rather different from the previous work. In particular, we do not use a lower Ricci curvature bound for a background or auxiliary metric, which is so essential in most of the past papers. Our argument is based on the revision of recently developed techniques that allows to obtain a rather neat control of constants in the estimates for the measure of disjoint sets.

1.3 The Berger inequality for minimal submanifolds

Now, we consider closed Riemannian manifolds (Σ^n, g_Σ) that can be isometrically immersed into (M, g) as minimal submanifolds. In the sequel, we might endow such a manifold Σ^n with another metric h and denote by

$$0 = \lambda_0(\Sigma^n, h) < \lambda_1(\Sigma^n, h) \leqslant \lambda_2(\Sigma^n, h) \leqslant \ldots \leqslant \lambda_k(\Sigma^n, h) \leqslant \ldots$$

its Laplace eigenvalues, repeated with respect to multiplicity. Our next result shows that conformal eigenvalue bounds in Theorem 1.1 extend naturally to minimal submanifolds $\Sigma^n \subset M$.

Theorem 1.2. Let (M,g) be a closed Riemannian manifold whose sectional curvatures are not greater than $\delta \geqslant 0$ and $\Sigma^n \subset M$ a closed immersed minimal submanifold of dimension n. Then, for any Riemannian metric h on Σ^n conformal to g_Σ its Laplace eigenvalues satisfy the inequalities

$$\lambda_k(\Sigma^n, h) Vol_h(\Sigma^n)^{2/n} \leqslant C_6(n) \left(\frac{Vol_g(\Sigma^n)}{\mathrm{rad}(g)^n} \right)^{1+2/n} k^{2/n}$$

for any $k \geqslant 1$, where $\mathrm{rad}(g)$ is the ambient quantity $\min\{\mathrm{inj}(g), \pi/(2\sqrt{\delta})\}$ and $C_6(n)$ is a positive constant that depends on the dimension n only. In particular, the Laplace eigenvalues of the metric g_Σ satisfy the inequalities

$$\lambda_k(\Sigma^n, g_\Sigma) \leqslant C_6(n) \frac{Vol_g(\Sigma^n)}{\mathrm{rad}(g)^{n+2}} k^{2/n} \tag{1.5}$$

for any $k \geqslant 1$.

Similar to the discussion after Theorem 1.1, we note that even the inequality for the 1st nonzero Laplace eigenvalue in (1.5) is new. Passing to the limit as $k \to +\infty$ in inequalities (1.5), we obtain the lower bound for

$$Vol_g(\Sigma^n) \geqslant C_6(n) \mathrm{rad}(g)^n \tag{1.6}$$

the volume of an immersed minimal submanifold Σ^n. This geometric inequality can be independently obtained from comparison monotonicity theorems for minimal submanifolds; see the discussion in Section 2. When the sectional curvatures of M are nonpositive, inequality (1.6) can be already derived from the work of Anderson [2]. When the upper bound δ for sectional curvatures of M is positive, to our knowledge, it is unknown whether the quantity used by Anderson is monotonic; see [20] for a related discussion. For this case, we prove monotonicity of a different quantity, which might be of independent interest. These monotonicity theorems yield two-sided volume bounds for the volumes of extrinsic balls and play a crucial role in the proof of Theorem 1.2.

Theorem 1.2 can be extended to the case when M is complete but not necessarily compact. If the injectivity radius $\mathrm{inj}(g)$ of M is positive, then the statement of

Theorem 1.2 continues to hold for closed minimal submanifolds $\Sigma^n \subset M$. If $\mathrm{inj}(g) = 0$, then the injectivity radius in the formula for $\mathrm{rad}(g)$ should be replaced by the quantity $\inf\{\mathrm{inj}_p(g) : p \in \Sigma^n\}$. If Σ^n is not closed, then one can consider boundary value problems for domains $\Omega \subset \Sigma^n$. In this case, the statement of Theorem 1.2 is amenable to extensions to the Neumann eigenvalue problem. Below, we give a sample version of such a result. For the sake of simplicity, we assume that the ambient manifold M is a Cartan–Hadamard space, that is, a complete simply connected space with nonpositive sectional curvatures. First, we introduce more notation.

Let Σ^n be a complete minimal submanifold in a Cartan–Hadamard space M. By the monotonicity theorem of Anderson [2], the ratio $Vol(B(p,r) \cap \Sigma^n)/(\omega_n r^n)$ is a nondecreasing function of $r > 0$, where $B(p,r)$ is a ball of radius r in M and ω_n is the volume of a unit ball in the Euclidean space $\mathbb{R}^n$. By $\theta(\Sigma^n)$, we denote the (possibly infinite) quantity

$$\theta(\Sigma^n) = \lim_{r\to+\infty} \frac{Vol_g(B(p,r) \cap \Sigma^n)}{\omega_n r^n};$$

it does not depend on a reference point $p \in M$ and is called the *density at infinity* of Σ^n. We have the following version of Theorem 1.2.

Theorem 1.3. Let (M,g) be a Cartan–Hadamard manifold and $\Sigma^n \subset M$ a complete properly immersed minimal submanifold. Then, for any precompact domain $\Omega \subset \Sigma^n$ and any Riemannian metric h on Ω conformal to g_Σ its Neumann eigenvalues satisfy the inequalities

$$\lambda_k(\Omega,h)Vol_h(\Omega)^{2/n} \leqslant C_7(n)\theta(\Sigma^n)^{1+2/n}k^{2/n}$$

for any $k \geqslant 1$, where $C_7(n)$ is a positive constant that depends on the dimension n only.

We end this discussion on the Neumann problem with the following two remarks. First, when M is a Euclidean space $\mathbb{R}^m$, there are many examples when $\theta(\Sigma^n)$ is finite—this is always the case when Σ^n has finite total curvature; see the discussion in [30]. More precisely, by the classical results of Osserman [31, 32], Chern and Osserman [12], and Anderson [3], such manifolds have finite topological type, that is, they are diffeomorphic to smooth compact manifolds with finitely many points removed. These points correspond to the ends of a minimal submanifold Σ^n, and the density at infinity $\theta(\Sigma^n)$ coincides with their number counted with multiplicity. When $n \geqslant 3$, by [3], each end of Σ^n is embedded and its multiplicity equals one. In other words, when $n \geqslant 3$, the density at infinity of such a minimal submanifolds is precisely the number of ends. Thus, Theorem 1.3 yields *topological eigenvalue bounds* for domains in minimal submanifolds $\Sigma^n \subset \mathbb{R}^m$ of finite total curvature.

Second, to our knowledge, no upper bounds for Neumann eigenvalues of domains in minimal submanifolds $\Sigma^n \subset \mathbb{R}^m$ is known until now, unless Σ^n is an affine subspace. The situation is in contrast with the Dirichlet problem, where (in this case more natural lower) bounds for the Dirichlet eigenvalues have been known since 1984; see [10, 29]. Thus, Theorem 1.3 gives an answer to the question that appears to have been open for some time.

1.4 Ambient bounds for Laplace eigenvalues of minimal submanifolds

There is another version of Theorem 1.2 that leads to bounds for Laplace eigenvalues of minimal submanifolds in terms of geometry of the ambient space.

Theorem 1.4. Let (M, g) be a closed Riemannian manifold whose sectional curvatures are not greater than $\delta \geqslant 0$ and $\Sigma^n \subset M$ a closed immersed minimal submanifold of dimension n. Then, for any Riemannian metric h on Σ^n conformal to g_Σ, its Laplace eigenvalues satisfy the inequalities

$$\lambda_k(\Sigma^n, h) Vol_h(\Sigma^n)^{2/n} \leqslant C_4(m) \left(\frac{Vol_g(M)}{\mathrm{rad}(g)^{m+2}} \right) Vol_g(\Sigma^n)^{2/n} k^{2/n}$$

for any $k \geqslant 1$, where $\mathrm{rad}(g)$ is the ambient quantity $\min\{\mathrm{inj}(g), \pi/(2\sqrt{\delta})\}$ and $C_4(m)$ is the constant from Theorem 1.1. In particular, the Laplace eigenvalues of the metric g_Σ satisfy the inequalities

$$\lambda_k(\Sigma^n, g_\Sigma) \leqslant C_4(m) \frac{Vol_g(M)}{\mathrm{rad}(g)^{m+2}} k^{2/n} \tag{1.7}$$

for any $k \geqslant 1$.

We proceed with one more related result. It also gives eigenvalue bounds in terms of geometry of the ambient space but has an extra, more traditional hypothesis—we additionally assume that the Ricci curvature of the ambient space is bounded below.

Theorem 1.5. Let (M, g) be a closed Riemannian manifold whose sectional curvatures are not greater than $\delta \geqslant 0$, and Ricci curvature is bounded below, $Ricci \geqslant -(m-1)\kappa$, where $\kappa \geqslant 0$. Let $\Sigma^n \subset M$ be a closed immersed minimal submanifold of dimension n. Then, for any Riemannian metric h on Σ^n conformal to g_Σ, its Laplace eigenvalues satisfy the inequalities

$$\lambda_k(\Sigma^n, h) Vol_h(\Sigma^n)^{2/n} \leqslant C_8(m) \max\{\kappa, \mathrm{rad}(g)^{-2} k^{2/n}\} Vol_g(\Sigma^n)^{2/n}$$

for any $k \geqslant 1$, where $\mathrm{rad}(g)$ is the ambient quantity $\min\{\mathrm{inj}(g), \pi/(2\sqrt{\delta})\}$ and $C_8(m)$ is a positive constant that depends on the dimension m only. In particular, the Laplace eigenvalues of the metric g_Σ satisfy the inequalities

$$\lambda_k(\Sigma^n, g_\Sigma) \leqslant C_8(m) \max\{\kappa, \mathrm{rad}(g)^{-2} k^{2/n}\} \tag{1.8}$$

for any $k \geqslant 1$.

To our knowledge, Theorems 1.4 and 1.5 are the 1st results in the literature that give upper bounds for Laplace eigenvalues in terms of ambient geometry. Previously, spectral properties (mostly related to the 1st nonzero eigenvalue) of minimal submanifolds have been studied in rank one symmetric spaces only; see [18, 25, 27] and the references therein. Note also that any complex submanifold of a Kähler manifold is minimal, and hence, the theorems above yield eigenvalue bounds for all complex submanifolds in terms of geometry of the ambient Kähler manifold. It is extremely interesting to know whether such upper bounds for complex submanifolds can be extended to all Kähler metrics with cohomologous Kähler forms. For projective submanifolds, such results are obtained in [26].

Concerning lower bounds for minimal submanifolds, we mention the following result due to Cheng and Tysk [11]: for any closed minimal submanifold $\Sigma^n \subset M$, its Laplace eigenvalues satisfy the inequalities

$$C(n, M) k^{2/n} \leqslant \lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n}$$

for any $k \geqslant \bar{C}(n, M) Vol_g(\Sigma^n)$, where $C(n, M)$ and $\bar{C}(n, M)$ are positive constants that depend on the dimension n of Σ^n and the geometry of M in a rather implicit way. It is important to note that, in contrast with these lower bounds, the scale-invariant quantities $\lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n}$ *cannot be bounded above* in terms of the ambient geometry only. To see this, recall that by [13] for any so-called bumpy metric g on a closed ambient manifold M of dimension m, where $3 \leqslant m \leqslant 7$, there is a sequence of closed connected embedded minimal hypersurfaces $\{\Sigma_i^{m-1}\}$ whose volumes tend to $+\infty$. As is known [35, 36], bumpy metrics form a dense subset in the set of all metrics on M, and in particular, we may choose a bumpy metric g of positive Ricci curvature. Then, by the result of Choi and Wang [14], we conclude that

$$\lambda_1(\Sigma_i^{m-1}, g) Vol_g(\Sigma_i^{m-1})^{2/(m-1)} \geqslant C Vol_g(\Sigma_i^{m-1})^{2/(m-1)} \to +\infty \qquad \text{when} \quad i \to +\infty,$$

where $C > 0$ is a constant that depends on the lower bound for the Ricci curvature. Thus, no ambient upper bound for $\lambda_k(\Sigma^n, g) Vol_g(\Sigma^n)^{2/n}$ for any $k \geqslant 1$ may exist.

We end with a brief discussion of the following corollary of Theorem 1.5, which gives particularly simple estimates for Laplace eigenvalues of minimal submanifolds in certain positively curved spaces.

Corollary 1.1. Let (M, g) be a compact Riemannian manifold such that one of the following holds:

(i) either M is even-dimensional and its sectional curvatures satisfy the bounds

$$0 < K_p(\sigma) \leqslant \delta \qquad \text{for any plane } \sigma \in T_pM,$$

for any point $p \in M$;

(ii) or M is simply connected and has $\frac{1}{4}$-pinched sectional curvatures,

$$\frac{1}{4}\delta \leqslant K_p(\sigma) \leqslant \delta \qquad \text{for any plane } \sigma \in T_pM,$$

for any point $p \in M$, where $\delta > 0$.

Let $\Sigma^n \subset M$ be a closed immersed minimal submanifold. Then, for any Riemannian metric h on Σ^n conformal to g_Σ its Laplace eigenvalues satisfy the inequalities

$$\lambda_k(\Sigma^n, h) Vol_h(\Sigma^n)^{2/n} \leqslant C_9(m) \delta Vol_g(\Sigma^n)^{2/n} k^{2/n}$$

for any $k \geqslant 1$, where $C_9(m)$ is a positive constant that depends on the dimension m of M only. In particular, the Laplace eigenvalues of a metric g_Σ satisfy the inequalities

$$\lambda_k(\Sigma^n, g) \leqslant C_9(m) \delta k^{2/n}$$

for any $k \geqslant 1$.

Corollary 1.1 is a direct consequence of Theorem 1.5 and Klingenberg's bounds for the injectivity radius; see [1, 34]. The most significant difference between the two cases in it is the pinching condition that appears in the latter. Note that it imposes strong topological restrictions on M: assuming that M is simply connected, as in case (ii) of Corollary 1.1, it has to be diffeomorphic to a compact symmetric space of rank one; see [1, 7]. As examples with geodesics and minimal tori in the Berger spheres show, when M is odd-dimensional, this condition is essential for an upper bound for the Laplace eigenvalues. More generally, the statements above suggest that the relationship between the injectivity radius of M and Laplace eigenvalues of minimal submanifolds might be interesting on its own. In dimension one, it traces to the classical relationship between the injectivity radius and the lengths of closed geodesics; see [34].

1.5 Organisation of the paper

The paper is organised in the following way. In Section 2, we discuss volume comparison theorems and closely related volume monotonicity theorems for minimal submanifolds in Riemannian manifolds whose sectional curvatures are bounded above. In Section 3, we revisit the recent constructions, due to [16, 19, 21], of disjoint sets with controlled amount of measure in pseudo-metric spaces. The improvements obtained there are necessary for our main results. The proofs of Theorems 1.1– 1.5 are collected in Section 4. The arguments in all proofs follow the same strategy but use different ingredients from Sections 2 and 3. There is also a certain logical dependence between the proofs of different statements—in one of them, we may refer to the notation or argument used in another. The paper has a short appendix, where we prove inequalities (1.3), extending to higher Laplace eigenvalues the inequality for the 1st eigenvalue found by Croke [17] in 1980.

2 Preliminaries

2.1 Volume comparison and its consequences

Let (M, g) be a complete m-dimensional Riemannian manifold whose sectional curvatures are not greater than δ, where $\delta \in \mathbb{R}$. We start with recalling the background material on volume comparison theorems for such manifolds. First, we introduce the necessary notation. Below, by sn_δ, we denote the real-valued function given by the formula

$$\mathrm{sn}_\delta(t) = \begin{cases} (1/\sqrt{\delta})\sin(\sqrt{\delta}t), & \text{if } \delta > 0, \\ t, & \text{if } \delta = 0, \\ (1/\sqrt{|\delta|})\sinh(\sqrt{|\delta|}t), & \text{if } \delta < 0. \end{cases} \tag{2.1}$$

Then, for any $0 < r < \pi/\sqrt{\delta}$, we have the following relations for the volumes of a geodesic sphere and a geodesic ball of radii r in a simply connected m-dimensional space of constant sectional curvature δ:

$$A_\delta(r) = m\omega_m \mathrm{sn}_\delta^{m-1}(r), \qquad V_\delta(r) = m\omega_m \int_0^r \mathrm{sn}_\delta^{m-1}(t)\,\mathrm{d}t, \tag{2.2}$$

where ω_m is the volume of a unit ball in the m-dimensional Euclidean space. We always assume that $\pi/\sqrt{\delta} = +\infty$ when δ is nonpositive.

Let (t, ξ) be geodesic spherical coordinates around a point $p \in M$, where $t \in (0, \mathrm{inj}_p)$ and ξ is a unit vector in T_pM. Let $A_p(t, \xi)$ be the density of the volume measure in these coordinates, that is,

$$A_p(t, \xi) = t^{m-1} \det(D_{t\xi} \exp_p),$$

where $\exp_p : T_pM \to M$ is the exponential map; see [9]. Recall that the Günther–Bishop comparison theorem [9, Theorem III.4.1] says that the function

$$t \longmapsto \frac{A_p(t, \xi)}{\mathrm{sn}_\delta^{m-1}(t)}, \qquad \text{where} \quad 0 < t < \min\left\{\mathrm{inj}_p(g), \frac{\pi}{\sqrt{\delta}}\right\}$$

is nondecreasing for any unit vector $\xi \in T_pM$. The following statement is a consequence of this result, which does not seem to appear explicitly in the literature. For reader's convenience, we sketch a proof below.

Proposition 2.1. Let (M, g) be a complete m-dimensional Riemannian manifold whose sectional curvatures are not greater than δ, where $\delta \in \mathbb{R}$. Then, for any point $p \in M$, the function

$$r \longmapsto \frac{1}{V_\delta(r)} Vol_g(B(p, r)), \qquad \text{where} \quad 0 < r < \min\left\{\mathrm{inj}_p(g), \frac{\pi}{\sqrt{\delta}}\right\},$$

is nondecreasing. Besides, if it equals one at some value r, then the ball $B(p,r)$ is isometric to a ball of radius r in the space form of constant curvature δ.

Proof. Integrating the function $A_p(t,\xi)/\mathrm{sn}_\delta^{m-1}(t)$ over unit vectors $\xi \in T_pM$, by the Günther–Bishop theorem, we conclude that the function

$$t \longmapsto \frac{Area(S(p,t))}{A_\delta(t)}, \qquad \text{where} \quad 0 < t < \min\left\{\mathrm{inj}_p(g), \frac{\pi}{\sqrt{\delta}}\right\},$$

and $S(p,t)$ is a geodesic sphere of radius t, is nondecreasing. Now, note that if for positive real-valued functions $f(t)$ and $g(t)$ of one variable the ratio f/g is a nondecreasing function, then the ratio $\int_0^r f/\int_0^r g$ is also a nondecreasing function. Taking as $f(t)$ the function $Area(S(p,t))$, and as $g(t)$ the function $A_\delta(t)$, we arrive at the 1st statement of Proposition 2.1. The 2nd statement—the equality case—follows from the equality case in the standard volume comparison theorem [9, Theorem III.4.2]. ■

Recall that a classical result by Berger [6] says that for any closed m-dimensional Riemannian manifold the inequality

$$Vol_g(M) \geqslant (m+1)\omega_{m+1}(\mathrm{inj}(g)/\pi)^m$$

holds, and the equality occurs if and only if after rescaling M is isometric to the unit round sphere. As a direct consequence of the volume comparison theorems, we also have the comparison version of this result:

$$Vol_g(M) \geqslant V_\delta(\mathrm{rad}(g)), \tag{2.3}$$

where $\mathrm{rad}(g)$ is $\min\{\mathrm{inj}(g), \pi/(2\sqrt{\delta})\}$, and the function $V_\delta(\cdot)$ is given by the 2nd relation in (2.2). This is a sharper inequality, if $\delta \leqslant 0$. One can also characterise the case of equality—it occurs if and only if $\delta > 0$ and after scaling M is isometric to the unit round sphere.

For the sequel, we need the following consequence of the volume comparison theorems.

Corollary 2.2. Let (M,g) be a closed m-dimensional Riemannian manifold whose sectional curvatures are not greater than δ, where $\delta \geqslant 0$. Then, for any point $p \in M$

the volume of a geodesic ball $B(p,r)$ satisfies the inequalities

$$2^{1-m}\omega_m r^m \leqslant Vol_g(B(p,r)) \leqslant 2^{m-1}\frac{Vol_g(M)}{\mathrm{rad}(g)^m}r^m, \tag{2.4}$$

where $\mathrm{rad}(g)$ stands for $\min\{\mathrm{inj}(g), \pi/(2\sqrt{\delta})\}$, and $0 < r \leqslant \mathrm{rad}(g)$.

Proof. Indeed, by Proposition 2.1, we obtain

$$1 \leqslant \frac{Vol_g(B(p,r))}{V_\delta(r)} \leqslant \frac{Vol_g(B(p,\mathrm{rad}))}{V_\delta(\mathrm{rad})} \tag{2.5}$$

for any $0 < r \leqslant \mathrm{rad}(g)$, where $\mathrm{rad} = \mathrm{rad}(g)$. When $\delta = 0$, we have $V_\delta(r) = \omega_m r^m$, and the statement follows directly from (2.5). Now, suppose that $\delta > 0$. Then, from the inequalities

$$\frac{1}{2}t \leqslant \mathrm{sn}_\delta(t) \leqslant t \qquad \text{for any} \quad 0 \leqslant t \leqslant \frac{\pi}{2\sqrt{\delta}},$$

we obtain

$$\frac{1}{2^{m-1}}\omega_m r^m \leqslant V_\delta(r) \leqslant \omega_m r^m \qquad \text{for any} \quad 0 \leqslant r \leqslant \frac{\pi}{2\sqrt{\delta}}.$$

Combining the last relations with the inequalities in (2.5), we arrive at the statement of the corollary. ■

2.2 Monotonicity theorems for minimal submanifolds

Let Σ^n be an n-dimensional immersed minimal submanifold in a Riemannian manifold (M,g); we assume that the sectional curvatures of M are not greater than δ, where $\delta \in \mathbb{R}$. As above, we use the notation

$$A_\delta^{n-1}(r) = n\omega_n \mathrm{sn}_\delta^{n-1}(r), \qquad V_\delta^n(r) = n\omega_n \int_0^r \mathrm{sn}_\delta^{n-1}(t)\,\mathrm{d}t, \tag{2.6}$$

where $0 < r < \pi/\sqrt{\delta}$, for the volumes of a geodesic sphere and a geodesic ball of radii r in an n-dimensional space form of curvature δ.

The following *volume monotonicity* theorem can be viewed as an extension of Proposition 2.1 to minimal submanifolds. When $\delta \leqslant 0$, it is due to Anderson [2]. For $\delta > 0$, the statement appears to be new.

Proposition 2.3. Let (M, g) be a complete Riemannian manifold whose sectional curvatures are not greater than δ, where $\delta \in \mathbb{R}$, and let Σ^n be an n-dimensional properly immersed minimal submanifold in M. Then, for any point $p \in M$, the following holds:

(i) if $\delta \leqslant 0$, the function

$$r \longmapsto \frac{1}{V_\delta^n(r)} Vol_g(B(p,r) \cap \Sigma^n), \qquad \text{where} \quad 0 < r < \mathrm{inj}_p(g),$$

is nondecreasing;

(ii) if $\delta > 0$, the function

$$r \longmapsto \frac{1}{A_\delta^n(r)} Vol_g(B(p,r) \cap \Sigma^n), \qquad \text{where} \quad 0 < r < \min\left\{\mathrm{inj}_p(g), \frac{\pi}{\sqrt{\delta}}\right\},$$

is nondecreasing.

Remark 2.1. Under the hypotheses of Proposition 2.3, consider the case when $\delta > 0$. To our knowledge, the answer to the following question, also implicitly raised in [20], is unknown: is the function

$$r \longmapsto \frac{1}{V_\delta^n(r)} Vol_g(B(p,r) \cap \Sigma^n), \qquad \text{where} \quad 0 < r < \min\left\{\mathrm{inj}_p(g), \frac{\pi}{\sqrt{\delta}}\right\}$$

nondecreasing?

Proposition 2.3 immediately implies comparison inequalities for the volumes of extrinsic balls $B(p,r) \cap \Sigma^n$; see also [10, 29], where these inequalities are obtained from the heat kernel comparison theorems. In particular, we obtain volume bounds for any immersed minimal submanifold $\Sigma^n \subset M$; for example, if $\delta \leqslant 0$, then

$$Vol_g(\Sigma^n) \geqslant V_\delta^n(\mathrm{rad}(g)). \tag{2.7}$$

By the results in [10, 29], the above inequality continues to hold for the case $\delta > 0$ also, while Proposition 2.3 gives a weaker result in this case. Inequality (2.7) can be viewed as a version of comparison inequality (2.3), inherited by minimal submanifolds. Proposition 2.3 also implies the following version of Corollary 2.2.

Corollary 2.4. Let (M, g) be a complete Riemannian manifold whose sectional curvatures are not greater than δ, where $\delta \geqslant 0$, and let Σ^n be an n-dimensional immersed closed minimal submanifold in M. Then, for any point $p \in \Sigma^n$ the volume of an extrinsic ball $B(p,r) \cap \Sigma^n$ in Σ^n satisfies the inequalities

$$2^{-n} n\omega_n r^n \leqslant Vol_g(B(p,r) \cap \Sigma^n) \leqslant 2^n \frac{Vol_g(\Sigma^n)}{\mathrm{rad}(g)^n} r^n, \tag{2.8}$$

where $\mathrm{rad}(g)$ stands for $\min\{\mathrm{inj}(g), \pi/(2\sqrt{\delta})\}$, and $0 < r \leqslant \mathrm{rad}(g)$.

The proof of Corollary 2.4 follows the line of the argument in the proof of Corollary 2.2. The rest of the section is devoted to the proof of Proposition 2.3. Our argument borrows some observations from the exposition in [28], where the authors describe other monotonic quantities for the case $\delta \leqslant 0$. Let us also mention that for the case $\delta > 0$ a monotonic quantity different from the one in Proposition 2.3 is used in [20].

We start with a number of auxiliary lemmas. The 1st statement underlines the difference in the cases $\delta \leqslant 0$ and $\delta > 0$. Its proof is elementary and therefore is omitted.

Lemma 2.5. For any positive integer n the function $\mathrm{sn}_\delta(r)$, defined by (2.1), satisfies the following relations:

(i) if $\delta \leqslant 0$, then

$$(n-1)\mathrm{sn}'_\delta(r)\int_0^r \mathrm{sn}_\delta^{n-1}(t)\,\mathrm{d}t \leqslant \mathrm{sn}_\delta^n(r) \leqslant n\mathrm{sn}'_\delta(r)\int_0^r \mathrm{sn}_\delta^{n-1}(t)\,\mathrm{d}t$$

for any $r > 0$;

(ii) if $\delta > 0$, then

$$n\mathrm{sn}'_\delta(r)\int_0^r \mathrm{sn}_\delta^{n-1}(t)\,\mathrm{d}t \leqslant \mathrm{sn}_\delta^n(r)$$

for any $0 < r < \pi/\sqrt{\delta}$.

For the sequel, we need the following consequence of Lemma 2.5.

Corollary 2.6. For any positive integer n, the function $\alpha_\delta(r) = V_\delta^n(r)/A_\delta^{n-1}(r)$ is nondecreasing for any $\delta \in \mathbb{R}$. Moreover, it is concave for $\delta \leqslant 0$ and is convex for $\delta > 0$, where $0 < r < \pi/\sqrt{\delta}$.

Proof. Differentiating $\alpha_\delta(r)$, we obtain

$$\alpha'_\delta(r) = 1 - (n-1)\frac{\mathrm{sn}'_\delta}{\mathrm{sn}^n_\delta}(r)\int_0^r \mathrm{sn}_\delta^{n-1}(t)\,\mathrm{d}t, \tag{2.9}$$

and by Lemma 2.5, conclude that $\alpha'_\delta(r) \geqslant 0$. To prove the 2nd statement of the corollary it is sufficient to consider the cases when δ equals -1, 0, and 1. We give an argument for the case $\delta = 1$; the others are considered similarly. A direct computation gives

$$\alpha''_1(r) = -\frac{n-1}{(\sin r)^{n+1}}\left((\sin r)^n\cos r - (n-1)(\cos r)^2\int_0^r(\sin t)^{n-1}\,\mathrm{d}t - \int_0^r(\sin t)^{n-1}\,\mathrm{d}t\right).$$

Denote by $\omega(r)$ the expression in the brackets on the right-hand side; we claim that it is nonpositive, $\omega(r) \leqslant 0$. Computing its derivative, we obtain

$$\omega'(r) = 2\sin r\left(-(\sin r)^n + (n-1)\cos r\int_0^r(\sin t)^{n-1}\,\mathrm{d}t\right) \leqslant 0$$

for $0 < r < \pi$, where in the last inequality we used Lemma 2.5. Since $\omega(0) = 0$, we conclude that $\omega(r)$ is indeed nonpositive, and hence, the function $\alpha''_1(r)$ is nonnegative on the interval $(0, \pi)$. ■

We proceed with the following consequence of the Hessian comparison theorem.

Lemma 2.7. Under the hypotheses of Proposition 2.3, let $r(x)$ be a distance function $\mathrm{dist}(p, x)$ to a point $p \in M$ restricted to a minimal submanifold Σ^n. Then, the relation

$$\Delta_{\Sigma^n} r(x) \geqslant \frac{\mathrm{sn}'_\delta}{\mathrm{sn}_\delta}(r(x))(n - |\nabla r(x)|^2)$$

holds for any $x \in \Sigma^n$ such that $0 < r(x) < \min\{\mathrm{inj}_p(g), \pi/\sqrt{\delta}\}$.

Proof. Let ϕ be a smooth function on M and φ be its restriction to Σ^n. Note that $\mathrm{grad}_x\varphi$ is the tangential (lying in $T_x\Sigma^n$) component of $\mathrm{grad}_x\phi$ and a straightforward calculation shows that

$$\mathrm{Hess}_x\phi(X, X) = \mathrm{Hess}_x^{\Sigma^n}\varphi(X, X) - \langle \mathrm{grad}_x\phi, B_x(X, X)\rangle$$

for any vector $X \in T_x\Sigma^n$, where $B_x(\cdot,\cdot)$ is the 2nd fundamental form of Σ_n. As a consequence of this relation, we obtain the following identity for an arbitrary submanifold $\Sigma^n \subset M$:

$$\Delta_{\Sigma^n}\varphi(x) = \sum_{i=1}^{n} \mathrm{Hess}_x\phi(X_i, X_i) + \langle \mathrm{grad}_x\phi, H_x\rangle,$$

where H_x is the mean curvature vector of Σ^n at x and $\{X_i\}$ is an orthonormal basis of $T_x\Sigma^n$. Recall that the Hessian comparison theorem, see [34], says that

$$\mathrm{Hess}_x r(V,W) \geqslant \frac{\mathrm{sn}_\delta'}{\mathrm{sn}_\delta}(r(x))\,(\langle V,W\rangle - \langle(\partial/\partial r), V\rangle\langle(\partial/\partial r), W\rangle)$$

for any vectors V, $W \in T_xM$, where $0 < r(x) < \min\{\mathrm{inj}_x(g), \pi/\sqrt{\delta}\}$. Now, combining the last two relations together with the assumption that Σ^n is minimal, we arrive at the statement of the lemma. ■

Finally, we need the following well-known application of the co-area formula. We omit its proof and refer to [28, 33] where related details can be found.

Lemma 2.8. Under the hypotheses of Proposition 2.3, the function $V(r) = Vol_g(B(p,r)\cap \Sigma^n)$ is differentiable almost everywhere and $V'(r) \geqslant Vol_g(\partial B(p,r) \cap \Sigma^n)$.

Now, we proceed with a proof of Proposition 2.3.

Proof of Proposition 2.3. Consider the function

$$f(r) = \int_0^r V_\delta^n(t)/A_\delta^{n-1}(t)\,\mathrm{d}t, \qquad \text{where} \quad 0 < r < \pi/\sqrt{\delta}.$$

Note that it satisfies the relations

$$f''(r) + (n-1)\frac{\mathrm{sn}_\delta'}{\mathrm{sn}_\delta}(r)f'(r) = 1, \qquad f(0) = 0, \quad f'(0) = 0. \tag{2.10}$$

The 1st relation above is another form of relation (2.9). Define the function ψ on $B(p,r)\cap \Sigma^n$ by the formula $\psi(x) = f \circ r(x)$, where $r(x) = \mathrm{dist}(p,x)$. Computing the Laplacian of ψ, we obtain

$$\Delta_{\Sigma^n}\psi = f''(r)\,|\nabla r|^2 + f'(r)\Delta_{\Sigma^n} r \geqslant f''(r)\,|\nabla r|^2 + f'(r)\frac{\mathrm{sn}'_\delta}{\mathrm{sn}_\delta}(r)(n-|\nabla r|^2)$$
$$= 1 + (1-|\nabla r|^2)\left(f'(r)\frac{\mathrm{sn}'_\delta}{\mathrm{sn}_\delta}(r) - f''(r)\right), \quad (2.11)$$

where we used Lemma 2.7 in the inequality above and identity (2.10) in the last relation. The term in the brackets on the right-hand side can be rewritten in the form

$$f'(r)\frac{\mathrm{sn}'_\delta}{\mathrm{sn}_\delta}(r) - f''(r) = n\frac{\mathrm{sn}'_\delta}{\mathrm{sn}^n_\delta}(r)\int_0^r \mathrm{sn}^{n-1}_\delta(t)\,\mathrm{d}t - 1. \quad (2.12)$$

Now, we consider cases when $\delta \leqslant 0$ and $\delta > 0$ separately.

Case (i). When $\delta \leqslant 0$, by Lemma 2.5, we see that the quantity in (2.12) is nonnegative, and hence, by relation (11), we conclude that $\Delta_{\Sigma^n}\psi \geqslant 1$. Using the divergence theorem, we obtain

$$V(r) = Vol_g(B(p,r)\cap\Sigma^n) \leqslant \int\limits_{B_r\cap\Sigma^n} \Delta_{\Sigma^n}\psi\,\mathrm{d}Vol_g = \int\limits_{\partial B_r\cap\Sigma^n} \langle \mathrm{grad}\psi, \nu\rangle \leqslant f'(r)Vol_g(\partial B(p,r)\cap\Sigma^n),$$

where ν is a unit normal vector, and we used the relation $|\nabla r| \leqslant 1$ in the last inequality. Note that the use of the divergence theorem above is justified by the hypothesis that Σ^n is immersed properly in M. Now, by Lemma 2.8, we get

$$V(r) \leqslant f'(r)V'(r) = \frac{V^n_\delta(r)}{A^{n-1}_\delta(r)}V'(r).$$

The latter inequality is equivalent to the hypothesis that the ratio $V(r)/V^n_\delta(r)$ is a nondecreasing function of r, where $0 < r < \mathrm{inj}_p(g)$.

Case (ii). When $\delta > 0$, by Lemma 2.5, the quantity in (2.12) is nonpositive. Introducing the new notation

$$\epsilon_\delta(r) = 1 - n\frac{\mathrm{sn}'_\delta}{\mathrm{sn}^n_\delta}(r)\int_0^r \mathrm{sn}^{n-1}_\delta(t)\,\mathrm{d}t \geqslant 0,$$

we can rewrite relation (11) in the form

$$1 \leqslant \Delta_{\Sigma^n}\psi + (1-|\nabla r|^2)\epsilon_\delta(r) \leqslant \Delta_{\Sigma^n}\psi + \epsilon_\delta(r).$$

Further, using Corollary 2.6, one can conclude that $\epsilon_\delta(r)$ is a nondecreasing function as r ranges in the interval $(0, \pi/\sqrt{\delta})$. The latter can be seen as the consequence of the relation

$$\epsilon_\delta'(r) = \frac{n}{n-1}\alpha_\delta''(r);$$

see identity (2.9), where α_δ is a function from Corollary 2.6. This observation together with the argument in Case (i) above yields the inequality

$$V(r) \leqslant \frac{V_\delta^n(r)}{A_\delta^{n-1}(r)} V'(r) + \epsilon_\delta(r)V(r),$$

where $V(r)$ is the volume $Vol(B(p,r) \cap \Sigma^n)$. By the definition of $\epsilon_\delta(r)$, we obtain

$$nV(r)\frac{\mathrm{sn}_\delta'}{\mathrm{sn}_\delta^n}(r)\int_0^r \mathrm{sn}_\delta^{n-1}(t)\,\mathrm{d}t \leqslant V'(r)\frac{1}{\mathrm{sn}_\delta^{n-1}(r)}\int_0^r \mathrm{sn}_\delta^{n-1}(t)\,\mathrm{d}t,$$

where $0 < r < \pi/\sqrt{\delta}$. The latter is equivalent to

$$\frac{(\mathrm{sn}_\delta^n)'}{\mathrm{sn}_\delta^n}(r) = n\frac{\mathrm{sn}_\delta'}{\mathrm{sn}_\delta}(r) \leqslant \frac{V'(r)}{V(r)},$$

and we conclude that the ratio $V(r)/\mathrm{sn}_\delta^n(r)$ is nondecreasing. ∎

Remark 2.2. Note that in the course of the proof of Proposition 2.3, we established the following isoperimetric inequalities

$$\frac{A_\delta^{n-1}(r)}{V_\delta^n(r)} \leqslant \frac{Vol_g(\partial B(p,r) \cap \Sigma^n)}{Vol_g(B(p,r) \cap \Sigma^n)}, \qquad \text{when} \quad \delta \leqslant 0,$$

$$n\frac{\mathrm{sn}_\delta'}{\mathrm{sn}_\delta}(r) \leqslant \frac{Vol_g(\partial B(p,r) \cap \Sigma^n)}{Vol_g(B(p,r) \cap \Sigma^n)}, \qquad \text{when} \quad \delta > 0.$$

The 1st inequality has an explicit comparison flavour. Similar results are also obtained, by a different method, in [33], but under more restrictive hypotheses—the author assumes that the intersection $B(p,r) \cap \Sigma^n$ is connected and a point p lies in Σ^n.

3 Revisiting Constructions of Disjoint Sets in Metric Measure Spaces

3.1 Covers refinement functions

In this section, we revisit the so-called *decomposition theorems*, that is, the constructions of disjoint sets in pseudo-metric measure spaces with controlled amount of measure. Such results originate in the work of Buser [8] and Korevaar [23] and are essential for obtaining upper bounds for the whole spectrum; see, for example, [15, 16, 21, 22, 24–26]. The known constructions rely heavily on covering properties by balls of the underlying pseudo-metric space. For our applications, it is important to keep track of the bound for the radii of the balls in covers and record how refined covers are used. These considerations motivate the definitions below. Throughout this section, by (X, d), we denote a separable pseudo-metric space and $B(p, r)$ stands for an open ball $\{x \in X : d(p, x) < r\}$ in X.

Definition 3.1 (Small balls). A nondecreasing function $N : (1, +\infty) \to \mathbb{R}^+$ is called the *small cover refinement function* for a pseudo-metric space (X, d), if for any $\rho > 1$ each ball $B(p, r)$ with $0 < r \leqslant 1$ can be covered by at most $N(\rho)$ balls of radius r/ρ.

Definition 3.2 (Arbitrary balls). A nondecreasing function $N : (1, +\infty) \to \mathbb{R}^+$ is called the *cover refinement function*, if for any $\rho > 1$ each pseudo-metric ball $B(p, r)$ with $r > 0$ can be covered by at most $N(\rho)$ balls of radius r/ρ.

The distinction between considering covers of arbitrary balls and only small balls is important for our applications; see also [21, 22]. Note that if for some $\rho_0 > 1$ each pseudo-metric ball $B(p, r)$ with $0 < r \leqslant 1$ can be covered by N_0 balls with radius r/ρ_0, then each $B(p, r)$ can be covered by $N(\rho)$ balls with radius r/ρ for any $\rho > 1$; see [19, Lemma 3.4]. Moreover, the argument in the proof of [19, Lemma 3.4] shows that the number $N(\rho)$ of such balls in the covering can be chosen so that the function $\rho \mapsto N(\rho)$ is nondecreasing. In other words, if such a covering property holds for some $\rho_0 > 1$, then a small cover refinement function exists. Unlike many previous papers, see for example [19, 21, 22, 25] and the references therein, where the mere fact whether such a covering property holds for some $\rho_0 > 1$ was used, for our purposes the (small) cover refinement function itself is important. For these reasons, we restate and sharpen some of the key results from [19, 21]. First, we recall the necessary notation.

By an annulus A in a pseudo-metric space (X, d), we mean a subset of the following form:

$$\{x \in X : r \leqslant d(x, a) < R\},$$

where $a \in X$ and $0 \leqslant r < R < +\infty$. These real numbers r and R are often referred to as the *inner* and *outer* radii, respectively, and the point a—as the centre of an annulus A.

By $2A$, we denote the annulus

$$\{x \in X : r/2 \leqslant d(x,a) < 2R\}.$$

Recall that a measure μ on a pseudo-metric space (X,d) is called *nonatomic* if for any point $p \in X$ the mass $\mu(B(p,r)) \to 0$ as $r \to 0+$. When (X,d) is a metric space, this is equivalent to saying that the measure does not charge a single point in X.

The following statement follows by examining the proof of [19, Theorem 3.5]; it is stated in the form reminiscent to [19, Corollary 3.12].

Proposition 3.1. Let (X,d) be a separable pseudo-metric space such that all balls $B(p,r)$ are precompact and $N(\rho)$ a cover refinement function for it. Then, for any finite nonatomic measure μ on X and any positive integer k, there exists a collection of k annuli $\{A_i\}$ such that the annuli $\{2A_i\}$ are pair-wise disjoint and

$$\mu(A_i) \geqslant \mu(X)/(ck) \qquad \text{for any} \quad 1 \leqslant i \leqslant k,$$

where $c = 8N(1600)$.

Note that the existence of a cover refinement function is one of the hypotheses in Proposition 3.1. We also need a statement with the weaker hypothesis—the existence of a small cover refinement function. It can be obtained by revisiting [16, 21]. The following proposition is a sharpened version of [21, Theorem 2.1].

Proposition 3.2. Let (X,d) be a separable pseudo-metric space such that all balls $B(p,r)$ are precompact and $N(\rho)$ a small cover refinement function for it. Then, for any finite nonatomic measure μ on X and any positive integer k there exists a collection of k bounded Borel sets $\{A_i\}$ such that

$$\mu(A_i) \geqslant \mu(X)/(ck) \qquad \text{for any} \quad 1 \leqslant i \leqslant k,$$

where $c = 64N(1600)$, and one of the following possibilities hold:

(i) either all the A_is are annuli and then the annuli $2A_i$ are pair-wise disjoint and their outer radii are not greater than one;

(ii) or the r_0-neighbourhoods

$$A_i^{r_0} = \left\{x \in X : \mathrm{dist}(x, A_i) \leqslant r_0\right\}$$

are pair-wise disjoint, where $r_0 = 1600^{-1}$.

An important new point in Proposition 3.2 is the linear dependence of the constant c on the refinement function N. The proof of Proposition 3.2 follows the idea in [21]; it relies on the argument in the proof of [19, Theorem 3.5] and an improved version of a statement from [16]. We discuss it in more detail at the end of the section.

Now, we consider the main example that is used in the sequel—pseudo-metric measure spaces with homogeneous bounds on the measure of balls. We describe it in the form of the following lemma; its proof is rather standard, but we include it for the sake of completeness.

Lemma 3.3. Let (X, d) be a pseudo-metric space equipped with a measure ν such that

$$C_1 r^\alpha \leqslant \nu(B(p,r)) \leqslant C_2 r^\alpha \qquad \text{for any} \quad p \in X \text{ and } 0 < r \leqslant 3,$$

where C_1, C_2, and α are positive constants. Then, the function $N(\rho) = (6\rho)^\alpha C_2/C_1$ is a small cover refinement function for X. If the above inequalities hold for any $r > 0$ and any $p \in X$, then the function $N(\rho)$ is a cover refinement function for X.

Proof. We prove the 1st statement of the lemma; the 2nd statement for arbitrary balls follows by the same argument. For a given value $\rho > 1$ and a ball $B(p, r)$ with $0 < r \leqslant 1$, let $\{B(p_i, r/(2\rho))\}$ be a maximal collection of disjoint balls of radii $r/(2\rho)$ centred at a point $p_i \in B(p, r)$, where $i = 1, \dots, \ell$. It is straightforward to see that the family of balls $\{B(p_i, r/\rho)\}$, where $i = 1, \dots, \ell$, covers the ball $B(p, r)$. Thus, for a proof of the statement, it is sufficient to show that the cardinality ℓ of this cover is not greater than $(6\rho)^\alpha C_2/C_1$.

Let i_0 be an index such that the measure $\nu(B(p_{i_0}, r/(2\rho)))$ is the least value among all measures $\nu(B(p_i, r/(2\rho)))$, where i ranges over $1, \dots, \ell$. Then, we obtain

$$\ell \nu(B(p_{i_0}, r/(2\rho))) \leqslant \sum_{i=1}^{\ell} \nu(B(p_i, r/(2\rho))) \leqslant \nu(B(p, 2r)) \leqslant \nu(B(p_{i_0}, 3r)), \tag{3.1}$$

where in the 2nd inequality, we used the inclusion $B(p_i, r/(2\rho)) \subset B(p, 2r)$, and in the 3rd, the inclusion $B(p, 2r) \subset B(p_{i_0}, 3r)$. Thus, using the hypotheses on the lemma, we obtain

$$\ell \leqslant \frac{\nu(B(p_{i_0}, 3r))}{\nu(B(p_{i_0}, r/(2\rho)))} \leqslant \frac{C_2(3r)^\alpha}{C_1(r/2\rho)^\alpha} = (6\rho)^\alpha \frac{C_2}{C_1}$$

and finish the proof of the 1st statement. ∎

3.2 On the proof of Proposition 3.2

A new ingredient in the proof of Proposition 3.2 is the following improved version of [16, Corollary 3.12]; see also [15, Lemma 2.1].

Lemma 3.4. Let (X, d) be a separable pseudo-metric space, $r > 0$ a real number, and N a positive integer such that any ball of radius $4r$ in X can be covered by N balls of radius r. Let μ be a finite Borel measure and k a positive integer such that

$$\mu(B(p, r)) \leqslant \frac{\mu(X)}{4Nk} \qquad \text{for any} \quad p \in X.$$

Then, there exists a collection of k bounded Borel subsets $\{A_i\}$ such that

$$\mu(A_i) \geqslant \frac{\mu(X)}{2Nk} \qquad \text{for any} \quad 1 \leqslant i \leqslant k,$$

and the r-neighbourhoods $\{A_i^r\}$s are pair-wise disjoint.

The proof of this lemma is based on the following statement.

Claim 3.5. Let (X, d) be a separable pseudo-metric space, $r > 0$ a real number, and N a positive integer such that any ball of radius $4r$ in X can be covered by N balls of radius r. Let μ be a finite Borel measure and $\beta < \mu(X)$ a positive real number such that

$$\mu(B(p, r)) \leqslant \frac{\beta}{2} \qquad \text{for any} \quad p \in X. \tag{3.2}$$

Then, there exist bounded Borel subsets $A \subset D$ in X such that

$$\beta \leqslant \mu(A) \leqslant \mu(D) \leqslant 2N\beta,$$

and $\operatorname{dist}(A, D^c) \geqslant 3r$.

Proof. For a positive integer ℓ, let $\mathcal{U}_\ell$ be the collection of all subsets in X that can be written as unions of at most ℓ balls of radius r, that is,

$$\mathcal{U}_\ell = \left\{ \bigcup_{j=1}^{\ell} B(x_j, r) : x_1, \ldots, x_\ell \in X \right\}.$$

By ξ_ℓ, we denote the supremum $\sup\{\mu(U) : U \in \mathscr{U}_\ell\}$. Note that $\mathscr{U}_\ell \subset \mathscr{U}_{\ell+1}$, and hence, the sequence ξ_ℓ is nondecreasing, $\xi_\ell \leqslant \xi_{\ell+1}$. Since X is a separable pseudo-metric space, it is straightforward to see that there exists a sequence of subsets $\{U_\ell\}$ such that $U_\ell \in \mathscr{U}_\ell$, $U_\ell \subset U_{\ell+1}$ for each ℓ, and $\cup_\ell U_\ell = X$. Thus, we conclude that the sequence ξ_ℓ converges to the value $\mu(X)$. Since by (3.2), we have $\xi_1 \leqslant \beta/2$, then there exists an integer $k \geqslant 2$ such that

$$\xi_{k-1} \leqslant \beta < \xi_k.$$

The 2nd inequality implies that there exists a set $A \in \mathscr{U}_k$ such that $\mu(A) > \beta$. The set A has the form $\cup B(p_j, r)$ for some points $p_j \in X$, and then we define the set $D \subset X$ as the union

$$D = \bigcup_{j=1}^{k} B(p_j, 4r).$$

It is straightforward to see that $\mathrm{dist}(A, D^c)$ is at least $3r$. Thus, for a proof of the claim it remains to show that $\mu(D) \leqslant 2N\beta$.

To prove the last inequality, note that each ball $B(p_j, 4r)$ can be covered by N balls of radius r. Thus, the set D can be covered by kN balls of radius r, that is, $D \subset W$, where $W \in \mathscr{U}_{kN}$. Since $kN \leqslant 2(k-1)N$, we see that W can be represented as the union

$$W = \bigcup_{j=1}^{2N} W_j, \qquad \text{where} \quad W_j \in \mathscr{U}_{k-1},$$

and we obtain

$$\mu(D) \leqslant \mu(W) \leqslant \sum_{j=1}^{2N} \mu(W_j) \leqslant 2N\xi_{k-1} \leqslant 2N\beta.$$

Thus, the claim is proved. ∎

Proof of Lemma 3.4. Equipped with Claim 3.5, we can now prove the lemma by following the line of the argument in [15, Section 4]. More precisely, taking $\beta = \mu(X)/(2Nk)$, one can construct inductively k pairs (A_j, D_j), where $1 \leqslant j \leqslant k$, such that

$$A_j \subset D_j, \qquad \mathrm{dist}(A_j, (\cup_{i\leqslant j} D_i)^c) \geqslant 3r,$$

the inequalities

$$\beta \leqslant \mu(A_j) \leqslant \mu(D_j) \leqslant 2N\beta = \frac{\mu(X)}{k}$$

hold, and additionally, $A_j \subset (\cup_{i<j} D)^c$. The above claim is used in the induction step. Then, the family $\{A_j\}$ satisfies the conclusion of Lemma 3.4. Indeed, we have

$$\mu(A_j) \geqslant \beta = \frac{\mu(X)}{2Nk},$$

and since

$$\mathrm{dist}(A_l, A_j) \geqslant \mathrm{dist}(A_l, (\cup_{i\leqslant l} D_i)^c) \geqslant 3r$$

for $l < j$, we see that the r-neighbourhoods $\{A_j^r\}$ are pair-wise disjoint.

To make the exposition more self-contained, we describe briefly the induction argument for the existence of such pairs (A_j, D_j). Taking $\beta = \mu(X)/(2Nk)$, by the hypotheses of the lemma, we see that Claim 3.5 applies, and there are bounded Borel sets $A_1 \subset D_1$ such that

$$\beta \leqslant \mu(A_1) \leqslant \mu(D_1) \leqslant 2N\beta = \frac{\mu(X)}{k},$$

and $\mathrm{dist}(A_1, D_1^c) \geqslant 3r$. Now, suppose that for $1 \leqslant j < k$ the desired pairs $\{(A_i, D_i)\}$, where $i = 1, \ldots, j$, are constructed. Denote by μ_{j+1} the measure on X, obtained by restricting μ to the complement $(\cup_{i\leqslant j} D_i)^c$. Note that for any ball $B(p, r)$ the inequalities

$$\mu_{j+1}(B(p,r)) \leqslant \mu(B(p,r)) \leqslant \frac{\mu(X)}{4Nk} = \frac{\beta}{2}$$

hold. By the induction hypotheses, we also have

$$\mu_{j+1}(X) \geqslant \mu(X) - \sum_{i=1}^{j} \mu(D_j) \geqslant \mu(X)\left(1 - \frac{j}{k}\right) \geqslant \frac{\mu(X)}{k},$$

and hence, see that

$$\beta = \frac{\mu(X)}{2Nk} \leqslant \frac{\mu_{j+1}(X)}{2N} < \mu_{j+1}(X).$$

Thus, Claim 3.5 applies to the measure μ_{j+1} on X, and there are sets $A \subset D$ in X such that

$$\beta \leqslant \mu_{j+1}(A) \leqslant \mu_{j+1}(D) \leqslant 2N\beta = \frac{\mu(X)}{k},$$

and $\mathrm{dist}(A, D^c) \geqslant 3r$. The pair (A_{j+1}, D_{j+1}) is defined by setting

$$A_{j+1} = A \cap (\cup_{i\leqslant j} D_i)^c \quad \text{and } D_{j+1} = D \cap (\cup_{i\leqslant j} D_i)^c.$$

It is straightforward to check that these sets satisfy the required hypotheses. ■

Now, the proof of Proposition 3.2 follows the scheme in [21, Section 2] with necessary adjustments for the constants involved. It relies on the argument in the proof of [19, Theorem 3.5] and uses Lemma 3.4 above in place of [21, Lemma 2.3].

4 Proofs

4.1 Proof of Theorem 1.1

Let (M, g) be an m-dimensional Riemannian manifold that satisfies the hypotheses of Theorem 1.1 and $\mathrm{dist}_g(\cdot,\cdot)$ a distance function on it. Scaling the metric g, we may assume that $\mathrm{rad}(g)$ equals three. Then, the combination of Lemma 3.3 and Corollary 2.2 implies that the function

$$N(\rho) = C_{11}(m)\frac{Vol_g(M)}{\mathrm{rad}(g)^m}\rho^m, \tag{4.1}$$

where $C_{11}(m) = 24^m/\omega_m$ is a small cover refinement function for the metric space (M, dist_g). For a given metric $\tilde{g}$ conformal to g, denote by μ its volume measure $Vol_{\tilde{g}}$ on M. Then, by Proposition 3.2, for any given positive integer k, there exists a collection of $2(k+1)$ bounded Borel sets $\{A_i\}$ such that

$$\mu(A_i) \geqslant \mu(M)/(2c(k+1)) \geqslant \mu(M)/(4ck) \tag{4.2}$$

for all $i = 1, \dots, 2(k+1)$, where $c = 64N(1600)$, and one of the following possibilities hold:

(i) either all the A_is are annuli and the annuli $2A_i$s are pair-wise disjoint and their outer radii are not greater than one;

(ii) or the r_0-neighbourhoods of the A_is, where $r_0 = 1600^{-1}$, are pair-wise disjoint.

Note that, using formula (4.1), the estimate for $\mu(A_i)$ in relation (4.2) can be re-written in the form

$$Vol_{\tilde{g}}(A_i) \geqslant \frac{Vol_{\tilde{g}}(M)}{k}\left(256(1600)^m C_{11}(m) Vol_g(M)/\mathrm{rad}(g)^m\right)^{-1} \tag{4.3}$$

for all $i = 1, \dots, 2(k+1)$. Now, we consider two cases corresponding to the two possibilities (i) and (ii) above.

Case (i). Since the annuli $2A_i$s are pair-wise disjoint, we have

$$\sum_{i=1}^{2(k+1)} \mu(2A_i) \leqslant \mu(M),$$

and hence, there exists at least $(k+1)$ sets A_i such that

$$\mu(2A_i) \leqslant \mu(M)/(k+1) \leqslant \mu(M)/k. \tag{4.4}$$

After reordering, we may assume that the above relation holds for $i = 1, \dots, k+1$. For such an i, we denote by u_i the test-function constructed in the following way: it vanishes on the complement of the exterior annulus $2A_i$, equals one on the interior annulus $A_i = B(a_i, R_i)\backslash B(a_i, r_i)$, and is given by the formula

$$u_i(x) = \begin{cases} \dfrac{2}{r_i}\mathrm{dist}(x, a_i) - 1, & \text{if } x \in B(a_i, r_i)\backslash B(a_i, r_i/2), \\ 2 - \dfrac{1}{R_i}\mathrm{dist}(x, a_i), & \text{if } x \in B(a_i, 2R_i)\backslash B(a_i, R_i), \end{cases}$$

on the complement $2A_i\backslash A_i$. It is straightforward to see that each u_i is a Lipschitz function, and moreover, on the complement $2A_i\backslash A_i$ its gradient satisfies the inequalities

$$\left|\nabla u_i\right| \leqslant 2/r_i \qquad \text{on} \quad B(a_i, r_i)\backslash B(a_i, r_i/2), \tag{4.5}$$

$$\left|\nabla u_i\right| \leqslant 1/R_i \qquad \text{on} \quad B(a_i, 2R_i)\backslash B(a_i, R_i). \tag{4.6}$$

Now, we estimate the Dirichlet energy of u_i with respect to the metric $\tilde{g}$. By the Hölder inequality, we obtain

$$
\begin{aligned}
\int_M |\nabla u_i|^2_{\tilde{g}}\, \mathrm{d}Vol_{\tilde{g}} &\leqslant Vol_{\tilde{g}}(2A_i)^{1-2/m}\left(\int_{B(a_i,2R_i)} |\nabla u_i|^m_{\tilde{g}}\, \mathrm{d}Vol_{\tilde{g}}\right)^{2/m}\\
&= Vol_{\tilde{g}}(2A_i)^{1-2/m}\left(\int_{B(a_i,2R_i)} |\nabla u_i|^m_{g}\, \mathrm{d}Vol_{g}\right)^{2/m}\\
&\leqslant Vol_{\tilde{g}}(2A_i)^{1-2/m}\left((2/r_i)^m Vol_g(B(a_i,r_i)) + (1/R_i)^m Vol_g(B(a_i,2R_i))\right)^{2/m},
\end{aligned}
$$

where in the equality above, we used the conformal invariance of $\int |\nabla u|^m\, \mathrm{d}Vol$, and in the last relation inequalities (4.5)–(4.6). Now, since by Proposition 3.2 the outer radii satisfy the inequality $2R_i \leqslant 1 < \mathrm{rad}(g)$, the volume bounds in Corollary 2.2 apply and we obtain

$$
\begin{aligned}
\int_M |\nabla u_i|^2_{\tilde{g}}\, \mathrm{d}Vol_{\tilde{g}} &\leqslant 16 Vol_{\tilde{g}}(2A_i)^{1-2/m}\left(Vol_g(M)/\mathrm{rad}(g)^m\right)^{2/m}\\
&\leqslant 16 (Vol_{\tilde{g}}(M)/k)^{1-2/m}\left(Vol_g(M)/\mathrm{rad}(g)^m\right)^{2/m}, \qquad (4.7)
\end{aligned}
$$

where in the last inequality, we used relation (4.4). Combining inequalities (4.3) and (7), we can now estimate the Rayleigh quotient:

$$
\begin{aligned}
\mathscr{R}_{\tilde{g}}(u_i) &= \left(\int_M |\nabla u_i|^2_{\tilde{g}}\, \mathrm{d}Vol_{\tilde{g}}\right) / \left(\int_M u_i^2 \mathrm{d}Vol_{\tilde{g}}\right)\\
&\leqslant C_{12}(m)(Vol_{\tilde{g}}(M)/k)^{-2/m}\left(Vol_g(M)/\mathrm{rad}(g)^m\right)^{1+2/m}\\
&= C_{12}(m)(Vol_{\tilde{g}}(M))^{-2/m}\left(Vol_g(M)/\mathrm{rad}(g)^m\right)^{1+2/m} k^{2/m},
\end{aligned}
$$

where $i = 1, \dots, k+1$. Since the u_is form a system of $W^{1,2}$-orthogonal functions, by the variational principle, we conclude that

$$
\lambda_k(\tilde{g}) Vol_{\tilde{g}}(M)^{2/m} \leqslant C_{12}(m)\left(Vol_g(M)/\mathrm{rad}(g)^m\right)^{1+2/m} k^{2/m}.
$$

Thus, the statement of the theorem is proved in this case.

Case (ii). Since the r_0-neighbourhoods of the A_is are pair-wise disjoint, as in the 1st case, we may assume that

$$\mu(A_i^{r_0}) \leqslant \mu(M)/k \qquad \text{for any} \quad i = 1, \dots, k+1. \tag{4.8}$$

For such an i, we denote by u_i the test-function supported in the r_0-neighbourhood $A_i^{r_0}$ that is given by the formula

$$u_i(x) = \begin{cases} 1, & \text{if } x \in A_i, \\ 1 - r_0^{-1}\mathrm{dist}(x, A_i), & \text{if } x \in A_i^{r_0} \backslash A_i, \end{cases}$$

where $\mathrm{dist}(\cdot, A)$ stands for the distance to a subset A. It is straightforward to see that u_i is a Lipschitz function such that $\left|\nabla u_i\right| \leqslant r_0^{-1}$ on $A_i^{r_0} \backslash A_i$. Thus, following the line of argument above, we obtain

$$\int_M \left|\nabla u_i\right|_{\tilde g}^2 \mathrm{d}Vol_{\tilde g} \leqslant Vol_{\tilde g}(A_i^{r_0})^{1-2/m} \left(\int_{A_i^{r_0}} \left|\nabla u_i\right|_{\tilde g}^m \mathrm{d}Vol_{\tilde g} \right)^{2/m}$$
$$= Vol_{\tilde g}(A_i^{r_0})^{1-2/m} \left(\int_{A_i^{r_0}} \left|\nabla u_i\right|_{g}^m \mathrm{d}Vol_{g} \right)^{2/m} \leqslant (Vol_{\tilde g}(M)/k)^{1-2/m} Vol_g(M)^{2/m} r_0^{-2},$$

where in the last inequality, we used relation (4.8). Recall that by our normalisation assumption, we have

$$r_0 = \frac{1}{1600} = \frac{1}{4800}\mathrm{rad}(g).$$

Hence, the bound above for the Dirichlet energy of u_i can be rewritten in the form

$$\int_M \left|\nabla u_i\right|_{\tilde g}^2 \mathrm{d}Vol_{\tilde g} \leqslant 4800^2 (Vol_{\tilde g}(M)/k)^{1-2/m} \left(Vol_g(M)/\mathrm{rad}(g)^m \right)^{2/m}, \tag{4.9}$$

where $i = 1, \dots, k+1$. Now, combining inequalities (4.3) and (4.9), we arrive at the estimate

$$\mathscr{R}_{\tilde g}(u_i) \leqslant C_{13}(m)(Vol_{\tilde g}(M))^{-2/m} \left(Vol_g(M)/\mathrm{rad}(g)^m \right)^{1+2/m} k^{2/m},$$

for all $i = 1, \dots, k+1$. Thus, by the variational principle, we conclude that the desired inequalities for the eigenvalues of $\lambda_k(\tilde g)$ hold in this case as well.

Remark 4.1. Note that choosing the sets A_i in the argument in Case (ii) more carefully, such that in addition to relation (4.8) the following inequalities hold:

$$Vol_g(A_i^{r_0}) \leqslant Vol_g(M)/k \qquad \text{for any} \quad i = 1, \dots, k+1,$$

one can show that the eigenvalue $\lambda_k(\tilde{g})$ is bounded independently of k in this case. However, this observation does not give any improvement to the final result.

4.2 Proof of Theorem 1.2

The proof of Theorem 1.2 follows the strategy used in the proof of Theorem 1.1. However, the way we use the decomposition theorem, Proposition 3.2, as well as a few ingredients involved, are different.

Let (Σ^n, g) be a manifold isometrically immersed to M, via $\iota : \Sigma^n \to M$, as a proper minimal submanifold. Below, we denote by g the metric on both manifolds Σ^n and M. We equip Σ^n with a pseudo-metric $\bar{d}(\cdot,\cdot)$ obtained by restricting the distance function $\mathrm{dist}_g(\cdot,\cdot)$ on M to the image $\iota(\Sigma^n)$. A metric ball $\bar{B}(\bar{p}, r)$ in this pseudo-metric can be viewed as the pre-image $\iota^{-1}(B(p,r))$, where $\iota(\bar{p}) = p$ and $B(p,r)$ is a metric ball in (M, dist_g). Abusing the notation, it is also denoted by $B(p,r) \cap \Sigma^n$ in Section 2. A measure $\bar{\mu}$ on Σ^n is nonatomic with respect to $\bar{d}(\cdot,\cdot)$, see Section 3, if and only if the pushforward measure $\iota_*\bar{\mu}$ is nonatomic on M. Since $\iota : \Sigma^n \to M$ is an immersion, it is straightforward to see that for any metric h on Σ^n its volume measure is nonatomic with respect to the pseudo-metric $\bar{d}(\cdot,\cdot)$.

As in the proof of Theorem 1.1, we assume that the metric g on M is scaled such that $\mathrm{rad}(g)$ equals three. Then, the combination of Lemma 3.3 and Corollary 2.4 implies that the function

$$\bar{N}(\rho) = C_{14}(n) \frac{Vol_g(\Sigma^n)}{\mathrm{rad}(g)^n} \rho^n, \tag{4.10}$$

where $C_{14}(n) = 24^n/(n\omega_n)$ is a small cover refinement function for the pseudo-metric space $(\Sigma^n, \bar{d})$. Now, let h be a metric on Σ^n that is conformal to g and $\bar{\mu}$ its volume measure. By the discussion above, Proposition 3.2 applies to the pseudo-metric space $(\Sigma^n, \bar{d})$ equipped with $\bar{\mu}$. Thus, for any positive integer k, there exists a collection of $2(k+1)$ bounded Borel sets $\{\bar{A}_i\}$ in Σ^n such that

$$\bar{\mu}(\bar{A}_i) \geqslant \bar{\mu}(\Sigma^n)/(2c(k+1)) \geqslant \bar{\mu}(\Sigma^n)/(4\bar{c}k), \tag{4.11}$$

for all $i = 1, \dots, 2(k+1)$, where $\bar{c} = 64\bar{N}(1600)$, and one of the following possibilities hold:

(i) either all the $\bar{A}_i$s are annuli for the pseudo-metric $\bar{d}(\cdot,\cdot)$, and the annuli $2\bar{A}_i$s are pair-wise disjoint and their outer radii are not greater than one;

(ii) or the r_0-neighbourhoods of the $\bar{A}_i$s, where $r_0 = 1600^{-1}$, are pair-wise disjoint.

Now, the cases (i) and (ii) can be considered following the line of argument in the proof of Theorem 1.1. The test-functions are constructed similarly but using the pseudo-metric $\bar{d}(\cdot,\cdot)$. A new ingredient in the estimate of their Dirichlet energies is one of the inequalities in Corollary 2.4. Below, we briefly sketch the key points of the argument. In the sequel, we use estimate (4.11) for $\bar{\mu}(\bar{A}_i)$ in the following form:

$$Vol_h(\bar{A}_i) \geqslant \frac{Vol_h(\Sigma^n)}{k}\left(256(1600)^n C_{14}(n) Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{-1}. \tag{4.12}$$

It follows by combination of the relation $\bar{c} = 64\bar{N}(1600)$ with formula (4.10) for a small cover refinement function.

Case (i). As in the proof of Theorem 1.1, we may assume that

$$\bar{\mu}(2\bar{A}_i) \leqslant \bar{\mu}(\Sigma^n)/(k+1) \leqslant \bar{\mu}(\Sigma^n)/k \tag{4.13}$$

for $i = 1, \dots, k+1$. For each such i the test-function $\bar{u}_i$ is set to equal one on the interior annulus $\bar{A}_i = \bar{B}(\bar{a}_i, R_i)\backslash\bar{B}(\bar{a}_i, r_i)$ and zero on the complement of the exterior annulus $2\bar{A}_i$. On the complement $2\bar{A}_i\backslash\bar{A}_i$, it is given by the formula

$$\bar{u}_i(x) = \begin{cases} \dfrac{2}{r_i}\bar{d}(x,\bar{a}_i) - 1, & \text{if } x \in \bar{B}(\bar{a}_i, r_i)\backslash\bar{B}(\bar{a}_i, r_i/2), \\ 2 - \dfrac{1}{R_i}\bar{d}(x,\bar{a}_i), & \text{if } x \in \bar{B}(\bar{a}_i, 2R_i)\backslash\bar{B}(\bar{a}_i, R_i). \end{cases} \tag{4.14}$$

It is straightforward to see that $\left|\nabla\bar{d}(x,\cdot)\right| \leqslant 1$ for any point $x \in \Sigma^n$, and hence, the gradient of $\bar{u}_i$ satisfies the inequalities

$$\left|\nabla\bar{u}_i\right| \leqslant 2/r_i \qquad \text{on} \quad \bar{B}(\bar{a}_i, r_i)\backslash\bar{B}(\bar{a}_i, r_i/2),$$

$$\left|\nabla\bar{u}_i\right| \leqslant 1/R_i \qquad \text{on} \quad \bar{B}(\bar{a}_i, 2R_i)\backslash\bar{B}(\bar{a}_i, R_i).$$

Arguing as in the proof of Theorem 1.1, we can now estimate the Dirichlet energy of $\bar{u}_i$. In more detail, we obtain

$$\int_{\Sigma^n} \left|\nabla \bar{u}_i\right|_h^2 \mathrm{d}Vol_h \leqslant Vol_h(2\bar{A}_i)^{1-2/n}\left((2/r_i)^n Vol_g(\bar{B}(\bar{a}_i, r_i)) + (1/R_i)^n Vol_g(\bar{B}(\bar{a}_i, 2R_i))\right)^{2/n}$$
$$\leqslant 16 Vol_h(2\bar{A}_i)^{1-2/n}\left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n}$$
$$\leqslant 16 (Vol_h(\Sigma^n)/k)^{1-2/n}\left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n},$$

where we used Corollary 2.4 to estimate volumes of extrinsic balls in the 2nd inequality and relation (4.13) in the 3rd. Combining the last inequality with relation (4.12), we obtain the following estimate for the Rayleigh quotient of $\bar{u}_i$:

$$\mathscr{R}_h(\bar{u}_i) = \left(\int_{\Sigma^n} \left|\nabla \bar{u}_i\right|_h^2 \mathrm{d}Vol_h\right) \Big/ \left(\int_{\Sigma^n} \bar{u}_i^2 \mathrm{d}Vol_h\right)$$
$$\leqslant C_{15}(n)(Vol_h(\Sigma^n))^{-2/n}\left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{1+2/n} k^{2/n}$$

for any $i = 1, \dots, k+1$. By the variational principle, these estimates immediately yield the desired inequality for the Laplace eigenvalue $\lambda_k(\Sigma^n, h)$.

Case (ii). As in the proof of Theorem 1.1, we may assume that

$$\mu(\bar{A}_i^{r_0}) \leqslant \mu(\Sigma^n)/k \qquad \text{for any} \quad i = 1, \dots, k+1. \tag{4.15}$$

The test-function $\bar{u}_i$, supported in the r_0-neighbourhood $\bar{A}_i^{r_0}$, is defined by the formula

$$\bar{u}_i(x) = \begin{cases} 1, & \text{if } x \in \bar{A}_i, \\ 1 - r_0^{-1}\overline{\mathrm{dist}}(x, \bar{A}_i), & \text{if } x \in \bar{A}_i^{r_0} \backslash \bar{A}_i, \end{cases}$$

where $\overline{\mathrm{dist}}(\cdot, \bar{A})$ is the distance to a subset in the sense of pseudo-metric $\bar{d}(\cdot,\cdot)$. As above, we see that $\left|\nabla \bar{u}_i\right| \leqslant r_0^{-1}$ on the complement $\bar{A}_i^{r_0} \backslash \bar{A}_i$ and estimate its Dirichlet energy in the following way:

$$\int_{\Sigma^n} \left|\nabla \bar{u}_i\right|_h^2 \mathrm{d}Vol_h \leqslant Vol_h(\bar{A}_i^{r_0})^{1-2/n}\left(\int_{\bar{A}_i^{r_0}} \left|\nabla \bar{u}_i\right|_g^n \mathrm{d}Vol_g\right)^{2/n}$$
$$\leqslant (Vol_h(\Sigma^n)/k)^{1-2/n} Vol_g(\Sigma^n)^{2/n} r_0^{-2}$$
$$= 4800^2 (Vol_h(\Sigma^n)/k)^{1-2/n}\left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n},$$

where we used relation (4.15) in the 2nd inequality and the scaling assumption $\mathrm{rad}(g)=3$ together with $r_0 = 1600^{-1}$ in the last relation. Combining this estimate with relation (4.12), we obtain

$$\mathscr{R}_h(\bar{u}_i) \leqslant C_{16}(n)(Vol_h(\Sigma^n))^{-2/n} \left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{1+2/n} k^{2/n}$$

for any $i = 1, \dots, k+1$. Now, the desired inequality for the Laplace eigenvalue $\lambda_k(\Sigma^n, h)$ follows from the variational principle.

4.3 Proof of Theorem 1.3

As in the proof of Theorem 1.2, we consider a pseudo-metric space $(\Sigma^n, \bar{d})$, where a pseudo-metric $\bar{d}(\cdot,\cdot)$ is obtained by restricting the distance function $\mathrm{dist}_g(\cdot,\cdot)$ to the image of an immersed submanifold Σ^n. For a point $\bar{p} \in \Sigma^n$ the volume of a pseudo-metric ball $\bar{B}(\bar{p}, r)$ satisfies the inequalities

$$\omega_n r^n \leqslant Vol_g(\bar{B}(\bar{p}, r)) \leqslant \omega_n \theta(\Sigma^n) r^n \tag{4.16}$$

for any $r > 0$, where ω_n is the volume of a unit ball in the Euclidean space $\mathbb{R}^n$ and $\theta(\Sigma^n)$ is the density at infinity. These inequalities are direct consequences of the volume monotonicity for minimal submanifolds; see Proposition 2.3. By Lemma 3.3, inequalities (4.16) imply that the function $\bar{N}(\rho) = \theta(\Sigma^n)(6\rho)^n$ is a cover refinement function for this pseudo-metric space.

Let h be a metric conformal to g on a domain $\Omega \subset \Sigma^n$ and $\bar{\mu}$ its volume measure restricted to Ω. As in the proof of Theorem 1.2, we conclude that the measure $\bar{\mu}$ is nonatomic with respect to $\bar{d}(\cdot,\cdot)$ and Proposition 3.1 applies. Thus, for any positive integer k, there exists a collection of $2(k+1)$ annuli $\{\bar{A}_i\}$ in Σ^n such that the annuli $\{2\bar{A}_i\}$ are pair-wise disjoint and

$$\bar{\mu}(\bar{A}_i) \geqslant \bar{\mu}(\Sigma^n)/(2c(k+1)) \geqslant \bar{\mu}(\Sigma^n)/(4\bar{c}k)$$

for all $i = 1, \dots, 2(k+1)$, where

$$\bar{c} = 8\bar{N}(1600) = C_{17}(n)\theta(\Sigma^n).$$

Let $\bar{u}_i$ be a test-function constructed as in Case (i) of the proof of Theorem 1.2; it is supported in the annulus $2\bar{A}_i$. Then, using inequalities (4.16) in place of Corollary 2.4, one can repeat the argument in the proof of Theorem 1.2 to show that

$$\mathscr{R}_h(\bar{u}_i) = \left(\int_\Omega |\nabla \bar{u}_i|_h^2 \,\mathrm{d}Vol_h\right) / \left(\int_\Omega \bar{u}_i^2 \mathrm{d}Vol_h\right) \leqslant C_{18}(n)(Vol_h(\Omega))^{-2/n}\theta(\Sigma^n)^{1+2/n}k^{2/n}$$

for some $k+1$ test-functions. Since these test-functions are supported in pair-wise disjoint sets, by the variational principle, we obtain the corresponding inequalities for the Neumann eigenvalues $\lambda_k(\Omega, h)$.

4.4 Proof of Theorem 1.4

The proof of the theorem uses ingredients from the proofs of both Theorems 1.1 and 1.2. The idea is to apply Proposition 3.2 to the metric space (M, dist_g) equipped with the pushforward measure $\mu_* = \iota_* Vol_h$, where $\iota : \Sigma^n \to M$ is an immersion. The test-functions on Σ^n are obtained by pulling back the test-functions u_i that are used in the proof of Theorem 1.1, and their Dirichlet energies are estimated following the line of argument in the proof of Theorem 1.2.

In more detail, let h be a metric on Σ^n conformal to g and μ_* the pushforward volume measure $\iota_* Vol_h$. It is straightforward to see that μ is nonatomic. Scaling the metric g on M, we may assume that $\mathrm{rad}(g)$ equals three. Applying Proposition 3.2 to the metric space (M, dist_g), for any positive integer k, we obtain a collection of $2(k+1)$ bounded Borel sets $\{A_i\}$ in M such that

$$\mu_*(A_i) \geqslant \mu_*(M)/(4ck) \qquad \text{for all} \quad i = 1, \ldots, 2(k+1), \tag{4.17}$$

where $c = 64N(1600)$, the function $N(\rho)$ is given by formula (4.1), and one of the following possibilities hold:

(i) either all the A_is are annuli and the annuli $2A_i$s are pair-wise disjoint and their outer radii are not greater than one;

(ii) or the r_0-neighbourhoods of the A_is, where $r_0 = 1600^{-1}$, are pair-wise disjoint.

In the sequel, we also use the notation $\bar{A}_i$ for the Borel set $\iota^{-1}(A_i)$ in Σ^n. Then, relation (4.17) can be rewritten in the form

$$Vol_h(\bar{A}_i) \geqslant \frac{Vol_h(\Sigma^n)}{k}(256(1600)^m C_{11}(m)Vol_g(M)/\mathrm{rad}(g)^m)^{-1} \tag{4.18}$$

for all $i = 1, \ldots, 2(k+1)$. Now, we briefly describe the arguments for the cases (i) and (ii), corresponding to the different properties of the sets A_i.

Case (i). As in the proof of Theorem 1.1, without loss of generality, we may assume that

$$\mu_*(2A_i) \leqslant \mu_*(M)/(k+1) \leqslant \mu_*(M)/k$$

for all $i = 1, \ldots, k+1$. Let u_i be a test-function constructed in Case (i) in the proof of Theorem 1.1. By $\bar{u}_i$, we denote the test-function supported in $2\bar{A}_i = \iota^{-1}(2A_i)$, given by $\bar{u}_i = u_i \circ \iota$. Note that the sets $\bar{A}_i = \iota^{-1}(A_i)$ and $2\bar{A}_i = \iota^{-1}(2A_i)$ are annuli in the pseudo-metric space $(\Sigma^n, \bar{d})$, and using the notation in the proof of Theorem 1.2, our test-functions $\bar{u}_i$ can be also described by formula (4.14). In particular, we may repeat the argument in the proof of Theorem 1.2 to obtain the estimate

$$\int_{\Sigma^n} \left|\nabla \bar{u}_i\right|_h^2 \mathrm{d}Vol_h \leqslant 16(Vol_h(\Sigma^n)/k)^{1-2/n} \left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n}$$

for any $i = 1, \ldots, k+1$. Combining the latter with relation (4.18), we arrive at the following estimate for the Rayleigh quotient:

$$\begin{aligned}\mathscr{R}_h(\bar{u}_i) &= \left(\int_{\Sigma^n} \left|\nabla \bar{u}_i\right|^2 \mathrm{d}Vol_h\right) / \left(\int_{\Sigma^n} \bar{u}_i^2 \mathrm{d}Vol_h\right) \\ &\leqslant C_{12}(m)(Vol_h(\Sigma^n)/k)^{-2/n} \left(Vol_g(M)/\mathrm{rad}(g)^m\right) \left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n} \\ &= C_{12}(m)(Vol_h(\Sigma^n))^{-2/n} \left(Vol_g(M)/\mathrm{rad}(g)^{m+2}\right) Vol_g(\Sigma^n)^{2/n} k^{2/n}\end{aligned}$$

for any $i = 1, \ldots, k+1$. Since the $\bar{u}_i$s are supported in the pair-wise disjoint sets $2\bar{A}_i = \iota^{-1}(2A_i)$ in Σ^n, they form a $W^{1,2}$-orthogonal system, and the inequalities for $\lambda_k(\Sigma^n, h)$ now follow from the variational principle.

Case (ii). As in the proof of Theorem 1.1, we may assume that

$$\mu_*(A_i^{r_0}) \leqslant \mu_*(M)/k \qquad \text{for any} \quad i = 1, \ldots, k+1.$$

Let u_i be a test-function constructed in Case (ii) in the proof of Theorem 1.1. By $\bar{u}_i$, we denote the test-function supported in $\bar{A}_i^{r_0} = \iota^{-1}(A_i^{r_0})$, given by $\bar{u}_i = u_i \circ \iota$. As above, we see that

$$\left|\nabla \bar{u}_i\right| \leqslant \left|\nabla(u_i \circ \iota)\right| \leqslant r_0^{-1} \qquad \text{on} \quad \iota^{-1}(A_i^{r_0} \backslash A_i),$$

and arguing as in the proof of Theorem 1.2, we obtain

$$\int_{\Sigma^n} \left|\nabla \bar{u}_i\right|_h^2 \mathrm{d}Vol_h \leqslant (Vol_h(\Sigma^n)/k)^{1-2/n} Vol_g(\Sigma^n)^{2/n} r_0^{-2}$$
$$= 4800^2 (Vol_h(\Sigma^n)/k)^{1-2/n} \left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n}.$$

Combining the latter with relation (4.18), we arrive at the following estimate:

$$\mathscr{R}_h(\bar{u}_i) \leqslant C_{13}(m)(Vol_h(\Sigma^n)/k)^{-2/n} \left(Vol_g(M)/\mathrm{rad}(g)^m\right) \left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n}$$
$$= C_{13}(m)(Vol_h(\Sigma^n))^{-2/n} \left(Vol_g(M)/\mathrm{rad}(g)^{m+2}\right) Vol_g(\Sigma^n)^{2/n} k^{2/n}$$

for any $i = 1, \ldots, k+1$ and the inequalities for $\lambda_k(\Sigma^n, h)$ now follow from the variational principle.

4.5 Proof of Theorem 1.5

As in the proof of Theorem 1.4, the strategy is to apply Proposition 3.2 to the metric space (M, dist_g) equipped with the pushforward measure $\mu_* = \iota_* Vol_h$, where $\iota : \Sigma^n \to M$ is an immersion. However, using the lower Ricci curvature bound, we can construct a different, from the one used before, small cover refinement function on (M, dist_g).

In more detail, a standard application of the Bishop–Gromov relative volume comparison theorem for spaces with a lower Ricci curvature bound, see [9], yields the inequality

$$\frac{Vol_g(B(p,R))}{Vol_g(B(p,r))} \leqslant \left(\frac{R}{r}\right)^m e^{(m-1)\sqrt{\kappa}R} \tag{4.19}$$

for any $0 < r \leqslant R$, where $B(p,t)$ stands for a metric ball of radius $t > 0$ in the space (M, dist_g). Scaling the metric g on M, we may assume that

$$\min\left\{\frac{1}{\sqrt{\kappa}}, \mathrm{rad}(g)\right\} = 3. \tag{4.20}$$

Using relation (4.19), we can repeat the argument in the proof of Lemma 3.3 to conclude that the function

$$N_0(\rho) = (6\rho)^m e^{(m-1)}$$

is a small cover refinement function on (M, dist_g).

Now, let h be a metric on Σ^n conformal to g and μ_* be the pushforward measure $\iota_* Vol_h$. As in the proof of Theorem 1.4, the measure μ_* is nonatomic and Proposition 3.2 applies to the metric space (M, dist_g). Thus, for any positive integer k, we can find a collection of $3(k+1)$ bounded Borel sets $\{A_i\}$ in M such that

$$\mu_*(A_i) \geqslant \mu_*(M)/(3c(k+1)) \geqslant \mu_*(M)/(6ck) \tag{4.21}$$

for all $i = 1, \ldots, 3(k+1)$, where $c = 64N_0(1600)$, and one of the following possibilities occur:

(i) either all the A_is are annuli and the annuli $2A_i$s are pair-wise disjoint and their outer radii are not greater than one;

(ii) or the r_0-neighbourhoods of the A_is, where $r_0 = 1600^{-1}$, are pair-wise disjoint.

Using the notation $\bar{A}_i$ for the Borel set $\iota^{-1}(A_i)$ in Σ^n, relation (4.21) can be rewritten in the form

$$Vol_h(\bar{A}_i) \geqslant \frac{Vol_h(\Sigma^n)}{k} C_{19}(m) \tag{4.22}$$

for all $i = 1, \ldots, 3(k+1)$. Now, we consider the cases (i) and (ii).

Case (i). As in the proof of Theorem 1.1, without loss of generality, we may assume that

$$\mu_*(2A_i) \leqslant \mu_*(M)/(k+1) \leqslant \mu_*(M)/k$$

for all $i = 1, \ldots, k+1$. Let $\bar{u}_i$ be a test-function supported in $\iota^{-1}(2A_i)$ from the proof of Theorem 1.4; see Case (i). As was shown there, the Dirichlet energy of $\bar{u}_i$ satisfies the inequality

$$\int_{\Sigma^n} \left|\nabla \bar{u}_i\right|_h^2 \mathrm{d}Vol_h \leqslant 16(Vol_h(\Sigma^n)/k)^{1-2/n} \left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n}$$

for any $i = 1, \ldots, k+1$; the argument uses the inequality $\mathrm{rad}(g) \geqslant 3$; see relation (4.20). Combining this estimate with relation (4.22), we obtain

$$\mathscr{R}_h(\bar{u}_i) = \left(\int_{\Sigma^n} \left|\nabla \bar{u}_i\right|_h^2 \mathrm{d}Vol_h\right) \Big/ \left(\int_{\Sigma^n} \bar{u}_i^2 \mathrm{d}Vol_h\right)$$
$$\leqslant C_{20}(m)(Vol_h(\Sigma^n))^{-2/n} \left(Vol_g(\Sigma^n)/\mathrm{rad}(g)^n\right)^{2/n} k^{2/n}$$

for any $i = 1, \ldots, k+1$. Now, by the variational principle, we conclude that

$$\lambda_k(\Sigma^n, h) Vol_h(\Sigma^n)^{2/n} \leqslant C_{20}(m) \mathrm{rad}(g)^{-2} Vol_g(\Sigma^n)^{2/n} k^{2/n}.$$

Case (ii). Denote by ν the pushforward measure $\iota_* Vol_g$ on M. Since all sets A_i are pairwise disjoint, we can choose $(k+1)$ sets such that

$$\mu_*(A_i^{r_0}) \leqslant \mu_*(M)/k \qquad \text{and} \qquad \nu(A_i^{r_0}) \leqslant \nu(M)/k. \tag{4.23}$$

Indeed, there exists at least $2(k+1)$ sets such that the 1st inequalities occur. Among these sets, we can choose further $(k+1)$ sets such that the 2nd inequalities for the measure ν hold. Without loss of generality, we may assume that both inequalities in (4.23) hold for $i = 1, \ldots, k+1$. Let $\bar{u}_i$ be a test-function supported in $\bar{A}_i^{r_0} = \iota^{-1}(A_i^{r_0})$ from the proof of Theorem 1.4; see Case (ii). Recall that its gradient satisfies the relation $\left|\nabla \bar{u}_i\right| \leqslant r_0^{-1}$ on $\iota^{-1}(A_i^{r_0} \backslash A_i)$. Thus, we obtain

$$\begin{aligned}
\int_{\Sigma^n} |\nabla \bar{u}_i|_h^2 \, \mathrm{d}Vol_h &\leqslant Vol_h(\bar{A}_i^{r_0})^{1-2/n} \left(\int_{\bar{A}_i^{r_0}} |\nabla \bar{u}_i|_g^n \, dVol_g \right)^{2/n} \\
&\leqslant Vol_h(\bar{A}_i^{r_0})^{1-2/n} Vol_g(\bar{A}_i^{r_0})^{2/n} r_0^{-2} \leqslant (Vol_h(\Sigma^n)/k)^{1-2/n} (Vol_g(\Sigma^n)/k)^{2/n} r_0^{-2} \\
&= \frac{4800^2}{k} Vol_h(\Sigma^n)^{1-2/n} Vol_g(\Sigma^n)^{2/n} \max\{\kappa, \mathrm{rad}(g)^{-2}\},
\end{aligned}$$

where we used relations (4.23) in the 3rd inequality and the scaling assumption (4.20) in the last equality. Combining this estimate with relation (4.22), we obtain

$$\mathscr{R}_h(\bar{u}_i) \leqslant C_{21}(m) Vol_h(\Sigma^n)^{-2/n} Vol_g(\Sigma^n)^{2/n} \max\{\kappa, \mathrm{rad}(g)^{-2}\}$$

for any $i = 1, \ldots, k+1$. Applying the variational principle, we get the inequalities

$$\lambda_k(\Sigma^n, h) Vol_h(\Sigma^n)^{2/n} \leqslant C_{21}(m) Vol_g(\Sigma^n)^{2/n} \max\{\kappa, \mathrm{rad}(g)^{-2}\}.$$

Comparing the latter with the eigenvalue inequalities in Case (i) above, we conclude that in both cases the Laplace eigenvalues $\lambda_k(\Sigma^n, h)$ satisfy

$$\lambda_k(\Sigma^n, h) Vol_h(\Sigma^n)^{2/n} \leqslant C_8(m) \max\{\kappa, \mathrm{rad}(g)^{-2} k^{2/n}\} Vol_g(\Sigma^n)^{2/n}$$

for any $k \geqslant 1$, where $C_8(m)$ equals $\max\{C_{20}(m), C_{21}(m)\}$.

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A Appendix: Croke's Bounds for Higher Laplace Eigenvalues

The purpose of this appendix is to give a proof of the following statement.

Proposition A.1. Let (M,g) be a closed Riemannian manifold of dimension m. Then, its Laplace eigenvalues $\lambda_k(g)$ satisfy the inequalities

$$\lambda_k(g) \leqslant C_3(m)\frac{Vol_g(M)^2}{\mathrm{conv}(g)^{2m+2}}k^{2m}$$

for any $k \geqslant 1$, where $\mathrm{conv}(g)$ is the convexity radius of (M,g) and $C_3(m)$ is the constant that depends on the dimension m only.

For $k = 1$ the inequality in Proposition A.1 is due to Croke [17, Corollary 19]. Its proof is based on the following upper bound for the 1st Dirichlet eigenvalue of a geodesic ball $B(p,r)$ in M:

$$\lambda_0(B(p,r)) \leqslant \bar{C}_3(m)\frac{Vol_g(B(p,r))^2}{r^{2m+2}}, \tag{A.1}$$

where $0 < r \leqslant \mathrm{conv}(g)$ and $\bar{C}_3(m)$ is a constant that depends on m only; see [17, Theorem 18]. Below, we demonstrate how inequality (A.1) can be used to prove Proposition A.1.

Proof of Proposition A.1. Pick an arbitrary point $p \in M$, and let q be a point from the cut locus of p. Thus, we have

$$\mathrm{dist}_g(p,q) \geqslant \mathrm{inj}(g) \geqslant \mathrm{conv}(g).$$

Denote by L the distance $\mathrm{dist}_g(p,q)$, and let $\gamma : [0,L] \to M$ be a shortest unit speed geodesic joining p and q. For a given positive integer k consider geodesic balls $B(p_i,r)$, where $r = L/(4k)$, the p_is are the points $\gamma(iL/(2k))$ on the geodesic γ, and $i = 0,\ldots,2k$. It is straightforward to see that these balls are pair-wise disjoint, and hence,

$$\sum_{i=0}^{2k} Vol_g(B(p_i,r)) \leqslant Vol_g(M).$$

Thus, there exists at least $(k+1)$ points p_i such that

$$Vol_g(B(p_i,r)) \leqslant Vol_g(M)/(k+1) \leqslant Vol_g(M)/k.$$

Combining the last inequality with Croke's inequality (A.1), we obtain

$$\lambda_0(B(p_i,r)) \leqslant \bar{C}_3(m)\frac{(Vol_g(M)/k)^2}{r^{2m+2}} \leqslant 4^{2m+2}\bar{C}_3(m)\frac{Vol_g(M)^2}{\mathrm{conv}(g)^{2m+2}}k^{2m},$$

where in the last inequality, we used the relation $r \geqslant \mathrm{conv}(g)/(4k)$. Now, let φ_i be a Dirichlet λ_0-eigenfunction on the ball $B(p_i,r)$ extended to M, by setting it to be equal to zero on the complement $M\backslash B(p_i,r)$. The above inequalities show that the Rayleigh quotients on M of at least $(k+1)$ such functions φ_i satisfy the inequality

$$\mathscr{R}_g(\varphi_i) \leqslant C_3(m)\frac{Vol_g(M)^2}{\mathrm{conv}(g)^{2m+2}}k^{2m},$$

where we set $C_3(m) = 4^{2m+2}\bar{C}_3(m)$. Since the supports of these φ_is are disjoint, by the variational principle, we conclude that the desired inequalities for the Laplace eigenvalues $\lambda_k(g)$ hold indeed. ■