TRANSPORT OF STRUCTURE IN HIGHER HOMOLOGICAL ALGEBRA

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ABSTRACT. We fill a gap in the literature regarding 'transport of structure' for $(n+2)$-angulated, n-exact, n-abelian and n-exangulated categories appearing in (classical and higher) homological algebra. As an application of our main results, we show that a skeleton of one of these kinds of categories inherits the same structure in a canonical way, up to equivalence. In particular, it follows that a skeleton of a weak $(n+2)$-angulated category is in fact what we call a strong $(n+2)$-angulated category. When $n = 1$ this clarifies a technical concern with the definition of a cluster category. We also introduce the notion of an n-exangulated functor between n-exangulated categories. This recovers the definition of an $(n+2)$-angulated functor when the categories concerned are $(n+2)$-angulated, and the higher analogue of an exact functor when the categories concerned are n-exact.

1. INTRODUCTION

'Transport of structure' refers to the situation in which an object gains structure by being isomorphic to an object with some pre-existing structure (see [6, §IV.1]). Instances of this well-known idea appear, for example, in: Lie theory [1], [31], [37]; operational calculus [9]; algebraic geometry and number theory [7], [20], [34]; the theory of concrete geometrical categories [14]; modular representation theory [38]; and the theory of algebraic structures [23]. These examples involve transporting some algebraic structure across a function between sets.

Analogously, one can ask which categorical structures can be transported across equivalences. For example, it is well-known that any category equivalent to an additive category is also additive (see Theorem 3.1). Other examples found in the literature include the transport of monoidal [13], model [16] and triangulated [11] structures. See also [19], [28], [29].

The motivation for writing this note arose in the context of triangulated categories. Suppose $(\mathcal{C}, \Sigma, \mathcal{T})$ is a triangulated category, where the suspension Σ is assumed to be an automorphism of $\mathcal{C}$; see Example 2.6. Let $\mathcal{C}'$ be any category and assume $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is an equivalence. Suppose one wants to find an automorphism Σ' of $\mathcal{C}'$ (and a collection $\mathcal{T}'$ of triangles in $\mathcal{C}'$), such that $(\mathcal{C}, \Sigma', \mathcal{T}')$ is a triangulated category and that $\mathscr{F}$ is a triangle equivalence. A canonical choice for Σ' is $\mathscr{F}\Sigma\mathscr{G}$, where $\mathscr{G} \colon \mathcal{C}' \to \mathcal{C}$ is a quasi-inverse for $\mathscr{F}$. Unfortunately, when one tries to prove this choice is an automorphism of $\mathcal{C}'$, one encounters problems that seem difficult (or, perhaps, impossible) to resolve. Bernhard Keller pointed out that it is more natural to ask only that Σ is an autoequivalence in the definition of a triangulated category, and we call these 'weak' triangulated categories here in order to make a distinction; see Definition 2.4. After this modification, by which nothing is lost (see [27, §2]), we can obtain a clean 'transport of triangulated structure' statement. Although well-known, we could not find the details of this claim. Theorem 3.4 fills this gap in the literature.

Recently, homological algebra has seen the introduction of *higher (dimensional)* analogues of categories that have been a core part of the classical theory for many years. Let $n \geqslant 1$ be a positive integer. The notion of an $(n+2)$-*angulated* category (see Definition 2.5) was introduced in [18], in a way so that one recovers the definition of a triangulated category by setting $n = 1$. Similarly, higher versions of exact and abelian categories were introduced in [25]; see Definitions 2.16 and 2.17. Parallel to this, Nakaoka and Palu defined an *extriangulated* category, which is a simultaneous generalisation of a triangulated category and an exact

category; see [35]. And, naturally, a higher analogue, called an *n-exangulated* category, of an extriangulated category has since been introduced in [22]; see Definition 2.28.

As a consequence of these developments, we include transport of structure results for $(n + 2)$-angulated, n-exact, n-abelian and n-exangulated categories. That is, we show that one can transport these categorical structures across equivalences; see Theorems 3.4, 3.5, 3.6 and 3.9. Moreover, in order to state the latter three results in the same fashion as Theorem 3.4, we introduce the notion of an *n-exact* functor (see Definition 2.18) and an *n-exangulated* functor (see Definition 2.32). We show in Theorem 2.33 that a (covariant) functor between $(n+2)$-angulated categories is an n-exangulated functor if and only if it is an $(n+2)$-*angulated* functor, in the sense of Definition 2.7. Similarly, in Theorem 2.34 we show that a functor between n-exact categories is an n-exangulated functor if and only if it is an n-exact functor.

An application of Theorem 3.5 (respectively, Theorem 3.6, Theorem 3.9) shows that any skeleton of an n-exact (respectively, n-abelian, n-exangulated) category is again n-exact (respectively, n-abelian, n-exangulated). Furthermore, in the $(n + 2)$-angulated case we deduce a stronger conclusion: we show that each skeleton of a *weak* $(n + 2)$-angulated category (see Definition 2.4) is what we call a *strong* $(n + 2)$-angulated category (i.e. with an $(n + 2)$-suspension functor that is an automorphism); see Corollary 4.6. In this case, the natural choice for the $(n + 2)$-suspension functor of the skeleton is an automorphism; see Proposition 4.4.

This paper is organised as follows. In §2 we recall the definitions of the categories from higher homological algebra that we work with in the sequel. In particular, we propose a definition for an n-exact functor in §2.2 and an n-exangulated functor in §2.3. Moreover, we show that an n-exangulated functor simultaneously generalises the notions of an $(n + 2)$-angulated functor and an n-exact functor. In §3 we prove our main results on transport of structure. In §4 we provide an improvement of transport of $(n+2)$-angulated structure when transporting the structure to a skeletal category. We also explain how our results from §4 may be used to show how the definition of a cluster category is made rigorous.

2. Higher structures

Throughout §2, let $n \geqslant 1$ be a positive integer and let $\mathcal{C}$ be an additive category. In §§2.1–2.2 we recall the higher analogues of the classical (i.e. $n = 1$) theory, and in §2.3 we recall the more recent theory of n-exangulated categories.

2.1. $(n+2)$**-angulated categories.** We assume throughout §2.1 that $\mathcal{C}$ is equipped with an autoequivalence $\Sigma \colon \mathcal{C} \to \mathcal{C}$. We recall the definition of an $(n+2)$-angulated category (see Definition 2.5), which was introduced in [18] and we follow the exposition therein.

Remark 2.1. Definitions 2.2–2.4 below are essentially [18, Def. 2.1]. However, there is one key difference. In [18] the endofunctor $\Sigma \colon \mathcal{C} \to \mathcal{C}$ is assumed to be an automorphism, whereas here we only assume that Σ is an autoequivalence.

Definition 2.2. A sequence of morphisms of the form $\quad X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0 \quad$ in $\mathcal{C}$ is called an $(n + 2)$-Σ-*sequence.*

Definition 2.3. A *morphism of* $(n + 2)$-Σ-*sequences* from

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0$$

to

$$Y^0 \xrightarrow{d_Y^0} Y^1 \xrightarrow{d_Y^1} \cdots \xrightarrow{d_Y^n} Y^{n+1} \xrightarrow{d_Y^{n+1}} \Sigma Y^0$$

is a tuple $(f^0, \ldots, f^{n+1})$ of morphisms $f^i \colon X^i \to Y^i$ in $\mathcal{C}$ such that

$$
\begin{array}{ccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^n} & X^{n+1} & \xrightarrow{d_X^{n+1}} & \Sigma X^0 \\
\downarrow{f^0} & & \downarrow{f^1} & & & & \downarrow{f^{n+1}} & & \downarrow{\Sigma f^0} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \xrightarrow{d_Y^n} & Y^{n+1} & \xrightarrow{d_Y^{n+1}} & \Sigma Y^0
\end{array}
$$

commutes in $\mathcal{C}$.

Definition 2.4. Recall that $n \geqslant 1$ is a positive integer and $\mathcal{C}$ is an additive category equipped with an autoequivalence Σ. Let $\mathcal{T}$ be a class of $(n+2)$-Σ-sequences in $\mathcal{C}$. Then we call $(\mathcal{C}, \Sigma, \mathcal{T})$ a *weak pre-$(n+2)$-angulated category* if the following axioms are satisfied.

(F1) (a) The class $\mathcal{T}$ is closed under finite taking direct sums and direct summands.

   (b) For each $X \in \mathcal{C}$, the $(n+2)$-Σ-sequence $\ X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X \ $ is in $\mathcal{T}$.

   (c) For each morphism d_X^0 there is an $(n+2)$-Σ-sequence in $\mathcal{T}$ whose first morphism is d_X^0.

(F2) An $(n+2)$-Σ-sequence $\ X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0 \ $ lies in $\mathcal{T}$ if and only if its *left rotation*

$$
X^1 \xrightarrow{\ d_X^1 \ } X^2 \xrightarrow{\ d_X^2 \ } \cdots \xrightarrow{\ d_X^n \ } X^{n+1} \xrightarrow{\ d_X^{n+1} \ } \Sigma X^0 \xrightarrow{(-1)^n \Sigma d_X^0} \Sigma X^1
$$

lies in $\mathcal{T}$.

(F3) Each commutative diagram

$$
\begin{array}{ccccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \xrightarrow{d_X^n} & X^{n+1} & \xrightarrow{d_X^{n+1}} & \Sigma X^0 \\
\downarrow{f^0} & & \downarrow{f^1} & & & & & & & & \downarrow{\Sigma f^0} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & Y^2 & \xrightarrow{d_Y^2} & \cdots & \xrightarrow{d_Y^n} & Y^{n+1} & \xrightarrow{d_Y^{n+1}} & \Sigma Y^0
\end{array}
\tag{2.1}
$$

with rows in $\mathcal{T}$ can be completed to a morphism

$$
\begin{array}{ccccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \xrightarrow{d_X^n} & X^{n+1} & \xrightarrow{d_X^{n+1}} & \Sigma X^0 \\
\downarrow{f^0} & & \downarrow{f^1} & & \downarrow{f^2} & & & & \downarrow{f^{n+1}} & & \downarrow{\Sigma f^0} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & Y^2 & \xrightarrow{d_Y^2} & \cdots & \xrightarrow{d_Y^n} & Y^{n+1} & \xrightarrow{d_Y^{n+1}} & \Sigma Y^0
\end{array}
\tag{2.2}
$$

of $(n+2)$-Σ-sequences.

In this case, we say: $\mathcal{T}$ is a *pre-$(n+2)$-angulation*; the members of $\mathcal{T}$ are *$(n+2)$-angles*; and Σ is an *$(n+2)$-suspension*.

Moreover, if $(\mathcal{C}, \Sigma, \mathcal{T})$ satisfies (F1)–(F3) and (F4) below, then we call $(\mathcal{C}, \Sigma, \mathcal{T})$ a *weak $(n+2)$-angulated category*. In this case, we refer to $\mathcal{T}$ as an *$(n+2)$-angulation*.

(F4) In the situation of (F3) above, (2.1) can be completed to (2.2) using morphisms $f^2, \ldots, f^{n+1}$ such that the *cone* $C^\bullet = C(f)^\bullet$ lies in $\mathcal{T}$, where $C^\bullet$ is the $(n+2)$-Σ-sequence

$$
X^1 \oplus Y^0 \xrightarrow{d_C^0} X^2 \oplus Y^1 \xrightarrow{d_C^1} \cdots \xrightarrow{d_C^n} \Sigma X^0 \oplus Y^{n+1} \xrightarrow{d_C^{n+1}} \Sigma X^1 \oplus \Sigma Y^0,
$$

in which

$$
d_C^i := \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix},
$$

for $i \in \{0, \ldots, n+1\}$, where $d_X^{n+2} := \Sigma d_X^0$ and $f^{n+2} := \Sigma f^0$.

The definition of a (pre-)$(n+2)$-angulated category, as originally given in [18], is the following.

Definition 2.5. [18, Def. 2.1] Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a weak (pre-)$(n+2)$-angulated category. We call $(\mathcal{C}, \Sigma, \mathcal{T})$ a *(pre-)$(n+2)$-angulated category* if Σ is an automorphism of $\mathcal{C}$.

Sometimes will use the adjective 'strong' for a (pre-)$(n+2)$-angulated category (in the sense of Definition 2.5 above) in order to highlight that the $(n+2)$-suspension functor is an automorphism. As is normal practice,

we will usually just talk of a (weak) (pre-)$(n+2)$-angulated category $\mathcal{C}$ and assume the existence of $\mathcal{T}$ and Σ is understood.

Example 2.6. With $n = 1$, an $(n+2)$-angulated category (in the sense of Definition 2.5) is the same as a triangulated category in the sense of [24, Def. 3.1]. See [36, Def. 3.1, Def. 3.2, pp. 246–248].

Definition 2.7. [4, §4] Let $(\mathcal{C}, \Sigma, \mathcal{T})$ and $(\mathcal{C}', \Sigma', \mathcal{T}')$ be weak (pre-)$(n+2)$-angulated categories. An additive covariant functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ is called an $(n+2)$-*angulated* functor if there exists a natural isomorphism $\Theta = \{\Theta_X\}_{X \in \mathcal{C}}\colon \mathscr{F} \circ \Sigma \xRightarrow{\cong} \Sigma' \circ \mathscr{F}$, such that if

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0$$

is an $(n+2)$-angle in $\mathcal{C}$, then

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{X^0} \circ \mathscr{F}d_X^{n+1}} \Sigma'\mathscr{F}X^0$$

is an $(n+2)$-angle in $\mathcal{C}'$.

We remark that $(n+2)$-angulated functors as we refer to them here were called 'exact' functors in [25]. The last piece of terminology we need for this setting is the following.

Definition 2.8. An $(n+2)$-angulated functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ is called an $(n+2)$-*angle equivalence* if it is an equivalence of categories.

2.2. n-exact and n-abelian categories. In §2.2 we follow [25] in recalling some prerequisite concepts from higher homological algebra and the definitions of an n-exact category and an n-abelian category (see Definitions 2.16 and 2.17, respectively). At the end of §2.2 we define an n-exact functor (see Definition 2.18). We denote by Ab the category of all abelian groups.

Definition 2.9. [25, Def. 2.2] Let $d_X^n\colon X^n \to X^{n+1}$ be a morphism in $\mathcal{C}$. An n-*kernel* of d_X^n is a sequence

$$(d_X^0, \ldots, d_X^{n-1})\colon \quad X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n$$

of composable morphisms in $\mathcal{C}$, such that for all $Z \in \mathcal{C}$ the functor $\mathcal{C}(Z, -)$ yields an exact sequence

$$0 \longrightarrow \mathcal{C}(Z, X^0) \xrightarrow{d_X^0 \circ -} \mathcal{C}(Z, X^1) \xrightarrow{d_X^1 \circ -} \cdots \xrightarrow{d_X^{n-1} \circ -} \mathcal{C}(Z, X^n) \xrightarrow{d_X^n \circ -} \mathcal{C}(Z, X^{n+1})$$

in Ab. An n-*cokernel* is defined dually.

Let $\mathsf{C}_\mathcal{C}$ denote the *category of (cochain) complexes* in $\mathcal{C}$. We denote by $\mathsf{C}_\mathcal{C}^n$ the full subcategory of $\mathsf{C}_\mathcal{C}$ consisting of complexes which are concentrated in degrees $0, \ldots, n+1$. We denote a complex $X^\bullet \in \mathsf{C}_\mathcal{C}^n$ by

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1}.$$

Definition 2.10. [25, Def. 2.4] An n-*exact sequence* in $\mathcal{C}$ is a complex $X^\bullet \in \mathsf{C}_\mathcal{C}^n$, such that $(d_X^0, \ldots, d_X^{n-1})$ is an n-kernel of d_X^n and $(d_X^1, \ldots, d_X^n)$ is an n-cokernel of d_X^0.

Definition 2.11. [25, Def. 2.9] Let $X^\bullet, Y^\bullet \in \mathsf{C}_\mathcal{C}^n$ be n-exact sequences. A *morphism* $f^\bullet\colon X^\bullet \to Y^\bullet$ *of n-exact sequences* is just a morphism $f^\bullet \in \mathsf{C}_\mathcal{C}^n(X^\bullet, Y^\bullet)$ of complexes.

Definition 2.12. [25, Def. 2.11] Let $f^\bullet\colon X^\bullet \to Y^\bullet$ be a morphism of complexes in $\mathsf{C}_\mathcal{C}^{n-1}$. The *mapping cone* $C^\bullet := \mathrm{MC}(f)^\bullet$ is the complex

$$X^0 \xrightarrow{d_C^{-1}} X^1 \oplus Y^0 \xrightarrow{d_C^0} X^2 \oplus Y^1 \xrightarrow{d_C^1} \cdots \xrightarrow{d_C^{n-2}} X^n \oplus Y^{n-1} \xrightarrow{d_C^{n-1}} Y^n$$

in $\mathsf{C}_{\mathcal{C}}^n$, where $d_C^{-1} := \begin{pmatrix} -d_X^0 \\ f^0 \end{pmatrix}$, $d_C^{n-1} := (\, f^n \quad d_Y^{n-1})$, and

$$d_C^i := \begin{pmatrix} -d_X^{i+1} & 0 \\ f^{i+1} & d_Y^i \end{pmatrix}$$

for $i \in \{0, \dots, n-2\}$.

Definition 2.13. [25, Def. 2.11] Let $X^\bullet \in \mathsf{C}_{\mathcal{C}}^{n-1}$ be a complex. Suppose $g^n \colon Z^n \to X^n$ is a morphism in $\mathcal{C}$. An n-*pullback of* $(d_X^0, \dots, d_X^{n-1})$ *along* g^n is a morphism of complexes

$$
\begin{array}{ccccccccc}
Z^0 & \xrightarrow{d_Z^0} & Z^1 & \xrightarrow{d_Z^1} & \cdots & \xrightarrow{d_Z^{n-2}} & Z^{n-1} & \xrightarrow{d_Z^{n-1}} & Z^n \\
\downarrow{g^0} & & \downarrow{g^1} & & & & \downarrow{g^{n-1}} & & \downarrow{g^n} \\
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-2}} & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n,
\end{array}
$$

such that in the mapping cone

$$C^\bullet := \mathrm{MC}(g)^\bullet \colon \qquad Z^0 \xrightarrow{d_C^{-1}} Z^1 \oplus X^0 \xrightarrow{d_C^0} Z^2 \oplus X^1 \xrightarrow{d_C^1} \cdots \xrightarrow{d_C^{n-2}} Z^n \oplus X^{n-1} \xrightarrow{d_C^{n-1}} X^n$$

the sequence $(d_C^{-1}, \dots, d_C^{n-2})$ is an n-kernel of d_C^{n-1}. An n-*pushout* is defined dually.

Definition 2.14. [25, §4.1] Fix a class $\mathcal{X}$ of n-exact sequences in $\mathcal{C}$. The members of $\mathcal{X}$ are called $\mathcal{X}$-*admissible n-exact sequences*. If $X^\bullet$ is an $\mathcal{X}$-admissible n-exact sequence, then $d_X^0 \colon X^0 \rightarrowtail X^1$ is called an $\mathcal{X}$-*admissible monomorphism* and $d_X^n \colon X^n \twoheadrightarrow X^{n+1}$ is called an $\mathcal{X}$-*admissible epimorphism*.

Definition 2.15. [25, Def. 4.1] Let $f^\bullet \colon X^\bullet \to Y^\bullet$ be a morphism of n-exact sequences in $\mathsf{C}_{\mathcal{C}}^n$. If there exists $i \in \{0, 1, \dots, n+1\}$ such that f^i and f^{i+1} (where $f^{n+2} := f^0$) are isomorphisms, then we call $f^\bullet$ a *weak isomorphism*, and we say $X^\bullet$ and $Y^\bullet$ are *weakly isomorphic*.

We are now able to recall the definitions of an n-exact category and an n-abelian category.

Definition 2.16. [25, Def. 4.2] Let $n \geqslant 1$ be a positive integer and let $\mathcal{C}$ be an additive category. Suppose $\mathcal{X}$ is a class of n-exact sequences in $\mathcal{C}$ that is closed under weak isomorphisms. The pair $(\mathcal{C}, \mathcal{X})$ forms an n-*exact category* if the following axioms are satisfied.

(n-E0)   The sequence $0 \rightarrowtail 0 \to \cdots \to 0 \twoheadrightarrow 0$ is an $\mathcal{X}$-admissible n-exact sequence.

(n-E1)   The class of $\mathcal{X}$-admissible monomorphisms is closed under composition.

(n-E1$^{\mathrm{op}}$)   The class of $\mathcal{X}$-admissible epimorphisms is closed under composition.

(n-E2)   For each $\mathcal{X}$-admissible n-exact sequence $X^\bullet$ and each morphism $f^0 \colon X^0 \to Y^0$, there exists an n-pushout

$$
\begin{array}{ccccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-2}} & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
\downarrow{f^0} & & \downarrow{f^1} & & & & & & \downarrow{f^{n-1}} & & \downarrow{f^n} \\
Y^0 & \dashrightarrow & Y^1 & \dashrightarrow & \cdots & \dashrightarrow & Y^{n-1} & \dashrightarrow & Y^n & &
\end{array}
$$

of $(d_X^0, \dots, d_X^{n-1})$ along f^0 such that d_Y^0 is an $\mathcal{X}$-admissible monomorphism.

(n-E2$^{\mathrm{op}}$)   Dual of (n-E2).

Definition 2.17. [25, Def. 3.1] Let $n \geqslant 1$ be a positive integer. An additive category $\mathcal{C}$ is called n-*abelian* if the following axioms are satisfied.

(n-A0)   The category $\mathcal{C}$ has split idempotents.

(n-A1)   Every morphism in $\mathcal{C}$ has an n-kernel and an n-cokernel.

(n-A2)   For each monomorphism $d_X^0 \colon X^0 \hookrightarrow X^1$ in $\mathcal{C}$ and every n-cokernel $(d_X^1, \dots, d_X^n)$ of d_X^0, the sequence

$$X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1}$$

is n-exact.

(n-A2$^{\mathrm{op}}$) Dual of (n-A2).

It was shown in [25] that every n-abelian category is n-exact; see [25, Thm. 4.4].

For an additive covariant functor $\mathscr{F}\colon \mathcal{A} \to \mathcal{B}$ between additive categories, we denote by $\mathscr{F}_{\mathsf{C}}\colon \mathsf{C}_{\mathcal{A}} \to \mathsf{C}_{\mathcal{B}}$ the induced additive covariant functor between the categories of complexes. More precisely, for complexes $X^{\bullet}, Y^{\bullet} \in \mathsf{C}_{\mathcal{A}}$ and any morphism $f^{\bullet} \in \mathsf{C}_{\mathcal{A}}(X^{\bullet}, Y^{\bullet})$, the complex $\mathscr{F}_{\mathsf{C}}(X^{\bullet}) \in \mathsf{C}_{\mathcal{B}}$ is

$$\cdots \longrightarrow \mathscr{F}X^{-1} \xrightarrow{\mathscr{F}d_X^{-1}} \mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots$$

and the morphism $\mathscr{F}_{\mathsf{C}}(f^{\bullet}) \in \mathsf{C}_{\mathcal{B}}(\mathscr{F}_{\mathsf{C}}X^{\bullet}, \mathscr{F}_{\mathsf{C}}Y^{\bullet})$ is given by the commutative diagram

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathscr{F}X^{-1} & \xrightarrow{\mathscr{F}d_X^{-1}} & \mathscr{F}X^0 & \xrightarrow{\mathscr{F}d_X^0} & \mathscr{F}X^1 & \xrightarrow{\mathscr{F}d_X^1} & \cdots \\
& \downarrow{\scriptstyle \mathscr{F}f^{-1}} & & \downarrow{\scriptstyle \mathscr{F}f^0} & & \downarrow{\scriptstyle \mathscr{F}f^1} & & \\
\cdots \longrightarrow & \mathscr{F}Y^{-1} & \xrightarrow{\mathscr{F}d_Y^{-1}} & \mathscr{F}Y^0 & \xrightarrow{\mathscr{F}d_Y^0} & \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_Y^1} & \cdots.
\end{array}
$$

Now we introduce the higher version of an exact functor between exact categories; see [10, Def. 5.1].

Definition 2.18. Let $(\mathcal{C}, \mathcal{X})$ and $(\mathcal{C}', \mathcal{X}')$ be n-exact categories. An additive covariant functor $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ is called an *n-exact* functor if it sends $\mathcal{X}$-admissible n-exact sequences to $\mathcal{X}'$-admissible n-exact sequences, i.e. $\mathscr{F}_{\mathsf{C}}(\mathcal{X}) \subseteq \mathcal{X}'$. If $\mathscr{F}$ is n-exact and an equivalence, then we call $\mathscr{F}$ an *n-exact equivalence*.

Remark 2.19. We note here that the terminology 'n-exact functor' has appeared in [30, §4.1]. However, in [30] an 'n-exact functor' is an additive covariant functor from an n-abelian category to an *abelian* category that maps n-exact sequences to exact sequences. On the other hand, what we call an n-exact functor is a direct generalisation of an exact functor (between exact categories).

2.3. **n-exangulated categories.** In §2.3 we follow [22, §2] in order to recall the definition of an n-exangulated category (see Definition 2.28). We also recall how one can view $(n+2)$-angulated and n-exact categories as n-exangulated categories (see Examples 2.30 and 2.31, respectively). Lastly, we define an n-exangulated functor (see Definition 2.32), and we relate this notion to $(n+2)$-angulated functors and n-exact functors (see Theorems 2.33 and 2.34, respectively).

Setup 2.20. We assume $\mathcal{C}$ comes equipped with a biadditive functor $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ (which is the same as an 'additive bifunctor' in the sense of [15, §1.2, p. 649]). Thus, for fixed objects A, C in $\mathcal{C}$, the functors $\mathbb{E}(C, -)\colon \mathcal{C} \to \mathsf{Ab}$ and $\mathbb{E}(-, A)\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$ are additive. Furthermore, each morphism $f\colon X \to Y$ in $\mathcal{C}$ gives rise to abelian group homomorphisms $\mathbb{E}(C, f)\colon \mathbb{E}(C, X) \to \mathbb{E}(C, Y)$ and $\mathbb{E}(f^{\mathrm{op}}, A)\colon \mathbb{E}(Y, A) \to \mathbb{E}(X, A)$. To simplify notation, we will omit the superscript "op", writing $\mathbb{E}(f, A)$ instead of $\mathbb{E}(f^{\mathrm{op}}, A)$.

Definition 2.21. [22, Def. 2.1, Rem. 2.2, Def. 2.3] By an $\mathbb{E}$-*extension* (or simply an *extension*) we mean an element δ of $\mathbb{E}(C, A)$ for some objects A, C in $\mathcal{C}$.

For morphisms $x\colon A \to X$, $z\colon Z \to C$ in $\mathcal{C}$ and for an extension $\delta \in \mathbb{E}(C, A)$, we obtain new extensions as follows:

$$x_{\mathbb{E}}\delta := \mathbb{E}(C, x)(\delta) \in \mathbb{E}(C, X) \qquad \text{and} \qquad z^{\mathbb{E}}\delta := \mathbb{E}(z, A)(\delta) \in \mathbb{E}(Z, A).$$

Let $\delta \in \mathbb{E}(C, A)$ and $\varepsilon \in \mathbb{E}(D, B)$ be extensions. A *morphism of $\mathbb{E}$-extensions* $\delta \to \varepsilon$ is a pair (a, c) of morphisms $a\colon A \to B$ and $c\colon C \to D$ in $\mathcal{C}$, such that $a_{\mathbb{E}}\delta = c^{\mathbb{E}}\varepsilon$.

By the Yoneda Lemma, each $\mathbb{E}$-extension gives rise to two natural transformations as follows.

Definition 2.22. [22, Def. 2.11] For any A, C in $\mathcal{C}$ and any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$, there is an induced natural transformation $\delta_{\mathbb{E}}^{\sharp}\colon \mathcal{C}(A, -) \Rightarrow \mathbb{E}(C, -)$, which is given by $(\delta_{\mathbb{E}}^{\sharp})_B(a) = a_{\mathbb{E}}\delta$ for each object $B \in \mathcal{C}$ and each morphism $a\colon A \to B$. Dually, there is also a natural transformation $\delta_{\sharp}^{\mathbb{E}}\colon \mathcal{C}(-, C) \Rightarrow \mathbb{E}(-, A)$, which is given by $(\delta_{\sharp}^{\mathbb{E}})_D(d) = d^{\mathbb{E}}\delta$ for each object $D \in \mathcal{C}$ and each morphism $d\colon D \to C$.

We will be particularly interested in pairs $\langle X^\bullet, \delta \rangle$ of complexes and extensions for which the natural transformations $\delta_{\mathbb{E}}^\sharp$ and $\delta_\sharp^{\mathbb{E}}$ fit into exact sequences induced by Hom-functors.

Definition 2.23. [22, Def. 2.13] A pair $\langle X^\bullet, \delta \rangle$, where $X^\bullet \in \mathsf{C}_\mathcal{C}^n$ and $\delta \in \mathbb{E}(X^{n+1}, X^0)$, is called an *n-exangle* if, for every object Z in $\mathcal{C}$, the sequences

$$\mathcal{C}(Z, X^0) \xrightarrow{d_X^0 \circ -} \mathcal{C}(Z, X^1) \xrightarrow{d_X^1 \circ -} \cdots \xrightarrow{d_X^n \circ -} \mathcal{C}(Z, X^{n+1}) \xrightarrow{(\delta_\sharp^{\mathbb{E}})_Z} \mathbb{E}(Z, X^0)$$

and

$$\mathcal{C}(X^{n+1}, Z) \xrightarrow{- \circ d_X^n} \mathcal{C}(X^n, Z) \xrightarrow{- \circ d_X^{n-1}} \cdots \xrightarrow{- \circ d_X^0} \mathcal{C}(X^0, Z) \xrightarrow{(\delta_{\mathbb{E}}^\sharp)_Z} \mathbb{E}(X^{n+1}, Z)$$

in Ab are exact.

In [22] and [35] the notation $x_* \delta$, respectively $z^* \delta$, is used to label the extensions $x_{\mathbb{E}} \delta$, respectively $z^{\mathbb{E}} \delta$. The reason for our change in notation become apparent later; see Remark 3.7.

Definition 2.24. [22, Def. 2.17] Let A, C be objects in $\mathcal{C}$. We define the subcategory $\mathsf{C}_{(A,C)}^n$ of $\mathsf{C}_\mathcal{C}^n$ in the following way. The objects of $\mathsf{C}_{(A,C)}^n$ consist of those complexes $X^\bullet \in \mathsf{C}_\mathcal{C}^n$ for which $X^0 = A$ and $X^{n+1} = C$. For $X^\bullet, Y^\bullet \in \mathsf{C}_{(A,C)}^n$, we set

$$\mathsf{C}_{(A,C)}^n(X^\bullet, Y^\bullet) \coloneqq \{ f^\bullet \in \mathsf{C}_\mathcal{C}^n(X^\bullet, Y^\bullet) \mid f^0 = 1_A \text{ and } f^{n+1} = 1_C \}.$$

For morphisms $f^\bullet, g^\bullet \colon X^\bullet \to Y^\bullet$ in $\mathsf{C}_\mathcal{C}$, we write $f^\bullet \sim g^\bullet$ if $f^\bullet$ and $g^\bullet$ are *homotopic* in the usual sense. Recall that $\sim$ gives an equivalence relation on $\mathsf{C}_\mathcal{C}(X^\bullet, Y^\bullet)$ for all complexes $X^\bullet, Y^\bullet$. Now fix $A, C \in \mathcal{C}$ and let $X^\bullet, Y^\bullet \in \mathsf{C}_{(A,C)}^n$ be arbitrary. Note that the set $\mathsf{C}_{(A,C)}^n(X^\bullet, Y^\bullet)$ is no longer an abelian group (see [22, p. 540]). However, the homotopy relation $\sim$ on $\mathsf{C}_\mathcal{C}^n(X^\bullet, Y^\bullet)$ restricts to an equivalence relation on $\mathsf{C}_{(A,C)}^n(X^\bullet, Y^\bullet)$, which we again denote by $\sim$.

Definition 2.25. [22, §2.3] By $\mathsf{K}_{(A,C)}^n$ we denote the category whose objects are those of $\mathsf{C}_{(A,C)}^n$ and, for $X^\bullet, Y^\bullet \in \mathsf{K}_{(A,C)}^n$, we put

$$\mathsf{K}_{(A,C)}^n(X^\bullet, Y^\bullet) \coloneqq \mathsf{C}_{(A,C)}^n(X^\bullet, Y^\bullet)/\sim .$$

Given a morphism $f^\bullet \in \mathsf{C}_{(A,C)}^n(X^\bullet, Y^\bullet)$, we will call $f^\bullet$ a *homotopy equivalence* if it induces an isomorphism in $\mathsf{K}_{(A,C)}^n(X^\bullet, Y^\bullet)$. In this case, we will say the complexes $X^\bullet$ and $Y^\bullet$ are *homotopy equivalent*. We denote by $[X^\bullet]$ the homotopy equivalence class of $X^\bullet$ in $\mathsf{C}_{(A,C)}^n$. (Note that this may not coincide with the usual homotopy equivalence class of $X^\bullet$ in $\mathsf{C}_\mathcal{C}^n$; see [22, Rem. 2.18].)

Definition 2.26. [22, Def. 2.22] Let $\mathfrak{s}$ be a correspondence that, for each $A, C \in \mathcal{C}$, assigns to each extension $\delta \in \mathbb{E}(C, A)$ a homotopy equivalence class $\mathfrak{s}(\delta) = [X^\bullet]$ where $X^\bullet \in \mathsf{C}_{(A,C)}^n$. We say that $\mathfrak{s}$ is a *realisation of* $\mathbb{E}$ if

(R0) for all $\delta \in \mathbb{E}(C, A)$ and $\varepsilon \in \mathbb{E}(D, B)$ with $\mathfrak{s}(\delta) = [X^\bullet]$ and $\mathfrak{s}(\varepsilon) = [Y^\bullet]$, and for all morphisms $(a, c) \colon \delta \to \varepsilon$ of $\mathbb{E}$-extensions, there exists a morphism $f^\bullet = (f^0, \ldots, f^{n+1}) \in \mathsf{C}_\mathcal{C}^n(X^\bullet, Y^\bullet)$ with $f^0 = a$ and $f^{n+1} = c$.

In this case, we say $X^\bullet$ *realises* δ (or δ is *realised by* $X^\bullet$) and that $f^\bullet$ *realises* (a, c) (or (a, c) is *realised by* $f^\bullet$). For objects $A, C \in \mathcal{C}$, we write $_A0_C$ for the identity element in the abelian group $\mathbb{E}(C, A)$.

A realisation $\mathfrak{s}$ of $\mathbb{E}$ is said to be *exact* if

(R1) the pair $\langle X^\bullet, \delta \rangle$ is an *n-exangle* whenever $\mathfrak{s}(\delta) = [X^\bullet]$; and

(R2) for any object A in $\mathcal{C}$, we have that $\mathfrak{s}(_A0_0) = [\ A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\ ]$ and $\mathfrak{s}(_00_A) = [\ 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{1_A} A\ ]$.

Let us recall some useful terminology from [22].

Definition 2.27. [22, Def. 2.23] Let $\mathfrak{s}$ be an exact realisation of $\mathbb{E}$. Suppose $\mathfrak{s}(\delta) = [X^\bullet]$ for some extension $\delta \in \mathbb{E}(C, A)$ and $X^\bullet \in \mathsf{C}_{(A,C)}^n$. Then we say $\langle X^\bullet, \delta \rangle$ is an $\mathfrak{s}$-*distinguished n-exangle* and $X^\bullet$ is an $\mathfrak{s}$-*conflation*.

A morphism f in $\mathcal{C}$ is called an $\mathfrak{s}$-*inflation* (respectively, $\mathfrak{s}$-*deflation*) if there exists an $\mathfrak{s}$-conflation $X^\bullet$ with $d_X^0 = f$ (respectively, $d_X^n = f$).

We are now in a position to recall the definition of an n-exangulated category.

Definition 2.28. [22, Def. 2.32] Recall that $n \geqslant 1$ is a positive integer, $\mathcal{C}$ is an additive category and $\mathbb{E} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ is a biadditive functor. Let $\mathfrak{s}$ be an exact realisation of $\mathbb{E}$. Then $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called an *n-exangulated category* if the following axioms are satisfied.

(n-EA1)   The class of $\mathfrak{s}$-inflations is closed under composition. Dually, the class of $\mathfrak{s}$-deflations is closed under composition.

(n-EA2)   For each $\delta \in \mathbb{E}(D, A)$ and each $c \in \mathcal{C}(C, D)$, if $\mathfrak{s}(c^{\mathbb{E}}\delta) = [X^\bullet]$ and $\mathfrak{s}(\delta) = [Y^\bullet]$, then there exists a morphism $f^\bullet = (1_A, f^1, \ldots, f^n, c) \colon X^\bullet \to Y^\bullet$ realising $(1_A, c) \colon c^{\mathbb{E}}\delta \to \delta$ such that $\mathfrak{s}((d_X^0)_{\mathbb{E}}\delta) = [\mathrm{MC}(\widehat{f})^\bullet]$, where $\widehat{f}^\bullet = (f^1, \ldots, f^n, c)$. Such an $f^\bullet$ is called a *good lift* of $(1_A, c)$.

(n-EA2$^{\mathrm{op}}$)   Dual of (n-EA2).

Example 2.29. A category is 1-exangulated if and only if it is *extriangulated* (in the sense of [35]); see [22, Prop. 4.3].

Let us recall how $(n + 2)$-angulated and n-exact categories provide examples of n-exangulated categories.

Example 2.30. Suppose $(\mathcal{C}, \Sigma, \mathcal{T})$ is an $(n + 2)$-angulated category. From this information, we obtain a functor $\mathbb{E}_\Sigma \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ by setting $\mathbb{E}_\Sigma(X^{n+1}, X^0) \coloneqq \mathcal{C}(X^{n+1}, \Sigma X^0)$ for any X^0, X^{n+1} in $\mathcal{C}$, and defining the map $\mathbb{E}_\Sigma(f, g) \colon \mathbb{E}_\Sigma(X^{n+1}, X^0) \to \mathbb{E}_\Sigma(Y^{n+1}, Y^0)$ by $\mathbb{E}_\Sigma(f, g)(\delta) \coloneqq (\Sigma g) \circ \delta \circ f$ for any $f \colon Y^{n+1} \to X^{n+1}$ and $g \colon X^0 \to Y^0$. An exact realisation $\mathfrak{s}$ of $\mathbb{E}_\Sigma$ is given as follows: for an element $\delta \in \mathbb{E}_\Sigma(X^{n+1}, X^0)$, complete it to an $(n + 2)$-angle $X^0 \to \cdots \to X^{n+1} \xrightarrow{\delta} \Sigma X^0$ and set $\mathfrak{s}(\delta) = [X^0 \to \cdots \to X^{n+1}]$. Then $(\mathcal{C}, \mathbb{E}_\Sigma, \mathfrak{s})$ is an n-exangulated category. See [22, §4.2] for more details.

Example 2.31. Suppose $(\mathcal{C}, \mathcal{X})$ is an n-exact category. For any X^0, X^{n+1} in $\mathcal{C}$, denote by $\Lambda^n_{(X^0, X^{n+1})}$ the class of all homotopy equivalence classes of n-exact sequences in $\mathsf{C}^n_{(X^0, X^{n+1})}$. Then define $\mathbb{E}_\mathcal{X}(X^{n+1}, X^0)$ to be the subclass of $\Lambda^n_{(X^0, X^{n+1})}$ consisting of those classes $[X^\bullet]$ such that $X^\bullet \in \mathcal{X}$. We assume that $\mathbb{E}_\mathcal{X}(X^{n+1}, X^0)$ is a set for all $X^0, X^{n+1} \in \mathcal{C}$. For any $\mathcal{X}$-admissible n-exact sequence $X^\bullet$ and any morphism $f \colon X^0 \to Y^0$, we can form an n-pushout of $(d_X^0, \ldots, d_X^{n-1})$ along f to obtain a morphism

$$
\begin{array}{ccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^n} & & \| \\
Y^0 & \dashrightarrow[d_Y^0] & Y^1 & \dashrightarrow[d_Y^1] & \cdots & \dashrightarrow[d_Y^{n-1}] & Y^n & \xrightarrow{d_Y^n} & X^{n+1}
\end{array}
$$

of $\mathcal{X}$-admissible n-exact sequences by [25, Prop. 4.8]. Thus, set $\mathbb{E}_\mathcal{X}(X^{n+1}, f)([X^\bullet]) = [Y^\bullet]$. We can define $\mathbb{E}_\mathcal{X}(g, X^0)$ in a dual manner. Then $\mathbb{E}_\mathcal{X} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ is a well-defined biadditive functor. Lastly, the assignment $\mathfrak{s}(\delta) = [X^\bullet]$ for each $\delta = [X^\bullet] \in \mathbb{E}_\mathcal{X}(X^{n+1}, X^0)$ gives an exact realisation of $\mathbb{E}_\mathcal{X}$, and $(\mathcal{C}, \mathbb{E}_\mathcal{X}, \mathfrak{s})$ is an n-exangulated category. See [22, §4.3] for more details.

For any functor $\mathscr{F} \colon \mathcal{A} \to \mathcal{B}$, we denote by $\mathscr{F}^{\mathrm{op}}$ the *opposite* functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ given by $\mathscr{F}^{\mathrm{op}}(A) = \mathscr{F}A$ and $\mathscr{F}(f^{\mathrm{op}}) = (\mathscr{F}f)^{\mathrm{op}}$ for any objects $A, B \in \mathcal{A}$ and any morphism $f \in \mathcal{A}(A, B)$; see [33, §IV.5].

We now introduce the notion of an n-exangulated functor, which appears to be new.

Definition 2.32. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ be n-exangulated categories. An additive covariant functor $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is called an *n-exangulated* functor if there exists a natural transformation

$$
\Gamma = \{\Gamma_{(C,A)}\}_{(C,A) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}} \colon \mathbb{E}(-, -) \Longrightarrow \mathbb{E}'(\mathscr{F}^{\mathrm{op}}-, \mathscr{F}-)
$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$, such that $\mathfrak{s}(\delta) = [X^\bullet]$ implies $\mathfrak{s}'(\Gamma_{(X^{n+1}, X^0)}(\delta)) = [\mathscr{F}_\mathsf{C} X^\bullet]$. To simplify notation, we write $\mathbb{E}'(\mathscr{F}-, \mathscr{F}-)$ in place of $\mathbb{E}'(\mathscr{F}^{\mathrm{op}}-, \mathscr{F}-)$. An n-exangulated functor $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is called an *n-exangle equivalence* if it is an equivalence of categories.

In particular, we call a 1-exangulated functor an *extriangulated* functor.

The following two results show that this definition specialises to recover the definitions of an $(n + 2)$-angulated functor and an n-exact functor.

Theorem 2.33. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ and $(\mathcal{C}', \Sigma', \mathcal{T}')$ be $(n + 2)$-angulated categories. Following the notation of Example 2.30, set $\mathbb{E} := \mathbb{E}_\Sigma$ and $\mathbb{E}' := \mathbb{E}_{\Sigma'}$, and consider the n-exangulated categories $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$. Suppose $\mathscr{F} : \mathcal{C} \to \mathcal{C}'$ is an additive covariant functor. Then $\mathscr{F}$ is an $(n + 2)$-angulated functor if and only if $\mathscr{F}$ is an n-exangulated functor.*

Proof. ($\Rightarrow$) Suppose first that $\mathscr{F}$ is an $(n + 2)$-angulated functor. Then there exists a natural isomorphism $\Theta : \mathscr{F}\Sigma \overset{\cong}{\Longrightarrow} \Sigma'\mathscr{F}$, such that for any $(n + 2)$-angle $X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0$ in $\mathcal{C}$,

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{X^0} \circ \mathscr{F}d_X^{n+1}} \Sigma'\mathscr{F}X^0$$

is an $(n + 2)$-angle in $\mathcal{C}'$.

For each X^0, X^{n+1} in $\mathcal{C}$, let $\Gamma_{(X^{n+1}, X^0)}$ be the composition:

$$\mathcal{C}(X^{n+1}, \Sigma X^0) \xrightarrow{\mathscr{F}_{X^{n+1}, \Sigma X^0}} \mathcal{C}'(\mathscr{F}X^{n+1}, \mathscr{F}\Sigma X^0) \xrightarrow{\Theta_{X^0} \circ -} \mathcal{C}'(\mathscr{F}X^{n+1}, \Sigma'\mathscr{F}X^0).$$

This gives a homomorphism $\Gamma_{(X^{n+1}, X^0)} : \mathbb{E}(X^{n+1}, X^0) \to \mathbb{E}'(\mathscr{F}X^{n+1}, \mathscr{F}X^0)$. We claim that the collection $\Gamma = \{\Gamma_{(X^{n+1}, X^0)}\}_{(X^{n+1}, X^0) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}}$ gives a natural transformation $\mathbb{E}(-, -) \Longrightarrow \mathbb{E}'(\mathscr{F}-, \mathscr{F}-)$.

To this end, suppose $f : Y^{n+1} \to X^{n+1}$ and $g : X^0 \to Y^0$ are arbitrary morphisms in $\mathcal{C}$. Then we must show that the square

$$
\begin{array}{ccc}
\mathbb{E}(X^{n+1}, X^0) & \xrightarrow{\Gamma_{(X^{n+1}, X^0)}} & \mathbb{E}'(\mathscr{F}X^{n+1}, \mathscr{F}X^0) \\
\downarrow{\scriptstyle \mathbb{E}(f,g)} & & \downarrow{\scriptstyle \mathbb{E}'(\mathscr{F}f, \mathscr{F}g)} \\
\mathbb{E}(Y^{n+1}, Y^0) & \xrightarrow{\Gamma_{(Y^{n+1}, Y^0)}} & \mathbb{E}'(\mathscr{F}Y^{n+1}, \mathscr{F}Y^0)
\end{array}
\tag{2.3}
$$

commutes. Note that for any $\delta \in \mathbb{E}(X^{n+1}, X^0)$ we have

$$\mathbb{E}'(\mathscr{F}f, \mathscr{F}g)(\Gamma_{(X^{n+1}, X^0)}(\delta)) = \mathbb{E}'(\mathscr{F}f, \mathscr{F}g)(\Theta_{X^0}\mathscr{F}\delta) = (\Sigma'\mathscr{F}g)\Theta_{X^0}(\mathscr{F}\delta)(\mathscr{F}f).$$

On the other hand,

$$\Gamma_{(Y^{n+1}, Y^0)}(\mathbb{E}(f, g)(\delta)) = \Gamma_{(Y^{n+1}, Y^0)}((\Sigma g)\delta f) = \Theta_{Y^0}(\mathscr{F}\Sigma g)(\mathscr{F}\delta)(\mathscr{F}f),$$

which is equal to $(\Sigma'\mathscr{F}g)\Theta_{X^0}(\mathscr{F}\delta)(\mathscr{F}f)$ since $(\Sigma'\mathscr{F}g)\Theta_{X^0} = \Theta_{Y^0}(\mathscr{F}\Sigma g)$ as Θ is natural. So (2.3) commutes and Γ is indeed a natural transformation.

Lastly, suppose $\mathfrak{s}(\delta) = [X^\bullet]$. That is, $X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{\delta} \Sigma X^0$ is an $(n + 2)$-angle in $\mathcal{C}$. Then, as $\mathscr{F}$ is an $(n + 2)$-angulated functor,

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{X^0} \circ \mathscr{F}\delta} \Sigma'\mathscr{F}X^0$$

is an $(n + 2)$-angle in $\mathcal{C}'$. Since $\Gamma_{(X^{n+1}, X^0)}(\delta) = \Theta_{X^0} \circ \mathscr{F}\delta$, we see that $\mathfrak{s}'(\Gamma_{(X^{n+1}, X^0)}(\delta)) = [\mathscr{F}_\mathsf{C}X^\bullet]$. Hence, $\mathscr{F}$ satisfies Definition 2.32.

($\Leftarrow$) Conversely, assume $\mathscr{F}$ satisfies Definition 2.32 with natural transformation $\Gamma : \mathbb{E}(-, -) \Rightarrow \mathbb{E}'(\mathscr{F}-, \mathscr{F}-)$. Let $X \in \mathcal{C}$ be arbitrary. Setting $X^{n+1} = \Sigma X$ and $X^0 = X$, we obtain a morphism

$$\mathcal{C}(\Sigma X, \Sigma X) = \mathbb{E}(\Sigma X, X) \xrightarrow{\Gamma_{(\Sigma X, X)}} \mathbb{E}'(\mathscr{F}\Sigma X, \mathscr{F}X) = \mathcal{C}'(\mathscr{F}\Sigma X, \Sigma'\mathscr{F}X).$$

Set $\Theta_X := \Gamma_{(\Sigma X, X)}(1_{\Sigma X}) : \mathscr{F}\Sigma X \to \Sigma'\mathscr{F}X$.

We claim that $\Theta = \{\Theta_X\}_{X \in \mathcal{C}}$ is a natural isomorphism $\mathscr{F}\Sigma \Rightarrow \Sigma'\mathscr{F}$. Let $a : X \to X'$ be an arbitrary morphism in $\mathcal{C}$. Then we must show that $(\Sigma'\mathscr{F}a)\Theta_X = \Theta_{X'}(\mathscr{F}\Sigma a)$. Since (2.3) commutes for all $f : Y^{n+1} \to X^{n+1}$ and $g : X^0 \to Y^0$ in $\mathcal{C}$, we can first choose $f = 1_{\Sigma X}$ and $g = a$, and then, using $\delta = 1_{\Sigma X} \in \mathbb{E}(\Sigma X, X)$, we

obtain $(\Sigma'\mathscr{F}a)\Theta_X = \Gamma_{(\Sigma X, X')}(\Sigma a)$. Secondly, choosing $f = \Sigma a, g = 1_{X'}$ and $\delta = 1_{\Sigma X'} \in \mathbb{E}(\Sigma X', X')$ yields $\Gamma_{(\Sigma X, X')}(\Sigma a) = \Theta_{X'}(\mathscr{F}\Sigma a)$. Consequently, $(\Sigma'\mathscr{F}a)\Theta_X = \Theta_{X'}(\mathscr{F}\Sigma a)$ and Θ is a natural transformation.

To show Θ_X is an isomorphism for each $X \in \mathcal{C}$, consider the $(n+2)$-angle

$$X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X \xrightarrow{1_{\Sigma X}} \Sigma X$$

in $\mathcal{C}$. Denote by $Z^\bullet$ the complex $X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X$ in $\mathsf{C}^n_{(X, \Sigma X)}$. Since $\mathscr{F}$ is an n-exangulated functor and $\mathfrak{s}(1_{\Sigma X}) = [Z^\bullet]$, we have that $\mathfrak{s}'(\Gamma_{(\Sigma X, X)}(1_{\Sigma X})) = [\mathscr{F}_{\mathsf{C}} Z^\bullet]$. Hence, in $\mathcal{C}'$ we have the $(n+2)$-angle

$$\mathscr{F}X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathscr{F}\Sigma X \xrightarrow{\Theta_X} \Sigma'\mathscr{F}X \tag{2.4}$$

as $\Theta_X = \Gamma_{(\Sigma X, X)}(1_{\Sigma X})$. It follows that Θ_X is an isomorphism, because applying the functors $\mathcal{C}'(\Sigma'\mathscr{F}X, -)$ and $\mathcal{C}'(-, \mathscr{F}\Sigma X)$ to (2.4) yields long exact sequences in Ab.

Finally, suppose that $X^\bullet$ is an $(n+2)$-angle in $\mathcal{C}$. Then

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Gamma_{(X^{n+1}, X^0)}(d_X^{n+1})} \Sigma'\mathscr{F}X^0$$

is an $(n+2)$-angle in $\mathcal{C}'$. Note that the commutativity of (2.3) with $f = d_X^{n+1}, g = 1_{X^0}$ and $\delta = 1_{\Sigma X^0} \in \mathbb{E}(\Sigma X^0, X^0)$ implies $\Gamma_{(X^{n+1}, X^0)}(d_X^{n+1}) = \Gamma_{(\Sigma X^0, X^0)}(1_{\Sigma X^0}) \circ \mathscr{F}d_X^{n+1} = \Theta_{X^0} \circ \mathscr{F}d_X^{n+1}$. Hence,

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{X^0} \circ \mathscr{F}d_X^{n+1}} \Sigma'\mathscr{F}X^0$$

is an $(n+2)$-angle in $\mathcal{C}'$.

■

Theorem 2.34. *Let $(\mathcal{C}, \mathcal{X})$ and $(\mathcal{C}', \mathcal{X}')$ be n-exact categories. Following the notation of* Example 2.31, *set $\mathbb{E} := \mathbb{E}_{\mathcal{X}}$ and $\mathbb{E}' := \mathbb{E}_{\mathcal{X}'}$. Assume for all $X^0, X^{n+1} \in \mathcal{C}$ (respectively, $Z^0, Z^{n+1} \in \mathcal{C}'$) that $\mathbb{E}(X^{n+1}, X^0)$ (respectively, $\mathbb{E}'(Z^{n+1}, Z^0)$) is a set. Consider the n-exangulated categories $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$. Suppose $\mathscr{F} : \mathcal{C} \to \mathcal{C}'$ is an additive covariant functor. Then $\mathscr{F}$ is an n-exact functor if and only if $\mathscr{F}$ is an n-exangulated functor.*

Proof. ($\Rightarrow$) Suppose $\mathscr{F}$ is an n-exact functor. For X^0, X^{n+1} in $\mathcal{C}$, define $\Gamma_{(X^{n+1}, X^0)} : \mathbb{E}(X^{n+1}, X^0) \to \mathbb{E}'(\mathscr{F}X^{n+1}, \mathscr{F}X^0)$ by $\Gamma_{(X^{n+1}, X^0)}([X^\bullet]) := [\mathscr{F}_{\mathsf{C}} X^\bullet]$. Note that this is a well-defined map since the additive functor $\mathscr{F}_{\mathsf{C}}$ preserves homotopies and since $\mathscr{F}_{\mathsf{C}}(\mathcal{X}) \subseteq \mathcal{X}'$ as $\mathscr{F}$ is n-exact. Now let us show that the collection of homomorphisms $\Gamma = \{\Gamma_{(X^{n+1}, X^0)}\}_{(X^{n+1}, X^0) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}}$ gives a natural transformation $\mathbb{E}(-, -) \Rightarrow \mathbb{E}'(\mathscr{F}-, \mathscr{F}-)$. Let $f : Y^{n+1} \to X^{n+1}$ and $g : X^0 \to Y^0$ be arbitrary morphisms in $\mathcal{C}$. Note that $\mathbb{E}(f, g) = \mathbb{E}(Y^{n+1}, g) \circ \mathbb{E}(f, X^0)$ and $\mathbb{E}'(\mathscr{F}f, \mathscr{F}g) = \mathbb{E}'(\mathscr{F}Y^{n+1}, \mathscr{F}g) \circ \mathbb{E}'(\mathscr{F}f, \mathscr{F}X^0)$, so it is enough to show that the squares

$$\begin{array}{ccc}
\mathbb{E}(X^{n+1}, X^0) & \xrightarrow{\Gamma_{(X^{n+1}, X^0)}} & \mathbb{E}'(\mathscr{F}X^{n+1}, \mathscr{F}X^0) \\
{\scriptstyle \mathbb{E}(f, X^0)} \downarrow & & \downarrow {\scriptstyle \mathbb{E}'(\mathscr{F}f, \mathscr{F}X^0)} \\
\mathbb{E}(Y^{n+1}, X^0) & \xrightarrow{\Gamma_{(Y^{n+1}, X^0)}} & \mathbb{E}'(\mathscr{F}Y^{n+1}, \mathscr{F}X^0)
\end{array} \tag{2.5}$$

and

$$\begin{array}{ccc}
\mathbb{E}(Y^{n+1}, X^0) & \xrightarrow{\Gamma_{(Y^{n+1}, X^0)}} & \mathbb{E}'(\mathscr{F}Y^{n+1}, \mathscr{F}X^0) \\
{\scriptstyle \mathbb{E}(Y^{n+1}, g)} \downarrow & & \downarrow {\scriptstyle \mathbb{E}'(\mathscr{F}Y^{n+1}, \mathscr{F}g)} \\
\mathbb{E}(Y^{n+1}, Y^0) & \xrightarrow{\Gamma_{(Y^{n+1}, Y^0)}} & \mathbb{E}'(\mathscr{F}Y^{n+1}, \mathscr{F}X^0)
\end{array} \tag{2.6}$$

commute.

We only show that (2.5) commutes as the commutativity of (2.6) can be shown similarly. Thus, let $[X^\bullet] \in \mathbb{E}(X^{n+1}, X^0)$ be arbitrary. Then $\Gamma_{(X^{n+1}, X^0)}([X^\bullet]) = [\mathscr{F}_{\mathsf{C}} X^\bullet]$, and $\mathbb{E}'(\mathscr{F}f, \mathscr{F}X^0)([\mathscr{F}_{\mathsf{C}} X^\bullet]) = [Z^\bullet]$ is obtained by forming an n-pullback of $(\mathscr{F}d_X^1, \ldots, \mathscr{F}d_X^n)$ along $\mathscr{F}f$, where $Z^0 = \mathscr{F}X^0$ and $Z^{n+1} = \mathscr{F}Y^{n+1}$.

By the dual of [25, Prop. 4.8], this yields a morphism $h^\bullet \colon Z^\bullet \to \mathscr{F}_{\mathsf{C}} X^\bullet$ of $\mathcal{X}'$-admissible n-exact sequences, where $h^0 = 1_{\mathscr{F} X^0}$ and $h^{n+1} = \mathscr{F} f$.

On the other hand, $\mathbb{E}(f, X^0)([X^\bullet]) = [W^\bullet]$ is given by taking an n-pullback of $(d_X^1, \ldots, d_X^n)$ along f, where $W^0 = X^0$ and $W^{n+1} = Y^{n+1}$. By the dual of [25, Prop. 4.8], there is a morphism $f^\bullet \colon W^\bullet \to X^\bullet$ of $\mathcal{X}$-admissible n-exact sequences such that $f^0 = 1_{X^0}$ and $f^{n+1} = f$. Note that $\Gamma_{(Y^{n+1}, X^0)}([W^\bullet]) = [\mathscr{F}_{\mathsf{C}} W^\bullet]$ and that, since $\mathscr{F}$ is an n-exact functor, $\mathscr{F}_{\mathsf{C}} f^\bullet \colon \mathscr{F}_{\mathsf{C}} W^\bullet \to \mathscr{F}_{\mathsf{C}} X^\bullet$ is a morphism of $\mathcal{X}'$-admissible n-exact sequences.

By the dual of [25, Prop. 4.9], there exists a morphism $p^\bullet \colon Z^\bullet \to \mathscr{F}_{\mathsf{C}} W^\bullet$ such that $p^0 = h^0 = 1_{\mathscr{F} X^0}$, and there exists a morphism $q^\bullet \colon \mathscr{F}_{\mathsf{C}} W^\bullet \to Z^\bullet$ such that $q^0 = \mathscr{F} f^0 = 1_{\mathscr{F} X^0}$. In particular, we then have a morphism $q^\bullet p^\bullet \colon Z^\bullet \to Z^\bullet$ with $q^0 p^0 = 1_{\mathscr{F} X^0} = 1_{Z^0}$, and a morphism $p^\bullet q^\bullet \colon \mathscr{F}_{\mathsf{C}} W^\bullet \to \mathscr{F}_{\mathsf{C}} W^\bullet$ with $p^0 q^0 = 1_{\mathscr{F} X^0} = 1_{\mathscr{F}_{\mathsf{C}} W^0}$. Hence, by [25, Lem. 2.1], we have that $q^\bullet p^\bullet \sim 1_{Z^\bullet}$ and $p^\bullet q^\bullet \sim 1_{\mathscr{F}_{\mathsf{C}} W^\bullet}$. Thus, the homotopy equivalence classes $[Z^\bullet]$ and $[\mathscr{F}_{\mathsf{C}} W^\bullet]$ coincide, so (2.5) commutes and Γ is a natural transformation.

Lastly, suppose $\mathfrak{s}(\delta) = [X^\bullet]$, which means $\delta = [X^\bullet]$ by Example 2.31. As defined above, $\Gamma_{(X^{n+1}, X^0)}(\delta) = \Gamma_{(X^{n+1}, X^0)}([X^\bullet]) = [\mathscr{F}_{\mathsf{C}} X^\bullet]$ so $\mathfrak{s}'(\Gamma_{(X^{n+1}, X^0)}(\delta)) = [\mathscr{F}_{\mathsf{C}} X^\bullet]$.

($\Leftarrow$) Conversely, suppose $\mathscr{F}$ is an n-exangulated functor. We must show that $\mathscr{F}_{\mathsf{C}}(\mathcal{X}) \subseteq \mathcal{X}'$. Let

$$X^0 \xrightarrow{\ d^0\ } X^1 \xrightarrow{\ d^1\ } \cdots \xrightarrow{\ d^{n-1}\ } X^n \xrightarrow{\ d^n\ } X^{n+1}$$

be an arbitrary $\mathcal{X}$-admissible n-exact sequence. Set $\delta \coloneqq [X^\bullet] \in \mathbb{E}(X^{n+1}, X^0)$. Then $\mathfrak{s}(\delta) = [X^\bullet]$ implies $\mathfrak{s}'(\Gamma_{(X^{n+1}, X^0)}(\delta)) = [\mathscr{F}_{\mathsf{C}} X^\bullet] \in \mathbb{E}'(\mathscr{F} X^{n+1}, \mathscr{F} X^0)$ as $\mathscr{F}$ is n-exangulated. Hence, $\mathscr{F}_{\mathsf{C}} X^\bullet$ is an $\mathcal{X}'$-admissible n-exact sequence and we are done.

$\blacksquare$

We note here that the notion of an n-exangulated functor is currently being generalised. In ongoing with J. Haugland and M. H. Sandøy, the authors are considering functors from an n-exangulated category to an m-exangulated category, which are compatible with the respective structures and where n and m might differ. See [3].

3. Transport of structure

In §3, let $n \geqslant 1$ be a positive integer. We show that an equivalence of categories preserves the structures discussed in §2. In all cases, we will need the following well-known result, which is easily verified.

Theorem 3.1. *Let $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ be an equivalence of categories and suppose $\mathcal{C}$ is an additive category. Then $\mathcal{C}'$ is an additive category and $\mathscr{F}$ is an additive equivalence.*

For any category $\mathcal{A}$, we will denote the *identity functor* of $\mathcal{A}$ by $\mathbb{1}_{\mathcal{A}}$; see [5, p. 6].

Setup 3.2. Suppose $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is an equivalence. Then there exists a quasi-inverse $\mathscr{G} \colon \mathcal{C}' \to \mathcal{C}$ for $\mathscr{F}$. In this situation, the *unit* $\Phi \colon \mathbb{1}_{\mathcal{C}'} \Longrightarrow \mathscr{F}\mathscr{G}$ and the *counit* $\Psi \colon \mathscr{G}\mathscr{F} \Longrightarrow \mathbb{1}_{\mathcal{C}}$ are natural isomorphisms, and the *counit-unit equations* (or *triangle identities*)

$$\mathscr{F}\Psi \circ \Phi_{\mathscr{F}} = 1_{\mathscr{F}} \qquad \text{and} \qquad \Psi_{\mathscr{G}} \circ \mathscr{G}\Phi = 1_{\mathscr{G}}, \tag{3.1}$$

are satisfied; see [5, Prop. 3.4.1]. The natural transformations $\mathscr{F}\Psi, \Phi_{\mathscr{F}}, \Psi_{\mathscr{G}}$ and $\mathscr{G}\Phi$ are obtained from *whiskering*; see [2, p. 24].

We will refer to Setup 3.2 in the main results of §§3.1–3.3.

3.1. $(n+2)$**-angulated categories.** Recall that if $(\mathcal{C}, \Sigma, \mathcal{T})$ is a weak $(n+2)$-angulated category, then in this article we only require Σ to be an autoequivalence of $\mathcal{C}$. Lemma 3.3 is a key ingredient in the proof of Theorem 3.4.

Lemma 3.3. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a weak (pre-)$(n+2)$-angulated category and let $\mathcal{C}'$ be a category equipped with an autoequivalence Σ'. Suppose $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is an equivalence and assume that there is a natural isomorphism $\Theta \colon \mathscr{F}\Sigma \overset{\cong}{\Longrightarrow} \Sigma'\mathscr{F}$. Let $\mathcal{T}'$ be the collection of $(n+2)$-Σ'-sequences consisting of those isomorphic to any $(n+2)$-Σ'-sequence of the form*

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{X^0} \circ \mathscr{F}d_X^{n+1}} \Sigma'\mathscr{F}X^0, \tag{3.2}$$

where $X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0$ is an $(n+2)$-angle in $\mathcal{C}$. Then $(\mathcal{C}', \Sigma', \mathcal{T}')$ is a weak (pre-)$(n+2)$-angulated category and $\mathscr{F}$ is an $(n+2)$-angle equivalence.

Proof. Let us fix the notation as in Setup 3.2. By Theorem 3.1, we have that $\mathcal{C}'$ is an additive category and $\mathscr{F}$ is an additive equivalence. Note that this implies Σ' is additive (also by Theorem 3.1). Suppose $(\mathcal{C}, \Sigma, \mathcal{T})$ is a weak pre-$(n+2)$-angulated category, and let us first verify axioms (F1)–(F3) from Definition 2.4 for $(\mathcal{C}', \Sigma', \mathcal{T}')$.

One can easily verify that $\mathcal{T}'$ is closed under taking finite direct sums and direct summands, using that $\mathscr{F}$, Σ and Σ' are all additive functors and that $\mathcal{T}$ satisfies (F1).

Let $X' \in \mathcal{C}'$ be arbitrary. Then

$$\mathscr{G}X' \xrightarrow{1_{\mathscr{G}X'}} \mathscr{G}X' \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma\mathscr{G}X'$$

is an $(n+2)$-angle in $\mathcal{C}$. Thus,

$$\mathscr{F}\mathscr{G}X' \xrightarrow{1_{\mathscr{F}\mathscr{G}X'}} \mathscr{F}\mathscr{G}X' \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma'(\mathscr{F}\mathscr{G}X')$$

is an $(n+2)$-Σ'-sequence in $\mathcal{T}'$. Using the natural isomorphism $\Phi \colon \mathbb{1}_{\mathcal{C}'} \overset{\cong}{\Longrightarrow} \mathscr{F}\mathscr{G}$, we see that

$$X' \xrightarrow{1_{X'}} X' \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma'X'$$

is also an $(n+2)$-Σ'-sequence in $\mathcal{T}'$.

Let $d_{X'}^0 \colon X'^0 \to X'^1$ be an arbitrary morphism in $\mathcal{C}'$, and consider the morphism $\mathscr{G}d_{X'}^0 \colon \mathscr{G}X'^0 \to \mathscr{G}X'^1$ in $\mathcal{C}$. Using (F1)(c) for $\mathcal{C}$ we obtain an $(n+2)$-angle

$$\mathscr{G}X'^0 \xrightarrow{\mathscr{G}d_{X'}^0} \mathscr{G}X'^1 \xrightarrow{d_X^1} X^2 \xrightarrow{d_X^2} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma\mathscr{G}X'^0.$$

So, the $(n+2)$-Σ'-sequence

$$\mathscr{F}\mathscr{G}X'^0 \xrightarrow{\mathscr{F}\mathscr{G}d_{X'}^0} \mathscr{F}\mathscr{G}X'^1 \xrightarrow{\mathscr{F}d_X^1} \mathscr{F}X^2 \xrightarrow{\mathscr{F}d_X^2} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{\mathscr{G}X'^0}\mathscr{F}d_X^{n+1}} \Sigma'\mathscr{F}\mathscr{G}X'^0.$$

belongs to $\mathcal{T}'$. Set $a := \Theta_{\mathscr{G}X'^0}\mathscr{F}d_X^{n+1}$. Then the diagram

$$
\begin{array}{ccccccccccc}
\mathscr{F}\mathscr{G}X'^0 & \xrightarrow{\mathscr{F}\mathscr{G}d_{X'}^0} & \mathscr{F}\mathscr{G}X'^1 & \xrightarrow{\mathscr{F}d_X^1} & \mathscr{F}X^2 & \xrightarrow{\mathscr{F}d_X^2} & \cdots & \xrightarrow{\mathscr{F}d_X^n} & \mathscr{F}X^{n+1} & \xrightarrow{a} & \Sigma'\mathscr{F}\mathscr{G}X'^0 \\
\downarrow{\scriptstyle\Phi_{X'^0}{}^{-1}} & & \downarrow{\scriptstyle\Phi_{X'^1}{}^{-1}} & & \| & & & & \| & & \downarrow{\scriptstyle\Sigma'\Phi_{X'^0}{}^{-1}} \\
X'^0 & \xrightarrow{d_{X'}^0} & X'^1 & \xrightarrow{(\mathscr{F}d_X^1)\circ\Phi_{X'^1}} & \mathscr{F}X^2 & \xrightarrow{\mathscr{F}d_X^2} & \cdots & \xrightarrow{\mathscr{F}d_X^n} & \mathscr{F}X^{n+1} & \xrightarrow{(\Sigma'\Phi_{X'^0}{}^{-1})\circ a} & \Sigma'X'^0
\end{array}
$$

commutes and all the vertical morphisms are isomorphisms, since Φ is a natural isomorphism. Thus the diagram is an isomorphism of $(n+2)$-Σ'-sequences and, hence, (F1) holds for $(\mathcal{C}', \Sigma', \mathcal{T}')$.

By using suitable $(n+2)$-Σ'-sequence isomorphisms, we need only show the remaining axioms hold for the *defining* $(n+2)$-Σ'-sequences of $\mathcal{T}'$, i.e. $(n+2)$-Σ'-sequences of the form (3.2), namely

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{X^0} \circ \mathscr{F}d_X^{n+1}} \Sigma'\mathscr{F}X^0$$

where $X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0$ is an $(n+2)$-angle in $\mathcal{C}$. We may apply (F2) in $\mathcal{C}$ yielding the $(n+2)$-angle

$$X^1 \xrightarrow{d_X^1} X^2 \xrightarrow{d_X^2} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0 \xrightarrow{(-1)^n \Sigma d_X^0} \Sigma X^1$$

in $\mathcal{C}$. Then, setting $b := (-1)^n \Theta_{X^1} \mathscr{F} \Sigma d_X^0$, the $(n+2)$-angle

$$\mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \mathscr{F}X^2 \xrightarrow{\mathscr{F}d_X^2} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\mathscr{F}d_X^{n+1}} \mathscr{F}\Sigma X^0 \xrightarrow{b} \Sigma'\mathscr{F}X^1$$

is in $\mathcal{T}'$. In the diagram

$$\begin{array}{ccccccccccc}
\mathscr{F}X^1 & \xrightarrow{\mathscr{F}d_X^1} & \mathscr{F}X^2 & \xrightarrow{\mathscr{F}d_X^2} & \cdots & \xrightarrow{\mathscr{F}d_X^n} & \mathscr{F}X^{n+1} & \xrightarrow{\mathscr{F}d_X^{n+1}} & \mathscr{F}\Sigma X^0 & \xrightarrow{b} & \Sigma'\mathscr{F}X^1 \\
\| & & \| & & & & \| & & \downarrow{\Theta_{X^0}} & & \| \\
\mathscr{F}X^1 & \xrightarrow{\mathscr{F}d_X^1} & \mathscr{F}X^2 & \xrightarrow{\mathscr{F}d_X^2} & \cdots & \xrightarrow{\mathscr{F}d_X^n} & \mathscr{F}X^{n+1} & \xrightarrow{\Theta_{X^0}\mathscr{F}d_X^{n+1}} & \Sigma'\mathscr{F}X^0 & \xrightarrow{(-1)^n\Sigma'\mathscr{F}d_X^0} & \Sigma'\mathscr{F}X^1
\end{array}$$

the rightmost square commutes since Θ is a natural isomorphism. Hence, the diagram above is an $(n+2)$-Σ'-sequence isomorphism and the bottom row is in $\mathcal{T}'$. Thus, we conclude that if an $(n+2)$-Σ'-sequence is in $\mathcal{T}'$, then its left rotation also lies in $\mathcal{T}'$. The converse is similar, and so (F2) holds for $(\mathcal{C}', \Sigma', \mathcal{T}')$.

In order to show (F3) is satisfied, suppose we are given a commutative diagram

$$\begin{array}{ccccccccc}
\mathscr{F}X^0 & \xrightarrow{\mathscr{F}d_X^0} & \mathscr{F}X^1 & \xrightarrow{\mathscr{F}d_X^1} & \cdots & \xrightarrow{\mathscr{F}d_X^n} & \mathscr{F}X^{n+1} & \xrightarrow{\Theta_{X^0} \circ (\mathscr{F}d_X^{n+1})} & \Sigma'\mathscr{F}X^0 \\
\downarrow{f^0} & \circlearrowleft & \downarrow{f^1} & & & & & & \downarrow{\Sigma'f^0} \\
\mathscr{F}Y^0 & \xrightarrow{\mathscr{F}d_Y^0} & \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_Y^1} & \cdots & \xrightarrow{\mathscr{F}d_Y^n} & \mathscr{F}Y^{n+1} & \xrightarrow{\Theta_{Y^0} \circ (\mathscr{F}d_Y^{n+1})} & \Sigma'\mathscr{F}Y^0,
\end{array} \quad (3.3)$$

in which the rows are defining $(n+2)$-Σ'-sequences in $\mathcal{T}'$ and $f^1 \circ \mathscr{F}d_X^0 = (\mathscr{F}d_Y^0) \circ f^0$. Since $\mathscr{F}$ is full, we have that $f^0 = \mathscr{F}g^0$ and $f^1 = \mathscr{F}g^1$ for some $g^0 \colon X^0 \to Y^0$ and $g^1 \colon X^1 \to Y^1$. Therefore, the identity $f^1 \circ \mathscr{F}d_X^0 = (\mathscr{F}d_Y^0) \circ f^0$ becomes $\mathscr{F}(g^1 d_X^0) = \mathscr{F}(d_Y^0 g^0)$, and we see that $g^1 d_X^0 = d_Y^0 g^0$ in $\mathcal{C}$ as $\mathscr{F}$ is also faithful. Hence, by (F3) for $\mathcal{C}$, we obtain a morphism

$$\begin{array}{ccccccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \longrightarrow & X^{n+1} & \xrightarrow{d_X^{n+1}} & \Sigma X^0 \\
\downarrow{g^0} & \circlearrowleft & \downarrow{g^1} & & \downarrow{g^2} & & & & \downarrow{g^{n+1}} & & \downarrow{\Sigma g^0} \\
Y^0 & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & Y^2 & \xrightarrow{d_Y^2} & \cdots & \longrightarrow & Y^{n+1} & \xrightarrow{d_Y^{n+1}} & \Sigma Y^0
\end{array} \quad (3.4)$$

of $(n+2)$-Σ-sequences. Applying $\mathscr{F}_{\mathsf{C}}$ and using the natural isomorphism Θ to adjust the rightmost square, we have a morphism

$$\begin{array}{ccccccccccc}
\mathscr{F}X^0 & \xrightarrow{\mathscr{F}d_X^0} & \mathscr{F}X^1 & \xrightarrow{\mathscr{F}d_X^1} & \mathscr{F}X^2 & \xrightarrow{\mathscr{F}d_X^2} & \cdots & \longrightarrow & \mathscr{F}X^{n+1} & \xrightarrow{\Theta_{X^0}\mathscr{F}d_X^{n+1}} & \Sigma'\mathscr{F}X^0 \\
\downarrow{f^0} & \circlearrowleft & \downarrow{f^1} & & \downarrow{f^2} & & & & \downarrow{f^{n+1}} & & \downarrow{\Sigma'\mathscr{F}g^0 = \Sigma'f^0} \\
\mathscr{F}Y^0 & \xrightarrow{\mathscr{F}d_Y^0} & \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_Y^1} & \mathscr{F}Y^2 & \xrightarrow{\mathscr{F}d_Y^2} & \cdots & \longrightarrow & \mathscr{F}Y^{n+1} & \xrightarrow{\Theta_{Y^0}\mathscr{F}d_Y^{n+1}} & \Sigma'\mathscr{F}Y^0
\end{array}$$

of $(n+2)$-Σ'-sequences in $\mathcal{C}'$, where $f^i := \mathscr{F}g^i$ for $2 \leqslant i \leqslant n+1$, so (F3) holds for $(\mathcal{C}', \Sigma', \mathcal{T}')$.

Altogether, we have shown (F1)–(F3) hold for $(\mathcal{C}', \Sigma', \mathcal{T}')$ whenever these axioms hold for $(\mathcal{C}, \Sigma, \mathcal{T})$. That is, if $\mathcal{C}$ is weak pre-$(n+2)$-angulated then so too is $\mathcal{C}'$.

Suppose now that $(\mathcal{C}, \Sigma, \mathcal{T})$ also satisfies (F4); we will show $(\mathcal{C}', \Sigma', \mathcal{T}')$ satisfies (F4). To this end, fix morphisms f^0, f^1 in $\mathcal{C}'$ such that (3.3) commutes. As before, this induces diagram (3.4) in $\mathcal{C}$, and since (F4) holds for $\mathcal{C}$, we may assume that $g^2, \ldots, g^{n+1}$ were obtained in such a way that the cone $C^\bullet := C(g)^\bullet$

$$X^1 \oplus Y^0 \xrightarrow{d_C^0} X^2 \oplus Y^1 \xrightarrow{d_C^1} \cdots \xrightarrow{d_C^n} \Sigma X^0 \oplus Y^{n+1} \xrightarrow{d_C^{n+1}} \Sigma X^1 \oplus \Sigma Y^0$$

lies in $\mathcal{T}$. Therefore,

$$\mathscr{F}X^1 \oplus \mathscr{F}Y^0 \xrightarrow{\mathscr{F}d_C^0} \cdots \xrightarrow{\mathscr{F}d_C^n} \mathscr{F}\Sigma X^0 \oplus \mathscr{F}Y^{n+1} \xrightarrow{c} \Sigma'\mathscr{F}X^1 \oplus \Sigma'\mathscr{F}Y^0$$

lies in $\mathcal{T}'$, where

$$c := \begin{pmatrix} \Theta_{X^1} & 0 \\ 0 & \Theta_{Y^0} \end{pmatrix} \mathscr{F} d_C^{n+1} = \begin{pmatrix} -\Theta_{X^1}\mathscr{F}\Sigma d_X^0 & 0 \\ \Theta_{Y^0}\mathscr{F}\Sigma g^0 & \Theta_{Y^0}\mathscr{F} d_Y^{n+1} \end{pmatrix}.$$

One can then check that the diagram

$$\begin{array}{ccccccccc}
\mathscr{F}X^1 \oplus \mathscr{F}Y^0 & \xrightarrow{\mathscr{F}d_C^0} & \cdots & \xrightarrow{\mathscr{F}d_C^{n-1}} & \mathscr{F}X^{n+1} \oplus \mathscr{F}Y^n & \xrightarrow{\mathscr{F}d_C^n} & \mathscr{F}\Sigma X^0 \oplus \mathscr{F}Y^{n+1} & \xrightarrow{c} & \Sigma'\mathscr{F}X^1 \oplus \Sigma'\mathscr{F}Y^0 \\
\| & & & & \| & & \downarrow{\scriptstyle d} & & \| \\
\mathscr{F}X^1 \oplus \mathscr{F}Y^0 & \xrightarrow{\mathscr{F}d_C^0} & \cdots & \xrightarrow{\mathscr{F}d_C^{n-1}} & \mathscr{F}X^{n+1} \oplus \mathscr{F}Y^n & \xrightarrow{d \circ \mathscr{F}d_C^n} & \Sigma'\mathscr{F}X^0 \oplus \mathscr{F}Y^{n+1} & \xrightarrow{e} & \Sigma'\mathscr{F}X^1 \oplus \Sigma'\mathscr{F}Y^0,
\end{array}$$

where

$$d := \begin{pmatrix} \Theta_{X^0} & 0 \\ 0 & 1_{\mathscr{F}Y^{n+1}} \end{pmatrix} \qquad \text{and} \qquad e := \begin{pmatrix} -\Sigma'\mathscr{F}d_X^0 & 0 \\ \Sigma'\mathscr{F}g^0 & \Theta_{Y^0}\mathscr{F}d_Y^{n+1} \end{pmatrix},$$

gives an isomorphism of $(n+2)$-Σ'-sequences using the natural isomorphism Θ for morphisms d_X^0 and g^0. Since the top row lies in $\mathcal{T}'$ and the bottom row is precisely the cone $C(f)^\bullet$ of the morphism $(f^0, \ldots, f^{n+1})$ in $\mathcal{C}'$, we see $C(f)^\bullet$ belongs to $\mathcal{T}'$. Thus, (F4) holds for $(\mathcal{C}', \Sigma', \mathcal{T}')$ whenever it holds for $(\mathcal{C}, \Sigma, \mathcal{T})$.

Lastly, we immediately see that $\mathscr{F}$ is an $(n+2)$-angulated functor by the existence of the natural isomorphism $\Theta: \mathscr{F}\Sigma \xRightarrow{\cong} \Sigma'\mathscr{F}$ and by how we defined the (pre-)$(n+2)$-angulation $\mathcal{T}'$. Furthermore, $\mathscr{F}$ is thus an $(n+2)$-angle equivalence.

∎

We are now in a position to prove transport of structure for weak (pre-)$(n+2)$-angulated categories.

Theorem 3.4. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a weak (pre-)$(n+2)$-angulated category and suppose $\mathscr{F}: \mathcal{C} \to \mathcal{C}'$ is an equivalence. Then $\mathcal{C}'$ is a weak (pre-)$(n+2)$-angulated category and $\mathscr{F}$ is an $(n+2)$-angle equivalence.*

Proof. Let us fix the notation as in Setup 3.2. By Theorem 3.1, we have that $\mathcal{C}'$ is an additive category and $\mathscr{F}$ is an additive equivalence of categories.

Since Σ is an autoequivalence of $\mathcal{C}$, there exists a functor $\Sigma_-: \mathcal{C} \to \mathcal{C}$ and natural isomorphisms $\Gamma: \Sigma\Sigma_- \xRightarrow{\cong} \mathbb{1}_\mathcal{C}$ and $\Lambda: \Sigma_-\Sigma \xRightarrow{\cong} \mathbb{1}_\mathcal{C}$. Set $\Sigma' := \mathscr{F}\Sigma\mathscr{G}$ and $\Sigma'_- := \mathscr{F}\Sigma_-\mathscr{G}$. Then there is a natural isomorphism $\Sigma'\Sigma'_- \Rightarrow \mathbb{1}_{\mathcal{C}'}$ given by the composition

$$\Sigma'\Sigma'_- = \mathscr{F}\Sigma(\mathscr{G}\mathscr{F})\Sigma_-\mathscr{G} \xRightarrow{\mathscr{F}\Sigma\Psi_-\mathscr{G}} \mathscr{F}\Sigma\mathbb{1}_\mathcal{C}\Sigma_-\mathscr{G} = \mathscr{F}\Sigma\Sigma_-\mathscr{G} \xRightarrow{\mathscr{F}\Gamma\mathscr{G}} \mathscr{F}\mathscr{G} \xRightarrow{\Phi^{-1}} \mathbb{1}_{\mathcal{C}'}.$$

Similarly, there is also a natural isomorphism $\Sigma'_-\Sigma' \xRightarrow{\cong} \mathbb{1}_{\mathcal{C}'}$, and hence Σ' is an autoequivalence of $\mathcal{C}'$.

Note that there is also a natural isomorphism

$$\Theta := \mathscr{F}\Sigma\Psi^{-1}: \mathscr{F}\Sigma = \mathscr{F}\Sigma\mathbb{1}_\mathcal{C} \xRightarrow{\cong} \mathscr{F}\Sigma\mathscr{G}\mathscr{F} = \Sigma'\mathscr{F}.$$

Then, by Lemma 3.3, $(\mathcal{C}', \Sigma', \mathcal{T}')$ is a weak (pre-)$(n+2)$-angulated category, where $\mathcal{T}'$ is the collection of $(n+2)$-Σ'-sequences that are isomorphic to any $(n+2)$-Σ'-sequence of the form

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1} \xrightarrow{\Theta_{X^0} \circ \mathscr{F}d_X^{n+1}} \Sigma'\mathscr{F}X^0,$$

such that $X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \Sigma X^0$ is an $(n+2)$-angle in $\mathcal{C}$. Furthermore, $\mathscr{F}$ is a $(n+2)$-angle equivalence by the same result.

∎

3.2. n-exact and n-abelian categories.

In §3.2, we show that whenever a category is equivalent to an n-exact (respectively, n-abelian) category, it must also be n-exact (respectively, n-abelian). The idea in both results is that the property of a complex becoming exact under a Hom-functor is preserved by any fully faithful functor.

Theorem 3.5. *Let $(\mathcal{C}, \mathcal{X})$ be an n-exact category and suppose $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is an equivalence. Consider the class $\mathcal{X}'$ consisting of all sequences in $\mathcal{C}'$ that are weakly isomorphic to any sequence of the form*

$$\mathscr{F}X^0 \xrightarrow{\mathscr{F}d_X^0} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^n} \mathscr{F}X^{n+1}, \tag{3.5}$$

where $X^0 \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^n} X^{n+1}$ is an $\mathcal{X}$-admissible n-exact sequence. Then $(\mathcal{C}', \mathcal{X}')$ forms an n-exact category and $\mathscr{F}$ is an n-exact equivalence.

Proof. First, note that $\mathcal{X}'$ does indeed consist of n-exact sequences in $\mathcal{C}'$, because $\mathscr{F}$ is an equivalence of categories and so gives bijections on morphism sets. Second, by definition $\mathcal{X}'$ is closed under weak isomorphisms. Now let us show the axioms for an n-exact category are satisfied by the pair $(\mathcal{C}', \mathcal{X}')$.

Since $0 \rightarrowtail 0 \to \cdots \to 0 \twoheadrightarrow 0$ is an $\mathcal{X}$-admissible n-exact sequence and $\mathscr{F}$ is an additive functor, the sequence $0 \to 0 \to \cdots \to 0 \to 0$ is in $\mathcal{X}'$, so (n-E0) holds for $(\mathcal{C}', \mathcal{X}')$.

With the use of suitable isomorphisms of n-exact sequences, we are reduced to showing the remaining axioms for the *defining* sequences of the form (3.5). Thus, suppose we have two composable $\mathcal{X}'$-admissible monomorphisms $\mathscr{F}d_X^0 \colon \mathscr{F}X^0 \to \mathscr{F}X^1$ and $\mathscr{F}d_Y^0 \colon \mathscr{F}X^1 \to \mathscr{F}Y^1$, where d_Y^0 and d_X^0 are $\mathcal{X}$-admissible monomorphisms. Then the morphism $d_Y^0 d_X^0$ is an $\mathcal{X}$-admissible monomorphism as $(\mathcal{C}, \mathcal{X})$ is an n-exact category. Hence, $\mathscr{F}d_Y^0 \circ \mathscr{F}d_X^0 = \mathscr{F}(d_Y^0 d_X^0)$ is an $\mathcal{X}'$-admissible monomorphism and $(\mathcal{C}', \mathcal{X}')$ satisfies (n-E1). Dually, (n-E1$^{\mathrm{op}}$) is also satisfied.

Suppose we have a defining $\mathcal{X}'$-admissible n-exact sequence of the form (3.5) and an arbitrary morphism $f^0 \colon \mathscr{F}X^0 \to Y'$ in $\mathcal{C}'$. Since $\mathscr{F}$ is essentially surjective, there exists Y^0 in $\mathcal{C}$ and an isomorphism $h \colon Y' \to \mathscr{F}Y^0$. Then there exists a morphism $g^0 \colon X^0 \to Y^0$ such that $\mathscr{F}g^0 = hf^0$ as $\mathscr{F}$ is full. Applying (n-E2) in $\mathcal{C}$, we obtain an n-pushout

$$
\begin{array}{ccccccc}
X^0 & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n \\
\downarrow{g^0} & & \vdots{g^1} & & & & \vdots{g^n} \\
Y^0 & \dashrightarrow & Y^1 & \dashrightarrow & \cdots & \dashrightarrow & Y^n,
\end{array} \tag{3.6}
$$

$$\qquad\qquad\; d_Y^0 \qquad\quad d_Y^1 \qquad\quad d_Y^{n-1}$$

such that d_Y^0 is an $\mathcal{X}$-admissible monomorphism.

Applying $\mathscr{F}$ yields a morphism

$$
\begin{array}{ccccccc}
\mathscr{F}X^0 & \xrightarrow{\mathscr{F}d_X^0} & \mathscr{F}X^1 & \xrightarrow{\mathscr{F}d_X^1} & \cdots & \xrightarrow{\mathscr{F}d_X^{n-1}} & \mathscr{F}X^n \\
\mathscr{F}g^0 = hf^0 \downarrow & & \downarrow{\mathscr{F}g^1} & & & & \downarrow{\mathscr{F}g^n} \\
\mathscr{F}Y^0 & \xrightarrow{\mathscr{F}d_Y^0} & \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_Y^1} & \cdots & \xrightarrow{\mathscr{F}d_Y^{n-1}} & \mathscr{F}Y^n
\end{array} \tag{3.7}
$$

of $\mathcal{X}'$-admissible n-exact sequences by definition of $\mathcal{X}'$. We claim that this diagram is an n-pushout in $\mathcal{C}'$. As (3.6) is an n-pushout, we know that in the mapping cone $C^\bullet \coloneqq \mathrm{MC}(g)^\bullet$, the sequence $(d_C^0, \ldots, d_C^{n-1})$ is an n-cokernel for d_C^{-1}. Thus, for any $Z \in \mathcal{C}$, the induced commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{C}(Y^n, Z) & \xrightarrow{-\circ d_C^{n-1}} & \cdots & \xrightarrow{-\circ d_C^0} & \mathcal{C}(X^1 \oplus Y^0, Z) & \xrightarrow{-\circ d_C^{-1}} & \mathcal{C}(X^0, Z) \\
\| & & \cong \downarrow{\mathscr{F}_{Y^n, Z}} & & & & \cong \downarrow{\mathscr{F}_{X^1 \oplus Y^0, Z}} & & \cong \downarrow{\mathscr{F}_{X^0, Z}} \\
0 & \longrightarrow & \mathcal{C}'(\mathscr{F}Y^n, \mathscr{F}Z) & \xrightarrow{-\circ \mathscr{F}d_C^{n-1}} & \cdots & \xrightarrow{-\circ \mathscr{F}d_C^0} & \mathcal{C}'(\mathscr{F}(X^1 \oplus Y^0), \mathscr{F}Z) & \xrightarrow{-\circ \mathscr{F}d_C^{-1}} & \mathcal{C}'(\mathscr{F}X^0, \mathscr{F}Z)
\end{array}
$$

in Ab has an exact top row. The vertical homomorphisms are isomorphisms because $\mathscr{F}$ is fully faithful, which thus implies exactness of the bottom row. As $\mathscr{F}$ is essentially surjective, it follows that the sequence $(d_D^0, \ldots, d_D^{n-1})$ is an n-cokernel for d_D^{-1} in the mapping cone $D^\bullet \coloneqq \mathrm{MC}(\mathscr{F}_\mathsf{C} g)^\bullet$. Hence, (3.7) is an n-pushout, as claimed.

Since $h \colon Y' \to \mathscr{F} Y^0$ is an isomorphism, we can adjust (3.7) to get another n-pushout

$$
\begin{array}{ccccccc}
\mathscr{F} X^0 & \xrightarrow{\mathscr{F} d_X^0} & \mathscr{F} X^1 & \xrightarrow{\mathscr{F} d_X^1} & \cdots & \xrightarrow{\mathscr{F} d_X^{n-1}} & \mathscr{F} X^n \\
\downarrow{\scriptstyle f^0} & & \vdots{\scriptstyle \mathscr{F} g^1} & & & & \vdots{\scriptstyle \mathscr{F} g^n} \\
Y' & \dashrightarrow[(\mathscr{F} d_Y^0)h]{} & \mathscr{F} Y^1 & \dashrightarrow[\mathscr{F} d_Y^1]{} & \cdots & \dashrightarrow[\mathscr{F} d_Y^{n-1}]{} & \mathscr{F} Y^n,
\end{array}
$$

where the bottom row is isomorphic to $\mathscr{F}_{\mathsf{C}} Y^\bullet$. In particular, $(\mathscr{F} d_Y^0)h$ is an $\mathcal{X}'$-admissible monomorphism. Thus, $(n\text{-E2})$ holds in $(\mathcal{C}', \mathcal{X}')$. Dually, $(n\text{-E2}^{\mathrm{op}})$ is satisfied.

Hence, $(\mathcal{C}', \mathcal{X}')$ is an n-exact category. The final claim follows from the definition of $\mathcal{X}'$. ∎

Using similar methods as in the proof of Theorem 3.5, one may also show the following result, which we state without proof.

Theorem 3.6. *Let $\mathcal{C}$ be an n-abelian category and suppose $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is an equivalence. Then $\mathcal{C}'$ is an n-abelian category and $\mathscr{F}$ is an n-exact equivalence.*

3.3. **n-exangulated categories.** The goal of §3.3 is to show that one can transport an n-exangulated structure across an equivalence.

Remark 3.7. We now supplement the discussion after Definition 2.23. Suppose $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is a functor and $\mathbb{E} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Ab}$ and $\mathbb{E}' \colon \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}' \to \mathsf{Ab}$ are biadditive functors. In Lemma 3.8 and Theorem 3.9 below, it is necessary to distinguish between $\mathbb{E}$-extensions and $\mathbb{E}'$-extensions, and describe relationships between them. This necessity motivates the use of the symbol $\mathbb{E}$ in denoting $\delta_\sharp^\mathbb{E}$, $\delta_\mathbb{E}^\sharp$, $x_\mathbb{E}\delta$ and $z^\mathbb{E}\delta$. Note that these were denoted $\delta_\sharp$, $\delta^\sharp$, $x_*\delta$ and $z^*\delta$, respectively in [22] and [35].

Lemma 3.8. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category and suppose $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is an equivalence. Fix the notation as in* Setup 3.2. *Define the functor $\mathbb{E}' \colon \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}' \to \mathsf{Ab}$ to be $\mathbb{E}(\mathscr{G}-, \mathscr{G}-)$. For each $A, C \in \mathcal{C}'$ and each $\delta \in \mathbb{E}'(C, A) = \mathbb{E}(\mathscr{G}C, \mathscr{G}A)$, set $\mathfrak{s}'(\delta)$ to be the homotopy equivalence class*

$$
[\ A \xrightarrow{(\mathscr{F} d_X^0)\Phi_A} \mathscr{F} X^1 \xrightarrow{\mathscr{F} d_X^1} \cdots \xrightarrow{\mathscr{F} d_X^{n-1}} \mathscr{F} X^n \xrightarrow{\Phi_C^{-1} \circ \mathscr{F} d_X^n} C\ ]
$$

in $\mathsf{K}^n_{(A,C)}$, where $\mathfrak{s}(\delta) = [\ \mathscr{G}A \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} \mathscr{G}C\ ]$. Then $\mathbb{E}'$ is a biadditive functor and $\mathfrak{s}'$ is an exact realisation of $\mathbb{E}'$.

Proof. It is straightforward to show that $\mathbb{E}' \colon \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}' \to \mathsf{Ab}$ is a biadditive functor. Using that $\mathscr{F}$ is an equivalence, and in particular that Φ and Ψ are natural isomorphisms, one can also check that $\mathfrak{s}'$ is a well-defined assignment.

Let us show that $\mathfrak{s}'$ is a realisation of $\mathbb{E}'$. Let $A, B, C, D \in \mathcal{C}'$ and suppose we have extensions $\delta \in \mathbb{E}'(C, A)$ and $\varepsilon \in \mathbb{E}'(D, B)$. Suppose

$$
\mathfrak{s}'(\delta) = [\ A \xrightarrow{(\mathscr{F} d_X^0)\Phi_A} \mathscr{F} X^1 \xrightarrow{\mathscr{F} d_X^1} \cdots \xrightarrow{\mathscr{F} d_X^{n-1}} \mathscr{F} X^n \xrightarrow{\Phi_C^{-1}\mathscr{F} d_X^n} C\ ]
$$

and

$$
\mathfrak{s}'(\varepsilon) = [\ B \xrightarrow{(\mathscr{F} d_Y^0)\Phi_B} \mathscr{F} Y^1 \xrightarrow{\mathscr{F} d_Y^1} \cdots \xrightarrow{\mathscr{F} d_Y^{n-1}} \mathscr{F} Y^n \xrightarrow{\Phi_D^{-1}\mathscr{F} d_Y^n} D\ ],
$$

where

$$
\mathfrak{s}(\delta) = [\ \mathscr{G}A \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} \mathscr{G}C\ ]
$$

and

$$
\mathfrak{s}(\varepsilon) = [\ \mathscr{G}B \xrightarrow{d_Y^0} Y^1 \xrightarrow{d_Y^1} \cdots \xrightarrow{d_Y^{n-1}} Y^n \xrightarrow{d_Y^n} \mathscr{G}D\ ].
$$

Let $(a, c)\colon \delta \to \varepsilon$ be a morphism of $\mathbb{E}'$-extensions, where $a\colon A \to B$ and $c\colon C \to D$ are morphisms in $\mathcal{C}'$. Then we have

$$
\begin{aligned}
(\mathscr{G}a)_{\mathbb{E}}(\delta) &= \mathbb{E}(\mathscr{G}C, \mathscr{G}a)(\delta) \\
&= \mathbb{E}'(C, a)(\delta) \\
&= a_{\mathbb{E}'}\delta \\
&= c^{\mathbb{E}'}\delta \\
&= \mathbb{E}'(c, B)(\varepsilon) \\
&= \mathbb{E}(\mathscr{G}c, \mathscr{G}B)(\varepsilon) \\
&= (\mathscr{G}c)^{\mathbb{E}}(\varepsilon),
\end{aligned}
$$

which implies $(\mathscr{G}a, \mathscr{G}c)\colon \delta \to \varepsilon$ is a morphism of $\mathbb{E}$-extensions. Thus, since $\mathfrak{s}$ is a realisation, there is a morphism $f^{\bullet} = (\mathscr{G}a, f^1, \ldots, f^n, \mathscr{G}c)$ of complexes in $\mathsf{C}_{\mathcal{C}}^n$ depicted by

$$
\begin{array}{ccccccccc}
\mathscr{G}A & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & \mathscr{G}C \\
\downarrow{\scriptstyle \mathscr{G}a} & & \downarrow{\scriptstyle f^1} & & & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle \mathscr{G}c} \\
\mathscr{G}B & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & \cdots & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & \mathscr{G}D.
\end{array}
\tag{3.8}
$$

There are commutative squares

$$
\begin{array}{ccc}
A & \xrightarrow{\Phi_A} & \mathscr{F}\mathscr{G}A \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathscr{F}\mathscr{G}a} \\
B & \xrightarrow{\Phi_B} & \mathscr{F}\mathscr{G}B
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathscr{F}\mathscr{G}C & \xrightarrow{\Phi_C^{-1}} & C \\
\downarrow{\scriptstyle \mathscr{F}\mathscr{G}c} & & \downarrow{\scriptstyle c} \\
\mathscr{F}\mathscr{G}D & \xrightarrow{\Phi_D^{-1}} & D
\end{array}
\tag{3.9}
$$

from the natural isomorphism $\Phi\colon \mathbb{1}_{\mathcal{C}'} \Rightarrow \mathscr{F}\mathscr{G}$. Thus, applying $\mathscr{F}$ to (3.8) and adjusting the resultant diagram with the squares in (3.9), we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
A & \xrightarrow{(\mathscr{F}d_X^0)\Phi_A} & \mathscr{F}X^1 & \xrightarrow{\mathscr{F}d_X^1} & \cdots & \xrightarrow{\mathscr{F}d_X^{n-1}} & \mathscr{F}X^n & \xrightarrow{\Phi_C^{-1}\mathscr{F}d_X^n} & C \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \mathscr{F}f^1} & & & & \downarrow{\scriptstyle \mathscr{F}f^n} & & \downarrow{\scriptstyle c} \\
B & \xrightarrow{(\mathscr{F}d_Y^0)\Phi_B} & \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_Y^1} & \cdots & \xrightarrow{\mathscr{F}d_Y^{n-1}} & \mathscr{F}Y^n & \xrightarrow{\Phi_D^{-1}\mathscr{F}d_Y^n} & D
\end{array}
\tag{3.10}
$$

in $\mathcal{C}'$. That is, $(a, \mathscr{F}f^1, \ldots, \mathscr{F}f^n, c)$ is a morphism in $\mathsf{C}_{\mathcal{C}'}^n$, and hence $\mathfrak{s}'$ is a realisation of $\mathbb{E}'$.

Now let us show that $\mathfrak{s}'$ satisfies (R1). Suppose that

$$
\mathfrak{s}'(\delta) = [\; A \xrightarrow{(\mathscr{F}d_X^0)\Phi_A} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^{n-1}} \mathscr{F}X^n \xrightarrow{\Phi_C^{-1}\mathscr{F}d_X^n} C \;],
$$

where $\mathfrak{s}(\delta) = [\; \mathscr{G}A \xrightarrow{d_X^0} X^1 \xrightarrow{d_X^1} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} \mathscr{G}C \;]$. Let $Z \in \mathcal{C}'$ be arbitrary, and consider the following diagram

$$
\begin{array}{ccccccccc}
\mathcal{C}(\mathscr{G}Z, \mathscr{G}A) & \xrightarrow{d_X^0 \circ -} & \mathcal{C}(\mathscr{G}Z, X^1) & \xrightarrow{d_X^1 \circ -} & \cdots & \xrightarrow{d_X^n \circ -} & \mathcal{C}(\mathscr{G}Z, \mathscr{G}C) & \xrightarrow{(\delta_\sharp^{\mathbb{E}})_{\mathscr{G}Z}} & \mathbb{E}(\mathscr{G}Z, \mathscr{G}A) \\
\downarrow{\scriptstyle \mathscr{F}_{\mathscr{G}Z, \mathscr{G}A}} & & \downarrow{\scriptstyle \mathscr{F}_{\mathscr{G}Z, X^1}} & & & & \downarrow{\scriptstyle \mathscr{F}_{\mathscr{G}Z, \mathscr{G}C}} & & \parallel \\
\mathcal{C}'(\mathscr{F}\mathscr{G}Z, \mathscr{F}\mathscr{G}A) & \xrightarrow{\mathscr{F}d_X^0 \circ -} & \mathcal{C}'(\mathscr{F}\mathscr{G}Z, \mathscr{F}X^1) & \xrightarrow{\mathscr{F}d_X^1 \circ -} & \cdots & \xrightarrow{\mathscr{F}d_X^n \circ -} & \mathcal{C}'(\mathscr{F}\mathscr{G}Z, \mathscr{F}\mathscr{G}C) & & \\
\downarrow{\scriptstyle g^0} & & \downarrow{\scriptstyle g^1} & & & & \downarrow{\scriptstyle g^{n+1}} & & \parallel \\
\mathcal{C}'(Z, A) & \xrightarrow{(\mathscr{F}d_X^0)\Phi_A \circ -} & \mathcal{C}'(Z, \mathscr{F}X^1) & \xrightarrow{\mathscr{F}d_X^1 \circ -} & \cdots & \xrightarrow{\Phi_C^{-1}\mathscr{F}d_X^n \circ -} & \mathcal{C}'(Z, C) & \xrightarrow{(\delta_\sharp^{\mathbb{E}'})_Z} & \mathbb{E}'(Z, A),
\end{array}
$$

where the upper vertical maps are those induced by the functor $\mathscr{F}$, and $g^0 = \Phi_A{}^{-1} \circ - \circ \Phi_Z$, $g^i = - \circ \Phi_Z$ for each $i = 1, \ldots, n$, and $g^{n+1} = \Phi_C{}^{-1} \circ - \circ \Phi_Z$. It is easy to check that the whole diagram commutes, except possibly the far right rectangle. To this end, suppose $v \colon \mathscr{G}Z \to \mathscr{G}C$ is arbitrary. As $\mathscr{G}$ is full, there exists $u \colon Z \to C$ such that $\mathscr{G}u = v$. Since $\Phi \colon \mathbb{1}_{\mathcal{C}'} \Rightarrow \mathscr{F}\mathscr{G}$ is a natural isomorphism, we have that $u = \Phi_C{}^{-1}(\mathscr{F}\mathscr{G}u)\Phi_Z$. Then we see that

$$
\begin{aligned}
(\delta_\sharp^{\mathbb{E}'})_Z(g^{n+1}(\mathscr{F}_{\mathscr{G}Z,\mathscr{G}C}(v))) &= (\delta_\sharp^{\mathbb{E}'})_Z(\Phi_C{}^{-1}(\mathscr{F}\mathscr{G}u)\Phi_Z) \\
&= (\delta_\sharp^{\mathbb{E}'})_Z(u) \\
&= \mathbb{E}'(u, A)(\delta) \\
&= \mathbb{E}(\mathscr{G}u, \mathscr{G}A)(\delta) \\
&= \mathbb{E}(v, \mathscr{G}A)(\delta) \\
&= (\delta_\sharp^{\mathbb{E}})_{\mathscr{G}Z}(v).
\end{aligned}
$$

Note that all the vertical maps are bijections, because $\mathscr{F}$ is fully faithful and $\Phi_{X'}$ is an isomorphism for all $X' \in \mathcal{C}'$. Since $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$, the top row is exact and hence so is the bottom row.

Similarly, one can show that there is an exact sequence

$$
]\; \mathcal{C}'(C,Z) \xrightarrow{- \circ \Phi_C{}^{-1}\mathscr{F}d_X^n} \mathcal{C}'(\mathscr{F}X^n, Z) \xrightarrow{- \circ \mathscr{F}d_X^{n-1}} \cdots \xrightarrow{- \circ (\mathscr{F}d_X^0)\Phi_A} \mathcal{C}'(A,Z) \xrightarrow{(\delta_{\mathbb{E}'}^\sharp)_Z} \mathbb{E}'(C,Z),
$$

and thus $\langle\; A \xrightarrow{(\mathscr{F}d_X^0)\Phi_A} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^{n-1}} \mathscr{F}X^n \xrightarrow{\Phi_C{}^{-1}\mathscr{F}d_X^n} C\; , \delta \rangle$ is an n-exangle.

Lastly, we show that (R2) holds for $\mathfrak{s}'$. Let $A \in \mathcal{C}'$ be arbitrary. Then, as $\mathfrak{s}$ is an exact realisation of $\mathbb{E}$, we have

$$
\mathfrak{s}(_{\mathscr{G}A}0_{\mathscr{G}0}) = [\; \mathscr{G}A \xrightarrow{1_{\mathscr{G}A}} \mathscr{G}A \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathscr{G}0 \;].
$$

Note that the element $_{\mathscr{G}A}0_{\mathscr{G}0} \in \mathbb{E}(\mathscr{G}0, \mathscr{G}A)$ is the trivial element $_A 0_0$ in $\mathbb{E}'(0, A)$, so by definition $\mathfrak{s}'(_A 0_0) = \mathfrak{s}'(_{\mathscr{G}A}0_{\mathscr{G}0}) = [\; A \xrightarrow{\Phi_A} \mathscr{F}\mathscr{G}A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \;]$. Lastly, we see that

$$
\mathfrak{s}'(_A 0_0) = [\; A \xrightarrow{1_A} A \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \;],
$$

using the isomorphism $(1_A, \Phi_A, 0, \ldots, 0)$ in $\mathsf{C}^n_{(A,0)}$. Similarly, one can also show that

$$
\mathfrak{s}'(_0 0_A) = [\; 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow A \xrightarrow{1_A} A \;]
$$

for the trivial element $_0 0_A \in \mathbb{E}'(A, 0)$. Therefore, $\mathfrak{s}'$ is an exact realisation of $\mathbb{E}'$.

$\blacksquare$

Theorem 3.9. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category and suppose $\mathscr{F} \colon \mathcal{C} \to \mathcal{C}'$ is an equivalence. Then there is a biadditive functor $\mathbb{E}' \colon \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}' \to \mathsf{Ab}$ and an exact realisation $\mathfrak{s}'$ of $\mathbb{E}'$, such that $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is an n-exangulated category and $\mathscr{F}$ is an n-exangle equivalence.*

Proof. Fix the notation as in Setup 3.2. Let $\mathbb{E}' = \mathbb{E}(\mathscr{G}-, \mathscr{G}-) \colon \mathcal{C}'^{\mathrm{op}} \times \mathcal{C}' \to \mathsf{Ab}$ be the functor and $\mathfrak{s}'$ the assignment defined in Lemma 3.8. From Lemma 3.8, we have that $\mathfrak{s}'$ is an exact realisation of the biadditive functor $\mathbb{E}'$. It remains to show that the triple $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ satisfies axioms (n-EA1), (n-EA2) and (n-EA2$^{\mathrm{op}}$).

For (n-EA1), suppose we have morphisms $a \colon A \to B$ and $b \colon B \to C$ in $\mathcal{C}'$ that are $\mathfrak{s}'$-inflations. That is, there are complexes $U^\bullet, V^\bullet$ in $\mathsf{C}^n_{\mathcal{C}'}$ and $\mathbb{E}'$-extensions $\delta \in \mathbb{E}'(U^{n+1}, A)$, $\varepsilon \in \mathbb{E}'(V^{n+1}, B)$ such that $\mathfrak{s}'(\delta) = [U^\bullet]$, $\mathfrak{s}'(\varepsilon) = [V^\bullet]$, $a = d_U^0$ and $b = d_V^0$. Since δ is also an $\mathbb{E}$-extension, there is a complex $X^\bullet$ in $\mathsf{C}^n_{\mathcal{C}}$ with $X^0 = \mathscr{G}A$ and $X^{n+1} = \mathscr{G}U^{n+1}$, such that $\mathfrak{s}(\delta) = [X^\bullet]$. Thus, we have $\mathfrak{s}'(\delta) = [W^\bullet]$, where $W^\bullet$ is the complex

$$
A \xrightarrow{(\mathscr{F}d_X^0)\Phi_A} \mathscr{F}X^1 \xrightarrow{\mathscr{F}d_X^1} \cdots \xrightarrow{\mathscr{F}d_X^{n-1}} \mathscr{F}X^n \xrightarrow{\Phi_{U^{n+1}}{}^{-1}\mathscr{F}d_X^n} U^{n+1}.
$$

In particular, this implies that the complexes $U^\bullet$ and $W^\bullet$ are homotopy equivalent in $\mathsf{C}^n_{(A, U^{n+1})}$, and in turn implies that $\mathscr{G}_{\mathsf{C}} U^\bullet$ and $\mathscr{G}_{\mathsf{C}} W^\bullet$ are homotopy equivalent in $\mathsf{C}^n_{(\mathscr{G}A, \mathscr{G}U^{n+1})}$.

Next we show that the following diagram depicts an isomorphism of complexes $\mathscr{G}_\mathsf{C} W^\bullet \to X^\bullet$.

$$
\begin{array}{ccccccccc}
\mathscr{G}A & \xrightarrow{(\mathscr{G}\mathscr{F}d_X^0)\mathscr{G}\Phi_A} & \mathscr{G}\mathscr{F}X^1 & \xrightarrow{\mathscr{G}\mathscr{F}d_X^1} & \cdots & \xrightarrow{\mathscr{G}\mathscr{F}d_X^{n-1}} & \mathscr{G}\mathscr{F}X^n & \xrightarrow{\mathscr{G}\Phi_{U^{n+1}}^{-1}(\mathscr{G}\mathscr{F}d_X^n)} & \mathscr{G}U^{n+1} \\
\| & & \downarrow{\Psi_{X^1}} & & & & \downarrow{\Psi_{X^n}} & & \| \\
\mathscr{G}A & \xrightarrow{d_X^0} & X^1 & \xrightarrow{d_X^1} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & \mathscr{G}U^{n+1}.
\end{array}
\tag{3.11}
$$

For each $i = 0, \ldots, n$, the square

$$
\begin{array}{ccc}
\mathscr{G}\mathscr{F}X^i & \xrightarrow{\mathscr{G}\mathscr{F}d_X^i} & \mathscr{G}\mathscr{F}X^{i+1} \\
\downarrow{\Psi_{X^i}} & & \downarrow{\Psi_{X^{i+1}}} \\
X^i & \xrightarrow{d_X^i} & X^{i+1}
\end{array}
\tag{3.12}
$$

commutes since Ψ is a natural isomorphism. This shows that all except possibly the far left and far right squares in (3.11) commute. To see the far left square commutes, observe that

$$
\begin{aligned}
\Psi_{X^1}(\mathscr{G}\mathscr{F}d_X^0)\mathscr{G}\Phi_A &= d_X^0 \Psi_{\mathscr{G}A}\mathscr{G}\Phi_A && \text{by (3.12)} \\
&= d_X^0 \circ 1_{\mathscr{G}A} && \text{using the triangle identities (3.1)} \\
&= d_X^0.
\end{aligned}
$$

Similarly, one can show the far right square of (3.11) also commutes, and hence (3.11) is an isomorphism of complexes in $\mathsf{C}^n_{(\mathscr{G}A,\mathscr{G}U^{n+1})}$. This implies that $\mathfrak{s}(\delta) = [X^\bullet] = [\mathscr{G}_\mathsf{C}W^\bullet] = [\mathscr{G}_\mathsf{C}U^\bullet]$, so $\mathscr{G}a$ is an $\mathfrak{s}$-inflation.

In the same way, it can be shown that $\mathscr{G}b$ is also an $\mathfrak{s}$-inflation. Since $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfies axiom $(n\text{-EA1})$, we have that the composition $\mathscr{G}(ba) = \mathscr{G}b \circ \mathscr{G}a$ is an $\mathfrak{s}$-inflation. Thus, there exist $Z^{n+1} \in \mathcal{C}$, $\zeta \in \mathbb{E}(Z^{n+1}, \mathscr{G}A)$ and $Z^\bullet \in \mathsf{C}^n_{(\mathscr{G}A, Z^{n+1})}$, such that $\mathfrak{s}(\zeta) = [Z^\bullet]$, $Z^0 = \mathscr{G}A$, $Z^1 = \mathscr{G}C$ and $d_Z^0 = \mathscr{G}(ba)$. As $\mathscr{G}$ is essentially surjective, there is an object $T \in \mathcal{C}'$ and an isomorphism $h \colon \mathscr{G}T \to Z^{n+1}$. Denote by $j := h^{-1} \colon Z^{n+1} \to \mathscr{G}T$ the inverse for h. Since $\zeta = (1_{Z^{n+1}})^\mathbb{E}\zeta = (hj)^\mathbb{E}\zeta = j^\mathbb{E}h^\mathbb{E}\zeta$, we have a sequence

$$
\zeta = j^\mathbb{E}(h^\mathbb{E}\zeta) \xrightarrow{(1_{\mathscr{G}A}, j)} h^\mathbb{E}\zeta \xrightarrow{(1_{\mathscr{G}A}, h)} \zeta
$$

of morphisms of $\mathbb{E}$-extensions, which may be realised as follows

$$
\begin{array}{ccccccccccc}
Z^\bullet: & & \mathscr{G}A & \xrightarrow{\mathscr{G}(ba)} & \mathscr{G}C & \xrightarrow{d_Z^1} & Z^2 & \xrightarrow{d_Z^2} & \cdots & \xrightarrow{d_Z^{n-1}} & Z^n & \xrightarrow{d_Z^n} & Z^{n+1} \\
& \downarrow{j^\bullet} & & & \| & & \downarrow{j^1} & & \downarrow{j^2} & & & \downarrow{j^n} & \cong\downarrow{j} \\
D^\bullet: & & \mathscr{G}A & \longrightarrow & D^1 & \longrightarrow & D^2 & \longrightarrow & \cdots & \longrightarrow & D^n & \longrightarrow & \mathscr{G}T \\
& \downarrow{h^\bullet} & & & \| & & \downarrow{h^1} & & \downarrow{h^2} & & & \downarrow{h^n} & \cong\downarrow{h} \\
Z^\bullet: & & \mathscr{G}A & \xrightarrow{\mathscr{G}(ba)} & \mathscr{G}C & \xrightarrow{d_Z^1} & Z^2 & \xrightarrow{d_Z^2} & \cdots & \xrightarrow{d_Z^{n-1}} & Z^n & \xrightarrow{d_Z^n} & Z^{n+1}
\end{array}
$$

Note that $\langle D^\bullet, h^\mathbb{E}\zeta \rangle$ is an n-exangle as $\mathfrak{s}(h^\mathbb{E}\zeta) = [D^\bullet]$. Let $E^\bullet$ denote the complex

$$
\mathscr{G}A \xrightarrow{\mathscr{G}(ba)} \mathscr{G}C \xrightarrow{d_Z^1} Z^2 \xrightarrow{d_Z^2} \cdots \xrightarrow{d_Z^{n-1}} Z^n \xrightarrow{jd_Z^n} \mathscr{G}T
$$

in $\mathsf{C}^n_{(\mathscr{G}A, \mathscr{G}T)}$. It is straightforward to verify that $\langle E^\bullet, h^\mathbb{E}\zeta \rangle$ is also an n-exangle. Moreover, we have morphisms

$$
(1_{\mathscr{G}A}, j^1, \ldots, j^n, 1_{\mathscr{G}T}) \in \mathsf{C}^n_{(\mathscr{G}A, \mathscr{G}T)}(E^\bullet, D^\bullet) \qquad \text{and} \qquad (1_{\mathscr{G}A}, h^1, \ldots, h^n, 1_{\mathscr{G}T}) \in \mathsf{C}^n_{(\mathscr{G}A, \mathscr{G}T)}(D^\bullet, E^\bullet).
$$

Thus, by [22, Prop. 2.21], we have that $(1_{\mathscr{G}A}, j^1, \ldots, j^n, 1_{\mathscr{G}T})$ is a homotopy equivalence, and so $\mathfrak{s}(h^\mathbb{E}\zeta) = [D^\bullet] = [E^\bullet]$. Therefore, we have

$$
\mathfrak{s}'(h^\mathbb{E}\zeta) = [\, A \xrightarrow{(\mathscr{F}\mathscr{G}ba)\Phi_A} \mathscr{F}\mathscr{G}C \xrightarrow{\mathscr{F}d_Z^1} \mathscr{F}Z^2 \xrightarrow{\mathscr{F}d_Z^2} \cdots \xrightarrow{\mathscr{F}d_Z^{n-1}} \mathscr{F}Z^n \xrightarrow{d} T \,],
$$

where $d := \Phi_T^{-1}\mathscr{F}(jd_Z^n)$. Since there is an isomorphism

$$
\begin{array}{ccccccccccc}
A & \xrightarrow{(\mathscr{F}\mathscr{G}ba)\Phi_A} & \mathscr{F}\mathscr{G}C & \xrightarrow{\mathscr{F}d_Z^1} & \mathscr{F}Z^2 & \xrightarrow{\mathscr{F}d_Z^2} & \cdots & \xrightarrow{\mathscr{F}d_Z^{n-1}} & \mathscr{F}Z^n & \xrightarrow{d} & T \\
\| & & \downarrow{\Phi_C^{-1}} & & \| & & & & \| & & \| \\
A & \xrightarrow{ba} & C & \xrightarrow{(\mathscr{F}d_Z^1)\Phi_C} & \mathscr{F}Z^2 & \xrightarrow{\mathscr{F}d_Z^2} & \cdots & \xrightarrow{\mathscr{F}d_Z^{n-1}} & \mathscr{F}Z^n & \xrightarrow{d} & T
\end{array}
$$

of complexes in $\mathsf{C}^n_{(A,T)}$, we see that ba is indeed an $\mathfrak{s}'$-inflation. Dually, $\mathfrak{s}'$-deflations are closed under composition, and $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ satisfies (n-EA1).

Lastly, we show (n-EA2) holds for $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$; (n-EA2$^{\mathrm{op}}$) can be shown using a similar argument. Thus, let $\delta \in \mathbb{E}'(D, A)$ and $c\colon C \to D$ be arbitrary, where $A, C, D \in \mathcal{C}'$. Then we may realise $c^{\mathbb{E}'}\delta = (\mathscr{G}c)^{\mathbb{E}}\delta \in \mathbb{E}(\mathscr{G}C, \mathscr{G}A)$ by a complex

$$
\mathscr{G}A \xrightarrow{d_X^0} \mathscr{G}B \xrightarrow{d_X^1} X^2 \xrightarrow{d_X^2} \cdots \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} \mathscr{G}C
$$

in $\mathsf{C}^n_{\mathcal{C}}$ for some $B \in \mathcal{C}'$, using that $\mathscr{G}$ essentially surjective. Since $\mathscr{G}$ is also full, we have that $d_X^0 = \mathscr{G}a$ for some $a\colon A \to B$. Furthermore, $\delta \in \mathbb{E}(\mathscr{G}D, \mathscr{G}A)$ so we may realise this $\mathbb{E}$-extension by a complex

$$
\mathscr{G}A \xrightarrow{d_Y^0} Y^1 \xrightarrow{d_Y^1} Y^2 \xrightarrow{d_Y^2} \cdots \xrightarrow{d_Y^{n-1}} Y^n \xrightarrow{d_Y^n} \mathscr{G}D
$$

in $\mathsf{C}^n_{\mathcal{C}}$. By (n-EA2) for the n-exangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, we may realise the morphism $(1_{\mathscr{G}A}, \mathscr{G}c)\colon (\mathscr{G}c)^{\mathbb{E}}\delta \to \delta$ of $\mathbb{E}$-extensions by a good lift $f^\bullet = (1_{\mathscr{G}A}, f^1, f^2, \ldots, f^n, \mathscr{G}c)$. That is, $f^\bullet$ is a morphism

$$
\begin{array}{ccccccccccc}
\mathscr{G}A & \xrightarrow{d_X^0} & \mathscr{G}B & \xrightarrow{d_X^1} & X^2 & \xrightarrow{d_X^2} & \cdots & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & \mathscr{G}C \\
\| & & \downarrow{f^1} & & \downarrow{f^2} & & & & \downarrow{f^n} & & \downarrow{\mathscr{G}c} \\
\mathscr{G}A & \xrightarrow{d_Y^0} & Y^1 & \xrightarrow{d_Y^1} & Y^2 & \xrightarrow{d_Y^2} & \cdots & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & \mathscr{G}D
\end{array}
$$

of complexes such that $\mathfrak{s}((d_X^0)_{\mathbb{E}}\delta) = [M^\bullet]$, where $M^\bullet := \mathrm{MC}(\widehat{f})^\bullet$ is the mapping cone

$$
\mathscr{G}B \xrightarrow{d_M^{-1}} X^2 \oplus Y^1 \xrightarrow{d_M^0} \cdots \longrightarrow X^n \oplus Y^{n-1} \xrightarrow{d_M^{n-2}} \mathscr{G}C \oplus Y^n \xrightarrow{d_M^{n-1}} \mathscr{G}D
$$

of $\widehat{f} = (f^1, \ldots, f^n, \mathscr{G}c)$. As in Definition 2.12, the differentials are $d_M^{-1} := \begin{pmatrix} -d_X^1 \\ f^1 \end{pmatrix}$, $d_M^{n-1} := (\mathscr{G}c \quad d_Y^n)$, and

$$
d_M^i := \begin{pmatrix} -d_X^{i+2} & 0 \\ f^{i+2} & d_Y^{i+1} \end{pmatrix}
$$

for $i \in \{0, \ldots, n-2\}$. Then we see that the morphism $(1_A, c)\colon c^{\mathbb{E}'}\delta \to \delta$ of $\mathbb{E}'$-extensions is realised by the morphism

$$
\begin{array}{ccccccccccc}
A & \xrightarrow{(\mathscr{F}d_X^0)\Phi_A} & \mathscr{F}\mathscr{G}B & \xrightarrow{\mathscr{F}d_X^1} & \mathscr{F}X^2 & \xrightarrow{\mathscr{F}d_X^2} & \cdots & \xrightarrow{\mathscr{F}d_X^{n-1}} & \mathscr{F}X^n & \xrightarrow{\Phi_C^{-1}\mathscr{F}d_X^n} & C \\
\| & & \downarrow{\mathscr{F}f^1} & & \downarrow{\mathscr{F}f^2} & & & & \downarrow{\mathscr{F}f^n} & & \downarrow{c} \\
A & \xrightarrow{(\mathscr{F}d_Y^0)\Phi_A} & \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_Y^1} & \mathscr{F}Y^2 & \xrightarrow{\mathscr{F}d_Y^2} & \cdots & \xrightarrow{\mathscr{F}d_Y^{n-1}} & \mathscr{F}Y^n & \xrightarrow{\Phi_D^{-1}\mathscr{F}d_Y^n} & D
\end{array}
$$

of complexes in $\mathcal{C}'$ since $\Phi\colon \mathbb{1}_{\mathcal{C}'} \Rightarrow \mathscr{F}\mathscr{G}$ is a natural isomorphism. Using Φ again and that $\mathscr{G}a = d_X^0$, we can adjust the diagram above appropriately to obtain a morphism $g^\bullet := (1_A, (\mathscr{F}f^1)\Phi_B, \mathscr{F}f^2, \ldots, \mathscr{F}f^n, c)$

$$
\begin{array}{ccccccccccc}
A & \xrightarrow{a} & B & \xrightarrow{(\mathscr{F}d_X^1)\Phi_B} & \mathscr{F}X^2 & \xrightarrow{\mathscr{F}d_X^2} & \cdots & \xrightarrow{\mathscr{F}d_X^{n-1}} & \mathscr{F}X^n & \xrightarrow{\Phi_C^{-1}\mathscr{F}d_X^n} & C \\
\| & & \downarrow{(\mathscr{F}f^1)\Phi_B} & & \downarrow{\mathscr{F}f^2} & & & & \downarrow{\mathscr{F}f^n} & & \downarrow{c} \\
A & \xrightarrow{(\mathscr{F}d_Y^0)\Phi_A} & \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_Y^1} & \mathscr{F}Y^2 & \xrightarrow{\mathscr{F}d_Y^2} & \cdots & \xrightarrow{\mathscr{F}d_Y^{n-1}} & \mathscr{F}Y^n & \xrightarrow{\Phi_D^{-1}\mathscr{F}d_Y^n} & D,
\end{array}
$$

giving another realisation of $(1_A, c)\colon c^{\mathbb{E}'}\delta \to \delta$.

We claim that $\mathfrak{s}'(a_{\mathbb{E}'}\delta) = [\mathrm{MC}(\widehat{g})^{\bullet}]$, where $\widehat{g}^{\bullet} := ((\mathscr{F}f^1)\Phi_B, \mathscr{F}f^2, \ldots, \mathscr{F}f^n, c)$. Since $\mathfrak{s}((d_X^0)_{\mathbb{E}}\delta) = [M^{\bullet}]$ and $\mathscr{G}a = d_X^0$, we have that $\mathfrak{s}'(a_{\mathbb{E}'}\delta) = \mathfrak{s}'((\mathscr{G}a)_{\mathbb{E}}\delta) = \mathfrak{s}'((d_X^0)_{\mathbb{E}}\delta)$ is the homotopy equivalence class

$$[\ B \xrightarrow{(\mathscr{F}d_M^{-1})\Phi_B} \mathscr{F}X^2 \oplus \mathscr{F}Y^1 \xrightarrow{\mathscr{F}d_M^0} \cdots \xrightarrow{\mathscr{F}d_M^{n-3}} \mathscr{F}X^n \oplus \mathscr{F}Y^{n-1} \xrightarrow{\mathscr{F}d_M^{n-2}} \mathscr{F}\mathscr{G}C \oplus \mathscr{F}Y^n \xrightarrow{\Phi_D^{-1}\mathscr{F}d_M^{n-1}} D\ ].$$

Note that there is an isomorphism

$$
\begin{array}{ccccccccccc}
B & \xrightarrow{(\mathscr{F}d_M^{-1})\Phi_B} & \mathscr{F}X^2 \oplus \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_M^0} & \cdots & \xrightarrow{\mathscr{F}d_M^{n-3}} & \mathscr{F}X^n \oplus \mathscr{F}Y^{n-1} & \xrightarrow{\mathscr{F}d_M^{n-2}} & C \oplus \mathscr{F}Y^n & \xrightarrow{\ e\ } & D \\
\Vert & & \Vert & & & & \Vert & & \cong \downarrow \left(\begin{smallmatrix} \Phi_C & 0 \\ 0 & 1_{\mathscr{F}Y^n} \end{smallmatrix}\right) & & \Vert \\
B & \xrightarrow{(\mathscr{F}d_M^{-1})\Phi_B} & \mathscr{F}X^2 \oplus \mathscr{F}Y^1 & \xrightarrow{\mathscr{F}d_M^0} & \cdots & \xrightarrow{\mathscr{F}d_M^{n-3}} & \mathscr{F}X^n \oplus \mathscr{F}Y^{n-1} & \xrightarrow{\mathscr{F}d_M^{n-2}} & \mathscr{F}\mathscr{G}C \oplus \mathscr{F}Y^n & \xrightarrow{\Phi_D^{-1}\mathscr{F}d_M^{n-1}} & D
\end{array}
$$

of complexes, where $e := (\,c \quad \Phi_D^{-1}\mathscr{F}d_Y^n\,)$, in which the top row is precisely $\mathrm{MC}(\widehat{g})^{\bullet}$. Hence, we see that the $\mathbb{E}'$-extension $a_{\mathbb{E}'}\delta$ is realised by $\mathrm{MC}(\widehat{g})^{\bullet}$, and $g^{\bullet}$ is a good lift of $(1_A, c)$. Thus, $(n\text{-EA2})$ holds for $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ and we are done.

∎

Remark 3.10. In Theorem 3.9, the n-exangulated structure possessed by a category $\mathcal{C}$ is transported across an equivalence $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$, so that $\mathcal{C}'$ becomes an n-exangulated category and $\mathscr{F}$ becomes an n-exangle equivalence. We remark here that the n-exangulated structure acquired by $\mathcal{C}'$ is unique up to n-exangle equivalence. Analogous statements also hold for the settings of Theorems 3.4, 3.5 and 3.6.

Remark 3.11. Suppose $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ and $\mathscr{G}\colon \mathcal{C}' \to \mathcal{C}$ are functors such that $(\mathscr{F}, \mathscr{G})$ and $(\mathscr{G}, \mathscr{F})$ are both adjoint pairs. Recall that this holds whenever $\mathscr{F}$ is an equivalence and $\mathscr{G}$ is a quasi-inverse for $\mathscr{F}$, but that the converse is false in general[1]. We note here that our proof of Theorem 3.9 cannot be directly generalised to this setting: we rely on the assumption that $\mathscr{F}$ and $\mathscr{G}$ are equivalences, so that the unit and counit are both natural isomorphisms. This is needed in Lemma 3.8 to check that the assignment $\mathfrak{s}'$ is well-defined, as well as in both Lemma 3.8 and Theorem 3.9 to produce several isomorphisms between complexes.

4. Special case: skeletal categories

Suppose $\mathcal{C}'$ is a skeletal category equivalent to a weak (pre-)$(n+2)$-angulated category. Then we know $\mathcal{C}'$ is also weak (pre-)$(n+2)$-angulated by Theorem 3.4. The goal of §4 is to show that the induced $(n+2)$-suspension functor of $\mathcal{C}'$ is actually an *automorphism*, not just an autoequivalence, thereby showing $\mathcal{C}'$ is what we call a strong (pre-)$(n+2)$-angulated category.

First, let us recall what it means for a category to be skeletal and how a skeleton of a category is defined.

Definition 4.1. [32, §IV.4] A category $\mathcal{A}$ is said to be *skeletal* if $X \neq Y$ implies X is not isomorphic to Y in $\mathcal{A}$ for all $X, Y \in \mathcal{A}$.

Definition 4.2. [32, §IV.4] Let $\mathcal{A}$ be a category. A *skeleton* of $\mathcal{A}$ is a full, skeletal subcategory $\mathcal{B}$ of $\mathcal{A}$ such that the (fully faithful) canonical inclusion $\mathscr{F}\colon \mathcal{B} \hookrightarrow \mathcal{A}$ is essentially surjective.

Remark 4.3. Note that $\mathcal{A}$ may have many skeletons, but they are all isomorphic; see [32, Exer. IV.4.1].

The following proposition is a key tool in the proof of the main result of §4. We include the details for the convenience of the reader.

---

[1] We refer the reader to an example provided by user 'Hanno' on an online forum; see https://math.stackexchange.com/q/1095196.

Proposition 4.4. [17, §I, 1.364][2] *Suppose $\mathcal{A}$ and $\mathcal{B}$ are both skeletal categories and that $\mathscr{F}\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of categories. Then $\mathscr{F}$ is an isomorphism of categories.*

Proof. We need to define a functor $\mathscr{G}\colon \mathcal{B} \to \mathcal{A}$ such that $\mathscr{G}\mathscr{F} = \mathbb{1}_{\mathcal{A}}$ and $\mathscr{F}\mathscr{G} = \mathbb{1}_{\mathcal{B}}$. In order to do this, we first show that $\mathscr{F}$ is bijective on objects. Let $X \in \mathcal{B}$ be arbitrary. As $\mathscr{F}$ is an equivalence, $\mathscr{F}$ is essentially surjective and so there exists $A \in \mathcal{A}$ such that $\mathscr{F}A \cong X$. Since $\mathcal{B}$ is skeletal, we must have $\mathscr{F}A = X$. Thus, $\mathscr{F}$ is surjective on objects. Now suppose we have $A, B \in \mathcal{A}$ such that $\mathscr{F}A = X = \mathscr{F}B$ in $\mathcal{B}$. Consider the identity morphism $1_X \in \mathcal{B}(\mathscr{F}A, \mathscr{F}B) = \mathcal{B}(\mathscr{F}B, \mathscr{F}A)$. Since $\mathscr{F}$ is full, there exist morphisms $f\colon A \to B$ and $g\colon B \to A$ such that $\mathscr{F}f = 1_X$ and $\mathscr{F}g = 1_X$. Then $\mathscr{F}(gf) = 1_X = \mathscr{F}(1_A)$ and $\mathscr{F}(fg) = 1_X = \mathscr{F}(1_B)$, whence $gf = 1_A$ and $fg = 1_B$ as $\mathscr{F}$ is faithful. Thus, $A \cong B$ and we have $A = B$ as $\mathcal{A}$ is skeletal. Hence, $\mathscr{F}$ is injective on objects also.

Now define $\mathscr{G}\colon \mathcal{B} \to \mathcal{A}$ as follows. For an object $X \in \mathcal{B}$, there is precisely one object $A \in \mathcal{A}$ such that $\mathscr{F}A = X$ (as $\mathscr{F}$ is bijective on objects). In this case, put $\mathscr{G}(X) = A$. Let $f\colon X \to Y$ be an arbitrary morphism in $\mathcal{B}$. Then $X = \mathscr{F}A$ and $Y = \mathscr{F}B$ for some uniquely determined $A, B \in \mathcal{A}$, so $f \in \mathcal{B}(\mathscr{F}A, \mathscr{F}B)$. As $\mathscr{F}$ is fully faithful, there is a unique morphism $a \in \mathcal{A}(A, B)$ such that $\mathscr{F}a = f$. Thus, set $\mathscr{G}(f) = a$.

Let us show that $\mathscr{G}$ is indeed a functor. Let $X \in \mathcal{B}$ be arbitrary and consider the identity morphism 1_X. Then $X = \mathscr{F}A$ for some $A \in \mathcal{A}$ and $1_X = \mathscr{F}(1_A)$, so $\mathscr{G}(1_X) = 1_A = 1_{\mathscr{G}X}$ is the identity morphism. Now suppose we have morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{B}$. Then $f = \mathscr{F}a\colon X = \mathscr{F}A \to \mathscr{F}B = Y$ and $g = \mathscr{F}b\colon Y = \mathscr{F}B \to \mathscr{F}C = Z$, where A, B, C and a, b are all uniquely determined. Note that $gf = (\mathscr{F}b)(\mathscr{F}a) = \mathscr{F}(ba)$, so $\mathscr{G}(gf) = ba = (\mathscr{G}g)(\mathscr{G}f)$ and $\mathscr{G}$ respects composition.

Lastly, let us show that $\mathscr{F}$ and $\mathscr{G}$ are mutually inverse functors. Let $A \in \mathcal{A}$ be arbitrary. By definition of $\mathscr{G}$, the object $\mathscr{G}(\mathscr{F}A)$ in $\mathcal{A}$ is the object $A' \in \mathcal{A}$ such that $\mathscr{F}A' = \mathscr{F}A$. Since $\mathscr{F}$ is injective on objects, we must have $A' = A$ and $\mathscr{G}(\mathscr{F}A) = A$. Now let $a\colon A \to B$ be an arbitrary morphism in $\mathcal{A}$, and consider $\mathscr{F}a\colon \mathscr{F}A \to \mathscr{F}B$. Then $\mathscr{G}(\mathscr{F}a)$ is the unique morphism $a'\colon A \to B$ such that $\mathscr{F}a' = \mathscr{F}a$. Using that $\mathscr{F}$ is faithful, we obtain $a' = a$ and $\mathscr{G}(\mathscr{F}a) = a$. Hence, we have shown $\mathscr{G}\mathscr{F} = \mathbb{1}_{\mathcal{A}}$. Similarly, one can also show $\mathscr{F}\mathscr{G} = \mathbb{1}_{\mathcal{B}}$.

∎

We are now in a position to improve Theorem 3.4 when transporting a weak (pre-)$(n+2)$-angulated structure to a skeletal category.

Theorem 4.5. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a weak (pre-)$(n+2)$-angulated category and let $\mathcal{C}'$ be any skeletal category. Suppose $\mathscr{F}\colon \mathcal{C} \to \mathcal{C}'$ is an equivalence. Then $\mathcal{C}'$ is a strong (pre-)$(n+2)$-angulated category and $\mathscr{F}$ is an $(n+2)$-angle equivalence.*

Proof. An application of Theorem 3.4 shows that $\mathcal{C}'$ has the structure of a weak $(n+2)$-angulated category, for some $(n+2)$-suspension functor $\Sigma'\colon \mathcal{C}' \to \mathcal{C}'$ that is an autoequivalence, and that $\mathscr{F}$ is an $(n+2)$-angle equivalence. Since Σ' is an equivalence between skeletal categories, we have that Σ' is actually an isomorphism of categories by Proposition 4.4. That is, the $(n+2)$-suspension functor Σ' of $\mathcal{C}'$ is an automorphism and $\mathcal{C}'$ is a strong (pre-)$(n+2)$-angulated category.

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As an immediate consequence, we have the following.

Corollary 4.6. *Let $(\mathcal{C}, \Sigma, \mathcal{T})$ be a weak (pre-)$(n+2)$-angulated category. Then any skeleton $\mathcal{C}^{\mathrm{skel}}$ of $\mathcal{C}$ is a strong (pre-)$(n+2)$-angulated category.*

Proof. Let $\mathscr{F}\colon \mathcal{C}^{\mathrm{skel}} \hookrightarrow \mathcal{C}$ be the inclusion functor associated to a skeleton $\mathcal{C}^{\mathrm{skel}}$ of $\mathcal{C}$. Then $\mathscr{F}$ is an equivalence of categories, and we may apply Theorem 4.5 with $\mathcal{C}' = \mathcal{C}^{\mathrm{skel}}$.

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[2]Although this result is stated in [17], we would like to acknowledge that user 'Eric Wofsey' provides the idea behind our proof on an online forum; see `https://math.stackexchange.com/q/1658991`.

We conclude §4 by clarifying some details involved in the construction of the cluster category, a certain orbit category which was first defined in [8]. Let $\mathcal{A}$ be an additive category and suppose $\mathscr{G}$ is an automorphism of $\mathcal{A}$. The *orbit category* $\mathcal{A}/\mathscr{G}$ has the same objects as $\mathcal{A}$, and for $X, Y \in \mathcal{A}/\mathscr{G}$ we set

$$(\mathcal{A}/\mathscr{G})(X, Y) := \coprod_{i \in \mathbb{Z}} \mathcal{A}(\mathscr{G}^i X, Y);$$

see [12, Def. 2.3], [26, §1]. Note that in order to have well-defined composition of morphisms in $\mathcal{A}/\mathscr{G}$, we need $\mathscr{G}$ to be an automorphism.

Example 4.7. Let k be a field and let Λ be a finite-dimensional, hereditary k-algebra. Let $\mathsf{D} := \mathsf{D}^b(\Lambda - \mathsf{mod})$ denote the bounded derived category of finite-dimensional left Λ-modules. Then D is a triangulated ($=(1+2)$-angulated) category (see [39, Prop. 3.5.40]) and denote its suspension functor by [1]. In addition, D has Auslander-Reiten triangles (see [21, Thm. I.4.6]), and let τ^{-1} denote a quasi-inverse for the Auslander-Reiten translation (see [21, p. 42]). Note that τ^{-1} is an autoequivalence of D, so the composite endofunctor $\mathscr{F} := \tau^{-1} \circ [1]$ is also an autoequivalence of D.

The *cluster category* of Λ is defined to be the orbit category $\mathcal{C}_\Lambda := \mathsf{D}/\mathscr{F}$ (see [8, p. 576]). Recall that for an orbit category $\mathcal{A}/\mathscr{G}$, the functor $\mathscr{G}$ is required be to an automorphism. However, $\mathscr{F}$ is only an autoequivalence, and not an automorphism, of D. Thus, the definition of $\mathcal{C}_\Lambda$ just stated masks some details that we now discuss. We first take a skeleton $\mathsf{D}^{\mathrm{skel}}$ of D. Then $\mathscr{F}$ induces an autoequivalence $\mathscr{F}'$ of $\mathsf{D}^{\mathrm{skel}}$ and, by Proposition 4.4, $\mathscr{F}'$ is actually an automorphism. (This is the 'standard construction' alluded to in [26, §1].) Hence, we may consider the cluster category of Λ to be the orbit category $\mathsf{D}^{\mathrm{skel}}/\mathscr{F}'$. Moreover, $\mathsf{D}^{\mathrm{skel}}$ is triangulated by Corollary 4.6, and we may thus apply [26, Thm. 1] to conclude $\mathsf{D}^{\mathrm{skel}}/\mathscr{F}'$ is triangulated.

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