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HIRING FROM A POOL OF WORKERS

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Abstract. In many countries and institutions around the world, the hiring of workers is made through open competitions. In them, candidates take tests and are ranked based on scores in exams and other predetermined criteria. Those who satisfy some eligibility criteria are made available for hiring from a "pool of workers." In each of an ex-ante unknown number of rounds, vacancies are announced, and workers are then hired from that pool. When the scores are the only criterion for selection, the procedure satisfies desired fairness and independence properties. We show that when affirmative action policies are introduced, the established methods of reserves and procedures used in Brazil, France, and Australia, fail to satisfy those properties. We then present a new rule, which we show to be the unique rule that extends static notions of fairness to problems with multiple rounds while satisfying aggregation independence, a consistency requirement. Finally, we show that if multiple institutions hire workers from a single pool, even minor consistency requirements are incompatible with variations in the institutions' rules.

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1. Introduction

While most companies are free to use almost any criteria to decide which workers to hire and when, that is not the case in many governments and institutions around the world. To reduce the agency problems of government institutions and increase the transparency of the hiring process, those institutions have to follow clear and strict criteria for selecting workers. In particular, when the number of workers hired is large (such as police officers, tax agents, etc.), the selection procedure may consist of several steps, such as written exams, physical and psychological tests, interviews, and so on, which may also be time consuming. Due to the high costs of executing such selection procedures, these hirings often occur in two phases: the evaluation phase, in which workers apply for the job and take part in the above-mentioned tests and exams, and the second phase, in which the institutions select, over time and on a need basis, workers from the "pool" of workers who took part in the first phase. After a certain period, the pool of workers is renewed, with new ones coming through a new evaluation phase. As described by the Public Service Commission of the New South Wales government:

> "A talent pool is a group of suitable candidates (whether or not existing Public Service employees) who have been assessed against capabilities at certain levels. (...) Using a talent pool enables you to source a candidate without advertising every time a vacancy occurs. You can either directly appoint from the pool without further assessment (for example, to fill a shorter-term vacancy), or conduct a capability-based behavioral interview with one or more candidates from the pool to ensure a fit with organizational, team and role requirements (or additional assessment for agency, role specific or specialist requirements – this is recommended for longer term or ongoing vacancy). This considerably reduces the time and costs associated with advertising."[1]

The main characteristics of these procedures, which will be essential to our analysis, are that (i) the selection of workers to hire at any time, follows a well-defined rule, which is a systematic way of selecting workers to fill a specified number of positions, (ii) workers are hired in *rounds*, on a need-basis, and must be selected from the pool of workers who took part in the evaluation phase, and (iii) the institutions do not necessarily know ex-ante the number of workers they will hire during the pool's validity. Therefore, in general, not all workers in the pool will be hired. This aspect is emphasized in the description of the selection process used for all personnel hiring in the European Union institutions:

[1]Source: Public Service Commission of the New South Wales (http://www.psc.nsw.gov.au/employmentportal/recruitment/recruitment/guide/planning/talent-pools)

> "The list is then sent to the EU institutions, which are responsible for recruiting successful candidates from the list. **Being included on a reserve list does not mean you have any right or guarantee of recruitment.**" [emphasis from the original article]European Union (2015)

Notice that the quote above refers to a "reserve list". A reserve list is a group of candidates who are not hired initially but may (or may not) be hired later. Procedures that mention reserve lists are equivalent to those with a pool of workers: the pool consists of the first set of workers hired together with the reserve list.

A vast number of hirings occur around the world following this type of procedure. Most developed countries use them when hiring public sector workers to the best of our knowledge. Below we provide three examples, which are informative about the number of jobs involved.

All hirings for the U.K. Civil Service occur using open competitions which result in a "order of merit list", with a reserve list valid for 12 months.[2] In 2019, there were 445,480 civil servants in the U.K., with 44,570 of them being hired in 2018.[3]

In Brazil, the federal constitution mandates that the hiring of public sector workers in all government levels (Federal, State, and Municipal), and state-owned companies, are made through open competition. It moreover states that their results are "valid for two years", and that the workers with a non-expired competition result have priority over those with later results.[4] In 2017, there were more than 11.37 million public sector workers in Brazil.[5]

In France, most public sector workers' hiring is also made through annual open competitions (*concours*). These result in an order of merit list and must also include a "complementary list", with a number of candidates that is at most 200% of the number of positions hired in the first round.[6] In 2018, there were 5.48 million public sector workers in France. In 2016, 40,209 workers at the federal level were hired using these procedures.[7]

Very often, the rules used for hiring workers involve scores in the selection process. This is not uncommon: the criteria that are used mostly consist of a weighted average of performance points in multiple dimensions, such as written exam results, education

[2]Source: Civil Service Commission. 2018. "Recruitment Principles."

[3]Source: Civil Service Statistics 2019, Cabinet Office National Statistics

[4]Source: Brazilian Federal Constitution (1988), Article 37. In practice, this article implies that after a public competition, workers who "pass" the competition are put on hold and might be hired for two years. Those who are not hired have to reapply to a new open competition to be considered.

[5]Source: Atlas do Estado Brasileiro, IPEA

[6]Décret n°2003-532 du 18 juin 2003 relatif à l'établissement et à l'utilisation des listes complémentaires d'admission aux concours d'accès aux corps de la fonction publique de l'Etat

[7]Source: Ministère de l'action et des comptes publics

level, etc.[8] When the workers' scores constitutes the sole element for determining which workers to hire, a very natural rule, namely *sequential priority*, is commonly used: if q workers are to be hired, hire the q workers with the highest scores among those who remain in the pool. This rule is simple but has many desirable characteristics. First, it is fair in the sense that every worker who is not (yet) hired has a lower score than those who were hired. This adds a vital element of transparency to the process: if the worker can see, as is often the case, the scores of those who were hired (or at least the lowest score among those who were hired), then she has a clear understanding of why she was not hired. Secondly, it responds to the agency problem: an institution cannot arbitrarily select low-scoring workers before selecting all those who have a score higher than that worker. Finally, the selected workers' quality and identity do not depend on the number of rounds and vacancies in each round. That is, selecting 20 workers in four rounds with five workers in each results in selecting the same workers as if 20 workers were selected at once. We denote this last property by *aggregation independence*. One implication of this requirement is that the set of selected workers is independent of the number of rounds and vacancies in each round: selecting 10 workers in two rounds of five workers in each results in the same selection as selecting two workers in each of five rounds.

While sequential priority satisfies those desirable properties, the criteria used for hiring workers often combine scores with other compositional objectives, in the form of a desired proportion of workers belonging to some subset of the population, such as ethnic minorities, people with disabilities, or women. In section 4 we formalize these objectives, noting that these cannot be achieved by the sequential priority rule even if scores are designed to incorporate them, and show that minority reserves, which is arguably the best method for implementing these objectives in static problems, does not satisfy desirable properties when used multiple times over a single pool of workers.

In section 5 we present our main contribution, which is a new rule for hiring workers, that is the unique rule that satisfies natural concepts of fairness for this family of problems. We also show that it is essentially the only rule that extends static notions of fairness with compositional objectives to a problem with sequential hirings while being aggregation independent.

In section 6, we evaluate rules used in real-life applications in different parts of the world, combining scores with compositional objectives. These include "quotas" for individuals with physical or mental disabilities in public sector jobs in France, for black workers in public sector workers in Brazil, and the gender-balanced hiring of firefighters in the Australian province of New South Wales. We show that these fail most of the time to satisfy natural concepts of fairness and aggregation independence.

[8]Real-life examples of selection rules based on the ranking of workers are the selection of policemen in Berlin and public sector workers in Brazil and France.

Finally, in section 8 we consider the cases where there are multiple institutions (or locations, departments, etc.) hiring from a single pool of workers. While this scenario is widespread, our main result shows that a mild requirement, saying that the order in which firms hire workers should not change whether some of the institutions hire a worker, essentially leaves us with a single rule, which says that all institutions must hire workers following a single common priority over them.

Other than the sections described above, the rest of the paper is organized as follows. In section 2 we introduce the basic model of hiring by rules and justify the desirability of aggregation independence. In section 3 we restrict our focus to rules that are based on scores associated with each worker, in section 7 we evaluate the properties of the rules evaluated when they are used for a single round of hiring, and in section 9 we conclude. Proofs and formal descriptions of the rules absent from the main text are found in the appendix.

1.1. **Related literature.** The structure and functioning of the hiring process for public sector workers have many elements that makes it a clear target for market design: salaries and terms of employment are often not negotiable, the criteria for deciding who should be hired are exogenously given (or "designed") and there is a clear concern with issues of fairness and transparency. This paper is, to the extent of our knowledge, the first to evaluate from a theoretical perspective this type of hiring that occurs in the public sector, in which workers are sequentially hired following a predetermined criterion.

There are a few branches of the literature, however, that are related to our analysis. First, the description and analysis of methods for hiring public sector workers around the world and the incentives involved. Sundell (2014) evaluates to what extent the use of examinations constitutes a meritocratic method for recruiting in the public sector. The author observes that exams may not be the most adequate way to identify fitness for each function, but that the patronage risk involved when using more subjective criteria such as interviews and CV screening often overcomes those losses. In fact, in an empirical analysis in different ministries in the Brazilian federal government, Bugarin and Meneguin (2016) found a positive relationship between corruption cases and the proportion of employees hired by using subjective criteria.

The property of aggregation independence, which we propose is important for this problem, is related to consistency (Thomson, 1990; Tadenuma and Thomson, 1991; Thomson, 1994) notions. Loosely speaking, an allocation rule is consistent if whenever agents leave the problem with their own allocations, the residual problem's solution makes the same allocation among the remaining agents. On the other hand, aggregation independence says that the order (or timing) in which the allocation of a given number of jobs occurs does not change the identity of those who will get the jobs. Different notions of consistency have been used in other matching and allocation problems based on priorities as well (Ergin, 2002; Klaus and Klijn, 2013).

Finally, a big part of our analysis concerns what we denote compositional objectives: objectives regarding characteristics that some portions of the workers hired should have, such as a minimum proportion of workers with disabilities, ethnic minorities, or certain genders. Sönmez *et al.* (2019) evaluated the constitutionally mandated affirmative action policy used in the hiring of public sector workers in India. Similar to the cases we study, workers are also selected based on open competitions that result in an order of merit of the candidates. They also identify some shortcomings that result from how the rules are used to implement affirmative action objectives and propose solutions for them. However, they do not consider the cases in which hirings occur in multiple rounds,[9] which, as we show in this paper, might have significant consequences.

Most of the positive and normative literature on the market design consequences of affirmative action policies focus on college admissions and school choice. Kojima (2012) and Hafalir *et al.* (2013) evaluate the use of maximum quotas (which limit the number of non-minority students who can be admitted in a school) with minority reserves in the context of a centralized school choice procedure. They show that majority quotas may paradoxically hurt minority students, while minority reserves improve upon this problem. However, these welfare results have no parallel in our analysis, in which workers are either hired or not. Several other papers also evaluate affirmative action procedures currently used to select students into schools or universities, identifying shortcomings and proposing alternative procedures. Aygun and Bó (Forthcoming) show how the affirmative action procedure used in university admissions in Brazil results in fairness and incentive problems. Dur *et al.* (2020) studied the allocation of students to Chicago's elite public high schools and compared various reservation policies.[10] All of them, however, treat the problem from a static perspective: either only one choice is made, or a complete allocation is produced once and for all.

2. Hiring by rules and aggregation independence

A **rule** determines which workers an institution should hire, given a number of workers to hire, a pool of workers, and, potentially, the workers that the institution hired before. Each time an institution attempts to hire workers from the pool is denoted as a **round**.

Let A be the set of workers hired in previous rounds, and W be the set of workers available. For each (W, A, q), a rule φ determines which q workers from W should be

[9]The Indian civil service, like the Brazilian one, uses reserve lists that are valid for two years. Source: Indian Union Public Service Commission

[10]Other papers, such as Abdulkadiroğlu and Sönmez (2003); Echenique and Yenmez (2015); Bo (2016); Abdulkadiroğlu (2005) evaluated affirmative action policies in school and college matching.

hired. That is, for each (W, A, q), $\varphi(W, A, q) \subseteq W$ and $|\varphi(W, A, q)| = \min\{q, |W|\}$. For simplicity of notation, we will sometimes use the following shorthand:

$$\varphi(W, A, \langle q_1, \dots, q_t \rangle) \equiv \varphi\left(W^0, A^0, q_1\right) \cup \varphi\left(W^1, A^1, q_2\right) \cup \dots \cup \varphi\left(W^{t-1}, A^{t-1}, q_t\right)$$

Where $A^0 = \emptyset$, $W^0 = W$ and for $i > 0$, $A^i = A^{i-1} \cup \varphi\left(W^{i-1}, A^{i-1}, q_i\right)$ and $W^i = W \backslash A^i$. Furthermore, for simplicity, we will use the shorter notation $\varphi(W, q)$ when $A = \emptyset$. Unless stated explicitly, none of our results rely on situations in which there are not enough workers, either in general or with some characteristics, to be hired. That is, in all of our results, we will assume that the number of workers in W is at least as large as $\sum q_i$, and the same holds for the cases that we will evaluate in which some workers belong to minority groups.

One crucial property of the process of hiring by rules is that the sequence of hires $\lambda = \langle q_1, \dots, q_t \rangle$ is ex-ante unknown. That is, every round may or may not be the last one. The total number of workers who will be hired is also unknown. Therefore, the properties that we will deem as desirable should hold at any point in time. In this context, a critical property that a rule should satisfy is **aggregation independence**. A rule is aggregation independent if the total set of workers hired after a certain number of rounds does not depend on how they are distributed among rounds.

Definition 1. A rule φ is **aggregation independent** if for any $q \geq q_1 \geq 0$ and sets of workers W and A, $\varphi(W, A, q) = \varphi(W, A, \langle q_1, q - q_1 \rangle)$.

Therefore, when the rule being used is aggregation independent, an institution that hires q_1 workers in the first round and q_2 in the second will select the same workers that it would by hiring $q_1 + q_2$ in a single round. One can easily check that if a rule is aggregation independent, this extends to any combination of rounds: if $\sum_i q_i = \sum_j q'_j$, a sequence of hires $q_1, \dots, q_n$ will select the same workers as $q'_1, \dots, q'_m$.

We now provide three reasons to justify aggregation independence as a strongly desired property for rules for hiring by rules: transparency, non-manipulability, and robustness.

Transparency

One of the main reasons driving governments and institutions to use hiring by rules is that, for those who are not hired, the reason that happens is made clear and straightforward. For example, take the rule that consists of always hiring the workers with the highest exam scores. By knowing the rule and observing the hired workers (and their scores), any worker who was not hired knows that there was no obscure reason why she was not yet hired: it is merely because her score was lower than all those who were hired.

Suppose, however, that the rule that is used is not aggregation independent. Then, a worker who was not hired, by just looking at the set of workers who were hired, may not be able to easily understand why she was not hired, even understanding the rule that was used, because it would also be necessary for her to know the precise sequence of the number of workers that were hired in each round.

Non-manipulability

While many times the rules which govern the hiring process are chosen in a way that reduces the ability of managers to make arbitrary choices of whom to hire, they may have freedom in choosing the sequence of hires. For example, instead of hiring four workers in one month, she may choose to hire two workers first and then two additional workers.

If the rule is aggregation independent, different choices of sequences of workers hired will not lead to different sets of workers hired. However, if the rule is not aggregation independent, that may not be the case, and a manager may choose a specific sequence of hiring decisions, which will allow a particular worker to be hired, whereas she would not be, absent the specific sequence chosen. An aggregation independent rule, by definition, is not manipulable by the choice of the sequence of hires.

Robustness

The third reason why aggregation independence is a desirable property is that the degree to which the set of workers hired satisfies the objectives represented in the rule is robust to uncertainty or bad planning on the part of the manager in terms of the number of workers that is needed. In other words, assuming that the criterion for choosing workers which is set by the rule represents the desirability of the workers it chooses (for example, it chooses the most qualified set of workers subject to some constraint), an aggregation independent rule will always choose the best set of workers, whether the manager makes hiring decisions all at once or continually re-evaluates the number of workers to be hired. Aggregation independent rules do not have that problem: managers may hire workers based on demand, and that will not result in a less desirable set of workers hired.

In Section 4.1 we show specific examples of how aggregation independence relates to non-manipulability and robustness.

3. Score-based rules

A common way workers are selected when hiring by rules is through a scoring of all workers. Using criteria such as written exams, evaluation of diplomas, certificates, and experience, workers receive a score (or a number of points). These scores become the deciding factor of who to hire: when hiring q workers, hire the q workers with the highest scores from the pool. For a set of workers W, let $\boldsymbol{s_W} = (s_w)_{w \in W}$ be the score profile of workers in W, where for all $w \neq w'$, $s_w \neq s_{w'}$.[11] Denote by $top_q(W)$ the q

[11]Our assumption that no two workers have the same score is based on how the procedures that we consider work in real life. Even when discretized scores are used for evaluating the candidates, the process always results in a strict ordering of these workers. This can be seen in how the legislation

workers with the highest scores in W.[12] A natural property for a score-based rule is for it to be **fair**. That is, after any number of rounds, if a worker w was hired and w' was not, then $s_w > s_{w'}$.

Definition 2. A rule φ is **fair** if for any W, A and $\lambda = \langle q_1, \ldots, q_t \rangle$, $w \in \varphi(W, A, \lambda)$ and $w' \notin \varphi(W, A, \lambda)$ implies that $s_w > s_{w'}$.

A natural rule for these kinds of problems is what we denote by **sequential priority**. When hiring q workers, it consists of selecting the q workers with the highest score from the pool of workers. If the pool contains less than q workers, then hire all of them. The following remark comes immediately from the definition of the rule.

Remark 1. The sequential priority rule is aggregation independent and fair.

When the selection of workers is based on scores, which is a very typical setup, the sequential priority rule gives us all we need: it is fair and aggregation independent.

4. Compositional objectives

It is common for hiring processes based on rules to combine the use of scores with compositional objectives, such as affirmative action. Typically, the objective is to reserve some of the jobs for workers with a certain characteristic, sometimes those belonging to an ethnic minority or those who possess some type of disability. Denote by M the set of workers who belong to the minority group (that is, $M \subseteq W$) and $\omega(W)$ be the number of minority workers in W. The affirmative action policy also has a **minority ratio** m, where $0 \leq m \leq 1$, which represents the proportion of hires that should be based on affirmative action.

As argued in Section 2, the desirable properties associated with affirmative action should also hold after any number of rounds. Our first requirement is that, when possible, the proportion of selected minorities should be at least m after each round.

Definition 3. A rule φ **respects minority rights** if, for any W and sequence of hires $\lambda = \langle q_1, \ldots, q_t \rangle$, (i) when $|M| \geq m \times \sum_{i=1}^{t} q_i$ we have $\omega(\varphi(W, \lambda)) / |\varphi(W, \lambda)| \geq m$, or (ii) when $|M| < m \times \sum_{i=1}^{t} q_i$ we have $M \subset \varphi(W, \lambda)$.

Remark 2. The sequential priority rule does not respect minority rights. In general, fairness is incompatible with respecting minority rights.[13]

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refers to an "order of merit"', or in the details of the hiring posts, which often describe multiple (deterministic) methods for breaking ties.

[12] Although W is a set, for simplicity of notation we will consider s_W following the order in which the elements of W are written. For example, if we denote $W = \{w_2, w_1, w_3\}$, $s_W = (10, 20, 30)$ implies that worker w_2 has a score of 10.

[13] Assume that there are three workers: one minority (call him K) and two non-minority (L and V), where the scores are as follows $s_L > s_V > s_K$. If the rule needs to select two workers and m is 0.3, then in order to *respect minority rights*, the rule should select K and L, which is not *fair*, as *fairness* requires L and V to be selected.

Therefore, we define a weaker notion of fairness, which takes into account the minority restriction. A rule is minority fair if, conditional on respecting minority rights, the hiring decision is based on scores.

Definition 4. A rule φ is **minority fair** if, for any W, $M \subseteq W$ and $\lambda = \langle q_1, \ldots, q_t \rangle$, where $H = \varphi(W, \lambda)$:

(i) for each $w, w' \in W \setminus M$ or $w, w' \in M$, if $w \in H$ and $w' \notin H$, then $s_w > s_{w'}$,

(ii) for each $w \in W \setminus M$ and $w' \in M$, if $s_w < s_{w'}$ and $w \in H$, then $w' \in H$,

(iii) if there is $w \in W \setminus M$ and $w' \in M$ with $s_w > s_{w'}$, $w \notin H$ and $w' \in H$, then $\omega(H) / |H| \leq m$.

In words, a rule is minority fair if it (i) chooses between workers from the same group (minorities or non-minorities) based on their scores, (ii) does not hire low-scoring non-minorities while higher-scoring minorities are available, and (iii) only hires low-scoring minorities over higher-scoring non-minorities when that is necessary to bring the ratio of minorities closer to m from below.

One natural question one might have is whether a carefully designed "standardized" scoring system could turn the sequential priority rule into one that respects minority rights and/or is minority fair. In other words, is there a transformation s'_W of the scores s_W, for which the sequential priority rule under s'_W respects minority rights and is minority fair under s_W? The answer for the first part is yes, and for the second, no. To see this, let $\overline{s}$ be the highest score any worker might have, and s'_W be a scoring system that replicates s_W for non-minorities, and that adds $\overline{s}$ to the scores of minorities. That is, s'_W makes all minorities have higher scores than non-minorities but keep all relative scores the same otherwise. It is clear that the sequential priority rule under this scoring system respects minority rights but might not be minority fair since it will always hire minority candidates whenever some are still available, regardless of the value of m.

To see that no transformation of a scoring system can respect minority rights and be minority fair, consider the following problem: $W = \{w_1, w_2, w_3, w_4\}$, where $M = \{w_2, w_4\}$, $s_W = (100, 90, 80, 70)$ and $m = 0.5$. Let s'_W be the scoring of the workers in W derived from this transformation. Since the sequential priority rule under s'_W respects minority rights and is minority fair, it is easy to check that it must satisfy $s_{w_2} > s_{w_1} > s_{w_4} > s_{w_3}$. Imagine, however, that the set of workers is $W = \{w_1, w_3, w_4\}$. In this case, the sequential priority under s'_W would imply that if $q_1 = 1$, worker w_1 would be hired, which is a violation of minority rights since it would require w_4 to be hired instead.[14]

[14] An alternative question that one might ask is whether there is a scoring function, defined for each given set of workers and original scores, which respects minority rights and is minority fair. The answer for this one will be yes, since one can produce a ranking of workers from any aggregation independent rule by choosing one worker at a time, and as we will show later in this paper, such a rule exists. In terms of interpretation, we believe that a scoring rule that is endogenous to the set of workers being

4.1. **Sequential use of minority reserves.** Perhaps the most natural candidate for a hiring rule for this family of problems is the use of *reserves*. With this method, in each period in which there are vacancies to be filled, the institution uses a *choice procedure generated by reserves* (Hafalir *et al.*, 2013; Echenique and Yenmez, 2015), with a proportion m of vacancies reserved for minority workers.

Given a set of workers W, of minorities $M \subseteq W$, a number of reserved positions q^m and of hires q, a choice generated by reserves consists of hiring the top $\min\{q^m, |M|\}$ workers from M and then filling the remaining $q - \min\{q^m, |M|\}$ positions with the top workers in W still available. In a static setting, this procedure is shown to have desirable fairness and efficiency properties while satisfying the compositional objectives (Hafalir *et al.*, 2013). We denote the sequential use of minority reserves rule by φ^{SM}.

In our setting, therefore, the sequential use of minority reserves rule consists of, in a round r, hiring q_r workers, reserving $m \times q_r$ of them for minorities.

Proposition 1. *The sequential use of minority reserves respects minority rights. However, it is neither minority fair nor aggregation independent.*

The next example shows the problems associated with this rule.

Example 1. Let $W = \{w_1, w_2, w_3, w_4, w_5\}$, $M = \{w_1, w_2, w_5\}$, and $s_W = (100, 90, 80, 50, 20)$. Let $r = 2$, $q_1 = q_2 = 2$ and $m = 0.5$. In the first round, the top worker from M and the top from $W \setminus \{w_1\}$ are hired, that is, $\{w_1, w_2\}$. In the second round, the pools of remaining workers are $W^2 = \{w_3, w_4, w_5\}$ and $M^2 = \{w_5\}$. The top worker from M^2 , that is, $\{w_5\}$, and the top from $W^2 \setminus \{w_5\}$ are hired. Therefore, $\{w_3, w_5\}$ are hired in the second round and $\varphi^{SM}(W, \langle q_1, q_2 \rangle) = \{w_1, w_2, w_3, w_5\}$.

Now consider the case where $q = q_1 + q_2 = 4$. Then in the first and unique round, the two top workers from M, $\{w_1, w_2\}$, and the top two workers from $W \setminus \{w_1, w_2\}$ are hired, that is, $\{w_3, w_4\}$. Therefore, $\varphi^{SM}(W, q_1 + q_2) = \{w_1, w_2, w_3, w_4\}$. Therefore, the φ^{SM} rule is *not aggregation independent*. Moreover, note that $w_4 \in \varphi^{SM}(W, \langle q_1, q_2 \rangle)$, $w_5 \in \varphi^{SM}(W, \langle q_1, q_2 \rangle)$ and $s_{w_4} = 50 > 20 = s_{w_5}$ while $\omega\left(\varphi^{SM}(W, \langle q_1, q_2 \rangle)\right) / \left|\varphi^{SM}(W, \langle q_1, q_2 \rangle)\right| = 0.75 > 0.5 = m$, implying that the φ^{SM} rule is *not minority fair*.

The sequential use of minority reserves rule is a good rule for providing examples of problems associated with rules that are not aggregation independent. First, consider the issue of **manipulability**. Take the example 1 above and suppose that the manager prefers to hire the worker w_5. If she hires the four workers that she needs all at once, w_5 would not be hired. If, instead, she chooses to hire first two workers, and then later two more workers, w_5 will be hired. That is, by choosing a sequence of hires strategically, the manager can hire the person she wanted.

considered is more of a technical property that results from aggregation independence than a criterion that can be described as a scoring method for hiring before knowing who will apply for the jobs.

Next, we show that the lack of aggregation independence may lead to the hiring of a group of workers who are not in line with some common objectives of desirability, (the issue of **robustness**, as described in section 2). To see how this can be a problem, consider the example below:

Example 2. Let $W = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$, $M = \{w_5, w_6, w_7, w_8\}$, $s_W = (100, 90, 80, 70, 60, 50, 40, 30)$ and $m = 0.5$.

If workers are hired in four rounds, where $q_1 = q_2 = q_3 = q_4 = 1$, the set of workers hired will be $\{w_5, w_6, w_7, w_8\}$. If, on the other hand, workers are hired all at once, with $q_1 = 4$, the set of workers hired will be $\{w_1, w_2, w_5, w_6\}$

Assuming that the scores are a good representation of the degree of desirability of a worker for a task, the example above shows that a lack of planning could lead to hiring a set of workers that are substantially less qualified.

5. Sequential adjusted minority reserves

We now present a new rule, *sequential adjusted minority reserves*, denoted by φ^{SA}. It consists of the sequential minority reserves rule in which the number of vacancies reserved for minorities is adjusted in response to hires made in previous rounds. More specifically, the rule works as follows:[15]

Round 1 Let $m^1 = m$, $M^1 = M$ and $W^1 = W$. The top $m^1 \times q_1$ workers from M^1 are hired. Denoted those workers by A^*. Additionally, the top $(1 - m^1) \times q_1$ workers from $W^1 \backslash A^*$ are hired. Let M^2 be the workers in M^1 who were not yet hired, and W^2 be the workers in W^1 who were not yet hired.

Round $r \geq 1$ Let $A^r = \varphi^{SA}(W, \langle q_1, \ldots, q_{r-1} \rangle)$ and $m^r = \max\left\{m - \frac{\omega(A^r)}{\sum_{i=1}^{r} q^i}, 0\right\}$. The $m^r \times q_r$ top scoring minority workers in M^{r-1} are hired. Denote those workers by A^*. Additionally, the top $(1 - m^r) \times q_r$ workers from $W^r \backslash A^*$ are hired. Let M^{r+1} be the workers in M^r who were not yet hired, and W^{r+1} be the workers in W^r who were not yet hired.

Therefore, the sequential adjusted minority reserves adapts the set of workers hired according to those hired in previous rounds. This makes sense: if we do not take into account, for example, that after the last round, the number of minority workers greatly exceeded the minimum required, some high-scoring non-minority workers may not be hired, leading to a violation of minority fairness. The theorem below shows that this is essentially the only way of achieving these objectives.

Theorem 1. *If φ is a rule that is minority fair and respects minority rights, then for every set of workers W and sequence of hires λ, $\varphi(W, \lambda) = \varphi^{SA}(W, \lambda)$.*

[15] For simplicity, the description below assumes that the number of workers in M and W is large enough so that in every round there is a sufficient number of them to be hired. A more general description can be found in the appendix.

The sequential adjusted minority reserves is also the only rule that extends "static" notions of fairness to this dynamic setting while being aggregation independent. To see that, we first define the static counterparts of definitions 3 and 4.

Definition 5. A rule φ **respects static minority rights** if, for any W and $q > 0$, (i) when $|M| \geq m \times q$ we have $\omega(\varphi(W, \langle q \rangle)) / |\varphi(W, \langle q \rangle)| \geq m$, or (ii) when $|M| < m \times q$ we have $M \subset \varphi(W, \langle q \rangle)$.

Definition 6. A rule φ is **static minority fair** if, for any W, $M \subseteq W$ and $q > 0$, where $H = \varphi(W, \langle q \rangle)$:

(i) for each $w, w' \in W \setminus M$ or $w, w' \in M$, if $w \in H$ and $w' \notin H$, then $s_w > s_{w'}$,

(ii) for each $w \in W \setminus M$ and $w' \in M$, if $s_w < s_{w'}$ and $w \in H$, then $w' \in H$,

(iii) if there is $w \in W \setminus M$ and $w' \in M$ with $s_w > s_{w'}$, $w \notin H$ and $w' \in H$, then $\omega(H) / |H| \leq m$.

In other words, a rule satisfies the static versions of these two notions if they hold when there is only one round of hiring. As a result, a rule that respects minority rights also respects static minority rights, and a rule that is minority fair is also static minority fair. However, the converse does not apply: definitions 5 and 6 restrict only the first set of workers hired in a sequence of hires.

The following result shows that the sequential adjusted minority reserves is essentially[16] the only rule that extends these static notions to multiple hires while being aggregation independent.

Theorem 2. *If φ is a rule that respects static minority rights, is static minority fair, and aggregation independent, then for every set of workers W and sequence of hires λ, $\varphi(W, \lambda) = \varphi^{SA}(W, \lambda)$.*

Moreover, notice that the first round of hiring in the sequential adjusted minority reserves is equivalent to the one in the sequential use of minority reserves. Therefore, the sequential adjusted minority reserves is also the aggregation independent extension of the static *minority reserves* rule (Hafalir *et al.*, 2013).

Corollary 1. *If φ is a rule for which $\varphi(W, \langle q \rangle) = \varphi^{SM}(W, \langle q \rangle)$ and φ is aggregation independent, then for every set of workers W and sequence of hires λ, $\varphi(W, \lambda) = \varphi^{SA}(W, \lambda)$.*

6. Hiring rules around the world

In the following sections, we present the rules currently being used in France, Brazil, and Australia, and show that they suffer from different issues.

[16] Since the definition of a rule includes an arbitrary set of previous hires, there can be more rules that can satisfy those properties by defining them differently, for example, for sets of previous hires that could not result from using the rule from the beginning. If we restrict ourselves to the case in which $A = \emptyset$ (the rule is used for every hire ever made), this is a uniqueness result.

6.1. **Public sector workers in France.** By law, every vacancy in the French public sector must be filled through an *open competition*. When vacancies are announced, a document explaining deadlines, job specifications, and the criteria that will be used to rank the applicants is published. Workers who satisfy some stated requirements then take written, oral, and/or physical exams. In some cases, diplomas or other certifications can also be used for evaluating the workers. At the end of this process, all workers' results in these tests are combined in a predetermined way, to produce a ranking over all workers. If the number of vacancies announced was q, then the top q workers are hired. An additional number of workers are put on a "waiting list." These workers may be hired if some of the top q workers reject the job offer or if additional vacancies need to be filled before a new open competition is set.

The French law also establishes that at least 6% of the vacancies should be filled by people with physical or mental disabilities. Instead of incorporating the selection of those workers into the hiring procedure in a unified framework, the institutions instead open, with unclear regularity, vacancies exclusive for workers who have those disabilities.[17] The hiring of workers over time continues following the same procedure as the open positions described above. However, nothing prevents workers with disabilities from applying for open positions. In fact, the authorities provide some accommodation for these workers during the selection, such as, for example, allowing for extra time to write down the exams. These are meant as an attempt to make up for some disadvantages that those workers have with respect to those without disabilities, and not to give any advantage.

Let W^* be the set of workers who applied for the open competitions, and M^* those who applied for the competitions reserved for candidates with disabilities. Workers in M^* *must* have a disability, but workers with disabilities might also apply for open competitions. Therefore, $M^* \subseteq M$, and in general $M^* \cap W^* \neq \emptyset$. Since these constitute different competitions, there are scores for the workers in each competition, and more specifically, a worker who applies to both competitions might obtain different scores in each. Therefore, we denote by s_w^O the score obtained by worker w in the open competition and by s_w^D the score that worker w obtained in the competition for workers with disabilities. Since W^* and M^* are usually different sets of workers, some workers might have a score in only one competition, but some workers with disabilities might have in both.

The number of vacancies that are open for workers with disabilities, and when they are opened, is not determined by any law and is mostly done in an ad hoc manner. To evaluate the consequences of the method used in France in a formal way, however, we

[17] In some cases, different procedures are also used, such as making candidates with disabilities compete for the same vacancies as those without disabilities, but giving them a "bonus" in their scores. Similar methods are used in affirmative action policies elsewhere, such as in local universities in Brazil. In this paper, however, we focus on examples involving quotas and reserved vacancies.

will consider two alternative policies.[18] In both cases, we will assume that there are two pools of workers, W^* and M^*, and given a number of positions to be hired q, a total of q workers from these pools must be hired. In what follows, we consider an arbitrary sequence of hires $\langle q_1, q_2, \ldots, \rangle$.

Policy 1: This policy consists of first hiring the top q_1 workers from W^* and then adjusting the number of workers in M^* hired in later rounds. For example, say that $q_1 = 100$, but only four workers among the top 100 workers in W^* (with respect to s_W^O) hired have disabilities. Then, considering the objective of hiring at least 6% workers with disabilities, if $q_2 = 50$, then open five vacancies exclusive for workers in M^* (selected with respect to s_W^D) and 45 for those in W^* (selected with respect to s_W^O). As a result, by the end of the second round, at least 9 workers with disabilities, or $m \times (q_1 + q_2)$,[19] will be hired.

Policy 2: This policy consists of first hiring $m \times q_1$ from M^* (selected with respect to s_W^D), $(1-m) \times q_1$ workers from W^* (selected with respect to s_W^O) and then adjusting the number of workers in M^* hired in later rounds. For example, say that $q_1 = 100$. Then the policy will result in hiring six workers from M^* and 94 from W^* in the first round. At least 6% of the workers hired would be among those with a disability, therefore, but potentially more. Suppose that eight workers with disabilities were hired in the first round and that $q_2 = 50$. Then in the second round, two vacancies exclusive for workers in M^* would be open, and the remaining 48 would be open for all workers in W^*.

While the two policies above represent what we believe are the best efforts to satisfy the objectives stated in the law under the current existing procedures, Policy 1 differs in that when only one round of hiring is done, the proportion of workers with a disability hired might be below the minimal proportion stated in the law. This fact will more evident in Proposition 2.

Whenever necessary, we will refer to the rules defined by policies 1 and 2 by φ^{F_1} and φ^{F_2}. Since under the French assignment rule each worker may have one or two scores, what constitutes minority fairness is less clear in this context. However, the example below shows that policy 2 may lead to outcomes that clearly violate the spirit of minority fairness.

Example 3. Let $W^* = \{w_3, w_4, w_5\}$ and $M^* = \{w_1, w_2, w_3\}$, with scores $s_W^O = (50, 40, 30)$ and $s_W^D = (50, 40, 30)$ and $m = 0.5$. If $q = 2$, then $\varphi^{F_2}(\{W^*, M^*\}, q)$ will select $\{w_1, w_3\}$. Worker w_2, however, has a disability and a better score than w_3 in the competition in which both participated.

[18] We have no evidence that any of these policies constitute actual practice by French institutions, but we believe that they represent the two most natural attempts at satisfying the legal requirements under the current rules.

[19] For simplicity, here and in the rest of the main text we will assume that every expression involving numbers of workers or vacancies are integers. In the appendix, we relax that restriction and show that none of the results presented depend on that.

Notice that if worker w_2 also applied for the open vacancies and in that competition obtained a score that is also better than the one obtained by w_3, she would have been hired instead of w_3. If the relative rankings of the workers in both competitions are different, more intricate violations of the spirit of minority fairness can also occur. If we make the (strong) assumptions that all workers with disabilities apply to both competitions and that the relative rankings between those workers in both competitions are the same, we can obtain a clear distinction between both policies, as shown below.

Proposition 2. *Suppose that $M^* \subseteq W^*$ and that for every $w, w' \in M^*$, $s^O_w > s^O_{w'} \iff s^D_w > s^D_{w'}$. Policy 1 of the French assignment rule does not respect minority rights and is not aggregation independent. Policy 2 respects minority rights, is aggregation independent, and minority fair.*

It is crucial to notice, however, that the result in proposition 2 depends on the **relative rankings of the workers with disabilities being the same in both competitions**, but perhaps most importantly, on **workers with disabilities participating in both competitions**. This is not a minor issue, since these competitions often involve a significant amount of time and effort.

6.2. **Quotas for black public sector job workers in Brazil.** The rules for the hiring of public sector workers in Brazil work essentially in the same way as in France: vacancies are filled with open competitions that result in scores associated with the workers, and workers are hired in each period by following their scores in descending order. Differently from France, however, there is no quota for workers with disabilities, but instead, since 2014, there are quotas for black workers.

The use of racial and income-based quotas has increased significantly in many areas of the Brazilian public sector and higher education. At least 50% of the seats in federal universities, for example, are reserved for students who are black, low-income, or studied in a public high-school (Aygün and Bo, 2013). Many municipalities also employ quotas for black workers in the jobs that they offer. One of the most significant recent developments, however, is a law which establishes that 20% of the vacancies offered in each job opening should give priority to black workers.[20]

Differently from France, the quotas for black workers are explicitly incorporated into the hiring process. More specifically, the rule currently used in Brazil (denoted the φ^B rule) works as follows. Let k be a number that is higher than any expected number of hires to be made.

Initial step Workers are partitioned into two groups: (i) Top Minority (TM) and (ii) Others (O). The TM group consists of the highest scoring $\lceil m \times k \rceil$ workers

[20]Lei N. 12.990, de 9 de junho de 2014.

from M, and O be the top $k - \lceil m \times k \rceil$ workers in $W \backslash TM$.[21] Let $TM^1 = TM$ and $O^1 = O$.

Round $r \geq 1$ The $\lceil m \times q_r \rceil$ top scoring minority workers from TM^r, and the top $q - \lceil m \times q \rceil$ workers from O^r are hired. By removing these workers hired, we obtain TM^{r+1} and O^{r+1}.

For the Brazilian law specifically, $m = 0.2$. The example below shows that the Brazilian rule is not minority fair.

Example 4. Let $W = \{w_1, w_2, w_3, w_4\}$, $M = \{w_1, w_2\}$, and $s_W = (100, 90, 80, 50)$. Let $q = 2$, $m = 0.5$ and $k = 4$. Then $TM^1 = \{w_1, w_2\}$ and $O^1 = \{w_3, w_4\}$. The Brazilian rule states that, when hiring two workers, the top worker from TM^1 and the top from O^1 should be hired. Therefore, $\varphi^B(W, q) = \{w_1, w_3\}$. Since $w_2 \notin \varphi^B(W, q)$ and $s_{w_2} > s_{w_3}$, the Brazilian rule is not *minority fair*.[22]

Notice that in this example, worker w_2, who is part of the minority, has a higher score than w_3, who is not a minority. Worker w_3 is hired, while w_2 is not. Given that the affirmative action rules were designed with the intent of increasing the access that minorities have to these jobs, this type of lack of fairness is especially undesirable, since if the hiring process was purely merit-based, worker w_2 would have been hired.

Aygün and Bo (2013) describe the implementation of affirmative action in the admission to Brazilian public universities. There, as here, the problems arise from the fact that positions (in that case seats) and workers are partitioned between those reserved for minorities and the open positions. Differently from there, however, unfair outcomes may not be prevented by workers even if they strategically manipulate their minority status. In the example above, even if w_2 applied as a non-minority he would not be hired.

Proposition 3. *The Brazilian rule is aggregation independent and respects minority rights. However, it is not minority fair.*

6.3. **Gender balance in the hiring of firefighters in New South Wales.** The hiring of firefighters in the Australian province of New South Wales attempts to achieve a gender-balanced workforce by following a simple rule:

[21]Notice that the set $W \backslash TM$, in general, contains both minority and non-minority workers. As a result, if there are not enough minority workers, the remaining positions are filled with the top non-minority workers.

[22]One may conjecture that the scenario above is very unexpected, since the affirmative action law must have been enacted in response to minority workers not being hired based solely on scores. As shown in Aygün and Bo (2013), however, this conjecture may be misleading. For example, even if the average score obtained by minority workers is lower, one can have situations in which the preferences of the higher achieving minority workers are correlated, leading to the top minority workers in the entire population applying to a specific job.

> "Candidates who have successfully progressed through the recruitment stages may then be offered a place in the Firefighter Recruitment training program. Written offers of employment will be made to an equal number of the most meritorious male and female candidates based on performance at interview and the other components of the recruitment process combines."[23]

We denote this rule by *NSW rule*, or φ^{NSW}. Although not stated explicitly in the institution's website, we will assume that if there are not enough individuals of some gender, the remaining hirings will be made among those candidates available, based on their scores. Moreover, to avoid results that rely simply on whether q is odd or even, we assume it is always even. The example below shows the problems involved in that rule:

Example 5. Let $W = \{w_1, w_2, w_3, w_4\}$, and $W^F = \{w_1, w_2\}$ and $W^M = \{w_3, w_4\}$ be the set of female and male workers, respectively. Suppose that the scores are $s_W = (100, 90, 80, 50)$. Let $q = 2$. Then $\varphi^{NSW}(W, q) = \{w_1, w_3\}$. Since w_2 is not hired but $s_{w_2} > s_{w_3}$, the NSW rule is not fair. Moreover, it is easy to see that if either gender is considered a minority, the rule is also not minority fair.

The result below summarizes the properties of the NSW rule.

Proposition 4. *The NSW rule is aggregation independent but not fair. If one of the genders is deemed as a minority, then it respects minority rights but is not minority fair.*

7. Single-round hiring

Until now, we evaluated rules from the perspective of whether they satisfy the desirable properties we introduced in the previous sections: aggregation independence, respecting minority rights, and minority fairness. While our analysis focuses on hirings potentially involving multiple rounds, one might wonder whether some of these problems would be present when the hiring is made in a single round.

The property of aggregation independence does not imply anything regarding a single round of hiring. As we mentioned in section 5, since the properties of respecting minority rights and minority fairness are only satisfied when they are satisfied for any number of rounds, the rules for which they are satisfied will also satisfy them for a single round of hiring. For the French policies, the results do not change.

Remark 3. Suppose, as in Proposition 2, that $M^* \subseteq W^*$, and that for every $w, w' \in M^*$, $s^O_w > s^O_{w'} \iff s^D_w > s^D_{w'}$. Policy 1 of the French assignment rule does not respect static minority rights, and Policy 2 respects static minority rights and is static minority fair.

[23]Source: Fire & Rescue NSW (https://www.fire.nsw.gov.au/page.php?id=9126)

Next, consider the (sequential) use of minority reserves. When only a single round of hiring occurs, the rule satisfies all the desirable characteristics. Moreover, as we mentioned in section 5, when there is a single round of hiring, it is equivalent to the sequential adjusted minority reserves.

Remark 4. The (sequential) use of minority reserves respects static minority rights and is static minority fair.

The problems with minority reserves are not present under single hiring. As shown in Example 1, the issues reside on the fact that minority fairness requires that an asymmetric priority is given to minority candidates only when their proportion among those hired is below m. The sequential use of minority rights "lacks memory", in the sense that it always gives this asymmetric priority to minority workers, regardless of how much it is still needed given past hires. When only one round of hiring occurs, that is not a problem.

Regarding the Brazilian rule, the variable that determines its characteristics is k, that is, what is the number of workers from W what will be put in the sets TM and O. To see this, let q_1 be the number of hires in the single-round hiring, and let moreover $k = q_1$. One can easily verify that the hiring that will be made is the same as the one done by the (sequential) use of minority reserves. If $k > q_1$, on the other hand, we can have the situation shown in Example 4. Therefore:

Remark 5. Let q_1 be the number of hires that occur in a single round using the Brazilian rule. If $k = q_1$, then the rule respects static minority rights and is static minority fair. If $k > q_1$, then the rule respects static minority rights, but is not static minority fair.

Finally, since the negative results for the NSW rule are based on a single round of hiring, we have the following remark:

Remark 6. When only one round of hiring occurs, the NSW rule is not fair. If one of the genders is deemed as a minority, then it respects static minority rights but is not static minority fair.

8. Multiple institutions

While often the hiring processes that we describe involve one or more positions in a single job specification – and therefore the pool of applicants, or reserve list, being used only for that position[24] – in many cases, a pool of workers is shared between multiple

[24]As we noted in section 1, most of the hiring in the public sector in France and Brazil, for example, requires the use of order of merit lists and reserve lists. While for some positions, such as police officers, it is natural to expect that the workers might be matched to different locations, many other positions are more specific and do not result in a pool of candidates shared by more than one job. Examples include the hiring of doctors with a specific specialty for a municipality with a single hospital, the role of economist in state companies, who only work at the headquarters, the entry-level diplomatic career, etc.

institutions or locations. For example, in the hiring process for the Brazilian Federal police, workers may be allocated to different locations.[25] In the selection process for the New Zealand police, the candidate's preference is also taken into account when deciding which district a worker who will be hired from the pool will go to:

> "The candidate pool is not a waiting list. The strongest candidates are always chosen according to the needs and priorities of the districts. The time it takes to get called up to college depends on your individual strengths and the constabulary recruitment requirements in your preferred districts. (...) We will look to place you into your preferred district but you may also be given the option to work in another district where recruits are needed most."[26]

In this section, we evaluate how the fact that workers may be hired by more than one institution affects the attainability of basic desirable properties. Now, in addition to the set of workers W, there is also a set of institutions $I = \{i_1, \ldots, i_\ell\}$, where $|I| \geq 3$. Institutions make sequences of hires, and there is no simultaneity in their hires: in each round only one institution may hire workers. Therefore, when we describe a round, we now must determine not only how many workers are hired, but also which institution those workers will be assigned to. Some additional notation will be, therefore, necessary. A **matching** μ is a function from $I \cup W$ to subsets of $I \cup W$ such that:

- $\mu(w) \in I \cup \{\emptyset\}$ and $|\mu(w)| = 1$ for every worker i,[27]
- $\mu(i) \subseteq W$ for every institution i,
- $\mu(w) = i$ if and only if $w \in \mu(i)$.

At the end of each round $r \geq 1$, we define the **matching** of workers to institutions as a function μ^r.

A **plural sequence of hires** Λ is a list of pairs (i, q), where $i \in I$, and q is the number of workers hired. A plural sequence of hires $\Lambda = \langle (i_1, 3), (i_3, 2), (i_1, 2) \rangle$, for example, represents the case in which in the first round institution i_1 hires three workers, in the second round institution i_3 hires two workers, and then in the third round institution i_1 hires two workers.

When considering hiring with multiple institutions, a rule, therefore, can be generalized to produce matchings instead of allocations. Given a pool of workers W, an initial matching μ^0, and a plural sequence of hires $\Lambda = \langle (i_1, q_1), (i_2, q_2), \ldots, (i_k, q_k) \rangle$, a hiring rule Φ is derived from a set of institutional rules $(\Phi_i)_{i \in I}$ by returning the matching combining all institutional rules. That is, if $\Phi(W, \mu^0, \Lambda) = \mu$, then $\mu(i) = \Phi_i(W, \mu^0, \Lambda)$.

[25]Source: Brazilian Department of Federal Police.

[26]Source: New Zealand Police (https://www.newcops.co.nz/recruitment-process/candidate-pool), accessed in March 8th 2018.

[27]We abuse notation and consider $\mu(w)$ to be an element of I, instead of a set with an element of I.

Given W and some matching μ^{t-1}, $\Phi_i\left(W, \mu^{t-1}, \langle (i,q) \rangle\right) \subseteq W \setminus \bigcup_{i\in I} \mu^{t-1}(i)$. That is, it selects workers from W who are not yet matched to some institution in μ^{t-1}. Moreover:

$$\Phi_i\left(W, \mu^0, \Lambda\right) = \bigcup_{t=1}^{k} \Phi_i\left(W, \mu^{t-1}, \langle (i_t, q_t) \rangle\right)$$

For any $t > 0$, the matching μ^t is such that for all $i \neq i_t$, $\mu^t(i) = \mu^{t-1}(i)$, but $\mu^t(i_t) = \mu^{t-1}(i_t) \cup \Phi_{i^t}\left(W, \mu^{t-1}, \langle (i_t, q_t) \rangle\right)$. We assume that $\Phi_i\left(W, \mu, \langle (i', q) \rangle\right) = \emptyset$ whenever $i \neq i'$. That is, one institution cannot "hire for another institution". We also assume that if μ and μ' are such that $\cup_{i\in I}\mu(i) = \cup_{i\in I}\mu'(i)$ and $\mu(i^*) = \mu'(i^*)$, then $\Phi_{i^*}\left(W, \mu, \langle (i^*, q) \rangle\right) = \Phi_{i^*}\left(W, \mu', \langle (i^*, q) \rangle\right)$. That is, an institution i^*'s hiring decision can depend on the set of workers who were not yet hired and the workers who were already hired by i^*, but not on the identity of the workers who were hired by each other institution.

Let $\mu^{\emptyset}$ be a matching in which for all $i \in I$, $\mu^{\emptyset}(i) = \emptyset$. We denote by $\Phi_i(W, \Lambda)$ and $\Phi(W, \Lambda)$ the values of $\Phi_i\left(W, \mu^{\emptyset}, \Lambda\right)$ and $\Phi\left(W, \mu^{\emptyset}, \Lambda\right)$. Finally, we abuse notation and if $\Lambda = \langle (i_1, q_1), (i_2, q_2), \ldots, (i_k, q_k) \rangle$, we can append the plural sequence of hires with the notation $\langle \Lambda, (i^*, q^*) \rangle \equiv \langle (i_1, q_1), (i_2, q_2), \ldots, (i_k, q_k), (i^*, q^*) \rangle$.

The example below clarifies these points.

Example 6. Consider a set of workers $W = \{w_1, w_2, w_3, w_4, w_5\}$ with scores $s_W = (100, 90, 80, 50, 20)$, a set of institutions $I = \{i_1, i_2, i_3\}$ and let Φ be a rule that, in any round, matches the highest scoring workers to the institution in that round. Then if $\Lambda = \langle (i_1, 1), (i_3, 2), (i_1, 1) \rangle$, the matchings μ^1, μ^2 and μ^3 produced at the end of each round are:

$$\mu^1 = \begin{pmatrix} i_1 & i_2 & i_3 \\ w_1 & \emptyset & \emptyset \end{pmatrix} \quad \mu^2 = \begin{pmatrix} i_1 & i_2 & i_3 \\ w_1 & \emptyset & \{w_2, w_3\} \end{pmatrix} \quad \mu^3 = \begin{pmatrix} i_1 & i_2 & i_3 \\ \{w_1, w_4\} & \emptyset & \{w_2, w_3\} \end{pmatrix}$$

We will consider two properties for rules when there are multiple institutions. The first is related to the desirability of workers.

Definition 7. A rule Φ satisfies **common top** if there exists a worker $w^* \in W$ such that, for every institution $i \in I$, $w^* \in \Phi_i\left(W, \langle (i, 1) \rangle\right)$.

In words, common top requires that there is at least one worker that, whenever available, every institution would hire.

Next, we consider a weak notion of consistency across the hirings made by the institutions.

Definition 8. A rule Φ satisfies **permutation independence** if for any plural sequence of hires Λ and any permutation of its elements $\sigma(\Lambda)$, $\bigcup_{i\in I} \Phi_i(W, \Lambda) = \bigcup_{i\in I} \Phi_i(W, \sigma(\Lambda))$.

Permutation independence, therefore, simply requires that the set of workers hired, regardless of where, should not change if we adjust the order of hiring.

We also adapt the notion of aggregation independence to multiple institutions by requiring that each institution's rules are aggregation independent.

Definition 9. A rule Φ is **aggregation independent** if for any $q \geq q_1 \geq 0$, sets of workers W, institution $i \in I$, and matching μ, $\Phi_i\left(W, \mu, \langle(i, q)\rangle\right) = \Phi_i\left(W, \mu, \langle(i, q_1), (i, q - q_1)\rangle\right)$.

The family of rules that we will use in our next result is elementary but also very restrictive.

A rule Φ is **single priority** if there exists a strict ranking $\succ$ of the workers in W such that when Λ is any plural sequence of hires in which at least two different institutions make hires, for every $i \in I$:

$$\Phi_i\left(W, \langle\Lambda, (i, q)\rangle\right) = \Phi_i\left(W, \Lambda\right) \cup \max_{\succ}^{q} W \backslash \Phi_i\left(W, \Lambda\right)$$

Where, given a set X, $\max_{\succ}^{q} X$ is the set with the top q elements of X with respect to the ordering $\succ$. In words, a rule is single priority if, whenever more than one institution make hires, all hirings from all institutions consist of hiring the top workers, among the remaining ones, when all of these institutions share a common ranking.

The result below shows that, for a wide range of applications, having multiple institutions is incompatible with most objectives a policymaker may have.

Theorem 3. *A rule satisfies common top, aggregation independence and permutation independence if and only if it is a single priority rule.*

Theorem 3 is a fundamentally negative result. It shows that aggregation and permutation independence, both arguably simple desirable characteristics, are incompatible with institutions following different criteria when evaluating these candidates. This is true even when all institutions share the same scores for workers, but institutions might have different values of m (the proportion of minorities that must be hired).[28]

Notice, however, that if the compositional objectives are interpreted as being applied to the entire set of workers hired by these institutions, as a whole, then we can simply use the sequential adjusted minority reserves, as defined in section 5, every time an institution wants to hire a given number of workers. This procedure satisfies aggregation independence and is also permutation independent. Moreover, it satisfies a natural adaptation of what it means to respect minority rights and minority fairness. Instead of applying to whether workers are hired by a specific institution, it applies to being hired at *some* institution.

[28]Take, for example, $I = \{i_1, i_2\}$, $W = \{w_1, w_2, w_3, w_4, w_5\}$, $M = \{w_1, w_2, w_5\}$, $s_{w_1} > s_{w_2} > s_{w_3} > s_{w_4} > s_{w_5}$, and the value of m for the two institutions being $m_1 = 0.5$ and $m_2 = 0$. If both institutions use the sequential adjusted minority reserves and i_1 hires two workers before i_2 also hires two, worker w_3 is hired and w_5 is not. If the order is that i_2 hires first, then w_5 is hired and w_3 is not. A violation of permutation independence.

9. Conclusion

In this paper, we evaluate a hiring method that is widely used around the world, especially for public sector jobs, where institutions select their workers over time from a pool of eligible workers. While the simple and natural rule of sequential priority satisfies all desirable characteristics, the addition of compositional objectives such as affirmative action policies increases the complexity of the procedures. We show that the rules being used in practical hiring processes, as well as the direct application of minority reserves, fail fairness or aggregation independence. When the compositional objectives can be modeled as affirmative action for minorities, the sequential adjusted minority reserves, which we introduced, is therefore the unique solution that satisfies those desirable properties.

If multiple institutions hire from the same pool of applicants, however, we show that the space for different hiring criteria between institutions, is highly restricted when a minimal requirement of independence is imposed.

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Appendix

Formal description of the rules. For the descriptions in this section, consider as given a set W of workers, a set $M \subseteq W$ of minority workers, a set A of workers previously hired, a sequence of hires $q^r = \langle q_1, q_2, \ldots, q_k \rangle$, and a score profile $\mathbf{s_W}$.

Sequential Priority (SP rule).

Round 1: Let $W_1 = W$. The highest scoring q_1 workers in W_1 are selected. Let A_1, be the set of selected workers, where for each $w \in A_1$ and each $w' \in W_1 \setminus A_1$ we have $s_w > s_{w'}$, and $|A_1| = q_1$.

Round $\boldsymbol{k > 1}$: Let $W_k = W_{k-1} \setminus A_{k-1}$. The highest scoring q_k workers in W_k are selected. Let A_k be the set of selected workers, where for each $w \in A_k$ and each $w' \in W_k \setminus A_k$ we have $s_w > s_{w'}$, and $|A_k| = q_k$.

The assignment selected by SP rule is

$$\varphi^{SP}(W, \langle q_1, \ldots, q_r \rangle) = \bigcup_{a \leq r} A_a.$$

Sequential Adjusted Minority Reserves (SA rule).

Round 1:

Step 1.1: Let $W_{1,1} = W$, $M_{1,1} = M \cap W_{1,1}$ and $q_{1,1} = \lceil m \times q_1 \rceil$. The highest scoring $\min\{q_{1,1}, |M_{1,1}|\}$ workers in $M_{1,1}$ are selected. Let $A_{1,1}$ be the set of selected workers, where $A_{1,1} \subseteq M_{1,1}$.

Step 1.2: Let $W_{1,2} = W_{1,1} \backslash A_{1,1}$, $M_{1,2} = M \cap W_{1,2}$ and $q_{1,2} = q_1 - |A_{1,1}|$. The highest scoring $q_{1,2}$ workers in $W_{1,2}$ are selected. Let $A_{1,2}$ be the set of selected workers.

Round $k > 1$:

Step k.1: Let $W_{k,1} = W_{k-1,2} \backslash A_{k-1,2}$, $M_{k,1} = M \cap (W_{k-1,2} \backslash A_{k-1,2})$ and $q_{k,1} = \lceil \min\{\max\{m - \frac{\omega(A_{1,2}) + \ldots + \omega(A_{k-1,2})}{q_k}, 0\} \times q_k, |M_{k,1}|\} \rceil$. The highest scoring $q_{k,1}$ workers in $M_{k,1}$ are selected. Let $A_{k,1}$ be the set of selected workers.

Step k.2: Let $W_{k,2} = W_{k,1} \backslash A_{k,1}$, $M_{k,2} = M \cap W_{k,2}$ and $q_{k,2} = q_k - |A_{k,1}|$. The highest scoring $q_{k,2}$ workers are selected from $W_{k,2}$. Let $A_{k,2}$ be the set of selected workers.

The assignment selected by the SA rule is
$$\varphi^{SA}(W, A, \langle q_1, \ldots, q_r \rangle) = \bigcup_{\substack{a \leq r \\ i \in \{1,2\}}} A_{a,i}.$$

Sequential use of minority reserves (SM rule).

Round 1:

Step 1.1: Let $W_{1,1} = W$, $M_{1,1} = M \cap W_{1,1}$ and $q_{1,1} = \lceil m \times q_1 \rceil$. The highest scoring $\min\{q_{1,1}, |M_{1,1}|\}$ workers are selected from $M_{1,1}$. Let $A_{1,1}$ be the set of selected workers.

Step 1.2: Let $W_{1,2} = W \setminus A_{1,1}$, $M_{1,2} = M \cap W_{1,2}$ and $q_{1,2} = q_1 - |A_{1,1}|$. The highest scoring $q_{1,2}$ workers are selected from $W_{1,2}$. Let $A_{1,2}$ be the set of selected workers.

Round $k > 1$:

Step k.1: Let $W_{k,1} = W_{k-1,2} \setminus A_{k-1,2}$, $M_{k,1} = M \cap (W_{k-1,2} \setminus A_{k-1,2})$ and $q_{k,1} = \lceil m \times q_k \rceil$. The highest scoring $\min\{q_{k,1}, |M_{k,1}|\}$ workers from $M_{k,1}$ are selected. Let $A_{k,1}$ be the set of selected workers.

Step k.2: Let $W_{k,2} = W \setminus A_{k,1}$, $M_{k,2} = M \cap W_{k,2}$ and $q_{k,2} = q_k - |A_{k,1}|$. The highest scoring $q_{k,2}$ workers from $W_{k,2}$ are selected. Let $A_{k,2}$ be the set of selected workers.

The assignment produced by the SM rule is $\varphi^{SM}(W, q^r) = \bigcup_{\substack{a \leq r \\ i \in \{1,2\}}} A_{a,i}$.

Brazilian assignment rule (B rule).

The rule first identifies a large number k (which is larger than the total number of vacancies to be filled but no larger than $|W|$). Then two groups are identified: (i) TM, which is the set with the top $k \times m$ minority workers: $TM \subseteq M$ with $|TM| = \lceil k \times m \rceil$ such that for each $w \in TM$ and each $w' \in M \setminus TM$, we have $s_w > s_{w'}$ and (ii) O, which is the set with the top $k(1-m)$ workers among those who were not chosen in (i), that is: $O \subseteq W \setminus TM$ such that $|O| = \lfloor k(1-m) \rfloor$ and for each $w \in O$ and $w' \in W \setminus (O \cup TM)$, we have $s_w > s_{w'}$. Within each round $a \leq r$, we have two steps.

Round 1:

Step 1.1: Let $O_{1,1} = O$, $TM_{1,1} = TM$ and $q_{1,1} = \lceil m \times q_1 \rceil$. The highest scoring $\min\{q_{1,1}, |TM_{1,1}|\}$ minority workers are selected from $TM_{1,1}$. Let$A_{1,1}$ be the set of selected workers.

Step 1.2: Let $O_{1,2} = O_{1,1}$, $TM_{1,2} = TM \setminus A_{1,1}$ and $q_{1,2} = q_1 - |A_{1,1}|$. The highest scoring $q_{1,2}$ workers are selected from $O_{1,2}$. Let $A_{1,2}$ be the set of selected workers.

Round k > 1:

Step k.1: Let $O_{k,1} = O_{k-1,2} \setminus A_{k-1,2}$, $TM_{k,1} = TM_{k-1,2}$ and $q_{k,1} = \lceil m \times q_k \rceil$. The highest scoring $\min\{q_{k,1}, |TM_{k,1}|\}$ minority workers are selected from $TM_{k,1}$. Let $A_{k,1}$ be the set of selected workers.

Step k.2: Let $O_{k,2} = O_{k,1}$, $TM_{k,2} = TM \setminus A_{k,1}$ and $q_{k,2} = q_k - |A_{k,1}|$. The highest scoring $q_{k,2}$ workers are selected from $O_{k,2}$. Let$A_{k,2}$ be the set of selected workers.

The assignment produced by the B rule is $\varphi^B(W, q^r) = \bigcup_{\substack{a \leq r \\ i \in \{1,2\}}} A_{a,i}$.

French assignment rule (F rule).

Let m be the target ratio of people with disabilities, $\mathbf{s_W^O}$ be a scoring profile for workers in the open competition and $\mathbf{s_W^D}$ be a scoring profile for workers in the competition for workers with disabilities.

Round 1:

Policy 1: Let $W_{1,1} = W$, $M_{1,1} = M \cap W_{1,1}$. The highest scoring $\min\{q_{1,1}, |W_{1,1}|\}$ workers, with respect to s_W^O, are selected from $W_{1,1}$. Let A_1 be the set of selected workers.

Policy 2: Let $W_{1,1} = W$, $M_{1,1} = M$. The highest scoring $\min\{\lfloor(1-m) \times q_{1,1}\rfloor, |M_{1,1}|\}$ workers, with respect to s_W^D, are selected from $M_{1,1}$. Let $A_{1,1}$ be the set of workers selected in this step. Then, the highest scoring $\min\{\lceil m \times q_{1,1}\rceil, |W_{1,1} \setminus A_{1,1}|\}$ workers, with respect to s_W^O, are selected from $W_{1,1} \setminus A_{1,1}$. Let $A_{1,2}$ be the set of selected workers in this step, and let $A_1 = A_{1,1} \cup A_{1,2}$.

Round k > 1:

Step k.1: Let $W_{k,1} = W_{k-1,2} \setminus A_{k-1,2}$, $TA_{k,1} = \bigcup_{i=1}^{k-1} A_i$. Let $q_{k,1} = min\left\{max\left\{m \times \left(\sum_{i=1}^{k} q_i\right) - \omega\left(TA_{k,1}\right), 0\right\}, |M_{k,1}|\right\}$. The highest scoring $q_{k,1}$ workers, with respect to s_W^D, are selected from $M_{k,1}$. Let $A_{k,1}$ be the set of workers selected in this step.

Step k.2: Let $W_{k,2} = W_{k,1} \setminus A_{k,1}$, and $q_{k,2} = q_k - |A_{k,1}|$. The highest scoring $q_{k,2}$ workers, with respect to s_W^O, are selected from $W_{k,2}$. Let $A_{k,2}$ be the set of selected workers, and $A_k = A_{k,1} \cup A_{k,2}$.

The assignment produced by the F rule is $\varphi^F(W, q^r) = \bigcup\limits_{a \leq r} A_a$.

Proofs.

Proof of Proposition 1. Example 1 shows that the sequential use of minority reserves is neither aggregation independent nor fair. To see that it respects minority rights, notice that every time q workers are hired, **at least** $m \times q$ minority workers are among them. As a result, a proportion of at least m of the workers hired, at any point, is in M and therefore the rule respects minority rights. □

Proof of Theorem 1. First, we show that the SA rule respects minority rights and is minority fair.

By definition, the SA rule *respects minority rights*, at the step $k.1$ of each round k, selects minority workers to satisfy the minimum requirement up to that round. Note that when there are not enough minority workers, SA selects all the available minority workers.

Now, we show that the rule is minority fair.

Let $A \equiv \varphi^{SA}(W, \langle q_1, \ldots, q_r \rangle)$ be the selection made for the problem. We want to show that (i) for each $w, w' \in W \setminus M$, if $w \in A$ and $w' \notin A$, then $s_w > s_{w'}$, (ii) for each $w, w' \in M$, if $w \in A$ and $w' \notin A$, then $s_w > s_{w'}$. (iii) for each $w \in W \setminus M$ and $w' \in M$, if $s_w < s_{w'}$ and $w \in A$, then $w' \in A$, (iv) if there is $w \in W \setminus M$ and $w' \in M$ with $s_w > s_{w'}$, $w \notin A$ and $w' \in A$, then $\omega(A)/|A| \leq m$.

First note that cases (i), (ii) and (iii) hold trivially as at step $k.1$ of each round k, the rule selects the highest scoring workers in M, and in step $k.2$ it selects the highest scoring workers.

Suppose, for contradiction, that there is $w \in W \setminus M$ and $w' \in M$ with $s_w > s_{w'}$, $w \notin A$ and $w' \in A$, but $\omega(A)/|A| > m$. Note that w' cannot be selected at step $k.2$ of any round k, as w would have been selected as well. The only case in which the candidate w' is selected is during step $\ell.1$ of some round ℓ. Since $s_w > s_{w'}$, $w \notin A$ and $w' \in A$, then $|top_q(W) \cap M| < m \times q$, where $q = \sum_{a \leq r} q_a$.[29] Thus, at step $r.1$ of the last round r,

[29]That is, the only way to hire a minority worker with a lower score and not the non-minority with a higher score, is to satisfy the minority requirements. As we mentioned earlier, the worker w' is hired during step $\ell.1$ of some round ℓ, where selection occurs among minorities only.

a selection is made so that $|(\bigcup_{\substack{a<r \\ i\in\{1,2\}}} A_a^i) \cup A_k^1| = m \times q$. Since $|top_q(W) \cap M| < m \times q$, we have $A_r^2 \cap M = \emptyset$. Thus, we obtain $|A \cap M| = m \times q$ which contradicts our assumption.

For the next part, we need to introduce some concepts. First, we will say that given a set of workers W, minority workers $M \subseteq W$, score profile $\mathbf{s_W}$, $q \geq 0$ and $m \geq 0$, a set $W^* \subseteq W$ *respects minority rights* if W^* satisfies the same conditions that $\varphi(W, \langle q \rangle)$ must satisfy when φ respects static minority rights (definition 5). Similarly, a set $W^* \subseteq W$ is *minority fair* if W^* satisfies the same conditions that $\varphi(W, \langle q \rangle)$ must satisfy when φ is static minority fair (definition 6).

To show that if a rule is minority fair and respects minority rights it is the SA rule, we show that, for any given number of workers to be hired q, there is only one set of workers of that size that is minority fair and respects minority rights.

Lemma 1. *For any given set of workers W, minority workers $M \subseteq W$, score profile $\mathbf{s_W}$, $q \geq 0$ and $m \geq 0$, there exists only one set $W^* \subseteq W$ that respects minority rights, is minority fair and such that $|W^*| = q$.*

Proof. First, note that property (i) of minority fairness implies that if a set W^* is minority fair, it contains the set $top_{\omega(W^*)}(M)$, that is, the top $\omega(W^*)$ highest scoring workers in M, and the set $top_{q-\omega(W^*)}(W \backslash M)$, the $q - \omega(W^*)$ highest scoring workers in $W \backslash M$, both with respect to $\mathbf{s_W}$.

Suppose, for contradiction, that there are $W^1 \subseteq W$ and $W^2 \subseteq W$, where both W^1 and W^2 respect minority rights and are minority fair, $|W^1| = |W^2| = q$, and $W^1 \neq W^2$.

Note first that if $|M| < m \times q$, respecting minority rights implies that $M \subset W^1$ and $M \subset W^2$. Minority fairness implies, moreover, that $top_{q-|M|}(W \backslash M) \subseteq W^1$ and $top_{q-|M|}(W \backslash M) \subseteq W^2$. But then $W^1 = W^2$, a contradiction. It must be, therefore, that $|M| \geq m \times q$.

Next, note that minority fairness implies that $\omega(W^1) \neq \omega(W^2)$. To see that, notice that if $\omega(W^1) = \omega(W^2) = m^*$, $top_{m^*}(M) \subseteq W^1$, $top_{q-m^*}(W \backslash M) \subseteq W^1$, $top_{m^*}(M) \subseteq W^2$, and $top_{q-m^*}(W \backslash M) \subseteq W^2$. But this would imply that $W^1 = W^2$, a contradiction.

Suppose now, without loss of generality, that $\omega(W^1) > \omega(W^2)$. Since W^2 respects minority rights, $\omega(W^2) \geq m \times q$, and therefore $\omega(W^1) > \omega(W^2) \geq m \times q$. Therefore, there is a worker $w_1^* \in W \backslash M$ such that $w_1^* \in W^2$ and $w_1^* \notin W^1$, and a worker $w_2^* \in M$ such that $w_2^* \in W^1$ and $w_2^* \notin W^2$.

We have two cases to consider. First, suppose that $s_{w_1^*} > s_{w_2^*}$. This would violate W^1 being minority fair, since $m(W^1) > m \times q$ and $w_1^* \notin W^1$. Then it must be that $s_{w_2^*} > s_{w_1^*}$. But then, since W^2 is minority fair, condition (ii) implies that $w_2^* \in W^2$, a contradiction.

We conclude, therefore, that $W^1 \neq W^2$ is false, proving uniqueness. $\square$

Since the SA rule respects minority rights and is minority fair, lemma 1 implies that this is the only such rule.

□

Proof of Theorem 2. Let φ be a rule that is static minority fair, satisfies static minority rights, and is aggregation independent. Let λ^* be any sequence of hires.

We will follow by induction on the rounds in λ^*. First, the base $\langle q_1 \rangle$: from lemma 1, there is a unique set $W^1 \subseteq W$ that is minority fair and respects minority rights. Both φ and φ^{SA} are static minority fair and respect static minority rights. Therefore, $\varphi(W, \langle q_1 \rangle) = \varphi^{SA}(W, \langle q_1 \rangle) = W^1$.

For the induction step, assume that $\varphi(W, \langle q_1, q_2, \ldots, q_\ell \rangle) = \varphi^{SA}(W, \langle q_1, q_2, \ldots, q_\ell \rangle)$. Since φ is aggregation independent, the following is true:

$$\varphi\left(W, \langle q_1, q_2, \ldots, q_\ell \rangle\right) = \varphi\left(W, \langle q \rangle\right)$$

where $q = \sum_{i=1}^{\ell} q_i$. Let $H = \varphi\left(W, \langle q \rangle\right)$. Aggregation independence of φ implies, moreover, that:

$$\varphi\left(W, \langle q \rangle\right) \cup \varphi\left(W, H, \langle q_{\ell+1} \rangle\right) = \varphi\left(W, \langle q + q_{\ell+1} \rangle\right) \;\; (*)$$

Since both φ and φ^{SA} are static minority fair and respect static minority rights, our claim above implies that $\varphi\left(W, \langle q \rangle\right) = \varphi^{SA}\left(W, \langle q \rangle\right)$ and $\varphi\left(W, \langle q + q_{\ell+1} \rangle\right) = \varphi^{SA}\left(W, \langle q + q_{\ell+1} \rangle\right)$. Since workers cannot be hired more than once, $\varphi\left(W, \langle q \rangle\right) \cap \varphi\left(W, H, \langle q_{\ell+1} \rangle\right) = \emptyset$.

Therefore, there is a unique value of $\varphi\left(W, H, \langle q_{\ell+1} \rangle\right)$ that satisfies the equality $(*)$ above, implying that $\varphi\left(W, H, \langle q_{\ell+1} \rangle\right) = \varphi^{SA}\left(W, H, \langle q_{\ell+1} \rangle\right)$, and therefore that:

$$\varphi\left(W, \langle q_1, q_2, \ldots, q_\ell, q_{\ell+1} \rangle\right) = \varphi^{SA}\left(W, \langle q_1, q_2, \ldots, q_\ell, q_{\ell+1} \rangle\right)$$

finishing our proof. □

Proof of Proposition 2. Let $W^* = \{w_1, w_2, w_3, w_4, w_5\}$ with scores $s_W = (50, 40, 30, 20, 10)$. For simplicity, we will use $m = 0.5$.

Consider first the case $M^* = \{w_3, w_4\}$. If $q = 2$, $\varphi^{F_1}(\{W^*, M^*\}, q) = \{w_1, w_2\}$, which fails to satisfy minority rights.

Consider now the case $M^* = \{w_4, w_5\}$. Consider two possibilities: $q_1 = q_2 = 2$ and $q = 4$. Then $\varphi^{F_1}\left(\{W^*, M^*\}, \langle q_1, q_2 \rangle\right) = \{w_1, w_2, w_4, w_5\}$ but $\varphi^{F_1}\left(\{W^*, M^*\}, q\right) = \{w_1, w_2, w_3, w_4\}$, a violation of *aggregation independence.*

It is easy to see that the rule that results from policy 2, under the given assumptions, is equivalent to the sequential adjusted minority reserves rule. Therefore, Policy 2 of the French assignment rule respects minority rights, is aggregation independent, and is minority fair. □

Proof of Proposition 3. We will show that the Brazilian rule respects minority rights and is aggregation independent, but fails to be minority fair.

By assumption, no more than k workers may be hired in total. Therefore, for any q workers to be hired in any given round there should be at least $\lceil q \times m \rceil$ minority workers in TM and $q - \lceil q \times m \rceil$ workers in O. As a result, the Brazilian rule acts as two parallel sequential priority rules: one in TM and one in O. Therefore, the combination of both is evidently aggregation independent. Next, notice that again because of the assumption on the value of k, $|M| \geq m \times \sum_{i=1}^{t} q_i$. Moreover, since for any $q \in \{q_1, \ldots, q_t\}$ there are at least $\lceil q \times m \rceil$ minority workers in TM, $\omega\left(\varphi\left(W, \langle q_1, \ldots, q_t \rangle\right)\right) \geq m \times \sum q_i$ and by assumption on k, $\left|\varphi\left(W, \langle q_1, \ldots, q_t \rangle\right)\right| = \sum q_i$ therefore $\omega\left(\varphi\left(W, \langle q_1, \ldots, q_t \rangle\right)\right) / \left|\varphi\left(W, \langle q_1, \ldots, q_t \rangle\right)\right| \geq m$, implying that the Brazilian rule respects minority rights. Finally, example 4 shows that the rule is not minority fair. □

Proof of Proposition 4. Example 5 shows that the NSW rule is neither fair nor minority fair. Moreover, since in our results we assume that the number of men and women are always large enough, the NSW consists of two parallel sequential priority hirings (one for males, the other for female workers), and therefore satisfies aggregation independence. Finally, it respects minority rights, since the number of male and female workers hired is always the same. □

Proof of Theorem 3. The single priority rule satisfying common top, aggregation independence and permutation independence is straightforward to see.

Denote by *sequence of single hirings* a plural sequence of hires of the form $\Lambda = \langle (i^1, 1), (i^2, 1), (i^3, 1), \ldots \rangle$. That is, every hire made by any institution in any round consists of only one worker.

Claim 1. *Let W be a set of workers, and Φ be a rule that satisfies common top and permutation independence. There exists a ranking $\succ^*$ over W such that for any sequence of single hirings Λ, $\Phi(W, \Lambda) = \Phi^{\succ^*}(W, \Lambda)$, where $\Phi^{\succ^*}$ is the single priority rule that uses $\succ^*$.*

Proof. We will prove by induction on the number of hires in a plural sequence of hires. That is, we will show that there exists a ranking $\succ^*$, independent of the sequence of hires, that is followed by Φ as a single priority.

In the remaining steps of the proof, the set W and the rule Φ are given, and so for any plural sequence of hires Λ, we will use the notation $\{\Lambda\}$ to represent the set $\bigcup_{i \in I} \Phi_i (W, \Lambda)$. That is, $\{\Lambda\}$ is the set of workers in W hired by some institution under Φ after the sequence of hires Λ. Since we will only look at single hirings, we will represent plural sequences of hires as sequences of institutions and use $\langle i_1, i_2, \ldots \rangle$ to represent $\langle (i_1, 1), (i_2, 1), \ldots \rangle$.

We will use **(PI)** to indicate that we used the property of *permutation independence* of Φ, **(AI)** to indicate that we used *aggregation independence*, and **(CT)** for *common top*.

Moreover, we will use **(P*)** to indicate that we are using the following fact:

> If Λ_1, Λ_2, and $i \in I$ are such that $\{\Lambda_1\} = \{\Lambda_2\}$ and $\Phi_i(W, \Lambda_1) = \Phi_i(W, \Lambda_2)$, then $\Phi_i(W, \langle \Lambda_1, i\rangle) = \Phi_i(W, \langle \Lambda_2, i\rangle)$. That is, if Λ_1 and Λ_2 are such that institution i hires the same set of workers, and the set of workers remaining after all of the hires in both plural sequences of hires is the same, then i would hire the same worker after both Λ_1 and Λ_2. This comes directly from the definition of a hiring rule Φ_i.

Induction Base The induction base is the case where the smallest number of hires is made while still having at least two institutions hiring. Therefore $|\Lambda| = 2$. Suppose that the claim is not true. That is, there might be plural sequences of hires with two hires that cannot be explained by a ranking $\succ^*$ over W. That implies that there are $\Lambda_1 \neq \Lambda_2$, where $\Lambda_1 = \langle i_1, i_2\rangle$, $\Lambda_2 = \langle i_3, i_4\rangle$, and $\{\Lambda_1\} \neq \{\Lambda_2\}$.

Since the sequences of hires involve at least two institutions, $i_1 \neq i_2$ and $i_3 \neq i_4$. Since $\Lambda_1 \neq \Lambda_2$, there are two cases to consider: (i) $i_1 \neq i_3$, and (ii) $i_2 \neq i_4$. Consider (i). By **(PI)**, $\{\langle i_1, i_2\rangle\} = \{\langle i_2, i_1\rangle\}$. By **(P*)**, **(CT)** and the fact that $i_1 \neq i_3$, $\{\langle i_2, i_1\rangle\} = \{\langle i_2, i_3\rangle\}$. By **(PI)**, $\{\langle i_2, i_3\rangle\} = \{\langle i_3, i_2\rangle\}$. By **(P*)**, **(CT)** and the fact that $i_3 \neq i_4$, $\{\langle i_3, i_2\rangle\} = \{\langle i_3, i_4\rangle\}$. But then $\{\langle i_1, i_2\rangle\} = \{\langle i_3, i_4\rangle\}$, a contradiction. For case (ii), **(PI)** implies that $\{\langle i_1, i_2\rangle\} = \{\langle i_2, i_1\rangle\}$ and $\{\langle i_3, i_4\rangle\} = \{\langle i_4, i_3\rangle\}$, which makes this case equivalent to (i).

Induction Step

We now assume that for every sequence of single hirings Λ such that $|\Lambda| \leq k$, the rule Φ hires according to the ranking $\succ^*$. We will use **(IA)** to indicate that we are using this *induction assumption*.

Suppose now that the claim is not true. That is, there are sequences of single hirings Λ_1, Λ_2, such that $|\Lambda_1| = |\Lambda_2| = k$, and institutions $i_1, i_2 \in I$, for which $\{\langle \Lambda_1, i_1\rangle\} \neq \{\langle \Lambda_2, i_2\rangle\}$. There are two cases to consider.

Case (i): $i_1 \neq i_2$**.**

Let Λ_1^a be a sequence of single hires, such that:

- $|\Lambda_1^a| = k$
- The rounds in which i_1 hires in Λ_1^a are exactly the same in which i_1 hires in Λ_1, if any.
- For the rounds in which i_1 does not hire:
  - Let i_3 be the institution hiring at the first round in which i_1 doesn't hire (note that this must exist, since Λ_1 has hirings from at least two institutions).
  - Let i_2 be the institution hiring at every other round of Λ_1^a, if any.

In Λ_1^a, therefore, hires made by i_1, if any, are the same as in Λ_1, there is exactly one hire by i_2, and the remaining hires, if any, are made by i_3.

By **(IA)** and **(P*)**, $\{\langle \Lambda_1, i_1 \rangle\} = \{\langle \Lambda_1^a, i_1 \rangle\}$. Next, let Λ_1^b be exactly as Λ_1^a, except that the single place where i_3 is is replaced by i_1. By **(PI)**, $\{\langle \Lambda_1^a, i_1 \rangle\} = \left\{\langle \Lambda_1^b, i_3 \rangle\right\}$. Notice that Λ_1^b contains all the hires made by i_1 in Λ_1, in addition to one extra hire from i_1. All other hires in Λ_1^b are made by i_2. That is, there is no hire from i_3 in Λ_1^b.

Next, let Λ_1^c a sequence of hires where the hires in Λ_1^c are exactly the same as Λ_2, except that:

- Every round in which i_3 hires in Λ_2, the hire is made by i_1 instead,
- Denote by t^* the first round in which i_3 does not hire in Λ_2. Note that this must exist, since Λ_2 has hirings from at least two institutions. Let i_2 hire in round t^* in Λ_1^c instead.

Notice, therefore, that there is no hire from i_3 in Λ_1^c. By **(IA)** and **(P*)**, therefore, $\left\{\langle \Lambda_1^b, i_3 \rangle\right\} = \{\langle \Lambda_1^c, i_3 \rangle\}$.

Next, let Λ_1^d be exactly as Λ_1^c, except that the i_2 in round t^* is replaced by i_3. By **(PI)**, $\{\langle \Lambda_1^c, i_3 \rangle\} = \left\{\langle \Lambda_1^d, i_2 \rangle\right\}$. Notice that the rounds in which i_2 hires in Λ_1^d are exactly the same as in Λ_2, and as a result, **(P*)** and **(IA)** imply that the last hire made by i_2 in $\langle \Lambda_1^d, i_2 \rangle$ is the same as in $\langle \Lambda_2, i_2 \rangle$. Not only that, **(IA)** implies that the set of workers hired in the first k hires are the same, and therefore $\left\{\langle \Lambda_1^d, i_2 \rangle\right\} = \{\langle \Lambda_2, i_2 \rangle\}$, implying that $\{\langle \Lambda_1, i_1 \rangle\} = \{\langle \Lambda_2, i_2 \rangle\}$, a contradiction.

Case (ii): $i_1 = i_2$.

We will use three institutions in the following steps: i, i_a, i_b, where $i = i_1 = i_2$, and $i \neq i_a \neq i_b$. Let Λ_1^a be a sequence of single hires, such that $|\Lambda_1^a| = k$, the rounds in which i hires in Λ_1 are exactly the same in which i hires in Λ_1^a, if any. Moreover, let i_b be the institution hiring at the first round in which i doesn't hire in Λ_1 (note that this must exist, since Λ_1 has hirings from at least two institutions), and i_a be the institution hiring in every other rounds, if any. By **(IA)** and **(P*)**, $\{\langle \Lambda_1, i \rangle\} = \{\langle \Lambda_1^a, i \rangle\}$.

Next, let Λ_1^b be exactly as Λ_1^a, replacing the single place where i_b is by i. By **(PI)**, $\{\langle \Lambda_1^a, i \rangle\} = \left\{\langle \Lambda_1^b, i_b \rangle\right\}$.

Next, let Λ_1^c be exactly as Λ_2, except that every hire made by i_b, if any, is made instead by i_2. Moreover, let i be the institution hiring at the first round in which i_b doesn't hire in Λ_2 (note that this must exist, since Λ_2 has hirings from at least two institutions). Denote this round by t^*. Notice, therefore, that there is no hire from i_b in Λ_1^c. By **(IA)** and **(P*)**, therefore, $\left\{\langle \Lambda_1^b, i_b \rangle\right\} = \{\langle \Lambda_1^c, i_b \rangle\}$.

Next, let Λ_1^d be exactly as Λ_1^c, except that the i in round t^* is replaced by i_b. By **(PI)**, $\{\langle \Lambda_1^c, i_b \rangle\} = \left\{\langle \Lambda_1^d, i \rangle\right\}$.

Notice that the rounds in which i hires in Λ_1^d are exactly the same as in Λ_2, and as a result, **(P*)** and **(IA)** imply that the last hire made by i in $\langle \Lambda_1^d, i \rangle$ is the same as in $\langle \Lambda_2, i \rangle$. Not only that, **(IA)** implies that the set of workers hired in the first k hires are the same, and therefore $\left\{\langle\Lambda_1^d, i\rangle\right\} = \{\langle\Lambda_2, i\rangle\}$, implying that $\{\langle\Lambda_1, i\rangle\} = \{\langle\Lambda_2, i\rangle\}$, a contradiction. $\square$

Finally, let $\Lambda = \langle(i_1, q_1), (i_2, q_2), \ldots, (i_k, q_k)\rangle$ be any plural sequence of hires and Φ^* be a rule that satisfies common top, permutation independence, and aggregation independence. By **(AI)**:

$$\Phi^*(W, \Lambda) = \Phi^*\left(W, \langle\underbrace{(i_1, 1), \ldots, (i_1, 1)}_{q_1 \text{ times}}, \ldots, \underbrace{(i_k, 1), \ldots, (i_k, 1)}_{q_k \text{ times}}\rangle\right)$$

That is, aggregation independence implies that each hire from an institution can be split into single hires without changing the workers that are chosen, round by round.[30] Our claim above implies, therefore, that the rule Φ^* must be single priority, finishing our proof.

$\square$

[30]Notice that the property of aggregation independence holds for any initial matching μ.