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SYMMETRIC SHIFT-INVARIANT SUBSPACES AND HARMONIC MAPS

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Abstract. The Grassmannian model represents harmonic maps from Riemann surfaces by families of shift-invariant subspaces of a Hilbert space. We impose a natural symmetry condition on the shift-invariant subspaces that corresponds to considering an important class of harmonic maps into symmetric and k-symmetric spaces. Using an appropriate description of such symmetric shift-invariant subspaces we obtain new results for the corresponding extended solutions, including how to obtain primitive harmonic maps from certain harmonic maps into the unitary group.

1. Summary of results

We characterize shift-invariant subspaces which are *k-symmetric* in terms of certain filtrations (Proposition 3.1 and Proposition 3.2). In Theorem 4.2, we give a general form for the corresponding *extended solutions*. In Theorem 5.1 we see how k-symmetric extended solutions correspond to *primitive harmonic maps into a k-symmetric space*. The combination of these results shows how to obtain primitive harmonic maps from certain harmonic maps into the unitary group, thus reversing a well-known [10, Ch. 21, Sec. IV] construction (see Remark 6.3). Finally, in Theorem 7.1, we see how our correspondences are given in terms of holomorphic potentials.

2. Introduction and Preliminaries

Recall that a smooth map φ between two Riemannian manifolds (M, g) and (N, h) is said to be *harmonic* if it is a critical point of the energy functional

$$E(\varphi, D) = \frac{1}{2}\int_D |d\varphi|^2 \omega_g$$

for any relatively compact D in M, where ω_g is the volume measure, and $|d\varphi|^2$ is the Hilbert–Schmidt norm of the differential of φ; this functional being the natural generalization of the classical Dirichlet integral.

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In this paper we continue our study [1] of harmonic maps from a Riemann surface M into the group $\mathrm{U}(n)$ of unitary matrices of order n and their relation with shift-invariant subspaces of Hilbert space. For background, largely aimed at the functional analysis community, see [1]; see also [9, 20] for the general theory and [18, 21] for some background relevant to this paper.

Recall that K. Uhlenbeck introduced [19] the notion of an *extended solution*, which is a smooth map $\Phi : S^1 \times M \to \mathrm{U}(n)$ satisfying $\Phi(1, \cdot) = I$ and such that, for every local (complex) coordinate z on M, there are $\mathfrak{gl}(n, \mathbb{C})$-valued maps A_z and $A_{\bar{z}}$ for which

$$\Phi(\lambda, \cdot)^{-1} d\Phi(\lambda, \cdot) = (1 - \lambda^{-1}) A_z dz + (1 - \lambda) A_{\bar{z}} d\bar{z}. \tag{2.1}$$

We can consider Φ as a map from M into the *loop group* of $\mathrm{U}(n)$ defined by $\Omega\,\mathrm{U}(n) = \{\gamma : S^1 \to \mathrm{U}(n) \text{ smooth} : \gamma(1) = I\}$. If Φ is an extended solution, then $\varphi = \Phi(-1, \cdot)$ is a harmonic map with the matrix-valued 1-form $\frac{1}{2}\varphi^{-1} d\varphi := A^\varphi_z dz + A^\varphi_{\bar{z}} d\bar{z}$ given by $A^\varphi_z = A_z$ and $A^\varphi_{\bar{z}} = A_{\bar{z}}$. Conversely, for a given harmonic map $\varphi : M \to \mathrm{U}(n)$, an extended solution with the property that

$$\Phi^{-1}(\lambda, \cdot) d\Phi(\lambda, \cdot) = (1 - \lambda^{-1}) A^\varphi_z dz + (1 - \lambda) A^\varphi_{\bar{z}} d\bar{z}$$

is said to be *associated* to φ, and we have

$$\Phi(-1, \cdot) = u\varphi$$

for some constant $u \in \mathrm{U}(n)$. If M is simply connected, the existence of extended solutions is equivalent to harmonicity, see [19]; the solution is unique up to multiplication from the left by a *constant loop*, i.e., a $\mathrm{U}(n)$-valued function on S^1, independent of $z \in M$. Moreover (see [19, Thm 2.2] and [1, §3.1]) the extended solution can be chosen to be a smooth map, or even holomorphic in $\lambda \in \mathbb{C} \setminus \{0\}$ and real analytic in M.

We again use the *Grassmannian model* [17], which associates to an extended solution Φ the family of closed subspaces $W(z)$, $z \in M$, of the Hilbert space $L^2(S^1, \mathbb{C}^n)$, defined by

$$W(z) = \Phi(\cdot, z)\mathcal{H}_+, \tag{2.2}$$

where $\mathcal{H}_+$ is the usual Hardy space of $\mathbb{C}^n$-valued functions, i.e., the closed subspace of $L^2(S^1, \mathbb{C}^n)$ consisting of Fourier series whose negative coefficients vanish. Note that the subspaces $W(z)$ form the fibres of a smooth bundle W over the Riemann surface (which is, in fact, a *subbundle* of the *trivial bundle* $\underline{\mathcal{H}} := M \times L^2(S^1, \mathbb{C}^n)$ see, for example, [1, §3.1]).

We denote by S the forward shift on $L^2(S^1, \mathbb{C}^n)$:

$$(Sf)(\lambda) = \lambda f(\lambda), \qquad \lambda \in S^1,$$

and by ∂_z and $\partial_{\bar{z}}$ differentiation with respect to z and $\bar{z}$ respectively, where z is a local coordinate on M; note that all equations below will be independent of the choice of local coordinate. If $f : S^1 \times M \to \mathbb{C}^n$ is differentiable in the second variable and satisfies $f(\cdot,z)\in W(z),\ z\in M$, it follows from (2.1) that

$$S\partial_z f(\cdot,z)\in W(z),\quad \partial_{\bar z} f(\cdot,z)\in W(z), \tag{2.3}$$

i.e., in terms of differentiable sections we have

$$S\partial_z W(z)\subseteq W(z),\quad \partial_{\bar z} W(z)\subseteq W(z), \tag{2.4}$$

which we shall often abbreviate to $S\partial_z W\subseteq W$ and $\partial_{\bar z}W\subseteq W$; in fact, these equations are equivalent to (2.1) see [17, 10].

The Iwasawa decomposition of loop groups [16, Theorem (8.1.1)] implies that $W(z)=\Phi(\cdot,z)\mathcal{H}_+$, with $\Phi: S^1\times M\to \mathrm{U}(n)$ smooth; given such a Φ, (2.3) implies that $\Phi\,\Phi^{-1}(1,\cdot)$ is an extended solution.

We continue to explore the connection between harmonic maps which possess extended solutions, and the associated infinite-dimensional family (i.e., bundle) $W=W(z)$ of shift-invariant subspaces (2.2). By extension we shall call the family $W(z)$ an *extended solution* as well.

In our previous paper [1] we studied a new criterion for finiteness of the uniton number; in the present paper we turn our attention to *symmetry.* Specifically, we impose the following symmetry condition on W:

$$\text{if } f\in W \text{ then } f_\omega\in W, \text{ where we set } f_\omega(\lambda)=f(\omega\lambda) \text{ for } \lambda\in S^1; \tag{2.5}$$

here $\omega=\omega_k$ is the primitive kth root of unity for some $k\in\{2,3,\dots\}$. A shift-invariant subspace W is said to be *k-symmetric* if it satisfies condition (2.5) for $\omega=\omega_k$; W is said to be *S^1-invariant* if it satisfies (2.5) for any $\omega\in S^1$.

The k-symmetric extended solutions correspond to an important class of harmonic maps into symmetric spaces and a generalization of those, the *primitive harmonic maps* into *k-symmetric spaces* [4, 10]. In §3, we establish a one-to-one correspondence between k-symmetric shift-invariant subspaces and filtrations $V_0\subseteq V_1\subseteq\ldots\subseteq V_{k-1}$ of shift-invariant subspaces satisfying $SV_{k-1}\subseteq V_0$. Moreover, we prove (see Proposition 4) that this correspondence induces a one-to-one correspondence between k-symmetric extended solutions W and *λ-cyclic superhorizontal sequences* of length k, that is, sequences $V_0\subseteq V_1\subseteq\ldots\subseteq V_{k-1}$ of extended solutions satisfying the superhorizontality condition

$$\partial_z V_j\subseteq V_{j+1}\quad\text{for}\quad j=0,\dots,k-2, \tag{2.6}$$

and the further condition $S\partial_z V_{k-1}\subseteq V_0$. This leads to Theorem 4.2, where we give a new general form for k-symmetric extended solutions. Theorem 4.2 also explains (see Remark 6.3) under what conditions a well-known method [10, Ch. 21, Sec. IV] of obtaining harmonic maps into $\mathrm{U}(n)$ from primitive harmonic maps can be reversed in order to obtain primitive harmonic maps from certain harmonic maps into $\mathrm{U}(n)$. Finally, in §7 we describe this construction in terms of holomorphic potentials (Theorem 7.1), and some examples are given.

3. k-symmetric shift-invariant subspaces

In this section, we describe all k-symmetric shift-invariant subspaces which are relevant for this work, for any $k \in \{2, 3, \dots\}$. The description will follow from the general form for shift-invariant subspaces [11] and some algebraic manipulations.

As before, $\mathcal{H}_+$ stands for the usual Hardy space of $\mathbb{C}^n$-valued functions, and S for the shift. As we did before, we sometimes write, by abuse of notation, λf instead of Sf, $f \in L^2(S^1, \mathbb{C}^n)$. Recall from §2 that a *k-symmetric* shift-invariant subspace W is one which is invariant with respect to the unitary map $\hat{\omega} : L^2(S^1, \mathbb{C}^n) \to L^2(S^1, \mathbb{C}^n)$, induced by the primitive kth root of unity ω, and defined by $\hat{\omega}(f)(\lambda) = f_\omega(\lambda) = f(\omega\lambda)$. The following result gives the spectral theorem for the restriction $\hat{\omega}|W$.

Proposition 3.1. *Let W be a k-symmetric shift-invariant subspace.*

(i) *For $0 \le j \le k-1$, the subspace*

$$W_j = \{f \in W : \; f_\omega = \omega^j f\} = \{g \in W : \; g(\lambda) = \sum_{l=0}^{k-1} \omega^{-lj} f(\omega^l \lambda), \; f \in W\}$$

is closed and

$$W = \bigoplus_{j=0}^{k-1} W_j. \tag{3.7}$$

(ii) *For $0 \le j \le k-1$, there exist closed shift-invariant subspaces V_j of $L^2(S^1, \mathbb{C}^n)$ such that $SV_{k-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1}$, and*

$$W_j = S^j \{g \in W : \; g(\lambda) = f(\lambda^k), \; f \in V_j\}. \tag{3.8}$$

(iii) *If $W = \Phi\mathcal{H}_+$ with Φ measurable and $\mathrm{U}(n)$-valued a.e. on S^1, then $V_{k-1} = \Psi\mathcal{H}_+$ with Ψ measurable and $\mathrm{U}(n)$-valued a.e. on S^1. Moreover, there exist subspaces $\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_{k-2} \subseteq \mathbb{C}^n$ with orthogonal projections π_{α_j}, $0 \le j \le k-2$, such that*

$$V_j = \Psi(\pi_{\alpha_j} + \lambda \pi_{\alpha_j}^\perp)\mathcal{H}_+ = \Psi(\alpha_j + \lambda\mathcal{H}_+),$$

and

$$W = \Psi(\lambda^k, \cdot)(\alpha_0 + \lambda\alpha_1 + \cdots \lambda^{k-2}\alpha_{k-2} + \lambda^{k-1}\mathcal{H}_+).$$

Proof. Part (i) is straightforward, as well as the representation of W_j in (ii). The rest of (ii) follows directly from the shift-invariance of W. To see (iii), note that the representation $V_{k-1} = \Psi\mathcal{H}_+$, with Ψ unitary-valued a.e., follows (see [11, Lecture VI]), once we show that V_{k-1} is not invariant for the inverse of the shift and

$$\overline{\bigvee_{n \ge 0} S^{-n} V_{k-1}} = L^2(S^1, \mathbb{C}^n). \tag{3.9}$$

If V_{k-1} is invariant for the inverse of the shift, then $SV_{k-1} = V_{k-1}$; hence by (ii), $V_{k-1} = V_0 = V_j$, $0 < j < k-1$, and thus $W_j = S^j W_0$, and we arrive easily at the contradiction $S^{-1}W \subseteq W$. Moreover, if (3.9) fails, there exists a $g \in L^2(S^1, \mathbb{C}^n) \setminus \{0\}$ with inner product

$$\langle h(\lambda), g(\lambda)\rangle = 0,$$

a.e., for all $h \in V_{k-1}$. This leads to

$$\langle f(\lambda), g(\lambda^k)\rangle = 0,$$

a.e., for all $f \in W$ and contradicts the hypothesis $W = \Phi\mathcal{H}_+$. Thus $V_{k-1} = \Psi\mathcal{H}_+$ with Ψ $\mathrm{U}(n)$-valued a.e., and from the inclusions $\lambda V_{k-1} \subseteq V_j \subseteq V_{k-1}$ we obtain that $\Psi^{-1}V_j$ consists of functions whose first Fourier coefficient lies in a given subspace α_j of $\mathbb{C}^n$. These subspaces α_j are nested since the subspaces V_j are. Then

$$\Psi^{-1}V_j = \alpha_j + \lambda\mathcal{H}_+,$$

and the remaining assertions follow. □

Proposition 3.2. *With the notations of Proposition 3.1, the correspondence between k-symmetric shift-invariant subspaces W and filtrations $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1}$ satisfying $SV_{k-1} \subseteq V_0$ is one-to-one.*

Proof. If W and W' are two k-symmetric shift-invariant subspaces with the same filtration $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1}$, then by (3.7) and (3.8), we must have $W = W'$.

Conversely, if $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1}$ is a filtration satisfying $SV_{k-1} \subseteq V_0$, consider the subspace W defined by (3.7) and (3.8). Clearly, W is shift-invariant and k-symmetric. Moreover, the eigenspace decomposition of W induces the given filtration. □

As pointed out in [1, §3.1], the unitary-valued function Ψ in Proposition 3.1 is unique up to multiplication from the right by a constant unitary matrix (see [13]), which affects the subspaces α_j as well. However, if $W = \Phi\mathcal{H}_+$, there is a natural choice of Ψ which relates it to the function Φ, as follows.

Proposition 3.3. *Let W be a k-symmetric shift-invariant subspace such that $W = \Phi\mathcal{H}_+$ with Φ measurable and $\mathrm{U}(n)$-valued a.e. on S^1. Then there exists a constant $\varphi_k \in \mathrm{U}(n)$ with $\varphi_k^k = I$ such that*

$$\Phi(\omega\lambda) = \Phi(\lambda)\varphi_k. \tag{3.10}$$

If $\beta_j = \ker(\varphi_k - \omega^j I)$, and π_j denotes the orthogonal projection from $\mathbb{C}^n$ onto β_j, then

$$\Phi_k(\lambda) = \Phi(\lambda)\sum_{j=0}^{k-1}\pi_j\lambda^{-j} \tag{3.11}$$

is a function of λ^k and Proposition 3.1(iii) holds with $\Psi(\lambda) = \Phi_k(\lambda^{1/k})$ and

$$\alpha_j = \bigoplus_{l=0}^{j}\beta_l. \tag{3.12}$$

In particular, if $W = \Phi(\cdot, z)\mathcal{H}_+$*, where* $\Phi : S^1 \times M \to \mathrm{U}(n)$ *is smooth,* k*-symmetric and has* $\Phi(1,\cdot) = I$*, then* Ψ *is a smooth map on* $S^1 \times M$ *with* $\Psi(1,\cdot) = I$*, and* α_j*,* $0 \le j < k-1$*, are smooth subbundles of the trivial bundle* $\underline{\mathbb{C}}^n := M \times \mathbb{C}^n$ *on* M*.*

Proof. The equality (3.10), with φ_k constant, follows as above from [13] and $\Phi(\lambda)\mathcal{H}_+ = \Phi(\omega\lambda)\mathcal{H}_+$. A repeated application of it gives $\varphi_k^k = I$. Since $\varphi_k \pi_j = \omega^j \pi_j$, Φ_k defined by (3.11) is clearly a function of λ^k.

From the identity (3.10) it follows that the subspaces $W_j,\ 0 \le j \le k-1$, introduced in Proposition 3.1(i) can be written as

$$W_j = \{f \in W :\ f_\omega = \omega^j f\} = \Phi\{g \in \mathcal{H}_+ :\ \varphi_k g_\omega = \omega^j g\}.$$

A function $g \in \mathcal{H}_+$ with Fourier coefficients $g_m,\ m \ge 0$, satisfies $\varphi_k g_\omega = \omega^j g$ if and only if, for $m = sk + l,\ 0 \le l \le k-1$, we have

$$\varphi_k g_m = \omega^{j-l} g_m,$$

or equivalently, $g_m \in \beta_{j-l}$ when $j \ge l$ and $g_m \in \beta_{k+j-l}$ when $l > j$. For $m = sk + l,\ 0 \le l \le k-1$, set

$$h_s = \sum_{l=0}^{k-1} g_{ks+l}$$

and note that, since the β_l are pairwise orthogonal, we have

$$g(\lambda) = \left(\sum_{l \le j} \pi_{j-l}\lambda^l + \sum_{l > j} \pi_{k+j-l}\lambda^l \right) \sum_{s \ge 0} h_s \lambda^{ks}.$$

The argument is clearly reversible and we obtain

$$\big\{g \in \mathcal{H}_+ :\ \varphi_k g_\omega = \omega^j g\big\} = \left(\sum_{l \le j} \pi_{j-l}\lambda^l + \sum_{l > j} \pi_{k+j-l}\lambda^l \right) \big\{h(\lambda^k) :\ h \in \mathcal{H}_+\big\}.$$

Consequently,

$$W_j = \lambda^j \Phi \left(\sum_{l \le j} \pi_{j-l}\lambda^{l-j} + \sum_{l > j} \pi_{k+j-l}\lambda^{l-j} \right) \big\{h(\lambda^k) :\ h \in \mathcal{H}_+\big\}.$$

In particular,

$$W_{k-1} = \lambda^{k-1}\Phi_k\{h(\lambda^k) :\ h \in \mathcal{H}_+\}.$$

Set $\Psi(\lambda) = \Phi_k(\lambda^{1/k})$. Using again the pairwise orthogonality of the $\beta_l,\ 0 \le l \le k-1$, we see that $\Phi_k(\lambda^{1/k})$ is $\mathrm{U}(n)$-valued a.e. and

$$\Psi(\lambda^k)^{-1}\Phi \left(\sum_{l \le j} \pi_{j-l}\lambda^{l-j} + \sum_{l > j} \pi_{k+j-l}\lambda^{l-j} \right) = \sum_{l \le j} \pi_{j-l} + \sum_{l > j} \pi_{k+j-l}\lambda^k.$$

On the other hand, in view of Proposition 3.1, we have

$$\lambda^{-j}\Psi(\lambda^k)^{-1} W_j = \alpha_j + \lambda^k \mathcal{H}_+,$$

and equation (3.12) follows.

Finally, if Φ is smooth on $S^1 \times M$ then φ_k is smooth on M, hence each $\pi_j,\ 0 \leq j \leq k-1$, is smooth on M since it is a polynomial in φ_k:

$$\prod_{\substack{i=0\\ i\neq j}}^{k-1} (\varphi_k - \omega^i I) = \prod_{\substack{i=0\\ i\neq j}}^{k-1} (\omega^j - \omega^i)\pi_j.$$

The result follows. □

4. k-SYMMETRIC EXTENDED SOLUTIONS

We assume throughout that

$$W = \Phi\mathcal{H}_+,$$

with $\Phi : S^1 \times M \to \mathrm{U}(n)$ smooth and $\Phi(1,\cdot) = I$. As we said before, Φ can be considered as a map from M into the loop group $\Omega\,\mathrm{U}(n)$.

We are interested in the case when W is an extended solution corresponding to a harmonic map defined on a Riemann surface M. We use the same notations as in Proposition 3.1.

Proposition 4.1. *Let W be k-symmetric. The following are equivalent*:

(i) *W is an extended solution*;

(ii) *$V_0 \subseteq V_1 \subseteq \ldots \subseteq V_{k-1}$ is a* λ-cyclic superhorizontal sequence, *that is, $V_j,\ 0 \leq j \leq k-1$, are extended solutions, $\partial_z V_j \subseteq V_{j+1},\ 0 \leq j < k-1$, and $\lambda\partial_z V_{k-1} \subseteq V_0$.*

Proof. W is an extended solution if and only if each $W_j,\ 0 \leq j \leq k-1$, satisfies $\partial_{\bar z} W_j \subseteq W,\ \lambda\partial_z W_j \subseteq W$. But by the definition of W_j this is equivalent to $\partial_{\bar z} W_j \subseteq W_j$, $\lambda\partial_z W_j \subseteq W_{j+1}$ if $0 \leq j < k-1$, and $\lambda\partial_z W_{k-1} \subseteq W_0$. Clearly, this is equivalent to (ii). □

An immediate consequence is that the function Ψ defined in Proposition 3.3 must be an extended solution if Φ is. Moreover, the general form of an extended solution Φ with the property that $\Phi(\omega\lambda, z) = \Phi(\lambda, z)\varphi_k(z)$ (that is, $W = \Phi\mathcal{H}_+$ is k-symmetric) is

$$\Phi(\lambda, z) = \Psi(\lambda^k, z)\sum_{j=0}^{k-1} \pi_j \lambda^j, \tag{4.13}$$

where π_j is the orthogonal projection onto the subbundle β_j, defined pointwise as in Proposition 3.3, and

$$\sum_{j=0}^{k-1} \pi_j \lambda^j \mathcal{H}_+ = \alpha_0 + \lambda\alpha_1 + \ldots + \lambda^{k-2}\alpha_{k-2} + \lambda^{k-1}\mathcal{H}_+$$

is S^1-invariant (see §2), but not necessarily an extended solution. In fact, we can characterize this situation in terms of the function Ψ and the subbundles α_j, as follows; see, for example, [1, §4.3] for more information on the operator $D^{\psi}_{\bar z}$.

Theorem 4.2. *Let $\Psi : S^1 \times M \to \mathrm{U}(n)$ be an extended solution (with $\Psi(1,\cdot) = I$), let $\psi = \Psi(-1,\cdot)$, and*

$$A^{\psi}_z = \frac{1}{2}\psi^{-1}\partial_z\psi.$$

If $\alpha_0 \subseteq \ldots \subseteq \alpha_{k-2}$ are smooth subbundles of the trivial bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$, then

$$W = \Psi(\lambda^k,\cdot)(\alpha_0 + \lambda\alpha_1 + \ldots + \lambda^{k-2}\alpha_{k-2} + \lambda^{k-1}\mathcal{H}_+) \tag{4.14}$$

is an extended solution if and only if the following conditions hold:

(i) *for $0 \le j < k-2$ we have $\partial_z\alpha_j \subseteq \alpha_{j+1}$;*
(ii) *$\alpha_{k-2} \subseteq \ker A^{\psi}_z$ and $\mathrm{Im}\, A^{\psi}_z \subseteq \alpha_0$;*
(iii) *for $0 \le j \le k-2$, α_j is closed under $D^{\psi}_{\bar z} := \partial_{\bar z} + A^{\psi}_{\bar z}$.*

Proof. Note that $V_j = \Psi(\alpha_j + \lambda\mathcal{H}_+)$, $0 \le j \le k-2$, and $V_{k-1} = \Psi\mathcal{H}_+$. The condition $\mathrm{Im}\, A^{\psi}_z \subseteq \alpha_0$ is equivalent to $\lambda\partial_z V_{k-1} \subseteq V_0$ and, if it holds, then $\partial_z V_j \subseteq V_{j+1}$, $0 \le j \le k-2$, become equivalent to $\alpha_j \subseteq \ker A^{\psi}_z$, $\partial_z\alpha_j \subseteq \alpha_{j+1}$. Finally, condition (iii) is equivalent to $\partial_{\bar z}V_j \subseteq V_j$, $0 \le j \le k-2$. Indeed, a direct calculation shows that, for $0 \le j \le k-2$, we have $\partial_{\bar z}V_j \subseteq V_j$ if and only if, for every section s in α_j, we have $\partial_{\bar z}s + A^{\psi}_{\bar z}s \in \alpha_j$. □

Remark 4.3. (a) By taking the adjoints, we see that the condition (iii) in Theorem 4.2 is equivalent to the following: for $0 \le j \le k-2$, we have $A^{\psi}_z = \partial_z\pi_{\alpha_j}$ on $\alpha_j^{\perp}$. If $k = 2$, condition (i) is empty.

(b) In Theorem 4.2, if $\Psi = I$, then conditions (i)—(iii) are equivalent to (α_i) *is a sequence of holomorphic subbundles which satisfies the superhorizontality condition* (2.6). In that case, the extended solution $W = \Phi\mathcal{H}_+$ given by (4.14) is S^1-invariant.

(c) The harmonic map $\varphi = \Phi(-1,\cdot)$ is given by $\varphi = \varphi_k^{k/2}$ if k is even (if k is odd this is more complicated), where $\varphi_k = \Phi(\omega,\cdot) = \sum_{j=0}^{k-1}\pi_j\omega^j$, as defined pointwise in Proposition 3.3. In §5 we shall see that φ_k corresponds to a *primitive harmonic map* into a certain *flag manifold* and that φ corresponds to a harmonic map into a certain complex Grassmannian. In Theorem 5.1, we shall consider the more general case $\varphi_k^{k/s}$, with s a divisor of k.

(d) Condition (ii) in Theorem 4.2 implies that

$$(A^{\psi}_z)^2 = 0; \tag{4.15}$$

thus its trace also vanishes, which is easily seen to be the condition for *(weak) conformality* (cf. [21]) of ψ.

(e) Conditions (ii) and (iii) imply that each α_j is a *basic and antibasic uniton* with respect to ψ, i.e., $\alpha_j \subseteq \ker A^{\psi}_z$ and $\mathrm{Im}\, A^{\psi}_z \subseteq \alpha_j$ (cf. [18, Example 3.2]).

(f) The extended solution $W = \Phi\mathcal{H}_+$ given by (4.14) is always k-symmetric. If $k > 2$ and Ψ, α_j are as above, we can easily construct l-symmetric extended solutions for $2 \le l < k$. We simply choose $0 \le j_0 < j_1 < \ldots < j_{l-2} \le$

$k-2$ and set

$$W = \Psi(\lambda^l, \cdot)(\alpha_{j_0} + \lambda\alpha_{j_1} + \ldots + \lambda^{l-2}\alpha_{j_{l-2}} + \lambda^{l-1}\mathcal{H}_+). \tag{4.16}$$

In Remark 5.2(b) we shall discuss the corresponding primitive harmonic maps.

If ψ satisfies (4.15), we shall say that ψ is 2-*nilconformal*. A slightly different notion of 'nilorder' is given by F.E. Burstall [3] for maps into Grassmannians. In the next proposition, we give a complete characterization of 2-nilconformal harmonic maps into a Grassmannian. We first recall some definitions for such maps, see [1, §4.3], [6] and the references therein for more details.

We represent smooth maps $\psi : M \to G_m(\mathbb{C}^n)$ from a surface into the Grassmannian of complex m-dimensional subspaces of $\mathbb{C}^n$ as subbundles, denoted by the same letter, of the trivial bundle $\underline{\mathbb{C}}^n = M \times \mathbb{C}^n$. We define the *second fundamental form* A'_ψ by $A'_\psi(s) = \pi_{\psi^\perp}\partial_z s$, $s \in \Gamma(\psi)$; this formula defines a linear bundle map from ψ to $\psi^\perp$. We can embed the Grassmannian $G_m(\mathbb{C}^n)$ into $\mathrm{U}(n)$ via the *Cartan embedding*, cf. [1, §4.3], then a map $\psi : M \to G_m(\mathbb{C}^n)$ is harmonic if and only if its composition $\psi : M \to \mathrm{U}(n)$ with the Cartan embedding is harmonic. Further, the above second fundamental form for a map $\psi : M \to G_m(\mathbb{C}^n)$ is related to A_z^ψ (see §2) by $A'_\psi = -A_z^\psi|\psi$, and $A'_{\psi^\perp} = -A_z^\psi|\psi^\perp$.

By a *harmonic diagram*, we shall mean a *diagram* in the sense of [6] of mutually orthogonal subbundles ψ_i with sum $\underline{\mathbb{C}}^n$ and arrows between them; the arrow from ψ_i to ψ_j represents the ψ_j-component $A'_{\psi_i,\psi_j} := \pi_{\psi_j} \circ A'_{\psi_i}$ of A'_{ψ_i}, the absence of that arrow indicating that A'_{ψ_i,ψ_j} is known to be zero. For a *harmonic* map ψ, we define the *Gauss bundle* $G^{(1)}(\psi) = G'(\psi)$ as the image of A'_ψ completed to a bundle by *filling out zeros* [6]; we iterate this construction to give the *ith Gauss bundle* $G^{(i)}(\psi)$ for $i = 1, 2, \ldots$. Then the *isotropy order* of a harmonic map $\psi : M \to G_m(\mathbb{C}^n)$ into a (complex) Grassmannian is defined to be the greatest value of $t \in \{1, 2, \ldots, \infty\}$ such that ψ is orthogonal to $G^{(i)}(\psi)$ for all i with $1 \leq i \leq t$.

Note that *any* 2-*nilconformal harmonic map* ψ *into a Grassmannian has isotropy order at least* 2; indeed, the image of $(A_z^\psi)^2|\psi$ is $\pi_\psi\big(G^{(2)}(\psi)\big)$.

Proposition 4.4. *Suppose that we have a harmonic diagram of the form*

$$\begin{tikzcd} \psi_0 \arrow[r, "A'_{\psi_0}"'] & \psi_1 \arrow[r, "A'_{\psi_1}"'] & \cdots \arrow[r, "A'_{\psi_{t-2}}"'] & \psi_{t-1} \arrow[r, "A'_{\psi_{t-1}}"'] & \psi_t \arrow[llll, bend right] \end{tikzcd} \tag{4.17}$$

where $t \geq 3$ *(possibly infinite) and, for* $0 \leq i \leq t$*, the bundle* ψ_i *corresponds to a harmonic map* $M \to G_{m_i}(\mathbb{C}^n)$ *into a Grassmannian.*

Then $\psi := \psi_0 \oplus \psi_1 : M \to G_{m_0+m_1}(\mathbb{C}^n)$ *is a* 2*-nilconformal harmonic map of isotropy order at least* $t-1$*. Moreover, all* 2*-nilconformal harmonic maps into a Grassmannian are given this way.*

Proof. If we have a diagram (4.17) with $t \geq 3$, then $\psi := \psi_0 \oplus \psi_1$ has a diagram

$$\begin{tikzcd} \psi = \tilde{\psi}_0 \arrow[r] & \tilde{\psi}_1 \arrow[r] & \cdots \arrow[r] & \tilde{\psi}_{t-2} \arrow[r] & \tilde{\psi}_{t-1} \arrow[llll, bend right] \end{tikzcd} \tag{4.18}$$

with $\tilde{\psi}_i = \psi_{i+1}$ for $1 \leq i \leq t-1$. Since $A'_\psi|\psi_1 = A'_{\psi_1}$ and $A'_\psi|\psi_0 = 0$, A'_ψ is holomorphic if A'_{ψ_1} is (see [6, Proposition 1.2(iii)]), and so the harmonicity of ψ follows directly from [6, Lemma 1.3 (b)]. Moreover, ψ has isotropy order at least $t-1$ and clearly satisfies (4.15).

Conversely, suppose that ψ is 2-nilconformal. Then, as remarked above, it has isotropy order at least 2 and so has a diagram (4.18) with $t \geq 3$. As ψ is 2-nilconformal, $\operatorname{Im} A'_{\tilde{\psi}_{t-1}} \subseteq \ker A'_{\tilde{\psi}_0}$. Write $\tilde{\psi}_0 = \psi_0 \oplus \psi_1$, where $\psi_0 = \ker A'_{\tilde{\psi}_0}$. It follows from [6, Theorem 2.4] that the subbundles ψ_0 and ψ_1 of ψ correspond to harmonic maps into Grassmannians. Clearly, $\operatorname{Im} A'_{\psi_0} \subseteq \psi_1$ and we have a diagram of the form (4.17), with $\psi_{i+1} = \tilde{\psi}_i$ for $i \geq 1$. □

Remark 4.5. (a) For any diagram of the form (4.17) with $t \geq 2$, the maps represented by the subbundles ψ_i are automatically harmonic by [6, Proposition 1.6].

(b) Given any harmonic map ψ_0 of finite isotropy order $t \geq 2$, there is a diagram (4.17) with the $\psi_i = G^{(i)}(\psi)$ for $i = 0, \ldots, t-1$, cf. [1, §4.3]. If ψ_0 has infinite isotropy order, there are diagrams (4.17) with varying values of t and some subbundles or arrows zero.

Example 4.6. Given a harmonic diagram (4.17), and an integer d with $1 \leq d \leq t-2$, we can combine the vertices $\psi_1 + \ldots + \psi_d$ to give a subbundle and a diagram (4.17) with $t - d + 2 \geq 4$ vertices. By [6, Proposition 1.6] $\psi_1 + \ldots + \psi_d$ represents a harmonic map. The construction in Proposition 4.4 then gives a 2-nilconformal harmonic map $\psi = \psi_0 + \ldots + \psi_d$. Then, for any k with $2 \leq k \leq \min(d+1, t-d)$, the subbundles

$$\alpha_j := \sum_{i=0}^{j} \psi_i \oplus \psi_{d+i+1}, \quad i = 0, \ldots, k-2$$

satisfy the conditions of Theorem 4.2 for the harmonic map ψ.

Example 4.7. Suppose that $\psi_0 : \mathbb{C} \to \mathbb{C}P^{n-1}$ is a Clifford solution (see [1, Example 4.14] and references therein). In homogeneous coordinates we have $\psi_0 = [F]$ where $F = (F_0, \ldots, F_{n-1}) : \mathbb{C} \to \mathbb{C}^n$ is given by

$$F_i(z) = (1/\sqrt{n})\, \mathrm{e}^{\omega^i z - \overline{\omega}^i \overline{z}}$$

with $\omega = \mathrm{e}^{2\pi \mathrm{i}/n}$. This is a harmonic map with isotropy order $t = n-1$. Consider the harmonic diagram with vertices $\psi_i = G^{(i)}(\psi)$ for $i = 0, \ldots, n-1$, as in (b) of Remark 4.5.

In view of Example 4.6, if we take $n \geq 4$ and $d = 1$, we must have $k = 2$. We then construct the 2-nilconformal harmonic map $\psi = \psi_0 \oplus G^{(1)}(\psi_0)$. The subbundle $\alpha_0 = \psi_0 \oplus G^{(2)}(\psi_0)$ satisfies the conditions of Theorem 4.2.

For $n \geq 5$ and $d = 2$, we obtain the 2-nilconformal harmonic map $\psi = \psi_0 \oplus G^{(1)}(\psi_0) \oplus G^{(2)}(\psi_0)$. In this case, if $n = 5$, we must have $k = 2$. But if $n > 5$, we can take $k = 2$ or $k = 3$. For $n > 5$ and $k = 3$, the subbundles

$$\alpha_0 = \psi_0 \oplus G^{(3)}(\psi_0), \quad \alpha_1 = \psi_0 \oplus G^{(1)}(\psi_0) \oplus G^{(3)}(\psi_0) \oplus G^{(4)}(\psi_0) \tag{4.19}$$

satisfy the conditions of Theorem 4.2.

Example 4.8. Let $\psi : M \to \mathbb{C}P^{n-1} \hookrightarrow \mathrm{U}(n)$ be a full holomorphic, and so harmonic map. We clearly have $(A_z^{\psi})^2 = 0$. Observe that we can consider a harmonic diagram of the form (4.17) with $\psi_0 = 0$, $\psi_1 = \psi$ and $\psi_i = G^{(i-1)}(\psi)$ for $2 \leq i \leq n$. Now we have no arrow from ψ_n to ψ_0 nor from ψ_0 to ψ_1. Following the procedure of Proposition 4.4, we write $\psi = \psi_0 \oplus \psi_1$. Moreover, the bundles

$$\alpha_j = G^{(1)}(\psi) \oplus G^{(2)}(\psi) \oplus \ldots \oplus G^{(j+1)}(\psi),$$

with $0 \leq j \leq k-2$ satisfy the conditions of Theorem 4.2 for any k with $2 \leq k \leq n$.

Recall from [1, 19] that a harmonic map $\varphi : M \to \mathrm{U}(n)$ has *finite uniton number* if there exists an extended solution Φ associated to φ which is defined on the whole of M and is a trigonometric polynomial in $\lambda \in S^1$. Regarding this issue, we have the following.

Proposition 4.9. *Let Φ be a k-symmetric extended solution, and let Ψ be the extended solution given by Proposition 3.3. Then $\varphi = \Phi(-1, \cdot)$ has finite uniton number if and only if $\psi = \Psi(-1, \cdot)$ has.*

Proof. It follows directly from the equality (4.13) that Φ is polynomial up to left multiplication by a constant loop if and only if Ψ is also polynomial up to left multiplication by a constant loop. □

5. Primitive harmonic maps into k-symmetric spaces

A *(regular) k-symmetric space of a compact semisimple Lie group G* is a homogeneous space G/K such that $(G^\tau)_0 \subseteq K \subseteq G^\tau$ for some automorphism $\tau : G \to G$ of finite order $k \geq 2$; here G^τ denotes the fixed point set of τ and $(G^\tau)_0$ its identity component. For $k = 2$, this is just a *symmetric space* of G. In this section we shall explain how k-symmetric extended solutions correspond to primitive harmonic maps into a k-symmetric space. For further details on primitive harmonic maps, we refer the reader to [4].

Given positive integers $r_0, \ldots, r_{k-1}$ with $r_0 + \ldots + r_{k-1} = n$, let $F_{r_0,\ldots,r_{k-1}}$ be the flag manifold of ordered sets $(A_0, \ldots, A_{k-1})$ of complex vector subspaces of $\mathbb{C}^n$, with $\mathbb{C}^n = \bigoplus_{i=0}^{k-1} A_i$ and $\dim A_i = r_i$. The unitary group $\mathrm{U}(n)$ acts transitively on $F = F_{r_0,\ldots,r_{k-1}}$ with isotropy subgroups conjugate to $\mathrm{U}(r_0) \times \ldots \times \mathrm{U}(r_{k-1})$. Fix a point $x_0 = (A_0, \ldots, A_{k-1}) \in F$. For each

$i \in \{0, \ldots, k-1\}$, let π_{A_i} denote the orthogonal (Hermitian) projection onto A_i. Let $s \in \Omega\,\mathrm{U}(n)$ be defined by

$$s(\lambda) = \sum_{i=0}^{k-1} \lambda^i \pi_{A_i} \tag{5.20}$$

and consider the loop $\sigma(\lambda) = \mathrm{Ad}_{s(\lambda)}$ of inner automorphisms of $\mathfrak{u}(n)$ defined by

$$\sigma(\lambda)(X) = s(\lambda)Xs(\lambda)^{-1}, \quad X \in \mathfrak{u}(n).$$

Set $\omega = e^{2\pi i/k}$ and $\tau = \sigma(\omega)^{-1}$.

The automorphism τ induces an eigenspace decomposition $\mathfrak{gl}(n,\mathbb{C}) = \bigoplus_{i\in\mathbb{Z}_k} \mathfrak{g}^i$, where

$$\mathfrak{g}^i = \bigoplus_{j\in\mathbb{Z}_k} \mathrm{Hom}(A_j, A_{j-i}) \tag{5.21}$$

is the ω^i-eigenspace of τ. Clearly, $\overline{\mathfrak{g}^i} = \mathfrak{g}^{-i}$. The automorphism τ exponentiates to give an order k automorphism of $\mathrm{U}(n)$, also denoted by τ, whose fixed-set subgroup $\mathrm{U}(n)^\tau$ is precisely the isotropy group at x_0. Hence, F has a canonical structure of a k-symmetric space. Moreover, F can be embedded in $\mathrm{U}(n)$ as a connected component of $\sqrt[k]{I}$ via the *(generalized) Cartan embedding* $\iota : F \to \sqrt[k]{I} \subseteq \mathrm{U}(n)$ defined by $\iota(gx_0) = gs(\omega)g^{-1}$ (note that when $k > 2$, this is not totally geodesic).

A smooth map $\varphi : M \to F$ is said to be *primitive* (see [4] for further details) if, given a lift $\psi : M \to \mathrm{U}(n)$ with $\varphi = \psi x_0$ (such lifts always exist locally), the following holds: $\psi^{-1}\psi_z$ takes values in $\mathfrak{g}^0 \oplus \mathfrak{g}^{-1}$. Since such a lift is unique up to right multiplication by some smooth map $K : M \to \mathrm{U}(n)^\tau$, this definition of primitive map does not depend on ψ. If $k \geq 3$, then any primitive map $\varphi : M \to F$ is harmonic with respect to the metric on F induced by the Killing form of $\mathfrak{u}(n)$ (as a matter of fact, φ is harmonic with respect to all invariant metrics on F for which $\mathfrak{g}^{-1}$ is isotropic [2]). For $k = 2$, all smooth maps into F are primitive. By *primitive harmonic map* into F we mean a primitive map if $k \geq 3$ and a harmonic map if $k = 2$.

Let $\varphi : M \to F$ be a primitive harmonic map and $\psi : M \to \mathrm{U}(n)$ a lift. Consider the $\mathfrak{gl}(n,\mathbb{C})$-valued 1-form $\alpha = \psi^{-1}d\psi$ on M and let $\alpha = \alpha' + \alpha''$ be the type decomposition of α into a $(1,0)$-form and a $(0,1)$-form on M. Since φ is primitive, we can write uniquely $\alpha' = \alpha'_{-1} + \alpha'_0$ and $\alpha'' = \alpha''_1 + \alpha''_0$ where α'_0, α'_{-1} are $\mathfrak{g}^0, \mathfrak{g}^{-1}$-valued, respectively, and α''_0, α''_1 are $\mathfrak{g}^0, \mathfrak{g}^1$-valued, respectively. The loop of 1-forms $\alpha_\lambda = \alpha'_{-1}\lambda^{-1} + \alpha_0 + \alpha''_1\lambda$, with $\alpha_0 = \alpha'_0 + \alpha''_0$, takes values in the Lie algebra of the infinite-dimensional Lie group

$$\Lambda_\tau\,\mathrm{U}(n) = \{\gamma : S^1 \to \mathrm{U}(n) \text{ smooth}: \ \tau\big(\gamma(\lambda)\big) = \gamma(\omega\lambda)\} \tag{5.22}$$

and satisfies the integrability condition $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$. This means that we can integrate to obtain a smooth map $\Psi : M \to \Lambda_\tau\,\mathrm{U}(n)$ such that $\Psi(1,\cdot) = \psi$ and, for each $\lambda \in S^1$, $\varphi_\lambda = \Psi(\lambda,\cdot)x_0$ is a primitive harmonic map; Ψ is called an *extended framing* associated to φ.

Moreover, as in [7], $\Phi = s\Psi\Psi(1,\cdot)^{-1}$ is an extended solution, and a short calculation shows that the original map is recovered via the Cartan embedding by evaluating Φ at $\lambda = \omega$, that is, $\iota \circ \varphi = \Phi(\omega, \cdot)$. Observe that this extended solution takes values in

$$\Omega^\omega \mathrm{U}(n) = \{\gamma \in \Omega \mathrm{U}(n) : \ \gamma(\lambda)\gamma(\omega) = \gamma(\omega\lambda)\}. \tag{5.23}$$

Clearly, given $\gamma \in \Omega^\omega \mathrm{U}(n)$, the corresponding shift-invariant subspace satisfies the symmetry condition (2.5). Then the extended solution $W = \Phi\mathcal{H}_+$ is k-symmetric.

Conversely, by Theorem 4.2, we see that any k-symmetric extended solution W corresponds to a smooth map $\Phi : M \to \Omega^\omega \mathrm{U}(n)$ of the form

$$\Phi(\lambda, \cdot) = \Psi(\lambda^k, \cdot) \sum_{j=0}^{k-1} \pi_j \lambda^j,$$

where π_j is the orthogonal projection onto $\beta_j = \alpha_j \cap \alpha_{j-1}^\perp$ (here we take α_{-1} to be the zero vector bundle and α_{k-1} to be the trivial bundle $M \times \mathbb{C}^n$).

Evaluating at $\lambda = \omega$, we obtain the map

$$\Phi(\omega, \cdot) = \sum_{j=0}^{k-1} \pi_j \omega^j,$$

which can be identified via the Cartan embedding with the map φ with values in $F_{r_0, r_1, \ldots, r_{k-1}}$ given by

$$\varphi = (\beta_0, \beta_1, \ldots, \beta_{k-2}, \beta_{k-1}),$$

where $r_i = \dim \beta_i$. Conditions (i)—(iii) in Theorem 4.2 imply that φ is primitive harmonic map. This can be slightly generalized as follows.

Theorem 5.1. *Let $W = \Phi\mathcal{H}_+$ be a k-symmetric extended solution and let l be a divisor of k. Consider the vector bundles $\beta_i^l = \bigoplus_{j=i \mod l} \beta_j$, and set $s_i = \dim \beta_i^l$. Then*

$$\varphi_l = (\beta_0^l, \beta_1^l, \ldots, \beta_{l-1}^l) : M \to F_{s_0, \ldots, s_{l-1}} \tag{5.24}$$

is a primitive harmonic map.

Proof. If $W = \Phi\mathcal{H}_+$ is a k-symmetric extended solution associated to the primitive harmonic map φ, then for any divisor l of k, $W = \Phi\mathcal{H}_+$ can also be seen as an l-symmetric extended solution. Let $\omega_l := \omega^{k/l}$ be the primitive lth root of unity. Then the smooth map

$$\varphi_l := \Phi(\omega_l, \cdot) = \sum_{i=0}^{l-1} \omega_l^i \sum_{j=i \mod l} \pi_j$$

takes values in a connected component of $\sqrt[l]{I}$ and can be identified, via the Cartan embedding of $F_{s_0, \ldots, s_{l-1}}$, with φ_l given by (5.24). By the previous discussion, φ_l is a primitive harmonic map. $\square$

Remark 5.2. (a) If k is even, the smooth map

$$\varphi_2 = \Phi(\omega,\cdot)^{k/2} = \sum_{j=0}^{k/2-1} \big(\pi_{2j} - \pi_{2j+1}\big)$$

corresponds to a harmonic map φ_2 into the complex Grassmannian $G_m(\mathbb{C}^n)$, with $m = \sum r_{2j}$. In this case, we have $\varphi_2 = p \circ \varphi$, where p is the *canonical homogeneous projection* (see [5, Ch. 4]) of the k-symmetric space $F_{r_0,r_1,\dots,r_{k-1}}$ onto the 2-symmetric space $G_m(\mathbb{C}^n)$ of $\mathrm{U}(n)$.

(b) We point out that, in general, the primitive maps φ_l are different from those of Remark 4.3(f). As a matter of fact, for any $l \leq k$, choose $0 \leq j_0 < j_1 < \ldots < j_{l-2} < j_{l-1} = k-1$. The primitive harmonic map $\tilde{\varphi}_l$ associated to the l-symmetric extended solution (4.16) is given by $\tilde{\varphi}_l = (\tilde{\beta}^l_0, \tilde{\beta}^l_1, \dots, \tilde{\beta}^l_{l-1}) : M \to F_{\tilde{s}_0,\dots,\tilde{s}_{l-1}}$ where

$$\tilde{\beta}^l_i = \bigoplus_{j=j_{i-1}+1}^{j_i} \beta_j, \quad \tilde{s}_i = \dim \tilde{\beta}^l_i.$$

Observe that the isotropy subgroup $\mathrm{U}(\tilde{s}_0)\times\ldots\times\mathrm{U}(\tilde{s}_{l-1})$ of $F_{\tilde{s}_0,\dots,\tilde{s}_{l-1}}$ contains the isotropy subgroup of $F_{r_0,\dots,r_{k-1}}$ and that $\tilde{\varphi}_l = \tilde{p}\circ\varphi$, where $\tilde{p} : F_{r_0,\dots,r_{k-1}} \to F_{\tilde{s}_0,\dots,\tilde{s}_{l-1}}$ is the corresponding homogeneous projection.

Example 5.3. Consider a full holomorphic map $\psi : M \to \mathbb{C}P^3 \hookrightarrow \mathrm{U}(4)$, and let π_ψ denote the orthogonal projection onto ψ. The corresponding extended solution is $\Psi(\lambda,\cdot) = \pi_\psi + \lambda\pi_\psi^\perp$ and we have $(A^\psi_z)^2 = 0$. Set $\alpha_0 = G^{(1)}(\psi)$ and $\alpha_1 = G^{(1)}(\psi) \oplus G^{(2)}(\psi)$. As observed in Example 4.8, these subbundles satisfy the conditions of Theorem 4.2 with $k = 3$. Then we get a 3-symmetric extended solution

$$W = \big(\pi_\psi + \lambda^3\pi_\psi^\perp\big)\big(G^{(1)}(\psi) + \lambda(G^{(1)}(\psi) \oplus G^{(2)}(\psi)) + \lambda^2\mathcal{H}_+\big).$$

Writing $W = \Phi\mathcal{H}_+$, on putting $\lambda = \omega_3$ we get

$$\Phi(\omega_3,\cdot) = \pi_{G^{(1)}(\psi)} + \omega_3\pi_{G^{(2)}(\psi)} + \omega_3^2\pi_{\psi+G^{(3)}(\psi)}$$

which corresponds to the primitive harmonic map

$$\varphi : M \to F_{1,1,2}, \qquad \varphi = \big(G^{(1)}(\psi), G^{(2)}(\psi), \psi \oplus G^{(3)}(\psi)\big).$$

However, W is S^1-invariant; in fact, multiplying out we see that

$$W = \lambda^2\big\{\psi + \lambda(\psi \oplus G^{(1)}(\psi)) + \lambda^2(\psi \oplus G^{(1)}(\psi) \oplus G^{(2)}(\psi)) + \lambda^3\mathcal{H}_+\big\},$$

hence $W = \Phi\mathcal{H}_+$ is k-symmetric for any $k \geq 2$. Now, for any n and k with $2 \leq k \leq n$, there are k-symmetric quotients of $\mathrm{U}(n)$ given by flag manifolds and we can interpret Φ as the Cartan embedding of a primitive harmonic map into such a flag manifold. In the present example, with $k = 4$, $\Phi(\omega_4,\cdot)$ is the primitive harmonic map

$$\varphi : \mathbb{C} \to F_{1,1,1,1}, \qquad \varphi = (\psi, G^{(1)}(\psi), G^{(2)}(\psi), G^{(3)}(\psi));$$

with $k=2$, $\Phi(\omega_2,\cdot)$ is the (primitive) harmonic map given by $\psi\oplus G^{(2)}(\psi)$, in accordance with Remark 5.2(a).

Example 5.4. Let $\psi_0:\mathbb{C}\to\mathbb{C}P^5$ be a Clifford solution, as in Example 4.7. Fix $\psi=\psi_0\oplus G^{(1)}(\psi_0)\oplus G^{(2)}(\psi_0)$ and the bundles α_0 and α_1 given by (4.19), which satisfy the conditions of Theorem 4.2 with respect to ψ and $k=3$. By applying Theorem 5.1 with $l=k$, these define the primitive harmonic map

$$\varphi=\big(\psi_0\oplus G^{(3)}(\psi_0),G^{(1)}(\psi_0)\oplus G^{(4)}(\psi_0),G^{(2)}(\psi_0)\oplus G^{(5)}(\psi_0)\big):\mathbb{C}\to F_{2,2,2}.$$

6. Loop group description

Recall the definitions of $\Lambda_\tau\,\mathrm{U}(n)$ and $\Omega^\omega\,\mathrm{U}(n)$ given by (5.22) and (5.23) respectively. There is a well-known method for obtaining harmonic maps into Lie groups from primitive harmonic maps (see [10, Ch. 21, Sec. IV] and references therein) which makes use of the isomorphism (see also [14, Lemma 5.1]) $\Gamma_\tau:\Lambda\,\mathrm{U}(n)\to\Lambda_\tau\,\mathrm{U}(n)$ given by

$$\Gamma_\tau(\gamma)(\lambda)=\mathrm{Ad}_{s(\lambda)^{-1}}\gamma(\lambda^k)=s(\lambda)^{-1}\gamma(\lambda^k)s(\lambda)$$

with inverse $\Gamma_\tau^{-1}:\Lambda_\tau\,\mathrm{U}(n)\to\Lambda\,\mathrm{U}(n)$ given by

$$\Gamma_\tau^{-1}(\gamma)(\lambda)=\mathrm{Ad}_{s(\lambda^{1/k})}\gamma(\lambda^{1/k})=s(\lambda^{1/k})\gamma(\lambda^{1/k})s(\lambda^{-1/k}).$$

We shall now establish how the subspace V_{k-1} associated to a shift-invariant k-symmetric space W as in Proposition 3.1 can be expressed in terms of Γ_τ. We denote by $\Omega_\tau\,\mathrm{U}(n)$ the subset of $\Omega^\omega\,\mathrm{U}(n)$ defined by: $\Phi\in\Omega_\tau\,\mathrm{U}(n)$ if $\Phi(\omega,\cdot)$ lies in the connected component of $\sqrt[k]{I}$ containing $s(\omega)$.

Lemma 6.1. *The correspondence Θ between left cosets of $\mathrm{U}(n)^\tau$ in $\Lambda_\tau\,\mathrm{U}(n)$ and loops in $\Omega_\tau\,\mathrm{U}(n)$ given by $\Theta\big(\tilde\Phi\,\mathrm{U}(n)^\tau\big)=s\tilde\Phi\tilde\Phi(1)^{-1}$ is bijective.*

Proof. Given $\Phi\in\Omega_\tau\,\mathrm{U}(n)$, there exists $g\in\mathrm{U}(n)$ such that $\Phi(\omega)=gs(\omega)g^{-1}$. It is easy to check that $\tilde\Phi=s^{-1}\Phi g$ is a loop in $\Lambda_\tau\,\mathrm{U}(n)$ and $\Theta\big(\tilde\Phi\,\mathrm{U}(n)^\tau\big)=\Phi$. Thus Θ is surjective.

If $\tilde\Phi,\tilde\Phi'\in\Lambda_\tau\,\mathrm{U}(n)$ are such that $\Theta\big(\tilde\Phi\,\mathrm{U}(n)^\tau\big)=\Theta\big(\tilde\Phi'\,\mathrm{U}(n)^\tau\big)$, then we have $\tilde\Phi(1)^{-1}\tilde\Phi'(1)=\tilde\Phi^{-1}(\lambda)\tilde\Phi'(\lambda)$ for each $\lambda\in S^1$. Applying τ to both sides, we get

$$\tau\big(\tilde\Phi(1)^{-1}\tilde\Phi'(1)\big)=\tilde\Phi^{-1}(\omega\lambda)\tilde\Phi'(\omega\lambda)=\tilde\Phi(1)^{-1}\tilde\Phi'(1),$$

hence $\tilde\Phi(1)^{-1}\tilde\Phi'(1)\in\mathrm{U}(n)^\tau$. This implies that $\tilde\Phi\,\mathrm{U}(n)^\tau=\tilde\Phi'\,\mathrm{U}(n)^\tau$, that is, Θ is injective. □

Proposition 6.2. *Let $W=\Phi\mathcal{H}_+$ be a k-symmetric shift invariant subspace with $\Phi\in\Omega_\tau\,\mathrm{U}(n)$. Take $\tilde\Phi\in\Lambda_\tau\,\mathrm{U}(n)$ such that $\Phi=\Theta\big(\tilde\Phi\,\mathrm{U}(n)^\tau\big)$. Then $V_{k-1}=\Gamma_\tau^{-1}(\tilde\Phi)\mathcal{H}_+$.*

Proof. For $W=\Phi\mathcal{H}_+$ with $\Phi\in\Omega_\tau\,\mathrm{U}(n)$, the element φ_k in Proposition 3.3 is precisely $\Phi(\omega)$ and, by Lemma 6.1, we can write $\Phi=s\tilde\Phi\tilde\Phi(1)^{-1}$ for some $\tilde\Phi\in\Lambda_\tau\,\mathrm{U}(n)$.

Since $\tilde{\Phi} \in \Lambda_\tau \mathrm{U}(n)$, it satisfies $\tau\big(\tilde{\Phi}(\lambda)\big) = \tilde{\Phi}(\lambda\omega)$. Evaluating at $\lambda = 1$, we get $s(\omega)^{-1}\tilde{\Phi}(1)s(\omega) = \tilde{\Phi}(\omega, \cdot)$. Hence, $\Phi(\omega, \cdot) = \tilde{\Phi}(1)s(\omega)\tilde{\Phi}(1)^{-1}$, and we have

$$\sum_{j=0}^{k-1} \pi_{\beta_j}\lambda^{-j} = \tilde{\Phi}(1)s(\lambda)^{-1}\tilde{\Phi}(1)^{-1}$$

with the β_j as in Proposition 3.3. Using this, we obtain

$$\begin{aligned} V_{k-1} = \Phi_k(\lambda^{1/k})\mathcal{H}_+ &= \Phi(\lambda^{1/k})\tilde{\Phi}(1)s(\lambda^{-1/k})\tilde{\Phi}(1)^{-1}\mathcal{H}_+ \\ &= s(\lambda^{1/k})\tilde{\Phi}(\lambda^{1/k})s(\lambda^{-1/k})\mathcal{H}_+ = \Gamma_\tau^{-1}(\tilde{\Phi})\mathcal{H}_+. \end{aligned}$$

□

Remark 6.3. It was already known [10, Ch. 21] that Γ_τ^{-1} is well-behaved with respect to harmonic maps, in the sense that if $\tilde{\Phi} : M \to \Lambda_\tau \mathrm{U}(n)$ is an extended framing (corresponding to a certain primitive harmonic map), then, setting $F := \Gamma_\tau^{-1}(\tilde{\Phi})$, the smooth map $FF_1^{-1} : M \to \Omega\,\mathrm{U}(n)$ is an extended solution (corresponding to a harmonic map into the group $\mathrm{U}(n)$). Our results of §3 and §4 provide a more complete picture of this. In fact, on using Proposition 6.2 to interpret Γ_τ^{-1} in terms of the Grassmannian model and setting $V_{k-1} = \Gamma_\tau^{-1}(W)$, we have the following: The isomorphism Γ_τ^{-1} can be extended to an one-to-one correspondence between k-symmetric shift-invariant subspaces and filtrations $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_{k-1}$ satisfying $\lambda V_{k-1} \subseteq V_0$; this correspondence induces a one-to-one correspondence between k-symmetric extended solutions and λ-cyclic superhorizontal sequences of length k; Theorem 4.2 explains under what conditions the method of obtaining harmonic maps into $\mathrm{U}(n)$ from primitive harmonic maps by making use of Γ_τ^{-1} can be reversed in order to obtain primitive harmonic maps from 2-nilconformal harmonic maps into $\mathrm{U}(n)$.

7. Holomorphic potentials.

In this section, we shall describe how the extended solutions V_j arise via the Dorfmeister, Pedit and Wu [8] method of obtaining harmonic maps from certain holomorphic forms.

Consider the following space of loops:

$$\Lambda_{-1,\infty} = \{\xi \in \Lambda\mathfrak{gl}(n, \mathbb{C})\} : \lambda\xi \text{ extends holomorphically to } |\lambda| < 1\}.$$

A $\Lambda_{-1,\infty}$-valued holomorphic 1-form μ on a simply connected Riemann surface M is called a *holomorphic potential* [8]. In terms of a local coordinate z, we can write $\mu = \xi dz$, for some holomorphic function

$$\xi = \sum_{i=-1}^{\infty} \xi_i\lambda^i : M \to \Lambda_{-1,\infty}.$$

The holomorphicity of μ is equivalent to $\bar{\partial}\mu = 0$. On the other hand, since $\partial\mu$ and $[\mu \wedge \mu]$ are $(2,0)$-forms on a surface, they are both zero. Hence,

$d\mu + \frac{1}{2}[\mu \wedge \mu] = \bar{\partial}\mu = 0$. This means that we can integrate

$$(g^\mu)^{-1}dg^\mu = \mu, \quad g^\mu(0) = I \tag{7.25}$$

to obtain a unique holomorphic map $g^\mu : M \to \Lambda\mathrm{GL}(n, \mathbb{C})$.

Consider the Iwasawa decomposition

$$\Lambda\mathrm{GL}(n, \mathbb{C}) = \Omega\,\mathrm{U}(n)\Lambda^+\mathrm{GL}(n, \mathbb{C}), \tag{7.26}$$

where $\Lambda^+\mathrm{GL}(n, \mathbb{C})$ is the subgroup of loops $\gamma \in \Lambda\mathrm{GL}(n, \mathbb{C})$ which extend holomorphically to $|\lambda| < 1$. We can decompose $g^\mu = \Phi^\mu b^\mu$ according to the Iwasawa decomposition; then $\Phi^\mu : M \to \Omega\,\mathrm{U}(n)$ is an extended solution (see [7, 8]).

The holomorphic potential $\mu = \sum_{i=-1}^{\infty} \xi_i \lambda^i dz$ is called *τ-twisted* if

$$\tau(\xi(\lambda)) = \xi(\omega\lambda).$$

This condition is independent of the choice of local coordinate and equivalent to the following: $\xi_i \in \mathfrak{g}^{i \bmod k}$ for all $i \geq -1$. Now, if we start with a holomorphic τ-twisted potential and proceed as above, we obtain an extended solution Φ^μ satisfying

$$\tau\big(\Phi^\mu(\lambda, \cdot)\big) = \Phi^\mu(\omega\lambda, \cdot)\,\big(\Phi^\mu(\omega, \cdot)\big)^{-1}.$$

Hence, $\Phi = s\Phi^\mu$ takes values in $\Omega^\omega\,\mathrm{U}(n)$. Since Φ is obtained from Φ^μ by left multiplication by a constant loop in $\Omega\,\mathrm{U}(n)$, Φ is also an extended solution. Moreover, since $\Phi^\mu(\cdot, 0) = I$, then $\Phi(\omega, 0) = s(\omega)$, which implies that $\Phi(\omega, \cdot)$ takes values in the connected component of $\sqrt[k]{I}$ containing $s(\omega)$, that is, it corresponds via the Cartan embedding to a primitive harmonic map in $F = F_{r_0,\ldots,r_{k-1}}$, as explained in §5. Observe that, since $b^\mu\mathcal{H}_+ = \mathcal{H}_+$, then the corresponding shift-invariant subspaces are given by

$$W = \Phi\mathcal{H}_+ = sg^\mu\mathcal{H}_+. \tag{7.27}$$

Theorem 7.1. *Consider the k-symmetric space $F = F_{r_0,\ldots,r_{k-1}}$ with base point $x_0 = (A_0, \ldots, A_{k-1})$, $s \in \Omega\,\mathrm{U}(n)$ as in (5.20) and canonical automorphism τ. Let μ be a τ-twisted potential and let $W = \Phi\mathcal{H}_+$ be the corresponding k-symmetric extended solution, with $\Phi = s\Phi^\mu$. For each $0 \leq j \leq k-1$, the V_j of Proposition 4.1 are given by*

$$V_j = \gamma_j \Phi^{\bar{\mu}_j}\mathcal{H}_+$$

where $\bar{\mu}_j = \gamma_j^{-1}\bar{\mu}\gamma_j$,

$$\bar{\mu}(\lambda) = s(\lambda^{1/k})\mu(\lambda^{1/k})s(\lambda^{-1/k}) \tag{7.28}$$

and $\gamma_j(\lambda) = \pi_{\bar{A}_j} + \lambda\pi_{\bar{A}_j}^{\perp}$, with $\bar{A}_j = A_0 \oplus A_1 \oplus \ldots \oplus A_j$. In particular, taking $j = k-1$, since $\gamma_{k-1} = I$, the map Ψ defined pointwise by Proposition 3.1 is given by $\Psi = \Phi^{\bar{\mu}}$.

Proof. In a local coordinate write $\mu = \xi dz$ where $\xi = \sum_{i \geq -1} \xi_i \lambda^i$. For each i, we can write uniquely $i = a_i + m_i k$, with $a_i \in \{0, 1, \ldots, k-1\}$ and $m_i \in \mathbb{Z}$.

If $a_i \neq 0$, we can decompose $\xi_i = \xi_i^+ + \xi_i^-$ accordingly to the decomposition $\mathfrak{g}^{a_i} = \mathfrak{g}_{a_i} \oplus \mathfrak{g}_{a_i-k}$, where

$$\mathfrak{g}_{a_i} = \bigoplus_{j=a_i}^{k-1} \mathrm{Hom}(A_j, A_{j-a_i}), \quad \mathfrak{g}_{a_i-k} = \bigoplus_{j=0}^{a_{i-1}} \mathrm{Hom}(A_j, A_{j+k-a_i}).$$

The automorphism $\sigma(\lambda) = \mathrm{Ad}_{s(\lambda)}$ acts as λ^{-a_i} on $\mathfrak{g}_{a_i}$ and as λ^{k-a_i} on $\mathfrak{g}_{a_i-k}$. Hence,

$$s(\lambda)\xi(\lambda)s^{-1}(\lambda) = \sum_{i\neq m_i k} (\lambda^{m_i k}\xi_i^+ + \lambda^{(1+m_i)k}\xi_i^-) + \sum_{i=m_i k} \lambda^{m_i k}\xi_{m_i k}.$$

Since $m_i \geq -1$ (with equality if and only if $i = -1$), we see that $\bar{\mu}$ as defined above is well defined and takes values in $\Lambda_{-1,\infty}$. The bottom term of $\bar{\mu}$ is given by ξ_{-1}^+.

We also have

$$g^{\bar{\mu}}(\lambda) = s(\lambda^{1/k})g^{\mu}(\lambda^{1/k})s(\lambda^{-1/k}). \tag{7.29}$$

Let $f(\lambda) \in W_j$. Taking Proposition 3.1(i) and equation (7.27) into account, we see that, for some $h \in \mathcal{H}_+$,

$$\begin{aligned} f(\lambda) &= \sum_{l=0}^{k-1} \omega^{-lj} s(\lambda\omega^l)g^{\mu}(\omega^l\lambda)h(\omega^l\lambda) \\ &= \sum_{l=0}^{k-1} \omega^{-lj} s(\lambda)s(\omega^l)g^{\mu}(\omega^l\lambda)s(\omega^{-l})s(\lambda^{-1})s(\lambda)s(\omega^l)h(\omega^l\lambda) \\ &= g^{\bar{\mu}}(\lambda^k)\sum_{i=0}^{k-1} \omega^{-lj} s(\lambda)s(\omega^l)h(\omega^l\lambda). \qquad (7.30) \end{aligned}$$

For the last equality we have used (7.29) and the fact that g^{μ} is τ-twisted, which implies that $s(\omega^l)g^{\mu}(\omega^l\lambda)s(\omega^{-l}) = g^{\mu}(\lambda)$. Now, writing $\pi_{A_i}h(\lambda) = \sum_{r\geq 0} h_{ir}\lambda^r$, we have

$$\begin{aligned} \sum_{l=0}^{k-1} \omega^{-lj} s(\lambda)s(\omega^l)h(\omega^l\lambda) &= \lambda^j \sum_{i,l=0}^{k-1} \omega^{l(i-j)}\lambda^{i-j}\pi_{A_i}h(\omega^l\lambda) \\ &= \lambda^j \sum_{r\geq 0}\sum_{i=0}^{k-1} \lambda^{i-j+r}h_{ir}\sum_{l=0}^{k-1}\omega^{l(i-j+r)}. \end{aligned}$$

Since $\sum_{l=0}^{k-1} \omega^{l(i-j+r)}$ equals k if $i-j+r$ is a multiple of k and 0 otherwise, we see that

$$\sum_{l=0}^{k-1} \omega^{-lj} s(\lambda)s(\omega^l)h(\omega^l\lambda) = \lambda^j(\pi_{\bar{A}_j} + \lambda^k\pi_{\bar{A}_j}^{\perp})\tilde{h}(\lambda^k) \tag{7.31}$$

for some $\tilde{h} \in \mathcal{H}_+$. Hence, from (7.30) and (7.31), we see that any $f(\lambda) \in W_j$ can be written as

$$f(\lambda) = \lambda^j g^{\bar{\mu}}(\lambda^k)(\pi_{\bar{A}_j} + \lambda^k \pi_{\bar{A}_j}^{\perp})\tilde{h}(\lambda^k)$$

for some $\tilde{h} \in \mathcal{H}_+$. According to the definition of V_j, this means that

$$V_j = g^{\bar{\mu}}(\lambda)(\pi_{\bar{A}_j} + \lambda \pi_{\bar{A}_j}^{\perp})\mathcal{H}_+.$$

Finally, observe that $\gamma_j^{-1}\bar{\mu}\gamma_j$ takes values in $\Lambda_{-1,\infty}$. In fact, the λ^{-2}-Fourier coefficient of $\gamma_j^{-1}\bar{\mu}\gamma_j$ is $\pi_{\bar{A}_j}^{\perp}\xi_{-1}^{+}\pi_{\bar{A}_j}$, which is zero since

$$\xi_{-1}^{+} \in \mathfrak{g}_{k-1} = \operatorname{Hom}(A_{k-1}, A_0).$$

Hence, $V_j = \gamma_j g^{\gamma_j^{-1}\bar{\mu}\gamma_j}\mathcal{H}_+$. □

Assume now that M is an open subset of $\mathbb{C}$ and consider the class of holomorphic potentials $\mu = \xi dz$ with $\xi \in \Lambda_{-1,\infty}$ constant. In this case, $g^{\mu} = \exp(\xi z)$. If additionally ξ has a finite Fourier expansion, then the corresponding harmonic map is said to be of *finite type*. The harmonic maps of finite type can also be obtained by using integrable systems methods from a certain Lax-type equation [4, 10] and they play an important role in the theory of harmonic maps from tori into symmetric spaces. For example, it is known (see [15] and references therein) that all non-constant harmonic tori in the n-dimensional Euclidean sphere S^n or the complex projective space $\mathbb{C}P^n$ are either of finite type or of finite uniton number. The following is a direct consequence of Theorem 7.1.

Corollary 7.2. (i) *W corresponds to a constant potential if and only if each V_j corresponds to a constant potential.*
(ii) *W is of finite type if and only if each V_j is of finite type.*

Example 7.3. Consider the harmonic map $\varphi : \mathbb{C} \to \mathbb{C}P^2$ defined in homogeneous coordinates by $\varphi = [F]$ where $F = (F_0, F_1, F_2) : \mathbb{C} \to \mathbb{C}^3$ is given by $F_i(z) = (1/\sqrt{3})\,\mathrm{e}^{\omega^i z - \overline{\omega}^i \overline{z}}$ with $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$.

This is the Clifford solution discussed in [12], see [1, Example 4.14]. A simple calculation shows that the first and second ∂'-Gauss bundles of φ are given by $G^{(1)}(\varphi) = [F^{(1)}]$ and $G^{(2)}(\varphi) = [F^{(2)}]$, respectively, where $F^{(j)}$ stands for the jth derivative of F with respect to z. Moreover, $G^{(3)}(\varphi) = \varphi$.

Let $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2$ be the canonical basis of $\mathbb{C}^3$. For each $j = 0, 1, 2$, let A_j be the one-dimensional complex subspace spanned by $\mathbf{u}_j$. Consider the 3-symmetric space $F_{1,1,1}$ with base point $x_0 = (A_0, A_1, A_2)$, $s \in \Omega\,\mathrm{U}(n)$ and canonical automorphism τ, as defined in §5. Let $g(z)$ be the 3×3 matrix whose $(j+1)$st column is $F^{(j)}(z)$; this defines a lift $g : \mathbb{C} \to \mathrm{U}(3)$ for φ, that is, $\varphi = [g\mathbf{u}_0]$. Moreover, by a direct calculation we see that A_z^g $(= \frac{1}{2}g^{-1}g_z)$ is the constant normal matrix A whose only non-zero entries are $a_{ij} = 1/2$ when $i - j = 1 \mod 3$. Hence A_z^g lies in the eigenspace $\mathfrak{g}^{-1}$ (see (5.21)) of

τ, which means that the map $\phi : \mathbb{C} \to F_{1,1,1}$ given by

$$\phi = gx_0 = (\varphi, G^{(1)}(\varphi), G^{(2)}(\varphi))$$

is a primitive harmonic map associated to the potential $\mu = \lambda^{-1}Adz$. The map g^μ satisfying (7.25) is given by $g^\mu(z) = \exp\big(\lambda^{-1}zA\big)$ and the corresponding extended solution is the *vacuum solution* (as in [1, §4.2]) given by

$$\Phi^\mu(\lambda, z) = \exp\big(z(\lambda^{-1} - 1)A - \bar{z}(\lambda - 1)A^*\big).$$

We recall from §5 that by evaluating $\Phi := s\Phi^\mu$ at $\lambda = \omega$ we obtain the Cartan embedding of the primitive harmonic map $g(0)^{-1}\phi : \mathbb{C} \to F_{1,1,1}$, and

$$g(0)^{-1}\phi(z) = \exp\big(zA - \bar{z}A^*\big)x_0. \tag{7.32}$$

The constant holomorphic potentials $\bar{\mu}_j$ of Theorem 7.1, associated to the extended solutions $V_j = \gamma_j \exp(z\xi_j)\mathcal{H}_+$, with $j = 0, 1, 2$, are then given by $\bar{\mu}_j = \xi_j dz$ where

$$\xi_0 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ \lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \xi_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \end{pmatrix},\ \xi_2 = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & \lambda^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

note that $\xi_j(1) = A$. In particular, with the notations of Theorem 4.2 and Theorem 7.1, we can find the Iwasawa decomposition (7.26) $g^{\bar{\mu}} = \Phi^{\bar{\mu}}b^{\bar{\mu}}$ with extended solution

$$\Psi(\lambda, z) = \Phi^{\bar{\mu}}(\lambda, z) = \exp\big(z\xi_2 - \bar{z}\xi_2^*\big)\exp\big(-zA + \bar{z}A^*\big), \tag{7.33}$$

where $\bar{\mu} = \bar{\mu}_2$. Consider the corresponding harmonic map $\psi = \Psi(-1, \cdot) : \mathbb{C} \to \mathrm{U}(3)$. From (7.33), we compute $A_z^\psi = \frac{1}{2}\psi^{-1}\partial_z\psi$:

$$A_z^\psi = \exp\big(zA - \bar{z}A^*\big)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\exp\big(-zA + \bar{z}A^*\big). \tag{7.34}$$

On the other hand, the smooth subbundles $\alpha_0 \subseteq \alpha_1$ of the trivial bundle $\mathbb{C} \times \mathbb{C}^3$, as defined in Theorem 4.2, are necessarily given by $\alpha_0 = \mathrm{Im}\, A_z^\psi$ and $\alpha_1 = \ker A_z^\psi$. Hence, in view of (7.32) and (7.34), we have

$$\alpha_0 = g(0)^{-1}\varphi, \quad \alpha_1 = g(0)^{-1}\big(\varphi \oplus G^{(1)}(\varphi)\big).$$

In order to find the holomorphic potential $\tilde{\mu} = \tilde{\xi}dz$ of the Clifford solution $\varphi : \mathbb{C} \to \mathbb{C}P^2$, we can either (i) consider the type decomposition $\alpha = \alpha' + \alpha''$ of $\alpha = g^{-1}dg$, write $\alpha' = \alpha'_{-1} + \alpha'_0$ accordingly to the decomposition of $\mathfrak{gl}(n, \mathbb{C})$ induced by the structure of 2-symmetric space of $\mathbb{C}P^2$, as in §5, and take $\tilde{\mu} = \lambda^{-1}\alpha'_{-1} + \alpha'_0$, or (ii), in view of Remark 4.3(f) and Remark 5.2(b), with $l = 2$ and $j_0 = 0$, we can start with the potential $\bar{\mu}_2 = \frac{1}{2}\xi_2 dz$ associated to ψ and reverse (7.28). This gives

$$\tilde{\xi} = \gamma_0(\lambda)^{-1}\xi_2(\lambda^2)\gamma_0(\lambda) = \tfrac{1}{2}\begin{pmatrix} 0 & 0 & \lambda^{-1} \\ \lambda^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

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